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Interesting Theta Function Identities Related to Jacobi Triple-Product

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I. Introduction

Since Jacobi introduce the so-called theta functions, they have been extensively investigated and have many applications in diverse areas for example number theory, quantum physics, quadratic forms and elliptic functions (see the references here and references cited therein) . We recall the well known Jacobi triple product (see, e.g., [4]; see also 1, 2, 3, 5, 6, 7, 8, 9, 10] as follows:

$$\sum_{i=-\infty}^{\infty} y^i x^{i^2} = \prod_{i=1}^{\infty}(1-x^{2i})(1+yx^{2i-1})(1+y^{-1}x^{2i-1}) \qquad (|x|<1, y\neq 0) \quad (1)$$

We define the following three fundamental theta functions (see, e.g., [2, 3, 7]):

$$f(-x) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{n(3n-1)}{2}} = \prod_{i=1}^{\infty}(1-x^i) \quad (2)$$

$$\phi(x) = \sum_{n=-\infty}^{\infty} x^{n^2} = \prod_{i=1}^{\infty}(1+x^{2i+1})(1+x^{2i-1})(1-x^{2i}) \quad (3)$$

$$\psi(x) = \sum_{n=0}^{\infty} x^{\frac{n(n+1)}{2}} = \prod_{i=1}^{\infty}\frac{(1-x^{2i})}{(1-x^{2i-1})} \quad (4)$$

The main object of the present article is to prove two (presumably new) identities involving the three functions given in (2), (3) and (4) in elementary way.

II. The Main Results

In this section, we begin by expressing the functions $f(-x)$, $\phi(x)$ and $\psi(x)$ in rising powers of x as follows:

$$f(-x) = 1+\sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{n(3n-1)}{2}}+x^{\frac{n(3n+1)}{2}}\right)$$

$$= 1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-\cdots; \quad (5)$$

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Ref

[4]. G. J. Jacobi, *Fundamental Nova Theoriae Functionum Ellipticarum*, Sumptibus Fratrum Borntrger, Regiomonti, 1829.

$$\phi(x) = 1 + 2\sum_{n=1}^{\infty} x^{n^2} = 1 + 2x + 2x^4 + 2x^9 + 2x^{16} + \cdots; \tag{6}$$

and

$$\psi(x) = 1 + \sum_{n=1}^{\infty} x^{\frac{n(n+1)}{2}} = 1 + x + x^3 + x^6 + x^{10} + x^{15} + \cdots. \tag{7}$$

Now the main results state in the following Theorem.

Theorem. *Each of the following relationships holds true:*

$$f(-x^3)[\psi(x) - x\psi(x^9)] = f(-x^6)\phi(-x^9) \tag{8}$$

and

$$\begin{aligned}[\phi(x^2) - \phi(x^{18})][\phi(x^6) - \phi(x^{54})][\psi(x^4) - x^4 \quad (x^{36})][\psi(x^{12}) - x^{12} \quad (x^{108})] \\ \psi \\ = x^5[\psi^2(x^{27})\{\psi(x) - x\psi(x^9)\}^2 - \psi^2(-x^{27})\{\psi(-x) + x\psi(-x^9)\}^2]\end{aligned} \tag{9}$$

where the functions $f(-x)$, $\phi(x)$ and $\psi(x)$ are given by (2), (3), (4), respectively.

Proof. First of all, we shall prove our first identity (8). Let η_1 and ξ_1 denote the left-hand and the right-hand sides of the identity (8), respectively. Then, in order to compute the value for η_1, by using (5) [for $x \mapsto x^3$] and (7) [for $x \mapsto x^9$] respectively, we have:

$$\eta_1 = (1 - x^3 - x^6 + x^{15} + x^{21} - x^{36} - \psi\cdot)[(1 + x\psi + +x^3 + x^6 + +x^{10}$$

$$+x^{15} + \cdots) - x(1 + x^9 + x^{27} + x^{54} + x^{90} + x^{135} + \cdots)]$$

which, after multiplication and further simplification, yields

$$\begin{aligned}\eta_1 = 1 - x^6 - 2x^9 - x^{12} + 2x^{15} + 2x^{21} + x^{30} + 2x^{36} - -2x^{39} - x^{42} \\ -2x^{48} - 2x^{51} + 2x^{66} - x^{72} + 2x^{78} + 2x^{87} - x^{90} + +2x^{93} + 2x^{99} \\ -2x^{108} - 2x^{111} - 2x^{123} - \cdots\end{aligned} \tag{10}$$

In a similar way, we can compute the value for ξ_1, by applying (5) [for $x \mapsto x^6$] and (6) [for $x \mapsto -x^9$] as follows:

$$\xi_1 = (1 - x^6 - x^{12} + x^{30} + x^{42} - x^{72} - x^{90} + x^{132} + \cdots)\times$$

$$\times(1 - 2x^9 + 2x^{36} - 2x^{81} + 2x^{144} - 2x^{225} + 2x^{324} - \cdots)$$

after simplifications and using little algebra, we obtain:

$$\begin{aligned}\xi_1 = 1 - x^6 - 2x^9 - x^{12} + 2x^{15} + 2x^{21} + x^{30} + 2x^{36} - 2x^{39} - x^{42} \\ -2x^{48} - 2x^{51} + 2x^{66} - x^{72} + 2x^{78} + 2x^{87} - x^{90} + 2x^{93} + 2x^{99} - \\ -2x^{108} - 2x^{111} - 2x^{123} - 2x^{126} + \cdots\end{aligned} \tag{11}$$

By comparing equations (10) and (11), we readily arrive at the identity (8).

We next prove the second identity (9). Let η_2 and ξ_2 denote the left-hand

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and the right-hand sides of the identity (9), respectively. Then, in order to compute the value for η_2, by using (6) [for $x \mapsto x^2$, $x \mapsto x^6$, $x \mapsto x^{18}$ and $x \mapsto x^{54}$] and (7) [for $x \mapsto x^4$, $x \mapsto x^{12}$, $x \mapsto x^{36}$ and $x \mapsto x^{108}$], as follows:

$$\eta_2 = [2x^2 + 2x^8 + 2x^{32} + 2x^{50} + 2x^{98} + 2x^{128} + 2x^{200} + \cdots][2x^6 + 2x^{24} +$$
$$+ 2x^{96} + 2x^{150} + 2x^{294} + \cdots][1 + x^{12} + x^{24} + x^{60} + x^{84} + x^{144} +$$
$$+ x^{180} + \cdots][1 + x^{36} + x^{72} + x^{108} + x^{252} + \cdots]$$

which, after simplification and by using algebraic manipulation, yields

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$$\eta_2 = 4x^8[1 + x^6 + x^{12} + 2x^{18} + 2x^{24} + 3x^{30} + 2x^{36} + 3x^{42} + 4x^{48} + 3x^{54} +$$
$$+ 5x^{60} + 5x^{66} + 4x^{72} + 5x^{78} + 6x^{84} + 6x^{90} + 5x^{96} + 5x^{102} + x^{104} + \cdots] \quad (12)$$

Now we have to compute the value ξ_2, using (7) [for $x \mapsto -x^{27}$, $x \mapsto -x^9$, $x \mapsto -x$, $x \mapsto x^9$ and $x \mapsto x^{27}$] as follows:

$$\xi_2 = x^5[(1 + 2x^3 + 3x^6 + 2x^9 + x^{12} + 2x^{15} + 2x^{18} + 4x^{21} + 2x^{24} + 4x^{27} +$$
$$+ 5x^{30} + 6x^{33} + 8x^{36} + 4x^{39} + 7x^{42} + 6x^{45} + 10x^{48} + 8x^{51} + x^{54} +$$
$$+ 6x^{57} + \cdots) - (1 - 2x^3 + 3x^6 - 2x^9 + x^{12} - 2x^{15} + 2x^{18} - 4x^{21} +$$
$$+ 2x^{24} - 4x^{27} + 5x^{30} - 6x^{33} + 8x^{36} - 4x^{39} + 7x^{42} - 6x^{45} + 10x^{48} -$$
$$- 8x^{51} + x^{54} - 6x^{57} + \cdots)]$$

after simplifications, we have:

$$\xi_2 = 4x^8[1 + x^6 + x^{12} + 2x^{18} + 2x^{24} + 3x^{30} + 2x^{36} + 3x^{42} + 4x^{48} + 3x^{54} +$$
$$+ 5x^{60} + 5x^{66} + 4x^{72} + 5x^{78} + 6x^{84} + 6x^{90} + 5x^{96} + 5x^{102} + x^{104} + \cdots] \quad (13)$$

The identity (9) now follows upon comparing the equations (12) and (13).

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