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A New Subclass of Univalent Functions

By Gagandeep Singh, Gurcharanjit Singh & Harjinder Singh
Majha College for Women

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ANEWSUBCLASSOFUNIVALENTFUNCTIONS

Strictly as per the compliance and regulations of:

A New Subclass of Univalent Functions

Gagandeep Singh [α], Gurcharanjit Singh [σ] & Harjinder Singh [ρ]

Abstract- In this paper, a new subclass $\chi_t(A,B)$ of close-to-convex functions, defined by means of subordination is investigated. Some results such as coefficient estimates, inclusion relations, distortion theorems, radius of convexity and Fekete- Szegö problem for this class are derived. The results obtained here is extension of earlier known work.

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I. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{1.1}$$

which are analytic and univalent in the open unit disc $E=\{z:|z|<1\}$.

Let U be the class of bounded functions

$$w(z)=\sum_{n=1}^{\infty}c_n z^n, \tag{1.2}$$

which are regular in the unit disc and satisfying the conditions

$$w(0)=0 \quad \text{and} \quad |w(z)|<1 \text{ in } E.$$

For functions f and g analytic in E, we say that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function $w(z)\in U$, $w(z)$ analytic in E with $w(0)=0$ and $|w(z)|<1$ in E, such that $f(z)=g(w(z))$.

By S, S^* and C we denote subclass of A, consisting of functions which are univalent, starlike and convex in E.

Gao and Zhou [1] discussed the following subclass K_S of analytic functions, which is indeed a subclass of close-to-convex functions.

Author α: Department of Mathematics, Majha College for Women, Tarn-Taran (Punjab), India. e-mail: kamboj.gagandeep@yahoo.in
Author σ: Department of Mathematics, Guru Nanak Dev University College, Chungh, Tarn-Taran (Punjab), India.
e-mail: dhillongs82@yahoo.com
Author ρ: Assistant Director, Directorate of Public Instructions Punjab, Chandigarh (Punjab), India. e-mail: harjindpreet@gmail.com

Ref

1. C.Y. Gao and S. Q. Zhou, On a class of analytic functions related to the starlike functions, Kyungpook Math. J. 45(2005), 123-130.

Let K_S denote the class of functions $f(z)$ of the form (1.1) and satisfying the conditions

$$\operatorname{Re}\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > 0 \tag{1.3}$$

where $g(z) \in S^*\left(\frac{1}{2}\right)$.

Knwalczyk and Les-Bomba [5] extended the class K_S by introducing the following subclass of analytic functions.

A function $f \in A$ is said to be in the class $K_S(\gamma), 0 \le \gamma < 1$, if there exist a function $g(z) \in S^*\left(\frac{1}{2}\right)$, such that

$$\operatorname{Re}\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > \gamma.$$

Recently Prajapat [7] introduced the following subclass of analytic functions.

A function $f \in A$ is said to be in the class $\chi_t(\gamma)(|t| \le 1, t \ne 0, 0 \le \gamma < 1)$, if there exist a function $g(z) \in S^*\left(\frac{1}{2}\right)$, such that

$$\operatorname{Re}\left(\frac{tz^2 f'(z)}{g(z)g(tz)}\right) > \gamma.$$

Motivated by above defined classes, we introduce the following subclass of analytic functions.

Let $\chi_t(A,B)\,(|t| \le 1, t \ne 0)$, denote the class of functions $f(z)$ of the form (1.1) and satisfying the conditions

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, \quad z \in E \tag{1.4}$$

where $g(z) \in S^*\left(\frac{1}{2}\right)$.

In particular,

$$\chi_t(1-2\gamma, -1) \equiv \chi_t(\gamma).$$

$$\chi_{-1}(1-2\gamma, -1) \equiv K_S(\gamma).$$

$$\chi_{-1}(1, -1) \equiv K_S.$$

By definition of subordination it follows that $f(z) \in \chi_t(A,B)$ if and only if $f(z)$ can be represented in the form

Ref

5. J. Kowalczyk and E.Les-Bomba, On a subclass of close-to-convex functions, Appl. Math. Letters, 23(2010), 1147-1151.

$$\frac{tz^2 f'(z)}{g(z)g(tz)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad w(z) \in U, \quad -1 \le B < A \le 1, \qquad z \in E. \tag{1.5}$$

In the present work, we obtained coefficient estimates, inclusion relation, distortion theorems, radius of convexity and Fekete- Szegö problem for functions in the functional class $\chi_t(A,B)$. Results obtained here extend the known results due to various authors.

Notes

Throughout our present discussion, to avoid repetition , we lay down once for all that $-1 \le B < A \le 1, 0 < |t| \le 1, t \ne 0$, $z \in E$.

II. Coefficient Estimates

Lemma 2.1 ([2]) Let

$$\frac{tz^2 f'(z)}{g(z)g(tz)} = P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

then
$$|p_n| \le (A-B), \quad n \ge 1. \tag{2.2}$$

The bounds are sharp, being attained for the functions

$$P_n(z) = \frac{1+A\delta z^n}{1+B\delta z^n}, \quad |\delta| = 1.$$

Lemma 2.2 ([8]) As $g(z) \in S^*\left(\frac{1}{2}\right)$, then $G(z) = \frac{g(z)g(tz)}{tz} = z + \sum_{n=2}^{\infty} d_n z^n \in S^*$, so

$$|d_n| \le n. \tag{2.3}$$

Theorem 2.3 If $f(z) \in \chi_t(A,B)$, then

$$|a_n| \le 1 + \frac{(n-1)(A-B)}{2}. \tag{2.4}$$

Proof. As $f(z) \in \chi_t(A,B)$, therefore (1.5) can be expressed as

$$\frac{zf'(z)}{G(z)} = P(z). \tag{2.5}$$

Using (1.1), (2.1) and (2.3), (2.5) yields

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} n d_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right) \tag{2.6}$$

Equating the coefficients of z^{n-1} in (2.6), we have

$$n a_n = d_n + d_{n-1} p_1 + d_{n-2} p_2 + \ldots + d_2 p_{n-2} + p_{n-1}. \tag{2.7}$$

Therefore using (2.2) and (2.3), it gives

$$n|a_n| \le n + (A-B)[(n-1) + (n-2) + \ldots + 2 + 1]. \tag{2.8}$$

Hence from (2.8), we have

$$|a_n| \le 1 + \frac{(n-1)(A-B)}{2}.$$

On putting $A = 1-2\gamma, B = -1$ in Theorem 2.3, the following result due to Prajapat [7] is obvious:

Corollary 2.4 If $f(z) \in \chi_t(\gamma)$, then

$$|a_n| \le 1 + (n-1)(1-\gamma).$$

Again for $A = 1, B = -1, t = -1$, Theorem 2.3 gives the following result:

Corollary 2.5 If $f(z) \in K_S$, then $|a_n| \le n-1$.

III. Inclusion Relation

Lemma 3.1 ([8]) Let $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$. Then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Theorem 3.2 Let $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$. Then

$$\chi_t(A_1, B_1) \subset \chi_t(A_2, B_2).$$

Proof. As $f(z) \in \chi_t(A_1, B_1)$, so

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Since $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, by Lemma 3.1, we have

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z},$$

it follows that $f(z) \in \chi_t(A_2, B_2)$ which proves the inclusion relation.

IV. Distortion Theorems

Theorem. 4.1 If $f(z) \in \chi_t(A, B)$, then for $|z| = r$, $0 < r < 1$, we have

$$\frac{(1-Ar)}{(1-Br)(1+r)^2} \le |f'(z)| \le \frac{(1+Ar)}{(1+Br)(1-r)^2} \tag{4.1}$$

and

$$\int_0^r \frac{(1-At)}{(1-Bt)(1+t)^2} dt \le |f(z)| \le \int_0^r \frac{(1+At)}{(1+Bt)(1-t)^2} dt. \tag{4.2}$$

Proof. From (2.5), we have

Notes

$$|f'(z)| = \frac{|G(z)|}{|z|}\left|\frac{1+Aw(z)}{1+Bw(z)}\right|, \quad w(z) \in U. \tag{4.3}$$

It is easy to show that the transformation

$$\frac{zf'(z)}{G(z)} = \frac{1+Aw(z)}{1+Bw(z)}$$

Notes

maps $|w(z)| \le r$ onto the circle

$$\left|\frac{zf'(z)}{G(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{(1-B^2r^2)}, \quad |z| = r.$$

This implies that

$$\frac{1-Ar}{1-Br} \le \left|\frac{1+Aw(z)}{1+Bw(z)}\right| \le \frac{1+Ar}{1+Br}. \tag{4.4}$$

Since by Lemma 2.2, $G(z)$ is a starlike function. It is well known that,

$$\frac{r}{(1+r)^2} \le |G(z)| \le \frac{r}{(1-r)^2}. \tag{4.5}$$

(4.3) together with (4.4) and (4.5) yields (4.1). On integrating (4.1) from 0 to r, (4.2) follows.

For $A = 1-2\gamma, B = -1$, Theorem 4.1 gives the following result due to Prajapat [7]:

Corollary 4.2 If $f(z) \in \chi_t(\gamma)$, then

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \le |f'(z)| \le \frac{1+(1-2\gamma)r}{(1-r)^3}$$

and

$$\int_0^r \frac{1-(1-2\gamma)t}{(1+t)^3}dt \le |f(z)| \le \int_0^r \frac{1+(1-2\gamma)t}{(1-t)^3}dt.$$

V. Radius of Convexity

Theorem. 5.1. Let $f(z) \in \chi_t(A,B)$, then $f(z)$ is convex in $|z| < r_1$, where r_1 is the smallest positive root in (0,1) of

$$ABr^3 - A(B-2)r^2 - (2B-1)r - 1 = 0. \tag{5.1}$$

Proof. As $f(z) \in \chi_t(A,B)$, we have

$$zf'(z) = G(z)p(z). \tag{5.2}$$

After differentiating (5.2) logarithmically, we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)}. \tag{5.3}$$

Now for $G(z)\in S^*$, we have

$$\operatorname{Re}\left(\frac{zG'(z)}{G(z)}\right)\geq\frac{1-r}{1+r}.$$

Therefore (5.3) yields,

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\geq\frac{1-r}{1+r}-\left|\frac{zp'(z)}{p(z)}\right|$$

$$\geq\frac{1-r}{1+r}-\frac{r(A-B)}{(1+Ar)(1+Br)}$$

$$\geq\frac{-ABr^3+A(B-2)r^2+(2B-1)r+1}{(1+r)(1+Ar)(1+Br)}.$$

Hence $f(z)$ is convex in $|z|<r_1$, where r_1 is the smallest positive root in $(0,1)$ of

$$ABr^3-A(B-2)r^2-(2B-1)r-1=0.$$

Taking $A=1-2\gamma, B=-1$, Theorem 5.1 gives the following result due to Prajapat [7]:

Corollary 5.2 If $f(z)\in\chi_t(\gamma)$, then $f(z)$ is convex in $|z|<r_0=2-\sqrt{3}.$

VI. Fekete-Szegö Problem

Lemma 6.1 ([3],[6]) If $p(z)=1+p_1z+p_2z^2+p_3z^3+\ldots$ is a function with positive real part, then for any complex number μ,

$$\left|p_2-\mu p_1^2\right|\leq 2\max\{1,|2\mu-1|\}$$

and the result is sharp for the functions given by $p(z)=\dfrac{1+z^2}{1-z^2}$ and $p(z)=\dfrac{1+z}{1-z}$.

Lemma 6.2 ([4]) If $G(z)=z+\sum_{n=2}^{\infty}d_nz^n\in S^*$, then for any complex number λ,

$$\left|d_3-\lambda d_2^2\right|\leq\max\{1,|3-4\lambda|\}$$

and the result is sharp for the Koebe function k if $\left|\lambda-\frac{3}{4}\right|\geq\frac{1}{4}$ and for $\left(k\left(z^2\right)\right)^{\frac{1}{2}}=\dfrac{z}{1-z^2}$ if $\left|\lambda-\frac{3}{4}\right|\leq\frac{1}{4}$.

Theorem. 6.3 Let $f(z)\in\chi_t(A,B)$, then for $\mu\in C$,

$$\left|a_3-\mu a_2^2\right|\leq\frac{(A-B)}{3}\max\{1,|2\gamma_1-1|\}+\frac{1}{3}\max\{1,|3-4\mu_1|\}+2(A-B)\left|\frac{1}{3}-\frac{\mu}{2}\right|,\qquad(6.1)$$

Notes

where $$\gamma_1 = \frac{(1+B)}{2} + \frac{3(A-B)\mu}{8} \quad \text{and} \quad \mu_1 = \frac{3\mu}{4}.$$

Proof. As $f(z) \in \chi_t(A,B)$, from (1.5) we have

$$\frac{zf'(z)}{G(z)} = \frac{1+Aw(z)}{1+Bw(z)}.$$

Let $h(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$, then $\text{Re}(h(z)) > 0$ and $h(0) = 1$.

So $$\frac{zf'(z)}{G(z)} = \frac{1-A+h(z)(1+A)}{1-B+h(z)(1+B)}. \tag{6.2}$$

On expanding (6.2), we have

$$1 + (2a_2 - d_2)z + (3a_3 - 2a_2 d_2 - d_3 + d_2^2)z^2 + \ldots = 1 + \frac{p_1(A-B)z}{2} + \frac{(A-B)}{2}\left\{p_2 - p_1^2\left(\frac{1+B}{2}\right)\right\}z^2 + \ldots \tag{6.3}$$

Equating coefficients of z and z^2 on both sides of (6.3), we get

$$a_2 = \frac{2d_2 + p_1(A-B)}{4}$$

and $$a_3 = \frac{1}{3}\left\{d_3 + \frac{(A-B)}{2}\left(p_1 d_2 + p_2 - \frac{p_1^2(1+B)}{2}\right)\right\}.$$

Therefore, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)}{6}\left|p_2 - \gamma_1 p_1^2\right| + \frac{1}{3}\left|d_3 - \mu_1 d_2^2\right| + \frac{(A-B)}{2}\left|d_2\right|\left[\frac{1}{3} - \frac{\mu}{2}\right]\left|p_1\right|.$$

Hence using Lemma 6.1 and Lemma 6.2, the desired result follows.

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Notes