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Laplace Transform of Fourier Series of Periodic Functions of a Period P = 2 π

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Keywords: laplace transforms, fourier series.

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7. Differential Equations for Engineers, Wei-Chau Xie, 2010

Laplace Transform of Fourier Series of Periodic Functions of a Period P $= 2\pi$

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Abstract- The authors establish a set of presumably new results, Which transform a periodic functions of period p $= 2\pi$ to new functions. So in this paper, the authors tries to evaluate Laplace transform of the discontinuous and periodic function that involves in some applications of non-homogeneous differential equations in Physics, electrical engineering, and other many disciplines. Hence, such type of functions expands to a Fourier series, which represent complicated and discontinuous regarding of simpler continuous and periodic functions of cosines and sines.

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I. Introduction

The Laplace Transform is a transformation, that it changes a function into a new function. Because of some its properties, it is very important in studying linear differential equations. Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform (now called the z-transform) in his work on probability theory.[2] The current widespread use of the transform (mainly in engineering) came about during and soon after World War II [3] although it had been used in the 19th century by Abel, Lerch, Heaviside,and Bromwich. The early history of methods having some similarity to Laplace transform is as follows. From 1744, Leonhard Euler investigated integrals of the form as solutions of differential equations but did not pursue the matter very far.[4] Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form which some modern historians have interpreted within modern Laplace transform theory.[5][6][Clarication needed] These types of integrals attracted Laplace's attention in 1782 where he was following in the spirit of Euler in using the integrals themselves as solutions of equations. [7] However, in 1785, Laplace took the critical step forward, rather than just looking for a solution in the form of an integral. He started to apply the transforms in the sense that was later to become accepted and transform the whole of a difference equation, to look for solutions to the transformed equation. He then went on to apply the Laplace transform in the same way and started to derive some of its properties, beginning to appreciate its potential power. [8] Laplace also recognized that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply

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to a limited region of space because those solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space.[9] When we solve some applications of non-homogeneous differential equations that involves Physics, electrical engineering, and other many disciplines by using Laplace transformations like Forced oscillations of a body of mass m on a spring of modulus k in Physics, mathematically defined by using ordinary differential equation as:
Where $r(t)$ is a discontinuous and periodic function defined as:

$$r(t) = t + \tfrac{\pi}{2}, \text{ if } -\pi < t < 0 \text{ and } r(t) = -t + \tfrac{\pi}{2}, \text{ if } 0 < t < \pi$$

where, $r(t+2\pi) = r(t)$
Hence, such type of functions expand to a Fourier series, which represent a complicated and discontinuous functions in terms of simpler continuous functions and periodic functions of cosines and sines.

a) Definition

The Laplace Transform of a function $f(t)$ defined for all $t \geq 0$, is the integral

$$F(s) = \int_0^{\infty} e^{-st} f(t)dt$$

The function F(s) is called the Laplace transform of the function $f(t)$. Denoted by $L(f(t)$.
where $s \in \Re$ is any real number such that the integral above converges.
The operation which yields $F(s)$ from a given $f(t)$ is called Laplace transformation. $f(t)$ is called the inverse transform or inverse of $F(s)$.

b) Definition

A Fourier series of a periodic functions of a period, $p = 2\pi$ is given by:

$$f(t) = a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt]$$

where a_0, a_k and b_k are called the Fourier coefficients and defined as:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt$$

Where, $k = 1, 2, 3, ...$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt$$

Where, $k = 1, 2, 3, ...$

II. Laplace Transform Of Fourier Series With A Period $P = 2\pi$

Theorem 1: (Laplace Transform of Fourier series with a period $p = 2\pi$
Suppose $f(t)$ a periodic function with a period $p = 2\pi$ is given,then its Fourier series expansion is:

$$f(t) = a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt]$$

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9. Historically, how and why was the Laplace Transform invented? Written 18 Oct 2015 From Wikipedia.

Notes

Such that, the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \frac{s}{s^2+k^2} + b_k \frac{k}{s+k^2}]$$

Where, a_0, a_k and b_k are the Fourier coefficients defined in definition1.2.

proof

Suppose $f(t)$ is a periodic with $p = 2\pi$ and has a Fourier series expansion; Then,

$$f(t) = a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt]$$

$$\Rightarrow L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$

$$= L(a_0) + L(\Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$

$$= \int_0^{\infty} e^{st} a_0 dt + \int_0^{\infty} e^{st}(\Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])dt$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[\int_0^{\infty} e^{st}(a_k \cos kt + b_k \sin kt)dt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[\int_0^{\infty} e^{st} a_k \cos ktdt + \int_0^{\infty} e^{st} b_k \sin ktdt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \int_0^{\infty} e^{st} \cos ktdt + b_k \int_0^{\infty} e^{st} \sin ktdt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \frac{s}{s^2+k^2} + b_k \frac{k}{s+k^2}]$$

Theorem 2: (Laplace Transform of Fourier Cosine and Sine series with a period $p = 2\pi$

A) Suppose $f(t)$ is even periodic function with a period $p = 2\pi$,then its Fourier Cosine series expansion is:

$$f(t) = a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt$$

Where,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(kt)dt$$

Such that, the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt)$$

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$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty} a_k \frac{s}{s^2+k^2}$$

Proof

Assume $f(t)$ is an even periodic function with a period of $p = 2\pi$ then its Fourier coefficients are defined as:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_k = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos ktdt$$

Where, $k = 1, 2, 3, \ldots$

$$b_k = 0$$

Hence, $f(t)$ has a Fourier cosine series expansion;
Then,

$$f(t) = a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt$$

$$\Rightarrow L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt)$$

$$= L(a_0) + L(\Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$

$$= \int_0^{\infty} e^{st} a_0 dt + \int_0^{\infty} e^{st}(\Sigma_{k=1}^{\infty} a_k \cos kt)dt$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[\int_0^{\infty} e^{st} a_k \cos ktdt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \int_0^{\infty} e^{st} \cos ktdt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty} a_k \frac{s}{s^2+k^2}$$

B) Suppose $f(t)$ is an odd periodic function with a period $p = 2\pi$,then its Fourier sine series expansion is:

$$f(t) = \Sigma_{k=1}^{\infty} b_k \sin kt$$

Such that, the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = L(\Sigma_{k=1}^{\infty} b_k \sin kt)$$

$$= \Sigma_{k=1}^{\infty} b_k \frac{k}{s^2+k^2}$$

Proof

Assume $f(t)$ is odd periodic function with a period of $p = 2\pi$, then its Fourier coefficients are defined as:
$a_0 = 0$ and $a_k = 0$

Notes

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin ktdt$$

Where, $k = 1, 2, 3, ...$

Hence, $f(t)$ has a Fourier sine series expansion;
Then,

$$f(t) = \Sigma_{k=1}^{\infty} b_k \sin kt$$

Notes

$$\Rightarrow L(f(t)) = L(\Sigma_{k=1}^{\infty} a_k \sin kt)$$

$$= \int_0^{\infty} e^{st}(\Sigma_{k=1}^{\infty} b_k \sin kt)dt$$

$$= \Sigma_{k=1}^{\infty}[\int_0^{\infty} e^{st} b_k \sin ktdt]$$

$$= \Sigma_{k=1}^{\infty}[b_k \int_0^{\infty} e^{st} \sin ktdt]$$

$$= \Sigma_{k=1}^{\infty} b_k \frac{k}{s^2+k^2}$$

III. Conclusion

When we solve some applications of non-homogeneous differential equations that involves in Physics, electrical engineering and other many disciplines by using Laplace transformations like Forced oscillations, electrical circuit, periodic rectangular wave and half-wave rectifier and so on, are expands to a Fourier series, which represent a complicated and discontinuous functions regarding simpler continuous and periodic functions of cosines and sines. Therefore it is possible to solve a differential equations that involve such type of functions by using Laplace transformation by expanding them in Fourier series.

Therefore the results on Laplace transform of Fourier series are summarized as follows;

1. If $f(t)$ a periodic function with a period $p = 2\pi$ is given,then the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \frac{s}{s^2+k^2} + b_k \frac{k}{s+k^2}]$$

2. I $f(t)$ is an even periodic function with a period $p = 2\pi$,then the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) == \frac{a_0}{s} + \Sigma_{k=1}^{\infty} a_k \frac{s}{s^2+k^2}$$

3.I $f(t)$ is odd periodic function with a period $p = 2\pi$,then the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = \Sigma_{k=1}^{\infty} b_k \frac{k}{s^2+k^2}$$

References Références Referencias

1. Ordinary Differential equations, GABRIEL NAGY, Mathematics Department, SEPTEMBER 14, 2015
2. Introduction to Ordinary Differential Equations and Some Applications, Edward Burkard
3. Advanced Engineering Mathematics, Erwin Kreyszing, Herbert Kreyszing, Edward J. Norminton 10th Edition
4. A first Course in Differential equations, Rudolph E. Longer, 1954
5. Differential Equation and Integral Equations, Peter J. Collins, 2006
6. Differential Equations, James R. Brannan, William E. Boyce, 2nd edition
7. Differential Equations for Engineers, Wei-Chau Xie, 2010
8. Advanced Engineering Mathematics 7th Edition, PETER V. ONEIL
9. Historically, how and why was the Laplace Transform invented? Written 18 Oct 2015 From Wikipedia:

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