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On A Subclass of Certain Convex Harmonic Univalent Functions Related to Q-Derivative

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ONASUBCLASSOFCERTAINCONVEXHARMONICUNIVALENTFUNCTIONSRELATEDTOQDERIVATIVE

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On A Subclass of Certain Convex Harmonic Univalent Functions Related to Q-Derivative

Hamid Shamsan [α] & S. Latha [σ]

Abstract- We define and investigate a new class of harmonic functions defined by q -derivative. We give univalence criteria and sufficient coefficient conditions for normalized q -harmonic functions that are convex of order $\beta, 0 \leq \beta < 1$. We obtain coefficient inequalities, extreme points distortion bounds, convolution and convex combination condition, and covering theorems for these functions. Further, we obtain the closure property of this class under integral operator.

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I. Introduction

Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics (e.g. see Choquet [2], Dorff [4] , Duren [5]). A continuous complex-valued function $f = u + iv$ defined in a domain $\mathscr{D} \subseteq \mathbb{C}$ is a harmonic in $\mathscr{D}$ if u and v are real harmonic in $\mathscr{D}$. We call h the analytic part and g the co-analytic part of f. In any simply connected domain we can write $f = h + \bar{g}$, where g and h are analytic and $\bar{g}$ denotes the function $z \to \overline{g(z)}$. Clunie and Sheil-Small [3] pointed out that a necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathscr{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathscr{D}$. Denote by H the class of functions f of the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \, , \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad |b_1| < 1, \tag{1}$$

that are harmonic univalent and sense-preserving in the unit disk $\mathscr{U} = \{z : z \in \mathbb{C} \, , \, |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$.

We note that the family H of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions in $\mathscr{U}$, if the co-analytic part of f is identically zero, that is $g \equiv 0$. For $0 \leq \beta < 1$, Let $K_H(\beta)$ be the subclass of H consisting of harmonic convex functions of order β. We further denote by $K_{\overline{H}}(\beta)$ the subclass of $K_H(\beta)$ such that the functions h and g in $f = h + \overline{g}$ are of the form

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2. G. Choquet, Sur un type de transformation analytique gnralisant la reprsentation conforme et dfinie au moyen de fonctions harmoniques, Bull. Sci. Math., 89(1945), 156-165.

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$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = -\sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{2}$$

Jackson[7] initiated q-calculus and developed the concept of the q-integral and q-derivative. For a function $f \in S$ given by (1) and $0 < q < 1$, the q-derivative of f is defined by

Definition 1.1.

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad , \quad where\ (0 < q < 1) \tag{3}$$

Equivalently (3), may be written as

$$\partial_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad z \neq 0$$

where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1 \\ k, & q = 1 \end{cases}$$

Note that as $q \to 1$, $[k]_q \to k$.

As a right inverse, Jackson[6] presented the q-integral of a function f as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k),$$

provided that the series converges. For a function $f(z) = z^k$, we note that

$$\int_0^z f(t) d_q t = \int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} \qquad (k \neq -1)$$

and

$$\lim_{q \to 1^-} \int_0^z f(t) d_q t = \lim_{q \to 1^-} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z f(t) dt,$$

Ref

7. F. H. Jackson, On q-functions and a certain difference operator, Trans. Royal Soc. Edinburgh., 46(1909), 253-281.

where $\int_0^z f(t)dt$ is the ordinary integral.

The Jacobian of f by q-derivative is given by

$$J_f(z) = |\partial_q h(z)|^2 - |\partial_q g(z)|^2$$

Notes

The mapping $z \to f(z)$ is locally one-to-one if $J_f(z) \neq 0$ in $\mathscr{U}$. Also the converse is true for harmonic mappings, and therefore $z \to f(z)$ is locally one-to-one and sense preserving if, and only if, $|\partial_q h(z)| > |\partial_q g(z)|$.

II. Main Results

Definition 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables and $0 < q < 1$, the partial q-derivatives at $(x,y) \in \mathbb{R}^2$ can be defined as follows*

$$\frac{\partial_q f(x,y)}{\partial_q x} = \frac{f(qx,y)-f(x,y)}{x(q-1)},$$

$$\frac{\partial_q f(x,y)}{\partial_q y} = \frac{f(x,qy)-f(x,y)}{y(q-1)},$$

and

$$\frac{\partial_q f(x,y)}{\partial_q x \partial_q y} = \frac{f(qx,qy)-f(qx,y)-f(x,qy)+f(x,y)}{xy(q-1)^2}.$$

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be partially q-differentiable on $\mathbb{R}^2$ if $\frac{\partial_q f(x,y)}{\partial_q x}$, $\frac{\partial_q f(x,y)}{\partial_q y}$ and $\frac{\partial_q^f (x,y)}{\partial_q x \partial_q y}$ exist for all $(x,y) \in \mathbb{R}^2$. We can similarly define higher order partial q-derivatives.

For $0 \leq \beta < 1$, let $K_H(\beta, q)$ denote the subclass of H consisting of q-harmonic convex functions of order β.

Definition 2.2. *A function f of the form (1) is said to be in the class $K_H(\beta,q)$ β, $0 \leq \beta < 1$, for $|z| = r < 1$ if*

$$\frac{\partial_q}{\partial_q \theta}\left\{\arg\left(\partial_q f(re^{i\theta})\right)\right\} = Im\left\{\frac{\partial_q}{\partial_q \theta}\log\left(f(re^{i\theta})\right)\right\}$$

$$= Re\left(\frac{\lambda_q|\ln q| z\partial_q h(z) + \lambda_q|\ln q| z^2 \partial_q^2 h(z) + \overline{\lambda_q|\ln q| z\partial_q g(z) + \lambda_q|\ln q| z^2 \partial_q^2 g(z)}}{(1-q)z\partial_q h(z) - \overline{(1-q)z\partial_q g(z)}}\right) \geq \beta, \tag{4}$$

where $\lambda_q = \sum_{k=0}^{\infty} \frac{(i\theta(1-q))^n}{(n+1)!}$, $\lambda_q \to 1$ as $q \to 1^-$, $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $\partial_q f = \partial_q h + \overline{\partial_q g}$.

Theorem 2.3. *Let $f = h + \overline{g}$ be given by (1). If*

$$\sum_{k=1}^{\infty}\left(\frac{[k]_q\left[\lambda_q|\ln q|[k]_q - \beta(1-q)\right]}{\lambda_q|\ln q| - \beta(1-q)}|a_k| + \frac{[k]_q\left[\lambda_q|\ln q|[k]_q + \beta(1-q)\right]}{\lambda_q|\ln q| - \beta(1-q)}|b_k|\right) \leq 2, \tag{5}$$

where $\lambda_q = \sum_{k=0}^{\infty} \frac{(i\theta(1-q))^n}{(n+1)!}$, $\lambda_q \to 1$ *as* $q \to 1^-$, $a_1 = 1$, $0 \leq \beta < 1$ *and* $0 < q < 1$. *Then* f *is* q*-harmonic univalent in* $\mathscr{U}$, *and* $f \in K_H(\beta, q)$.

Proof. Note first that

$$|\partial_q h(z)| \geq 1 - \sum_{k=1}^{\infty}[k]_q|a_k|r^{k-1} > 1 - \sum_{k=1}^{\infty}[k]_q|a_k| \geq 1 - \sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q - \beta(1-q)\right]}{\lambda_q|\ln q| - \beta(1-q)}|a_k|$$

$$\geq \sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q + \beta(1-q)\right]}{\lambda_q|\ln q| - \beta(1-q)}|b_k| \geq \sum_{k=1}^{\infty}[k]_q|b_k| > \sum_{k=1}^{\infty}[k]_q|a_k|r^{k-1} \geq |\partial_q g(z)|.$$

So that f is locally univalent and sense-preserving in $\mathscr{U}$. If $g \equiv 0$, the univalence of f will follow from its starlikeness. Otherwise if $z_1, z_2 \in \mathscr{U}$, $z_1 \neq z_2$. Since $\mathscr{U}$ is simply connected and convex, we have $z(t) = (1-t)z_1 + tz_2 \in \mathscr{U}$, where $0 \leq t \leq 1$. So we can write

$$f(z_2) - f(z_1) = \int_0^1 \left[(z_2 - z_1)\partial_q h(z(t)) + \overline{(z_2 - z_1)\partial_q g(z(t))}\right] d_q t.$$

Therefore

$$Re\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \int_0^1 Re\left[\partial_q h(z(t)) + \frac{\overline{z_2 - z_1}}{z_2 - z_1}\overline{\partial_q g(z(t))}\right] d_q t$$

$$> \int_0^1 \left[Re\partial_q h(z(t)) - \left|\partial_q g(z(t))\right|\right] d_q t. \tag{6}$$

Moreover

$$Re\partial_q h(z(t)) - \left|\partial_q g(z(t))\right| \geq Re\partial_q h(z(t)) - \sum_{k=1}^{\infty}[k]_q|b_k|$$

$$\geq 1 - \sum_{k=2}^{\infty}[k]_q|a_k| - \sum_{k=1}^{\infty}[k]_q|b_k|$$

$$\geq 1 - \sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q - \beta(1-q)\right]}{\lambda_q|\ln q| - \beta(1-q)}|a_k| - \sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q + \beta(1-q)\right]}{\lambda_q|\ln q| - \beta(1-q)}|b_k|$$

$$\geq 0, \qquad by (5). \tag{7}$$

Hence (6) and (7) leads to the univalence of f.

Now we show that $f \in K_H(\beta, q)$. It suffices to show that $\frac{\partial_q}{\partial_q \theta}\left\{\arg\left(\frac{\partial_q}{\partial_q \theta}f(re^{i\theta})\right)\right\} \geq \beta$, $0 \leq \beta < 1$, $0 \leq \theta < 2\pi$ and $0 < r < 1$.

Using the fact that $Re\omega \geq \beta$ if and only if $|1 - \beta + \omega| \geq |1 + \beta - \omega|$, it suffices to show that

$$|A(z) + (1-\beta)B(z)| - |A(z) - (1+\beta)B(z)| \geq 0, \tag{8}$$

Notes

where $A(z)=\lambda_q|\ln q|z\partial_q h(z)+\lambda_q|\ln q|z^2\partial_q^2 h(z)+\overline{\lambda_q|\ln q|z\partial_q g(z)+\lambda_q|\ln q|z^2\partial_q^2 g(z)}$

and $B(z)=(1-q)z\partial_q h(z)-\overline{(1-q)z\partial_q g(z)}$. Substituting for $A(z)$ and$B(z)$ in (8), we get

$|A(z)+(1-\beta)B(z)|-|A(z)-(1+\beta)B(z)|$

$$=\left|(\lambda_q|\ln q|+(1-\beta)(1-q))\,z\partial_q h(z)+\lambda_q|\ln q|z^2\partial_q^2 h(z)+\overline{(\lambda_q|\ln q|+(1-\beta)(1-q))\,z\partial_q g(z)+\lambda_q|\ln q|z^2\partial_q^2 g(z)}\right|$$

$$-\left|(\lambda_q|\ln q|-(1+\beta)(1-q))\,z\partial_q h(z)+\lambda_q|\ln q|z^2\partial_q^2 h(z)+\overline{(\lambda_q|\ln q|+(1+\beta)(1-q))\,z\partial_q g(z)+\lambda_q|\ln q|z^2\partial_q^2 g(z)}\right|$$

$$=\left|(-q+\beta q-\beta+\lambda_q|\ln q|+1)z+\sum_{k=2}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q-q+\beta q-\beta+1)\,a_k z^k\right.$$
$$\left.+\overline{\sum_{k=1}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q+q-\beta q+\beta-1)\,b_k z^k}\right|$$

$$-\left|(q+\beta q-\beta+\lambda_q|\ln q|+1)z+\sum_{k=2}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q+q+\beta q-\beta-1)\,a_k z^k\right.$$
$$\left.+\overline{\sum_{k=1}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q-q-\beta q+\beta+1)\,b_k z^k}\right|$$

$$\geq(-q+\beta q-\beta+\lambda_q|\ln q|+1)|z|-\sum_{k=2}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q-q+\beta q-\beta+1)\,|a_k||z^k|$$

$$-\overline{\sum_{k=1}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q+q-\beta q+\beta-1)\,|b_k||z^k|}+(q+\beta q-\beta+\lambda_q|\ln q|+1)|z|$$

$$-\sum_{k=2}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q+q+\beta q-\beta-1)\,|a_k||z^k|-\overline{\sum_{k=1}^{\infty}[k]_q\,(\lambda_q|\ln q|[k]_q-q-\beta q+\beta+1)\,b_k z^k}$$

$$=2\,[\lambda_q|\ln q|-\beta(1-q)]\,|z|\left\{1-\sum_{k=2}^{\infty}\frac{[k]_q\,[\lambda_q|\ln q|[k]_q-\beta(1-q)]}{\lambda_q|\ln q|-\beta(1-q)}|a_k||z|^{k-1}\right.$$
$$\left.-\sum_{k=1}^{\infty}\frac{[k]_q\,[\lambda_q|\ln q|[k]_q+\beta(1-q)]}{\lambda_q|\ln q|-\beta(1-q)}|b_k||z|^{k-1}\right\}$$

$$\geq 2\,[\lambda_q|\ln q|-\beta(1-q)]\,|z|\left\{1-\left(\sum_{k=2}^{\infty}\frac{[k]_q\,[\lambda_q|\ln q|[k]_q-\beta(1-q)]}{\lambda_q|\ln q|-\beta(1-q)}|a_k|\right.\right.$$
$$\left.\left.+\sum_{k=1}^{\infty}\frac{[k]_q\,[\lambda_q|\ln q|[k]_q+\beta(1-q)]}{\lambda_q|\ln q|-\beta(1-q)}|b_k|\right)\right\}\geq 0,\ by(5).$$

The starlike harmonic mappings

$$f(z)=z+\sum_{k=2}^{\infty}\frac{[k]_q\,[\lambda_q|\ln q|[k]_q-\beta(1-q)]}{\lambda_q|\ln q|-\beta(1-q)}x_k z^k+\sum_{k=1}^{\infty}\frac{[k]_q\,[\lambda_q|\ln q|[k]_q+\beta(1-q)]}{\lambda_q|\ln q|-\beta(1-q)}\bar{y}_k\bar{z}^k, \tag{9}$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=2}^{\infty}|y_k|=1$, show that the coefficient bound given by (5) is sharp. Therefore

$$\sum_{k=1}^{\infty}\left(\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_k|+\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_k|\right)=1+\sum_{k=2}^{\infty}|x_k|+\sum_{k=2}^{\infty}|y_k|=2.$$

Hence $f\in K_H(\beta,q)$.

Note that when $\beta=0$ in the above theorem, we get the following corollary

Corollary 2.4. *Let $f=h+\overline{g}$ be given by* (1). *If*

$$\sum_{k=1}^{\infty}[k]_q^2\left(|a_k|+|b_k|\right)\le 2, \tag{10}$$

where $a_1=1$ and $0<q<1$. Then f is q-harmonic univalent in $\mathscr{U}$, and $f\in K_H(q)$.

Special choices, $\beta=b_1=0$, and as $q\to 1^-$ yield the following result, proved by Avci and Zlotkiewicz [1]

Corollary 2.5. *Let $f=h+\overline{g}$ be given by* (1). *If*

$$\sum_{k=1}^{\infty}k^2\left(|a_k|+|b_k|\right)\le 2, \tag{11}$$

where $a_1=1$ and $b_1=0$, Then f is harmonic univalent in $\mathscr{U}$, and $f\in K_H^0$.

As $q\to 1^-$ and for we get the following result, proved by J. M. Jahangiri [8]

Corollary 2.6. *Let $f=h+\overline{g}$ be given by* (1). *If*

$$\sum_{k=1}^{\infty}\left(\frac{k(k-\beta)}{1-\beta}|a_k|+\frac{k(k+\beta)}{1-\beta}|b_k|\right)\le 2, \tag{12}$$

where $a_1=1$, $0\le\beta<1$ Then f is q-harmonic univalent in $\mathscr{U}$, and $f\in K_H(\beta)$..

Theorem 2.7. *Let $f=h+\overline{g}$ be given by* (2). *Then $f\in K_{\overline{H}}(\beta,q)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_k|+\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_k|\right)\le 2, \tag{13}$$

where $\lambda_q=\sum_{k=0}^{\infty}\frac{(i\theta(1-q))^n}{(n+1)!}$, $\lambda_q\to 1$ as $q\to 1^-$, $a_1=1$, $0\le\beta<1$ and $0<q<1$.

Proof. Since $f\in K_{\overline{H}}(\beta,q)\subset K_H(\beta,q)$, we only need to prove the necessary part of the theorem. Assume that $f\in K_{\overline{H}}(\beta,q)$, then by (4) we have

$$Re\left(\frac{\lambda_q|\ln q|z\partial_q h(z)+\lambda_q|\ln q|z^2\partial_q^2 h(z)+\overline{\lambda_q|\ln q|z\partial_q g(z)+\lambda_q|\ln q|z^2\partial_q^2 g(z)}}{(1-q)z\partial_q h(z)-\overline{(1-q)z\partial_q g(z)}}\right)-\beta$$

Ref

1. Y. Avci and E. Zlotkiewicz, On harmonic univalent mappings, Ann. Univ. Mariae CurieSklodowska Sect. 44(1990), 1-7.

$$= Re \frac{[\lambda_q|\ln q| - \beta(1-q)]\,z - \sum_{k=2}^{\infty}[k]_q\,[\lambda_q|\ln q|[k]_q - \beta(1-q)]\,|a_k|z^k - \sum_{k=1}^{\infty}[k]_q\,[\lambda_q|\ln q|[k]_q + \beta(1-q)]\,|b_k|\bar{z}^k}{(1-q)z - (1-q)\sum_{k=2}^{\infty}[k]_q|a_k|z^k + (1-q)\sum_{k=1}^{\infty}[k]_q|b_k|\bar{z}^k}$$

$$\geq 0.$$

The above condition must hold for all values of $z \in \mathscr{U}$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$ we must have

$$\frac{[\lambda_q|\ln q| - \beta(1-q)] - \sum_{k=2}^{\infty}[k]_q\,[\lambda_q|\ln q|[k]_q - \beta(1-q)]\,|a_k|r^{k-1} - \sum_{k=1}^{\infty}[k]_q\,[\lambda_q|\ln q|[k]_q + \beta(1-q)]\,|b_k|r^{k-1}}{(1-q)\,(1 - \sum_{k=2}^{\infty}[k]_q|a_k|r^{k-1} + \sum_{k=1}^{\infty}[k]_q|b_k|r^{k-1})} \geq 0. \quad (14)$$

If (13) does not hold, then the numerator in (14) is negative for r sufficiently close to 1. Therefore, there exists a point $z_0 = r_0$ in $(0,1)$ for which the quotient in (14) is negative. This contradicts our assumption that $f \in K_{\overline{H}}(\beta,q)$. This completes the proof of Theorem.

As $q \to 1^-$ and for we get the following result, proved by J. M. Jahangiri [8]

Corollary 2.8. *Let $f = h + \overline{g}$ be given by* (2)*. Then $f \in K_{\overline{H}}(\beta)$ if and only if*

$$\sum_{k=1}^{\infty}\left(\frac{k(k-\beta)}{1-\beta}|a_k| + \frac{k(1+\beta)}{1-\beta}|b_k|\right) \leq 2, \quad (15)$$

where $a_1 = 1$ and $0 \leq \beta < 1$.

Now we determine the extreme points of the closed convex hulls of $K_{\overline{H}}(\beta,q)$, denoted by $clcoK_{\overline{H}}(\beta,q)$.

Theorem 2.9. *Let f given by* (2)*. Then $f \in clcoK_H(\beta,q)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k) \quad (16)$$

where $h_1(z) = z$, $h_K(z) = z - \frac{[\lambda_q|\ln q| - \beta(1-q)]}{[k]_q[\lambda_q|\ln q|[k]_q - \beta(1-q)]}z^k$ $(k = 2,3,\ldots)$

$$\text{and } g_K(z) = z - \frac{[\lambda_q|\ln q| - \beta(1-q)]}{[k]_q[\lambda_q|\ln q|[k]_q + \beta(1-q)]}\bar{z}^k$$

$(k = 1,2,3,\ldots)$, $\sum_{k=1}^{\infty}(X_k + Y_k) = 1$, $X_k \geq 0$ *and* $Y_k \geq 0$. *In particular, the extreme points of* $K_{\overline{H}}(\beta,q)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. Let f be written as (16). Then we have

$$f(z) = \sum_{k=1}^{\infty}(X_k h_k + Y_k g_k) = \sum_{k=1}^{\infty}(X_k + Y_k)\,z - \sum_{k=2}^{\infty}\frac{\lambda_q|\ln q| - \beta(1-q)}{[k]_q\,[\lambda_q|\ln q|[k]_q - \beta(1-q)]}x_k z^k$$

Ref

8. J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients. Ann. Univ. Mariae Curie-Skodowska Sect. A, 52(2) (1998), 57-66.

$$-\sum_{k=1}^{\infty}\frac{\lambda_q|\ln q|-\beta(1-q)}{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}\bar{y}_k\bar{z}^k.$$

$$=z-\sum_{k=2}^{\infty}a_n z^n-\sum_{k=1}^{\infty}b_n\bar{z}^n.$$

Then

$$\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}\left(\frac{\lambda_q|\ln q|-\beta(1-q)}{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}x_k\right)$$

$$+\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}\left(\frac{\lambda_q|\ln q|-\beta(1-q)}{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}y_k\right).$$

$$=\sum_{k=2}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1-X_1\leq 1$$

and so $f\in clcoK_{\overline{H}}(\beta,q)$. Conversely, assume that $f\in clcoK_{\overline{H}}(\beta,q)$. Putting

$$X_k=\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_k|,\qquad k=2,3,\ldots$$

$$Y_k=\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_k|,\qquad k=1,2,3,\ldots$$

and

$$X_1=1-\sum_{k=2}^{\infty}X_k-\sum_{k=1}^{\infty}Y_k,$$

then $\sum_{k=1}^{\infty}(X_k+Y_k)=1,\ 0\leq X_k\leq 1(k=2,3,\ldots),0\leq Y_k\leq 1(k=1,2,3,\ldots)$. Consequently, we obtain $f(z)=\sum_{k=1}^{\infty}(X_kh_k+Y_kg_k)$ as required.

Finally we give the distortion bounds for functions in $K_{\overline{H}}(\beta,q)$ which yield a covering result for $K_{\overline{H}}(\beta,q)$.

Theorem 2.10. *If $f\in K_{\overline{H}}(\beta,q)$ then*

$$|f(z)|\leq(1+|b_1|)r+\frac{1}{[2]_q}\left(\frac{\lambda_q|\ln q|-\beta(1-q)}{[2]_q\lambda_q|\ln q|-\beta(1-q)}-\frac{\lambda_q|\ln q|+\beta(1-q)}{[2]_q\lambda_q|\ln q|-\beta(1-q)}|b_1|\right)r^2,\quad |z|=r<1,$$

and

$$|f(z)|\geq(1-|b_1|)r-\frac{1}{[2]_q}\left(\frac{\lambda_q|\ln q|-\beta(1-q)}{[2]_q\lambda_q|\ln q|-\beta(1-q)}-\frac{\lambda_q|\ln q|+\beta(1-q)}{[2]_q\lambda_q|\ln q|-\beta(1-q)}|b_1|\right)r^2,\quad |z|=r<1.$$

Proof. Assume that $f\in K_{\overline{H}}(\beta,q)$. Then we have

Notes

$$|f(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k$$

$$\leq (1+|b_1|)r + \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$

$$= (1+|b_1|)r + \frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}\sum_{k=2}^{\infty}\left(\frac{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|a_k| + \frac{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|b_k|\right)r^2$$

$$\leq (1+|b_1|)r + \frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}\sum_{k=2}^{\infty}\left(\frac{[k]_q(\lambda_q|\ln q|[k]_q - \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|a_k| + \frac{[k]_q(\lambda_q|\ln q|[k]_q + \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|b_k|\right)r^2$$

$$\leq (1+|b_1|)r + \frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}\left(1 - \frac{\lambda_q|\ln q| + \beta(1-q)}{\lambda_q|\ln q| - \beta(1-q)}|b_1|\right)r^2, \quad by (5),$$

$$= (1+|b_1|)r + \frac{1}{[2]_q}\left(\frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)} - \frac{\lambda_q|\ln q| + \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)}|b_1|\right)r^2.$$

and

$$|f(z)| \geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^k$$

$$\geq (1-|b_1|)r - \sum_{k=2}^{\infty}(|a_k|+|b_k|)r^2$$

$$= (1-|b_1|)r - \frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}\sum_{k=2}^{\infty}\left(\frac{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|a_k| + \frac{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|b_k|\right)r^2$$

$$\geq (1-|b_1|)r - \frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}\sum_{k=2}^{\infty}\left(\frac{[k]_q(\lambda_q|\ln q|[k]_q - \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|a_k| + \frac{[k]_q(\lambda_q|\ln q|[k]_q + \beta(1-q))}{\lambda_q|\ln q| - \beta(1-q)}|b_k|\right)r^2$$

$$\geq (1-|b_1|)r - \frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q([2]_q\lambda_q|\ln q| - \beta(1-q))}\left(1 - \frac{\lambda_q|\ln q| + \beta(1-q)}{\lambda_q|\ln q| - \beta(1-q)}|b_1|\right)r^2, \quad by (5),$$

$$= (1-|b_1|)r - \frac{1}{[2]_q}\left(\frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)} - \frac{\lambda_q|\ln q| + \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)}|b_1|\right)r^2.$$

As $q \to 1^-$ and for we get the following result, proved by J. M. Jahangiri [8]

Corollary 2.11. *If $f \in K_{\overline{H}}(\beta)$ then*

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{2}\left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)r^2, \qquad |z| = r < 1,$$

and

$$|f(z)| \geq (1-|b_1|)r - \frac{1}{2}\left(\frac{1-\beta}{2-\beta} - \frac{1+\beta}{2-\beta}|b_1|\right)r^2, \qquad |z| = r < 1.$$

Ref

8. J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients. Ann. Univ. Mariae Curie-Skodowska Sect. A, 52(2) (1998), 57-66.

The bounds given in Theorem 2.10 for the function $f = h + \bar{g}$ of the form (2) also hold for functions of the form (1) if the coefficient condition (5) is satisfied. The functions

$$f(z) = (1+|b_1|)\bar{z} + \frac{1}{[2]_q}\left(\frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)} - \frac{\lambda_q|\ln q| + \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)}|b_1|\right)\bar{z}^2$$

and

$$f(z) = (1-|b_1|)z - \frac{1}{[2]_q}\left(\frac{\lambda_q|\ln q| - \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)} - \frac{\lambda_q|\ln q| + \beta(1-q)}{[2]_q\lambda_q|\ln q| - \beta(1-q)}|b_1|\right)z^2$$

for $|b_1| \le \frac{1}{[2]_q}\left(\frac{\lambda_q|\ln q|-\beta(1-q)}{[2]_q\lambda_q|\ln q|-\beta(1-q)} - \frac{\lambda_q|\ln q|+\beta(1-q)}{[2]_q\lambda_q|\ln q|-\beta(1-q)}|b_1|\right)$ show that the bounds given in Theorem 2.6 are sharp.

Theorem 2.12. *If $f \in K_{\overline{H}}(\beta,q)$ then*

$$\left\{\omega : |\omega| < \frac{1}{[2]_q}\frac{\lambda_q|\ln q|([2]_q-1)\left[([2]_q+1)-\beta(1-q)\right]}{[2]_q\lambda_q|\ln q|-\beta(1-q)} - \frac{1}{[2]_q}\frac{\lambda_q|\ln q|([2]_q+1)\left[([2]_q-1)-\beta(1-q)\right]}{[2]_q\lambda_q|\ln q|-\beta(1-q)}|b_1|\right\} \subset f(\mathscr{U}). \tag{17}$$

Proof. Letting $r \to 1^-$ in the left hand inequality in Theorem 2.10 and collecting the like terms we obtain (17).

The condition (17) for $\beta = b_1 = 0$ yields the following

Corollary 2.13. *if $f \in K^0_{\overline{H}}(0,q)$ then*

$$\{\omega : \omega < \frac{[2]_q^2 - 1}{[2]_q^2}\} \subset f(\mathscr{U}).$$

As $q \to 1^-$ and $\beta = b_1 = 0$, we get the following result, proved by Jay M. Jahangiri [8]

Corollary 2.14. *if $f \in K^0_{\overline{H}}(0)$ then*

$$\{\omega : \omega < \frac{3}{4}\} \subset f(\mathscr{U}).$$

For harmonic functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \bar{z}^k$ and $g(z) = z + \sum_{n=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_n \bar{z}^k$ we define the Hadamard product of f and g as

$$(f*g)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k \bar{z}^k.$$

Theorem 2.15. *For $0 \le \alpha \le \beta < 1$, let $f \in K_{\overline{H}}(\beta,q)$ and $g \in K_{\overline{H}}(\alpha,q)$. Then $f*g \in K_H(\beta,q) \subset K_H(\alpha,q)$.*

Ref

8. J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients. Ann. Univ. Mariae Curie-Skodowska Sect. A, 52(2) (1998), 57-66.

Proof. Putting $f(z)=z-\sum_{k=2}^{\infty}|a_k|z^k-\sum_{k=1}^{\infty}|b_k|\overline{z}^k$ and $g(z)=z-\sum_{k=2}^{\infty}|A_k|z^k+\sum_{k=1}^{\infty}|B_k|\overline{z}^k$. Then the Hadamard product of f and g is

$$(f*g)(z)=z+\sum_{k=2}^{\infty}|a_k||A_k|z^k+\sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k.$$

Notes

Since $|A_k|\leq 1$ and $|B_k|\leq 1$, we can write

$$\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_k||A_k|+\sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_k||B_k|$$

$$\leq\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_k|+\sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_k|.$$

The right hand side of the above inequality is bounded by 1 because $f\in K_{\overline{H}}(\beta,q)$ .Therefore $f*g\in K_H(\beta,q)\subset K_H(\alpha,q)$.

Theorem 2.16. *The class $K_{\overline{H}}(\beta,q)$ is closed under convex combination.*

Proof. For $i=1,2,3,\ldots$ assume that $f_i\in K_{\overline{H}}(\beta,q)$ where f_i is given by

$$f_i(z)=z-\sum_{k=2}^{\infty}|a_{i_k}|z^k-\sum_{k=1}^{\infty}|b_{i_k}|\overline{z}^k.$$

Then by (13)

$$\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_{i_k}|+\sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_{i_k}|\leq 1. \tag{18}$$

For $\sum_{i=1}^{\infty}t_i=1$, $0\leq t_i\leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{i_k}|\right)z^n-\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{i_k}|\right)\overline{z}^n.$$

Then by (18),

$$\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}\left|\sum_{i=1}^{\infty}t_i|a_{i_k}|\right|+\sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}\left|\sum_{i=1}^{\infty}t_i|b_{i_k}|\right|$$

$$\sum_{i=1}^{\infty}t_i\left\{\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_{i_k}|+\sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_{i_k}|\right\}\leq\sum_{i=1}^{\infty}t_i=1,$$

and so $\sum_{i=1}^{\infty}t_if_i(z)\in K_{\overline{H}}(\beta,q)$.

Now, we consider the closer property of the class $K_{\overline{H}}(\beta,q)$ under the Bernardi integral operator $F_a(z)$, which is defined by

$$F_a(z) = \frac{[a+1]_q}{z^a}\int_0^z t^{a-1}f(t)d_qt + \overline{\frac{[a+1]_q}{z^a}\int_0^z t^{a-1}f(t)d_qt}\ (a>-1). \tag{19}$$

Theorem 2.17. *Let $f \in K_{\overline{H}}(\beta,q)$. Then $F_a(z) \in K_{\overline{H}}(\beta,q)$.*

Proof. From the representation of $F_a(z)$, we have

$$F_a(z) = \sum_{k=1}^{\infty}\left(\frac{[a+1]_q}{[a+n]_q}\right)a_kz^k + \sum_{k=1}^{\infty}\left(\frac{[a+1]_q}{[a+k]_q}\right)\overline{b_k}\overline{z}^k \tag{20}$$

Now

$$\sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}\left(\frac{[a+1]_q}{[a+k]_q}|a_k|\right) + \sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}\left(\frac{[a+1]_q}{[a+k]_q}|b_k|\right) \tag{21}$$

$$\leq \sum_{k=2}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q-\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|a_k| + \sum_{k=1}^{\infty}\frac{[k]_q\left[\lambda_q|\ln q|[k]_q+\beta(1-q)\right]}{\lambda_q|\ln q|-\beta(1-q)}|b_k| \leq 1.$$

Therefore $F_a(z) \in K_{\overline{H}}(\beta,q)$.

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Notes