Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 7 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Fourier Transform of Power Series

By Shiferaw Geremew Kebede, Awel Seid Gelete, Dereje Legesse Abaire & Mekonnen Gudeta Gizaw
Madda Walabu University

Abstract- The authors establish a set of presumably new results, which provide Fourier transform of power series. So in this paper the author try to evaluate Fourier transform of some challenging functions by expressing them as a sum of infinitely terms. Hence, the method is useful to find the Fourier transform of functions that difficult to obtain their Fourier transform by ordinary method or using definition of Fourier transformations.

Keywords: *fourier transforms, power series, taylor's and maclaurin series and gamma function.*

GJSFR-F Classification: *FOR Code: MSC 2010: 35S30*

FOURIERTRANSFORMOFPOWERSERIES

Strictly as per the compliance and regulations of:

Notes

Fourier Transform of Power Series

Shiferaw Geremew Kebede [α], Awel Seid Gelete [σ], Dereje Legesse Abaire [ρ] & Mekonnen Gudeta Gizaw [ω]

Abstract- The authors establish a set of presumably new results, which provide Fourier transform of power series. So in this paper the author try to evaluate Fourier transform of some challenging functions by expressing them as a sum of infinitely terms. Hence, the method is useful to find the Fourier transform of functions that difficult to obtain their Fourier transform by ordinary method or using definition of Fourier transformations.

Keywords: fourier transforms, power series, taylor's and maclaurin series and gamma function.

I. Introduction

The Fourier transform is one of the most important integral transforms. Be-cause of a number of special properties, it is very useful in studying linear differential equations.

Fourier analysis has its most important applications in mathematical modeling, physical and engineering and solving partial differential equations (PDEs) related to boundary and initial value problems of Mechanics, heat flow, electro statistics and other fields. Daniel Bernoulli (1700-1782) and Leonhard Euler (1707-1783), Swiss mathematicians, and Jean-Baptiste D Alembert (1717-1783), a French mathematician, physicist, philosopher, and music theorist, were all prominent in the ensuing mathematical music debate. In 1751, Bernoullis memoir of 1741-1743 took Rameaus findings into account, and in 1752, DAlembert published Elements of theoretical and practical music according to the principals of Monsieur Rameau, clarified, developed, and simplified. DAlembert was also led to a differential equation from Taylors problem of the vibrating string,

$$\frac{\partial^2 y}{\partial x^2} = \alpha^2 \frac{\partial^2 y}{\partial^2 t^2}$$

The current widespread use of the transform (mainly in engineering) came about during and soon after World War II, although it had been used in the 19th century by Abel, Lerch, Heaviside, and Bromwich.

Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply to a limited region of space because those solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space.

a) Definition

The Fourier transform of the function $f(x)$ is given by:

$$F(f(x)) = \frac{1}{\sqrt{2\Pi}} \int_0^\infty f(x) e^{-ix\omega} dx$$

b) Definition

A Power series is a series defined of the form:

Author: Department of Mathematics, Madda Walabu University, Bale Robe Ethiopia. e-mail: yerosenshiferaw@gmail.com

$$f(x) = \Sigma_{n=0}^{\infty} a_n (x-c)^n$$

$$= a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + ... + a_n(x-c)^n$$

where c is any constant $c \in \Re$

c) Definition

If $f(x)$ has a power series expansion at c,where c is any constant $c \in \Re$. It's Taylor's series expansion is:

$$f(x) = \Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!}$$

d) Definition

Maclaurin Series expansion of the function $f(x)$ is:

$$f(x) = \Sigma_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

$$= f(0) + f^{'}(0)x + \frac{f^{''}(0)}{2!}x^2 + \frac{f^{'''}(0)}{3!}x^3 + ...$$

e) Definition

The gamma function, whose symbol $\Gamma(s)$ is defined when $s > 0$ by the formula

$$\Gamma = \int_0^{\infty} e^{-x} x^{n-1} dx$$

II. Fourier Transform of Power Series

Theorem 1: (Fourier Transform of power series)

If$f(x)$ has a Power series expansion at c,where c is any constant $c \in \Re$.

$$f(x) = \Sigma_{n=0}^{\infty} a_n (x-c)^n$$

then the Fourier transform of $f(x)$ is given in the form of power series as:

$$F(f(x)) = F(\Sigma_{n=0}^{\infty} a_n (x-c)^n)$$

$$= \frac{1}{\sqrt{2\Pi} e^{ic\omega}} \Sigma_{n=0}^{\infty} a_n \frac{1}{(i\omega)^{n+1}} \frac{\Gamma(n+1)}{s^{n+1}}$$

Proof

Suppose $f(x)$ has a Power series expansion at c,where c is any constant $c \in \Re$.

i.e

$$f(x) = \Sigma_{n=0}^{\infty} a_n (x-c)^n$$

Then, By using the definition of Fourier transforms,

Notes

Notes

$$F(f(x)) = F(\Sigma_{n=0}^{\infty} a_n (x-c)^n)$$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\Sigma_{n=0}^{\infty} a_n (x-c)^n] e^{-ix\omega} dx$$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \Sigma_{n=0}^{\infty} a_n (x-c)^n e^{-ix\omega} dx$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} a_n e^{-ix\omega} (x-c)^n dx$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-ix\omega} (x-c)^n dx$$

Let,$x = t + c \Longleftrightarrow dx = dt$
So,

$$F(f(x)) = F(\Sigma_{n=0}^{\infty} a_n (x-c)^n)$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-i(t+c)\omega} t^n dt$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-it\omega} e^{-ic\omega} t^n dt$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n e^{-ic\omega} \int_{-\infty}^{\infty} e^{-it\omega} t^n dt$$

Let, $v = it\omega \Longleftrightarrow t = \frac{v}{i\omega} \Rightarrow dt = \frac{1}{i\omega} dv$
Hence,

$$F(f(x)) = \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n e^{-ic\omega} \int_{-\infty}^{\infty} e^{-v} [\frac{v}{i\omega}]^n \frac{1}{i\omega} dv$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n e^{-ic\omega} \int_{-\infty}^{\infty} e^{-v} \frac{v^n}{(i\omega)^n} \frac{1}{i\omega} dv$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} e^{-ic\omega} \int_{-\infty}^{\infty} e^{-v} v^n dv$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} e^{-ic\omega} [2 \int_{0}^{\infty} e^{-v} v^n dv]$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} e^{-ic\omega} [2 \frac{\Gamma(n+1)}{s^{n+1}}]$$

$$= \frac{2}{\sqrt{2\Pi} e^{ic\omega}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} \frac{\Gamma(n+1)}{s^{n+1}}$$

In particular, for $n = 1, 2, 3, ...$

$$\Gamma(n+1) = n!$$

Such that,

$$F(\Sigma_{n=0}^{\infty} a_n (x-c)^n) = \frac{2}{\sqrt{2\Pi} e^{ic\omega}} \Sigma_0^{\infty} a_n \frac{1}{[i\omega]^{n+1}} \frac{n!}{s^{n+1}}$$

Theorem 2:) (Fourier Transform of Taylor's Series)
If $f(x)$ has a power series expansion at c,where c is any constant $c \in \Re$. It's Taylor's series expansion is:

$$f(x) = \Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!}$$

then the Fourier transform of $f(x)$ is given in the form of power series as:

$$F(f(x)) = F[\Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!}]$$

$$= \frac{2}{\sqrt{2\Pi} e^{ic\omega}} f^{(n)}(c) \Sigma_{n=0}^{\infty} a_n \frac{1}{n!(i\omega)^{n+1}} \frac{\Gamma(n+1)}{s^{n+1}}$$

Proof

Suppose $f(x)$ has a Power series expansion at c,where c is any constant $c \in \Re$.
Hence, the Taylor's series expansion of $f(x)$ is:

$$f(x) = \Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!}$$

Then, By using the definition of Fourier transforms,

$$F(f(x)) = F(\Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!})$$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!}] e^{-ix\omega} dx$$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!} e^{-ix\omega} dx$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} a_n f^{(n)}(c) \frac{1}{n!} e^{-ix\omega} (x-c)^n dx$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^{\infty} a_n f^{(n)}(c) \frac{1}{n!} \int_{-\infty}^{\infty} e^{-ix\omega} (x-c)^n dx$$

Let,$x = t + c \Longleftrightarrow dx = dt$
So,

$$F(f(x)) = F(\Sigma_{n=0}^{\infty} a_n f^{(n)}(c) \frac{(x-c)^n}{n!}$$

Notes

Global Journal of Science Frontier Research (F) Volume XVIII Issue VII Version I Year 2018

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} \int_{-\infty}^{\infty} e^{-i(t+c)\omega} t^n dt$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} \int_{-\infty}^{\infty} e^{-it\omega} e^{-ic\omega} t^n dt$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} e^{-ic\omega} \int_{-\infty}^{\infty} e^{-it\omega} t^n dt$$

Notes

Let, $v = it\omega \Longleftrightarrow t = \frac{v}{i\omega} \Rightarrow dt = \frac{1}{i\omega} dv$
Hence,

$$F(f(x)) = \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} e^{-ic\omega} \int_{-\infty}^{\infty} e^{-v} [\frac{v}{i\omega}]^n \frac{1}{i\omega} dv$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} e^{-ic\omega} \int_{-\infty}^{\infty} e^{-v} \frac{v^n}{(i\omega)^n} \frac{1}{i\omega} dv$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} \frac{1}{[i\omega]^{n+1}} e^{-ic\omega} \int_{-\infty}^{\infty} e^{-v} v^n dv$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} \frac{1}{[i\omega]^{n+1}} e^{-ic\omega} [2 \int_0^{\infty} e^{-v} v^n dv]$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} \frac{1}{[i\omega]^{n+1}} e^{-ic\omega} [2 \frac{\Gamma(n+1)}{s^{n+1}}]$$

$$= \frac{2}{\sqrt{2\Pi} e^{ic\omega}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} \frac{1}{[i\omega]^{n+1}} \frac{\Gamma(n+1)}{s^{n+1}}$$

In particular, for $n = 1, 2, 3, ...$

$$\Gamma(n+1) = n!$$

Such that,

$$F(\Sigma_{n=0}^\infty a_n f^{(n)}(c) \frac{(x-c)^n}{n!}) = \frac{2}{\sqrt{2\Pi} e^{ic\omega}} \Sigma_0^\infty a_n f^{(n)}(c) \frac{1}{n!} \frac{1}{[i\omega]^{n+1}} \frac{n!}{s^{n+1}}$$

Theorem 3:) (Fourier Transform of Maclaurin Series)
In particular if $f(x)$ has a power series expansion at 0, then, the power series expansion of f(x) is given by:

$$f(x) = \Sigma_{n=0}^\infty a_n x^n$$

which is known as Maclaurin series, then the Fourier transform of $f(x)$ is defined by:

$$F(f(x)) = F(\Sigma_{n=0}^\infty a_n x^n)$$

$$= \frac{2}{\sqrt{2\Pi}} f^{(n)}(c) \Sigma_{n=0}^\infty a_n \frac{1}{(i\omega)^{n+1}} \frac{\Gamma(n+1)}{s^{n+1}}$$

proof

suppose $f(x)$ has the power series expansion at 0
i.e

$$f(x) = \Sigma_{n=0}^{\infty} a_n x^n$$

By using the definition of Fourier transforms,

$$F(f(x)) = F(\Sigma_{n=0}^{\infty} a_n x^n)$$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} [\Sigma_{n=0}^{\infty} a_n x^n] e^{-ix\omega} dx$$

$$= \frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{\infty} \Sigma_{n=0}^{\infty} a_n x^n e^{-ix\omega} dx$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{n=0}^{\infty} \int_{-\infty}^{\infty} a_n e^{-ix\omega} x^n dx$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-ix\omega} x^n dx$$

Let, $t = ix\omega \iff x = \frac{t}{i\omega} \Rightarrow dx = \frac{1}{i\omega} dt$ Hence,

$$F(f(x)) = \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-t} [\frac{t}{i\omega}]^n \frac{1}{i\omega} dt$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \int_{-\infty}^{\infty} e^{-t} \frac{t^n}{(i\omega)^n} \frac{1}{i\omega} dt$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} \int_{-\infty}^{\infty} e^{-t} t^n dt$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} [2 \int_{0}^{\infty} e^{-t} t^n dt]$$

$$= \frac{1}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} [2 \frac{\Gamma(n+1)}{s^{n+1}}]$$

$$= \frac{2}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} \frac{\Gamma(n+1)}{s^{n+1}}$$

Note: In particular, for $n = 1, 2, 3,$

$$\Gamma(n+1) = n!$$

Hence,

$$F(\Sigma_{n=0}^{\infty} a_n x^n) = \frac{2}{\sqrt{2\Pi}} \Sigma_{0}^{\infty} a_n \frac{1}{[i\omega]^{n+1}} \frac{n!}{s^{n+1}}$$

Notes

III. Conclusion

The results on Fourier transform of power series are summarized as follows;

Some functions like $e^{t^2}, \frac{sint}{t}$ and son on are difficult to get their Fourier transform. Hence it is possible to find Fourier transform such functions by expanding them into power series, Taylor's series and Maclaurin series form as:

Notes

$$F(f(x)) = \frac{2}{\sqrt{2\Pi}e^{ic\omega}}\Sigma_0^\infty a_n \frac{1}{[i\omega]^{n+1}}\frac{\Gamma(n+1)}{s^{n+1}}$$

where

$$f(x) = \Sigma_{n=0}^\infty a_n(x-c)^n, c \in \Re$$

2.

$$F(f(x)) = \frac{2}{\sqrt{2\Pi}e^{ic\omega}}\Sigma_0^\infty a_n f^{(n)}(c)\frac{1}{n!}\frac{1}{[i\omega]^{n+1}}\frac{\Gamma(n+1)}{s^{n+1}}$$

where

$$f(x) = \Sigma_{n=0}^\infty a_n f^{(n)}(c)\frac{(x-c)^n}{n!}$$

3.

$$F(f(x)) = \frac{2}{\sqrt{2\Pi}}\Sigma_0^\infty a_n \frac{1}{[i\omega]^{n+1}}\frac{\Gamma(n+1)}{s^{n+1}}$$

where

$$f(x) = \Sigma_{n=0}^\infty a_n t^n$$

References Références Referencias

1. Shiferaw Geremew Kebede, Properties of Fourier cosine and sine transforms, Basic and Applied Research IJSBAR, 35(3) (2017) 184-193
2. Shiferaw Geremew Kebede, Properties of Fourier Cosine and Sine Integrals with the Product of Power and Polynomial Functions, International Journal of Sciences: Basic and Applied Research (IJSBAR) (2017) Volume 36, No 7, pp 1-17
3. Janelle k. Hammond, UW-L Journal of undergraduate Research XIV(2011)
4. Ordinary Differential equations, GABRIEL NAGY, Mathematics Department, SEPTEMBER 14, 2015
5. Advanced Engineering Mathematics, Erwin Kreyszing, Herbert Kreyszing,Edward J. Norminton 10th Edition
6. A first Course in Di˙erential equations, Rudolph E. Longer, 1954
7. Differential Equation and Integral Equations, Peter J. Collins, 2006
8. Differential Equations, James R. Brannan, William E. Boyce, 2nd edition
9. Differential Equations for Engineers, Wei-Chau Xie, 2010
10. Advanced Engineering Mathematics 7th Edition, PETER V. ONEIL
11. Historically, how and why was the Laplace Transform invented? Written 18 Oct 2015 From Wikipedia: