Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 3 Version 1.0 Year 2018
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Certain Study of Bicomplex Matrices and a New Composition of Bicomplex Matrices

By Prabhat Kumar & Akhil Prakash
Dr. B. R. Ambedkar University

Abstract- In this paper, we have studied Orthogonal and Unitary matrix in C_2, some theorems and properties related to bicomplex matrix. We have defined the new concept over the bicomplex matrix, relation between bicomplex matrix and its complex component matrix, algebraic structure of bicomplex matrix in new system as well as the new definition of inverse of matrix in C_2 and some properties in new system. A similar relation between two bicomplex matrices is also defined in this paper.

Keywords: orthogonal bicomplex matrix, unitary bicomplex matrices, algebraic structure in new system, tranjugate matrix, inverse.

GJSFR-F Classification: MSC 2010: 15 B 57, 30 G 35

CERTAINSTUDYOFBICOMPLEXMATRICESANDANEWCOMPOSITIONOFBICOMPLEXMATRICES

Strictly as per the compliance and regulations of:

Certain Study of Bicomplex Matrices and a New Composition of Bicomplex Matrices

Prabhat Kumar [α] & Akhil Prakash [σ]

Abstract- In this paper, we have studied Orthogonal and Unitary matrix in C_2, some theorems and properties related to bicomplex matrix. We have defined the new concept over the bicomplex matrix, relation between bicomplex matrix and its complex component matrix, algebraic structure of bicomplex matrix in new system as well as the new definition of inverse of matrix in C_2 and some properties in new system. A similar relation between two bicomplex matrices is also defined in this paper.

Keywords: orthogonal bicomplex matrix, unitary bicomplex matrices, algebraic structure in new system, tranjugate matrix, inverse.

I. Introduction

In 1892, Corrado Segre (1860-1924) published a paper [9] in which he treated an infinite set of Algebras whose elements he called bicomplex numbers, tricomplex numbers,…., n-complex numbers. A number which can expressed in the form of $x_1+i_1x_2+i_2x_3+i_1i_2x_4$, $i_p^2 = -1$,for all p=1, 2 and $i_1i_2 = i_2i_1$ as well as $x_1,\ldots,x_4$ are real numbers, is called a bicomplex number. Segre showed that every bicomplex number $z_1+i_2z_2$ can be represented as the complex combination

$$(z_1\text{-}i_1z_2)\left[\frac{1+i_1i_2}{2}\right]+(z_1+i_1z_2)\left[\frac{1-i_1i_2}{2}\right]$$

Shrivastava [10] introduced the notations $^1\xi$ and $^2\xi$ for the idempotent components of the bicomplex number $\xi = z_1+i_2z_2$, so that

$$\xi = {}^1\xi.\frac{1+i_1i_2}{2}+{}^2\xi.\frac{1-i_1i_2}{2}$$

Michiji Futagawa seems to have been the first to consider the theory of functions of a bicomplex variable [2, 3] in 1928 and 1932.

The hyper complex system of Ringleb [8] is more general than the Algebras; he showed in 1933 that Futagawa system is a special case of his own.

In 1953 James D. Riley published a paper [7] entitled "Contributions to theory of functions of a bicomplex variable".

In the entire work, the symbols C_2, C_1 and C_0 denote the set of all bicomplex, complex and real numbers respectively.

In C_2-besides 0 and 1- there are exactly two non-trivial idempotent elements denoted as e_1 and e_2 and defined as

Author: Department of Mathematics, Institute of basic Science, Khandari, Dr. B. R. Ambedkar University, Agra-282002, India.
e-mails: prabhatchaudhary553@gmail.com, akhil.sharma140@gmail.com

Ref

8. Ringleb, F.: Beiträge Zur Funktionentheorie in hyperkomlexon System, I, Rend. Circ. Mat. Palermo, 57 (1933) 311-340

$$e_1 = \frac{1+i_1 i_2}{2} \text{ and } e_2 = \frac{1-i_1 i_2}{2}$$

Obviously $(e_1)^n = e_1$, $(e_2)^n = e_2$

$$e_1 + e_2 = 1,\ e_1.e_2 = 0$$

Every bicomplex number ξ has unique idempotent representation as complex combination of e_1 and e_2 as follows

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$$

The complex numbers $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called idempotent component of ξ, and are denoted by ${}^1\xi$ and ${}^2\xi$ respectively (cf. Srivastava [10]).

Thus $\xi = {}^1\xi\ e_1 + {}^2\xi\ e_2$

The idempotent representation is perfectly consistent with the Algebraic structure of C_2 in the following sense

$$\xi \pm \eta = ({}^1\xi\ e_1 + {}^2\xi\ e_2) \pm ({}^1\eta\ e_1 + {}^2\eta\ e_2)$$
$$= ({}^1\xi \pm {}^1\eta)\ e_1 + ({}^2\xi \pm {}^2\eta)\ e_2$$

So that ${}^1(\xi \pm \eta) = {}^1\xi \pm {}^1\eta$ and ${}^2(\xi \pm \eta) = {}^2\xi \pm {}^2\eta$

$$a.\xi = a.\ ({}^1\xi\ e_1 + {}^2\xi\ e_2)$$
$$= (a.{}^1\xi)\ e_1 + (a.{}^2\xi)\ e_2$$

So that ${}^1(a.\xi) = a.{}^1\xi$ and ${}^2(a.\xi) = a.{}^2\xi$

$$\xi.\eta = ({}^1\xi\ e_1 + {}^2\xi\ e_2).\ ({}^1\eta\ e_1 + {}^2\eta\ e_2)$$
$$= ({}^1\xi.{}^1\eta)\ e_1 + ({}^2\xi.{}^2\eta)\ e_2$$

So that ${}^1(\xi.\eta) = {}^1\xi.{}^1\eta$ and ${}^2(\xi.\eta) = {}^2\xi.{}^2\eta$

$\xi / \eta = ({}^1\xi / {}^1\eta)\ e_1 + ({}^2\xi / {}^2\eta)\ e_2$; provided $\eta \notin O_2$

So that ${}^1(\xi / \eta) = {}^1\xi / {}^1\eta$ and ${}^2(\xi / \eta) = {}^2\xi / {}^2\eta$,

where O_2 = set of all singular element in C_2

a) Singular elements and Norm of a bicomplex number

There are infinite numbers of element in C_2 which do not possess multiplicative inverse. A bicomplex number $\xi = z_1 + i_2 z_2$ is singular iff $\left|z_1^2 + z_2^2\right| = 0$. Evidently a nonzero bicomplex number ξ is singular if and only if either ${}^1\xi = 0$ or ${}^2\xi = 0$. In fact C_2 is not a field while C_1 is a field.

The norm of a bicomplex number ξ is defined as

$$\|\xi\| = \|z_1 + i_2 z_2\|$$
$$= \left\{|z_1|^2 + |z_2|^2\right\}^{1/2}$$
$$= \{\tfrac{1}{2}(|{}^1\xi|^2 + |{}^2\xi|^2)\}^{1/2}$$

C_2 forms a modified Banach algebra. i.e. Banach algebra with modified consistency of the norm of product of two bicomplex number is less than or equal to $\sqrt{2}$ time of product of their individual norm i.e. $\|\xi\eta\| \le \sqrt{2}\|\xi\|\|\eta\|$

Ref

10. Srivastava, Rajiv K.: Bicomplex Numbers: Analysis and applications, Math. Student, 72 (1-4) 2003, 69-87.

Notes

b) Some special properties and subsets of bicomplex space

Every bicomplex number ξ possesses three types of conjugates called i_1-conjugate, i_2-conjugate and i_1i_2-conjugate corresponding to i_1, i_2 and i_1i_2 independent vectors respectively represented by $\bar{\xi}, \xi^{\sim}$ and $\xi^{\#}$. Thus

$$\bar{\xi} = (x_1 - i_1x_2) + i_2(x_3 - i_1x_4) = \bar{z}_1 + i_2\bar{z}_2 = ({}^2\bar{\xi})e_1 + ({}^1\bar{\xi})e_2$$

$$\xi^{\sim} = (x_1 + i_1x_2) - i_2(x_3 + i_1x_4) = z_1 - i_2z_2 = ({}^2\xi)e_1 + ({}^1\xi)e_2$$

$$\xi^{\#} = (x_1 - i_1x_2) - i_2(x_3 - i_1x_4) = \bar{z}_1 - i_2\bar{z}_2 = ({}^1\bar{\xi})e_1 + ({}^2\bar{\xi})e_2$$

We shall use specific notations for some special subset of C_2 that are given below.

$$C(i_1) = \{a + i_1b: a, b \in C_0\}$$

$$C(i_2) = \{a + i_2b: a, b \in C_0\}$$

$$H = \{a + i_1i_2b: a, b \in C_0\}$$

c) Representation of bicomplex matrix

A matrix 'A' whose entries are bicomplex numbers is called bicomplex matrix i.e.

$$A = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn} \end{bmatrix}, \ \forall \xi_{pq} \text{ in } C_2,\ 1 \le p \le m \text{ and } 1 \le q \le n$$

According to three types of representation of a bicomplex number, there are three types of representation of a bicomplex matrix as real representation, complex representation and idempotent representation.

A square bicomplex matrix "A" is said to be non-singular if $|A| \notin O_2$ otherwise the matrix will be singular.

II. Certain Results on Bicomplex Matrices

a) Algebraic structure and Inversion of Bicomplex matrices[1]

2.1.1 Algebraic structure

Let M be the set of all square and non-singular bicomplex matrices of order n then the set M with operations addition "+" coordinate wise, multiplication "×" is term by term multiplication as well as scalar multiplication "." is also coordinate wise, forms an algebra over the field of complex number.

2.1.2 Determinant and Adjoint of a bicomplex matrix

Let A= $[\xi_{ij}]_{n\times n}$ be the bicomplex square matrix of order n where n is the positive integer. The determinant of A is defined by

$$|A| = |[\xi_{ij}]|,\ \xi_{ij} \in C_2$$

$$= \sum_{j^i=1}^{n} \pm \xi_{1j^1}\xi_{2j^2}\cdots\xi_{nj^n}$$

where ± sign is taken according to even and odd permutation of suffixes of ξ.

Global Journal of Science Frontier Research (F) Volume XVIII Issue III Version I Year 2018

Let A = $[\xi_{ij}]_{n\times n}$ be a bicomplex square matrix and $[\zeta_{ij}]_{n\times n}$ denote the co-factor matrix of A then the transpose of the matrix $[\zeta_{ij}]_{n\times n}$ is defined as Adjoint of A and denoted by Adj.A.

Some Results-

(a) $|A| = |{}^1A|\, e_1 + |{}^2A|\, e_2$

(b) If $|A| \notin O_2 \Leftrightarrow |{}^1A| \neq 0 \;\&\; |{}^2A| \neq 0$

(c) Adj.A = Adj.(^{1}A) e_1 + Adj.(^{2}A) e_2

2.1.3 Inversion of Bicomplex matrix by two techniques

Anjali [1] has developed two techniques to determine the inverse of bicomplex matrix.

a. *Adjoint technique*

Let A = $[\xi_{ij}]_{n\times n}$ be a square and non-singular matrix whose elements are in C_2 then Inverse of A is defined as

$$A^{-1} = \frac{A.(AdjA)}{|A|}$$

b. *Idempotent technique*

Suppose M = ^{1}M e_1 + ^{2}M e_2=$[\xi_{ij}]_{n\times n}$ be a square and nonsingular bicomplex matrix of order n. Let $[z_{ij}]_{n\times n}$ and $[w_{ij}]_{n\times n}$be the inverse of ^{1}M and ^{2}M respectively then Inverse of M is defined as

$$M^{-1} = [z_{ij}]_{n\times n}\, e_1 + [w_{ij}]_{n\times n}\, e_2 = [\eta_{ij}]_{n\times n} \text{ (say)}$$

b) Hermitian and Skew-Hermitian matrix in C_2 [1]

2.2.1 Tranjugate of a bicomplex matrix

Analogous to three types of conjugate element in C_2 we have three types of conjugate of a matrix in C_2 viz. are i_1 conjugate matrix, i_2 conjugate matrix and i_1i_2 conjugate matrix. The transpose of the conjugate matrix is called tranjugate of the matrix. There are three types of tranjugates of a matrix in C_2.

a. *i_1 tranjugate of a bicomplex matrix*

Let A = $[a_{ij}]_{n\times n}$ be any bicomplex matrix and $\bar{A}$ denotes the i_1 conjugate of A obtained by taking i_1 conjugate of each entry of A. Transposing $\bar{A}$, we get the tranjugate of A. Simply denoted by $[\bar{A}]^T$ or A^{θ_1}

b. *i_2 tranjugate of a bicomplex matrix*

The i_2 conjugate of a bicomplex matrix A denoted by $\tilde{A}$ is the matrix obtained by taking i_2 conjugate of each entry of A. On taking transpose of $\tilde{A}$ then we obtain $[\tilde{A}]^T$ which is known as i_2 tranjugate of A and denoted by A^{θ_2}.

c. *i_1i_2 tranjugate of a bicomplex matrix*

The i_1i_2 conjugate of a bicomplex matrix A denoted by $A^{\#}$ is the matrix obtained from A by taking i_1i_2 conjugate of each entry of A. On taking transpose of $A^{\#}$, we obtain $[A^{\#}]^T$ which is known as i_1i_2 tranjugate of A and denoted by A^{θ_3}.

Properties of a bicomplex matrix [1]-

For all 'k' in C_2 and A, B of $C_2^{n\times n}$ then

(1) $\overline{[\bar{A}]} = A$

Ref

1. Anjali: Certain results on bicomplex matrices, M. Phil. Dissertation, Dr. B. R. Ambedker University, Agra (2011).

Notes

(2) $[\tilde{A}]^{\sim} = A$

(3) $[A^{\#}]^{\#} = A$

(4) $(\overline{A+B}) = \bar{A} + \bar{B}$

(5) $(A+B)^{\tilde{}} = A^{\tilde{}} + B^{\tilde{}}$

(6) $(A+B)^{\#} = A^{\#} + B^{\#}$

(7) $\overline{kA} = \bar{k}\bar{A}$

(8) $[k\ A]^{\tilde{}} = k^{\tilde{}}.A^{\tilde{}}$

(9) $[k\ A]^{\#} = k^{\#}.A^{\#}$

(10) $[\overline{[(\bar{A})^T]}]^T = A$

(11) $[[(\tilde{A})^T]^{\sim}]^T = A$

(12) $[[[A^{\#}]^T]^{\#}]^T = A$

(13) $(\overline{kA})^T = \bar{k}.[\bar{A}]^T$

(14) $[[kA]^{\tilde{}}]^T = k^{\tilde{}}.[A^{\tilde{}}]^T$

(15) $[[kA]^{\#}]^T = k^{\#}.[A^{\#}]^T$

2.2.2 Symmetric and Skew-symmetric matrix in C_2 [1]

A square bicomplex matrix "A" is symmetric if $A^T = A$ or $a_{ij} = a_{ji}$ for all i, j and if $A^T = -A$ or $a_{ij} = -a_{ji}$ for all i, j then it is called a skew symmetric matrix.
In skew symmetric matrix all principal diagonal elements are zero.

2.2.3 Hermitian and Skew-Hermitian matrix in C_2

Since three types of conjugate elements exist in C_2 and each conjugate will introduce Hermitian matrix, so that in C_2, there will be three types of Hermitian matrices.

a. *i_1-Hermitian matrix*

A bicomplex square matrix A is said to be i_1-Hermitian matrix if $A = [\bar{A}]^T$.
The element of the principal diagonal of i_1-Hermitian matrix are the member of $C(i_2)$ i.e. i_2-complex number.

b. *i_2-Hermitian matrix*

A bicomplex square matrix A is said to be i_2-Hermitian matrix if $A = [\tilde{A}]^T$.

The element of the principal diagonal of i_2-Hermitian matrix are the member of $C(i_1)$ i.e. i_1-complex number.

c. *i_1i_2-Hermitian matrix*

A bicomplex square matrix A is said to be i_1i_2-Hermitian matrix if $A = [A^{\#}]^T$.

The elements of the principal diagonal of i_1i_2-Hermitian matrix are the member of H (set of hyperbolic numbers).
There are three types of skew Hermitian matrix in C_2.

- *i_1-Skew Hermitian matrix*

A bicomplex square matrix A is said to be i_1-skew Hermitian matrix If $A = -[\bar{A}]^T$.

The element of the principal diagonal of i_1-skew Hermitian matrix are the member of the type $i_1(s)$, $s \in C(i_2)$.

- *i_2-Skew Hermitian matrix*

A bicomplex square matrix A is said to be i_2-skew Hermitian matrix if $A = -[\tilde{A}]^T$.

The elements of the principal diagonal of i_2-skew Hermitian matrix are the member of the type $i_2(s)$, $s \in C(i_1)$.

➢ *i_1i_2-Skew Hermitian matrix*

A bicomplex square matrix A is said to be i_1i_2 - skew Hermitian matrix if $A = -[A^{\#}]^T$ or $(A^{\#})^T = -A$.

The elements of the principal diagonal of i_1i_2 - skew Hermitian matrix are the member of the type $i_1(s)$, $s \in H$

Notes

III. Study of Bicomplex Matrix under Traditional and New System

In this section we present the work which has been done by us. In this section we have studied Orthogonal and Unitary matrix in C_2 and defined the new concept over the bicomplex matrix. A similar relation between two bicomplex matrices is also defined in this section.

a) Orthogonal and Unitary Bicomplex matrices

3.1.1. Orthogonal Bicomplex matrix

Let A be any square and invertible bicomplex matrix then A is said to be orthogonal bicomplex matrix if

$$A^T A = I = A A^T$$

i.e .$A^{-1} = A^T$
where A^T is the transpose of A and I is the identity matrix.

3.1.2 Unitary bicomplex matrices

Corresponding to three types of tranjugate of any bicomplex matrix, there are three types of bicomplex Unitary matrix.

a. *i_1 Unitary matrix*

Let A be any square bicomplex matrix which is invertible and $\bar{A}$ denote the i_1 conjugate of A and $[\bar{A}]^T$ is the transpose of i_1 conjugate of A. We shall use A^{θ_1} in place of $[\bar{A}]^T$ in entire work.

The matrix A is called i_1Unitary matrix if $A^{\theta_1} A = I = A A^{\theta_1}$

i.e. $A^{-1} = A^{\theta_1}$

Thus, the matrix $[\zeta_{ij}]_{n\times n}$ is an i_1 Unitary matrix if

$[\overline{\zeta_{ji}}]_{n\times n} . [\zeta_{ij}]_{n\times n} = I_{n\times n}$,where $I_{n\times n}$is the identity matrix.

b. *i_2 Unitary matrix*

Let A be any square bicomplex matrix which is invertible and $A^{\sim}$ be the i_2 conjugate of A and $[A^{\sim}]^T$ be transpose of i_2 conjugate matrix of A and we shall use A^{θ_2} in place of $[A^{\sim}]^T$ in entire work.

If $A A^{\theta_2} = I = A^{\theta_2} A$

ie. $A^{-1} = A^{\theta_2}$

Then A is called i_2Unitary matrix.

c. *i_1i_2 Unitary matrix*

Let A be any square bicomplex matrix which is invertible and $A^{\#}$ be the i_1i_2 conjugate of A and $[A^{\#}]^T$ be transpose of i_1i_2 conjugate matrix of A. We shall use A^{θ_3} in place of $[A^{\#}]^T$in entire work. If $A A^{\theta_3} = I = A^{\theta_3} A$

i.e. $A^{-1} = A^{\theta_3}$

then A is called $i_1 i_2$ Unitary matrix.

Remark: If A is any real Unitary matrix then it will obviously be an i_1 Unitary matrix, i_2 Unitary matrix and $i_1 i_2$ Unitary matrix.

3.1.3 Theorem

If A and B are two $i_1 i_2$ Unitary matrices of same order then AB will be $i_1 i_2$ Unitary matrix similarly i_1AB, i_2AB, $i_1 i_2$AB will also be $i_1 i_2$ Unitary matrices.

Notes

Proof:

By the definition of $i_1 i_2$Unitary matrix

$A^{\theta_3}.A = I$, similarly $B^{\theta_3}.B = I$

$$\begin{aligned}(AB)^{\theta_3}.AB &= \{{}^1(AB)e_1 + {}^2(AB)e_2\}^{\theta_3}.AB \\ &= \{{}^1(AB)^{\theta_3} e_1 + {}^2(AB)^{\theta_3} e_2\}.AB \\ &= \left[{}^1(B^{\theta_3} A^{\theta_3})e_1 + {}^2(B^{\theta_3} A^{\theta_3})e_2\right]AB \\ &= (B^{\theta_3} A^{\theta_3}).AB \\ &= B^{\theta_3} A^{\theta_3}.AB \\ &= B^{\theta_3}(A^{\theta_3}.A)B \\ &= B^{\theta_3}.I.B \qquad (\because A \text{ is } i_1 i_2 \text{ unitary}) \\ &= B^{\theta_3}.B = I \qquad (\because B \text{ is } i_1 i_2 \text{ unitary})\end{aligned}$$

Now we find out the nature of i_1AB, i_2AB, $i_1 i_2$AB

Therefore

$$\left.\begin{aligned}(i_1 AB)^{\theta_3}.i_1 AB &= \bar{i}_1 B^{\theta_3} A^{\theta_3}.i_1 AB \\ &= B^{\theta_3} IB \\ &= I\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}(i_2 AB)^{\theta_3}.i_2 AB &= \bar{i}_2 B^{\theta_3} A^{\theta_3}.i_2 AB \\ &= B^{\theta_3} IB \\ &= I\end{aligned}\right\} \quad (2)$$

$$\left.\begin{aligned}(i_1 i_2 AB)^{\theta_3}.i_1 i_2 AB &= \bar{i}_1 \bar{i}_2 B^{\theta_3} A^{\theta_3}.i_1 i_2 AB \\ &= (-i_1)(-i_2), i_1 i_2 B^{\theta_3} IB \\ &= I\end{aligned}\right\} \quad (3)$$

Proof of the theorem is complete.

Remark:

$$(AB)^{\theta_S} \neq B^{\theta_S} A^{\theta_S}, \quad where \quad S = 1,2$$

Therefore analogues of theorem 3.1.3 is not true for i_1Unitary and i_2Unitary matrices.

3.1.4 Theorem

The adjoint of $i_1 i_2$ tranjugate of a bicomplex square matrix is equal to the $i_1 i_2$ tranjugate of the adjoint of the matrix.

$$Adj(A^{\theta_3}) = (Adj\, A)^{\theta_3}$$

Proof:

$$(Adj\ A)^{\theta_3} = (Adj\ ^1A\, e_1 + Adj\ ^2A\, e_2)^{\theta_3} [by\ 2.1.2(c)]$$
$$= (Adj\ ^1A)^{\theta_3} e_1 + (Adj\ ^2A)^{\theta_3} e_2$$
$$= Adj\,(^1A^{\theta_3})e_1 + Adj\,(^2A^{\theta_3})e_2$$
$$\because (Adj\ A)^{\theta} = Adj\,(A^{\theta}),\ for\ A\ in\ C_1\ therefore$$
$$(Adj\ A)^{\theta_3} = Adj\,[(^1A^{\theta_3})e_1 + (^2A^{\theta_3})e_2]$$
$$= Adj\,(A^{\theta_3})$$

Remark:

Since i_1 and i_2 conjugates of e_1 is e_2 and e_2 is e_1, we get $(Adj\ A)^{\theta_S} \neq Adj\,(A^{\theta_S})\ \ where\ S = 1,2$

3.1.5 Theorem

Let A and B be two square bicomplex matrices of order n, such that $|A| \notin O_2$ and $|B| \notin O_2$, then their product (AB) will be invertible, and the inverse of AB will be $B^{-1}A^{-1}$.

Proof:

Since the bicomplex matrices A and B both are nonsingular i.e.$|A| \notin O_2$ and $|B| \notin O_2$, that means

$$|A| = |^1A|e_1 + |^2A|e_2 \notin O_2$$
$$\Leftrightarrow |^1A| \neq 0\ and\ |^2A| \neq 0$$
$$similarly\ |^1B| \neq 0\ and\ |^2B| \neq 0$$
$$\Rightarrow |^1A|.|^1B| \neq 0\ and\ |^2A|.|^2B| \neq 0$$
$$\Rightarrow (|^1A||^1B|)e_1 + (|^2A||^2B|)e_2 \notin O_2$$
$$\Rightarrow |^1(AB)|e_1 + |^2(AB)|e_2 \notin O_2$$
$$\Rightarrow |AB| \notin O_2$$

⇒ AB is invertible.

The inverse of the matrix (AB) is $(AB)^{-1}$therefore

Further

$$(AB)^{-1} = [^1(AB)e_1 + ^2(AB)e_2]^{-1}$$
$$= {}^1(AB)^{-1}e_1 + {}^2(AB)^{-1}e_2$$
$$= (^1B^{-1}\, ^1A^{-1})e_1 + (^2B^{-1}\, ^2A^{-1})e_2\ (\text{since } (PQ)^{-1} = Q^{-1}P^{-1} \text{ in } C_1)$$
$$= [^1B^{-1}e_1 + ^2B^{-1}e_2]\,[^1A^{-1}e_1 + ^2A^{-1}e_2]$$
$$= B^{-1}A^{-1}$$

3.1.6 Theorem

Let A, B be two square bicomplex matrices then determinant of their product will be equal to product of their individual determinant.

Proof:

$$|AB| = |^1(AB)|e_1 + |^2(AB)|e_2$$
$$= (|^1A||^1B|)e_1 + (|^2A||^2B|)e_2$$
$$= (|^1A|e_1 + |^2A|e_2)\ (|^1B|e_1 + |^2B|e_2)$$
$$\Rightarrow |AB| = |A|.|B|$$

Notes

Notes

3.1.7 Some Properties of bicomplex matrices

(i) If A is any bicomplex square matrix of order n then det A and the det of transpose A are equal.

(ii) A and B are two bicomplex matrices of order n such that B is obtained from interchanging any two row /column only of A then $|A| = -|B|$.

(iii) If any one of the row/column in a square bicomplex matrix has each element in O_2 then matrix will be singular or non - invertible.

Proofs of these results are straight forward.

b) Study under new system

If A is any bicomplex matrix, then it can be written as

$$A = A_0 + i_2 A_1, \text{ where } A_s \in C_1^{m\times n}, \; s = 0,1$$

The i_2 independent part and dependent part of bicomplex matrix A is denoted by A_0 and A_1 respectively. The matrices A_0 and A_1 are known as complex component of matrix A.

We define a new binary composition "ʘ" between two arbitrary square biocomplex matrices A and B as follow

$$\forall \; A, B \in C_2^{n\times n} \text{ then}$$

$$A \odot B = (A_0 + i_2 A_1) \odot (B_0 + i_2 B_1)$$

$$= (A_0 B_0 + i_2 A_1 B_1)$$

$$= (C_0 + i_2 C_1) \in C_2^{n\times n}, \text{ where } C_s \in C_1^{n\times n} \; \forall \; s = 0, 1$$

It is a new definition of product of two bicomplex matrices (specially) and the procedures of both, addition and scalar multiplication, will be the same as traditional system procedures.

Thus the three operations will be as follow

$$\forall A, B \in C_2^{n\times n}$$

$$\text{"+"} \rightarrow A+B = [A_0 + i_2 B] + [B_0 + i_2 B_1]$$

$$= [A_0 + B_0] + i_2 [A_1 + B_1]$$

$$\text{"}\odot\text{"} \rightarrow A \odot B = [A_0 + i_2 A_1] \odot [B_0 +_2 B_1]$$

$$= A_0 B_0 + i_2 A_1 B_1$$

$$\text{and "}\bullet\text{"} \rightarrow \alpha . A = \alpha \, [A_0 + i_2 A_1]$$

$$= \alpha \, A_0 + i_2 \alpha \, A_1$$

$$\xi . \, A = [\alpha + i_2 \beta] . \, [A_0 + i_2 A_1]$$

$$= \alpha A_0 + i_2 \alpha A_1 - \beta A_1 + i_2 \beta A_0$$

$$= \alpha \, A_0 - \beta A_1 + i_2 \, [\alpha A_1 + \beta A_0]$$

3.2.1 Relation between the bicomplex matrix and its complex component matrices

We define addition on the set $C_1^{m\times n} \times C_1^{m\times n}$ as follow.

If (A_0, A_1) and (B_0, B_1) are two arbitrary element of $C_1^{m\times n} \times C_1^{m\times n}$ then $(A_0, A_1) + (B_0, B_1) = (A_0+B_0, A_1+B_1)$. Further (A_0, A_1) and (B_0, B_1) are said to be equal if and only if $A_0 = B_0$ and $A_1 = B_1$ The set $C_1^{m\times n} \times C_1^{m\times n}$ is an abelian group w.r.t. addition '+'.

We define a function $f : C_2^{m\times n} \to C_1^{m\times n} \times C_1^{m\times n}$

Such that $f(A) = (A_0, A_1)$

3.2.2 Theorem

If $f: C_2^{m\times n} \to C_1^{m\times n} \times C_1^{m\times n}$ is the function Such that $f(A) = (A_0, A_1)$ then f is an on to isomorphism i.e. $C_2^{m\times n} \cong C_1^{m\times n} \times C_1^{m\times n}$

Proof:

f is one-one:

$$\forall\ A, B \in C_2^{m\times n}$$

$$\text{Let}\quad f(A) = f(B)$$

$$\Rightarrow (A_0, A_1) = (B_0, B_1)$$

$$\Rightarrow A_0 = B_0 \text{ and } A_1 = B_1$$

$$\Rightarrow A = B$$

f is onto:

Let (A_0, A_1) be the arbitrary element of $C_1^{m\times n} \times C_1^{m\times n}$.

Corresponding to (A_0, A_1) there exist a bicomplex matrix $A = A_0+i_2A_1$ such that

$$f(A) = f(A_0+i_2A_1)$$

$$= (A_0, A_1)$$

Therefore A is the preimage of (A_0, A_1) in $C_2^{m\times n}$.

f is homomorphism:

$$\forall\ A, B \in C_2^{m\times n}$$

$$f(A+B) = f\,[(A_0+ B_0) + i_2\,(A_1+B_1)]$$

$$= (A_0+B_0,\ A_1+B_1)$$

$$= [(A_0, A_1) + (B_0, B_1)]$$

$$= f(A) + f(B)$$

Hence f is an on to isomorphism i.e. $C_2^{m\times n} \cong C_1^{m\times n} \times C_1^{m\times n}$

3.2.3 Theorem

Let M be the set of all square bicomplex matrix of order n. If we introduce the operation "⊙" with set M over new system and binary operation addition "+"taken coordinate wise and scalar multiplication "•" is term by term then the structure [M,"+","•","⊙"] forms an algebra with identity $(I + i_2\, I)$.

Proof:

$$\forall\ A, B, C \in M^{n\times n}$$

Notes

Notes

therefore $A= A_0+i_2A_1$, $B= B_0+ i_2B_1$, $C=C_0+i_2C_1$

(M,+) is an abelian group:

Closure:

$$A+B = [A_0+i_2A_1] + [B_0+ i_2B_1]$$
$$= [A_0+B_0] +i_2[A_1+B_1] \in M$$

Associativity:

$$A + (B + C) = (A + B) + C \text{ (Hold)}$$

Additive identity:

$$\forall\ A \in M,\ A = [A_0 + i_2A_1]\ \exists \text{an } (0 + i_20)$$
$$A + (0 + i_20) = [A_0 + i_2A_1] + [0 + i_20]$$
$$= [A_0 + 0] + i_2\ [A_1 + 0]$$
$$= A_0 + i_2A_1 = A$$

Hence $(0 + i_20)$ is the additive identity.

Inverse property:

$\forall\ A \in M,\ \exists\ -A\in M$, such that

$$[A_0+i_2A_1] - [A_0+ i_2A_1] = [A_0 - A_0] + i_2\ [A_1 - A_1] =[0 + i_20]$$

Commutativity:

$$A + B = B + A\ \ \forall\ A, B \in M$$

[M, +, ʘ] is ring structure:

Closure under new multiplication 'ʘ':

$$\forall\ A, B \in M$$
$$A \odot B = (A_0\ B_0) + i_2\ (A_1\ B_1) \in M$$

Associativity:

$$A \odot [B \odot C] = [A_0+i_2A_1] \odot [(B_0\ C_0) + i_2\ (B_1\ C_1)]$$
$$= A_0\ [B_0\ C_0] + i_2\ A_1\ [B_1\ C_1]$$

$\because$ Complex matrix are associative and A_s, B_S, C_S in $C_1^{n \times n}$, $\forall\ S = 0, 1$

$$A \odot [B \odot C] = [A_0\ B_0]\ C_0 + i_2\ [A_1\ B_1]\ C_1$$
$$= [(A_0+i_2A_1) \odot (B_0+i_2B_1)] \odot (C_0+i_2C_1)$$
$$= [A \odot B] \odot C$$

Distribution property:

$$\forall\ A, B, C \in M$$
$$A \odot [B + C] = [A_0+i_2A_1] \odot [(B_0 + C_0) + i_2\ (B_1 + C_1)]$$

$$= A_0\,(B_0 + C_0) + i_2\,A_1\,[B_1 + C_1)$$

$$= [A_0B_0 + A_0C_0] + i_2\,[A_1B_1 + A_1C_1]$$

(Distributive laws for complex matrices)

$$= [A_0\,B_0 + A_0\,C_0] + [i_2\,A_1\,B_1 + i_2\,A_1\,C_1]$$

$$= [A_0B_0 + i_2\,A_1B_1] + [A_0C_0 + i_2\,A_1C_1]$$

$$= \quad A \odot B + A \odot C$$

Notes

Linear space:

Closed w.r.t scalar multiplication:

$$\forall\ \alpha \in C_0 \rightarrow \alpha.A = \alpha\,[A_0 + i_2A_1]$$

$$= \alpha\,A_0 + \alpha\,i_2A_1$$

$$\alpha.A = \alpha \begin{bmatrix} z_{11} & z_{12} & ..... & z_{1n} \\ z_{21} & z_{22} & ..... & z_{2n} \\ ..... & ..... & ..... & ..... \\ z_{n1} & z_{n2} & ..... & z_{nn} \end{bmatrix} + i_2\,\alpha \begin{bmatrix} w_{11} & w_{12} & ..... & w_{1n} \\ w_{21} & w_{22} & ..... & w_{2n} \\ ..... & ..... & ..... & ..... \\ w_{n1} & w_{n2} & ..... & w_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha z_{11} & \alpha z_{12} & ..... & \alpha z_{1n} \\ \alpha z_{21} & \alpha z_{22} & ..... & \alpha z_{2n} \\ ..... & ..... & ..... & ..... \\ \alpha z_{n1} & \alpha z_{n2} & ..... & \alpha z_{nn} \end{bmatrix} + i_2 \begin{bmatrix} \alpha w_{11} & \alpha w_{12} & ..... & \alpha w_{1n} \\ \alpha w_{21} & \alpha w_{22} & ..... & \alpha w_{2n} \\ ..... & ..... & ..... & ..... \\ \alpha w_{n1} & \alpha w_{n2} & ..... & \alpha w_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha z_{11} + i_2\alpha w_{11} & \alpha z_{12} + i_2\alpha w_{12} & ..... & \alpha z_{1n} + i_2\alpha w_{1n} \\ \alpha z_{21} + i_2\alpha w_{21} & \alpha z_{22} + i_2\alpha w_{22} & ..... & \alpha z_{2n} + i_2\alpha w_{2n} \\ ..... & ..... & ..... & ..... \\ \alpha z_{n1} + i_2\alpha w_{n1} & \alpha z_{n2} + i_2\alpha w_{n2} & ..... & \alpha z_{nn} + i_2\alpha w_{nn} \end{bmatrix} \in M.$$

Again

$$\forall\ A \in M \text{ and } 1 \in C_0,\ 1.A = 1.[\,A_0 + i_2A_1]$$

$$= 1.A_0 + i_2\,1.A_1$$

$$= A_0 + i_2A_1 = A$$

$$\forall\ \alpha, \beta \in F, \quad (\alpha + \beta)A = (\alpha + \beta)\ [\,A_0 + i_2A_1]$$

$$= (\alpha + \beta)A_0 + i_2(\alpha + \beta)A_1$$

$$= (\alpha A_0 + \beta A_0) + i_2(\alpha A_1 + \beta A_1)$$

$$= [\alpha\,A_0 + i_2\,\alpha\,A_1] + [\beta\,A_0 + i_2\,\beta\,A_1]$$

$$= \alpha\,A + \beta\,A$$

$$\forall\ A, B \in M, \text{ and } \alpha \in F$$

$$\alpha\,[\,A + B] = \alpha[(A_0 + B_0) + i_2\,(A_1 + B_1)]$$

$$= \alpha\,(A_0 + B_0) + i_2\,\alpha\,(A_1 + B_1)$$

$$= (\alpha\,A_0 + \alpha B_0) + i_2\,(\alpha A_1 + \alpha B_1)$$

Notes

$= \alpha A + \alpha B$

$(\alpha\,\beta)\, A = \alpha\,\beta\,(A_0) + i_2\,\alpha\,\beta\,(A_1)$

$= \alpha\,(\beta\,A_0) + i_2\alpha\,(\beta\,A_1)$

$= \alpha\,[\beta\,A_0 + i_2\beta\,A_1]$

$= \alpha.\,[\beta.\,A]$

Consistency (compatibility) between ⊙ and •:

$$\forall\ A, B \in M, \text{ and } \alpha \in F$$

$$\alpha\,[\,A \odot B] = \alpha[(A_0\,B_0)+i_2(A_1\,B_1)]$$

$$= \alpha\,(A_0\,B_0) + i_2\,\alpha\,(A_1\,B_1)$$

$$= (\alpha\,A_0)\,B_0 + i_2\,(\alpha A_1)\,B_1$$

$$= (\alpha A_0 + i_2\alpha A_1) \odot (B_0 + i_2\,B_1)$$

$$= (\alpha.A) \odot B$$

$$= (A_0\alpha)\,B_0 + i_2\,(A_1\alpha)\,B_1$$

$$= A_0\,(\alpha\,B_0) + i_2\,A_1\,(\alpha\,B_1)$$

$$= [A_0+i_2A_1] \odot [\alpha\,B_0+i_2\,\alpha\,B_1]$$

$$= A \odot (\alpha B)$$

Hence M (set of all square bicomplex matrix of order n) is an algebra.
Identity:

$A_0 + i_2A_1= A,\ \exists\,(I + i_2\,I)$such that

$$[A_0 + i_2A_1] \odot [I + i_2\,I] = A_0I + i_2A_1I = A = (I + i_2\,I).A$$

$\Rightarrow (I + i_2I)$ will be the identity under new system
Moreover for all ξ in C_2

$$\xi.\ A = (z_1 + i_2z_2)\,(A_0 + i_2A_1)$$

$$= z_1\,A_0 + i_2z_1\,A_1 + i_2\,z_2\,A_0 - z_2\,A_1$$

3.2.4 Definition: New inversion of a square bicomplex matrix

Let $A = A_0 + i_2\,A_1 \in C_2^{n\times n}$ be given bicomplex matrix where A_0, A_1 are complex matrix of same order if the inverse of A_0 and A_1 both exist then bicomplex matrix A is said to be invertible and inverse of A is written as $A^{-} = A_0^{-}+i_2A_1^{-}$, where A_0^{-} and A_1^{-} are the inverse of A_0 and A_1 respectively as well as A^{-} is the inverse of A or reciprocal of A.

3.2.5 Theorem

Let A be any square bicomplex matrix which is invertible in new system, then the inverse of the bicomplex matrix A, will be $\left(\frac{adj\ A_0}{|A_0|}\right)+i_2\left(\frac{adj\ A_1}{|A_1|}\right)$

Proof:

$$A = A_0 + i_2 A_1$$

Let A_0 and A_1 both has inverse A_0^- and A_1^-

Inverse of A = $A^- = A_0^- + i_2 A_1^-$(by definition)

Since A_0 and A_1 both are complex matrices therefore the inverse of A_0 and A_1 are

$$\left(\frac{adj\ A_0}{|A_0|}\right) \text{ and } \left(\frac{adj\ A_1}{|A_1|}\right) \text{ respectively}$$

Thus $$A^- = \left(\frac{adj\ A_0}{|A_0| \neq 0}\right) + i_2\left(\frac{adj\ A_1}{|A_1| \neq 0}\right) \quad (4)$$

Next from here we shall use M in place of $C_2^{n\times n}$

3.2.6 Some properties under new system

Property: 1

The multiplication is not commutative in general

$$A \odot B = (A_o + i_2 A_1) \odot (B_0 + i_2B_1)$$
$$= A_0 B_0 + i_2 A_1B_1$$
$$B \odot A = (B_o + i_2 B_1) \odot (A_0 + i_2A_1)$$
$$= B_0 A_0 + i_2 B_1A_1$$

Let $A \odot B = B \odot A$

$\Rightarrow A_0 B_0 + i_2 A_1B_1 = B_0 A_0 + i_2 B_1A_1$

$\Rightarrow A_0 B_0 = B_0 A_0$ and $A_1B_1 = B_1A_1$

$\Rightarrow$ Complex matrix is commutative which is contradicted.

$\Rightarrow A \odot B \neq B\odot A$

Counter example:

$$A=\begin{bmatrix}1 & i\\ -i & 2\end{bmatrix} \text{ and } B=\begin{bmatrix}i & 3\\ 2i & 5i\end{bmatrix} \text{ then } AB=\begin{bmatrix}i-2 & -2\\ 1+4i & -7i\end{bmatrix}$$

$$\text{but } BA=\begin{bmatrix}-2i & 5\\ 2i+5 & -2+10i\end{bmatrix} \Rightarrow AB \neq BA$$

Property: 2

$$(A \odot B)^T = (A_0 B_0 + i_2 A_1 B_1)^T$$
$$= (A_0 B_0)^T + i_2 (A_1 B_1)^T$$
$$= (B_0^T A_0^T) + i_2 (B_1^T A_1^T)$$

Notes

Notes

Since $(A\ B)^T = B^T A^T$ true in C_1

$$\text{Therefore } (A \odot B)^T = (B_0^T + i_2 B_1^T) \odot (A_0^T + i_2 A_1^T) = B^T \odot A^T$$

Property: 3

If A is any bicomplex square and invertible matrix whose inverse is A^{-} under new system then $(A^{-})^{-} = A$

Proof:

$$(A^{-})^{-} = [(A_0 + i_2 A_1)^{-}]^{-}$$

$$= (A_0^{-} + i_2 A_1^{-})^{-} \qquad (\because A^{-} = A_0^{-} + A_1^{-}, \text{ by definition})$$

$$= (C_0 + i_2 C_1)^{-} \qquad (\text{say } C_0 = A_0^{-} \text{ and } C_1 = A_1^{-})$$

$$= (C_0^{-} + i_2 C_1^{-}) \qquad (\text{by using again definition})$$

$$= (A_0^{-})^{-} + i_2 (A_1^{-})^{-}$$

$$= (A_0 + i_2 A_1)$$

$$= A$$

Property: 4

$$(A^{-})^k = (A_0^{-} + A_1^{-})^k$$

$$= (A_0^{-} + i_2 A_1^{-}) \odot (A_0^{-} + i_2 A_1^{-})^{k-1}$$

$$= [(A_0^{-})^2 + i_2 (A_1^{-})^2] \odot (A_0^{-} + i_2 A_1^{-})^{k-2}$$

$$= [(A_0^{-})^k + i_2 (A_1^{-})^k] \qquad (5)$$

$$= (A_0^{-} A_0^{-} \ldots\ldots k \text{ times}) + i_2 (A_1^{-} A_1^{-} \ldots\ldots k \text{ times})$$

$$= (A_0 A_0 \ldots\ldots k \text{ times})^{-} + i_2 (A_1 A_1 \ldots\ldots k \text{ times})^{-}$$

$$= (A_0^k)^{-} + i_2 (A_1^k)^{-}$$

$$= (A^k)^{-}$$

Property: 5

The inverse of the product of two bicomplex matrices A and B is equal to product of their inverses in reverse order

Proof:

Let $\forall\ A, B \in M$ then

$$(A \odot B)^{-} = [(A_0 + i_2 A_1) \odot (B_0 + i_2 B_1)]^{-}$$

$$= [A_0 B_0 + i_2 A_1 B_1]^{-}$$

$$= [A_0 B_0]^{-} + [A_1 B_1]^{-}$$

$$= [B_0^{-} A_0^{-}] + i_2 [B_1^{-} A_1^{-}]$$

(Since A_0, B_0, A_1 and B_1 are in $C_1^{n\times n}$ and $(A\ B)^- = B^- A^-$)

$$= [B^-_0 + i_2 B^-_1] \odot [A^-_0 + i_2\ A^-_1]$$

Therefore $[(A \odot B)]^- = B^- \odot A^-$

3.2.7 Theorem

If $A_1\ A_2 \ldots\ldots A_n$ are the invertible bicomplex matrix then the inverse of product of $A_1\ A_2 \ldots\ldots A_n$, will be equal to the individual product of their inverse in reverse order.

Proof:

Let $A_1\ A_2 \ldots\ldots A_n$ be the invertible bicomplex matrix then

$$(A_1 \odot A_2 \odot A_3 \odot \ldots\ldots\ldots \odot A_n)^-$$

$$= [(A_{10}+i_2\ A_{11}) \odot (A_{20}+i_2 A_{21}) \odot \ldots\ldots\ldots \odot (A_{n0}+i_2 A_{n1})]^-$$

$$= [(A_{10}\ A_{20} \ldots\ldots A_{n0})+i_2(A_{11}\ A_{21} \ldots\ldots A_{n1})]^-$$

$$= [(A_{10}\ A_{20} \ldots\ldots A_{n0})^- + i_2\ (A_{11}\ A_{21} \ldots\ldots\ A_{n1})^-]$$

$$= (A_{n0}^-\ A_{n-1\,0}^- \ldots\ldots A_{10}^-) + (A_{n1}^-\ A_{n-1\,1}^- \ldots A_{11}^-)$$

$$= (A_{n0}^- + i_2\ A_{n1}^-) \odot (A_{n-10}^- + i_2\ A_{n-1\,1}^-) \odot \ldots\ldots (A_{10}^- + i_2\ A_{n1}^-)$$

$$= (A_n^-) \odot (A_{n-1}^-) \ldots\ldots \odot (A_1^-).$$

Thus it is clear the inversion of the bicomplex matrix under new definition has the same fundamental properties as those under the traditional algebraic system.

c) Some definitions and theorems related to bicomplex matrix in both systems

3.3.1 Idempotent bicomplex matrix

Let A be a square bicomplex matrix and if $A^2 = A$, then A is called the idempotent bicomplex matrix, obviously identity matrices in C_2 will be idempotent matrix in their individual systems.

Remark:

The identity matrix is always idempotent bicomplex matrices in their own systems as well as the Null matrix is also idempotent matrix.

Example:

(1) $\begin{bmatrix} \frac{e_1}{2} & \frac{e_1}{2} \\ \frac{e_1}{2} & \frac{e_1}{2} \end{bmatrix}$ *is the example of idempotent matrix.*

(2) $\begin{bmatrix} \xi & {}^2\xi \\ {}^1\xi & {}^1\xi\, e_1 \end{bmatrix}$ $\forall\ \xi \neq 0 \in C_2$ is not idempotent matrix.

(3) $\begin{bmatrix} e_1 & 0 \\ 0 & e_1 \end{bmatrix}$ is also an idempotent matrix.

Notes

(4) $\left[\begin{bmatrix} 1 & 0 \\ i_1 & 0 \end{bmatrix} + i_2 \begin{bmatrix} 1 & 0 \\ 6i_1 & 0 \end{bmatrix}\right]$ is an idempotent matrix in new system.

3.3.2 Theorem

A is an idempotent bicomplex square matrix of order n if and only if both complex component matrixes A_0 and A_1 are idempotent complex matrix simultaneously.

Notes

Proof:

$\because$ A is an idempotent matrix i.e. by definition $A^2 = A$

$\Leftrightarrow$ $(A_0 + i_2A_1) \odot (A_0 + i_2A_1) = A_0 + i_2A_1$

$\Leftrightarrow$ $A_0^2 + i_2A_1^2 = A_0 + i_2A_1$

$\Leftrightarrow$ $A_0^2 = A_0$ and $A_1^2 = A_1$

$\Leftrightarrow$ Both A_0 and A_1 are idempotent complex matrix.

3.3.3 Involutory bicomplex matrix

Let A be any bicomplex square matrix if $A^2 = I$ matrix then A is known as involutory bicomplex matrix. i.e. the inverse of the given matrix A will be itself.

Clearly the identity matrices will be involutory bicomplex matrices in their own systems.

Remark:

All idempotent bicomplex matrices which are not identity matrix will not be involutory. Example:

(1) $\begin{bmatrix} e_2 & e_1 \\ e_1 & e_2 \end{bmatrix}^2 = I$

(2) $\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}^2 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3.3.4 Theorem

A is an involutory bicomplex matrix if and only if both complex component matrix A_0 and A_1 (under the new system) are involutory.

Proof:

By new definition of product of bicomplex matrix

i.e. $A^2 = A_0^2 + i_2A_1^2$, where $A_0, A_1 \in C_1$

$\because$ A is an involutory bicomplex matrix

$\therefore$ $A^2 = I + i_2 I \rightarrow$ (identity under new system)

$\Leftrightarrow$ $A_0^2 + i_2A_1^2 = I + i_2 I$

$\Leftrightarrow$ $A_0^2 = I$ and $A_1^2 = I$

$\Leftrightarrow$ Both A_0 and A_1 are involutory.

Remark:

Under new product definition $(I + i_2 I)$ is always an involutory matrix i.e.

$$(I + i_2 I) \odot (I + i_2 I) = (I + i_2 I)$$

3.3.5 Similar bicomplex matrix

Let A, B $\in$M if $\exists$ an invertible matrix P $\in$M such that

$A = P^{-1}BP$ then A and B are said to be similar bicomplex matrix and denoted by A˜ B

If A and B are similar in new system then

$$A = P^{-} \odot B \odot P \tag{6}$$

And if A, B $\in C_1^{n \times n}$ then equation (6) will be equivalent to $A = P_0^{-} B P_0$, where $P = P_0 + i_2 P_1$

3.3.6 Theorem

Let A and B be two square bicomplex matrices, and an invertible matrix P such that $A = P^{-1}BP$ or $A \sim B$ then

$$|A| = |B|$$

Proof:

$\because$ $A = P^{-1}BP$, where P is an invertible bicomplex matrix

Now $|A| = |P^{-1}B\,P|$

$$= |{}^{1}(P^{-1}BP)|e_1 + |{}^{2}(P^{-1}B\,P)|e_2$$

$$= |{}^{1}P^{-1}||{}^{1}B||{}^{1}P|\,e_1 + |{}^{2}P^{-1}||{}^{2}B||{}^{2}P|\,e_2$$

$$= |{}^{1}P^{-1}||{}^{1}P||{}^{1}B|\,e_1 + |{}^{2}P^{-1}||{}^{2}P||{}^{2}B|\,e_2$$

$$= |{}^{1}P^{-1}\,{}^{1}P||{}^{1}B|\,e_1 + |{}^{2}P^{-1}\,{}^{2}P||{}^{2}B|\,e_2$$

$$= |{}^{1}B|e_1 + |{}^{2}B|e_2$$

therefore $|A| = |B|$

3.3.7 Theorem

The $i_1 i_2$ tranjugate of inverse of a matrix A is equal to the inverse of $i_1 i_2$ tranjugate of A. i.e.

$$\left(A^{-1}\right)^{\theta_3} = \left(A^{\theta_3}\right)^{-1}$$

Proof:

$$\left(A^{-1}\right) = {}^{1}A^{-1}e_1 + {}^{2}A^{-1}e_2$$

$$\left(A^{-1}\right)^{\theta_3} = \left[\left({}^{1}A^{-1}e_1 + {}^{2}A^{-1}e_2\right)^{\#}\right]^T$$

$$= \left(\overline{{}^{1}A^{-1}}\,\overline{e}_1 + \overline{{}^{2}A^{-1}}\,\overline{e}_2\right)^T$$

$$= \left(\overline{{}^{1}A^{-1}}\right)^T e_1 + \left(\overline{{}^{2}A^{-1}}\right)^T e_2$$

$$= conj\left({}^{1}A^{T}\right)^{-1} e_1 + conj\left({}^{2}A^{T}\right)^{-1} e_2$$

$$= \left(\overline{{}^{1}A^{T}}\right)^{-1} e_1 + \left(\overline{{}^{2}A^{T}}\right)^{-1} e_2$$

$$\text{i.e}\left(A^{-1}\right)^{\theta_3} = \left(A^{\theta_3}\right)^{-1}$$

Remark:

Since i_1 and i_2 conjugates of e_1 is e_2 and e_2 is e_1 therefore this result is not true for S = 1, 2 where $\left(A^{-1}\right)^{\theta_s} = \left(A^{\theta_s}\right)^{-1}$.

Notes

3.3.8 Theorem

In new system, the $i_1$tranjugate of inverse of a matrix A is equal to the inverse of i_1 tranjugate of A. i.e. $\left(A^{-}\right)^{\theta_1} = \left(A^{\theta_1}\right)^{-}$

Proof:

$$\left(A^{-}\right)^{\theta_1} = \left[\overline{\left(A_0^{-} + i_2 A_1^{-}\right)}\right]^T$$

$$\left(A^{-}\right)^{\theta_1} = \left[\overline{\left(A_0^{-}\right)}\right]^{T} + i_2 \left[\overline{\left(A_1^{-}\right)}\right]^T$$

$$\left(A^{-}\right)^{\theta_1} = \left(\overline{A}_0^{\,T}\right)^{-} + i_2 \left(\overline{A}_1^{\,T}\right)^{-}$$

$$\left(A^{-}\right)^{\theta_1} = \left(A^{\theta_1}\right)^{-}$$

In this system this result is not valid for S = 2, 3 where $\left(A^{-}\right)^{\theta_S} = \left(A^{\theta_S}\right)^{-}$.

By above two theorems it is evident that the given bicomplex matrix has same property by taken different conjugate in both different system.

3.3.9 Theorem

A will be an orthogonal bicomplex matrix in new system if and only if the complex component matrix A_0 and A_1 are orthogonal complex matrices.

Proof:

Since A $\in$M is an orthogonal matrix.

Therefore $$A^T = A^{-} \quad (7)$$

Since A can be express as the i_2 combination of two complex matrices A_0 and A_1 as follow

$$A = (A_0 + i_2 A_1)$$

$A^T = (A_0 + i_2 A_1)^T = A^{-}$(where A^{-} is the inverse of A)

And $A^{-} = A^{-}_0 + i_2 A^{-}_1$

From equation (7) we have

$$(A_0 + i_2 A_1)^T = A^{-}_0 + i_2 A^{-}_1$$

$\Leftrightarrow A_0^T + i_2 A_1^T = A^{-}_0 + i_2 A_1^{-}$

$\Leftrightarrow A_0^T = A_0^{-}$ and $A_1^T = A_1^{-}$

$\Leftrightarrow$ Both complex matrices A_0 and A_1 are orthogonal.

The proof of theorem 3.3.9 is complete.

If A is an orthogonal matrix in traditional system then

$$A^T = A^{-1}$$

$\Leftrightarrow {}^1A^T e_1 + {}^2A^T e_2 = {}^1A^{-1} e_1 + {}^2A^{-1} e_2$

$\Leftrightarrow {}^1A^T = {}^1A^{-1}$and ${}^2A^T = {}^2A^{-1}$

$\Leftrightarrow$ Both idempotent component matrices are orthogonal.

Notes

Hence A is an orthogonal matrix in traditional system if and only if both idempotent component matrices are orthogonal.

3.3.10 Theorem

A will be an i_1 Unitary bicomplex matrix if and only if the complex component matrix A_0 and A_1 of A are Unitary complex matrix but idempotent matrix ${}^1A$ and ${}^2A$ of A may or may not be Unitary.

Proof:

Part-1st

According to definition of i_1 Unitary bicomplex matrix

$$A^{\theta_1} = [(A_0 + i_2 A_1)^{-}]^T = [A_0^- + i_2 A_1^-]$$

$\Leftrightarrow \bar{A}_0^T + i_2 \bar{A}_1^T = A_0^- + i_2 A_1^-$

$\Leftrightarrow \bar{A}_0^T = A_0^-$ and $\bar{A}_1^T = A_1^-$

$\Leftrightarrow$ A_0 and A_1 are Unitary

Part-2nd

Note that

$$\left({}^1Ae_1 + {}^2Ae_2\right)^{\theta_1} = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$$

$\Leftrightarrow \left(\overline{{}^1A}\, e_2 + \overline{{}^2A}\, e_1\right)^T = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \left(\overline{{}^1A}^T e_2 + \overline{{}^2A}^T e_1\right) = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \overline{{}^2A}^T = {}^1A^{-1}$ and $\overline{{}^1A}^T = {}^2A^{-1}$

It is evident that if A is an i_1 Unitary bicomplex matrix then idempotent matrix ${}^1A$ and ${}^2A$ of A will be Unitary complex matrix only if A is a complex matrix.

It is clear from here that both component A_0 and A_1 as well as ${}^1A$ and ${}^2A$ be an unitary complex matrices then matrix A will be different type of unitary bicomplex matrix that means it has shown the different nature of representations of A.

3.3.11 Theorem

Let A be an i_1 Hermitian bicomplex matrix then the i_1 tranjugate of both ${}^1A$ and ${}^2A$ will be ${}^2A$ and ${}^1A$ respectively as well as A_0 and A_1 both will be Hermitian complex matrix.

Proof:

Part-1st

Since A is i_1 Hermitian $\Rightarrow$ $(\bar{A})^T = A$

$\Leftrightarrow \left(\overline{{}^1A^{-1} e_1 + {}^2A^{-1} e_2}\right)^T = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow \left(\overline{{}^1A}\, e_2 + \overline{{}^2A}\, e_1\right)^T = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow \overline{{}^2A}^T e_1 + \overline{{}^1A}^T e_2 = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow \left({}^2A^{\theta_1} = {}^1A\right)$ and $\left({}^1A^{\theta_1} = {}^2A\right)$

Notes

Notes

Part-2nd

$$A = A_0 + i_2 A_1$$

$$\left(\overline{A_0 + i_2 A_1}\right)^T = (\overline{A_0})^T + i_2 (\overline{A_1})^T$$

Since A is i_1 Hermitian $\Leftrightarrow (\bar{A})^T = A$

$\Leftrightarrow (\overline{A_0})^T + i_2 (\overline{A_1})^T = A_0 + i_2 A_1$

$\Leftrightarrow (\overline{A_0})^T = A_0$ and $(\overline{A_1})^T = A_1$

$\Leftrightarrow$ Both complex component matrix A_0 and A_1 of A are Hermitian.

3.3.12 Theorem

Let A be an i_2 Hermitian bicomplex matrix if and only if the transpose of ${}^2A$ and ${}^1A$ are ${}^1A$ and ${}^2A$ respectively as well as A_0 and A_1 are symmetric and Skew – Symmetric bicomplex matrix respectively.

Proof:

Part-1st

By definition of i_2 Hermitian in C_2

$[(A)^{\sim}]^T = A^{\theta_2} = A$

$\Leftrightarrow \left({}^1Ae_1 + {}^2Ae_2\right)^{\theta_2} = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow {}^1A^T e_2 + {}^2A^T e_1 = {}^1Ae_1 + {}^2Ae_2$

$\Leftrightarrow {}^2A^T = {}^1A$ and ${}^1A^T = {}^2A$

Part- 2nd

$$A = A_0 + i_2 A_1 \quad \forall\ A_s = C_1^{\,n\times n}, S = 0,1$$

$$A^{\theta_2} = (A_0 + i_2 A_1)^{\theta_2} = A_0^{\ T} - i_2 A_1^{\ T}$$

Since A is i_2 Hermitian $A^{\theta_2} = A$

$\Leftrightarrow A^{\theta_2} = A$

$\Leftrightarrow A_0^{\ T} - i_2 A_1^{\ T} = A_0 + i_2 A_1$

$\Leftrightarrow A_0^{\ T} = A_0$ and $A^T{}_1 = -A_1$

3.3.13 Theorem

A is a $i_1 i_2$ Hermitian matrix if and only if ${}^1A$ and ${}^2A$ both idempotent component matrix of A will be Hermitian as well as the complex component matrix A_0 and A_1 of A will be Hermitian and Skew – Hermitian respectively.

Proof:

Part-1st

$\because$ A is $i_1 i_2$ Hermitian

$$[A]^{\theta_3} = A$$

$\Leftrightarrow \left({}^1Ae_1 + {}^2Ae_2\right)^{\theta_3} = {}^1Ae_1 + {}^2Ae_2$

$\Leftrightarrow {}^1A^{\theta_3} = {}^1A$ *and* ${}^2A^{\theta_3} = {}^2A$

$\Leftrightarrow$ both ^1A *and* ^2A *are Hermitian*

Part- 2nd

$$A^{\theta_3} = (A_0 + i_2 A_1)^{\theta_3}$$

$\because$ A is Hermitian

$\Leftrightarrow (A_0 + i_2 A_1)^{\theta_3} = A_0 + i_2 A_1$

$\Leftrightarrow A_0^{\theta_3} - i_2 A_1^{\theta_3} = A_0 + i_2 A_1$

$\Leftrightarrow A_0^{\theta_3} = A_0$ and $A_1^{\theta_3} = -A_1$

$\Leftrightarrow$ A_0 is Hermitian and A_1 is Skew – Hermitian in C_1.

3.3.14 Theorem

Let A be an i_2 Unitary bicomplex matrix then idempotent component matrix ^1A and ^2A are not symmetric until $^1A^{-1} = {}^2A$ and $[A_0]^T = A_0^-$ and $[A_1]^T = -A_1^-$

Proof:

Part-1st

by definition of i_2 unitary bicomplex matrix $({}^1Ae_1 + {}^2Ae_2)^{\theta_2} = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \left[({}^1Ae_1 + {}^2Ae_2)^{\sim}\right]^T = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow [({}^1A)^{\sim}]^T e_1^{\sim} + [({}^2A)^{\sim}]^T e_2^{\sim} = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow [{}^1A]^T e_2 + [{}^2A]^T e_1 = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow [{}^2A]^T = {}^1A^{-1}$ and $[{}^1A]^T = {}^2A^{-1}$

Therefore it is clear that if $^1A^{-1} \neq {}^2A$ then ^1A and ^2A will never symmetric.

Part- 2nd

Since A is i_2 Unitary bicomplex matrix and

$A = (A_0 + i_2 A_1)$ *and* $A^- = (A_0^- + i_2 A_1^-)$ *therefore*

$[A_0 + i_2 A_1]^{\theta_2} = A_0^- + i_2 A_1^-$

$\Leftrightarrow [(A_0 + i_2 A_1)^{\sim}]^T = A_0^- + i_2 A_1^-$

$\Leftrightarrow [A_0^{\sim}]^T - i_2 [A_1^{\sim}]^T = A_0^- + i_2 A_1^-$

$\Leftrightarrow [A_0]^T = A_0^-$ and $[A_1]^T = -A_1^-$

3.3.15 Theorem

Let A be an $i_1 i_2$ Unitary bicomplex matrix then the idempotent component matrix ^1A and ^2A both are Unitary simultaneously but complex component matrix A_0 and A_1 are not Unitary simultaneously. Moreover A_0 will be Unitary but A_1 will not be Unitary.

Proof:

$$A = (A_0 + i_2 A_1)$$

Notes

$$A^{\theta_3} = [(A_0 + i_2 A_1)^{\#}]^T = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$$

$\Leftrightarrow \quad [\overline{{}^1A}]^T e_1 + [\overline{{}^2A}]^T e_2 = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \quad [\overline{{}^1A}]^T = [{}^1A^{-1}]$ and $[\overline{{}^2A}]^T = [{}^2A^{-1}]$

$\Leftrightarrow$ both idempotent component ${}^1A$ and ${}^2A$ are Unitary

Notes

and $A^{\theta_3} = [(A_0 + i_2 A_1)^{\#}]^T$

$$\because \; A^{\theta_3} = A^{-1} \Leftrightarrow [\bar{A}_0]^T - i_2[\bar{A}_1]^T = A_0^{-} + i_2 A_1^{-}$$

$\Leftrightarrow [\bar{A}_0]^T = A_0^{-}$ and $[\bar{A}_1]^T = -A_1^{-}$

Therefore A_0 is Unitary but A_1 is not Unitary.

3.3.16 Theorem

The similar relation ˜ between two arbitrary bicomplex square matrix A and B in new system i.e. A ˜ B then this relation will be an equivalence relation.

Proof:

Reflexive-

Since We know that $P = [I + i_2 I]$ is an invertible matrix such that $A = P^{-} \odot A \odot P$ i.e. every bicomplex square matrix is similar to itself.

Symmetric Relation-

Let A˜B then we have to prove B will be similar to A.

$\Rightarrow \quad A = P^{-} \odot B \odot P$ for some invertible $P \in M^{n \times n}$

$\Rightarrow \quad P \odot A = P \odot [P^{-} \odot B \odot P]$

$= [P \odot P^{-}] \odot [B \odot P]$

$= [I + i_2 I] \odot [B \odot P]$

$\Rightarrow \quad P \odot A = [B \odot P]$

i.e. $P \odot A \odot P^{-} = [B \odot P] \odot P^{-}$

$= B \odot [P \odot P^{-}]$

$= B \odot [I + i_2 I]$

$= [B_0 + i_2 B_1] \odot [I + i_2 I]$

$= B_0 + i_2 B_1$

$P \odot A \odot P^{-} = B$

$\Rightarrow \quad B \sim A$

Transitive-

If $A \sim B \Rightarrow \quad \exists$ an invertible Bicomplex matrix P

Such that $A = P^{-} \odot B \odot P$

And $B \sim C \Rightarrow \quad B = Q^{-} \odot C \odot Q$ for some $Q^{-} \in M^{n \times n}$

We have to show $A \sim C$

$\because$ $A = P^{-} \odot B \odot P$ and $B = Q^{-} \odot C \odot Q$

$A = P^{-} \odot [Q^{-} \odot C \odot Q] \odot P$

$= [P^{-} \odot Q^{-}] \odot [C \odot Q] \odot P$

$= E^{-} \odot C \odot E$, where $E = Q \odot P$

We have an invertible bicomplex matrix E
Such that $A = E^{-} \odot C \odot E$
therefore A is similar to C then Relation is transitive.
Hence the similar relation between bicomplex matrices is an equivalence relation.

Moreover the collection of bicomplex matrices similar to A forms a class denoted by [A] and is called the class of similar matrices of A.

Two classes [A] and [B] are either same or disjoint, in the sense that no matrix can belong to two different classes. Thus there exists a natural partition of the set of "All bicomplex square matrices".

The set of all square bicomplex matrices can also be viewed as the collection of all mutually disjoint equivalence classes with respect to a suitable defined equivalent relation.

References Références Referencias

1. Anjali: Certain results on bicomplex matrices, M. Phil. Dissertation, Dr. B. R. Ambedker University, Agra (2011).
2. Futagawa, M.: On the theory of functions of a quaternary variable, Tôhoku Math. J., 29(1928), 175-222.
3. Futagawa, M.: On the theory of functions of a quaternary variable, II, Tôhoku Math. J., 35 (1932), 69-120.
4. Kumar, Anil: Certain Tetralineartrans formations between the subsets of Bicomplex space, M.Phil. Dissertation, Dr. B.R. Ambedker University, Agra (2010).
5. Lipschutz, S. and Lipson, M.:Schaum's Outline theory and problem of linear algebra, Tata McGraw-Hill Education, 2005.
6. Price, G.B.: An introduction to multiomplex space and functions, Marcel Dekker Inc., New York, 1991.
7. Riley, J.D.: Contributions to the theory of function of a Bicomplex variable" Tôhoku math. J., 2ed series 5 (1953), 132- 165.
8. Ringleb, F.: Beiträge Zur Funktionentheorie in hyperkomlexon System, I, Rend. Circ. Mat. Palermo, 57 (1933) 311-340.
9. Segre,C.: Le Rappresentazioni Reali Delle Forme Complessee GlientiIperalgebrici, Math. Ann. 40 (1892), 413- 467.
10. Srivastava, Rajiv K.: Bicomplex Numbers: Analysis and applications, Math. Student, 72 (1-4) 2003, 69-87.
11. Srivastava, Rajiv K.: Certain topological aspects of the Bicomplex space, Bull. Pure & Appl. Maths., 2(2)(2008), 222-234.

Notes

Global Journals Guidelines Handbook 2018

www.GlobalJournals.org