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Variational Inequalities for Systems of Strongly Nonlinear Elliptic Operators of Infinite Order

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Abstract- We are concerned with the existence of weak solutions of strongly nonlinear variational inequalities for systems of infinite order elliptic operators of the form:

$$A^r(u)(x)+ Br(u)(x),\ \boldsymbol{x}\in\Omega,$$

where

$Ar(u)(x) = \Sigma(-\mathbf{1})|\boldsymbol{\alpha}|\infty|\boldsymbol{\alpha}|=\mathbf{0}\boldsymbol{D\alpha A\alpha r}(x,\boldsymbol{D\gamma u}(\boldsymbol{x}))$,

$(\mathbf{u})(\mathbf{x})\ \Sigma(-\mathbf{1})|\boldsymbol{\alpha}|\boldsymbol{D\alpha B\alpha r}|\boldsymbol{\alpha}|\le\boldsymbol{Mr}(x,\ \boldsymbol{D\alpha u}(\boldsymbol{x})),\ \boldsymbol{Mr}\in\mathbb{N}$ *fixed,*

$(\boldsymbol{x})|\boldsymbol{\partial}\Omega=\mathbf{0},|\boldsymbol{\omega}|=0,1,2,\ldots,$

Ω is a bounded domain in $\mathbf{R^N}$, $|\boldsymbol{\gamma}|\le|\boldsymbol{\alpha}|$ *and* $\mathbf{r}=\mathbf{1},2,\ldots\mathbf{m}$.

We require that the coefficients A_a^r satisfy only some growth and coerciveness conditions and B_α^r obey a sign condition.

Keywords: systems of strongly nonlinear elliptic operators of infinite order-variational inequalities.

GJSFR-F Classification: MSC 2010: 11J89

VARIATIONALINEQUALITIESFORSYSTEMSOFSTRONGLYNONLINEARELLIPTICOPERATORSOFINFINITEORDER

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$$(u)(x)\ \Sigma(-1)|\alpha|D\alpha B\alpha r|\alpha|\leq Mr(x,\ D\alpha u(x)),\ Mr\in\mathbb{N}\ fixed,$$

$$(x)|\partial\Omega=0, |\omega|=0,1,2,\ldots,$$

Ω is a bounded domain in $\mathbf{R}^N$, $|\gamma|\leq|\alpha|$ *and* $r=1, 2,\ldots m$.

We require that the coefficients A^r_a satisfy only some growth and coerciveness conditions and B^r_α obey a sign condition.

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I. Introduction

In a recent paper, Benkirane, Chrif and El-Manouni[1]considerd the existence of solutions for strongly nonlinear elliptic equations of the form

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} D^{\alpha}A_{\alpha}(x, D^{\gamma}u(x))+g(x,u)=f(x), x \in \Omega, |\gamma| \leq |\alpha|$$

where A_α are assumed to satisfy polynomial growth and coerciveness

Conditions and g is strongly nonlinear in the sense that no growth condition is imposed but only a sign condition and $f\in L^1(\Omega)$. They relaxed the monotonicity condition, but we cann't see this.

In this paper, we extend the result of[1] to the corresponding class of variational inequalities of the above system without assuming this condition.

II. Function Spaces

Let Ω be a bounded domain in $\mathbf{R}(N\geq 1)$having a locally Lipschitz property.

Let V be a closed linear subspace of $[W(a_\alpha, p_\alpha)(\Omega)]^m$ such that

$$[W_0^k(a_\alpha, p_\alpha)(\Omega)]^m \subseteq V \subseteq [W^k(a_\alpha, p_\alpha)(\Omega)]^m$$

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Ref

1. A. Benkirane, M. Chrif and S. El Manouni, Existence results for strongly nonlinear elliptic equations of infinite order, Z.Anal. Anwend.(J.Anal. Appl.)26, 2007, 303-313.

where $[W^k(a_\alpha,\mathrm{p}_\alpha)(\Omega)]^m = \Pi_{r=1}^m [W^k(a_{\alpha,r},\mathrm{p}_\alpha)(\Omega)]$, equipped with the norm

$$\|u\|_{k,\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^k a_{\alpha,r} \|D^\alpha u^r\|_{\mathrm{p}_\alpha}^{\mathrm{p}_\alpha}$$

and $[W_0^k(a_\alpha,\mathrm{p}_\alpha)(\Omega)]^m = \overline{[C_0^\infty(\Omega)]^m}^{\|\ \|_{k,\mathrm{p}_\alpha}}$

where $\{a_{\alpha,r}\}$ is an arbitrary sequence of nonnegative numbers and $\mathrm{p}_\alpha > 1$.

Denote by $\mathrm{p}'_\alpha = \frac{\mathrm{p}_\alpha}{\mathrm{p}_\alpha - 1}$, $|\alpha| \le k$.Put $W = V \cap [W^{k+1}(a_\alpha, s_\alpha)(\Omega)]^m$, $s_\alpha > \max\{N,,\mathrm{p}_\alpha\}$.
W is furnished with the norm

$$\|.\|_W = \max\{\|.\|_V\ ,\ \|.\|_{k+1,s_\alpha}\}$$

By the Sobolev embedding theorem

$$[W^{k+1}(a_\alpha, s_\alpha)(\Omega)]^m \to [C^k(\bar{\Omega})]^m \qquad (1)$$

Consider the m-product of Sobolev spaces of infinite order:

$$[W_0^\infty(a_\alpha,\mathrm{p}_\alpha)(\Omega)]^m = \Pi_{r=1}^m W_0^\infty(a_{\alpha,r},\mathrm{p}_\alpha)(\Omega)$$

$$\{u \in [C_0^\infty(\Omega)]^m : \|u\|_{\infty,\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty a_{\alpha,r} \|D^\alpha u\|_{\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} < \infty\},$$

$$[W^\infty(a_\alpha,\mathrm{p}_\alpha)(\Omega)]^m = \{u \in [C^\infty(\Omega)]^m : \|u\|_{\infty,\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty a_{\alpha,r} \|D^\alpha u\|_{\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} < \infty\}$$

$$[W^{-\infty}(a_\alpha,\mathrm{p}'_\alpha)(\Omega)]^m = \{h : h = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty (-1)^{|\alpha|} a_{\alpha,r} D^\alpha h_\alpha^r;\ h_\alpha^r \in L^{\mathrm{p}'_\alpha}(\Omega)\},$$

$$\|u\|_{-\infty,\mathrm{p}'_\alpha}^{\mathrm{p}'_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty a_{\alpha,r} \|h_\alpha^r\|_{\mathrm{p}'_\alpha}^{\mathrm{p}'_\alpha} < \infty$$

The nontriviality of these spaces are discussed by Dubinskii in[5]. So we choose $a_\alpha = (a_{\alpha,r})_{r=1}^m$ such that the nontriviality of these spaces holds.

III. Strongly Nonlinear Variational Inequalities of Finite Order

We start with the existence of weak solutions of strongly nonlinear variational inequalities for systems of the finite order elliptic operators:

$$\sum_{|\alpha|=0}^k (-1)^{|\alpha|} D^\alpha A_\alpha^r(x, D^\gamma u(x)) + \sum_{|\alpha|\le M_r} (-1)^{|\alpha|} D^\alpha B_\alpha^r(x, D^\alpha u(x)),\ x \in \Omega, \qquad (2)$$

To define the system(2)more precisely we introduce the following hypotheses:

A₁) $A_\alpha^r(x, \xi_\gamma) : \Omega \times \mathbb{R}^{N_0^1} \times \ldots \times \mathbb{R}^{N_0^m} \to \mathbb{R}$ are carathéodory functions.

There exist *a* constant $c_0 > 0$, independent of k and a function $K_1^r \in L^{\mathrm{p}'_\alpha}(\Omega)$ such that

$$|A_\alpha^r(x,\xi_\gamma)| \le c_0 a_{\alpha,r} |\xi_\gamma{}^r|^{\mathrm{p}_\alpha - 1} + K_1^r(x) \qquad \forall\, x \in \Omega,\ all\ \xi_\gamma{}^r,\ all\ r = 1,2,..m,\ |\gamma| \le |\alpha|$$

Ref

5. Yu. A. Dubinskii, Higher order parabolic differential equations, Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya, Vol.37,1990,89-166.

where $a_\alpha > 0$, $p_\alpha > 1$ are real numbers.

A_2) There exists a constant c_1, independent of k and a function $K_2 \in [L^1(\Omega)]^m$ Such that

$$\sum_{r=1}^{m}\sum_{|\alpha|=0}^{k} A_\alpha^r(x, \xi_\gamma)\, \xi_\alpha{}^r \geq c_1 \sum_{r=1}^{m}\sum_{|\alpha|=0}^{k} a_{\alpha,r}\, |\xi_\alpha{}^r|^{p_\alpha} + K_2(x),$$

$$\forall\, \xi_\gamma{}^r, \xi_\alpha{}^r \in \mathbb{R}^{N_0^r}, |\gamma| \leq |\alpha|$$

Notes

B) $B_\alpha^r(x,\eta)$ are carathéodory functions defined for all $x \in \Omega$, *all* $\eta_\alpha{}^r \in \mathbb{R}^{N_1^r}$, each $r = 1, 2, \dots m$ and α with $|\alpha| \leq M_r < k$ such that $B_\alpha^r(x,\eta)\eta_\alpha{}^r \geq 0$ and

$$\sup_{|\eta|\leq a} |B_\alpha^r(x,\eta)| \leq h_\alpha^r(x) \in L^1(\Omega)$$

Consider the nonlinear form

$$a(u,v) = \int_\Omega \sum_{r=1}^{m}\Big[\sum_{|\alpha|=0}^{k} A_\alpha^r(x, D^\gamma u(x)) D^\alpha v^r(x)dx +$$

$$\sum_{|\alpha|\leq M_r} B_\alpha^r(x, D^\alpha u(x))\, D^\alpha v^r(x)dx\Big]$$

which by A_1) and B) gives rise to a nonlinear mapping $S: K \cap W \to W^*$ such that

$$a(u,v) = (S(u), v) \quad (v \in K \cap W)$$

Theorem 1. Let the hypotheses A_1) - A_2) and B)be satisfied. Let K be a closed convex subset of V with $0 \in K$. Let $f \in V^*$ be given. Suppose that for some $R > 0$,

$$(S(v)-f, v) > 0 \ \text{ for all } \ v \in K \cap W, \|v\|_V = R,$$

Then there exists $u \in K \cap W$, $\|u\|_V \leq R$, such that

$$(S(u), v-u) \geq (f, v-u) \text{ for all } \ v \in K \cap W$$

Outline of proof.

Let Λ be the family of all finite dimensional linear subspaces F of W, which is a directed set under inclusion, and let F be provided with the norm $\|v\|_F = \|v\|_V$.

For each $F \in \Lambda$ let J_F be the injection mapping of F into W and $J_F^*: W^* \to F^*$ its adjoint.

In view of the compactness of the embedding(1) is easy to see that the the restriction of S to W is demicontinuous and moreover

$$(S_F(v)-f, v) > 0 \quad \text{for all } \ v \in F \cap K \quad \text{with } \|v\|_F = R.$$

Therefore by lemma 2 of [2] there exists $u_F \in F$ with $\|u_F\|_F \leq R$ such that

$$(S_F(u_F), v-u_F) - (J_F^* f, v-u_F) \geq 0 \ \text{ for all } \ v \in F \cap K \qquad (3)$$

For any $F' \in \Lambda$, let $U_{F'} = \{u_F : F \in \Lambda, F' \subset F, u_F$ as above$\}$. The family $\{(U_{F'}) : F \in \Lambda\}$ has the finite intersection property and by the reflexivity of V, there exists

$$u \in \bigcap_{F' \in \Lambda}\{weak\ cl._V (U_{F'})\}$$

with $\|u\|_V \leq R$. Since $u \in \{weak\, cl._V\,(U_{F'})\}$, then for each $F_0 \in \Lambda$ there exists a sequence $(Fn) \subset \Lambda$, whose union is dense in W, with $F_0 \subset F_1 \subset \ldots$, and for each $n \in \mathbb{N}$ an elemaent $u_n \in F_n$ such that $u_n \to u$ weakly in V[proposition 11 of[3]. Therefore for each $n \in \mathbb{N}$ we have from(3)

$$(S(u_n), v- u_n) - (f, v- u_n) \geq 0 \quad \text{for all } v \in F_n \cap K \tag{4}$$

Setting v=0 in (4) we conclude the uniform boundedness from above of the numerical sequence $\{(S(u_n), u_n)\}_{n\in\mathbb{N}}$. From the compactness of the embedding (1), we get

$$D^{\alpha}u_n(x) \to D^{\alpha}u(x) \text{ uniformly on } \bar{\Omega} \text{ for all } \alpha \text{ with } |\alpha| \leq k, \tag{5}$$

From A_1) and A_2), we obtain

$$\|u_n\|^{p_\alpha}_{k,p_\alpha} \leq c_2, \qquad \int_\Omega |A^r_\alpha(x, D^\gamma u_n(x))|^{p'_\alpha} \leq c_3.$$

From the inequality

$$|B^r_\alpha(x,\eta)| \leq \sup_{|\eta|\leq\delta^{-1}} |B^r_\alpha(x,\eta)| + \delta B^r_\alpha(x, Du_n(x)) D^\alpha u_n(x)$$

which is always true for each $\delta > 0, r = 1,2, \ldots m$ *and* all α with $|\alpha| \leq k_r$.
For any measurable subset A of Ω, we get from B)

$$\int_A |B^r_\alpha(x,\eta)|dx \leq c_4 \qquad (c_2 - c_4 \text{ are constants})$$

Now, allowing n→∞ in (4), taking these estimates into cosideration as well as Vitali's and dominated convergence theorems and fatou's lemma, the proof follows.

IV. Strongly Nonlinear Variational Inequalities of Infinite Order

Now we consider the existence of weak solutions of strongly nonlinear variational inequalities for systems of the infinite order elliptic operators:

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} D^\alpha A^r_\alpha(x, D^\gamma u(x)) + \sum_{|\alpha|\leq M_r}(-1)^{|\alpha|} D^\alpha B^r_\alpha(x, D^\alpha u(x)),\ x \in \Omega, \tag{6}$$

Theorem 2. Let the hypotheses A_1)- A_2) and B)be satisfied. Let K be a closed convex Subset of $[W^\infty_0(a_\alpha, p_\alpha)(\Omega)]^m$ with $0 \in K$. Let $f \in [W^{-\infty}(a_\alpha, p'_\alpha)(\Omega)]^m$ be given. Then there exists at least one solution u∈**K**, such that

$$(A^r(u) + B^r(u), v^r - u^r) \geq (f^r, v^r - u^r), r \in \{1,2,\ldots,m\} \tag{7}$$

Proof. We adopt the ideas of [6].Consider the auxiliary Dirchlet problem of order 2k, which may be thought as the partial sum of the series (6):

$$(A^r_{2k}(u_k), v^r - u^r_k) + (B^r(u_k), v^r - u^r_k) \geq (f^r_k, v^r - u^r_k),\ v \in K \cap W,\ r \in \{1,2,\ldots,m\} \tag{8}$$

where

$$A^r_{2k}(u_k)(x) = \sum_{|\alpha|=0}^{k}(-1)^{|\alpha|} D^\alpha A^r_\alpha(x, D^\gamma u_k(x)),$$

$$B^r(u_k)(x) = \sum_{|\alpha|\leq M_r<k}(-1)^{|\alpha|} D^\alpha B^r_\alpha(x, D^\alpha(u_k)(x))$$

Ref

6. A.T.El-dessouky,Variational inequalities of strongly nonlinear elliptic operators of infinite order, Publications de l'institut mathematique, Nouvelle serie tome(67),1993,81-87.

and

$$\mathbf{f}_k^{\mathbf{r}} = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} a_{\alpha,r} D^{\alpha} f_{\alpha}^{r} \in W^{-k}(a_{\alpha,r}, \mathbf{p}'_{\alpha})(\Omega)$$

The solvability of (8) in view of the hypotheses A_1)- A_2) and B) is a consequence of theorem1. Thus there exists $u_k \in \mathbf{K} \cap \mathbf{W}$ solving (8).

One of the fundamental roles in finding the solution of (8)is played by the so called a priori estimates. By A_2) and B), we get

$$\|\mathbf{u}_{\mathbf{k}}\|_{k,\mathbf{p}_{\alpha}}^{\mathbf{p}_{\alpha}} \leq c_2$$

Since $\mathbf{u_k} \in [W^k(\mathrm{a}_{\alpha,}\mathbf{p}_{\alpha})(\Omega)]^{\mathbf{m}}$ Implies $\mathbf{u_k} \in [W^1(a_{\alpha,}\mathbf{p}_{\alpha})(\Omega)]^{m}$ we get from the compactness of $[W^1(a_{\alpha,}\mathbf{p}_{\alpha})(\Omega)]^{m} \to [C(\bar{\Omega})]^{\mathbf{m}}$, the uniform convergence of $\mathbf{u_k}(\mathbf{x}) \to u(x)$ *on* $\bar{\Omega}$ *as* $k \to \infty$.

Similarly, by the compactness of $[W^k(a_{\alpha,}\mathbf{p}_{\alpha})(\Omega)]^{m} \to [C^{k-\ell}(\bar{\Omega})]^{m}$, for large enough k and $\ell \in \mathbb{N}$, we have $\mathbf{D}^{\alpha}\mathbf{u_k}(\mathbf{x}) \to \mathbf{D}^{\alpha}\mathbf{u}(\mathbf{x})$ uniformly on $\bar{\Omega}$ *as* $k \to \infty$.

By the definition of $[W_0^{\infty}(a_{\alpha,}\mathbf{p}_{\alpha})(\Omega)]^{m}$, ***we get*** $u \in [W_0^{\infty}(a_{\alpha,}\mathbf{p}_{\alpha})(\Omega)]^{m}$ and by closedness of **K,** $u \in K$. It remains to slow that u is a solution of (**7**).
For this aim it suffices to prove the following assertions:

$$\lim_{k\to\infty} (\mathrm{A}_{2k}^{\mathrm{r}}(u_k), z^r) = (\mathrm{A}^{\mathrm{r}}(\mathbf{u}), z^r) \tag{9}$$

$$\lim_{k\to\infty} (\mathrm{B}^{\mathrm{r}}(u_k), z^r) = (\mathrm{B}^{\mathrm{r}}(\mathbf{u}), z^r) \tag{10}$$

$$\liminf_{k\to\infty} (\mathrm{A}_{2k}^{\mathrm{r}}(u_k), {u_k}^r) \geq (\mathrm{A}^{\mathrm{r}}(\mathbf{u}), u^r) \tag{11}$$

$$\liminf_{k\to\infty} (\mathrm{B}^{\mathrm{r}}(u_k), {u_k}^r) \geq (\mathrm{B}^{\mathrm{r}}(\mathbf{u}), u^r \tag{12}$$

for all $\mathbf{z} \in K$, $r = 1, 2, \dots, m$.

As above, (9)and(10) are consequence of the uniform boundedness of

$$\{\sum_{r=1}^{m}\sum_{|\alpha|=0}^{k} \mathrm{A}_{\alpha}^{\mathrm{r}}(\mathbf{x}, \mathrm{D}^{\gamma}u_k)\, D^{\alpha}u_k\}_{k\in\mathbb{N}},\ \{\sum_{r=1}^{m}\sum_{|\alpha|=0}^{M_r} \mathrm{B}^{\mathrm{r}}(\mathbf{x}, D^{\alpha}u_k)\, D^{\alpha}u_k\}_{k\in\mathbb{N}} \text{ and}$$

uniform equi-integrability of

$$\{\sum_{r=1}^{m}\sum_{|\alpha|=0}^{k} \mathrm{A}_{\alpha}^{\mathrm{r}}(\mathbf{x}, \mathrm{D}^{\gamma}u_k)\},\ \{\sum_{r=1}^{m}\sum_{|\alpha|=0}^{M_r} \mathrm{B}^{\mathrm{r}}(\mathbf{x}, D^{\alpha}u_k)\} \text{ in } [L^1(\Omega)]^{\mathbf{m}}$$

in view of Vitali's and dominated convergence theorems as well as (5). Assertions(11) and(12)are direct consequences of Fatou's lemma and(5).

Example. As a particular example which can be handled by our result but fails outside the Scope of [4], we consider the nonlinear system

Ref

4. Yu. A. Dubinskii, Sobolev spaces of infinite order and the behavior of solutions of some boundary-value problems with unbounded increase of the order of the equation, Math. USSR Sbornik 72, 1972, 143-162.

$$\begin{cases} \sum_{j=0}^{\infty} \sum_{|\alpha|=j} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha,1} |D^{\alpha} u_1|^{p_\alpha - 2} D^{\alpha} u_1) + h_1(x) |u_2| e^{|u_2|} \\ \sum_{j=0}^{\infty} \sum_{|\alpha|=j} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha,2} |D^{\alpha} u_2|^{p_\alpha - 2} D^{\alpha} u_2) + h_2(x) |u_1| e^{|u_1|} \end{cases}$$

$(h_i(x))_{i=1}^{2}$ are arbitrary nonnegative $L^1(x)-$functions, $u = (u_1, u_2)$.

Notes

References Références Referencias

1. A. Benkirane, M. Chrif and S. El Manouni, Existence results for strongly nonlinear elliptic equations of infinite order, Z.Anal. Anwend.(J.Anal. Appl.)26, 2007, 303-313.
2. F. E. Browder: On the unification of the calculus of variation and the theory of monotone nonlinear operators in Banach spaces.Proc.Nat.Acad.Sc.U.S.A.,56, 1966, 419-425.
3. F. E. Browder, and P. Hess: Nonlinear mappings of monotone type in Banach spaces. J. of Functional Analysis Appl.43, 1973, 241-249.
4. Yu. A. Dubinskii, Sobolev spaces of infinite order and the behavior of solutions of some boundary-value problems with unbounded increase of the order of the equation, Math. USSR Sbornik 72, 1972, 143-162.
5. Yu. A. Dubinskii, Higher order parabolic differential equations, Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya, Vol.37,1990,89-166.
6. A.T.El-dessouky,Variational inequalities of strongly nonlinear elliptic operators of infinite order, Publications de l'institut mathematique, Nouvelle serie tome(67),1993,81-87.