Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Solutions on Generalized Non-Linear Cauchy-Euler ODE

By Wasihun Assefa Woldeyes & Beletu Worku Beyene
Madda Walabu University

Abstract- In this note, the authors generalize the linear Cauchy-Euler ordinary differential equations (ODEs) into nonlinear ODEs and provide their analytic general solutions. Some examples are presented in order to clarify the applications of interesting results.

Keywords: *Cauchy-Euler equation, nonlinear ODE.*

GJSFR-F Classification: *MSC 2010: 45E05*

SOLUTIONSONGENERALIZEDNONLINEARCAUCHYEULERODE

Strictly as per the compliance and regulations of:

Solutions on Generalized Non-Linear Cauchy-Euler ODE

Wasihun Assefa Woldeyes [α] & Beletu Worku Beyene [σ]

Abstract- In this note, the authors generalize the linear Cauchy-Euler ordinary differential equations (ODEs) into nonlinear ODEs and provide their analytic general solutions. Some examples are presented in order to clarify the applications of interesting results.

Keywords: Cauchy-Euler equation, nonlinear ODE.

I. Introduction and Preliminary Results

The Cauchy-Euler equation is often one of the first higher order differential equations with variable coefficients (see [1], p. 281, see also [2, 3, 4, 5]). If the independent variable is changed from x to t (via the transformation $x = e^t$), then the resulting equation becomes a linear constant coefficient ODE. The standard technique for solving a linear constant coefficient ODE is to look for exponential solutions. A special class of homogeneous second order Cauchy-Euler ODE has the form

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0, \qquad (1.1)$$

for constants a and b. This equation actually has what it called a singular point at $x = 0$ which yields trivial solution but we are focus to find non-trial solutions. To solve the equation, alternatively, a solution of the form $y = x^m$ can be tried directly into (1.1) and deduce the characteristic (or auxiliary or indicial) equation

$$m^2 + (a-1)m + b = 0. \qquad (1.2).$$

The general solution of (1.1) depends on the nature of the roots of (1.2) (see, e.g., [6, 7, 8, 9]). That is, if (1.2) has two distinct real roots say m_1 and m_2, then the general solution of (1.1) is $y = c_1x^{m_1} + c_2x^{m_2}$. If (1.2) has double real roots (or the unique real root) m, then the general solution of (1.1) is $y = c_1x^m + c_2x^m\ln x$. And also, if (1.2) has complex conjugates roots $m_{1,2} = \alpha \pm i\beta$, then the general solution of (1.1) is $y = x^\alpha(c_1\sin(\beta\ln x) + c_2\cos(\beta\ln x))$, where c_1 and c_2 are some constants.

Ref

1. Daniel Zwillinger, *Handbook of Differential Equations 3rd edition*, Academic Press, 1997.

Author α σ: Department of Mathematics, College of Natural and Computational Science, Madda Walabu University, Bale-Robe, Ethiopia. e-mails: wasihune2011@gmail.com, bityworku21@gmail.com

The linear none-homogeneous Cauchy-Euler equation has the form

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=r(x). \tag{1.3}$$

Lemma: Let $r(x)$ be a piecewise continuous function on I and let y_1 and y_2 be two linearly independent solutions of (1.1) on I. Then a particular solution y_p of (1.3), [7, 10, 11, 12] is given by

$$y_p=-y_1\int\frac{y_2r(x)}{x^2W(x;y_1,y_2)}dx+y_2\int\frac{y_1r(x)}{x^2W(x;y_1,y_2)}dx, \tag{1.4}$$

where $W(x;y_1,y_2)$ is the Wronskian of y_1 and y_2.

Proof. Let $u(x)$ and $v(x)$ be continuously differentiable functions (to be determined) for some x in I such that

$$y_p=uy_1+vy_2, \tag{1.5}$$

is a particular solution of (1.3). Differentiation of (1.5) leads to

$$\frac{dy_p}{dx}=u\frac{dy_1}{dx}+y_1\frac{du}{dx}+v\frac{dy_2}{dx}+y_2\frac{dv}{dx}. \tag{1.6}$$

We choose u and v so that

$$y_1\frac{du}{dx}+y_2\frac{dv}{dx}=0. \tag{1.7}$$

Using (1.7) in (1.6), we have

$$\frac{dy_p}{dx}=u\frac{dy_1}{dx}+v\frac{dy_2}{dx}\ \textit{and}\ \frac{d^2y_p}{dx^2}=u\frac{d^2y_1}{dx^2}+v\frac{d^2y_2}{dx^2}+\frac{du}{dx}\frac{dy_1}{dx}+\frac{dv}{dx}\frac{dy_2}{dx}. \tag{1.8}$$

Since y_p is a particular solution of (1.3), y_1 and y_2 are solutions of (1.1), using (1.5) and (1.8) in (1.3), we obtain the condition

$$\frac{du}{dx}\frac{dy_1}{dx}+\frac{dv}{dx}\frac{dy_2}{dx}=\frac{r(x)}{x^2}. \tag{1.9}$$

Thus from (1.7) and (1.9), we have system of equations as

$$\begin{cases} y_1\frac{du}{dx}+y_2\frac{dv}{dx}=0 \\ \frac{du}{dx}\frac{dy_1}{dx}+\frac{dv}{dx}\frac{dy_2}{dx}=\frac{r(x)}{x^2} \end{cases} \tag{1.10}$$

Applying Cramer's rule for (1.10) and after simplification, we get

$$u=-\int\frac{y_2r(x)}{x^2W(x;y_1,y_2)}dx\ \textit{and}\ v=\int\frac{y_1r(x)}{x^2W(x;y_1,y_2)}dx. \tag{1.11}$$

So (1.5) and (1.11) yield the desired results. We complete proof of lemma.

In the next section, we have showed that a general solution of the generalized second order nonlinear Cauchy-Euler equation and examples are presented to clarify the results.

Notes

II. The Main Results

Notes

Theorem. Let $f(y(x))$ be continuously differentiable function. The second order nonlinear homogeneous ordinary differential equation of the form

$$x^2\left(p(y)\frac{d^2y}{dx^2}+q(y)\left(\frac{dy}{dx}\right)^2\right)+axp(y)\frac{dy}{dx}+bf(y)=0, \qquad (2.1)$$

has the general solution $f(y)=c_1x^{m_1}+c_2x^{m_2}$ if (1.2) has two distinct real roots. And also (2.1) has the general solution $f(y)=c_1x^m+c_2x^m\ln x$ if (1.2) has double real roots; and (2.1) has the general solution $f(y)=x^\alpha(c_1\sin(\beta\ln x)+c_2\cos(\beta\ln x))$ whenever (1.2) has complex conjugates roots, where a, b, c_1 and c_2 are real constants, $p(y)=\frac{d}{dy}(f(y))$ and $q(y)=\frac{d}{dy}(p(y))$.

If $r(x)$ be a piecewise continuous function and $f_1(y(x))$ and $f_2(y(x))$ be two linearly independent solutions of (2.1), then the second order nonlinear non-homogeneous ODE of the form

$$x^2\left(p(y)\frac{d^2y}{dx^2}+q(y)\left(\frac{dy}{dx}\right)^2\right)+axp(y)\frac{dy}{dx}+bf(y)=r(x), \qquad (2.2)$$

has a particular solution $f_p(y(x))$ which is given by

$$f_p(y(x))=-f_1(y)\int\frac{f_2(y)r(x)}{x^2W(x;f_1,f_2)}dx+f_2(y)\int\frac{f_1(y)r(x)}{x^2W(x;f_1,f_2)}dx. \qquad (2.3)$$

Where a, b are real constants, $p(y)=\frac{d}{dy}(f(y))$, $q(y)=\frac{d}{dy}(p(y))$, and $W(x;f_1(y),f_2(y))\neq 0$ is the Wronskian of $f_1(y)$ and $f_2(y)$.

Proof. To prove the first identity (2.1), let $\xi=f(y)$, so that $\frac{d\xi}{dx}=p(y)\frac{dy}{dx}$ and $\frac{d^2\xi}{dx^2}=q(y)(\frac{dy}{dx})^2+p(y)\frac{d^2y}{dx^2}$. Then, we have

$$x^2\frac{d^2\xi}{dx^2}+ax\frac{d\xi}{dx}+b\xi=0. \qquad (2.4)$$

To solve the equation, plug $\xi=x^m$ into (2.4) and we get the characteristic equation (1.2). Applying (1.1), (1.2) and after simplification, we obtain the general corresponding solutions of (2.1). Hence we complete proof of (2.1).

Next, to prove the second identity (2.2), let $\xi=f(y)$, then we obtain

$$x^2\frac{d^2\xi}{dx^2}+ax\frac{d\xi}{dx}+b\xi=r(x). \qquad (2.5)$$

Let $\xi_1=f_1(y)$ and $\xi_2=f_2(y)$ be two linearly independent solutions of (2.4) so that they are also solutions for (2.1). Let $u(x)$ and $v(x)$ be continuously differentiable functions (to be determined) such that

$$\xi_p=f_p(y)=u\xi_1+v\xi_2, \qquad (2.6)$$

is a particular solution of (2.5) which is also a particular solution of (2.2). Then applying (1.6)-(1.10) and using (2.6) with its first and second derivatives in (2.5), we have system of equations as

$$\begin{cases} \xi_1 \frac{du}{dx} + \xi_2 \frac{dv}{dx} = 0 \\ \frac{du}{dx}\frac{d\xi_1}{dx} + \frac{dv}{dx}\frac{d\xi_2}{dx} = \frac{r(x)}{x^2} \end{cases} \tag{2.7}$$

Using (1.11) and (2.7) after simplification with little algebra, we get

$$u = -\int \frac{\xi_2 r(x)}{x^2 W(x;\xi_1,\xi_2)}dx \quad and \quad v = \int \frac{\xi_1 r(x)}{x^2 W(x;\xi_1,\xi_2)}dx;$$

$$u = -\int \frac{x^{-2} f_2(y) r(x)}{W(x; f_1(y), f_2(y))}dx \quad and \quad v = \int \frac{x^{-2} f_1(y) r(x)}{W(x; f_1(y), f_2(y))}dx. \tag{2.8}$$

Using (2.6) and (2.8), we arrive at (2.2).

III. Examples

Now let us show the usefulness of the theorem through some examples.

Example 3.1. Solve the following nonlinear ODE for $x \in (0, \frac{\pi}{2})$.

$$2x^2\left(\frac{1}{1+4y^2}\frac{d^2y}{dx^2} - \frac{8y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2\right) - \frac{4x}{1+4y^2}\frac{dy}{dx} + 2\arctan(2y) = x^3,$$

Solution. Let $\xi = \arctan(2y)$ so that

$$\frac{d\xi}{dx} = \frac{2}{1+4y^2}\frac{dy}{dx} \quad and \quad \frac{d^2\xi}{dx^2} = \frac{2}{1+4y^2}\frac{d^2y}{dx^2} - \frac{16y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2.$$

Then the given equation reduce to the form

$$x^2\frac{d^2\xi}{dx^2} - 2x\frac{d\xi}{dx} + 2\xi = x^3.$$

From this, the two linearly independent solutions of the corresponding homogeneous part are $\xi_1 = x = f_1(y)$ and $\xi_2 = x^2 = f_2(y)$. Here the Wronskian $W(x, x^2) = x^2 \neq 0$. Clearly $y_1(x) = \frac{1}{2}\tan(x)$ and $y_2(x) = \frac{1}{2}\tan(x^2)$ are the two linearly independent solutions of the corresponding homogeneous part.

From the above theorem using (2.3), a particular solution ξ_p is given by

$$\xi_p = f_p(y) = -x\int \frac{x^2.x^3}{x^2.x^2}dx + x^2\int \frac{x.x^3}{x^2.x^2}dx = \frac{x^3}{2}.$$

Thus by applying the above theorem (2.2), a particular solution of the given equation is $f_p(y) = \arctan(2y) = \frac{x^3}{2}$ implies $y_p = \frac{1}{2}\tan(\frac{x^3}{2})$.

Notes

Thus obviously the general solution of the given nonlinear non-homogeneous second order ODE is simply the sum of the general solutions of its corresponding homogeneous part and the particular solution of the non-homogeneous part. Hence $y(x) = c_1 \tan(x) + c_2 \tan(x^2) + \frac{1}{2}\tan(\frac{x^3}{2})$ is the required general solution, for some integral constants c_1 and c_2.

Example 3.2. Find a particular solution of the nonlinear ODE

$$2x^2\left(\frac{1}{1+4y^2}\frac{d^2y}{dx^2} - \frac{8y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2\right) - \frac{4x}{1+4y^2}\frac{dy}{dx} + 2\arctan(2y) = x^4.$$

Solution. Let $\xi = \arctan(2y)$ and using example (*3.1*), we obtain

$$x^2\frac{d^2\xi}{dx^2} - 2x\frac{d\xi}{dx} + 2\xi = x^4.$$

By the above theorem and example (*3.1*), a particular solution is $\xi_p = f_p(y) = \frac{x^4}{6}$, implies that $y_p = \frac{1}{2}\tan(\frac{x^4}{6})$ is the desired solution.

Example 3.3. Find a particular solution of the nonlinear ODE

$$2x^2\left(\frac{1}{1+4y^2}\frac{d^2y}{dx^2} - \frac{8y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2\right) - \frac{4x}{1+4y^2}\frac{dy}{dx} + 2\arctan(2y) = x^4 + x^3.$$

Solution. By the principle of superposition, a particular solution should be

$$y_p = y_{p1} + y_{p2} = \frac{1}{2}\tan\left(\frac{x^3}{2}\right) + \frac{1}{2}\tan\left(\frac{x^4}{6}\right)$$

is the desired solution.

Example 3.4. Find the general solution of the nonlinear ODE

$$x^2\left(\frac{2}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{1}{y^2}\frac{d^2y}{dx^2}\right) - \frac{7x}{y^2}\frac{dy}{dx} + \frac{13}{y} = \frac{4}{x^3}, \quad for \quad x \in (1, 2].$$

Solution. Let $\xi = \frac{1}{y}$. Using ξ and its first and second derivatives in the given equation, we obtain

$$x^2\frac{d^2\xi}{dx^2} + 7x\frac{d\xi}{dx} + 13\xi = \frac{4}{x^3},$$

which has two linearly independent solutions $\xi_1 = x^{-3}\cos(2\ln x)$ and $\xi_2 = x^{-3}\sin(2\ln x)$. Here the Wronskian $W(x; \xi_1, \xi_2) = 2x^{-7} \neq 0$ for $x \in (1, 2]$. Thus by the above theorem (2.2) and (2.3), we get a particular solution

$$\xi_p = x^{-3}\sin^2(2\ln x) - x^{-3}\cos^2(2\ln x).$$

$$y_p = \frac{x^3}{\sin^2(2\ln x) - \cos^2(2\ln x)}.$$

Notes

Hence the required general solution for some constants c_1 and c_2 is

$$y(x) = c_1 \frac{x^3}{\cos(2\ln x)} + c_2 \frac{x^3}{\sin(2\ln x)} + y_p.$$

IV. Concluding Remarks and Observations

In this paper besides having an important history background, it also has interesting applications. Using this paper, we can find the general solutions of a nonlinear Cauchy-Euler equation that can be reduced to the general form of a linear Cauchy-Euler equation [1], which is given as

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x),$$

by using appropriate methods. In particular, the ideas of this paper may be a base to obtain a generalized version of other first order ODEs. Moreover, the approach adopted in this paper was meant to reach both researchers and undergraduate students.

References Références Referencias

1. Daniel Zwillinger, *Handbook of Differential Equations 3rd edition,* Academic Press, 1997.
2. N. Finizio and G. Ladas, *Ordinary Differential Equations with Modern Applications,* Wadsworth Publishing Company, Belmont, CA, 1982.
3. L. Jodar, *Boundary value problems for second order operator differential equations,* Linear Algebra and Its Appls, 91, 1987, 1-12.
4. G. F. Simmons, *Differential Equations with Applications and Historical Notes*, McGraw-Hill Book Company, New York, 1972.
5. G. Valiron, *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions,* Math Sci Press, Brookline, MA, 1950.
6. Peeyush Chandra, A. K. Lal, V. Raghavendra, G. Santhanam, *Notes on Mathematics-102*[1], [1]Supported by a grant from MHRD.
7. Edward Burkard, *Introduction to Ordinary Differential Equations and Some Applications,* 2010.
8. Jeffrey R. Chasnov, *Introduction to Differential Equations: Lecture notes for MATH 2351/2352,* Hong Kong University, 2014.
9. Thomas Kecker, *On the singularity structure of differential equations in the complex plane,* University College London, 2014.
10. N. Euler, T. Wolf[1], PGL Leach[2] and M. Euler, *Linearisable Third Order Ordinary Differential Equations and Generalised Sundman Transformations*, Sweden, 2002.
11. P. K. Subramanian and Melisa *Hendrata, Lecture Notes on Ordinary Differential Equations,* California State University, Los Angeles, 2011.
12. Paul Dawkins, *Differential Equations*, 2007.

Ref

1. Daniel Zwillinger, *Handbook of Differential Equations 3rd edition,* Academic Press, 1997.