Bifurcation for a Class of Fourth-Order Stationary Kuramoto-Sivashinsky Equations under Navier Boundary Condition

By Imed Abid, Soumaya Sâanouni & Nihed Trabelsi

University of Tunis El Manar

Abstract- In this paper, we study the bifurcation of semilinear elliptic problem of fourth-order with Navier boundary conditions. We discuss the existence and the uniqueness of a positive solution and we also prove the existence of critical value and the uniqueness of extremal solutions. We take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point.

Keywords: bifurcation, regularity, stability, quasilinear.

GJSFR-F Classification: MSC 2010: 35B32, 35B65, 35B35, 35J62

© 2018. Imed Abid, Soumaya Sâanouni & Nihed Trabelsi. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org/licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.
Bifurcation for a Class of Fourth-Order Stationary Kuramoto-Sivashinsky Equations under Navier Boundary Condition

Imed Abid *, Soumaya Sâanouni * & Nihed Trabelsi *

Abstract: In this paper, we study the bifurcation of semilinear elliptic problem of fourth-order with Navier boundary conditions. We discuss the existence and the uniqueness of a positive solution and we also prove the existence of critical value and the uniqueness of extremal solutions. We take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point.

Keywords: bifurcation, regularity, stability, quasilinear.

1. Introduction

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \), the Kuramoto-Sivashinsky (KS) equation

\[
\partial_t u + \Delta^2 u - \gamma \Delta u + \delta |\nabla u|^2 = \lambda f(x,u) \quad \text{in} \quad \Omega,
\]

\[
\Delta u = u = 0 \quad \text{on} \quad \partial \Omega,
\]

arises in many applications from mathematical physics, which are usually used to describe some phenomena appearing in physics, engineering, and other sciences.

Moreover, the first bifurcation may be subcritical and bistability then occurs. The features qualitatively agree with the experiments. Finally, wave shapes are compared. The addition of the lighter order terms leads to better agreement with the experiments. The (KS) equation was the first non-linear wave equation to be proposed for describing long interfacial waves of two-layer couette and poiseuille flows. In agreement with previous studies, it is shown that each higher order term has a "laminarizing effect": the solutions of the (KS) equation simplify the benefit of stationary traveling waves.

In the stationary case and for \( \gamma = \delta = \beta = 0 \), various authors have studied the existence of weak solutions for the bifurcation problem

\[
(E_\lambda) \quad \left\{ \begin{array}{c}
\Delta^2 u = \lambda f(u) \quad \text{in} \quad \Omega, \\
\Delta u = u = 0 \quad \text{on} \quad \partial \Omega,
\end{array} \right.
\]
where \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \), \( n \geq 2 \). Abid and al. have proved in [1] that there exists \( 0 < \lambda^* < \infty \), a critical value of the parameter \( \lambda \), such as \((E_\lambda)\) has a minimal, positive, classical solution \( u_\lambda \) for \( 0 < \lambda < \lambda^* \) and does not have a weak solution for \( \lambda > \lambda^* \).

When \( \lim_{t \to \infty} \frac{f(t)}{t} = a < \infty \), it is proved also there exists a unique classical solution \( u^* \) of \((E_{\lambda^*})\) if and only if \( \lim_{t \to +\infty} (f(t) - at) < 0 \).

When \( \delta = \beta = 0 \), several researchers are interested in this type of phenomenon:

\[
\Delta^2 u - \gamma \Delta u = \lambda f(x, u) \quad \text{in} \quad \Omega, \\
\Delta u = u = 0 \quad \text{on} \quad \partial \Omega.
\]

In [17, 18], Lazer and Mckenna firstly proposed and studied the problem of periodic oscillations and traveling waves in a suspension bridge. It was pointed out in [23, 28] that the problem provides a good model for the study of the static deflection of an elastic plate in a fluid. Moreover, Ahmed and Harbi in [2] showed that the problem can also be applied to engineering, such as communication satellites, space shuttles and space stations equipped with large antennas mounted on long flexible beams. Such problems appear for example in the micro-electromechanical systems giving the modelization of electrostatic actuation for membranes deflecting on thin plates in the field of nanotechnology detection systems in [19, 24] where the parameter \( \gamma \) represents the constant tension rising in the stretching energy sector in the presence of elastic deformation. It arises in mechanics too, see [11] and in electricity, see [16].

Ben Omrane and Khenissy in [6] show the nonexistence of solutions with Dirichlet boundary conditions. They prove a dichotomy result giving the positivity preserving property for a biharmonic equation arising in MEMS models.

When \( \gamma \) is not constant, other results prove the existence of solutions and study the bifurcation problem with Navier boundary conditions,

\[
\Delta^2 u - \text{div}(\gamma(x) \nabla u) = \lambda f(u) \quad \text{in} \quad \Omega.
\]

In [26], Sâanouni and Trabelsi show how the critical problem behaves when it is considered with the Navier boundary condition. The function \( \gamma(x) \) is smooth positive on \( \overline{\Omega} \) and the \( L^\infty \)-norm of its gradient is small enough in order to assure the application of maximum principle [12].

Our main interest here will be in the study of a bifurcation problem in stationary case for \( \beta = 0 \) and \( \gamma, \delta, \lambda > 0 \), we consider the following problem

\[
(P_\lambda) \quad \left\{ \begin{array}{l}
\Delta^2 u - \gamma \Delta u + \delta u = \lambda f(u) \quad \text{in} \quad \Omega, \\
u > 0 \quad \text{in} \quad \Omega, \\
\Delta u = u = 0 \quad \text{on} \quad \partial \Omega,
\end{array} \right.
\]

where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n; (n \geq 2) \) and \( f \) is a positive, increasing and convex smooth function on \((0, +\infty)\), which verifies

\[
\lim_{t \to \infty} \frac{f(t)}{t} = a \in (0, \infty).
\]

This paper is organized as follows: In next section we state our main results contained in this work (see Theorems 1, 2 and 3). The content of section 3 is devoted to the proof of Theorem 1, concerns existence of minimal solutions. It is shown that there exists a limiting parameter \( \lambda^* \) such that one has existence of stable regular minimal solutions to
(\(P_\lambda\)) for \(\lambda \in (0, \lambda^*)\), while for \(\lambda > \lambda^*\), not even singular solutions exist. In Sections 4 and 5, we devoted to the proofs of Theorems 1 and 2, we take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point. Finally, in Section 6 we also give some hints on how to proceed for semilinear problems under Dirichlet boundary conditions.

II. Main Results

Throughout our paper, we denote by \(\|\cdot\|_2\), the \(L^2(\Omega)\)-norm, whereas we denote by \(\|\cdot\|\), the \(H^2(\Omega) \cap H_0^1(\Omega)\)-norm given by

\[
\|u\|^2 = \int_{\Omega} |\Delta u|^2.
\]

**Definition 1.** We say that \(u \in H^2(\Omega) \cap H_0^1(\Omega)\) is a weak solution of the problem \((P_\lambda)\), if \(f(u) \in L^1(\Omega)\) and

\[
\int_{\Omega} \Delta u \Delta \varphi + \gamma \int_{\Omega} \nabla u \nabla \varphi + \delta \int_{\Omega} u \varphi = \lambda \int_{\Omega} f(u) \varphi, \quad \forall \varphi \in C^2(\bar{\Omega}) \cap H^2(\Omega) \cap H_0^1(\Omega).
\]

Such solutions are usually known as weak energy solutions. For short, we will refer to them simply as solutions which is assured by the next lemma.

**Remark 1.** Since \(f(t) \leq at + f(0)\), if \(u \in H^2(\Omega) \cap H_0^1(\Omega)\) is a weak solution of \((P_\lambda)\) and \(u \in L^1(\Omega)\), we say that \(u\) is regular solution. It is easily seen by a standard bootstrap argument that \(u\) is always a classical solution.

For more detail, see [12, Proposition 7.15]. In the rest of this paper, we denote by a solution of problem \((P_\lambda)\) any weak or classical solution.

**Definition 2.** We say that \(u \in H^2(\Omega) \cap H_0^1(\Omega)\) is a supersolution of \((P_\lambda)\) if \(f(u) \in L^1(\Omega)\) and

\[
\Delta^2 u - \gamma \Delta u + \delta u \geq \lambda f(u) \text{ in } \mathcal{D}'(\Omega).
\]

Reversing the inequality one defines the notion of subsolution.

Next, we recall a version of the Maximum Principle for the biharmonic operator.

**Proposition 1.** Let \(u \in H^4(\Omega) \cap H_0^1(\Omega)\) be a function such that

\[
\Delta^2 u \geq 0 \text{ in } \Omega, \quad \Delta u = u = 0 \text{ on } \partial \Omega.
\]

Then

\[
u(x) \geq 0, \quad \Delta u(x) \leq 0 \text{ in } \Omega.
\]

We say that a solution \(u_\lambda\) of problem \((P_\lambda)\) is minimal if \(u_\lambda \leq u\) in \(\Omega\) for any solution \(u\) of \((P_\lambda)\).

Recall that \(\lambda_1 \in \mathbb{R}\), be the first eigenvalue and \(\varphi_1\) the first normalized eigenfunction of \((-\Delta)^2 - \gamma \Delta + \delta\) in \(\Omega\) with homogeneous Dirichlet boundary data.
Theorem 1. There exists a critical value \( \lambda^* \in (0, \infty) \) such that the following properties hold:

(i) For any \( \lambda \in (0, \lambda^*) \), problem \((P_\lambda)\) has a minimal solution \( u_\lambda \), which is the unique stable solution of \((P_\lambda)\).

(ii) For any \( \lambda \in (0, \lambda^*/a) \), \( u_\lambda \) is the unique solution of problem \((P_\lambda)\).

(iii) The function \( \lambda \mapsto u_\lambda \) is a \( C^1 \), convex, increasing function.

(iv) If \((P_{\lambda^*})\) has a solution, then \( u^* := \lim_{\lambda \to \lambda^*} u_\lambda \) and \( \eta_1(\lambda^*, u^*) = 0 \).

An eigenvalue \( \lambda_1 \) which is positive and that can be characterized as follows

\[
\lambda_1 = \min_{\varphi \in H^2(\Omega)} \frac{\int_{\Omega} (|\Delta \varphi|^2 + \gamma |\nabla \varphi|^2 + \delta \varphi^2)}{\int_{\Omega} |\varphi|^2};
\]

there exists a non-negative function \( \varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega) \), which is an eigenfunction corresponding to \( \lambda_1 \), attaining the minimum in (2), that is \( \|\varphi_1\|_2 = 1 \) and

\[
\lambda_1 = \int_{\Omega} \left( |\Delta \varphi_1|^2 + \gamma |\nabla \varphi_1|^2 + \delta \varphi_1^2 \right).
\]

A solution \( u \) of problem \((P_\lambda)\) is stable if and only if the first eigenvalue of the linearized operator

\[
v \mapsto L_{\lambda, u}(v) := \Delta^2 v - \gamma \Delta v + \delta v - \lambda f'(u) v,
\]

given by

\[
\eta_1(\lambda, u)(v) := \inf_{v \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \left( |\Delta v|^2 + \gamma |\nabla v|^2 + \delta v^2 \right) - \lambda \int_{\Omega} f'(u)v^2 dx}{\|v\|_2^2}
\]

is non-negative. In other words,

\[
\lambda \int_{\Omega} f'(u)v^2 dx \leq \int_{\Omega} \left( |\Delta v|^2 + \gamma |\nabla v|^2 + \delta v^2 \right), \quad \text{for any } v \in H^2(\Omega) \cap H_0^1(\Omega).
\]

If \( \eta(\lambda, u) < 0 \), the solution \( u \) is said to be unstable.

Next, we let

\[
\Lambda := \{ \lambda > 0 \mid (P_\lambda) \text{ admits a solution} \}
\]

and

\[
r_0 := \inf_{t > 0} \frac{f(t)}{t}.
\]
Bifurcation for a Class of Fourth-Order Stationary Kuramoto-Sivashinsky Equations under Navier Boundary Condition

The next natural obvious object of study gives us more precise information for \( \lambda^* \).
An important role in our arguments will be played by

\[
l := \lim_{t \to \infty} \left( f(t) - at \right).
\]

We distinguish two different situations strongly depending on the sign of \( l \).

**Theorem 2.** Assume that \( l \geq 0 \). We have three equivalent assertions:

(i) \( \lambda^* = \lambda_1/a \).
(ii) problem \( (P_{\lambda^*}) \) has no solution.
(iii) \( \lim_{\lambda \to \lambda^*} u_\lambda = \infty \) uniformly on compact subsets of \( \Omega \).

Again the question arises as to what happens when \( l < 0 \). The following was proved.

**Theorem 3.** Assume that \( l < 0 \). Then we have.

(i) the critical value \( \lambda^* \) belongs to \( (\lambda_1/a, \lambda_1/r_0) \)
(ii) \( (P_{\lambda^*}) \) has a unique solution \( u^* \).
(iii) The problem \( (P_\lambda) \) has an unstable solution \( v_\lambda \) for any \( \lambda \in (\lambda_1/a, \lambda^*) \) and the sequence \( (v_\lambda)_\lambda \) satisfies:

(a) \( \lim_{\lambda \to \lambda_1/a} v_\lambda = \infty \) uniformly on compact subsets of \( \Omega \),
(b) \( \lim_{\lambda \to \lambda^*} v_\lambda = u^* \) uniformly in \( \Omega \).

III. Proof of Theorem 1

We say that a solution \( u_\lambda \) of problem \( (P_\lambda) \) is *minimal* if \( u_\lambda \leq u \) in \( \Omega \) for any solution \( u \) of \( (P_\lambda) \).

**Lemma 1.** Problem \( (P_\lambda) \) has no solution for any \( \lambda > \lambda_1/r_0 \), but has at least one solution provided \( \lambda \) is positive and small enough.

**Proof:** First, to show that \( (P_\lambda) \) has a solution, we use the barrier method.

Since \( f(0) > 0 \), \( u \equiv 0 \) is a strict subsolution of \( (P_\lambda) \) for every \( \lambda > 0 \). To this aim, let \( \bar{w} \in H^4(\Omega) \) which satisfies

\[
\begin{cases}
\Delta^2 \bar{w} - \gamma \Delta \bar{w} + \delta \bar{w} = 1 & \text{in } \Omega, \\
\Delta \bar{w} = \bar{w} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The choice of \( \bar{w} \) implies that \( \bar{w} \) is a bounded supersolution of \( (P_\lambda) \) for small \( \lambda \), more precisely whenever \( \lambda < 1/f(\|\bar{w}\|_\infty) \).

Notice that for any \( \lambda > 0 \), the function \( w \equiv 0 \) is a sub-solution of \( (P_\lambda) \) since \( f(0) > 0 \). Next, we define a sequence \( w_n \in H^4(\Omega) \) by

\[
\begin{cases}
\Delta^2 w_{n+1} - \gamma \Delta w_{n+1} + \delta w_{n+1} = \lambda f(w_n) & \text{in } \Omega, \\
\Delta w_{n+1} = w_{n+1} = 0 & \text{on } \partial \Omega.
\end{cases}
\]

The maximum principle (see [7]) implies that

\( w \leq w_n \leq w_{n+1} \leq \bar{w} \) for all \( n \in \mathbb{N} \),

so that the sequence \( (w_n)_{n \geq 0} \) is increasing and bounded, then it converges. It follows that problem \( (P_\lambda) \) has a solution.
Assume now that \( u \) is a solution of \((P_\lambda)\) for some \( \lambda > 0 \). Using \( \varphi_1 \) given in (1) as a test function and integrating by parts, we get
\[
\lambda_1 \int_{\Omega} \varphi_1 \ u = \int_{\Omega} (\Delta^2 \varphi_1 - \gamma \Delta \varphi_1 + \delta \varphi_1) u = \int_{\Omega} \Delta^2 u \varphi_1 - \gamma \int_{\Omega} \Delta u \varphi_1 + \delta \int_{\Omega} u \varphi_1
\]
\[
= \lambda \int_{\Omega} f(u) \varphi_1 \geq \lambda \ r_0 \int_{\Omega} u \varphi_1.
\]
This yields
\[
(\lambda_1 - \lambda r_0) \int_{\Omega} \varphi_1 u \geq 0.
\]
Since \( \varphi_1 > 0 \) and \( u > 0 \), we conclude that the parameter \( \lambda \) should belong to \((0, \lambda_1/r_0)\).

As a consequence we have that \( \lambda^* \) is a real. Another useful result is stated in what follows.

**Lemma 2.** Assume that \((P_\lambda)\) is resolvable, then a minimal solution \( u_\lambda \) exists. Moreover, \((P_{\lambda'})\) is resolvable for any \( \lambda' \in (0, \lambda) \).

**Proof:** Fix \( \lambda \in (0, \lambda^*) \) and let \( u \) be a solution of \((P_\lambda)\). As above, we use the barrier method to obtain a minimal solution of \((P_\lambda)\). The basic idea is to prove by induction that the sequence \((w_n)_{n \geq 0}\) defined in (4) is increasing and bounded by \( u \), so it converges to some solution \( u_\lambda \). Since \( u_\lambda \) is independent of the choice of \( u \), then it is a minimal solution.

Now, if \( u \) is a solution of \((P_\lambda)\), then \( u \) is a super-solution for the problem \((P_{\lambda'})\) for any \( \lambda' \) in \((0, \lambda)\) and 0 can be used always as a sub-solution.

**Remark 2.** Thanks to lemmas 1 and 2, the set \( \Lambda \) is an interval not empty and bounded.

\textbf{a) Proof of Theorem 1.}

\textbf{i. Proof of (i).} First, we claim that \( u_\lambda \) is stable. Indeed, arguing by contradiction, i.e. the first eigenvalue \( \eta_1(\lambda, u_\lambda) \) is negative. Then, there exists an eigenfunction \( \varphi \in H^4(\Omega) \) such that
\[
\Delta^2 \varphi - \gamma \Delta \varphi + \delta \varphi - \lambda f'(u_\lambda) \varphi = \eta_1 \varphi \quad \text{in} \quad \Omega,
\]
\[
\varphi > 0 \quad \text{in} \quad \Omega,
\]
\[
\Delta \varphi = \psi = 0 \quad \text{on} \quad \partial \Omega.
\]
Consider \( u^\varepsilon := u_\lambda - \varepsilon \psi \). Hence, by linearity, we have
\[
\Delta^2 u^\varepsilon - \gamma \Delta u^\varepsilon + \delta u^\varepsilon - \lambda f(u^\varepsilon) = \lambda f(u_\lambda) - \varepsilon (\Delta^2 \varphi - \gamma \Delta \varphi + \delta \varphi) - \lambda f(u_\lambda - \varepsilon \psi)
\]
\[
= \lambda f(u_\lambda) - \varepsilon (\lambda f'(u_\lambda) \psi + \eta_1 \psi) - \lambda f(u_\lambda - \varepsilon \psi)
\]
\[
= \lambda \left[ f(u_\lambda) - f(u_\lambda - \varepsilon \psi) - \varepsilon f'(u_\lambda) \psi \right] - \varepsilon \eta_1 \psi
\]
\[
= \varepsilon \left( \lambda \ o_\varepsilon(\varepsilon) - \eta_1 \right).
\]
Since \( \eta_1(\lambda, u_\lambda) < 0 \), for \( \varepsilon > 0 \) small enough, we have
\[
\Delta^2 u^\varepsilon - \gamma \Delta u^\varepsilon + \delta u^\varepsilon - \lambda f(u^\varepsilon) \geq 0 \quad \text{in} \quad \Omega.
\]
Then, for \( \varepsilon > 0 \) small enough, we use the strong maximum principle (Hopf’s lemma, see [14]) to deduce that \( u^\varepsilon \geq 0 \) is a super-solution of \((P_\lambda)\). As before, we obtain a solution \( u \) such that \( u \leq u^\varepsilon \) and since \( u^\varepsilon < u_\lambda \), then we contradict the minimality of \( u_\lambda \).

Now, we show that \((P_\lambda)\) has at most one stable solution. Assume the existence of another stable solution \( v \neq u_\lambda \) of problem \((P_\lambda)\). Let \( w := v - u_\lambda \), then by maximum principle \( w > 0 \) and from (3) taking \( w \) as a test function, we have
\[
\lambda \int_{\Omega} f'(v) w^2 \leq \int_{\Omega} |\Delta w|^2 + \gamma \int_{\Omega} |\nabla w|^2 + \delta \int_{\Omega} w^2 \\
\leq \int_{\Omega} \Delta^2 w \ w - \gamma \int_{\Omega} \Delta w \ w + \delta \int_{\Omega} w \ w \\
\leq \int_{\Omega} \left[ \Delta^2 v - \gamma \Delta v + \delta v - \Delta^2 u_\lambda + \gamma \Delta u_\lambda - \delta u_\lambda \right] w \\
\leq \lambda \int_{\Omega} \left[ f(v) - f(u_\lambda) \right] w.
\]

Therefore

\[
\int_{\Omega} \left[ f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) \right] w \geq 0.
\]

Thanks to the convexity of \( f \), the term in the brackets is nonpositive, hence

\[
f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) = 0 \text{ in } \Omega,
\]

which implies that \( f \) is affine over \([u_\lambda, v]\) in \( \Omega \). So, there exists two real numbers \( \alpha \) and \( \beta \) such that

\[
f(x) = \alpha x + \beta \quad \text{in } [0, \max_{\Omega} v].
\]

Finally, since \( u_\lambda \) and \( v \) are two solutions to \( \Delta^2 w - \gamma \Delta w + \delta w = \lambda \omega w + \lambda \beta \), we obtain that

\[
0 = \int_{\Omega} \left( u_\lambda \Delta^2 v - v \Delta^2 u_\lambda \right) - \gamma \int_{\Omega} \left( u_\lambda \Delta v - v \Delta u_\lambda \right) + \delta \int_{\Omega} \left( u_\lambda v - v u_\lambda \right) = \lambda \beta \int_{\Omega} (u_\lambda - v).
\]

This is impossible since \( \beta = f(0) > 0 \) and \( w = v - u_\lambda \) is positive in \( \Omega \).

3.1.2. Proof of (ii). Recall that \( \lambda_1 \) is defined in (1). By the convexity of \( f \), we deduce that \( a = \sup_{\mathbb{R}_+} f'(t) \). Let \( u \) be a solution to \((P_\lambda)\) for \( \lambda \in (0, \lambda_1/a) \), we suppose that \( u \) is unstable. Then, we can take \( \varphi = \varphi_1 \in H^2(\Omega) \cap H^1_0(\Omega) \) which satisfy

\[
\lambda a \int_{\Omega} \varphi^2 \geq \lambda \int_{\Omega} f'(u) \varphi^2 > \int_{\Omega} |\Delta \varphi|^2 + \gamma \int_{\Omega} |\nabla \varphi|^2 + \delta \int_{\Omega} \varphi^2 = \lambda_1 \int_{\Omega} \varphi^2,
\]

which shows that

\[
(\lambda a - \lambda_1) \int_{\Omega} \varphi^2 > 0.
\]

Impossible for \( \lambda \in (0, \lambda_1/a) \). Then, \( \eta_1(\lambda, u) \geq 0 \), so we obtain the uniqueness of \( u \).

For the existence, we consider the minimization problem

\[
\min_{u \in H^2(\Omega) \cap H^1_0(\Omega)} J_\lambda(u),
\]

where

\[
J_\lambda(u) := \frac{1}{2} \int_{\Omega} \left( |\Delta u|^2 + \gamma |\nabla u|^2 + \delta u^2 \right) - \lambda \int_{\Omega} F(u), \quad \text{for all } u \in H^2(\Omega) \cap H^1_0(\Omega)
\]

with

\[
u^+ := \max(u, 0) \quad \text{and} \quad F(u) := \int_0^{u^+} f(s)ds.
\]

If \( \lambda \in (0, \lambda_1/a) \), there exist \( \varepsilon > 0 \) and \( A > 0 \) depending on \( \lambda \) such that

\[
2\lambda F(t) \leq (\lambda_1 - \varepsilon)t^2 + A, \quad \forall t \in \mathbb{R}.
\]
Standard arguments imply that $J_\lambda(u)$ is coercive, bounded from below and weakly lower semi-continuous in $H^2(\Omega) \cap H^1_0(\Omega)$ (see propositions 2.2 and 2.3 in [29]). It is easy to see that the minimum of $J_\lambda$ is attained by some function $u \in H^2(\Omega) \cap H^1_0(\Omega)$. So, the critical point $u$ of $J_\lambda$ gives a solution of $(P_\lambda)$.

3.1.3. **Proof of (iii).** By sub- and super-solution method, see Lemma 2 we obtain that the mapping $\lambda \mapsto u_\lambda$ is increasing and this proves (iii).

3.1.4. **Proof of (iv).** Now Consider the nonlinear operator

$$G : (0, +\infty) \times C^{4,\alpha}(\Omega) \cap E \longrightarrow C^{0,\alpha}(\Omega)$$

where $\alpha \in (0, 1)$ and $E$ is the function space defined by

$$E := \{ u \in W^{4,2}(\Omega) \mid \Delta u = u = 0 \text{ on } \partial \Omega \}. \quad (5)$$

Assume that $(P_{\lambda^*})$ has a solution $u$. Then for any $\lambda \in (0, \lambda^*)$, $u_\lambda \leq u$ in $\Omega$. Then for every $\lambda \in (0, \lambda^*)$ we have $u_\lambda \leq u^*$ in $\Omega$. Using the monotonicity of $u_\lambda$, we deduce that the function

$$u^* = \lim_{\lambda \to \lambda^*} u_\lambda$$

is well defined in $\Omega$ and is a semi-stable solution of problem $(P_{\lambda^*})$. Assuming that the first eigenvalue $\eta_1(\lambda^*, u^*)$ is positive, we can apply the implicit function theorem to the operator $G$. For any $\lambda$ in a neighborhood of $\lambda^*$ and $u$ in a neighborhood of $u^*$, we have $G(\lambda, u) = 0$, which proves that the problem $(P_\lambda)$ has a solution for $\lambda$ in a neighborhood of $\lambda^*$. But this contradicts the definition of $\lambda^*$. So, $\eta_1(\lambda^*, u^*) = 0$ and this completes the proof of Theorem 1.

**IV. PROOF OF THEOREM 2**

**Remark 3.** Thanks to Lemma 1 and (ii) of Theorem 1, the critical value $\lambda^*$ satisfies:

$$\lambda_1/a \leq \lambda^* \leq \lambda_1/r_0.$$ 

To prove this theorem, we show that the three assertions are equivalent. And finally, we prove that one holds. We shall use the following auxiliary result which is a reformulation of Theorem due to Hörmander [15].

**Lemma 3.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$ with smooth boundary. Let $(u_n)$ be a sequence of super-harmonic nonnegative functions defined on $\Omega$. Then the following alternative holds:

: (i) either $\lim_{n \to \infty} u_n = \infty$ uniformly on compact subsets of $\Omega$,

: (ii) or $(u_n)$ contains a subsequence which converges in $L^1_{loc}(\Omega)$ to some function $u$.

**Remark 4.** The result by Hörmander is also true if $(u_n)$ is a sequence of a super-harmonic nonnegative functions.

4.1. **Proof.** (i) $\Rightarrow$ (ii). By contradiction. We assume that $\lambda^* = \frac{\lambda_1}{a}$. If $(P_{\lambda^*})$ has a solution $u^*$ then, as we have already observed in (iv) of Theorem 1, $\eta_1(\lambda^*, u^*) = 0$. Thus, there exists $\psi \in H^4(\Omega)$ satisfying:

$$\Delta^2 \psi - \gamma \Delta \psi + \delta \psi - \lambda^* f'(u^*) \psi = 0 \quad \text{in } \Omega$$

$$\psi > 0 \quad \text{in } \Omega$$

$$\Delta \psi = \psi = 0 \quad \text{on } \partial \Omega.$$
Using $\varphi_1$, given in (1), as a test function and integrating by parts, we obtain
\[
\int_\Omega \left( \Delta^2 \varphi_1 - \gamma \Delta \varphi_1 + \delta \varphi_1 \right) - \lambda^* \int_\Omega f'(u^*) \psi \varphi_1 = 0
\]
therefore
\[
\int_\Omega \left( \lambda_1 - \lambda^* f'(u^*) \right) \psi \varphi_1 = 0.
\]
$\varphi_1 > 0$, $\psi > 0$, $\lambda^* = \frac{\lambda_1}{a}$ and $a = \sup_{t>0} f'(t)$, we have $\lambda_1 - \lambda^* f'(u^*) \geq 0$, the above equation forces $\lambda_1 - \lambda^* f'(u^*) = 0$. Hence
\[
f'(u^*) \equiv a \quad \text{in} \quad \Omega.
\]
This implies that $f(t) = at + b$ in $[0, \max_{\Omega} u^*]$ for some scalar $b > 0$. But there is no positive function in $\Omega$ such that $u = \Delta u = 0$ on $\partial \Omega$ and
\[
\Delta^2 u - \gamma \Delta u + \delta u = \lambda^* au + \lambda^* b \quad \text{in} \quad \Omega.
\]
If not, using $\varphi_1$ and integrating by parts, we have
\[
\int_\Omega \Delta^2 u \varphi_1 - \gamma \int_\Omega \Delta u \varphi_1 + \delta \int_\Omega u \varphi_1 = \lambda^* a \int_\Omega u \varphi_1 + \lambda^* b \int_\Omega \varphi_1
\]
then
\[
\int_\Omega \left( \Delta^2 \varphi_1 - \gamma \Delta \varphi_1 + \delta \varphi_1 \right) u = \lambda_1 \int_\Omega u \varphi_1 + \lambda^* b \int_\Omega \varphi_1
\]
i.e.
\[
0 = \lambda^* b \int_\Omega \varphi_1 \quad \text{which is impossible.}
\]
Hence, problem $(P_{\lambda^*})$ has no solution and (i) implies (ii).

4.2. Proof (ii) $\Rightarrow$ (iii). We assume that (ii) occurs and we claim that $\lim_{\lambda \to \lambda^*} u_\lambda = \infty$ uniformly on compact subsets of $\Omega$. By contradiction, suppose that (iii) doesn’t hold. By Lemma 4 and up to a subsequence, $(u_\lambda)$ converges locally in $L^1(\Omega)$ to $u^*$ as $\lambda \to \lambda^*$.

Lemma 4. The minimal solution $u_\lambda$ of the problem $(P_\lambda)$ is bounded in $L^2(\Omega)$.

Proof. If not, we define
\[
u_\lambda := k_\lambda w_\lambda,
\]
with
\[
\|w_\lambda\| = 1 \quad \text{and} \quad k_\lambda \to +\infty \quad \text{as} \quad \lambda \to \lambda^*.
\]
Since $f(t) \leq at + f(0)$, We have
\[
\int_\Omega |\Delta w_\lambda|^2 \leq \int_\Omega |\Delta w_\lambda|^2 + \gamma \int_\Omega |\nabla w_\lambda|^2 + \delta \int_\Omega w_\lambda^2
\]
\[
= \int_\Omega \Delta^2 w_\lambda w_\lambda - \gamma \int_\Omega \Delta w_\lambda w_\lambda + \delta \int_\Omega w_\lambda^2 w_\lambda = \int_\Omega \frac{\lambda f(u_\lambda)}{k_\lambda} w_\lambda
\]
\[
\leq \lambda^* \int_\Omega \left( a w_\lambda^2 + \frac{f(0)}{k_\lambda} w_\lambda \right) \leq \lambda^* a - c \int_\Omega w_\lambda
\]
\[
\leq \lambda^* a - c \sqrt{|\Omega|},
\]
where $c$ is a positive constant independent on $\lambda$. 


Recall that \( w_\lambda \) satisfies \( \Delta^2 w_\lambda - \gamma \Delta w_\lambda + \delta w_\lambda = \frac{\lambda f(k_\lambda w_\lambda)}{k_\lambda} \) and \( f \) is quasilinear. These facts imply that \( (w_\lambda) \) is bounded in \( H^4(\Omega) \). Hence, up to a subsequence, we have \( w_\lambda \rightharpoonup w \) weakly in \( H^4(\Omega) \) and \( w_\lambda \to w \) strongly in \( H^3(\Omega) \) as \( \lambda \to \lambda^* \).

Moreover, by the trace theorem, \( w = \Delta w = 0 \) on \( \partial \Omega \). We deduce that

\[
\Delta^2 w_\lambda - \gamma \Delta w_\lambda + \delta w_\lambda = \lambda^* f(u) \quad \text{in} \quad \Omega
\]

and this impossible by the hypothesis (ii). This shows that (ii) implies (iii). Moreover, this simply shows that (ii) and (iii) are equivalent.

4.3. **Proof (iii)⇒(i).** If \( (P_{\lambda^*}) \) has a solution \( u^* \) then the sequence \( (u_\lambda) \) converges to \( u^* \) as \( \lambda \) tends to \( \lambda^* \), which cannot happen in the case where \( \lim_{\lambda \to \lambda^*} u_\lambda = \infty \). Hence, (iii) implies (i).

Indeed, clearly if (ii) and (iii) occur, we have \( \lim_{\lambda \to \lambda^*} \| u_\lambda \|_2 = \infty \). Set

\[
u_\lambda = k_\lambda w_\lambda \quad \text{with} \quad \| w_\lambda \|_2 = 1.\]

Then, up to a subsequence, we obtain

\( w_\lambda \rightharpoonup w \) weakly in \( H^4(\Omega) \) and \( w_\lambda \to w \) strongly in \( H^3(\Omega) \) as \( \lambda \to \lambda^* \).

Moreover,

\[
\Delta^2 w_\lambda - \gamma \Delta w_\lambda + \delta w_\lambda \to \Delta^2 w - \gamma \Delta w + \delta w \quad \text{in} \quad \mathcal{D}'(\Omega) \quad \text{as} \quad \lambda \to \lambda^*
\]

and

\[
\frac{\lambda}{k_\lambda} f(k_\lambda w_\lambda) \to \lambda^* a w \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \lambda \to \lambda^*.
\]

Then,

\[
\begin{cases}
\Delta^2 w - \gamma \Delta w + \delta w = \lambda^* a w & \text{in} \ \Omega,
\Delta w = w = 0 & \text{on} \ \partial \Omega.
\end{cases}
\]

Multiplying by \( \varphi_1 \), which is defined in (1), we obtain

\[
\int_\Omega \lambda^* a w \varphi_1 = \int_\Omega \Delta^2 w \varphi_1 - \gamma \int_\Omega \Delta w \varphi_1 + \delta \int_\Omega w \varphi_1
\]

\[
= \int_\Omega \Delta^2 \varphi_1 w - \gamma \int_\Omega \Delta \varphi_1 w + \delta \int \varphi_1 w = \int_\Omega \lambda_1 \varphi_1 w.
\]
This proves (i). To finish the proof of Theorem 2, we need only to show that \((P_{\lambda_1/a})\) has no solution. Indeed, assume that \(u\) is a solution of \((P_{\lambda_1/a})\). Since \(f(t) - at \geq 0\), we have

\[
\Delta^2 u - \gamma \Delta u + \delta u = \frac{\lambda_1}{a} f(u) \geq \lambda_1 u \quad \text{in} \quad \Omega.
\]

Multiplying the previous equation by \(\varphi_1\) and integrating by parts, we get \(f(u) = au\) in \(\Omega\), which contradicts \(f(0) > 0\). This concludes the proof of Theorem 2.

**Remark 5.** Observe that the equivalence of the assertions of Theorem 2 does not depend on the sign of \(l\).

### V. Proof of Theorem 3

5.1. **Proof (i).** For the first part of Theorem 3, we have already seen in Remark 3 that \(\lambda_1/a \leq \lambda^* \leq \lambda_1/r_0\). Hence it suffices to prove that \(\lambda^* \neq \lambda_1/a\) and \(\lambda^* \neq \lambda_1/r_0\). First, assume that \(\lambda^* = \lambda_1/a\). By Remark 5, we have

\[
\lim_{\lambda \to \lambda^*} u_\lambda = \infty \quad \text{uniformly on compact subsets of} \quad \Omega.
\]

Let \(u_\lambda\) be the minimal solution to \((P_\lambda)\). Then, multiplying \((P_\lambda)\) by \(\varphi_1\) and integrating, we obtain

\[
0 = \int_\Omega \left( \lambda_1 u_\lambda - \lambda f(u_\lambda) \right) \varphi_1 = \int_\Omega \left( (\lambda_1 - a\lambda) u_\lambda - \lambda (f(u_\lambda) - au_\lambda) \right) \varphi_1
\]

and then

\[
\lambda \int_\Omega \varphi_1 (f(u_\lambda) - au_\lambda) \geq 0.
\]

Passing to the limit in the last inequality as \(\lambda\) tends to \(\lambda^*\), we find

\[
0 \leq l\lambda^* \int_\Omega \varphi_1 < 0,
\]

which is impossible and then \(\lambda^* \neq \frac{\lambda_1}{a} a\).

Now, assume that \(\lambda^* = \lambda_1/r_0\) and let \(u\) be a solution of problem \((P_{\lambda^*})\). Multiplying \((P_{\lambda^*})\) by \(\varphi_1\) and integrating by parts, we have

\[
\lambda_1 \int_\Omega u \varphi_1 = \frac{\lambda_1}{r_0} \int_\Omega f(u) \varphi_1
\]

that is

\[
\int_\Omega (f(u) - r_0 u) \varphi_1 = 0
\]

which forces \(f(u) = r_0 u\) in \(\Omega\), so that \(f(t) = r_0 t\) in \([0, \max_\Omega u]\). As above, this contradicts the fact that \(f(0) > 0\).

5.2. **Proof (ii).** Since \(\lambda^* > \lambda_1/a\), the existence of a solution to \((P_{\lambda^*})\) is assured by Remark 5. Then, it remains to prove the uniqueness. Assume that \(u\) is another solution to \((P_{\lambda^*})\) and let \(w := u - u^*\). Since \(u_\lambda < u\) and \(\lim_{\lambda \to \lambda^*} u_\lambda = u^*\), we have \(w \geq 0\). Then by convexity of \(f\) we have

\[
\Delta^2 w - \gamma \Delta w + \delta w = \lambda^* \left( f(u) - f(u^*) \right) \geq \lambda^* f'(u^*) w \quad \text{in} \quad \Omega.
\]
Recall that $\eta_1(\lambda^*, u^*) = 0$, so let be the corresponding eigenfunction. Multiplying the last inequality by $\lambda^*$ and integrating by parts, we find

$$0 = \int_{\Omega} \lambda^* \left( f(u) - f(u^*) - f'(u^*)w \right) \geq 0.$$ Therefore, we must have equality $f(u) - f(u^*) = f'(u^*)w$ in $\Omega$, which implies that $f$ is linear in $[0, \max u]$ and this leads a contradiction as in the proof of Theorem 1.

5.3. **Proof (iii).** concerning the existence of a non stable solution $v_\lambda$ of $(P_\lambda)$ will be proved by using the mountain pass theorem of Ambrosetti and Rabinowitz [3] in the following form:

**Theorem 4.** Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$. Assume that $J$ satisfies the Palais-Smale condition and the following geometric assumptions:

(*) there exist positive constants $R$ and $\rho$ such that

$$J(u) \geq J(u_0) + \rho, \text{ for all } u \in E \text{ with } \|u - u_0\| = R.$$ (***) there exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$.

Then the functional $J$ possesses at least a critical point. The critical value is characterized by

$$c := \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where

$$\Gamma := \{g \in \mathcal{C}([0,1], E) \mid g(0) = u_0, g(1) = v_0\}$$

and satisfies

$$c \geq J(u_0) + \rho.$$ In our case,

$$J_\lambda : E \rightarrow \mathbb{R}$$

$$u \mapsto \frac{1}{2} \left( \int_{\Omega} |\Delta u|^2 + \gamma |\nabla u|^2 + \delta u^2 \right) - \int_{\Omega} F(u),$$

where

$$F(t) = \lambda \int_0^t f(s) ds, \text{ for all } t \geq 0.$$ We take $u_0$ as the stable solution $u_\lambda$ for each $\lambda \in (\lambda_1/a, \lambda^*)$.

**Remark 6.** The energy functional $J_\lambda$ belongs to $C^1(E, \mathbb{R})$ and

$$\langle J'_\lambda(u), v \rangle = \int_{\Omega} \Delta u \Delta v + \gamma \int_{\Omega} \nabla u \nabla v + \delta \int_{\Omega} uv - \lambda \int_{\Omega} f(u)v dx, \text{ for all } u, v \in E.$$ Since $\eta_1(\lambda, u_\lambda) > 0$, the function $u_\lambda$ is a strict local minimum for $J_\lambda$, we apply the mountain pass theorem for $J_\lambda$. We show in the next lemma that $J_\lambda$ satisfies the Palais-Smale compactness condition.
Lemma 5. Let \((u_n) \subset E\) be a Palais-Smale sequence; that is,
\[
\sup_{n \in \mathbb{N}} |\mathcal{J}_\lambda(u_n)| < +\infty, \tag{6}
\]
\[
\|\mathcal{J}_\lambda'(u_n)\|_{E^*} \to 0 \quad \text{as} \quad n \to \infty. \tag{7}
\]

Then \((u_n)\) is relatively compact in \(E\).

Proof: Since any subsequence of \((u_n)\) verifies (6) and (7) it is enough to prove that \((u_n)\) contains a convergent subsequence. It suffices to prove that \((u_n)\) contains a bounded subsequence in \(E\). Indeed, suppose we have proved this. Then, up to a subsequence, \(u_n \to u\) weakly in \(E\), strongly in \(L^2(\Omega)\). Now (7) gives
\[
\Delta^2 u_n - \gamma \Delta u_n + \delta u_n - \lambda f(u_n) \to 0 \quad \text{in} \quad D'(\Omega)
\]
Note that \(f(u_n) \to f(u)\) in \(L^2(\Omega)\) because \(|f(u_n) - f(u)| \leq a|u_n - u|\). This shows that
\[
\Delta^2 u_n - \gamma \Delta u_n + \delta u_n \to \lambda f(u) \quad \text{in} \quad D'(\Omega).
\]
That is
\[
\Delta^2 u - \gamma \Delta u + \delta u - \lambda f(u) = 0.
\]
The above equality multiplied by \(u\) gives
\[
\int_{\Omega} |\Delta u|^2 + \gamma \int_{\Omega} |\nabla u|^2 + \delta \int_{\Omega} u^2 - \lambda \int_{\Omega} f(u)u = 0. \tag{8}
\]
Now (7) multiplied by \((u_n)\) gives
\[
\int_{\Omega} |\Delta u_n|^2 + \gamma \int_{\Omega} |\nabla u_n|^2 + \delta \int_{\Omega} u_n^2 - \lambda \int_{\Omega} f(u_n)u_n \to 0 \tag{9}
\]
in view of the boundedness of \((u_n)\) and the \(L^2(\Omega)\)-convergence of \(u_n\) and \(f(u_n)\), we have
\[
\lambda \int_{\Omega} f(u_n)u_n \to \lambda \int_{\Omega} f(u)u.
\]
Hence, (8) and (9) give
\[
\int_{\Omega} |\Delta u_n|^2 \to \int_{\Omega} |\Delta u|^2 \quad \text{and} \quad \gamma \int_{\Omega} |\nabla u_n|^2 \to \gamma \int_{\Omega} |\nabla u|^2
\]
which insures us that \(u_n \to u\) in \(E\).

Actually, it is enough to prove that \((u_n)\) is (up to a subsequence) bounded in \(L^2(\Omega)\). Indeed, the \(L^2(\Omega)\)-boundedness of \((u_n)\) implies that \(E\)-boundedness of \((u_n)\) as it can be seen by examining (6).

We shall conclude the proof obtaining a contradiction from the supposition that \(\|u_n\|_2 \to \infty\). Let \(u_n = k_n w_n\) with \(k_n > 0, k_n \to \infty\) and \(\|w_n\|_2 = 1\). Then
\[
0 = \lim_{n \to \infty} \frac{\mathcal{J}_\lambda(u_n)}{k_n^2} = \lim_{n \to \infty} \left[ \frac{1}{2} \int_{\Omega} |\Delta w_n|^2 + \gamma \int_{\Omega} |\nabla w_n|^2 + \delta \int_{\Omega} w_n^2 - \frac{1}{k_n^2} \int_{\Omega} F(u_n) \right]
\]
However, since \(|f(t)| \leq a|t| + b\), we have
\[
|F(u_n)| = |F(k_n w_n)| \leq \frac{a\lambda}{2} k_n^2 w_n^2 + b\lambda |k_n w_n|.
\]
This shows that
\[ \frac{1}{k_n^2} \int_{\Omega} F(u_n) \leq \frac{a\lambda}{2} \int_{\Omega} w_n^2 + \frac{b\lambda}{k_n} \int_{\Omega} w_n < \infty. \]

We claim that
\[ \Delta^2 w - \gamma \Delta w + \delta w = a\lambda w^+ \quad \text{where} \quad w^+ := \max\{0, w\}. \quad (10) \]

Indeed, (7) divided by \( k_n \) gives
\[ \int_{\Omega} \Delta w_n \cdot \Delta v + \gamma \int_{\Omega} \nabla w_n \cdot \nabla v + \delta \int_{\Omega} w_n v - \lambda \int_{\Omega} f(u_n) \frac{1}{k_n} v \to 0 \quad (11) \]
for each \( v \in E \). Now
\[ \int_{\Omega} \Delta w_n \cdot \Delta v + \gamma \int_{\Omega} \nabla w_n \cdot \nabla v + \delta \int_{\Omega} w_n v \to \int_{\Omega} \Delta w \cdot \Delta v + \gamma \int_{\Omega} \nabla w \cdot \nabla v + \delta \int_{\Omega} w v \]
Hence (10) can be concluded from (11) if we show that \( 1/k_n f(u_n) \) converges (up to a subsequence) to \( aw^+ \) in \( L^2(\Omega) \). Now \( 1/k_n f(u_n) = 1/k_n f(k_n w_n) \) and it is easy to see that the required limit is equal to \( aw \) in the set \( \{ x \in \Omega : w_n(x) \to w(x) \neq 0 \} \).

If \( w(x) = 0 \) and \( w_n(x) \to w(x) \), let \( \varepsilon > 0 \) and \( n_0 \) be such that \( |w_n(x)| < \varepsilon \) for \( n \geq n_0 \). Then
\[ \frac{f(k_n w_n)}{k_n} \leq a\varepsilon + \frac{b}{k_n} \quad \text{for such} \quad n, \]
that is the required limit is 0. Thus, \( f(u_n)/k_n \to aw^+ \) a.e. Here \( b = f(0) \). Now \( w_n \to w \) in \( L^2(\Omega) \) and, thus, up to a subsequence, \( w_n \) is dominated in \( L^2(\Omega) \) (see [7, Theorem IV.9]).

Since \( 1/k_n f(u_n) \leq a|w_n| + 1/k_n b \), it follows that \( 1/k_n f(u_n) \) is also dominated. Hence (10) is now obtained. Now (10) and the maximum principle imply that \( w \geq 0 \) and (10) becomes
\[ \Delta^2 w - \gamma \Delta w + \delta w = \lambda a w \quad \text{in} \quad \Omega, \quad (12) \]
\[ w \geq 0 \quad \text{in} \quad \Omega, \]
\[ \|w\|_2 = 1 \quad \text{in} \quad \Omega. \]

Thus from (1), we have \( \lambda a = \lambda_1 \) and \( w = \varphi_1 \), which contradicts the fact that \( \lambda \neq \lambda_1/a \). This contradiction finishes the proof of the lemma 5.

Now, we need only to check that the two geometric assumptions of theorem 4 are fulfilled.

First, since \( u_\lambda \) is a local minimum of \( J_\lambda \), there exists \( R > 0 \) such that for all \( u \in E \) satisfying \( \|u - u_\lambda\| = R \), we have \( J_\lambda(u) \geq J_\lambda(u_\lambda) \). Then
\[ J_\lambda(u) - J_\lambda(u_\lambda) = J_\lambda''(u_\lambda)(u - u_\lambda, u - u_\lambda) + \rho \quad \text{where} \quad \rho > 0. \]
This makes \( u_\lambda \) becomes a strict local minimal for \( J \), which proves (\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast\ast).\n
Recall that \( \lim_{t \to +\infty} (f(t) - at) \) is finite, then there exists \( \beta \in \mathbb{R} \) such that
\[ f(t) \geq at + \beta, \quad \forall t > 0. \]
Hence
\[ F(t) \geq \frac{a \lambda}{2} t^2 + \beta \lambda t, \quad \forall t > 0. \]
This yields, using the definition of $\varphi_1$ mentioned in (1),

$$J_{\lambda}(t\varphi_1) = \frac{\lambda_1 - a\lambda}{2} t^2 \int_{\Omega} \varphi_1^2 - \beta \lambda t \int_{\Omega} \varphi_1,$$

since $\|\varphi_1\|_2 = 1$, then we have

$$\frac{J_{\lambda}(t\varphi_1)}{t^2} = \frac{\lambda_1 - a\lambda}{2} - \frac{\beta \lambda}{t} \int_{\Omega} \varphi_1$$

which implies

$$\limsup_{t \to +\infty} \frac{1}{t^2} J_{\lambda}(t\varphi_1) \leq \frac{\lambda_1 - a\lambda}{2} < 0, \quad \forall \lambda > \lambda_1/a.$$

Therefore

$$\lim_{t \to +\infty} J_{\lambda}(t\varphi_1) = -\infty.$$

So, there exists $v_0 \in E$ such that $J_{\lambda}(v_0) \leq J_{\lambda}(u_{\lambda})$ and (**) is proved.

Finally, let $\tilde{v}$ (respectively $\tilde{c}$) be the critical point (respectively critical value) of $J_{\lambda}$, we recall that the function $\tilde{v}$ belongs to $E$ and satisfies

$$\Delta^2 \tilde{v} - \gamma \Delta \tilde{v} + \delta \tilde{v} = \lambda f(\tilde{v}) \quad \text{in} \quad \Omega \quad \text{and} \quad J_{\lambda}(\tilde{v}) = \tilde{c}.$$

The next lemma states that the limit of a sequence of unstable solutions is also unstable.

**Lemma 6.** Let $u_n \rightharpoonup u$ in $H^2(\Omega) \cap H^1_0(\Omega)$ and $\mu_n \to \mu$ be such that $\eta_1(\mu_n, u_n) < 0$. Then, $\eta(\mu, u) < 0$.

**Proof:** The fact that $\eta_1(\mu_n, u_n) < 0$ is equivalent to the existence of a $\varphi_n \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$\int_{\Omega} |\Delta \varphi_n|^2 + \gamma \int_{\Omega} |\nabla \varphi_n|^2 + \delta \int_{\Omega} \varphi_n^2 \leq \mu_n \int_{\Omega} f'(u_n) \varphi_n^2 \quad \text{with} \quad \int_{\Omega} \varphi_n^2 = 1 \quad (14)$$

Since $f' \leq a$, (14) shows that $(\varphi_n)$ is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$. Let $\varphi \in E$ be such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $H^2(\Omega) \cap H^1_0(\Omega)$. Then

$$\mu_n \int_{\Omega} f'(u_n) \varphi_n^2 \to \mu \int_{\Omega} f'(u) \varphi^2$$

This can be seen by extracting from $(\varphi_n)$ a subsequence dominated in $L^2(\Omega))$ as in [7, Theorem IV.9]. Now we have

$$\int_{\Omega} |\Delta \varphi|^2 \leq \liminf_{n \to \infty} \int_{\Omega} |\Delta \varphi_n|^2 \quad \text{and} \quad \int_{\Omega} |\nabla \varphi|^2 \leq \liminf_{n \to \infty} \int_{\Omega} |\nabla \varphi_n|^2$$

finally, since $\|\varphi\|_2 = 1$, we obtain

$$\int_{\Omega} |\Delta \varphi|^2 + \int_{\Omega} |\nabla \varphi|^2 + \delta \int_{\Omega} \varphi^2 \leq \mu \int_{\Omega} f'(u) \varphi^2.$$

Obviously, the fact that the function $v$ belongs to $C^4(\Omega) \cap E$ follows from a bootstrap argument.

Actually, the next paragraph said a good deal more, giving additional information on precisely the comportment of the instable solution $u_{\lambda}$. 
5.4. **Proof (iii) (a).** By contradiction, thanks to Lemma 3, there is a sequence of positives scalars \((\lambda_n)\) and a sequence \((v_n)\) of unstable solutions to \((P_\lambda)\) such that \(v_n \to v\) in \(L^1_{\text{loc}}(\Omega)\) as \(\lambda_n \to \lambda_1/a\) for some function \(v\).

We first claim that \((v_n)\) cannot be bounded in \(E\). Otherwise, let \(w \in E\) be such that, up to a subsequence,

\[ v_n \rightharpoonup w \quad \text{weakly in } E \quad \text{and} \quad v_n \to w \quad \text{strongly in } L^2(\Omega). \]

Therefore,

\[ \Delta^2 v_n - \gamma \Delta v_n + \delta v_n \to \Delta^2 w - \gamma \Delta w + \delta w \quad \text{in } \mathcal{D}'(\Omega), \]

\[ f(v_n) \to f(w) \quad \text{in } L^2(\Omega), \]

which implies that \(\Delta^2 w - \gamma \Delta w + \delta w = \frac{\lambda_n}{a} f(w)\) in \(\Omega\). It follows that \(w \in E\) and solves \((P_\lambda)\) with \(\lambda_1/a\) in stead of \(\lambda\). From Lemma 6, we deduce that

\[ \eta_1 \left( \frac{\lambda_1}{a}, w \right) \leq 0. \quad (15) \]

Relation (15) shows that \(w \neq u_{\lambda_1/a}\) which contradicts the fact that \((P_\lambda)\) with \(\lambda_1/a\) in stead of \(\lambda\) has a unique solution. Now, since \(\Delta^2 v_n - \gamma \Delta v_n + \delta v_n = \lambda_n f(v_n)\), the unboundedness of \((v_n)\) in \(E\) implies that this sequence is unbounded in \(L^2(\Omega)\), too. To see this, let

\[ v_n = k_n w_n, \quad \text{where } k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \to \infty. \]

Then

\[ \Delta^2 w_n - \gamma \Delta w_n + \delta w_n = \frac{\lambda_n}{k_n} f(v_n) \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega). \]

So, we have convergence also in the sense of distributions and \((w_n)\) is seen to be bounded in \(E\) with standard arguments. We obtain

\[ \Delta^2 w - \gamma \Delta w + \delta w = 0 \quad \text{and} \quad \|w\|_2 = 1. \]

The desired contradiction is obtained since \(w \in E\).

5.5. **Proof (iii) (b).** We end the proof by showing that \(v_\lambda\) tends to \(u^*\) uniformly in \(\Omega\) when \(\lambda\) tends to \(\lambda^*\).

As before, it is sufficient to prove the \(L^2(\Omega)\) boundedness of \(v_\lambda\) near \(\lambda^*\) and to use the uniqueness property of \(u^*\). Assume that \(\|v_n\|_2 \to \infty\) as \(\lambda_n \to \lambda^*\), where \(v_n\) is a solution to \((P_{\lambda_n})\). We write again \(v_n = k_n w_n\). Then,

\[ \Delta^2 w_n - \gamma \Delta w_n + \delta w_n = \frac{\mu_n}{k_n} f(v_n). \quad (16) \]

The fact that the right-hand side of (16) is bounded in \(L^2(\Omega)\) implies that \((w_n)\) is bounded in \(E\). Let \((w_n)\) be such that (up to a subsequence)

\[ w_n \rightharpoonup w \quad \text{weakly in } E \quad \text{and} \quad w_n \to w \quad \text{strongly in } L^2(\Omega). \]

A computation already done shows that

\[ \Delta^2 w - \gamma \Delta w + \delta w = \lambda^* a w, \quad w \geq 0 \quad \text{and} \quad \|w\|_2 = 1, \]

which forces \(\lambda^*\) to be \(\lambda_1/a\). This contradiction concludes the proof.
In conclusion, all these results give us a rather clear schema of solutions for the quasilinear case $a \in (0, +\infty)$. An important role in our arguments has played by $l := \lim_{t \to +\infty} (f(t) - at)$. We distinguish two different situations strongly depending on the sign of $l$.

**Fig. 1:** Behavior of the minimal solution, $l > 0$

**Fig. 2:** Bifurcation branches, $l < 0$

**VI. The Dirichlet Problem**

**References Références Referencias**


5. S. Baraket, M. Khtai fi, T. Ouni, *Singular limit for 4-dimensional general stationary q-Kuramoto-Sivashinsky (q-KS) equations with exponential nonlinearity*, accepted in Anal Stiinti ce ale Universitatii Ovidius Constanta.


