Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Bifurcation for a Class of Fourth-Order Stationary Kuramoto-Sivashinsky Equations under Navier Boundary Condition

By Imed Abid, Soumaya Sâanouni & Nihed Trabelsi
University of Tunis El Manar

Abstract- In this paper, we study the bifurcation of semilinear elliptic problem of fourth-order with Navier boundary conditions. We discuss the existence and the uniqueness of a positive solution and we also prove the existence of critical value and the uniqueness of extremal solutions. We take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point.

Keywords: *bifurcation, regularity, stability, quasilinear.*

GJSFR-F Classification: *MSC 2010: 35B32, 35B65, 35B35, 35J62*

BIFURCATIONFORACLASSOFFOURTHORDERSTATIONARYKURAMOTOSIVASHINSKYEQUATIONSUNDERNAVIERBOUNDARYCONDITION

Bifurcation for a Class of Fourth-Order Stationary Kuramoto-Sivashinsky Equations under Navier Boundary Condition

Imed Abid [α], Soumaya Sâanouni [σ] & Nihed Trabelsi [ρ]

Abstract- In this paper, we study the bifurcation of semilinear elliptic problem of fourth-order with Navier boundary conditions. We discuss the existence and the uniqueness of a positive solution and we also prove the existence of critical value and the uniqueness of extremal solutions. We take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point.

Keywords: bifurcation, regularity, stability, quasilinear.

I. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, the Kuramoto-Sivashinsky (KS) equation

$$\begin{array}{rcll} \partial_t u + \Delta^2 u - \gamma\Delta u + \delta u - \beta|\nabla u|^2 & = & \lambda f(x,u) & \text{in} \quad \Omega, \\ \Delta u = u & = & 0 & \text{on} \quad \partial\Omega, \end{array}$$

arises in many applications from mathematical physics, which are usually used to describe some phenomena appearing in physics, engineering, and other sciences.

Moreover, the first bifurcation may be subcritical and bistability then occurs. The features qualitatively agree with the experiments. Finally, wave shapes are compared. The addition of the lighter order terms leads to better agreement with the experiments. The (KS) equation was the first non-linear wave equation to be proposed for describing long interfacial waves of two-layer couette and poiseuille flows. In agreement with previous studies, it is shown that each higher order term has a "laminarizing effect": the solutions of the (KS) equation simplify the benefit of stationary traveling waves.

In the stationary case and for $\gamma = \delta = \beta = 0$, various authors have studied the existence of weak solutions for the bifurcation problem

$$(E_\lambda) \quad \begin{cases} \Delta^2 u = \lambda f(u) & \text{in} \quad \Omega, \\ \Delta u = u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

Author α: University of Tunis El Manar, higher Institut of medicals technologies of Tunis, 9 street Dr.Zouhair Essa_ 1006 Tunis, Tunisia. e-mail: abidimed7@gmail.com

Author σ: University of Tunis El Manar, Faculty of Sciences of Tunis, Department of Mathematics, University Campus 2092 Tunis,Tunisia. e-mail: saanouni.soumaya@yahoo.com

Author ρ: University of Tunis El Manar, higher Institut of medicals technologies of Tunis, 9 street Dr.Zouhair Essafi 1006 Tunis, Tunisia. e-mail: nihed.trabelsi78@gmail.com

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Abid and al. have proved in [1] that there exists $0 < \lambda^* < \infty$, a critical value of the parameter λ, such as (E_λ) has a minimal, positive, classical solution u_λ for $0 < \lambda < \lambda^*$ and does not have a weak solution for $\lambda > \lambda^*$. When $\lim_{t\to\infty} \frac{f(t)}{t} = a < \infty$, it is proved also there exists a unique classical solution u^* of (E_{λ^*}) if and only if $\lim_{t\to+\infty} (f(t) - at) < 0$.

When $\delta = \beta = 0$, several researchers are interested in this type of phenomenon:

$$\begin{array}{rcl} \Delta^2 u - \gamma\Delta u &=& \lambda f(x,u) \quad \text{in} \quad \Omega, \\ \Delta u = u &=& 0 \qquad\qquad \text{on} \quad \partial\Omega. \end{array}$$

In [17, 18], Lazer and Mckenna firstly proposed and studied the problem of periodic oscillations and traveling waves in a suspension bridge. It was pointed out in [23, 28] that the problem provides a good model for the study of the static deflection of an elastic plate in a fluid. Moreover, Ahmed and Harbi in [2] showed that the problem can also be applied to engineering, such as communication satellites, space shuttles and space stations equipped with large antennas mounted on long flexible beams. Such problems appear for example in the micro-electromechanical systems giving the modelization of electrostatic actuation for membranes deflecting on thin plates in the field of nanotechnology detection systems in [19, 24] where the parameter γ represents the constant tension rising in the stretching energy sector in the presence of elastic deformation. It arises in mechanics too, see [11] and in electricity, see [16].

Ben Omrane and Khenissy in [6] show the nonexistence of solutions with Dirichlet boundary conditions. They prove a dichotomy result giving the positivity preserving property for a biharmonic equation arising in MEMS models.

When γ is not constant, other results prove the existence of solutions and study the bifurcation problem with Navier boundary conditions,

$$\Delta^2 u - \operatorname{div}(\gamma(x)\nabla u) = \lambda f(u) \quad \text{in } \Omega.$$

In [26], Sâanouni and Trabelsi show how the critical problem behaves when it is considered with the Navier boundary condition. The function $\gamma(x)$ is smooth positive on $\overline{\Omega}$ and the L^∞-norm of its gradient is small enough in order to assure ensure the application of maximum principle [12].

Our main interest here will be in the study of a bifurcation problem in stationary case for $\beta = 0$ and γ, δ, $\lambda > 0$, we consider the following problem

$$(P_\lambda) \quad \left\{ \begin{array}{rcl} \Delta^2 u - \gamma\Delta u + \delta u &=& \lambda f(u) \quad \text{in} \quad \Omega, \\ u &>& 0 \qquad\quad \text{in} \quad \Omega, \\ \Delta u = u &=& 0 \qquad\quad \text{on} \quad \partial\Omega, \end{array} \right.$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$; $(n \geq 2)$ and f is a positive, increasing and convex smooth function on $(0, +\infty)$, which verifies

$$\lim_{t\to\infty} \frac{f(t)}{t} = a \in (0, \infty).$$

This paper is organized as follows: In next section we state our main results contained in this work (see Theorems 1, 2 and 3). The content of section 3 is devoted to the proof of Theorem 1, concerns existence of minimal solutions. It is shown that there exists a limiting parameter λ^* such that one has existence of stable regular minimal solutions to

Ref

2. NU. Ahmed, H. Harbi, *Mathematical analysis of dynamic models of suspension bridges.* SIAM J. Appl. Math. 58, 853-874 (1998).

(P_λ) for $\lambda \in (0,\lambda^*)$, while for $\lambda > \lambda^*$, not even singular solutions exist. In Sections 4 and 5, we devoted to the proofs of Theorems 1 and 2, we take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point. Finally, in Section 6 we also give some hints on how to proceed for semilinear problems under Dirichlet boundary conditions.

Notes

II. Main Results

Throughout our paper, we denote by $\|\cdot\|_2$, the $L^2(\Omega)$-norm, whereas we denote by $\|\cdot\|$, the $H^2(\Omega)\cap H_0^1(\Omega)$-norm given by

$$\|u\|^2 = \int_\Omega |\Delta u|^2.$$

Definition 1. *We say that $u \in H^2(\Omega)\cap H_0^1(\Omega)$ is a weak solution of the problem (P_λ), if $f(u)\in L^1(\Omega)$ and*

$$\int_\Omega \Delta u.\Delta\varphi + \gamma\int_\Omega \nabla u.\nabla\varphi + \delta\int_\Omega u\varphi = \lambda\int_\Omega f(u)\varphi, \quad \forall\varphi\in C^2(\overline{\Omega})\cap H^2(\Omega)\cap H_0^1(\Omega).$$

Such solutions are usually known as weak energy solutions. For short, we will refer to them simply as solutions which is assured by the next lemma.

Remark 1. *Since $f(t)\leq at+f(0)$, if $u\in H^2(\Omega)\cap H_0^1(\Omega)$ is a weak solution of (P_λ) and $u\in L^1(\Omega)$, we say that u is regular solution. It is easily seen by a standard bootstrap argument that u is always a classical solution.*

For more detail, see [12, Proposition 7.15]. In the rest of this paper, we denote by a solution of problem (P_λ) any weak or classical solution.

Definition 2. *We say that $u\in H^2(\Omega)\cap H_0^1(\Omega)$ is a supersolution of (P_λ) if $f(u)\in L^1(\Omega)$ and*

$$\Delta^2 u - \gamma\Delta u + \delta u \geq \lambda f(u) \ \textit{in}\ \mathcal{D}'(\Omega).$$

Reversing the inequality one defines the notion of subsolution.

Next, we recall a version of the Maximum Principle for the biharmonic operator.

Proposition 1. *Let $u\in H^4(\Omega)\cap H_0^1(\Omega)$ be a function such that*

$$\Delta^2 u \geq 0 \ \textit{in}\ \Omega,\ \ \Delta u = u = 0 \ \textit{on}\ \partial\Omega.$$

Then

$$u(x)\geq 0,\ \ \Delta u(x)\leq 0 \ \textit{in}\ \Omega.$$

We say that a solution u_λ of problem (P_λ) is *minimal* if $u_\lambda \leq u$ in Ω for any solution u of (P_λ).

Recall that $\lambda_1\in\mathbb{R}$, be the first eigenvalue and φ_1 the first normalized eigenfunction of $(-\Delta)^2-\gamma\Delta+\delta$ in Ω with homogeneous Dirichlet boundary data

$$\begin{cases} \Delta^2 \varphi_1 - \gamma \Delta \varphi_1 + \delta \varphi_1 &= \lambda_1 \varphi_1 \quad \text{in} \quad \Omega, \\ \Delta \varphi_1 = \varphi_1 &= 0 \qquad \text{in} \quad \partial\Omega \\ \|\varphi_1\|_2 &= 1. \end{cases} \tag{1}$$

an eigenvalue λ_1 which is positive and that can be characterized as follows

$$\lambda_1 = \min_{\varphi \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \left(|\Delta \varphi|^2 + \gamma |\nabla \varphi|^2 + \delta \varphi^2\right)}{\displaystyle\int_\Omega |\varphi|^2}; \tag{2}$$

there exists a non-negative function $\varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, which is an eigenfunction corresponding to λ_1, attaining the minimum in (2), that is $\|\varphi_1\|_2 = 1$ and

$$\lambda_1 = \int_\Omega \left(|\Delta \varphi_1|^2 + \gamma |\nabla \varphi_1|^2 + \delta \varphi_1^2\right).$$

A solution u of problem (P_λ) is stable if and only if the first eigenvalue of the linearized operator

$$v \mapsto L_{\lambda,u}(v) := \Delta^2 v - \gamma \Delta v + \delta v - \lambda f'(u) v,$$

given by

$$\eta_1(\lambda, u) := \inf_{v \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \eta_1(\lambda, u)(v),$$

where for any $v \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$

$$\eta_1(\lambda, u)(v) = \frac{\displaystyle\int_\Omega \left(|\Delta v|^2 + \gamma |\nabla v|^2 + \delta v^2\right) - \lambda \int_\Omega f'(u) v^2 dx}{\|v\|_2^2}$$

is non negative. In other words,

$$\lambda \int_\Omega f'(u) v^2 dx \le \int_\Omega \left(|\Delta v|^2 + \gamma |\nabla v|^2 + \delta v^2\right), \quad \text{for any } v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3}$$

If $\eta(\lambda, u) < 0$, the solution u is said to be *unstable*.

Next, we let

$$\Lambda := \{\lambda > 0 \,|\, (P_\lambda) \text{ admits a solution}\} \text{ and } \lambda^* := \sup \Lambda \le +\infty.$$

and

$$r_0 := \inf_{t > 0} \frac{f(t)}{t}.$$

The two values a and r_0 that we have already defined will be important in the bifurcation phenomena. More precisely, in the frame of the critical value λ^*.

We propose to show the following results, first main statement asserts the existence of the critical value λ^*.

Theorem 1. *There exists a critical value $\lambda^* \in (0, \infty)$ such that the following properties hold:*

(i) *For any $\lambda \in (0, \lambda^*)$, problem (P_λ) has a minimal solution u_λ, which is the unique stable solution of (P_λ).*
(ii) *For any $\lambda \in (0, \lambda_1/a)$, u_λ is the unique solution of problem (P_λ).*
(iii) *The function $\lambda \longmapsto u_\lambda$ is a $\mathcal{C}^1$, convex, increasing function.*
(iv) *If (P_{λ^*}) has a solution, then $u^* := \lim_{\lambda \to \lambda^*} u_\lambda$ and $\eta_1(\lambda^*, u^*) = 0$.*

Notes

The next natural obvious object of study gives us more precise information for λ^*. An important role in our arguments will be played by

$$l := \lim_{t\to\infty}\Big(f(t) - at\Big).$$

We distinguish two different situations strongly depending on the sign of l.

Notes

Theorem 2. *Assume that $l \geq 0$. We have three equivalent assertions:*

(i) $\lambda^* = \lambda_1/a$.

(ii) *problem (P_{λ^*}) has no solution.*

(iii) $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* Ω.

Again the question arises as to what happens when $l < 0$. The following was proved.

Theorem 3. *Assume that $l < 0$. Then we have.*

(i) *the critical value λ^* belongs to $(\lambda_1/a, \lambda_1/r_0)$*

(ii) *(P_{λ^*}) has a unique solution u^*.*

(iii) *The problem (P_λ) has an unstable solution v_λ for any $\lambda \in (\lambda_1/a, \lambda^*)$ and the sequence $(v_\lambda)_\lambda$ satisfies:*

(a) $\lim_{\lambda\to\lambda_1/a} v_\lambda = \infty$ *uniformly on compact subsets of* Ω,

(b) $\lim_{\lambda\to\lambda^*} v_\lambda = u^*$ *uniformly in* Ω.

III. Proof of Theorem 1

We say that a solution u_λ of problem (P_λ) is *minimal* if $u_\lambda \leq u$ in Ω for any solution u of (P_λ).

Lemma 1. *Problem (P_λ) has no solution for any $\lambda > \lambda_1/r_0$, but has at least one solution provided λ is positive and small enough.*

Proof: First, to show that (P_λ) has a solution, we use the barrier method.

Since $f(0) > 0$, $\underline{u} \equiv 0$ is a strict subsolution of (P_λ) for every $\lambda > 0$. To this aim, let $\overline{w} \in H^4(\Omega)$ which satisfies

$$\begin{cases} \Delta^2\overline{w} - \gamma\Delta\overline{w} + \delta\overline{w} &= 1 \text{ in } \Omega, \\ \Delta\overline{w} = \overline{w} &= 0 \text{ in } \partial\Omega. \end{cases}$$

The choice of $\overline{w}$ implies that $\overline{w}$ is a bounded supersolution of (P_λ) for small λ, more precisely whenever $\lambda < 1/f(\|\overline{u}\|_\infty)$.

Notice that for any $\lambda > 0$, the function $\underline{w} \equiv 0$ is a sub-solution of (P_λ) since $f(0) > 0$. Next, we define a sequence $w_n \in H^4(\Omega)$ by

$$\begin{cases} \Delta^2 w_{n+1} - \gamma\Delta w_{n+1} + \delta w_{n+1} &= \lambda f(w_n) \quad \text{in} \quad \Omega \\ \Delta w_{n+1} = w_{n+1} &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

The maximum principle (see [7]) implies that

$$\underline{w} \leq w_n \leq w_{n+1} \leq \overline{w} \text{ for all } n \in \mathbb{N},$$

so that the sequence $(w_n)_{n\geq 0}$ is increasing and bounded, then it converges. It follows that problem (P_λ) has a solution.

Assume now that u is a solution of (P_λ) for some $\lambda > 0$. Using φ_1 given in (1) as a test function and integrating by parts, we get

$$\begin{aligned}\lambda_1 \int_\Omega \varphi_1 \, u &= \int_\Omega (\Delta^2\varphi_1 - \gamma\Delta\varphi_1 + \delta\varphi_1)u = \int_\Omega \Delta^2 u \, \varphi_1 - \gamma \int_\Omega \Delta u \, \varphi_1 + \delta \int_\Omega u \, \varphi_1 \\ &= \lambda \int_\Omega f(u) \, \varphi_1 \geq \lambda \, r_0 \int_\Omega u \, \varphi_1.\end{aligned}$$

This yields

$$(\lambda_1 - \lambda r_0) \int_\Omega \varphi_1 u \geq 0.$$

Since $\varphi_1 > 0$ and $u > 0$, we conclude that the parameter λ should belong to $(0, \lambda_1/r_0)$.

As a consequence we have that λ^* is a real. Another useful result is stated in what follows.

Lemma 2. Assume that (P_λ) is resolvable, then a minimal solution u_λ exists. Moreover, $(P_{\lambda'})$ is resolvable for any $\lambda' \in (0, \lambda)$.

Proof: Fix $\lambda \in (0, \lambda^*)$ and let u be a solution of (P_λ). As above, we use the barrier method to obtain a minimal solution of (P_λ). The basic idea is to prove by induction that the sequence $(w_n)_{n\geq 0}$ defined in (4) is increasing and bounded by u, so it converges to some solution u_λ. Since u_λ is independent of the choice of u, then it is a minimal solution.

Now, if u is a solution of (P_λ), then u is a super-solution for the problem $(P_{\lambda'})$ for any λ' in $(0, \lambda)$ and 0 can be used always as a sub-solution.

Remark 2. Thanks to lemmas 1 and 2, the set Λ is an interval not empty and bounded.

a) Proof of Theorem 1.

i. *Proof of (i).* First, we claim that u_λ is stable. Indeed, arguing by contradiction, i.e. the first eigenvalue $\eta_1(\lambda, u_\lambda)$ is negative. Then, there exists an eigenfunction $\in H^4(\Omega)$ such that

$$\begin{aligned}\Delta^2\psi - \gamma\Delta\psi + \delta\psi - \lambda f'(u_\lambda)\,\psi &= \eta_1\psi \quad \text{in} \quad \Omega,\\ \psi &> 0 \quad \text{in} \quad \Omega\\ \Delta\psi = \psi &= 0 \quad \text{on} \quad \partial\Omega.\end{aligned}$$

Consider $u^\varepsilon := u_\lambda - \varepsilon\,\psi$. Hence, by linearity, we have

$$\begin{aligned}\Delta^2 u^\varepsilon - \gamma\Delta u^\varepsilon + \delta u^\varepsilon - \lambda f(u^\varepsilon) &= \lambda f(u_\lambda) - \varepsilon(\Delta^2\psi - \gamma\Delta\psi + \delta\psi) - \lambda f(u_\lambda - \varepsilon\psi)\\ &= \lambda f(u_\lambda) - \varepsilon(\lambda f'(u_\lambda)\psi + \eta_1\psi) - \lambda f(u_\lambda - \varepsilon\psi)\\ &= \lambda \left[f(u_\lambda) - f(u_\lambda - \varepsilon\psi) - \varepsilon f'(u_\lambda)\psi\right] - \varepsilon\,\eta_1\psi\\ &= \lambda\, o_\varepsilon(\varepsilon\psi) - \varepsilon\,\eta_1\psi\\ &= \varepsilon\,\psi\,(\lambda\, o_\varepsilon(1) - \eta_1).\end{aligned}$$

Since $\eta_1(\lambda, u_\lambda) < 0$, for $\varepsilon > 0$ small enough, we have

$$\Delta^2 u^\varepsilon - \gamma\Delta u^\varepsilon + \delta u^\varepsilon - \lambda f(u^\varepsilon) \geq 0 \quad \text{in } \Omega.$$

Then, for $\varepsilon > 0$ small enough, we use the strong maximum principle (Hopf's lemma, see [14]) to deduce that $u^\varepsilon \geq 0$ is a super-solution of (P_λ). As before, we obtain a solution u such that $u \leq u^\varepsilon$ and since $u^\varepsilon < u_\lambda$, then we contradict the minimality of u_λ.

Now, we show that (P_λ) has at most one stable solution. Assume the existence of another stable solution $v \neq u_\lambda$ of problem (P_λ). Let $w := v - u_\lambda$, then by maximum principle $w > 0$ and from (3) taking w as a test function, we have

Notes

$$\begin{aligned}
\lambda \int_\Omega f'(v)\, w^2 &\le \int_\Omega |\Delta w|^2 + \gamma \int_\Omega |\nabla w|^2 + \delta \int_\Omega w^2 \\
&\le \int_\Omega \Delta^2 w\; w - \gamma \int_\Omega \Delta w\; w + \delta \int_\Omega w\; w \\
&\le \int_\Omega \Big[\Delta^2 v - \gamma \Delta v + \delta v - \Delta^2 u_\lambda + \gamma \Delta u_\lambda - \delta u_\lambda\Big]\; w \\
&\le \lambda \int_\Omega \Big[f(v) - f(u_\lambda)\Big]\; w.
\end{aligned}$$

Notes

Therefore

$$\int_\Omega \Big[f(v) - f(u_\lambda) - f'(v)(v-u_\lambda)\Big] w \ge 0.$$

Thanks to the convexity of f, the term in the brackets is nonpositive, hence

$$f(v) - f(u_\lambda) - f'(v)(v-u_\lambda) = 0 \text{ in } \Omega,$$

which implies that f is affine over $[u_\lambda, v]$ in Ω. So, there exists two real numbers α and β such that

$$f(x) = \alpha x + \beta \quad \text{ in } [0, \max_\Omega v].$$

Finally, since u_λ and v are two solutions to $\Delta^2 w - \gamma \Delta w + \delta w = \lambda \alpha w + \lambda \beta$, we obtain that

$$0 = \int_\Omega \Big(u_\lambda\, \Delta^2 v - v\, \Delta^2 u_\lambda\Big) - \gamma \int_\Omega \Big(u_\lambda\, \Delta v - v\, \Delta u_\lambda\Big) + \delta \int_\Omega \Big(u_\lambda\, v - v\, u_\lambda\Big) = \lambda\beta \int_\Omega (u_\lambda - v).$$

This is impossible since $\beta = f(0) > 0$ and $w = v - u_\lambda$ is positive in Ω.

3.1.2. *Proof of (ii).* Recall that λ_1 is defined in (1). By the convexity of f, we deduce that $a = \sup_{\mathbb{R}_+} f'(t)$. Let u be a solution to (P_λ) for $\lambda \in (0, \lambda_1/a)$, we suppose that u is unstable. Then, we can take $\varphi = \varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ which satisfy

$$\lambda\; a \int_\Omega \varphi^2 \ge \lambda \int_\Omega f'(u)\varphi^2 > \int_\Omega |\Delta \varphi|^2 + \gamma \int_\Omega |\nabla \varphi|^2 + \delta \int_\Omega \varphi^2 = \lambda_1 \int_\Omega \varphi^2,$$

which shows that

$$(\lambda\; a - \lambda_1) \int_\Omega \varphi^2 > 0.$$

Impossible for $\lambda \in (0, \lambda_1/a)$. Then, $\eta_1(\lambda, u) \ge 0$, so we obtain the uniqueness of u.

For the existence, we consider the minimization problem

$$\min_{u \in H^2(\Omega)\cap H_0^1(\Omega)} \mathcal{J}_\lambda(u),$$

where

$$\mathcal{J}_\lambda(u) := \frac{1}{2} \int_\Omega \Big(|\Delta u|^2 + \gamma |\nabla u|^2 + \delta u^2\Big) - \lambda \int_\Omega F(u), \quad \text{for all} \quad u \in H^2(\Omega) \cap H_0^1(\Omega)$$

with

$$u^+ := \max(u, 0) \quad \text{and} \quad F(u) := \int_0^{u^+} f(s) ds.$$

If $\lambda \in (0, \lambda_1/a)$, there exist $\varepsilon > 0$ and $A > 0$ depending on λ such that

$$2\lambda F(t) \le (\lambda_1 - \varepsilon) t^2 + A, \qquad \forall\, t \in \mathbb{R}.$$

Standard arguments imply that $\mathcal{J}_\lambda(u)$ is coercive, bounded from below and weakly lower semi-continuous in $H^2(\Omega)\cap H_0^1(\Omega)$ (see propositions 2.2 and 2.3 in [29]). It is easy to see that the minimum of $\mathcal{J}_\lambda$ is attained by some function $u \in H^2(\Omega)\cap H_0^1(\Omega)$. So, the critical point u of $\mathcal{J}_\lambda$ gives a solution of (P_λ).

3.1.3. *Proof of (iii).* By sub- and super-solution method, see Lemma 2 we obtain that the mapping $\lambda \mapsto u_\lambda$ is increasing and this proves (iii).

3.1.4. *Proof of (iv).* Now Consider the nonlinear operator

$$\begin{array}{rcc} G : (0,+\infty)\times C^{4,\alpha}(\overline{\Omega})\cap E & \longrightarrow & C^{0,\alpha}(\overline{\Omega}) \\ (\lambda, u) & \longmapsto & \Delta^2 u - \gamma\Delta u + \delta u - \lambda f(u), \end{array}$$

where $\alpha \in (0,1)$ and E is the function space defined by

$$E := \{u \in W^{4,2}(\Omega) \mid \Delta u \;=\; u \;=\; 0 \text{ on } \partial\Omega\}. \tag{5}$$

Assume that (P_{λ^*}) has a solution u. Then for any $\lambda \in (0,\lambda^*)$, $u_\lambda \le u$ in Ω.

Then for every $\lambda \in (0,\lambda^*)$ we have $u_\lambda \le u^*$ in Ω. Using the monotonicity of u_λ, we deduce that the function

$$u^* = \lim_{\lambda\to\lambda^*} u_\lambda$$

is well defined in Ω and is a semi-stable solution of problem (P_{λ^*}). Assuming that the first eigenvalue $\eta_1(\lambda^*, u^*)$ is positive, we can apply the implicit function theorem to the operator G. For any λ in a neighborhood of λ^* and u in a neighborhood of u^*, we have $G(\lambda,u) = 0$, which proves that the problem (P_λ) has a solution for λ in a neighborhood of λ^*. But this contradicts the definition of λ^*. So, $\eta_1(\lambda^*,u^*) = 0$ and this completes the proof of Theorem 1.

IV. Proof of Theorem 2

Remark 3. *Thanks to Lemma 1 and (ii) of Theorem 1, the critical value λ^* satisfies:*

$$\lambda_1/a \le \lambda^* \le \lambda_1/r_0.$$

To prove this theorem, we show that the three assertions are equivalent. And finally, we prove that one hoolds. We shall use the following auxiliary result which is a reformulation of Theorem due to Hörmander [15].

Lemma 3. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \ge 2$ with smooth boundary. Let (u_n) be a sequence of super-harmonic nonnegative functions defined on Ω. Then the following alternative holds:*

- : *(i) either $\lim_{n\to\infty} u_n = \infty$ uniformly on compact subsets of Ω,*
- : *(ii) or (u_n) contains a subsequence which converges in $L^1_{loc}(\Omega)$ to some function u.*

Remark 4. *The result by Hörmander is also true if (u_n) is a sequence of a super-harmonic nonnegative functions.*

4.1. *Proof.* **(i)⇒(ii).** By contradiction. We assume that $\lambda^* = \frac{\lambda_1}{a}$. If (P_{λ^*}) has a solution u^* then, as we have already observed in (iv) of Theorem 1, $\eta_1(\lambda^*,u^*) = 0$. Thus, there exists $\in H^4(\Omega)$ satisfying:

$$\begin{aligned} \Delta^2\psi - \gamma\Delta\psi + \delta\psi - \lambda^* f'(u^*)\,\psi \;&= 0 \qquad \text{in } \Omega \\ \psi &> 0 \qquad \text{in } \Omega \\ \Delta\psi = \;\psi &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Notes

Using φ_1, given in (1), as a test function and integrating by parts, we obtain

$$\int_\Omega \Big(\Delta^2\varphi_1 - \gamma\Delta\varphi_1 + \delta\varphi_1\Big) \quad - \lambda^* \int_\Omega f'(u^*)\ \psi\ \varphi_1 = 0$$

therefore

$$\int_\Omega \Big(\lambda_1 - \lambda^* f'(u^*)\Big)\ \psi\ \varphi_1 = 0.$$

$\varphi_1 > 0$, $\psi > 0$, $\lambda^* = \frac{\lambda_1}{a}$ and $a = \sup_{t>0} f'(t)$, we have $\lambda_1 - \lambda^* f'(u^*) \geq 0$, the above equation forces $\lambda_1 - \lambda^* f'(u^*) = 0$. Hence

$$f'(u^*) \equiv a \quad \text{in} \quad \Omega.$$

This implies that $f(t) = at + b$ in $[0, \max_\Omega u^*]$ for some scalar $b > 0$. But there is no positive function in Ω such that $u = \Delta u = 0$ on $\partial\Omega$ and

$$\Delta^2 u - \gamma\Delta u + \delta u = \lambda^* a u + \lambda^* b \quad \text{in } \Omega.$$

If not, using φ_1 and integrating by parts, we have

$$\int_\Omega \Delta^2 u\ \varphi_1 - \gamma \int_\Omega \Delta u\ \varphi_1 + \delta \int_\Omega u\ \varphi_1 = \lambda^* a \int_\Omega u\ \varphi_1 + \lambda^* b \int_\Omega \varphi_1$$

then

$$\int_\Omega \Big(\Delta^2\varphi_1 - \gamma\Delta\varphi_1 + \delta\varphi_1\Big)\ u = \lambda_1 \int_\Omega u\ \varphi_1 + \lambda^* b \int_\Omega \varphi_1$$

i.e.

$$0 = \lambda^* b \int_\Omega \varphi_1 \quad \text{which is impossible.}$$

Hence, problem (P_{λ^*}) has no solution and (i) implies (ii).

4.2. *Proof* **(ii) $\Rightarrow$(iii).** We assume that (ii) occurs and we claim that $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ uniformly on compact subsets of Ω. By contradiction, suppose that (iii) doesn't hold. By Lemma 4 and up to a subsequence, (u_λ) converges locally in $L^1(\Omega)$ to u^* as $\lambda \to \lambda^*$.

Lemma 4. The minimal solution u_λ of the problem (P_λ) is bounded in $L^2(\Omega)$.

Proof. If not, we define

$$u_\lambda := k_\lambda w_\lambda,$$

with

$$\|w_\lambda\|_2 = 1 \quad \text{and} \quad k_\lambda \to +\infty \quad \text{as} \quad \lambda \to \lambda^*.$$

Since $f(t) \leq at + f(0)$, We have

$$\begin{aligned}
\int_\Omega |\Delta w_\lambda|^2 &\leq \int_\Omega |\Delta w_\lambda|^2 + \gamma \int_\Omega |\nabla w_\lambda|^2 + \delta \int_\Omega w_\lambda^2 \\
&= \int_\Omega \Delta^2 w_\lambda\ w_\lambda - \gamma \int_\Omega \Delta w_\lambda\ w_\lambda + \delta \int_\Omega w_\lambda\ w_\lambda = \int_\Omega \frac{\lambda f(u_\lambda)}{k_\lambda} w_\lambda \\
&\leq \lambda^* \int_\Omega \Big(a\, w_\lambda^2 + \frac{f(0)}{k_\lambda} w_\lambda\Big) \leq \lambda^* a - c \int_\Omega w_\lambda \\
&\leq \lambda^* a - c\sqrt{|\Omega|},
\end{aligned}$$

where c is a positive constant independent on λ.

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version 1 Year 2018

Recall that w_λ satisfies $\Delta^2 w_\lambda - \gamma\Delta w_\lambda + \delta w_\lambda = \dfrac{\lambda f(k_\lambda w_\lambda)}{k_\lambda}$ and f is quasilinear. These facts imply that (w_λ) is bounded in $H^4(\Omega)$. Hence, up to a subsequence, we have

$$w_\lambda \rightharpoonup w \text{ weakly in } H^4(\Omega) \text{ and } w_\lambda \to w \text{ strongly in } H^3(\Omega) \text{ as } \lambda \to \lambda^*.$$

Moreover, by the trace theorem, $w = \Delta w = 0$ on $\partial\Omega$. We deduce that

$$\Delta^2 w_\lambda - \gamma\Delta w_\lambda + \delta w_\lambda = \frac{\lambda f(u_\lambda)}{k_\lambda} \longrightarrow 0 \text{ in } L^1_{loc}(\Omega) \text{ as } \lambda \to \lambda^*.$$

This implies $\Delta^2 w - \gamma\Delta w_\lambda + \delta w_\lambda = 0$ in $\mathcal{D}'(\Omega)$. So, by Navier boundary condition, we deduce that $w \equiv 0$ in Ω. This contradicts the fact that $\|w\|_2 = \lim_{\lambda\to\lambda^*} \|w_\lambda\|_2 = 1$. This complete the proof of lemma.

Hence, (u_λ) is bounded in $L^2(\Omega)$ and by the same arguments as above, it is bounded in $H^4(\Omega)$ and up to a subsequence, we have

$$u_\lambda \rightharpoonup u \text{ weakly in } H^4(\Omega) \text{ and } u_\lambda \to u \text{ in } L^2(\Omega) \text{ as } \lambda \to \lambda^*.$$

$$\begin{cases} \Delta^2 u - \gamma\Delta u + \delta u &= \lambda^* f(u) \quad \text{in} \quad \Omega \\ \Delta u = u &= 0 \qquad\quad \text{on} \quad \partial\Omega \end{cases}$$

and this impossible by the hypothesis (ii). This shows that (ii) implies (iii). Moreover, this simply shows that (ii) and (iii) are equivalent.

4.3. *Proof* **(iii)⇒(i).** if (P_{λ^*}) has a solution u^* then the sequence (u_λ) converges to u^* as λ tends to $^*\lambda$, which cannot happen in the case where $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$. Hence, (iii) implies (i).

Indeed, clearly if (ii) and (iii) occur, we have $\lim_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Set

$$u_\lambda = k_\lambda\, w_\lambda \text{ with } \|w_\lambda\|_2 = 1.$$

Then, up to a subsequence, we obtain

$$w_\lambda \rightharpoonup w \text{ weakly in } H^4(\Omega) \text{ and } w_\lambda \to w \text{ strongly in } H^3(\Omega) \text{ as } \lambda \to \lambda^*.$$

Moreover,

$$\Delta^2 w_\lambda - \gamma\Delta w_\lambda + \delta w_\lambda \longrightarrow \Delta^2 w - \gamma\Delta w + \delta w \quad \text{in } \mathcal{D}'(\Omega) \text{ as } \lambda \to \lambda^*$$

and

$$\frac{\lambda}{k_\lambda} f(k_\lambda w_\lambda) \longrightarrow \lambda^* a w \quad \text{in } L^2(\Omega) \text{ as } \lambda \to \lambda^*.$$

Then,

$$\begin{cases} \Delta^2 w - \gamma\Delta w + \delta w &= \lambda^* a w \quad \text{in } \Omega, \\ \Delta w = w &= 0 \qquad\quad \text{on } \partial\Omega. \end{cases}$$

Multiplying by φ_1, which is defined in (1), we obtain

$$\begin{aligned} \int_\Omega \lambda^* a\, w\, \varphi_1 &= \int_\Omega \Delta^2 w\, \varphi_1 - \gamma \int_\Omega \Delta w\, \varphi_1 + \delta \int_\Omega w\, \varphi_1 \\ &= \int_\Omega \Delta^2 \varphi_1\, w - \gamma \int_\Omega \Delta v_1\, w + \delta \int_\Omega \varphi_1\, w = \int_\Omega \lambda_1\, \varphi_1\, w. \end{aligned}$$

Notes

This proves (i). To finish the proof of Theorem 2, we need only to show that $(P_{\lambda_1/a})$ has no solution. Indeed, assume that u is a solution of $(P_{\lambda_1/a})$. Since $f(t) - at \geq 0$, we have

$$\Delta^2 u - \gamma \Delta u + \delta u = \frac{\lambda_1}{a} f(u) \geq \lambda_1 u \text{ in } \Omega.$$

Multiplying the previous equation by φ_1 and integrating by parts, we get $f(u) = a\,u$ in Ω, which contradicts $f(0) > 0$. This concludes the proof of Theorem 2.

Notes

Remark 5. Observe that the equivalence of the assertions of Theorem 2 does not depend on the sign of l.

V. Proof of Theorem 3

5.1. *Proof* **(i).** For the first part of Theorem 3, we have already seen in Remark 3 that $\lambda_1/a \leq \lambda^* \leq \lambda_1/r_0$. Hence it suffices to prove that $\lambda^* \neq \lambda_1/a$ and $\lambda^* \neq \lambda_1/r_0$. First, assume that $\lambda^* = \lambda_1/a$. By Remark 5, we have

$$\lim_{\lambda \to \lambda^*} u_\lambda = \infty \text{ uniformly on compact subsets of } \Omega.$$

Let u_λ be the minimal solution to (P_λ). Then, multiplying (P_λ) by φ_1 and integrating, we obtain

$$0 = \int_\Omega \Big(\lambda_1 u_\lambda - \lambda f(u_\lambda)\Big)\varphi_1 = \int_\Omega \Big((\lambda_1 - a\lambda)u_\lambda - \lambda(f(u_\lambda) - au_\lambda)\Big)\varphi_1$$

and then

$$\lambda \int_\Omega \varphi_1 \Big(f(u_\lambda) - au_\lambda\Big) \geq 0.$$

Passing to the limit in the last inequality as λ tends to λ^*, we find

$$0 \leq l\lambda^* \int_\Omega \varphi_1 < 0,$$

which is impossible and then $\lambda^* \neq \frac{\lambda_1}{a} a$.

Now, assume that $\lambda^* = \lambda_1/r_0$ and let u be a solution of problem (P_{λ^*}). Multiplying (P_{λ^*}) by φ_1 and integrating by parts, we have

$$\lambda_1 \int_\Omega u\varphi_1 = \frac{\lambda_1}{r_0} \int_\Omega f(u)\varphi_1$$

that is

$$\int_\Omega (f(u) - r_0 u)\varphi_1 = 0$$

which forces $f(u) = r_0\, u$ in Ω, so that $f(t) = r_0 t$ in $[0, \max_\Omega u]$. As above, this contradicts the fact that $f(0) > 0$.

5.2. *Proof* **(ii).** Since $\lambda^* > \lambda_1/a$, the existence of a solution to (P_{λ^*}) is assured by Remark 5. Then, it remains to prove the uniqueness. Assume that u is another solution to (P_{λ^*}) and let $w := u - u^*$. Since $u_\lambda < u$ and $\lim_{\lambda \to \lambda^*} u_\lambda = u^*$, we have $w \geq 0$. Then by convexity of f we have

$$\Delta^2 w - \gamma \Delta w + \delta w = \lambda^* \Big(f(u) - f(u^*)\Big) \geq \lambda^* f'(u^*) w \text{ in } \Omega.$$

Recall that $\eta_1(\lambda^*, u^*) = 0$, so let be the corresponding eigenfunction. Multiplying the last inequality by and integrating by parts, we find

$$0 = \int_\Omega \lambda^* \Big(f(u) - f(u^*) - f'(u^*)w \Big) \geq 0.$$

Therefore, we must have equality $f(u) - f(u^*) = f'(u^*)w$ in Ω, which implies that f is linear in $[0, \max_\Omega u]$ and this leads a contradiction as in the proof of Theorem 1.

Notes

5.3. *Proof* **(iii).** concerning the existence of a non stable solution v_λ of (P_λ) will be proved by using the mountain pass theorem of Ambrosetti and Rabinowitz [3] in the following form:

Theorem 4. *Let E be a real Banach space and $J \in \mathcal{C}^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions:*
$(*)$ *there exist positive constants R and ρ such that*

$$J(u) \geq J(u_0) + \rho, \text{ for all } u \in E \text{ with } \|u - u_0\| = R.$$

$(**)$ *there exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$.*

Then the functional J possesses at least a critical point. The critical value is characterized by

$$c := \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where

$$\Gamma := \Big\{ g \in C([0,1], E) \,|\, g(0) = u_0,\ g(1) = v_0 \Big\}$$

and satisfies

$$c \geq J(u_0) + \rho.$$

In our case,

$$\begin{aligned} \mathcal{J}_\lambda : \quad E &\longrightarrow \mathbb{R} \\ u &\longmapsto \frac{1}{2}\Big(\int_\Omega |\Delta u|^2 + \gamma |\nabla u|^2 + \delta u^2 \Big) - \int_\Omega F(u), \end{aligned}$$

where

$$F(t) = \lambda \int_0^t f(s) ds, \quad \text{for all } t \geq 0.$$

We take u_0 as the stable solution u_λ for each $\lambda \in (\lambda_1/a, \lambda^*)$.

Remark 6. *The energy functional $\mathcal{J}_\lambda$ belongs to $\mathcal{C}^1(E, \mathbb{R})$ and*

$$\langle \mathcal{J}_\lambda'(u), v \rangle = \int_\Omega \Delta u \Delta v + \gamma \int_\Omega \nabla u \nabla v + \delta \int_\Omega uv - \lambda \int_\Omega f(u) v dx, \text{ for all } u,\ v \in E.$$

Since $\eta_1(\lambda, u_\lambda) > 0$, the function u_λ is a strict local minimum for $\mathcal{J}_\lambda$, we apply the mountain pass theorem for $\mathcal{J}_\lambda$. We show in the next lemma that $\mathcal{J}_\lambda$ satisfies the Palais-Smale compactness condition.

Lemma 5. Let $(u_n) \subset E$ be a Palais-Smale sequence; that is,

$$\sup_{n\in\mathbb{N}} |\mathcal{J}_\lambda(u_n)| < +\infty, \tag{6}$$

$$\|\mathcal{J}'_\lambda(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{7}$$

Then (u_n) is relatively compact in E.

Notes

Proof: Since any subsequence of (u_n) verifies (6) and (7) it is enough to prove that (u_n) contains a convergent subsequence. It suffices to prove that (u_n) contains a bounded subsequence in E. Indeed, suppose we have proved this. Then, up to a subsequence, $u_n \to u$ weakly in E, strongly in $L^2(\Omega)$. Now (7) gives

$$\Delta^2 u_n - \gamma\Delta u_n + \delta u_n - \lambda f(u_n) \to 0 \quad \text{in } \mathcal{D}'(\Omega)$$

Note that $f(u_n) \to f(u)$ in $L^2(\Omega)$ because $|f(u_n) - f(u)| \le a|u_n - u|$. This shows that

$$\Delta^2 u_n - \gamma\Delta u_n + \delta u_n \to \lambda f(u) \quad \text{in } \mathcal{D}'(\Omega).$$

That is

$$\Delta^2 u - \gamma\Delta u + \delta u - \lambda f(u) = 0.$$

The above equality multiplied by u gives

$$\int_\Omega |\Delta u|^2 + \gamma \int_\Omega |\nabla u|^2 + \delta \int_\Omega u^2 - \lambda \int_\Omega f(u)u = 0. \tag{8}$$

Now (7) multiplied by (u_n) gives

$$\int_\Omega |\Delta u_n|^2 + \gamma \int_\Omega |\nabla u_n|^2 + \delta \int_\Omega u_n^2 - \lambda \int_\Omega f(u_n)u_n \to 0 \tag{9}$$

in view of the boundedness of (u_n) and the $L^2(\Omega)$-convergence of u_n and $f(u_n)$, we have

$$\lambda \int_\Omega f(u_n)u_n \to \lambda \int_\Omega f(u)u$$

Hence, (8) and (9) give

$$\int_\Omega |\Delta u_n|^2 \to \int_\Omega |\Delta u|^2 \quad \text{and} \quad \gamma \int_\Omega |\nabla u_n|^2 \to \gamma \int_\Omega |\nabla u|^2$$

which insures us that $u_n \to u$ in E.

Actually, it is enough to prove that (u_n) is (up to a subsequence) bounded in $L^2(\Omega)$. Indeed, the $L^2(\Omega)$-boundedness of (u_n) implies that E-boundedness of (u_n) as it can be seen by examining (6).

We shall conclude the proof obtaining a contradiction from the supposition that $\|u_n\|_2 \to \infty$. Let $u_n = k_n w_n$ with $k_n > 0$, $k_n \to \infty$ and $\|w_n\|_2 = 1$. Then

$$0 = \lim_{n\to\infty} \frac{\mathcal{J}_\lambda(u_n)}{k_n^2} = \lim_{n\to\infty} \left[\frac{1}{2}\int_\Omega |\Delta w_n|^2 + \frac{\gamma}{2}\int_\Omega |\nabla w_n|^2 + \frac{\delta}{2}\int_\Omega w_n^2 - \frac{1}{k_n^2}\int_\Omega F(u_n)\right]$$

However, since $|f(t)| \le a|t| + b$, we have

$$|F(u_n)| = |F(k_n w_n)| \le \frac{a\lambda}{2} k_n^2 w_n^2 + b\lambda |k_n w_n|.$$

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version 1 Year 2018

This shows that

$$\frac{1}{k_n^2}\int_\Omega F(u_n) \leq \frac{a\lambda}{2}\int_\Omega w_n^2 + \frac{b\lambda}{k_n}\int_\Omega w_n < \infty.$$

We claim that

$$\Delta^2 w - \gamma\Delta w + \delta w = a\lambda w^+ \quad \text{where } w^+ := \max\{0, w\}. \tag{10}$$

Indeed, (7) divided by k_n gives

$$\int_\Omega \Delta w_n \cdot \Delta v + \gamma\int_\Omega \nabla w_n \cdot \nabla v + \delta\int_\Omega w_n\, v - \lambda\int_\Omega \frac{f(u_n)}{k_n} v \to 0 \tag{11}$$

for each $v \in E$. Now

$$\int_\Omega \Delta w_n \cdot \Delta v + \gamma\int_\Omega \nabla w_n \cdot \nabla v + \delta\int_\Omega w_n\, v \to \int_\Omega \Delta w \cdot \Delta v + \gamma\int_\Omega \nabla w \cdot \nabla v + \delta\int_\Omega w\, v$$

Hence (10) can be concluded from (11) if we show that $1/k_n f(u_n)$ converges (up to a subsequence) to aw^+ in $L^2(\Omega)$. Now $1/k_n f(u_n) = 1/k_n f(k_n w_n)$ and it is easy to see that the required limit is equal to aw in the set $\{x \in \Omega : w_n(x) \to w(x) \neq 0\}$.

If $w(x) = 0$ and $w_n(x) \to w(x)$, let $\varepsilon > 0$ and n_0 be such that $|w_n(x)| < \varepsilon$ for $n \geq n_0$. Then

$$\frac{f(k_n w_n)}{k_n} \leq a\varepsilon + \frac{b}{k_n} \quad \text{for such } \; n,$$

that is the required limit is 0. Thus, $f(u_n)/k_n \to aw^+$ a.e. Here $b = f(0)$. Now $w_n \to w$ in $L^2(\Omega)$ and, thus, up to a subsequence, w_n is dominated in $L^2(\Omega)$ (see [7, Theorem IV.9]).

Since $1/k_n f(u_n) \leq a|w_n| + 1/k_n b$, it follows that $1/k_n f(u_n)$ is also dominated. Hence (10) is now obtained. Now (10) and the maximum principle imply that $w \geq 0$ and (10) becomes

$$\begin{aligned} \Delta^2 w - \gamma\Delta w + \delta w &= \lambda a w \quad \text{in } \Omega, \\ w &\geq 0 \qquad \text{in } \Omega, \\ \|w\|_2 &= 1 \qquad \text{in } \Omega. \end{aligned} \tag{12}$$

Thus from (1), we have $\lambda a = \lambda_1$ and $w = \varphi_1$, which contradicts the fact that $\lambda \neq \lambda_1/a$. This contradiction finishes the proof of the lemma 5.

Now, we need only to check that the two geometric assumptions of theorem 4 are fulfilled.

First, since u_λ is a local minimum of $\mathcal{J}_\lambda$, there exists $R > 0$ such that for all $u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $\mathcal{J}_\lambda(u) \geq \mathcal{J}_\lambda(u_\lambda)$. Then

$$\mathcal{J}_\lambda(u) - \mathcal{J}_\lambda(u_\lambda) = \mathcal{J}''_\lambda(u_\lambda)(u - u_\lambda, u - u_\lambda) + \rho \quad \text{where } \; \rho > 0.$$

This makes u_λ becomes a strict local minimal for J, which proves $(*)$.

Recall that $\lim_{t\to+\infty}(f(t) - a\,t)$ is finite, then there exists $\beta \in \mathbb{R}$ such that

$$f(t) \geq a\,t + \beta, \quad \forall t > 0.$$

Hence

$$F(t) \geq \frac{a\,\lambda}{2}t^2 + \beta\lambda t, \quad \forall t > 0.$$

Notes

This yields, using the definition of φ_1 mentioned in (1),

$$\mathcal{J}_\lambda(t\varphi_1) = \frac{\lambda_1 - a\lambda}{2} t^2 \int_\Omega \varphi_1^2 - \beta\lambda t \int_\Omega \varphi_1,$$

since $\|\varphi_1\|_2 = 1$, then we have

$$\frac{\mathcal{J}_\lambda(t\varphi_1)}{t^2} = \frac{\lambda_1 - a\lambda}{2} - \frac{\beta\lambda}{t} \int_\Omega \varphi_1 \tag{13}$$

which implies

$$\limsup_{t\to+\infty} \frac{1}{t^2}\, \mathcal{J}_\lambda(t\varphi_1) \leq \frac{\lambda_1 - a\lambda}{2} < 0, \quad \forall\lambda > \lambda_1/a.$$

Therefore

$$\lim_{t\to+\infty} \mathcal{J}_\lambda(t\varphi_1) = -\infty.$$

So, there exists $v_0 \in E$ such that $\mathcal{J}_\lambda(v_0) \leq \mathcal{J}_\lambda(u_\lambda)$ and $(**)$ is proved.

Finally, let $\tilde{v}$ (respectively $\tilde{c}$) be the critical point (respectively critical value) of $\mathcal{J}_\lambda$, we recall that the function $\tilde{v}$ belongs to E and satisfies

$$\Delta^2\tilde{v} - \gamma\Delta\tilde{v} + \delta\tilde{v} = \lambda f(\tilde{v}) \quad \text{in } \Omega \quad \text{and} \quad \mathcal{J}_\lambda(\tilde{v}) = \tilde{c}.$$

The next lemma states that the limit of a sequence of unstable solutions is also unstable.

Lemma 6. Let $u_n \rightharpoonup u$ in $H^2(\Omega)\cap H_0^1(\Omega)$ and $\mu_n \to \mu$ be such that $\eta_1(\mu_n, u_n) < 0$. Then, $\eta_1(\mu, u) < 0$.

Proof: The fact that $\eta_1(\mu_n, u_n) < 0$ is equivalent to the existence of a $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega |\Delta\varphi_n|^2 + \gamma \int_\Omega |\nabla\varphi_n|^2 + \delta \int_\Omega \varphi_n^2 \leq \mu_n \int_\Omega f'(u_n)\varphi_n^2 \quad \text{with} \quad \int_\Omega \varphi_n^2 = 1 \tag{14}$$

Since $f' \leq a$, (14) shows that (φ_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. Let $\varphi \in E$ be such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Then

$$\mu_n \int_\Omega f'(u_n)\varphi_n^2 \to \mu \int_\Omega f'(u)\varphi^2$$

This can be seen by extracting from (φ_n) a subsequence dominated in $L^2(\Omega))$ as in [7, Theorem IV.9]. Now we have

$$\int_\Omega |\Delta\varphi|^2 \leq \liminf \int_\Omega |\Delta\varphi_n|^2 \text{ and } \int_\Omega |\nabla\varphi|^2 \leq \liminf \int_\Omega |\nabla\varphi_n|^2$$

finally, since $\|\varphi\|_2 = 1$, we obtain

$$\int_\Omega |\Delta\varphi|^2 + \int_\Omega |\nabla\varphi|^2 + \delta \int_\Omega \varphi^2 \leq \mu \int_\Omega f'(u)\varphi^2.$$

Obviously, the fact that the function v belongs to $C^4(\bar{\Omega}) \cap E$ follows from a bootstrap argument.

Actually, the next paragraph said a good deal more, giving additional information on precisely the comportment of the instable solution v_λ.

5.4. ***Proof* (iii) (a).** By contradiction, thanks to Lemma 3, there is a sequence of positives scalars (λ_n) and a sequence (v_n) of unstable solutions to (P_{λ_n}) such that $v_n \to v$ in $L^1_{\text{loc}}(\Omega)$ as $\lambda_n \to \lambda_1/a$ for some function v.

We first claim that (v_n) cannot be bounded in E. Otherwise, let $w \in E$ be such that, up to a subsequence,

$$v_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad v_n \to w \text{ strongly in } L^2(\Omega).$$

Therefore,

$$\Delta^2 v_n - \gamma\Delta v_n + \delta v_n \to \Delta^2 w - \gamma\Delta w + \delta w \text{ in } \mathcal{D}'(\Omega),$$

$$f(v_n) \to f(w) \text{ in } L^2(\Omega),$$

which implies that $\Delta^2 w - \gamma\Delta w + \delta w = \frac{\lambda_1}{a} f(w)$ in Ω. It follows that $w \in E$ and solves (P_λ) with λ_1/a in stead of λ. From Lemma 6, we deduce that

$$\eta_1\left(\frac{\lambda_1}{a}, w\right) \leq 0. \tag{15}$$

Relation (15) shows that $w \neq u_{\lambda_1/a}$ which contradicts the fact that (P_λ) with λ_1/a in stead of λ has a unique solution. Now, since $\Delta^2 v_n - \gamma\Delta v_n + \delta v_n = \lambda_n f(v_n)$, the unboundedness of (v_n) in E implies that this sequence is unbounded in $L^2(\Omega)$, too. To see this, let

$$v_n = k_n w_n, \quad \text{where } k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \to \infty.$$

Then

$$\Delta^2 w_n - \gamma\Delta w_n + \delta w_n = \frac{\lambda_n}{k_n} f(v_n) \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega).$$

So, we have convergence also in the sense of distributions and (w_n) is seen to be bounded in E with standard arguments. We obtain

$$\Delta^2 w - \gamma\Delta w + \delta w = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

The desired contradiction is obtained since $w \in E$.

5.5. ***Proof* (iii) b.** We end the proof by showing that v_λ tends to u^* uniformly in Ω when λ tends to λ^*.

As before, it is sufficient to prove the $L^2(\Omega)$ boundedness of v_λ near λ^* and to use the uniqueness property of u^*. Assume that $\|v_n\|_2 \to \infty$ as $\lambda_n \to \lambda^*$, where v_n is a solution to (P_{λ_n}). We write again $v_n = k_n w_n$. Then,

$$\Delta^2 w_n - \gamma\Delta w_n + \delta w_n = \frac{\mu_n}{k_n} f(v_n). \tag{16}$$

The fact that the right-hand side of (16) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in E. Let (w_n) be such that (up to a subsequence)

$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \to w \text{ strongly in } L^2(\Omega).$$

A computation already done shows that

$$\Delta^2 w - \gamma\Delta w + \delta w = \lambda^* a w, \quad w \geq 0 \text{ and } \|w\|_2 = 1,$$

which forces λ^* to be λ_1/a. This contradiction concludes the proof.

Notes

Notes

In conclusion, all these results give us a rather clear schema of solutions for the quasilinear case $a \in (0, +\infty)$. An important role in our arguments has played by $l := \lim_{t\to\infty}\Big(f(t) - at\Big)$. We distinguish two different situations strongly depending on the sign of l.

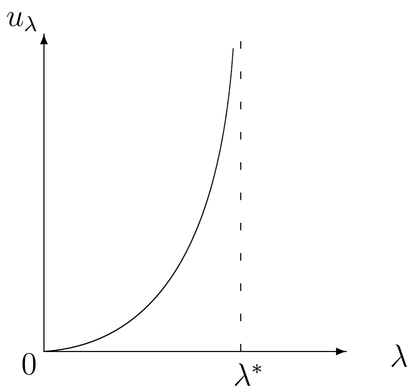

Fig. 1: Behavior of the minimal solution, l > 0

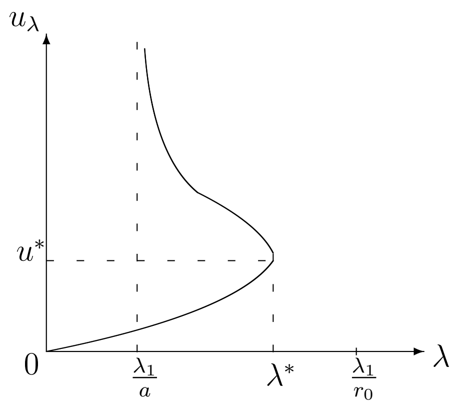

Fig. 2: Bifurcation branches, l < 0

VI. The Dirichlet Problem

References Références Referencias

1. Abid, M. Jleli and N. Trabelsi, *Weak solutions of quasilinear biharmonic problems with positive, increasing and convex nonlinearities.* Analysis and applications. vol.06, No. 03 pp-213-227 (2008).
2. NU. Ahmed, H. Harbi, *Mathematical analysis of dynamic models of suspension bridges.* SIAM J. Appl. Math. 58, 853-874 (1998).
3. A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Funct. Anal. 14 (1973), 349-381.
4. S. Baraket, M. Khtaifi, T. Ouni, *Singular limit solutions for 4-dimensional stationary Kuramoto-Sivashinsky equations with exponential nonlinearity,* EJDE, Vol. 2015 (2015), No. 187, pp. 1–26.
5. S. Baraket, M. Khtaifi, T. Ouni, *Singular limit for 4-dimensional general stationary q-Kuramoto-Sivashinsky (q-KS) equations with exponential nonlinearity,* accepted in Anal Stiinti˙ce ale Universitatii Ovidius Constanta.
6. H. Ben Omrane and S. Khenissy, *Positivity preserving results for a biharmonic equation under Dirichlet boundary conditions,* Opuscula Math. 34, no. 3 (2014), 601-608.
7. H. Brezis, *Analyse Fonctionnelle. Théeorie et Applications,* (Masson, Paris, 1992).
8. H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow up for* $u_t - \Delta u = g(u)$ *revisited,* Adv. Diff. Eq. 1 (1996), 73-90.
9. F.J.S. Correa, J.V.A. Goncalves, Angelo Roncalli, *On a class of fourth order nonlinear elliptic equations under Navier boundary conditions,* Analysis and applications. 8 (2010) 185-197
10. M. Filippakis and N. Papageorgiou, *Multiple solutions for nonlinear elliptic problems with a discontinuous nonlinearity,* Analysis and applications. (Singap.) 4 (2006), 1-18.
11. S. Fucik, A.Kufner, *Nonlinear Differential Equations.* Studies in Applied Mechanics 2. (Amsterdam-New York, Elsevier Scienti˙c Publ. Co. 1980. 360)

12. F. Gazzola, Hans-Christoph Grunau and G.Sweers, *Polyharmonic Boundary Value Problems, Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains,* Springer(1991).
13. M. Ghergu and V. R˘adulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis,* Oxford Lecture Series in Mathematics and its Applications, vol. 37, (Oxford University Press, 2008).
14. D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* (Springer-Verlag, Heidelberg, 2001).
15. L. Hörmander, *The Analysis of Linear Differential Operators I,* (Springer-Verlag, Berlin, 1983).
16. H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to Partial Differential Equations,* (Springer-Verlag, Berlin, 2003).
17. AC. Lazer, PJ. Mckenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis,* (SIAM Rev. 32(4), 537-578 (1990).
18. AC. Lazer, PJ. Mckenna, *Global bifurcation and a theorem of Tarantello,* Analysis and applications. 181(3), 648-655 (1994).
19. F.H. Lin, Y.S. Yang, *Nonlinear non-local elliptic equation modelling electrostatic actuation,* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. A 463 (2007), 1323-1337.
20. Y. Martel, *Uniqueness of weak solution for nonlinear elliptic problems,* Houston Journal Math. 23 (1997), 161-168.
21. P. Mironescu and V. R˘adulescu, *A bifurcation problem associated to a convex, asymtotically linear function,* C. R. Acad. Sci. Paris, Ser. I 316 (1993), 667-672.
22. P. Mironescu and V. R˘adulescu, *The study of a bifurcation problem associated to an asymptotically linear function,* Nonlinear Analysis 26 (1996), 857-875.
23. CV. Pao, *On fourth-order elliptic boundary value problems,* Proc. Am. Math. Soc. 128, 1023-1030 (2000).
24. J.A. Pelesko, D.H. Bernstein, *Modelling MEMS and NEMS,* New York, NY: Chapman and Hall/CRC, 2003.
25. V. Rădulescu, *Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations,* Contemporary Mathematics and Applications, Vol. 6 (Hindawi Publ. Corp., 2008).
26. S. Sâanouni, N. Trabelsi, *Bifurcation for elliptic forth-order problems with quasilinear source term* Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 92, pp. 1–16.
27. M. Sanchón, *Boundedness of the extremal solution of some p-Laplacian problems,* Nonlinear Anal. 67(1) (2007) 281-294. Universidade de Coimbra, Preprint No. 06-04.
28. G. Tarantella, *A note on a semilinear elliptic problem,* Differ. Integral Equ. 5(3), 561-565 (1992).
29. J. Velin, *A criterion for existence of a positive solution of a nonlinear elliptic system,* Analysis and applications. vol.6, No.3 (2008) 299-321.

Notes