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A Volume Preserving Map from Cube to Octahedron

By Adrian Holhoş
Universitatea Tehnică din Cluj-Napoca

Abstract- Using simple geometric reasoning we deduce a volume preserving map from the cube to the octahedron.

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A Volume Preserving Map from Cube to Octahedron

Adrian Holhoş

Abstract- Using simple geometric reasoning we deduce a volume preserving map from the cube to the octahedron.

I. Preliminaries

Consider the cube $\mathbb{C} = [-1,1]^3$ centered at the origin O and the regular octahedron $\mathbb{K}$ of the same volume, centered at O and with vertices on the coordinate axes

$$\mathbb{K} = \left\{(x,y,z) \in \mathbb{R}^3, |x| + |y| + |z| \le a\right\}.$$

Let L denote the edge of $\mathbb{K}$. Since the volume of the octahedron $\mathbb{K}$ is $\sqrt{2}L^3/3$, and this is equal to the volume of the cube $\mathbb{C}$, we have $8 = \sqrt{2}L^3/3$. Then, the distance from the origin to each vertex of $\mathbb{K}$ is

$$a = L/\sqrt{2} = \sqrt[3]{6}. \tag{1}$$

We will construct a map $\mathcal{U}\colon \mathbb{C} \to \mathbb{K}$ which preserves the volume, i.e.

$$Volume(D) = Volume(\mathcal{U}(D)), \qquad \text{for all } D \subseteq \mathbb{C},$$

where $Volume(D)$ denotes the volume of a domain D. For an arbitrary point $(x,y,z) \in \mathbb{C}$ we denote

$$(X,Y,Z) = \mathcal{U}(x,y,z) \in \mathbb{K}.$$

Author: Universitatea Tehnică din Cluj-Napoca. e-mail: adrian.holhos@math.utcluj.ro

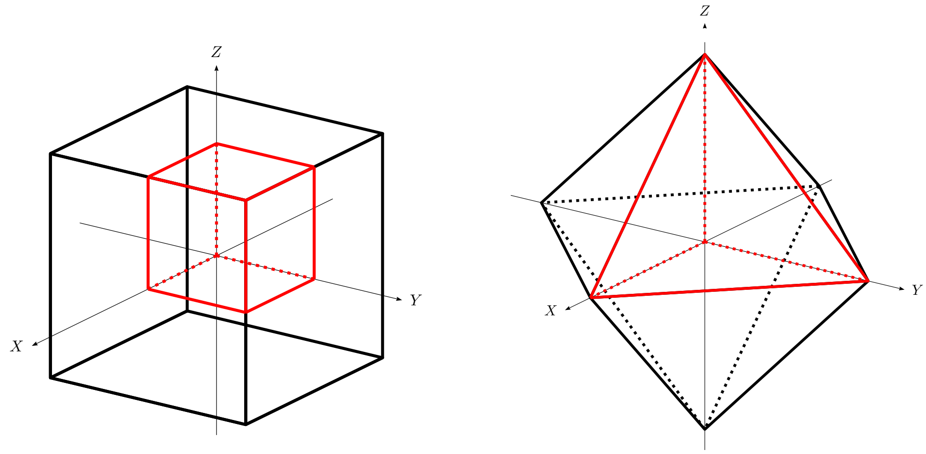

Figure 1: In the left, the cube $\mathbb{C}$ in black and the little cube $\mathbb{C}^1$ from the positive octant in red. In the right, the octahedron $\mathbb{K}$ in black and the tetrahedron K^1 in red

For the construction of $\mathcal{U}$, we split the cube into eight congruent cubes separated by the coordinate planes XOY, YOZ and ZOX, and thus, the construction of $\mathcal{U}$ can be reduced to the construction of its restriction to one of these cubes. We will denote $\mathbb{C}^1$ the eight part of $\mathbb{C}$ situated in the positive octant. We will denote by $\mathbb{K}^1$ the part of $\mathbb{K}$ situated in the positive octant. The map $\mathcal{U}$ will be constructed in such a way that $\mathbb{C}^1$ will be mapped in $\mathbb{K}^1$ and all the other seven cubes of $\mathbb{C}$ will be mapped to the corresponding tetrahedrons of $\mathbb{K}$.

II. Construction of the Volume Preserving Map $\mathcal{U}$

We focus on the region $\mathbb{C}^1$ of $\mathbb{C}$ situated in the positive octant

$$I_0^+ = \{(x, y, z) \in \mathbb{R}^3,\ x \geq 0,\ y \geq 0,\ z \geq 0\},$$

and we denote the vertices of the cube $\mathbb{C}^1$ as follows: $A = (1, 0, 0)$, $B = (1, 1, 0)$, $C = (0, 1, 0)$, $D = (0, 1, 1)$, $E = (0, 0, 1)$, $F = (1, 0, 1)$ and $G = (1, 1, 1)$, see Figure 2 (left). We also consider the following points in $\mathbb{K}^1 = \mathbb{K} \cap I_0$: $A' = (a, 0, 0)$, $B' = (a/2, a/2, 0)$, $C' = (0, a, 0)$, $D' = (0, a/2, a/2)$,

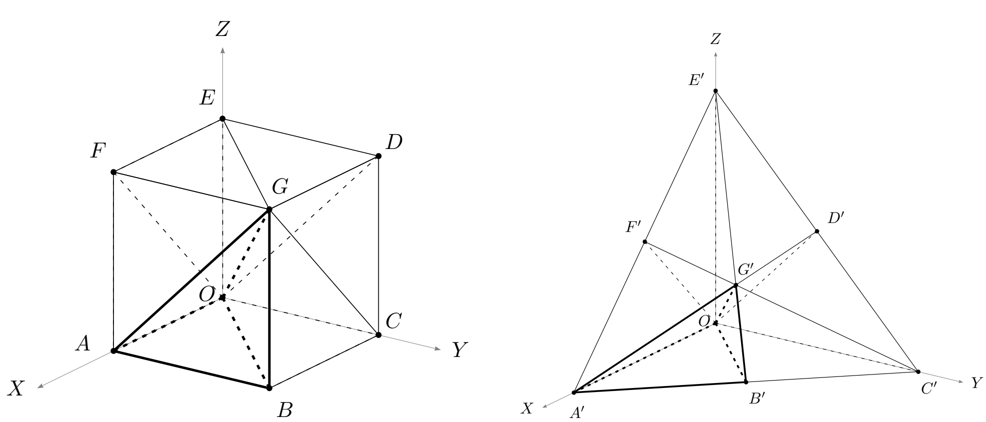

Figure 2: The cubical region $\mathbb{C}^1$ and its image $\mathbb{K}^1$

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$E' = (0,0,a)$, $F = (a/2, 0, a/2)$ and $G' = (a/3, a/3, a/3)$, see Figure 2 (right). We split the region $\mathbb{C}^1$ into six tetrahedrons of equal volume:

$$\begin{aligned}
OABG &= \{(x,y,z) \in I_0,\ 1 \geq x \geq y \geq z \geq 0\},\\
OBCG &= \{(x,y,z) \in I_0,\ 1 \geq y \geq x \geq z \geq 0\},\\
OCDG &= \{(x,y,z) \in I_0,\ 1 \geq y \geq z \geq x \geq 0\},\\
ODEG &= \{(x,y,z) \in I_0,\ 1 \geq z \geq y \geq x \geq 0\},\\
OEFG &= \{(x,y,z) \in I_0,\ 1 \geq z \geq x \geq y \geq 0\},\\
OFAG &= \{(x,y,z) \in I_0,\ 1 \geq x \geq z \geq y \geq 0\}.
\end{aligned}$$

They will be mapped onto the tetrahedrons $OA'B'G'$, $OB'C'G'$, $OC'D'G'$, $OD'E'G'$, $OE'F'G'$ and $OF'A'G'$, respectively.

We focus on the region $OABG$ and the corresponding region $OA'B'G'$

$$OA'B'G' = \{(X,Y,Z) \in \mathbb{R}^3 \,|\, 0 \leq Z \leq Y \leq X,\ X+Y+Z \leq a\}.$$

Consider a point $M(x,y,z)$ in $OABG$ and the corresponding point $M'(X,Y,Z)$ through the map $\mathcal{U}$. Consider the plane π_1 through M paralel with the plane ABG and P the point of intersection of π_1 with OA. Suppose this plane is mapped in the plane π_1' through M'paralel with $A'B'G'$ and P' is the corresponding point on OA', the intersection of the plane π_1' with OA'.

The ratio of the volumes of the two pyramids with the same vertex O and the bases on the planes π_1 and ABG must be equal with the ratio of the volumes of the two pyramids with the same vertex O and the bases on the planes π_1' and $A'B'G'$ and equal with the cube of the ratio OP/OA and equal with the cube of the ratio OP'/OA'. We obtain

$$OP' = OA' \cdot OP,$$

which is equivalent with

$$X + Y + Z = ax. \tag{2}$$

Consider the plane π_2 through M paralel with the plane OBG and Q the point of intersection of π_2 with OA. Suppose this plane is mapped in the plane π_2' through M'paralel with $OB'G'$ and Q' is the corresponding point on OA', the intersection of the plane π_2' with OA'.

The ratio of the volumes of the two pyramids with the same vertex A and the bases on the planes π_2 and OBG must be equal with the ratio of the volumes of the two pyramids with the same vertex A' and the bases on the planes π_2' and $OB'G'$ and equal with the cube of the ratio AQ/AO and equal with the cube of the ratio $A'Q'/A'O$. We obtain

$$A'Q' = OA' \cdot AQ,$$

which is equivalent with $OQ' = a \cdot OQ$. We obtain

$$X - Y = a(x - y). \tag{3}$$

Consider the plane π_3 through M paralel with the plane OAG and R the point of intersection of π_3 with AB. Suppose this plane is mapped in the plane π_3' through M'paralel with $OA'G'$ and R' is the corresponding point on $A'B'$, the intersection of the plane π_3' with $A'B'$.

The ratio of the volumes of the two pyramids with the same vertex B and the bases on the planes π_3 and OAG must be equal with the ratio of the volumes of the two pyramids with the same vertex B' and the bases on the planes π_3' and $OA'G'$ and equal with the cube of the ratio BR/BA and equal with the cube of the ratio $B'R'/B'A'$. We obtain

$$B'R' = A'B' \cdot BR,$$

which is equivalent with $A'R' = \frac{a\sqrt{2}}{2} \cdot AR$. We obtain

$$Y - Z = \frac{a\sqrt{2}}{2} \cdot \frac{(y-z)\sqrt{2}}{2} = \frac{a}{2}(y-z). \tag{4}$$

Solving the system of the three equations (2), (3) and (4), we obtain

$$\begin{aligned} X &= ax - \frac{a}{2}y - \frac{a}{6}z \\ Y &= \frac{a}{2}y - \frac{a}{6}z \\ Z &= \frac{a}{3}z, \end{aligned}$$

where the value of a is specified by (1).

More general, the equations for all eight octants can be obtain in the following way: let

$$\begin{aligned} M &= \max(|x|,|y|,|z|) & M' &= \max(|X|,|Y|,|Z|) \\ m &= \min(|x|,|y|,|z|) & m' &= \min(|X|,|Y|,|Z|) \\ c &= |x|+|y|+|z|-M-m & c' &= |X|+|Y|+|Z|-M'-m'. \end{aligned}$$

Using the same reasoning we obtain the following system of equations

$$\begin{aligned} |X|+|Y|+|Z| &= a \cdot M \\ M' - c' &= a \cdot (M - c) \\ c' - m' &= \frac{a}{2} \cdot (c - m) \end{aligned}$$

with the solution

$$\begin{aligned} M' &= aM - \frac{a}{2}c - \frac{a}{6}m \\ c' &= \frac{a}{2}c - \frac{a}{6}m \\ m' &= \frac{a}{3}m. \end{aligned}$$

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III. Applications

A volume preserving map can be useful in some statistics applications and for a decomposition of a 3D solid into smaller elements of equal volume.

For example, if we start with a uniform distribution of points in the solid cube we can obtain a uniform distribution of points in the solid octahedron (see [1] and the references therein, for a similar application in the case of planar domains).

For a cube is easy to obtain a uniform grid, a grid with all the elements having the same volume. By applying the map $\mathcal{U}$ we get a uniform grid for the octahedron. Using the map constructed in [2] we obtain a uniform grid for the ball. This method to obtain a uniform grid in the ball is different than the method described in [3].

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