Variational Inequalities for Systems of Strongly Nonlinear Elliptic Operators of Infinite Order

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Abstract- We are concerned with the existence of weak solutions of strongly nonlinear variational inequalities for systems of infinite order elliptic operators of the form:

$$A'(u)(x) + Br(u)(x), \quad x \in \Omega,$$

where

$$A_{\alpha}(u)(x) = \sum (-1)^{1} |\alpha| \infty |\alpha| = 0 D\alpha A_{\alpha}r(x, D\gamma u(x)),\quad (\alpha)(x) \Sigma (-1) |\alpha| D\alpha B_{\alpha}r |\alpha| \leq M r(x, D\alpha u(x)), \quad M r \in \mathbb{N} \quad \text{fixed},$$

$$|x| D\Omega = 0, \quad |\omega| = 0, 1, 2, \ldots,$$

$$\Omega$$ is a bounded domain in $$\mathbb{R}^n, \quad |\gamma| \leq |\alpha| \quad \text{and} \quad r = 1, 2, \ldots m.$$ We require that the coefficients $$A_{\alpha}$$ satisfy only some growth and coerciveness conditions and $$B_{\alpha}$$ obey a sign condition.

Keywords: systems of strongly nonlinear elliptic operators of infinite order-variational inequalities.

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Variational Inequalities for Systems of Strongly Nonlinear Elliptic Operators of Infinite Order

A. T. El-dessouky

Abstract: We are concerned with the existence of weak solutions of strongly nonlinear variational inequalities for systems of infinite order elliptic operators of the form:

\[ A_r(u)(x) + B_r(u)(x), x \in \Omega, \]

where

\[ A_r(u)(x) = \sum (\alpha) |\alpha|_\infty |\alpha|_D A_{\alpha} x D_{\alpha u}(x), \]

\[ (u)(x) \sum (\alpha) |\alpha| D_{\alpha} B_{\alpha} x \leq M_r x, D_{\alpha u}(x), M_r \in \mathbb{N} fixed, \]

\( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( |y| \leq |x| \) and \( r = 1, 2, \ldots, m \).

We require that the coefficients \( A_{\alpha} \) satisfy only some growth and coerciveness conditions and \( B_{\alpha} \) obey a sign condition.

Keywords: systems of strongly nonlinear elliptic operators of infinite order-variational inequalities.

1. Introduction

In a recent paper, Benkirane, Chrif and El-Manouni [1] considered the existence of solutions for strongly nonlinear elliptic equations of the form

\[ \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_{\alpha} x D^\gamma u(x) + g(x, u) = f(x), x \in \Omega, |\gamma| \leq |\alpha| \]

where \( A_{\alpha} \) are assumed to satisfy polynomial growth and coerciveness.

Conditions and \( g \) is strongly nonlinear in the sense that no growth condition is imposed but only a sign condition and \( f \in L^1(\Omega) \). They relaxed the monotonicity condition, but we can’t see this.

In this paper, we extend the result of [1] to the corresponding class of variational inequalities of the above system without assuming this condition.

2. Function Spaces

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^{(N \geq 1)} \) having a locally Lipschitz property.

Let \( V \) be a closed linear subspace of \( [W(a_{\alpha}, p_{\alpha})(\Omega)]^m \) such that

\[ [W^k_0(a_{\alpha}, p_{\alpha})(\Omega)]^m \subseteq V \subseteq [W^k(a_{\alpha}, p_{\alpha})(\Omega)]^m \]

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where \( W^k(\alpha, p, \alpha)(\Omega) \) is equipped with the norm
\[
\| u \|_{k,p,\alpha}^p = \sum_{r=1}^{m} \sum_{|\alpha|=0}^{k} a_{\alpha,r} \| D^\alpha u^r \|_{p,\alpha}^p
\]
and \( W_0^k(\alpha, p, \alpha)(\Omega) \) is an arbitrary sequence of nonnegative numbers and \( p_\alpha > 1 \).

Denote by \( p^r = \frac{p_\alpha}{p_\alpha - 1}, |\alpha| \leq k \). Put \( W = V \cap \left( W_0^{k+1}(\alpha, s_\alpha)(\Omega) \right)^m, s_\alpha > \max\{N, p_\alpha \} \).

\( W \) is furnished with the norm
\[
\| \cdot \|_W = \max\{\| \cdot \|_V, \| \cdot \|_{k+1,s_\alpha}\}
\]

By the Sobolev embedding theorem
\[
\left( W_0^{k+1}(\alpha, s_\alpha)(\Omega) \right)^m \rightarrow [C^k(\overline{\Omega})]^m
\]

Consider the \( m \)-product of Sobolev spaces of infinite order:
\[
\left( W_0^\infty(\alpha, p_\alpha)(\Omega) \right)^m = \prod_{r=1}^{m} W_0^\infty(\alpha, p_\alpha)(\Omega)
\]
\[
\{ u \in [C^\infty(\Omega)]^m: \| u \|_{\infty,p_\alpha}^{p_\alpha} = \sum_{r=1}^{m} \sum_{|\alpha|=0}^{\infty} a_{\alpha,r} \| D^\alpha u^r \|_{p_\alpha}^{p_\alpha} < \infty \}
\]
\[
\left( W^\infty(\alpha, p_\alpha)(\Omega) \right)^m = \{ u \in [C^\infty(\Omega)]^m: \| u \|_{\infty,p_\alpha}^{p_\alpha} = \sum_{r=1}^{m} \sum_{|\alpha|=0}^{\infty} a_{\alpha,r} \| D^\alpha u^r \|_{p_\alpha}^{p_\alpha} < \infty \}
\]
\[
\left( W^{-\infty}(\alpha, p^r_\alpha)(\Omega) \right)^m = \{ h : h = \sum_{r=1}^{m} \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_{\alpha,r} D^\alpha h^r_\alpha, h^r_\alpha \in L^{p^r_\alpha}(\Omega) \},
\]
\[
\| u \|_{-\infty,p^r_\alpha}^{p^r_\alpha} = \sum_{r=1}^{m} \sum_{|\alpha|=0}^{\infty} a_{\alpha,r} \| h^r_\alpha \|_{p^r_\alpha}^{p^r_\alpha} < \infty
\]

The nontriviality of these spaces are discussed by Dubinskii in [5]. So we choose \( \alpha = (\alpha, r) \) such that the nontriviality of these spaces holds.

### III. Strongly Nonlinear Variational Inequalities of Finite Order

We start with the existence of weak solutions of strongly nonlinear variational inequalities for systems of the finite order elliptic operators:
\[
\sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} \sum_{|\alpha| \leq M_r}^{M_r} D^\alpha A_\alpha^r(x, D^\gamma u(x)) + \sum_{|\alpha| \leq M_r}^{M_r} (-1)^{|\alpha|} D^\alpha B_\alpha^r(x, D^\alpha u(x)), x \in \Omega
\]

To define the system (2) more precisely we introduce the following hypotheses:

**A_{r}** \( A_\alpha^r(x, \xi) : \Omega \times \mathbb{R}^N_\alpha \times \ldots \times \mathbb{R}^N_\alpha \rightarrow \mathbb{R} \) are carathéodory functions.

There exist \( a \) constant \( c_0 > 0 \), independent of \( k \) and a function \( K_1^r \in L^{p^r_\alpha}(\Omega) \) such that
\[
|A_\alpha^r(x, \xi)| \leq c_0 a_{\alpha,r} |\xi|^{|p^r_\alpha - 1|} + K_1^r(x) \quad \forall x \in \Omega, all \ |\xi|^r, all \ r = 1,2,\ldots, m, |\gamma| \leq |\alpha|
\]
where \(a_\alpha > 0, p_\alpha > 1\) are real numbers.

A2) There exists a constant \(c_1\), independent of \(k\) and a function \(K_2 \in [L^1(\Omega)]^m\) such that

\[
\sum_{|\alpha|=0}^{m} \sum_{r=1}^{k} A_{\alpha}^r(x, \xi) \xi_\alpha^r \geq c_1 \sum_{|\alpha|=0}^{m} \sum_{r=1}^{k} a_{\alpha,r} |\xi_\alpha^r|^{p_\alpha} + K_2(x),
\]

\[\forall \xi_\alpha^r, \xi_\alpha^r \in \mathbb{R}, |y| \leq |\alpha|\]

B) \(B_{\alpha}^r(x, \eta)\) are carathéodory functions defined for all \(x \in \Omega, all \eta_\alpha^r \in \mathbb{R}^{N \Gamma}\), each \(r = 1, 2, \ldots, m\) and \(\alpha\) with \(|\alpha| \leq M_r < k\) such that \(B_{\alpha}^r(x, \eta) \eta_\alpha^r \geq 0\) and

\[
\sup_{|\eta| \leq a} |B_{\alpha}^r(x, \eta)| \leq h_{\alpha}^r(x) \in L^1(\Omega)
\]

Consider the nonlinear form

\[
a(u,v) = \int_{\Omega} \sum_{\alpha=1}^{m} \sum_{|\alpha|=0}^{k} A_{\alpha}^r(x, D^\alpha u(x)) D^\alpha v^r(x) \, dx + \sum_{|\alpha|=0}^{m} \sum_{r=1}^{k} B_{\alpha}^r(x, D^\alpha u(x)) D^\alpha v^r(x) \, dx
\]

which by A1) and B) gives rise to a nonlinear mapping \(S:K \cap W \to W^*\) such that

\[
a(u,v) = (S(u),v) \quad (v \in K \cap W)
\]

**Theorem 1.** Let the hypotheses A1) - A2) and B) be satisfied. Let \(K\) be a closed convex subset of \(V\) with \(0 \in K\). Let \(f \in V^*\) be given. Suppose that for some \(R > 0\),

\[
(S(v)-f,v) > 0 \quad \text{for all} \quad v \in K \cap W, \|v\|_V = R,
\]

Then there exists \(u \in K \cap W, \|u\|_V \leq R\), such that

\[
(S(u),v-u) \geq (f,v-u) \quad \text{for all} \quad v \in K \cap W
\]

Outline of proof.

Let \(\Lambda\) be the family of all finite dimensional linear subspaces \(F\) of \(W\), which is a directed set under inclusion, and let \(F\) be provided with the norm \(\|v\|_F = \|v\|_V\).

For each \(F \in \Lambda\) let \(J_F\) be the injection mapping of \(F\) into \(W\) and \(J_F^*:W^* \to F^*\) its adjoint.

In view of the compactness of the embedding (1) is easy to see that the the restriction of \(S\) to \(W\) is demicontinuous and moreover

\[
(S_F(v)-f,v) > 0 \quad \text{for all} \quad v \in F \cap K \quad \text{with} \quad \|v\|_F = R.
\]

Therefore by lemma 2 of [2] there exists \(u_F \in F\) with \(\|u_F\|_F \leq R\) such that

\[
(S_F(u_F),v-u_F)-(J_F^*f,v-u_F) \geq 0 \quad \text{for all} \quad v \in F \cap K
\]

For any \(F \in \Lambda, let U_F = \{ u_F : F \in \Lambda, F' \subset F, u_F \text{ as above}\}\). The family \(\{U_F: F \in \Lambda\}\) has the finite intersection property and by the reflexivity of \(V\), there exists

\[
u \in \cap_{F \in \Lambda} \{\text{weak cl}_V (U_F)\}\]
with \( \| u \|_V \leq R \). Since \( u \in \{ \text{weak} \ cl \_ V (U_t) \} \), then for each \( F_0 \in \Lambda \) there exists a sequence \( (F_n) \subset \Lambda \), whose union is dense in \( W \), with \( F_0 \subset F_1 \subset \ldots \), and for each \( n \in \mathbb{N} \) an element \( u_n \in F_n \) such that \( u_n \to u \) weakly in \( V \). From proposition 11 of [3]. Therefore for each \( n \in \mathbb{N} \) we have from (3)

\[
(S(u_n), v - u_n) - (f, v - u_n) \geq 0 \quad \text{for all } v \in F_n \cap K 
\]  

(4)

Setting \( v = 0 \) in (4) we conclude the uniform boundedness from above of the numerical sequence \( \{(S(u_n), u_n)\}_{n \in \mathbb{N}} \). From the compactness of the embedding (1), we get

\[
D^\alpha u_n(x) \to D^\alpha u(x) \text{ uniformly on } \overline{\Omega} \text{ for all } \alpha \text{ with } |\alpha| \leq k, 
\]

(5)

From (6), we obtain

\[
\| u_n \|_{P^\alpha_{k,p}} \leq c_2, \quad \int_{\Omega} |A^\alpha_{\alpha}(x,D^\alpha u_n(x))|^{p^\alpha_{\alpha}} \leq c_3.
\]

From the inequality

\[
|B^\alpha_{\alpha}(x,\eta)| \leq \sup_{|\eta| \leq \delta^{-1}} |B^\alpha_{\alpha}(x,\eta)| + \delta B^\alpha_{\alpha}(x,Du_n(x))D^\alpha u_n(x)
\]

which is always true for each \( \delta > 0 \), \( r = 1,2,\ldots,m \) and all \( \alpha \) with \( |\alpha| \leq k_r \).

For any measurable subset \( A \) of \( \Omega \), we get from (B)

\[
\int_A |B^\alpha_{\alpha}(x,\eta)|dx \leq c_4 \quad (c_2 - c_4 \text{ are constants})
\]

Now, allowing \( n \to \infty \) in (4), taking these estimates into consideration as well as Vitali’s and dominated convergence theorems and Fatou’s lemma, the proof follows.

IV. STRONGLY NONLINEAR VARIATIONAL INEQUALITIES OF INFINITE ORDER

Now we consider the existence of weak solutions of strongly nonlinear variational inequalities for systems of the infinite order elliptic operators:

\[
\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A^r_{\alpha}(x,D^r u(x)) + \sum_{|\alpha|\leq M_r} (-1)^{|\alpha|} D^\alpha B^r_{\alpha}(x,D^\alpha u(x)), \ x \in \Omega,
\]

(6)

**Theorem 2.** Let the hypotheses \( A_1 \)- \( A_2 \) and (B) be satisfied. Let \( K \) be a closed convex Subset of \( \{W^\infty_0 (a_\alpha p_\alpha)(\Omega)\}^m \) with \( 0 \in K \). Let \( f \in \{W^{\infty_0} (a_\alpha p_\alpha')(\Omega)\}^m \) be given. Then there exists at least one solution \( u \in K \), such that

\[
(A'(u) + B'(u), v - u') \geq (f_r, v - u'), r \in \{1,2,\ldots,m\}
\]

(7)

**Proof.** We adopt the ideas of [6]. Consider the auxiliary Dirichlet problem of order 2k, which may be thought as the partial sum of the series (6):

\[
(A^r_{2k}(u_k), v - u_k') + (B'(u_k), v - u_k') \geq (f', v - u_k'), \ v \in K \cap W, \ r \in \{1,2,\ldots,m\}
\]

(8)

where

\[
A^r_{2k}(u_k)(x) = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha A^r_{\alpha}(x,D^r u_k(x)),
\]

\[
B'(u_k)(x) = \sum_{|\alpha|\leq M_r} (-1)^{|\alpha|} D^\alpha B^r_{\alpha}(x,D^\alpha (u_k)(x))
\]
and

\[ f_k^r = \sum_{|\alpha|=0}^k (-1)^{|\alpha|} a_\alpha r D^\alpha f_k^r \in W^{-k}(a_\alpha r, p'_\alpha)(\Omega) \]

The solvability of (8) in view of the hypotheses A_1- A_2) and B) is a consequence of theorem 1. Thus there exists \( u_k \in K \cap W \) solving (8).

One of the fundamental roles in finding the solution of (8) is played by the so called a priori estimates. By A_2) and B), we get

\[
\begin{align*}
& f_k^r (u_k) = \sum (-1)^{|\alpha|} a_\alpha r D^\alpha f_k^r \\
& \quad \in W^{-k}(a_\alpha r, p'_\alpha)(\Omega) \end{align*}
\]

Since \( u_k \in [W^k(a_\alpha p_\alpha)(\Omega)]^m \) we get from the compactness of \([W^k(a_\alpha p_\alpha)(\Omega)]^m \rightarrow [C(\Omega)]^m\), the uniform convergence of \( u_k(x) \rightarrow u(x) \) on \( \overline{\Omega} \) as \( k \rightarrow \infty \).

Similarly, by the compactness of \([W^k(a_\alpha p_\alpha)(\Omega)]^m \rightarrow [C^k(\Omega)]^m\), for large enough \( k \) and \( \ell \in \mathbb{N} \), we have \( D^\alpha u_k(x) \rightarrow D^\alpha u(x) \) uniformly on \( \overline{\Omega} \) as \( k \rightarrow \infty \).

By the definition of \([W^k_0(a_\alpha p_\alpha)(\Omega)]^m\), we get \( u \in [W^k_0(a_\alpha p_\alpha)(\Omega)]^m \) and by closedness of \( K, u \in K \). It remains to show that \( u \) is a solution of (7).

For this aim it suffices to prove the following assertions:

\[
\begin{align*}
\lim_{k \rightarrow \infty} (A^r_{2k} (u_k), z^r) &= (A^r(u), z^r) \quad \text{(9)} \\
\lim_{k \rightarrow \infty} (B^r(u_k), z^r) &= (B^r(u), z^r) \quad \text{(10)} \\
\liminf_{k \rightarrow \infty} (A^r_{2k} (u_k), u^r_k) &\geq (A^r(u), u^r) \quad \text{(11)} \\
\liminf_{k \rightarrow \infty} (B^r(u_k), u^r_k) &\geq (B^r(u), u^r) \quad \text{(12)}
\end{align*}
\]

for all \( z \in K, r = 1, 2, \ldots, m \).

As above, (9) and (10) are consequence of the uniform boundedness of

\[
\begin{align*}
\{ \sum_{|\alpha|=0}^k A^r_\alpha (x, D^\alpha u_k) D^\alpha u_k \}_{k \in \mathbb{N}}, \quad \{ \sum_{|\alpha|=0}^m B^r (x, D^\alpha u_k) D^\alpha u_k \}_{k \in \mathbb{N}}
\end{align*}
\]

and uniform equi-integrability of

\[
\begin{align*}
\{ \sum_{|\alpha|=0}^k A^r_\alpha (x, D^\alpha u_k) \}, \quad \{ \sum_{|\alpha|=0}^m B^r (x, D^\alpha u_k) \} \in [L^1(\Omega)]^m
\end{align*}
\]

in view of Vitali’s and dominated convergence theorems as well as (5). Assertions (11) and (12) are direct consequences of Fatou’s lemma and (5).

Example. As a particular example which can be handled by our result but fails outside the Scope of [4], we consider the nonlinear system
\[
\left( \sum_{j=0}^{\infty} \sum_{|\alpha|=j} (-1)^{|\alpha|} D^\alpha (a_{\alpha,1}|D^\alpha u_1|^{p_a-2} D^\alpha u_1) + h_1(x)|u_2|e^{\|u_2\|} \right)
\left( \sum_{j=0}^{\infty} \sum_{|\beta|=j} (-1)^{|\beta|} D^\beta (a_{\beta,2}|D^\beta u_2|^{p_a-2} D^\beta u_2) + h_2(x)|u_1|e^{\|u_1\|} \right)
\]

\((h_i(x))_{i=1}^2\) are arbitrary nonnegative \(L^1(x)\)-functions, \(u = (u_1, u_2)\).

**References**