Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 18 Issue 6 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Path Integral Solutions of the PT-Symmetric and Non-Hermitian q-Deformed Eckart Plus Modified Hylleraas Potential

By Nalan Kandirmaz
Mersin University

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GJSFR-A Classification: *FOR Code: PACS: 31.15.Kb, 11.30Er*

PATHINTEGRALSOLUTIONSOFTHEPT-SYMMETRICANDNON-HERMITIANQ-DEFORMEDECKARTPLUSMODIFIEDHYLLERAASPOTENTIAL

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I. Introduction

One of the important problems of quantum mechanics is to obtain the exact analytical solution of Schrodinger Equation for various potentials. In recent years, so many methods have been developed to solve the exact or quasi exactly for various complicated potentials, such as (SUSY), factorization, asymptotic iteration, path integral etc. These methods have been applied to a great variety of quantum mechanical interactions as analytical methods, variational methods, numerical approaches, Fourier analysis, semi-classical estimates, quantum field theory and Lie group theoretical approaches [1 - 8].

Feynman's Path Integral Method has wide application in many areas of theoretical physics [9 - 22]. It allows the transition to quantum mechanics from the Lagrangian formalism via quantization. Duru and Kleinert calculated Green's function for H-atom using a new time parameter and using the transform the Coulomb path integral into a harmonic oscillator path integral. The energy spectrum and the normalized s-state eigenfunctions for the Hulthen Potential and Woods Saxon potentials are obtained using Green's function [12 -14]. By using path integrals, coherent states are constructed for various potentials [17, 18, 20]. Feynman's Path integral Formalism is powerful and challenging method but it has difficulties for the many of the quantum mechanical systems. Path integration is an alternative method to obtain the exact analytical solution of Schrodinger Equation. The path integral formulation builds on the propagator, which is the probability amplitude for making a transition between the initial position x' at time $t' = 0$ and the final position x'' at time t. It is also called kernel which express the time evolution of the initial state. The kernel contains all dynamical information about a quantum mechanical system [1]. The quantum evolution operator that generates the propagator is defined as $\widehat{U}(t_b, t_a) = \exp[-i\widehat{H}(t_b, t_a)]$. Here H is Hamiltonian which determines the evolution over time. In the quantum theory the Hamiltonian includes the symmetries of the systems. If the symmetry is represented by an $\widehat{A}$ linear operator, the $\widehat{A}$ operator comutes to Hamiltonian: $[\widehat{A}, \widehat{H}] = 0$. Two important symmetry operators in quantum mechanics are Parity operator: P and Time operator T. When they act on the position and momentum operators they lead to momentum as $P: x \to -x, p \to -p$ and $T: x \to x, p \to p, i \to -i$ Due to the increased interest in PT-Symmetric quantum mechanics, the cases of real or complex eigenvalues for the Hermitian and non-Hermitian hamiltonians of various potentials have been studied [25 - 27].

The object of this paper is to evaluate energy spectrum and wave functions of the PT-Symmetric the and non-Hermitian q-deformed Eckart plus modified Hylleraas potential via Feynman's path integral method. The organization of this paper is as follows. In Sec. II Kernel and energy-dependent Green's function of The q-deformed Eckart plus modified Hylleraas potential derived using the Duru and Kleinert method. In Sec. III energy eigenvalues and the corresponding wave functions are derived using Green's function in Sec II. In Sec IV Woods-Saxon, Rosen Morse, were discussed reduced with different parameters than the q-deformed Eckart plus modified Hylleraas potential.

II. The Kernel

The PT-Symmetric and non-Hermitian q-deformed Eckart plus modified Hylleraas potential is

Author: Department of Physics, Mersin University, Mersin, Turkey.
e-mail: nkandirmaz@mersin.edu.tr

$$V(x)=\frac{V_0}{b}\left(\frac{a-e^{-2i\alpha x}}{1-qe^{-2i\alpha x}}\right)-V_1\frac{e^{-2i\alpha x}}{1-qe^{-2i\alpha x}}+V_2\frac{e^{-2i\alpha x}}{(1-qe^{-2i\alpha x})^2}\ . \tag{1}$$

It determines by taking $\alpha\longrightarrow i\alpha$ in the q - deformed Eckart plus modified Hylleraas potential [28]. The potential satisfies $V^*(-x)=V(x)$ which shows that we obtain PT-Symmetric q-deformed Eckart plus modified Hylleraas potential. The parameters V_0, V_1 and V_2 are the depths of the potential well, a, and b Hylleraas parameters and α is the inverse of the range of the potential in Eq. (1). The kernel of one dimension potential between the initial position x' at time $t'=0$ and final position x'' at time t'' is given in Ref. [1] as

$$K(x_b,t_b;x_a,t_a)=\int\frac{DxDp}{2\pi}exp\{i\int dt[p\dot{x}-\frac{p^2}{2m}-V(x)]\} \tag{2}$$

Path integral express in terms of an integral over all paths in configuration space. Kernel is describes as

$$K(x_b,t_b;x_a,t_a)\ =\ \lim_{N\to\infty}\int\prod_{j=1}^{N}dx_j\prod_{j=1}^{N+1}\frac{dp_j}{2\pi\hbar}\sum_{j=1}^{N}\exp\frac{i}{\hbar}\left\{\sum_{j=1}^{N+1}p_j\Delta x_j\right. \tag{3}$$

$$\left.-\frac{V_0}{b}\left(\frac{a-e^{-2i\alpha x}}{1-qe^{-2i\alpha x}}\right)-V_1\frac{e^{-2i\alpha x}}{1-qe^{-2i\alpha x}}+V_2\frac{e^{-2i\alpha x}}{(1-qe^{-2i\alpha x})^2}\right\} \tag{4}$$

The partial action is expressed by

$$S(x_j,x_{j-1})=\sum_{j=1}^{N+1}p_j\Delta x_j-\left[\frac{p_j^2}{2m}+\frac{V_0}{b}\left(\frac{a-e^{-2i\alpha x}}{1-qe^{-2i\alpha x}}\right)-V_1\frac{e^{-2i\alpha x}}{1-qe^{-2i\alpha x}}+V_2\frac{e^{-2i\alpha x}}{(1-qe^{-2i\alpha x})^2}\right]\Bigg\} \tag{5}$$

where we shall $\hbar=m=1$. To solution via Feynman Path integral method we introduce the new angular variable $\theta\epsilon(0,\pi)$ to transform the radial variable $x\epsilon(0,\infty)$

$$x=\frac{1}{2i\alpha}\ln(-\frac{1}{q}\cot^2\theta)\qquad p_x=i\alpha\sin\theta\cos p_\theta \tag{6}$$

The contribution to Jacobien of this transformation becomes

$$\frac{DxDp}{2\pi}=i\alpha\sin\theta_b\cos\theta_b\quad . \tag{7}$$

By defining $\Delta x_j=x_j-x_{j-1}$, $\varepsilon=t_j-t_{j-1},\ t'=t_0=t_a,\ t''=t_N=t_b,\ (n+1)\varepsilon=T$ the Kernel in Eq.(3) can be written as

$$K(x_b,x_a;T)=i\alpha\sin\theta_b\cos\theta_b\int D\theta Dp_\theta\exp\left\{i\int_0^T dt\left[p_\theta\dot{\theta}+\alpha^2\sin^2\theta\cos^2\theta\frac{p_\theta^2}{2m}\right.\right.$$

$$\left.\left.(-\frac{V_0a}{b}\sin^2\theta+\frac{1}{q}(\frac{V_0}{b}+V_1)\cos^2\theta-\frac{V_2}{q}\sin^2\theta\cos^2\theta)\right]\right\}\ . \tag{8}$$

The coordinate transformation in Eq. (7) produces a factor $\alpha^2\sin^2\theta\cos^2\theta$ that is kinetic energy term. We need a new time parameter s for eliminating the $\alpha^2\sin^2\theta\cos^2\theta$ factor in the kinetic energy term [12 - 14] as

$$\frac{dt}{ds}=-\frac{1}{\alpha^2\sin^2\theta\cos^2\theta}\quad or\quad t=-\frac{1}{\alpha^2}\int\frac{ds'}{\sin^2\theta\cos^2\theta}\ . \tag{9}$$

If Fourier transformation of $\delta-$ function

$$1=\int dS\frac{1}{\alpha^2\sin^2\theta_b\cos^2\theta_b}\delta(-T-\int ds\frac{1}{\alpha^2\sin^2\theta\cos^2\theta})$$
$$=\int dS\int\frac{dE}{2\pi}\frac{1}{\alpha^2\sin^2\theta_b\cos^2\theta_b}exp[-i(ET-\int ds\frac{E}{\alpha^2\sin^2\theta\cos^2\theta})] \quad (10)$$

is added to the kernel it becomes

$$K(x_b,x_a;T)=\frac{i}{i\alpha\sin\theta_b\cos\theta_b}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iET}\int_0^{\infty}dS\int D\theta Dp_\theta\ e^{i\left(\frac{V_2}{q\alpha^2}\right)S}$$
$$\times exp[i\int_0^S ds\ (p_\theta\dot{\theta}-\frac{p_\theta^2}{2}-\left[\frac{\frac{V_0}{qb\alpha^2}+\frac{V_1}{q\alpha^2}+\frac{E}{\alpha^2}}{\sin^2\theta}\right]-\frac{\frac{V_0a}{b\alpha^2}+\frac{E}{\alpha^2}}{\cos^2\theta})]. \quad (11)$$

It can be symmetrized according to points a and b the factor in front of the in Eq.(10) that get to from Jacobian as following

$$\frac{1}{\sin\theta_b\cos\theta_b}=\frac{2}{\sqrt{\sin 2\theta_a\sin 2\theta_b}}\exp\left(-\frac{1}{2}\ln\frac{\sin 2\theta_b}{\sin 2\theta_a}\right)$$
$$=\frac{2}{\sqrt{\sin 2\theta_a\sin 2\theta_b}}\exp\left(i\int_0^S dsi\frac{\cos 2\theta}{\sin 2\theta}\dot{\theta}\right) \quad . \quad (12)$$

Thus Eq. (10) takes

$$K(x'',x';E)=\int_0^{\infty}dSe^{i\left(\frac{V_2}{q\alpha^2}\right)S}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iET}\frac{i}{\alpha\sqrt{\sin 2\theta_a\cos 2\theta_b}}K\left(\theta_b,\theta_a;S\right) \quad (13)$$

Where

$$K\left(\theta_b,\theta_a;S\right)=\int D\theta Dp_\theta\exp\left\{i\int_0^S ds\left[p_\theta\dot{\theta}-\frac{p_\theta^2}{2}-\frac{1}{2}\left(\frac{\kappa\left(\kappa-1\right)}{\sin^2\theta}+\frac{\gamma\left(\gamma-1\right)}{\cos^2\theta}\right)-\frac{ip_\theta\cos 2\theta}{2\sin 2\theta}\right]\right\} \quad (14)$$

and K and γ are

$$\kappa=\frac{1}{2}\left[1+\sqrt{1+8\left(\frac{V_0}{qb\alpha^2}+\frac{V_1}{q\alpha^2}+\frac{E}{\alpha^2}\right)}\right]$$

$$\gamma=\frac{1}{2}\left[1+\sqrt{1+8\left(\frac{V_0a}{b\alpha^2}+\frac{E}{\alpha^2}\right)}\right] \quad . \quad (15)$$

If we take the time division of the momentum variables from $j=0$ to $j=n$ instead of from $j=0$ to $j=n+1$, we have the quantum mechanical contributions $+\frac{ip_\theta \cos 2\theta}{2\sin 2\theta}$ in Eq. (13).

In the same way if this factor is symmetrized as Eq.(13) which only sign, of imaginary term will be change. Therefore the contributions to be kernel becomes

$$\frac{\theta_j-\theta_{j-1}}{\epsilon} \longrightarrow \frac{\theta_j-\theta_{j-1}}{\epsilon} \pm \frac{i\cos\theta_j}{2\sin\theta_j}. \tag{16}$$

So the problem is reduced the path integral for Pöschl - Teller potential which is known exact solution [13; 15]. K $(\theta_b,\theta_a;S)$ can be obtained as

$$K(\theta_b,\theta_a;S)=\int D\theta Dp_\theta \exp\left\{i\int_0^S ds\left[p_\theta\dot{\theta}-\frac{p_\theta^2}{2}-\frac{1}{2}\left(\frac{\kappa(\kappa-1)}{\sin^2\theta}+\frac{\gamma(\gamma-1)}{\cos^2\theta}\right)\right]\right\}. \tag{17}$$

Writing the kernel as

$$K(\theta_b,\theta_a;S)=\sum_{n=0}^{\infty}\exp\left[-i\varepsilon_n S\right]\quad_n(\theta_a)\quad{}_n^*(\theta_b) \tag{18}$$

$$\varepsilon_n=\frac{1}{2}(\kappa+\gamma+2n)^2 \tag{19}$$

Where

$$\psi_n(\theta)=\sqrt{2(\kappa+\gamma+2n)}\sqrt{\frac{\Gamma(n+1)\Gamma(\kappa+\gamma+n)}{\Gamma\left(\gamma+n+\frac{1}{2}\right)\Gamma\left(\kappa+n+\frac{1}{2}\right)}}$$
$$\times(\cos\theta)^\gamma(\sin\theta)^\kappa P_n^{(\kappa-1/2,\gamma-1/2)}\left(1-2\sin^2\theta\right) \tag{20}$$

Eq.(12) can be expressed

$$K(x'',x';T)=\frac{iq}{\alpha\sqrt{\sin 2\theta_a\cos 2\theta_b}}\int_{-\infty}^{\infty}\frac{dE}{2\pi}e^{iET}\int_0^\infty dSe^{i\left(\frac{V_2}{q\alpha^2}\right)S}K(\theta_b,\theta_a;S). \tag{21}$$

With integrating over dS Greens function for Eq.(1) can be obtained as

$$G(x'',x';E)=\frac{8i\mu q}{\alpha\sqrt{\sin 2\theta_a\cos 2\theta_b}}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{dE}{2\pi}\frac{e^{iET}}{(\kappa+\gamma+2n)^2-\left(\frac{V_2}{q\alpha^2}\right)}\quad_n(\theta_a)\quad{}_n^*(\theta_b) \tag{22}$$

Therefore the kernel of a physical system is rewritten as

$$K(x'',x';E)=\sum_{n=0}^{\infty}e^{-iE_nT}\varphi_n(x_a)\varphi_n^*(x_b). \tag{23}$$

III. Energy Eigenvalues and Wave Functions

We calculated Green's function and Kernel for the PT Symmetric and Non-Hermitian q-deformed Eckart plus modified Hylleraas potential in Sec.(2). If we can integrate over dE, we can get energy eigenvalues as

$$E_n = \frac{V_0 a}{b} - \frac{\alpha^2}{8}\left\{\frac{\left[\frac{V_2}{q\alpha^2} - (2n+1)\right]^2 - \frac{2V_0 a}{\alpha^2 b} + \frac{2V_0}{\alpha^2 qb} + \frac{2V_1}{\alpha^2 q}}{\frac{V_2}{q\alpha^2} - (2n+1)}\right\}^2 \tag{24}$$

and normalized wave functions in terms of Jacobi polynoms are

$$\phi(r) = 2\sqrt{2}\sqrt{(\kappa_n + \gamma_n + 2n)}\sqrt{\frac{\Gamma(n+1)\Gamma(\kappa_n+\gamma_n+n)}{\Gamma\left(\gamma_n+n+\frac{1}{2}\right)\Gamma\left(\kappa_n+n+\frac{1}{2}\right)}}$$

9where we got

$$\kappa_n = \frac{1}{2} + \frac{1}{\frac{V_2}{q\alpha^2} - (2n+1)}\left\{\left[\frac{V_2}{q\alpha^2} - (2n+1)\right]^2 + \left(\frac{2V_0 a}{\alpha^2 b} + \frac{2V_0}{\alpha^2 qb} + \frac{2V_1}{\alpha^2 q}\right)\right\}$$

$$\gamma_n = \frac{1}{2} + \frac{1}{\frac{V_2}{q\alpha^2} - (2n+1)}\left\{\left[\frac{V_2}{q\alpha^2} - (2n+1)\right]^2 - \left(\frac{2V_0 a}{\alpha^2 b} + \frac{2V_0}{\alpha^2 qb} + \frac{2V_1}{\alpha^2 q}\right)\right\} \tag{25}$$

We can write the terms of Hypergeometric functions in Eq.(19)

$$P_n^{(\alpha,\beta')}(z) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} F\left(-n, n+\alpha+\beta'+1, \beta'+1, \frac{1+z}{2}\right) \tag{26}$$

and wave functions can be obtained

$$\phi(x) = \sqrt{(\kappa_n+\gamma_n+2n)}\sqrt{\frac{\Gamma(n+1)\Gamma(\kappa_n+\gamma_n+n)\Gamma(n+\kappa_n+1/2)}{\Gamma\left(\gamma_n+n+\frac{1}{2}\right)}}$$

$$\times \frac{\left(-iqe^{-2i\alpha x}\right)^{\gamma-1/2}}{\alpha\left(1-qe^{-2i\alpha x}\right)^{(\kappa_n+\gamma_n-2)}\left(1+qe^{-2i\alpha x}\right)^{1/2}} F\left(-n, \kappa_n+\gamma_n+n, \gamma_n+\frac{1}{2}, -\frac{qe^{-2i\alpha x}}{1-qe^{-2i\alpha x}}\right) \tag{27}$$

Therefore we evaluated energy spectrum and wave functions for the PT-Symmetric and Non-Hermitian q-deformed Eckart plus modified Hylleraas potential has real energy spectra.

IV. Discussion

(i) Setting $V_0 = V_2 = 0,\ a = 0,\ b = 1$ and $q = -1$ the potential in Eq. (1) is reduced to PT-Symmetric and Non-Hermitian Woods-Saxon potential

$$V(x) = -\frac{V_1 e^{-2i\alpha x}}{1+e^{-2i\alpha x}} \tag{28}$$

Energy eigenvalues of (27) potential can be obtained as

$$E_n = -\frac{\alpha^2}{8}\left\{\frac{\left[\frac{2V_0}{\alpha^2} - (2n+1)^2\right]}{(2n+1)}\right\}^2. \tag{29}$$

and normalized wave functions can be written as

$$\phi(x)=\sqrt{(\kappa_n+\gamma_n+2n)}\sqrt{\frac{\Gamma\left(n+1\right)\Gamma\left(\kappa_n+\gamma_n+n\right)\Gamma\left(n+\kappa_n+1/2\right)}{\Gamma\left(\gamma_n+n+\frac{1}{2}\right)}}$$

$$\times\frac{\left(ie^{-2i\alpha x}\right)^{\gamma-1/2}}{\alpha\left(1+e^{-2i\alpha x}\right)^{(\kappa_n+\gamma_n-2)}\left(1-e^{-2i\alpha x}\right)^{1/2}}F\left(-n,\kappa_n+\gamma_n+n,\gamma_n+\frac{1}{2},\frac{e^{-2i\alpha x}}{1+e^{-2i\alpha x}}\right) \qquad (30)$$

(ii) Choosing $V_0=V_2=0$, and $b=1$, $a=-1$ and $q=1$, the potential in Eq. (1) is reduced to PT-Symmetric and Non-Hermitian Rosen Morse potential

$$V(x)=-\frac{V_0(1+e^{-2i\alpha x})}{1-e^{-2i\alpha x}}\,. \qquad (31)$$

So energy eigenvalues are

$$E_n=-\frac{\alpha^2}{8}\left\{\frac{4V_0-(2n+1)^2}{(2n+1)}\right\}^2 \qquad (32)$$

and normalized wave function is

$$\phi(x)=\sqrt{(\kappa_n+\gamma_n+2n)}\sqrt{\frac{\Gamma\left(n+1\right)\Gamma\left(\kappa_n+\gamma_n+n\right)\Gamma\left(n+\kappa_n+1/2\right)}{\Gamma\left(\gamma_n+n+\frac{1}{2}\right)}}$$

$$\times\frac{\left(-ie^{-2i\alpha x}\right)^{\gamma-1/2}}{\alpha\left(1-e^{-2i\alpha x}\right)^{(\kappa_n+\gamma_n-2)}\left(1+e^{-2i\alpha x}\right)^{1/2}}F\left(-n,\kappa_n+\gamma_n+n,\gamma_n+\frac{1}{2},-\frac{e^{-2i\alpha x}}{1-e^{-2i\alpha x}}\right)\,. \qquad (33)$$

V. Conclusion

In this work, we have investigated the Schrodinger Equation with The q-deformed Eckart Plus Modified Hylleraas potential for n quantum states. We used space-time transformation to obtain energy eigenvalues and corresponding wave functions. We expressed normalized wave functions in terms of Jacobi polynomials and Hypergeometric functions. We obtained exactly the energy eigenvalues and the corresponding eigenfunctions. We have seen that the potential has real eigenvalues. The energy eigenvalues and the eigenfunctions can be computed making different choices for the $V_0, V_1, V_2,\ \alpha, q, b$ parameters of the potential. Choosing appropriate parameters for the potential, we indicated energy spectrum and wave functions n states for Woods Saxon, Rosen Morse potentials..

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