Fractional Integration of the Product of Two Multivariable Gimel-Functions and A General Class of Polynomials

By Frederic Ayant

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Fractional Integration of the Product of two Multivariable Gimel-Functions and a General Class of Polynomials

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Abstract - A significantly large number of earlier works on the subject of fractional calculus give the interesting account of the theory and applications of fractional calculus operators in many different areas of mathematical analysis (such as ordinary and partial differential equations, integral equations, special functions, the summation of series, etc.). The object of the present paper is to study and develop the Saigo-Maeda operators. First, we establish four results that give the images of the product of two multivariable Gimel-functions and a general class of multivariable polynomials in Saigo-Maeda operators. On account of the general nature of the Saigo-Maeda operators, multivariable Gimel-functions and a class multivariable polynomials a large number of new and known theorems involving Riemann-Liouville and Erdelyi-Kober fractional integral operators and several special functions.

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I. INTRODUCTION AND PRELIMINARIES

The fractional integral operator involving various special functions has found significant importance and applications in Various subfields of applicable mathematical analysis. Since the last four decades, some workers like Love [17], McBride [20], Kalla [8,9], Kalla and Saxena [10,11], Saxena et al. [29], Saigo [24,25], Kilbas [12], Kilbas and Sebastian [14] and Kiryakova [16,17] have studied in depth the properties, applications and different extensions of Various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Samko, Kilbas, and Marichev [26], Miller and Ross [22], Kilbas, Srivastava, and Trujillo [15] and Debnath and Bhatta [6]. A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [23,24], has been introduced by Marichev [18], see Samko et al. [28] and also see Kilbas and Saigo [13] for more details. The generalized fractional integral operator of arbitrary order, involving Appell function $F_3$ in the kernel defined and studied by Saigo and Maeda [27, p. 393, Eq (4.12)] and (4.13)] in the following manner:

Let $\alpha, \alpha', \beta, \beta', \eta$ be complex numbers and, $x, \text{Re}(\eta) > 0$, we have, see Saigo and Maeda [28, p. 393, Eq (4.12)]

Definition 1

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\eta} f(x) = \frac{x^{-\alpha}}{\Gamma(\eta)} \int_0^x (t-x)^{\eta-1} t^{\eta-1} F_3 \left[ \alpha, \alpha', \beta, \beta', \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right] f(t) dt$$

and

Definition 2

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta} f(x) = \frac{x^{-\alpha}}{\Gamma(\eta)} \int_0^x (t-x)^{\eta-1} t^{\eta-1} F_3 \left[ \alpha, \alpha', \beta, \beta', \eta; 1 - \frac{t}{x}, 1 - \frac{x}{t} \right] f(t) dt$$

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We have the following two results due to Saigo [25] where \( Re(\eta) > 0 \)

**Definition 3**

\[
I^\alpha_{0+}\beta, \eta f(z) = \frac{x^{-\alpha-\beta}}{\Gamma(\eta)} \int_0^x (x-t)^{\alpha-1} F \left[ \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right] f(t) dt \tag{1.3}
\]

**Definition 4**

\[
I^\alpha_{0-}\beta, \eta f(z) = \frac{1}{\Gamma(\eta)} \int_x^\infty t^{-\alpha-\beta} (x-t)^{\alpha-1} F \left[ \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right] f(t) dt \tag{1.4}
\]

\( F \) is the Gaussian hypergeometric function. We obtain the following lemmas.

**Lemma 1.**

\[
\left( I^\alpha_{0,x} \alpha', \beta', \eta \mu^{-1} \right) = \frac{\Gamma(u)\Gamma(\mu + \eta - \alpha - \alpha' - \beta)(\mu + \beta' - \alpha')}{\Gamma(\mu + \eta - \alpha - \alpha')(\mu + \beta)(\mu + \beta')}(x^\mu)^{\alpha + \alpha' + \eta - 1} \tag{1.5}
\]

where \( \alpha, \alpha', \beta, \beta', \eta \in \mathbb{C}, Re(\mu) > \max\{0, \Re(\alpha + \alpha' + \beta - \eta), Re(\alpha' - \beta')\} \)

**Lemma 2.**

\[
\left( I^\alpha_{x;\infty} \alpha', \beta', \eta \mu^{-1} \right) = \frac{\Gamma(1 + \alpha + \alpha' - \eta - \mu)\Gamma(1 + \alpha + \beta' - \eta - \mu)\Gamma(1 - \beta - \mu)}{\Gamma(1 + \eta - \alpha + \alpha' + \beta')(1 + \alpha + \beta - \mu)}(x^\mu)^{\alpha + \alpha' + \eta - 1} \tag{1.6}
\]

where \( \alpha, \alpha', \beta, \beta', \eta \in \mathbb{C}, Re(\eta) > 0, Re(\mu) < \min\{Re(-\beta), \Re(\alpha + \alpha' - \eta), Re(\alpha' + \beta' - \eta)\} \)

**Lemma 3.**

\[
\left( I^\alpha_{0,x} \beta, \eta \mu^{-1} \right) = \frac{\Gamma(u)\Gamma(\mu + \eta - \beta)}{\Gamma(\mu + \eta + \alpha + \eta)(\mu + \beta)}(x^\mu)^{\beta - 1} \tag{1.7}
\]

where \( \alpha, \beta, \eta \in \mathbb{C}, Re(\beta - \eta), Re(\alpha' - \beta') \)

**Lemma 4.**

\[
\left( I^\alpha_{x;\infty} \beta, \eta \mu^{-1} \right) = \frac{\Gamma(\beta - \mu + 1)\Gamma(\eta - \mu + 1)}{\Gamma(1 - \mu)(\alpha + \beta + \eta - \mu + 1)}(x^\mu)^{\beta - 1} \tag{1.8}
\]

where \( \alpha, \beta, \eta \in \mathbb{C}, Re(\alpha) > 0, Re(\mu) < 1 + \min\{Re(\beta), \Re(\eta)\} \)

Recently, Gupta et al. [7] have obtained the images of the product of two \( H \)-functions in Saigo operator given by (1.3) and (1.4) and thereby generalized several results obtained earlier by Kilbas, Kilbas and Sebastian [14] and Saxena et al. [29] as mentioned in this paper cited above. It has recently become a subject of interest for many researchers in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators to obtain the image formulas. The aim of this paper is to obtain four results that give the theorems of the product of two multivariable Gimel functions and a general class of multivariable polynomials [30] in Saigo-Maeda operators and Saigo operators.

### II. Multivariable Gimel-Function

We throughout this paper, let \( \mathbb{C}, \mathbb{R}, \) and \( \mathbb{N} \) be set of complex numbers, real numbers and positive integers respectively. Also, \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). We define a generalized transcendental function of several complex variables.
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\[ \sum_{1}^{r} \psi(s_{1}, \ldots, s_{r}) \prod_{k=1}^{r} \theta_{k}(s_{k}) z_{k}^{s_{k}} \, ds_{1} \cdots ds_{r} \]  

(2.1)

with \( \omega = \sqrt{-1} \)

\[ \psi(s_{1}, \ldots, s_{r}) = \frac{\prod_{j=1}^{n} \Gamma^{A_{ij}}(1 - a_{ij} + \sum_{k=1}^{2} \alpha_{ij}^{(k)} s_{k})}{\sum_{i=1}^{R_{i}} \prod_{j=n+1}^{p_{i}} \Gamma^{A_{ij}}(1 - a_{ij} + \sum_{k=1}^{2} \alpha_{ij}^{(k)} s_{k})} \]

(2.2)

and

\[ \theta_{k}(s_{k}) = \frac{\prod_{j=1}^{n} \Gamma^{D_{ij}}(d_{ij}^{(k)} - \gamma_{ij}^{(k)} s_{k}) \prod_{j=1}^{p_{i}} \Gamma^{C_{ij}}(1 - c_{ij}^{(k)} + \delta_{ij}^{(k)} s_{k})}{\sum_{i=1}^{R_{i}} \prod_{j=m+1}^{R_{i}} \Gamma^{D_{ij}}(1 - d_{ij}^{(k)} + \delta_{ij}^{(k)} s_{k}) \prod_{j=n+1}^{p_{i}} \Gamma^{C_{ij}}(c_{ij}^{(k)} - \gamma_{ij}^{(k)} s_{k})} \]  

(2.3)
The contour $L_k$ is in the $s_k(k = 1, \ldots, r)$-plane and runs from $\sigma - i\infty$ to $\sigma + i\infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of $\Gamma^{A_{a_jj}} \left(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{2j}^{(k)} s_k\right)_{j = 1, \ldots, n_2}, \Gamma^{A_{a_{t_jj}}} \left(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{3j}^{(k)} s_k\right)_{j = 1, \ldots, n_3}\), \ldots, $ the right of the contour $L_k$ and the poles of $\Gamma^{D_{j}(k)} \left(d_{j}^{(k)} - \delta_{j}^{(k)} s_k\right)_{j = 1, \ldots, r(k)}(k = 1, \ldots, r)$ lie to the left of the contour $L_k$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as

$$|\arg(z_k)| < \frac{1}{2} A_{t_j}^{(k)} \pi$$

where

$$A_{t_j}^{(k)} = \sum_{j=1}^{n_1} D_{j}^{(k)} \delta_{j}^{(k)} + \sum_{j=1}^{n_2} C_{j}^{(k)} \tau_{j}^{(k)} - \tau_{t_j}^{(k)} \left(\sum_{j=m(k)+1}^{n_1} D_{j}^{(k)} \delta_{j}^{(k)} + \sum_{j=n(k)+1}^{n_2} C_{j}^{(k)} \tau_{j}^{(k)}\right)$$

Following the lines of Braaksma ([4] p. 278), we may establish the asymptotic expansion in the following convenient form

$$\mathcal{N}(z_1, \ldots, z_r) = 0( |z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r}, \max( |z_1|, \ldots, |z_r| ) \rightarrow 0$$

$$\mathcal{N}(z_1, \ldots, z_r) = 0( |z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r}, \min( |z_1|, \ldots, |z_r| ) \rightarrow \infty where i = 1, \ldots, r :$$

$$\alpha_i = \min_{1 \leq j \leq n(i)} \Re \left[D_{j}^{(i)} \left(\frac{C_{j}^{(i)}}{\delta_{j}^{(i)}} - 1\right)\right]$$

$$\beta_i = \max_{1 \leq j \leq n(i)} \Re \left[C_{j}^{(i)} \right]$$

Remark 1.

If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$, $A_{2j} = A_{2j_2} = \cdots = A_{rj} = A_{rj_2} = \cdots = A_{rj_2} = 1$, then the multivariable Gimel-function reduces in multivariable Aleph-function defined by Ayant [3].

Remark 2.

If $n_2 = \cdots = n_r = p_{i_2} = q_{i_2} = \cdots = p_{i_r} = q_{i_r} = 0$, $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i_2}^{(i)} = \cdots = \tau_{i_r}^{(i)} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in multivariable I-function defined by Prathima et al. [23].

Remark 3.

If $A_{2j} = A_{2j_2} = B_{2j_2} = \cdots = A_{rj} = A_{rj_2} = B_{rj_2} = 1$. $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i_2}^{(i)} = \cdots = \tau_{i_r}^{(i)} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [22].

Remark 4.

If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the H-function of several defined by Srivastava and Panda [32,33]. About the simplified notations, see Ayant ([4], page 248-255)

Now, we define the second Gimel function of $s$ variables, the parameters are identical to the Gimel function of $r$ variables with the prim sign and the validities conditions are equivalent.

The generalized polynomials of multivariable defined by Srivastava [30], is given in the following manner:

$$S_{N_1, \ldots, N_r}^{[N_1, \ldots, N_r]} [y_1, \ldots, y_r] = \sum_{K_1=0}^{N_1} \cdots \sum_{K_r=0}^{N_r} \left[(-N_1)_y y_1 K_1 \cdots (-N_r)_y y_r K_r \right] A[N_1, K_1; \cdots; N_r, K_r] y_1 K_1 \cdots y_r K_r (2.5)$$
where $\mathcal{M}_1, \cdots, \mathcal{M}_a$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_v, K_v]$ are arbitrary constants, real or complex.

We shall note $a_v = \frac{(-N_v)_{\mathcal{M}_v, K_v}}{K_v!} \cdots \frac{(-N_v)_{\mathcal{M}_v, K_v}}{K_v!} A[N_1, K_1; \cdots; N_v, K_v]$.

### III. Main Results

We shall note

$$U = 0, n_2; 0, n_3; \cdots; 0, n_{r-1}; 0, n_{r-1}'; 0, n_{s-1}'$$

$$V = m^{(1)}, n^{(1)}; n^{(2)}; \cdots; n^{(r)}, n^{(r)}; m'^{(1)}, n'^{(1)}; m'^{(2)}, n'^{(2)}; \cdots; m'^{(s)}, n'^{(s)}$$

$$X = p_{i_1}, q_{i_1}, \tau_i, R_i, \cdots; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}}, R_{i_{r-1}}; p_{i_{r-1}'}, q_{i_{r-1}'}, \tau_{i_{r-1}'}, R_{i_{r-1}'}$$

$$Y = p_{i}, q_{i}, \tau, R^{(1)}; \cdots; p_{i_{r+1}}, q_{i_{r+1}}, \tau_{i_{r+1}}, R^{(r)}; p_{i_{r+1}'}, q_{i_{r+1}'}, \tau_{i_{r+1}'}, R^{(r)}$$

Theorem 1.

$$\left(\begin{array}{c}
\frac{c_1 t^{\lambda_1} (b - at)^{-\delta_1}}{1} \\
\vdots \\
\frac{c_v t^{\lambda_v} (b - at)^{-\delta_v}}{1} \\
\end{array}\right) = b^{\nu} x^{-\alpha - \alpha' - \eta - 1} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_v=1}^{[N_v/M_v]} a_v c_{K_1} \cdots c_{K_v}$$

$$\left(\begin{array}{c}
\frac{z_1 t^{\sigma_1} (b - at)^{-\omega_1}}{1} \\
\vdots \\
\frac{z_v t^{\sigma_v} (b - at)^{-\omega_v}}{1} \\
\end{array}\right)$$

$$\left(\begin{array}{c}
\sum_{j=1}^{r} \frac{a_j}{1} \\
\sum_{j=1}^{r} \frac{\alpha_j}{1} \\
\sum_{j=1}^{r} \frac{\alpha_j}{1} \\
\end{array}\right)$$

$$\begin{pmatrix}
\mathcal{A}_1, \mathcal{A}_1, \mathcal{A}_1; \mathcal{A} \\
\vdots \\
\mathcal{B}, \mathcal{B}_1; \mathcal{B}, (0, 1, 1)
\end{pmatrix}$$

where

$$\mathcal{A} = (a_{2j}; \alpha_{2j}; \alpha_{2j}^{(1)}; A_2)_{1, n_2}; \tau_{i_1}, \alpha_{2j_i}, \alpha_{2j_i}^{(1)}; A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2} A_{2}$$

$$\mathcal{A}_{r-1} = \frac{a_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1}$$

$$\mathcal{A}_{r-1} = \frac{a_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1} \frac{\alpha_{(r-1)}}{1}$$
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\[ A_1 = (1 - v + \sum_{j=1}^{n} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1, 1), (1 + \alpha + \beta - v - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1, 1) \]

\[ (1 - \mu + \sum_{j=1}^{n} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1, 1), (1 + \alpha + \beta - \mu - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1, 1) \]

\[ A = \left( \left[ (a_{2j}; \alpha^{(2)}_1, \ldots, \alpha^{(2)}_r, 0; \ldots, 0; A_{rj}) \right]_{n+1, p_{rj}} \mid \left[ (a_{2j}; \alpha^{(1)}_1, \ldots, \alpha^{(1)}_r, 0; \ldots, 0; A_{rj}) \right]_{n+1, p_{rj}} \right)^{\prime} \]

\[ \left[ \alpha^{(4)}_1, \ldots, \alpha^{(4)}_s \right]_{n+1, p_{rj}} \left[ \alpha^{(3)}_1, \ldots, \alpha^{(3)}_s \right]_{n+1, p_{rj}} \]

\[ \cdots \]

\[ \left[ \alpha^{(1)}_1, \ldots, \alpha^{(1)}_s \right]_{n+1, p_{rj}} \left[ \alpha^{(4)}_1, \ldots, \alpha^{(4)}_s \right]_{n+1, p_{rj}} \]

\[ B = [\tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j})]_{1, q_{rj}} \left[ \tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j}) \right]_{1, q_{rj}} \left[ \tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j}) \right]_{1, q_{rj}} \cdots \]

\[ \left[ \tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j}) \right]_{1, q_{rj}} \left[ \tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j}) \right]_{1, q_{rj}} \left[ \tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j}) \right]_{1, q_{rj}} \cdots \]

\[ B = [\tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j})]_{1, q_{rj}} \left[ \tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j}) \right]_{1, q_{rj}} \left[ \tau_{r}(b_{2j}; \beta^{(1)}_1, \beta^{(2)}_1; B_{2j}) \right]_{1, q_{rj}} \cdots \]

\[ (1 - v + \sum_{j=1}^{n} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 0; 1), (1 + \alpha + \alpha' - v - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1, 1) \]

\[ (1 - \mu + \sum_{j=1}^{n} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1, 1), (1 + \alpha + \beta - \mu - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1, 1) \]
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\[ B = \left[ (d_j^{(1)}, d_j^{(1)}; D_j^{(1)}) \right]_{m(1)}, \left[ \tau_i^{(1)} (d_j^{(1)}, d_j^{(1)}; D_j^{(1)}) \right]_{m(1)+1, q_i^{(1)}} \cdots \]

\[ \left[ (d_j^{(r)}, d_j^{(r)}; D_j^{(r)}) \right]_{m+r}, \left[ \tau_i^{(r)} (d_j^{(r)}, d_j^{(r)}; D_j^{(r)}) \right]_{m+r+1, q_i^{(r)}} \]

\[ \left[ (d_j^{(s)}, d_j^{(s)}; D_j^{(s)}) \right]_{m+s}, \left[ \tau_i^{(s)} (d_j^{(s)}, d_j^{(s)}; D_j^{(s)}) \right]_{m+s+1, q_i^{(s)}} \cdots \]

In our investigation, we will use these simplified notations cited above.

Provided

\[ a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega_j \in \mathbb{C}, k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \ldots, s \]

\[ \lambda_k, \sigma_i, \sigma_j' > 0; \ k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \ldots, s \]

\[ |\arg(z_i)| < \frac{1}{2} \pi A_i^{(k)} \text{ and } A_i^{(k)} \text{ is defined by (2.4), } |\arg(z_j')| < \frac{1}{2} \pi A_j^{(k)} \; \frac{a}{b} < 1 \]

\[ \text{Re}(\mu) + \sum_{i=1}^{s} \sigma_i \min_{1 \leq j \leq m_1} \text{Re} \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{d_j^{(i)}} \right) \right] + \sum_{i=1}^{s} \sigma_j' \min_{1 \leq j \leq m_1} \text{Re} \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{d_j^{(i)}} \right) \right] > \max[0, \text{Re}(\alpha + \alpha' + \beta - \eta), \text{Re}(\alpha' - \beta')] \]

\[ \text{Re}(\nu) + \sum_{i=1}^{s} \omega_i \min_{1 \leq j \leq m_1} \text{Re} \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{d_j^{(i)}} \right) \right] + \sum_{i=1}^{s} \omega_j' \min_{1 \leq j \leq m_1} \text{Re} \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{d_j^{(i)}} \right) \right] > \max[0, \text{Re}(\alpha + \alpha' + \beta - \eta), \text{Re}(\alpha' - \beta')] \]

Proof

To prove (3.1), we first express the class of multivariable polynomials \( S_{N_1, \ldots, N_v}^{\alpha, \beta, \eta} \) in series with the help of (2.13), the multivariable Gimmel-functions regarding Mellin-Barnes type integrals contour with the help of (2.1). Now interchange the order of summations and two multiple Mellin-Barnes integrals contour, respectively and taking the fractional integral operator inside (which is permissible under the stated conditions) and make simplifications. Next, we express the terms \( (b - ax)^{r+s+1} - \sum_{i=1}^{v} \omega_i - \sum_{j=1}^{s} \omega_j \) in terms of Mellin-Barnes integrals contour (Srivastava et al. [31], page 18, (2.6.3) and after algebraic manipulations, we obtain

\[ L.H.S = b \sum_{R_1, \ldots, R_v = 0}^{R_1, \ldots, R_v \in L} y_1 K_{1} \ldots y_v K_{v} \epsilon_{e_1}^{K_{1}} \ldots e_{e_v}^{K_{v}} \left( \sum_{j=1}^{s} \delta_j K_j \right) \left( \frac{1}{(2\pi \omega)} \right)^{r+s+1} \]

\[ \frac{\prod_{k=1}^{s} \theta_k(s_k) z_k^{s_k} \prod_{j=1}^{s} \theta_j(s_j) t_j^{s_j} b^{-v} - \sum_{i=1}^{v} \omega_i + \sum_{j=1}^{s} \omega_j} {b^{v} - \sum_{i=1}^{v} \omega_i - \sum_{j=1}^{s} \omega_j} \]

Now using the lemma 1. Finally interpreting the resulting Mellin-Barnes integrals contour as a multivariable Gimmel-function of \((r+s+1)\)-variables, we obtain the desired result (3.1).

Let
\[ A_2 = (1 - v - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1), (\eta + \mu - \alpha' + \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \]

\[ (\beta + \mu + \sum_{j=1}^{v} \lambda_j K_j - \alpha - \alpha'; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1), (\mu + \eta - \alpha - \beta' + \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \]  

(3.13)

\[ B_2 = (\mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1), (\beta + \mu - \alpha + \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \]

\[ (1 - \mu - \beta' - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1), (1 + \alpha' + \beta - \mu - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \]  

(3.14)

We have the following resulting

Theorem 2.

\[
\left\{ \left( I_{\mathcal{S}, \mathcal{S'}, \mathcal{S''}} \right)^{a, \alpha', \beta', \eta, \mu - 1} (b - at)^{-\delta_1} \right\} \cdot \left\{ \left( \begin{array}{c} c_t t^{\lambda_1} (b - at)^{-\delta_1} \\ \vdots \\ c_v t^{\lambda_v} (b - at)^{-\delta_v} \end{array} \right) \right\} = (b - at)^{-\omega_1} \sum_{K_1=1}^{[N_1/M_1]} \ldots \sum_{K_v=1}^{[N_v/M_v]} \sum_{a_v} \sum_{c_v} K_1 \cdots c_v \right) \]

\[
\left( z_1^{\sigma_1} (b - at)^{-\omega_1} \right) \cdots \left( z_v^{\sigma_v} (b - at)^{-\omega_v} \right) \right) \right) \right) \right)
\]

Provided

\[ a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega_j' \in \mathbb{C}, k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \cdots, s \]

\[ \lambda_k, \sigma_i, \sigma'_j > 0; \ k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \cdots, s \]

\[ |\arg(z_i)| < \frac{1}{2} \pi \text{ and } A_i^{(k)} \text{ is defined by (2.4), } |\arg(z_i')| < \frac{1}{2} \pi A_i^{(k)}; \left| \frac{a}{b} \right| < 1 \]
\[
\text{Re}(\mu) - \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m(i)} \text{Re} \left[ D_{j}^{(i)} \left( \frac{d^{(i)}}{\delta^{(i)}} \right) \right] - \sum_{i=1}^{r} \sigma'_i \min_{1 \leq j \leq n^{(i)}} \text{Re} \left[ D'_{j}^{(i)} \left( \frac{d'_{j}^{(i)}}{\delta'_{j}^{(i)}} \right) \right] < 1 + \min \left[ \text{Re}(\beta), \text{Re}(\alpha + \alpha' - \eta), \text{Re}(\alpha + \beta' - \eta) \right]
\]

\[
\text{Re}(\nu) - \sum_{i=1}^{r} \omega_i \min_{1 \leq j \leq m(i)} \text{Re} \left[ D_{j}^{(i)} \left( \frac{d^{(i)}}{\delta^{(i)}} \right) \right] - \sum_{i=1}^{r} \omega'_i \min_{1 \leq j \leq n^{(i)}} \text{Re} \left[ D'_{j}^{(i)} \left( \frac{d'_{j}^{(i)}}{\delta'_{j}^{(i)}} \right) \right] < 1 + \max \left[ \text{Re}(\beta), \text{Re}(\alpha + \alpha' - \eta), \text{Re}(\alpha + \beta' - \eta) \right]
\]

To prove the equation (3.14), we use the similar method that formula (3.5) by using the lemma 2.

Let

\[
A_3 = (1 - \mu - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1), = (1 - v - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1),
\]

\[
(1 - \mu - \eta + \beta - \sum_{j=1}^{v} \delta_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1); \tag{3.15}
\]

\[
B_3 = (1 - v - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 0; 1), (1 + \beta - \mu - \sum_{j=1}^{K} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1)
\]

\[
(1 - \mu - \eta - \alpha - \sum_{j=1}^{K} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1) \tag{3.16}
\]

Theorem 3.

\[
\left\{ \left( {r_{0+}^{\alpha, \beta, \eta, \mu-1} (b - at)^{-\delta_1}} \right) \right\} \left( \begin{array}{ccc} c_1 t^\sigma_1 (b - at)^{-\delta_1} & \cdots & \left( \begin{array}{c} z_1 t^\sigma_1 (b - at)^{-\omega_1} \\ \vdots \\ z_r t^\sigma_r (b - at)^{-\omega_r} \end{array} \right) \\ \vdots \end{array} \right) \right\} (x) = d^{-\nu_x - \mu - \alpha' - \eta - 1} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_s=1}^{[N_s/M_s]} a_1 c_1^{K_1} \cdots c_v^{K_v}
\]

\[
\left( \begin{array}{c} z_1 x_1^{\frac{s_1}{z_1}} \\ \vdots \\ z_r x_r^{\frac{s_r}{z_r}} \end{array} \right) \left( \begin{array}{c} A_x, \tilde{A}_3, A : A \\ \vdots \\ \tilde{B}_B, B_3 : B (0, 1; 1) \end{array} \right) \tag{3.17}
\]

Provided

\[
a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega_j \in \mathbb{C}; k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \ldots, s
\]

\[
\lambda_k, \sigma_i, \sigma'_j > 0; k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \ldots, s
\]
\[ |\arg(z_i)| < \frac{1}{2} \pi A_i^{(k)} \text{ and } A_i^{(k)} \text{ is defined by } (2.4), |\arg(z'_i)| < \frac{1}{2} \pi A_i^{(k)}, |\frac{a}{b}| < 1 \]

\[ \text{Re}(\mu) + \sum_{i=1}^{v} \sigma_i \min_{1 \leq j < m_i} \text{Re} \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{d_j^{(i)}} \right) \right] + \sum_{i=1}^{v} \sigma'_i \min_{1 \leq j < m_i'} \text{Re} \left[ D_j'^{(i)} \left( \frac{d_j'^{(i)}}{d_j'^{(i)}} \right) \right] > \max[0, \text{Re}(\beta - \eta)] \]

\[ \text{Re}(\nu) + \sum_{i=1}^{v} \omega_i \min_{1 \leq j < m_i} \text{Re} \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{d_j^{(i)}} \right) \right] + \sum_{i=1}^{v} \omega'_i \min_{1 \leq j < m_i'} \text{Re} \left[ D_j'^{(i)} \left( \frac{d_j'^{(i)}}{d_j'^{(i)}} \right) \right] > \max[0, \text{Re}(\beta - \eta)] \]

To prove the formula (3.17), we use the similar method that the theorem 1 by using the lemma 3.

Let

\[ A_4 = (1 - \mu - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_v, \omega'_1, \cdots, \omega'_v, 1; 1); (-\eta + \mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_v, \sigma'_1, \cdots, \sigma'_v, 1; 1), \]

\[ (1 - \mu - \eta + \beta - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_v, \sigma'_1, \cdots, \sigma'_v, 1; 1) \]  

(3.18)

\[ B_4 = (-\nu - \sum_{j=1}^{v} \delta_j K_j; \eta_1, \cdots, \eta_v, \eta'_1, \cdots, \eta'_v, 0; 1), (\mu + \sum_{j=1}^{K} \lambda_j K_j; 1, \cdots, \sigma_v, \sigma'_1, \cdots, \sigma'_v, 1; 1) \]

\[ (-\alpha - \beta - \eta + \mu + \sum_{j=1}^{K} \lambda_j K_j; \sigma_1, \cdots, \sigma_v, \sigma'_1, \cdots, \sigma'_v, 1; 1) \]  

(3.19)

We have the formula.

Theorem 4.

\[ \{ (I_{x, \infty})^{-1} (b - at)^{-\frac{\omega}{N_1} + \cdots + \frac{\omega}{N_v}}, c_1 t^{\lambda_1} (b - at)^{-\delta_1}, \ldots, c_v t^{\lambda_v} (b - at)^{-\delta_v} \} \]

\[ \begin{pmatrix} z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\ \vdots \\ z_v t^{\sigma_v} (b - at)^{-\omega_v} \end{pmatrix} \]

\[ = b^{-\nu} x^{-\alpha - \alpha' + \eta - 1} \sum_{R_1 = 1}^{[N_1/M_1]} \ldots \sum_{R_v = 1}^{[N_v/M_v]} a_{R_1}^{K_1} \ldots a_{R_v}^{K_v} \]

\[ \begin{pmatrix} z_1^{\frac{\omega_1}{\omega}} \\ \vdots \\ z_v^{\frac{\omega_v}{\omega}} \end{pmatrix} \]

(3.20)
Provided

\[ a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega_j \in \mathbb{C}, k = 1, \ldots, v; i = 1 \ldots, r; j = 1, \ldots, s \]
\[ \lambda_k, \sigma_i, \sigma_j > 0; \ k = 1, \ldots, v; i = 1 \ldots, r; j = 1, \ldots, s \]

\[ |\arg(z_i)| < \frac{1}{2} \pi A_i^{(k)} \text{ and } A_i^{(k)} \text{ is defined by (2.4)}, |\arg(z_i^r)| < \frac{1}{2} \pi A_i^{(k)}; \left| \frac{a}{b} \right| < 1 \]

\[
Re(\mu) - \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] - \sum_{i=1}^{s} \omega_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] < 1 + \min[Re(\alpha - \beta), Re(\alpha + \alpha' - \eta), Re(\alpha + \beta' - \eta)]
\]

\[
Re(\nu) - \sum_{i=1}^{r} \omega_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] - \sum_{i=1}^{s} \omega_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] < 1 + \max[Re(\alpha - \beta), Re(\alpha + \alpha' - \eta), Re(\alpha + \beta' - \eta)]
\]

Provided

\[ a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega_j \in \mathbb{C}, k = 1, \ldots, v; i = 1 \ldots, r; j = 1, \ldots, s \]
\[ \lambda_k, \sigma_i, \sigma_j > 0; \ k = 1, \ldots, v; i = 1 \ldots, r; j = 1, \ldots, s \]

\[ |\arg(z_i)| < \frac{1}{2} \pi A_i^{(k)} \text{ and } A_i^{(k)} \text{ is defined by (2.4)}, |\arg(z_i^r)| < \frac{1}{2} \pi A_i^{(k)}; \left| \frac{a}{b} \right| < 1 \]

\[
Re(\mu) - \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] - \sum_{i=1}^{s} \omega_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] < 1 + \min[Re(\eta), Re(\beta)]
\]

\[
Re(\nu) - \sum_{i=1}^{r} \omega_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] - \sum_{i=1}^{s} \omega_i \min_{1 \leq j \leq m(i)} Re \left[ D_j^{(i)} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] < 1 + \max[Re(\beta), Re(\eta)]
\]

To prove the theorem 4, we use the similar method that the equation (3.5) by using the lemma 4.

IV. PARTICULAR CASES

In this section, we shall see four particular cases.

If we put \( \beta = -\alpha \) in the theorem three, we get

**Corollary 1.**

\[
\begin{pmatrix}
  c_1 t^{\lambda_1} (b - at)^{\delta_1} \\
  \vdots \\
  c_v t^{\lambda_v} (b - at)^{\delta_v}
\end{pmatrix}
\begin{pmatrix}
  z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\
  \vdots \\
  z_r t^{\sigma_r} (b - at)^{-\omega_r}
\end{pmatrix}
\]

\[
\left( \begin{pmatrix}
  z_1' t^{\sigma_1} (b - at)^{-\omega_1'} \\
  \vdots \\
  z_s' t^{\sigma_s} (b - at)^{-\omega_s'}
\end{pmatrix} \right)
\]

\[
\left( x \right) = b^{-\nu} x^{\mu - \beta - 1} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_s=1}^{[N_s/M_s]} a_v c_1^{K_1} \cdots c_v^{K_v}
\]
Fractional Integration of the Product of Two Multivariable Gimel-Functions and A General Class of Polynomials

\[ v = \sum_{j=1}^{\nu} \delta_j K_j \sum_{j=1}^{\nu'} \lambda_j K_j \mathcal{U}^{0,0,n_n + n'_n + 2,V,1,0} \]

\[ \mathcal{X}_{p_n + p'_n + q_n + q'_n + 2, \tau_n + \tau'_n} : R_n : R'_n : Y : 0,1 \]

\[ \left( \begin{array}{c}
    z_1 \frac{\sigma_1}{b_1} \\
    \vdots \\
    z_n \frac{\sigma_n}{b_n} \\
    z_1 \frac{\sigma'_1}{b_1'} \\
    \vdots \\
    z_n \frac{\sigma'_n}{b_n'} \\
    \end{array} \right) \]

\[ \begin{array}{c}
    A_n, A_n, A_n, A_n, B_n : B_{n,0},1;1 \\
    \end{array} \]

(4.1)

where

\[ A_3 = (1 - v - \sum_{j=0}^{v} \delta_j K_j, \omega_1, \cdots, \omega_n, \omega'_1, \cdots, \omega'_n, 1;1), (1 - \mu - \sum_{j=1}^{\nu} \lambda_j K_j, \sigma_1, \cdots, \sigma_n, \sigma'_1, \cdots, \sigma'_n, 1;1) \]

(4.2)

\[ A_3 = (1 - v - \sum_{j=0}^{v} \delta_j K_j, \omega_1, \cdots, \omega_n, \omega'_1, \cdots, \omega'_n, 0;1), B_3 = (1 - \mu - \alpha - \eta - \sum_{j=1}^{\nu} \lambda_j K_j, \sigma_1, \cdots, \sigma_n, \sigma'_1, \cdots, \sigma'_n, 1;1) \]

(4.3)

under the same existence conditions that formula (3.17) with \( \beta = -\alpha \).

If \( \beta = 0 \) in theorem three, we have

Corollary 2.

\[ (I_{\eta,\alpha} t^{u-1} (b - at)^{-1}) S_{N_1, \cdots, N_n} \left( \begin{array}{c}
    c_1 t^{\lambda_1} (b - at)^{-\delta_1} \\
    \vdots \\
    c_v t^{\lambda_v} (b - at)^{-\delta_v} \\
    \end{array} \right) \]

\[ \left( \begin{array}{c}
    z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\
    \vdots \\
    z_n t^{\sigma_n} (b - at)^{-\omega_n} \\
    \end{array} \right) \]

\[ \left( \begin{array}{c}
    z'_1 t^{\sigma'_1} (b - at)^{-\omega'_1} \\
    \vdots \\
    z'_n t^{\sigma'_n} (b - at)^{-\omega'_n} \\
    \end{array} \right) \]

\[ \left( \begin{array}{c}
    1 \\
    \vdots \\
    1 \\
    \end{array} \right) = b^{-v} x^{\mu-\beta-1} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_v=1}^{[N_v/M_v]} a_{c_1} c_{K_1} \cdots c_{K_v} \]

(4.4)

\[ \left( \begin{array}{c}
    z_1 \frac{\sigma_1}{b_1} \\
    \vdots \\
    z_n \frac{\sigma_n}{b_n} \\
    z_1 \frac{\sigma'_1}{b_1'} \\
    \vdots \\
    z_n \frac{\sigma'_n}{b_n'} \\
    \end{array} \right) \]

\[ \begin{array}{c}
    A_n, A_n, A_n, A_n, B_n : B_{n,0},1;1 \\
    \end{array} \]

(4.4)
where

\[ A_0 = (1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r; \omega'_1, \cdots, \omega'_s, 0; 1), (1 - \mu - \eta - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1) \]  \hspace{1cm} (4.5)

\[ B_0 = (1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r; \omega'_1, \cdots, \omega'_s, 0; 1), (1 - \mu - \alpha - \eta - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1) \]  \hspace{1cm} (4.6)

provided that

\[ a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega'_j \in \mathbb{C}, k = 1, \cdots, v; i = 1 \cdots r; j = 1, \cdots, s \]

\[ \lambda_k, \sigma_i, \sigma'_j > 0; \ k = 1, \cdots, v; i = 1 \cdots r; j = 1, \cdots, s \]

\[ |\arg(z_i)| < \frac{1}{2} \pi A_{(k)}^i \] and \( A_{(k)}^i \) is defined by (2.4), \(|\arg(z'_i)| < \frac{1}{2} \pi A_{(k)}^{i'}\) , \( \frac{a}{b} < 1 \)

\[ \text{Re}(\mu) + \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m^{(i)}} \text{Re} \left[ D^{(i)}_j \left( \frac{d^{(i)}_j}{\delta_j} \right) \right] + \sum_{i=1}^{s} \sigma'_i \min_{1 \leq j \leq m^{(i)}} \text{Re} \left[ D'^{(i)}_j \left( \frac{d'^{(i)}_j}{\delta'_j} \right) \right] > \max \{|0, \text{Re}(-\eta)| \} \]

\[ \text{Re}(\nu) + \sum_{i=1}^{r} \omega_i \min_{1 \leq j \leq m^{(i)}} \text{Re} \left[ D^{(i)}_j \left( \frac{d^{(i)}_j}{\delta_j} \right) \right] + \sum_{i=1}^{s} \omega'_i \min_{1 \leq j \leq m^{(i)}} \text{Re} \left[ D'^{(i)}_j \left( \frac{d'^{(i)}_j}{\delta'_j} \right) \right] > \max \{|0, \text{Re}(-\eta)| \} \]

If we put \( \beta = -\alpha \) in the equation (3.20), we get

Corollary 3.

\[ \{ \left( \left( \frac{t^\alpha}{\Gamma(\alpha)} \right)^{-1} S_{N_1, \cdots, N_r}^{a \in \mathbb{R}_+, \cdots, a \in \mathbb{R}_+} \right) \cdot \left( \begin{array}{c} c_1 t^\lambda (b - at)^{-\delta_1} \\ \vdots \\ c_v t^\lambda (b - at)^{-\delta_v} \\ \end{array} \right) \left( \begin{array}{c} z_1 t^\sigma (b - at)^{-\omega_1} \\ \vdots \\ z_v t^\sigma (b - at)^{-\omega_v} \\ \end{array} \right) \} \left( \begin{array}{c} z_1 t^\sigma (b - at)^{-\omega_1} \\ \vdots \\ z_v t^\sigma (b - at)^{-\omega_v} \\ \end{array} \right) \]  \hspace{1cm} (4.7)
where

$$A_T = (1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1), (\alpha + \mu + \sum_{j=1}^{v} \lambda_j K_j, \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1) \quad (4.8)$$

$$(4.9)$$

$$A_T = (1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s; 0, 1), B_T = (\mu + \sum_{j=1}^{v} \lambda_j K_j, \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1)$$

under the same existence conditions that formula (3.20) with $\beta = -\alpha$.

If $\beta = 0$ in theorem four, we have

**Corollary 4.**

Let

$$A_T = (1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1), (-\alpha - \eta + \sum_{j=1}^{v} \lambda_j K_j, \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1) \quad (4.10)$$

$$B_T = (1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s; 0, 1), (\mu - \alpha - \eta + \sum_{j=1}^{v} \lambda_j K_j, \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1) \quad (4.11)$$

$$(4.12)$$

$$\left\{ K_{\eta,\lambda}^{-(\mu - 1)}(b - at)^{-\delta} \right\}_{t^{N_1}, \ldots, t^{N_v}} \left( \begin{array}{c} c_1 t^{\lambda_1} (b - at)^{-\delta_1} \\ \vdots \\ c_v t^{\lambda_v} (b - at)^{-\delta_v} \end{array} \right) = \left( \begin{array}{c} z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\ \vdots \\ z_r t^{\sigma_r} (b - at)^{-\omega_r} \end{array} \right)$$

Provided

$$a, b, \alpha, \beta, \eta, \mu, \delta, \omega_j, \omega'_j \in \mathbb{C}, k = 1, \ldots, v; i = 1, \ldots, r; j = 1, \ldots, s$$

$$\lambda_k, \sigma_i, \sigma'_j > 0; k = 1, \ldots, v; i = 1, \ldots, r; j = 1, \ldots, s$$
under the same existence conditions that equation (3.20) with $\beta = 0$.

Remark: By the similar procedure, the results of this document can be extended to the product of any finite number of multivariable Gimel-functions and a class of multivariable polynomials defined by Srivastava [30]. Agarwal [1,2] has studied the fractional integration about the multivariable H-function.

V. Conclusion

In this paper, we have obtained several theorems of the generalized fractional integral operators given by Saigo-Maeda and Saigo. The images have been developed regarding the product of the two multivariable Gimel-functions and a general class of multivariable polynomials in a compact and elegant form with the help of Saigo-Maeda and Saigo operators. Most of the results obtained in this paper are useful in deriving the composition formulae involving Riemann–Liouville, Erdelyi–Kober fractional calculus operators and multivariable Gimel functions. The findings of this paper provide an extension of the results given earlier by Kilbas [12], Kilbas and Saigo [13], Kilbas and Sebastain [14], Saxena et al.[29] and Gupta et al.[7] as mentioned earlier.

References Références Referencias