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On Liouville Decompositions of Polyadic Integers

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On Liouville Decompositions of Polyadic Integers

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I. Introduction

In 1844 J. Liouville showed that if for any positive integer m there exists a rational number $\frac{p}{q}$, $(p,q)=1$ such that $\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^m}$, then α is a transcendental number (a so-called Liouville number). In 1962 P. Erdős [1], in particular, gave a simple proof, that any real number can be represented as a sum of two Liouville numbers. In 1996 his result was essentially generalized by E.D. Burger [2]. He also mentioned that one can prove the results on Liouville decompositions for direct products of local fields. In this note, we show, following the lines from [1], how any polyadic integer can be explicitly expressed as a sum of two polyadic Liouville numbers.

We first supply a brief introduction to the theory of polyadic integers. Let K be a commutative ring. A mapping v of K into non-negative real numbers is called a non-archimedean pseudo-valuation of K if it has the following properties:

1. $v(a)\geqslant 0$ for all $a\in K$, $v(a)=0$ if $a=0\in K$.
2. $v(ab)\leqslant v(a)v(b)$ for all $a,b\in K$.
3. $v(a\pm b)\leqslant \max(v(a),v(b))$ for all $a,b\in K$.

If for all $a,b\in K$ the stronger condition $v(ab)=v(a)v(b)$ holds, then v is called a valuation. For a prime p, the p-adic valuation $|\alpha|_p$ of an element $\alpha\in\mathbb{Q}$ is defined as follows. If $a\in\mathbb{Z}$ is divisible by p^f and is not divisible by p^{f+1}, then $|a|_p=p^{-f}$. For $\alpha=\frac{a}{b}$, $a\in\mathbb{Z}, b\in\mathbb{N}$ we have $|\alpha|_p=\frac{|a|_p}{|b|_p}$. As usual, $\mathbb{Q}_p$ denotes the corresponding completion of $\mathbb{Q}$. It is the field of p-adic numbers, and $\mathbb{Z}_p$ denotes the ring of p-adic integers which satisfy the inequality $|\alpha|_p\leqslant 1$. For $g=p_1^{r_1}\cdot\ldots\cdot p_s^{r_s}$, where p_i are primes and r_i are positive integers we can consider the g-adic pseudo-valuation which is defined in a similar way (cf. [3]). The corresponding completion is denoted by $\mathbb{Q}_g$. A well-known theorem by K. Hensel (cf. [3]) asserts that $\mathbb{Q}_g$ is a direct sum

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2. E.D. Burger. On Liouville decomposition in local fields. Proc.Amer.Math.Soc., v.124, Num.11, (1966),3305-3310.

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of the fields $\mathbb{Q}_{p_1}, \ldots, \mathbb{Q}_{p_s}$, so any element $A \in \mathbb{Q}_g$ can be expressed as $A = (A_1, \ldots, A_s)$ with $A_i \in \mathbb{Q}_{p_i}$ and for any polynomial $P(x) \in \mathbb{Z}[x]$ one has $P(A) = (P(A_1), \ldots, P(A_s))$. Recall that $\alpha \in K$ is called *algebraic* (over $\mathbb{Q}$) if there exists a nonzero polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\alpha) = 0$. Otherwise it is called *transcendental*. Therefore $A \in \mathbb{Q}_g$ is algebraic if $A_i \in \mathbb{Q}_{p_i}$, $i = 1, \ldots, s$ are algebraic.

We introduce a topology in the ring $\mathbb{Z}$ by considering the set of all ideals (m) as the system of vicinities of zero. Addition and multiplication are continuous with respect to this topology. The completion of this topological ring is called the ring of *polyadic integers*. (The detailed descriptions of the construction of polyadic numbers are presented in [4]). The elements of this ring have canonical representations of the form

$$\alpha = \sum_{n=0}^{\infty} a_n \cdot n!, \quad a_n \in \{0, 1, \ldots, n\}. \tag{1}$$

A series of this form converges in any $\mathbb{Q}_p$ and, for example, in any $\mathbb{Q}_p$ we have $\sum\limits_{n=1}^{\infty} n \cdot n! = -1$. One can prove that the ring of polyadic integers is a prime product of the rings $\mathbb{Z}_p$ over all primes p. Therefore any polyadic integer can be expressed as $\alpha = (\alpha_1, \ldots, \alpha_s, \ldots)$ where the components α_s belong to $\mathbb{Z}_{p_s}$, e.g. $\sum_{n=1}^{\infty} n \cdot n! = (-1, \ldots, -1, \ldots)$. This remark allows us to give the following definition: a polyadic integer α is called *algebraic*, if there exists a polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\alpha) = 0$ (where $0 = (0, \ldots, 0, \ldots)$), in other words, if for any s one has $P(\alpha_s) = 0$ in $\mathbb{Z}_{p_s}$. In terms of [5], [6] it means that α satisfies a global relation.

We call the polyadic integer α *transcendental*, if for any nonzero polynomial $P(x) \in \mathbb{Z}[x]$ there exists at least one prime p such that $P(\alpha_p) \neq 0$ in $\mathbb{Q}_p$. We call the polyadic integer α *infinitely transcendental*, if for any nonzero polynomial $P(x) \in \mathbb{Z}[x]$ there exist infinitely many primes p such that $P(\alpha_p) \neq 0$ in $\mathbb{Q}_p$. At last, a polyadic integer is *globally transcendental*, if for any nonzero polynomial $P(x) \in \mathbb{Z}[x]$ and for all primes p the inequality $P(\alpha_p) \neq 0$ holds in $\mathbb{Q}_p$. Of course, globally transcendental polyadic integers form a subset of infinitely transcendental polyadic integers which, in turn, form a subset of transcendental polyadic integers. Hensel's theorem mentioned above implies that there exist transcendental polyadic integers, which are not infinitely transcendental, and there exist infinitely transcendental polyadic integers, which are not globally transcendental.

The arithmetic properties of polyadic integers are studied in [5]-[12].

We call a polyadic integer α a *polyadic Liouville number*, if for any positive P, D and any prime p, $p \leqslant P$ there exists a positive integer A such that

$$|\alpha - A|_p < A^{-D}. \tag{2}$$

It is easy that for fixed positive P, D there exist infinitely many positive integers A satisfying this inequality. One can easily prove that the Liouville number is globally transcendental.

Let $\tau(k)$ be any integer-valued function defined on non-negative integers and satisfying

$$\frac{\tau(k+1)}{\tau(k) \ln \tau(k)} \to +\infty, \quad \text{as} \quad k \to +\infty. \tag{3}$$

We define the functions $l_1(n) = 1, l_2(n) = 0$ for all n with $\tau(k) \leqslant n < \tau(k+1)$ for $k = 0, 2, 4, \ldots$. We also put $l_1(n) = 0, l_2(n) = 1$ for all n with $\tau(k) \leqslant n < \tau(k+1)$ when $k = 1, 3, 5, \ldots$

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4. H. Prüfer, Neue Begründung der algebraischen Zahlentheorie, Math. Ann. V.94, 3-4 (1925), 198-243.

Notes

Theorem. For any polyadic integer (1) *we have* $\alpha = L_1 + L_2$, *where*

$$L_1 = \sum_{n=0}^{\infty} l_1(n) \cdot a_n \cdot n!, \quad L_2 = \sum_{n=0}^{\infty} l_2(n) \cdot a_n \cdot n! \tag{4}$$

are the two polyadic Liouville numbers.

Proof. The equation $\alpha = L_1 + L_2$ is evident. We now prove that L_1 is a polyadic Liouville number. The number L_2 can be treated in analogy. Let's denote

$$A_{1,k} = \sum_{n=0}^{\tau(k)} l_1(n) \cdot a_n \cdot n!. \tag{5}$$

Since $a_n \in \{0, 1, \ldots, n\}$, we have, using (5) and a rough estimate for factorial, the inequality

$$A_{1,k} < (\tau(k) + 1)! \leqslant e^{(\tau(k)+1)\ln((\tau(k)+1))} \leqslant e^{2\tau(k)\ln\tau(k)} \tag{6}$$

From (4) and (5) we get

$$L_1 - A_{1,k} = \sum_{n=\tau(k+1)}^{\infty} l_1 \cdot a_n \cdot n!$$

so, for any prime p, $p \leqslant P$ we obtain

$$|L_1 - A_{1,k}|_p \leqslant |\tau(k+1)!|_p = e^{-\frac{\ln p}{p-1}\left(\tau(k+1) - S_{\tau(k+1)}\right)} \leqslant e^{-\frac{\frac{1}{2}\ln p}{p-1}(\tau(k+1))} \tag{7}$$

for sufficiently large k. Here for any positive integer N the symbol S_N denotes the sum of digits in the p-adic representation of N and it's evident that $S_N \leqslant (p-1)(\log_p N + 1)$.

For any fixed P, D and for any prime p, $p \leqslant P$ from (3), (6), (7) we get that if k is sufficiently large, then

$$|L_1 - A_{1,k}|_p \leqslant e^{-\frac{\frac{1}{2}\ln p}{p-1}(\tau(k+1))} \leqslant \left(e^{2\tau(k)\ln\tau(k)}\right)^{-D} \leqslant (A_{1,k})^{-D}.$$

This means that (2) holds and that the theorem is proved.

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