Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 1 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Oscillations of Second order Impulsive Differential Equations with Advanced Arguments

By Ubon Akpan Abasiekwere, Ita Micah Esuabana, Idongesit Okon Isaac & Zsolt Lipcsey

University of Uyo

Abstract- A comparison theorem providing sufficient conditions for the oscillation of all solutions of a class of second order linear impulsive differential equations with advanced argument is formulated. A relation between the oscillation (non-oscillation) of second order impulsive differential equations with advanced arguments and the oscillation (non-oscillation) of the corresponding impulsive ordinary differential equations is established by means of the Lebesgue dominated convergence theorem. Obtained comparison principle essentially simplifies the examination of the studied equations.

Keywords: impulsive differential equations, comparison theorem, advanced arguments, second order, oscillation.

GJSFR-F Classification: MSC 2010: 31B35

OSCILLATIONSOFSECONDORDERIMPULSIVEDIFFERENTIALEQUATIONSWITHADVANCEDARGUMENTS

Strictly as per the compliance and regulations of:

Notes

Oscillations of Second order Impulsive Differential Equations with Advanced Arguments

Ubon Akpan Abasiekwere [α], Ita Micah Esuabana [σ], Idongesit Okon Isaac [ρ] & Zsolt Lipcsey [ω]

Abstract- A comparison theorem providing sufficient conditions for the oscillation of all solutions of a class of second order linear impulsive differential equations with advanced argument is formulated. A relation between the oscillation (non-oscillation) of second order impulsive differential equations with advanced arguments and the oscillation (non-oscillation) of the corresponding impulsive ordinary differential equations is established by means of the Lebesgue dominated convergence theorem. Obtained comparison principle essentially simplifies the examination of the studied equations.

Keywords: impulsive differential equations, comparison theorem, advanced arguments, second order, oscillation.

I. Introduction

Impulsive differential equations with deviating arguments are adequate mathematical models of numerous processes studied in physics, biology, electronics, etc. In spite of the great possibilities for applications, the theory of these equations is developing rather slowly due to the obstacles of technical and theoretical character arising in the investigation of impulsive differential equations.

Oscillation theory is one of the directions which initiated the investigations of the qualitative properties of differential equations. This theory started with the classical works of Sturm and Kneser, and still attracts the attention of many mathematicians as much for the interesting results obtained as for their various applications.

In recent few decades, the number of investigations of the oscillatory properties of the solutions of functional differential equations have been constantly growing (Ladde *et al.*, 1987, Samoilenko and Perestyuk, 1987, Gyori and Ladas, 1991, Erbe *et al.*, 1995,). In 1989 the paper of Gopalsamy and Zhang was published, where the first investigation on oscillatory properties of impulsive differential equations

Author α: (corresponding author) Department of Mathematics and Statistics, University of Uyo, P.M.B. 1017, Uyo, Akwa Ibom State, Nigeria. e-mail: ubeeservices@yahoo.com
Author σ: Department of Mathematics, University of Calabar, P.M.B. 1115, Calabar, Cross River State, Nigeria.
e-mail: esuabanaita@gmail.com
Author ρ: Department of Mathematics/Statistics, Akwa Ibom State University, P.M.B. 1167, Ikot Akpaden, Akwa Ibom State, Nigeria.
e-mail: idonggrace@yahoo.com
Author ω: Department of Mathematics, University of Calabar, P.M.B. 1115, Calabar, Cross River State, Nigeria.
e-mail: zlipcsey@yahoo.com

with deviating arguments was carried out (Gopalsamy and Zhang, 1989). However, not much has been done in the study of oscillation theory of second order impulsive differential equations with advanced arguments. This paper therefore is targeted at filling this gap by examining sufficient conditions for oscillation of the solutions of a class of second order linear impulsive differential equations with advanced arguments.

Before the formulation of the problem considered in this study, we present some basic definitions and concepts that will be useful throughout our discussions.

Usually, the solution $y(t)$ for $t\in[t_0,T)$ of a given impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k\in[t_0,T)$, $t_k\neq t$. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r\in N$, $D:=[T,\infty)\subset R$ and let $S:=\{t_k\}_{k\in E}$,where E is our subscript set which can be the set of natural numbers N or the set of integers Z, be fixed. Throughout this discussion,we will assume that the elements of the sequence $S:=\{t_k\}_{k\in E}$ are the moments of impulsive effects and satisfy the following properties:

C1.1: If $\{t_k\}$ is defined for all $k\in N$, then $0<t_1<t_2<\cdots$ and $\lim_{k\to\infty} t_k=+\infty$.

C1.2: If $\{t_k\}$ is defined for all $k\in Z$, then $t_0\leq 0<t_1$, $t_k<t_{k+1}$ for $k\in Z$, $k\neq 0$ and $\lim_{k\to\pm\infty} t_k=\pm\infty$.

We denote by $PC(D,R)$ the set of all values $\psi:D\to R$ which is continuous for all $t\in D$, $t\notin S$. They are functions from the left and have discontinuity of the first kind at the points for $t\in S$. By $PC^r(D,R)$, we denote the set of functions $\psi:D\to R$ having derivative $\frac{d^j\psi}{dt^j}\in PC(D,R)$, $0\leq j\leq r$ (Lashmikantham *et al.*, 1989, Bainov and Simeonov, 1998). To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S)$, $r\in N$ (Isaac and Lipcsey, 2009, Isaac and Lipcsey, 2010a, Isaac and Lipcsey, 2010b).

Now, let (P) be some property of the solution of $y(t)$ of an impulsive differential equation, which can be fulfilled for some $t\in R$. Hereafter, we shall say that the function $y(t)$ enjoys the property (P) *finally*, if there exists $T\in R$ such that $y(t)$ enjoys the property (P) for all $t\geq T$ (Bainov and Simeonov, 1998).

Definition 1.1: The solution $y(t)$ of an impulsive differential equation is said to be

i) Finally positive (finally negative) if there exist $T\geq 0$ such that $y(t)$ is defined and is strictlypositive(negative) for $t\geq T$ (Isaac *et al.*, 2011);
ii) Non-oscillatory, if it is either finally positive or finally negative; and
iii) Oscillatory, if it is neither finally positive nor finally negative (Bainov and Simeonov, 1998, Isaac and Lipcsey, 2010a).

II. Statement of the Problem

In this work, we seek sufficient conditions for the oscillation of the solutions of the linear impulsive differential equation with advanced argument of the form

Notes

$$\begin{cases}\left(r(t)y'(t)\right)'+q(t)y\left(\tau(t)\right)=0, \ t\notin S\\ \Delta\left(r(t_k)y'(t_k)\right)+q_k y\left(\tau(t_k)\right)=0, \ \forall\, t_k\in S,\end{cases} \tag{2.1}$$

where $0\le t_0<t_1<\cdots<t_k<\cdots$ with $\lim_{k\to+\infty} t_k=+\infty$, $\Delta y^{(i)}(t_k)=y^{(i)}(t_k^+)-y^{(i)}(t_k^-)$, $i=0,1$ and $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t=t_k$, respectively. For the sake of definiteness, we shall suppose that the functions $y(t)$ and $y'(t)$ are continuous from the left at the points t_k such that $y'(t_k^-)=y'(t_k)$, $y(t_k^-)=y(t_k)$ and $\Delta(r(t_k)y'(t_k))=r(t_k^+)y'(t_k^+)-r(t_k)y'(t_k)$.

Without further mentioning, we will assume throughout this paper the following conditions:

C2.1: $\tau\in C([t_0,\infty),R)$, τ is a non-decreasing function in R_+, $\tau(t)\ge t$ for $t\in R_+$ and $\lim_{t\to\infty}\tau(t)=+\infty$;

C2.2: $r\in PC^1\left([t_0,\infty),R_+\right)$ and $r(t)>0$, $r\left(t_k^+\right)>0$, for $t,t_k\in R_+$;

C2.3: $q\in PC\left([t_0,\infty),R_+\right)$ and $q_k\ge 0$, $k\in N$;

C2.4: $\int_0^\infty \frac{dt}{r(t)}=\infty.$.

Using the method of steps, the real valued function $y(t)$ is said to be the solution of equation (2.1) if there exists a number $t_0\in R$ such that $y(t)\in PC([t_0-\tau(t_0),\infty),R)$, the function $r(t)y'(t)$ is continuously differentiable for $t\ge t_0-\tau(t_0)$, $t\ne t_k$, $k\in N$ and $y(t)$ satisfies equation (2.1) for all $t\ge t_0-\tau(t_0)$. We remark that every solution $y(t)$ of equation (2.1) that is under consideration here, is continuable to the right and is nontrivial. That is, $y(t)$ is defined on some half-line $[T_y,\infty)$ and $\sup\left\{|y(t)|:\ t\ge T\right\}>0$ for all $T\ge T_y$. Such a solution is called a regular solution of equation (2.1). Equation (2.1) is said to be oscillatory if all is solutions are oscillatory.

Before proceeding, we establish the following lemmas which will be useful in proving the main results. The lemmas are extensions of Erbe '*s* work on pages 284-287 of his monograph (Erbe *et al.*, 1995).

Lemma 2.1: Assume that

$$\int_{t_0}^{\infty} q(t)dt+\sum_{t_0\le t_k<\infty} q_k<\infty. \tag{2.2}$$

Let $y(t)>0$, $t\ge t_1$, be a solution of equation (2.1). Set

$$\omega(t)=\frac{r(t)y'(t)}{y(t)}. \tag{2.3}$$

Notes

Then $\omega(t)>0,\ \lim_{t\to\infty}\omega(t)=0,$

$$\int_{t_1}^{\infty}\frac{\omega^2(t)}{r(t)}dt+\sum_{t_1\le t_k<\infty}\frac{\omega^2(t_k)}{r(t_k)}\le\infty \tag{2.4}$$

and

$$\omega(t)=\int_t^{\infty}\frac{\omega^2(s)}{r(s)}ds+\sum_{t\le t_k<\infty}\frac{\omega^2(t_u)}{r(t_k)}+\int_t^{\infty}q(s)exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)ds+$$

$$+\sum_{t\le t_k<\infty}q_k\,exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right). \tag{2.5}$$

Proof: From equation (2.1), bearing condition (2.2) in mind, we obtain

$$\begin{cases}\left[y(t)\omega(t)\right]'+q(t)y(\tau(t))=0,\ t\notin S\\ \Delta\left[y(t_k)\omega(t_k)\right]+q_k y(\tau(t_k))=0,\ \forall\, t_k\in S,\end{cases}$$

since

$$\frac{y(\tau(t))}{y(t)}=exp\left(\int_t^{\tau(t)}\frac{\omega(s)}{r(s)}ds+\sum_{t\le t_k<\tau(t)}\frac{\omega(t_k)}{r(t_k)}\right),$$

and

$$\omega'(t)+\frac{\omega^2(t)}{r(t)}+q(t)exp\left(\int_t^{\tau(t)}\frac{\omega(s)}{r(s)}ds+\sum_{t\le t_k<\tau(t)}\frac{\omega(t_k)}{r(t_k)}\right)=0. \tag{2.6}$$

Integrating equation (2.6) from t to T for $T\ge t\ge t_1$, we have

$$\omega(T)-\omega(t)+\int_t^{T}\frac{\omega^2(s)}{r(s)}ds+\sum_{t\le t_k<T}\frac{\omega^2(t_k)}{r(t_k)}+\int_t^{T}q(s)exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)ds+$$

$$+\sum_{t\le t_k<T}q_k\,exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)=0. \tag{2.7}$$

Because $r(t)y'(t)>0$, so $\omega(t)>0$. We shall show that $\lim_{t\to\infty}\omega(t)=0.$ In fact, if $\lim_{t\to\infty}r(t)y'(t)=c>0$, then there exists a $t_2\ge t_1$ such that for $t\ge t_2$,

Notes

$$y(t) \geq \left[y(t_2) + \int_{t_2}^{t} \frac{c}{2r(s)} ds + \sum_{t_2 \leq t_k < \tau} \frac{c}{2r(t_k)} \right] \to \infty, \quad t \to \infty,$$

and hence, $\lim_{t\to\infty} \omega(t) = 0$. If $\lim_{t\to\infty} r(t) y'(t) = 0$, then $\lim_{t\to\infty} \omega(t) = 0$ also. Letting $T \to \infty$, in equation(2.7), we obtain condition (2.5). This completes the proof of Lemma 2.1.

Notes

Lemma 2.2: Equation (2.1) has a non-oscillatory solution if and only if there exists a positive differential function $\phi(t)$ such that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \leq -q(t) exp\left(\int_t^{\tau(t)} \frac{\omega(s)}{r(s)} ds + \sum_{t \leq t_k < \tau(s)} \frac{\omega(t_k)}{r(t_k)} \right), \quad t \geq t_2. \tag{2.8}$$

Proof: The necessity follows from Lemma 2.1. Now we assume that inequality (2.8) holds. Then, $\phi'(t) < 0$ and hence $\lim_{t\to\infty} \phi(t) = -\infty$, a contradiction. Therefore, $\lim_{t\to\infty} \phi(t) = 0$. Integrating inequality (2.8) from t to ∞, we obtain

$$\int_t^{\infty} \frac{\phi^2(s)}{r(s)} ds + \sum_{t \leq t_k < \infty} \frac{\phi^2(t_k)}{r(t_k)} + \int_t^{\infty} q(s) exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)} du + \sum_{s \leq t_k < \tau(s)} \frac{\phi(t_k)}{r(t_k)} \right) +$$

$$+ \sum_{t \leq t_k < \infty} q_k \, exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)} ds + \sum_{s \leq t_k < \tau(s)} \frac{\phi(t_k)}{r(t_k)} \right) \leq \phi(t), \; t \geq t_2 \tag{2.9}$$

which implies that

$$\int_t^{\infty} \frac{\phi^2(s)}{r(s)} ds + \sum_{t \leq t_k < \infty} \frac{\phi^2(t_k)}{r(t_k)} < \infty$$

and

$$\int_t^{\infty} q(s) exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)} du + \sum_{s \leq t_k < \tau(s)} \frac{\phi(t_k)}{r(t_k)} \right) ds + \sum_{t \leq t_k < \infty} q_k \, exp\left(\int_s^{\tau(s)} \frac{\phi(u)}{r(u)} du + \sum_{s \leq t_k < \tau(s)} \frac{\phi(t_k)}{r(t_k)} \right) < \infty.$$

For all functions $x(t)$ satisfying $0 \leq x(t) \leq \phi(t), \; t \geq t_2$, define a mapping J by

$$(Jx)(t) = \int_t^{\infty} \frac{x^2(s)}{r(s)} ds + \sum_{t \leq t_k < \infty} \frac{x^2(t_k)}{r(t_k)} + \int_t^{\infty} q(s) exp\left(\int_s^{\tau(s)} \frac{x(u)}{r(u)} du + \sum_{s \leq t_k < \tau(s)} \frac{x(t_k)}{r(t_k)} \right) ds +$$

$$+ \sum_{t \leq t_k < \infty} q_k \, exp\left(\int_s^{\tau(s)} \frac{x(u)}{r(u)} du + \sum_{s \leq t_k < \tau(s)} \frac{x(t_k)}{r(t_k)} \right), \quad t \geq t_2.$$

Global Journal of Science Frontier Research (F) Volume XVIII Issue I Version I Year 2018

If is easy to see that $0\le x_1(t)\le x_2(t),\ \ t\ge t_2$, implies $(Jx_1)(t)\le(Jx_2)(t),\ t\ge t_2$.

Define $y_0(t)\equiv 0$ and $y_n(t)=(Jy_{n-1})(t),\ n=1,2,\cdots$. Then $y_{n-1}(t)\le y_n(t)\le\phi(t),\ n=1,2,\cdots$, and $\lim_{n\to\infty} y_n(t)=\omega(t)\le\phi(t)$. By the Lebesgue dominated convergence theorem, we have

$$\omega(t)=\int_t^{\infty}\frac{\omega^2(s)}{r(s)}ds+\sum_{t\le t_k<\infty}\frac{\omega^2(t_k)}{r(t_k)}+\int_t^{\infty}q(s)\exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)ds+$$

$$+\sum_{t\le t_k<\infty}q_k\exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right),\ t\ge t_2.$$

Global Journal of Science Frontier Research (F) Volume XVIII Issue I Version I Year 2018

Notes

Set

$$y(t)=\exp\left(\int_{t_2}^{t}\frac{\omega(u)}{r(u)}du+\sum_{t_2<t_k<t}\frac{\omega(t_k)}{r(t_k)}\right),\ t\ge t_2.$$

Then

$$\omega(t)=\frac{r(t)y'(t)}{y(t)}$$

and

$$\begin{cases}(r(t)y'(t))'+q(t)y(\tau(t))=0, & t\ge t_2,\ t\notin S\\ \Delta(r(t_k)y'(t_k))+q_k y(\tau(t_k))=0, & t_k\ge t_2,\ \forall\, t_k\in S,\end{cases}$$

that is, $y(t)$ is a non-oscillatory solution of equation (2.1). This completes the proof of Lemma 2.2.

In what follows we try to deduce the oscillatory conditions on equation taking advantage of the above lemmas.

III. Results

The following theorems are extensions of Theorem 4.9.1 and Theorem 4.9.2 as identified on pages 284 and 287 of the monograph by Erbe (Erbe *et al.*, 1995).Without loss of generality, we will deal only with the positive solutions of equation (2.1) where applicable.

Theorem 3.1: Assume that

$$\int_{t_0}^{\infty}q(t)dt+\sum_{t_0\le t_k<\infty}q_k=\infty. \tag{3.1}$$

Then every solution of equation (2.1) is oscillatory.

Proof: Let us assume, by contradiction, that $y(t)$ is a finally positive solution of equation (2.1). Onecan see that $r(t)y'(t)>0$ for $t\ge T\ge t_0$. Then

$$\int_{T}^{\infty} q(t)y(\tau(t))dt + \sum_{T\le t_k<\infty} q_k y(\tau(t_k)) < \infty \quad (3.2)$$

which contradicts equation (3.1). Therefore, every solution of equation (2.1) is oscillatory. This completes the proof of Theorem 3.1.

Notes

Theorem 3.2: If equation (2.1) has a non-oscillatory solution, then the second order linear impulsive differential equation

$$\begin{cases} (r(t)y'(t))' + q(t)y(t) = 0, & t \notin S \\ \Delta(r(t_k)y'(t_k)) + q_k y(t_k) = 0, & \forall\, t_k \in S \end{cases} \quad (3.3)$$

is non-oscillatory. Conversely, if equation (3.3) is oscillatory, then every solution of equation (2.1) is oscillatory.

Proof: Assume that equation (2.1) has a non-oscillatory solution. By Lemma 2.2, there exists a positive differential function $\phi(t)$ such that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \le -q(t)\exp\left(\int_{t}^{\tau(t)} \frac{\phi(u)}{r(u)}du + \sum_{t\le t_k<\tau(t)} \frac{\phi(t_k)}{r(t_k)}\right),\ t \ge t_2, \quad (3.4)$$

which implies that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \le -q(t). \quad (3.5)$$

Taking advantage of Lemma 2.2 for the case in which $h(t) \equiv t$, equation (3.3) is non-oscillatory.

Consequently, the second part of the theorem is immediately obtained. This completes the proof of Theorem 3.2.

IV. Conclusion

It has become imperative in recent times to determine the properties of the solutions of certain mathematical equations from the knowledge of associated equations. In this work, we established a comparison theorem which compares theimpulsive differential equation with advanced argument (2.1) with the impulsive ordinary differential equation (3.3), in the sense that the oscillation (non-oscillation) of the impulsive ordinary differential equation (3.3) guarantees the oscillation (non-oscillation) of the impulsive differential equation with advanced argument (2.1). The formulated comparison theorem essentially simplifies the examination of the oscillatory properties of equation (2.1) and enables us also to eliminate some conditions imposed on the given problem.

References Références Referencias

1. Bainov, D. D. & Simeonov, P. S. (1998). Oscillation Theory of Impulsive Differential Equations, International Publications, Orlando, Florida.
2. Erbe, L. H., Kong, Q. & Zhang, B. G. (1995). Oscillation Theory for Functional Differential Equations, Dekker, New York.

3. Gopalsamy, K. & Zhang, B. G. (1989). *On delay differential equations with impulses,* J. Math. Anal. Appl., 139, 110-122.
4. Gyori, I. & Ladas, G. (1991). Oscillation Theory of Delay Differential Equations with Applications, Clarendon, Oxford.
5. Isaac, I. O. & Lipcsey, Z. Linearized Oscillations in Nonlinear Neutral Delay Impulsive Differential Equations, Journal of Modern Mathematics and Statistics, 3, (2009), 17-21.
6. Isaac, I. O. & Lipcsey, Z. (2010a). Oscillations of Scalar Neutral Impulsive Differential Equations of the First Order with variable Coefficients, Dynamic Systems and Applications,19, 45-62.
7. Isaac, I. O. & Lipcsey, Z. (2010b). Oscillations in Linear Neutral Delay Impulsive Differential Equations with Constant Coefficients, Communications in Applied Analysis, 14, 123-136.
8. Isaac, I. O., Lipcsey, Z. & Ibok, U. J. (2011). Nonoscillatory and Oscillatory Criteria for First Order Nonlinear Neutral Impulsive Differential Equations; Journal of Mathematics Research; Vol. 3 Issue 2, 52-65.
9. Ladde, G. S., Lakshmikantham, V. & Zhang, B. G. (1987). Oscillation Theory of Differential Equations with Deviating Arguments, Dekker, New York.
10. Lakshmikantham, V., Bainov, D. D. & Simeonov, P. S. (1989). Theory of Impulsive Differential Equations, World Scientific, Singapore.
11. Samoilenko, A. N. & Perestynk, N. A. (1987). Differential Equations with Impulse Effect, Visca Skola, Kiev.

Notes