Laplace Transform of Fourier Series of Periodic Functions of a Period $P = 2\pi$

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GJSFR-F Classification: MSC 2010: 44A10

Strictly as per the compliance and regulations of:
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Abstract: The authors establish a set of presumably new results, which transform a periodic function of period p = 2π to new functions. So in this paper, the authors try to evaluate Laplace transform of the discontinuous and periodic function that involves in some applications of non-homogeneous differential equations in Physics, electrical engineering, and other many disciplines. Hence, such type of functions expands to a Fourier series, which represent complicated and discontinuous regarding of simpler continuous and periodic functions of cosines and sines.

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1. Introduction

The Laplace Transform is a transformation, that it changes a function into a new function. Because of some of its properties, it is very important in studying linear differential equations. Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform (now called the z-transform) in his work on probability theory.[2] The current widespread use of the transform (mainly in engineering) came about during and soon after World War II [3] although it had been used in the 19th century by Abel, Lerch, Heaviside, and Bromwich. The early history of methods having some similarity to Laplace transform is as follows. From 1744, Leonhard Euler investigated integrals of the form as solutions of differential equations but did not pursue the matter very far.[4] Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form which some modern historians have interpreted within modern Laplace transform theory.[5][6][Clarification needed] These types of integrals attracted Laplace’s attention in 1782 where he was following in the spirit of Euler in using the integrals themselves as solutions of equations. [7] However, in 1785, Laplace took the critical step forward, rather than just looking for a solution in the form of an integral. He started to apply the transforms in the sense that was later to become accepted and transform the whole of a difference equation, to look for solutions to the transformed equation. He then went on to apply the Laplace transform in the same way and started to derive some of its properties, beginning to appreciate its potential power. [8] Laplace also recognized that Joseph Fourier’s method of Fourier series for solving the diffusion equation could only apply.
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II. Laplace Transform of Fourier Series With A Period $P = 2\pi$

**Theorem 1:** (Laplace Transform of Fourier series with a period $p = 2\pi$)

Suppose $f(t)$ a periodic function with a period $p = 2\pi$ is given, then its Fourier series expansion is:

$$f(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt]$$
Such that, the Laplace transform of \(f(t)\) is given in the form of series as:

\[
L(f(t)) = L(a_0 + \sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt])
\]

\[
= \frac{a_0}{s} + \sum_{k=1}^{\infty} \left[ a_k \frac{s}{s^2 + k^2} + b_k \frac{k}{s + k^2} \right]
\]

Where, \(a_0, a_k\) and \(b_k\) are the Fourier coefficients defined in definition 1.2.

**proof**

Suppose \(f(t)\) is a periodic with \(p = 2\pi\) and has a Fourier series expansion; Then,

\[
f(t) = a_0 + \sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt]
\]

\[
\Rightarrow L(f(t)) = L(a_0 + \sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt])
\]

\[
= L(a_0) + L(\sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt])
\]

\[
= \int_0^{\infty} e^{st} a_0 dt + \int_0^{\infty} e^{st} (\sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt]) dt
\]

\[
= \frac{a_0}{s} + \sum_{k=1}^{\infty} \left[ \int_0^{\infty} e^{st} a_k \cos kt dt + \int_0^{\infty} e^{st} b_k \sin kt dt \right]
\]

\[
= \frac{a_0}{s} + \sum_{k=1}^{\infty} \left[ a_k \int_0^{\infty} e^{st} \cos kt dt + b_k \int_0^{\infty} e^{st} \sin kt dt \right]
\]

\[
= \frac{a_0}{s} + \sum_{k=1}^{\infty} \left[ a_k \frac{s}{s^2 + k^2} + b_k \frac{k}{s + k^2} \right]
\]

**Theorem 2:** (Laplace Transform of Fourier Cosine and Sine series with a period \(p = 2\pi\))

A) Suppose \(f(t)\) is even periodic function with a period \(p = 2\pi\), then its Fourier Cosine series expansion is:

\[
f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kt
\]

Where,

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt
\]

\[
a_k = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(kt) dt
\]

Such that, the Laplace transform of \(f(t)\) is given in the form of series as:

\[
L(f(t)) = L(a_0 + \sum_{k=1}^{\infty} a_k \cos kt)
\]
\[
L(f(t)) = \frac{a_0}{s} + \sum_{k=1}^{\infty} a_k \frac{s}{s^2 + k^2}
\]

**Proof**

Assume \( f(t) \) is an even periodic function with a period of \( p = 2\pi \) then its Fourier coefficients are defined as:

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt
\]

\[
a_k = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos kt dt
\]

Where, \( k = 1, 2, 3, ... \)

\( b_k = 0 \)

Hence, \( f(t) \) has a Fourier cosine series expansion;

Then,

\[
f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kt
\]

\( \Rightarrow L(f(t)) = L(a_0 + \sum_{k=1}^{\infty} a_k \cos kt) \)

\[
= L(a_0) + L(\sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt])
\]

\[
= \int_{0}^{\infty} e^{st} a_0 dt + \int_{0}^{\infty} e^{st} (\sum_{k=1}^{\infty} a_k \cos kt) dt
\]

\[
= \frac{a_0}{s} + \sum_{k=1}^{\infty} [\int_{0}^{\infty} e^{st} a_k \cos kt dt]
\]

\[
= \frac{a_0}{s} + \sum_{k=1}^{\infty} [a_k \int_{0}^{\infty} e^{st} \cos kt dt]
\]

\[
= \frac{a_0}{s} + \sum_{k=1}^{\infty} a_k \frac{s}{s^2 + k^2}
\]

**B)** Suppose \( f(t) \) is an odd periodic function with a period \( p = 2\pi \) then its Fourier sine series expansion is:

\[
f(t) = \sum_{k=1}^{\infty} b_k \sin kt
\]

Such that, the Laplace transform of \( f(t) \) is given in the form of series as:

\[
L(f(t)) = L(\sum_{k=1}^{\infty} b_k \sin kt)
\]

\[
= \sum_{k=1}^{\infty} b_k \frac{k}{s^2 + k^2}
\]

**Proof**

Assume \( f(t) \) is odd periodic function with a period of \( p = 2\pi \), then its Fourier coefficients are defined as:

\( a_0 = 0 \) and \( a_k = 0 \)
Conclusion

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\[
b_k = \frac{2}{\pi} \int_0^\pi f(t) \sin kt \, dt
\]

Where, \( k = 1, 2, 3, \ldots \)

Hence, \( f(t) \) has a Fourier sine series expansion;

Then,

\[
f(t) = \sum_{k=1}^\infty b_k \sin kt
\]

\[
\Rightarrow L(f(t)) = L(\sum_{k=1}^\infty a_k \sin kt)
\]

\[
= \int_0^\infty e^{st} (\sum_{k=1}^\infty b_k \sin kt) \, dt
\]

\[
= \sum_{k=1}^\infty \left[ \int_0^\infty e^{st} b_k \sin kt \, dt \right]
\]

\[
= \sum_{k=1}^\infty b_k \int_0^\infty e^{st} \sin kt \, dt
\]

\[
= \sum_{k=1}^\infty b_k \frac{k}{s^2 + k^2}
\]

III. Conclusion

When we solve some applications of non-homogeneous differential equations that involves in Physics, electrical engineering and other many disciplines by using Laplace transformations like Forced oscillations, electrical circuit, periodic rectangular wave and half-wave rectifier and so on, are expands to a Fourier series, which represent a complicated and discontinuous functions regarding simpler continuous and periodic functions of cosines and sines. Therefore it is possible to solve a differential equations that involve such type of functions by using Laplace transformation by expanding them in Fourier series.

Therefore the results on Laplace transform of Fourier series are summarized as follows;

1. If \( f(t) \) a periodic function with a period \( p = 2\pi \) is given, then the Laplace transform of \( f(t) \) is given in the form of series as:

\[
L(f(t)) = \frac{a_0}{s} + \sum_{k=1}^\infty \left[ a_k \frac{s}{s^2 + k^2} + b_k \frac{k}{s^2 + k^2} \right]
\]

2. If \( f(t) \) is an even periodic function with a period \( p = 2\pi \), then the Laplace transform of \( f(t) \) is given in the form of series as:

\[
L(f(t)) = \frac{a_0}{s} + \sum_{k=1}^\infty a_k \frac{s}{s^2 + k^2}
\]

3. If \( f(t) \) is odd periodic function with a period \( p = 2\pi \), then the Laplace transform of \( f(t) \) is given in the form of series as:

\[
L(f(t)) = \sum_{k=1}^\infty b_k \frac{k}{s^2 + k^2}
\]
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