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# Investigation of Classical Systems with Complex Energy in the Field of Quantum-Classical Correspondence

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INVESTIGATIONOFCLASSICALSYSTEMSWITHCOMPLEXENERGYINTHEFIELDOFQUANTUMCLASSICALCORRESPONDENCE

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# Investigation of Classical Systems with Complex Energy in the Field of Quantum-Classical Correspondence

Shahadat Hossain [α], Syed Badiuzzaman Faruque [σ] & M. S. Hossain [ρ]

*Abstract-* Classical mechanics and quantum mechanics contradict each other, and both are essential to explain the phenomena of the exclusively different realm of nature. On the other hand, Bohr's correspondence principle shows classical mechanics is the somewhat approximate version of quantum mechanics. Classically a particle with negative energy i. e. $E < V$ is not allowed go through a forbidden region or disappearing from one well to another well. This paper gives the numerical studies for the trajectory of the particle in a double-well potential and presents quantum mechanical behavior such as tunneling in the complex plane for different energies. Our findings provide a route to solve the classical system with complex energy.

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## I. Introduction

Quantum mechanical phenomena are completely different from classical mechanical in our physical world. The motion of a classical particle is deterministic and is described by Hamiltonian's equation. The position x(t) of a particle at any instant can be found by solving a local initial value problem for this differential equation. The energy of a particle which is numerically equal to the Hamiltonian is a constant and can have any values. Classically, the motion of the particle is confined or allowed to a region where energy E≥V(x) and forbidden where E<V(x). A classical particle may not travel through the barrier which separates two classical allowed regions.

Many researchers have astonished by extending both quantum mechanics and classical mechanics into the complex domain. In conventional quantum mechanics, all physical observable must be represented by Hermitian operators on Hilbert space so that the Hamiltonians have real energy eigen values and unitary time evolution[1]. But a class of physical allowable Hamiltonians may be extended to include non-Hermitian Hamiltonians that possess an unbroken PT(combined parity and time reversal) symmetry because these complex Hamiltonians also have real energy eigenvalues and generate unitary time evolution[2-12].In the recent years; some new surprising phenomena were revealed by PT symmetry quantum mechanics which observed in laboratory experiment[13-15]. Conventional classical mechanics is the study of the real solution to Hamiltonian's equation, and we find the exact trajectory of a particle. To understand PT-symmetric quantum mechanics, conventional classical mechanics is extended into the complex domain. In the complex classical mechanics, we study all solutions, real as well as complex of Hamiltonian's equation[16-18]i.e., the real and the complex trajectory for a system having real energy. Study of complex classical mechanics has provided an intuitive image of what is happening at the unbroken and broken PT-symmetric phase of PT-symmetric quantum mechanics, the classical trajectories are closed and periodic, but in the broken phase those are open[19].

A new area of research has been recently introduced which concerns the generalization of classical mechanics from real to complex energy[20]. Since the energy of a quantum particle cannot be determined precisely due to an infinite amount of time. According to the time-energy uncertainty principle in quantum mechanics $\Delta E\, \Delta t \geq \frac{h}{4\pi}$, the energy cannot be measured without an uncertainty of $\Delta E$. As a consequence of this argument, the uncertainty exists in classical mechanics and further, it, can be assumed as complex. The generalization from real to complex energy reveals many features of quantum mechanics by the classical system having complex energy. Carl M Bender, Dorje C Brody and Daniel W Hook have performed numerical studies in the conjectural paper[20] and they found some well-known quantum effects by the deterministic equations of classical mechanics(Newton's law) when these equations are solved in the complex plane for the systems having complex energy.

In the discussion of the conjectural paper[20], it has been concluded that the analogies between quantum mechanics and complex energy classical mechanics make further investigation worthwhile. In [21-22] the authors investigated the analogies between quantum mechanics and complex classical mechanics and also provide a procedure to obtain the trajectory

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when the energy of a deterministic classical particle is allowed to be complex in the double well potential.

In this work, we allowed a classical particle to have a complex energy in the double well potential $x^4 - x^2$, the corresponding system appears with a phenomenon that completely a feature of quantum mechanics such as tunneling.

## II. Methodology

In classical mechanics, the motion of a particle is modeled by the Hamiltonian of the for

$$H = \frac{1}{2}p^2 + V(x) \tag{2.1}$$

Such that the first derivative of the potential V(x) is a function of the position of the particle, $x$.The Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p} = p(= u) \tag{2a}$$

and

$$\dot{p} = -\frac{\partial H}{\partial x}(= a) \tag{2b}$$

$u$ and $a$ express the velocity and the acceleration of the particle at the point $x$.

If the energy $E$ is a given numerical value of $H$, which is a constant of motion, then we get from equation (1)

$$E = \frac{1}{2}p^2 + V(x) \tag{2.3}$$

Using equation (2) in (3) we obtain,

$$u = \pm\sqrt{2E - 2V(x)} \tag{2.4}$$

It is obvious from above two that $u$ is complex value when $E \le V(x)$, but $u$ is real value when $E \ge V(x)$. The positive and negative of $\pm\sqrt{2E-2V(x)}$ is denoted by $u_+$ and $u_-$respectively.

The particle starts its journey from $x_{+0}$ at the time $t = 0$ and if the time interval $\Delta t$ is infinitesimally small, we can consider $a$ as a constant. Hence, at the position $x_{+0}$, the velocity of the particle is,

$$u_{+0} = \sqrt{2E - 2V(x_{+0})} \tag{2.5}$$

Where particle moves with positive value of $u$ and the acceleration is

$$a_{+0} = -\frac{\partial H}{\partial x}(x_{+0}) \tag{2.6}$$

Now according to the well-known classical formula

$$x = x_0 + ut + \frac{1}{2}at^2, \tag{2.7}$$

we have

$$x_{+1} = x_{+0} + u_{+0}\Delta t + \frac{1}{2}a_{+0}(\Delta t)^2 \tag{2.8}$$

Because both the velocity and the acceleration depend on position of the particle, the velocity and acceleration at the position $x_{+1}$ are

$$u_{+1} = \sqrt{2E - 2V(x_{+1})} \tag{2.9}$$

and

$$a_{+1} = -\frac{\partial H}{\partial x}(x_{+1}) \tag{2.10}$$

respectively. Now the position of the particle for time, $0 + \Delta t + \Delta t$, i.e. $2\Delta t$ is

$$x_{+2} = x_{+1} + u_{+1}\Delta t + \frac{1}{2}a_{+1}(\Delta t)^2 \tag{2.11}$$

Similarly, we can obtain the position of the particle after each interval, $x_{+3}$, $x_{+4}$, $x_{+5}$

Let us take the final position $x_{+n}$ for the positive sign of $u$. Then the particle travels from $x_{+n}$ with the negative sign of $u$. Therefore, the velocity of the particle at $x_{+n}$ is

$$u_{-0} = \sqrt{2E - 2V(x_{+n})} \tag{2.12}$$

and

$$a_{-0} = -\frac{\partial H}{\partial x}(x_{+n}) \tag{2.13}$$

After the time interval $\Delta t$ from the $x_{+n}$ position of the particle is

$$x_{-1} = x_{+n} + u_{-0}\Delta t + \frac{1}{2}a_{-0}(\Delta t)^2 \tag{2.14}$$

Similarly using the above procedure we can obtain the positions of the particle $x_{-2}, x_{-3}, x_{-4}, \dots\dots\dots. x_{-n}$, for each interval $\Delta t$, where $x_{-n}$ is the final position of the particle with the negative sign of $u$. Then at the point, $x_{-n}$, then the particle travels with the positive sign of $u$.

The alternatively taking positive sign of $u$ and negative sign of $u$ of the particle continue endlessly, and we have an endless trajectory of the classical particle. If after $n\Delta t$ time, the particle returns to its initial position, the trajectory of the particle is closed and periodic with $n\Delta$t, where $n$ is a positive integer.

## III. Theoretical Calculation

*a) Motion of a Particle Having Real Energy in the Potential $(x^4 - x^2)$ in the Complex Domain*

Investigation of the classical trajectories of a particle having energy $E = 1$ with different initial conditions in the potential, $x^4 - x^2$, defined by the Hamiltonian $H = \frac{1}{2}p^2 + x^4 - x^2$ in the complex domain showed in figure 1. The solutions of the equation $V(x) = E$, i.e.$x^4 - x^2 = 1$ gives the classical turning points located at $x = \pm 1.2720, \pm 0.7862i$ and indicated by red dots. The so-called 'classical allowed region'(for which $E \geq V(x)$ i. e. $1 \geq x^4 - x^2$) is the portion of the real $x$ between $x = -1.2720$ to $x = 1.2720$, and a classical particle initially on this line segment moves parallel to the real axis and oscillate between real turning points. The classical forbidden regions (represented by $E < V(x)$ i.e.,$1 < x^4 - x^2$) are the portions of the real axis for which $x > 1.2720$ and $x < -1.2720$, and a particle having initial position in either one moving perpendicularly to the real axis. The particle then enters into the complex-x and makes a sharp turn about the imaginary turning points and return to its initial position. All orbits in figure 1 have the same period which is exactly 3.998. It was observed that two different trajectories never cross each other.

The trajectories of a classical particle having real negative energy,$E = -1$, in the potential, $x^4 - x^2$ of the Hamiltonian $H = \frac{1}{2}p^2 + x^4 - x^2$ with different initial conditions were shown in figure 2. The turning points belong to the energy $E = -1$ are located at $x = \pm(0.8660 - 0.5000i), \pm(0.8660 + 0.5000i)$, which are indicated by red dots in figure 2.We observed that the all trajectories are closed and periodic. The classical trajectories are always confined to either the right-half or left-half of the complex- $x$ plane and unable to go through imaginary axis, $x = 0$. Figure 2 shows the sixteen classical trajectories for energy $E = -1$. Eight trajectories lie in the right-half enclosing the turning points $x = 0.8660 + 0.5000i$ and $x = 0.8660 - 0.5000i$, and other eight trajectories lie in the left-half enclosing the turning points $x = -0.8660 + 0.5000i$ and $x = -0.8660 - 0.5000i$. No two trajectories cross each other. Thus for a particle having negative real energy the potential $x^4 - x^2$ act.

*b) Classical Trajectory of a Particle of Energy $2 + 0.2i$in the Double-Well Potential*

A single classical trajectory of a particle having energy $E = 2 + 0.2i$ in the potential, $x^4 - x^2$ defined by Hamiltonian $H = \frac{1}{2}p^2 + x^4 - x^2$ presnted in figure 3. The solution of the equation $V(x) = E$, i.e. $x^4 - x^2 = 2 + 0.2i$ gives classical turning points. Hence we have four turning points located at $x = 1.4149 + 0.0235i$, $-1.4149 - 0.0235i$, $0.0333 - 1.0013i$, $-0.0333 + 1.0013i$ which are indicated by red dots. The turning points are different from figure 1 due to the amount of adding energy $(0.2i)$ to real energy$(2)$. A particle whose initial position in any point in the complex- $x$ plane have an initial motion having two components, along with real axis and perpendicular to the real axis and the particle moves in the complex-$x$ plane. The trajectory spirals inward around the pair of turning points, $1.4149 + 0.0235i$ and $-1.4149 - 0.0235i$ and make a sharp turn about the other pair of turning points, $0.0333 - 1.0013i$ and$-0.0333 + 1.0013i$. The effect is that the trajectory still does not cross itself; the trajectory no longer needs to be closed and periodic. The trajectory, in this case, is open.

*c) Classical Trajectory of a Particle of Energy $2 - 0.2i$*

The single classical trajectory of a particle having energy $E = 2 - 0.2i$ in the potential, $x^4 - x^2$ defined by Hamiltonian $H = \frac{1}{2}p^2 + x^4 - x^2$ depicted in figure 4. The solution of the equation $V(x) = E$, i.e. $x^4 - x^2 = 2 - 0.2i$ gives classical turning points. Hence we have four turning points located at $x = 1.4149 - 0.0235i$, $-1.4149 + 0.0235i$, $0.0333 + 1.0013i$, $-0.0333 - 1.0013i$ which are indicated by red dots. The turning points are different from figure 1 due to the amount of subtracting energy $(0.2i)$ to real energy(2). A particle whose initial position in any point in the complex- $x$ plane have an initial motion having two components, along real axis and perpendicular to the real axis and the particle moves in the complex-$x$ plane. The trajectory spirals inward around the pair of turning points, $1.4149 - 0.0235i$ and $-1.4149 + 0.0235i$ and make a sharp turn about the other pair of turning points, $0.0333 + 1.0013i$ and $-0.0333 - 1.0013i$. The effect is that the trajectory still does not cross itself, the trajectory no longer need be closed and periodic. The trajectory, in this case, is open. The direction of motion of the particle, in this case, becomes reverse compared to figure 3 due to change of sign in the imaginary part of the energy.

*d) Classical Trajectory of a Particle of Energy $-1 - 2i$*

For the classical particle with complex energy $E = -1 - 2i$ in the double well potential, turning points are 1.2503 - 0.4804i, 0.6760 + 0.8885i, -1.2503 + 0.4804i, -0.6760 - 0.8885i which are indicated by red dots. A single classical trajectory of classical particle having energy $E = -1 - 2i$ in the complex-x plane sketched in figure 5. In figure 5 the trajectory begins at $x = 1$, and it spirals around the right pair of turning points of the right-half of double-well potential then it crosses imaginary axis $x = 0$ and enters into left-half of double well potential and spirals around the left pair of turning points. In figure 6 the trajectory begins at $x = -1$ and it spirals around the left pair of turning points of the left-half of double-well potential then it crosses imaginary axis $x = 0$ and enters into right-half

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of double well potential and spirals around the right pair of turning points. Both trajectories are not periodic because they are open. According to classical mechanics this type of motion is forbidden, but allowing the energy of a classical particle to be complex, we get a quantum mechanical phenomenon.

*e) Classical Trajectory of a Particle of Energy* $-1+2i$

The single classical trajectory of a classical particle having energy $\mathrm{E}=-1+2\mathrm{i}$ in the double-well potential in the complex domain pictured in figure 7. The four classical turning points are associated with this energy are 1.3077 + 0.4956i, 0.6754 - 0.9595i, -1.3077 - 0.4956i, -0.6754 + 0.9595i, which are indicated by red dots in the figure 7. The particle begins its motion from position x=1, and it shows classical tunneling through imaginary plane $x=0$. Although we changed the sign of imaginary part of energy, it does not change tunneling of the classical particle. The classical particle first spirals around the right pair of turning points then leaps to left-half of the complex-x plane. Then the particle spirals around the left pair of turning points.

*f) Classical Trajectory of a Particle of Energy* $-1-2.5i$

A single classical trajectory of a particle with energy $\mathrm{E}=-1-2.5\mathrm{i}$ in the double-well potential in the complex-x plane pictured in figure 8. The four turning points associated with energy $\mathrm{E}=-1-2.5\mathrm{i}$ are 1.3077 - 0.4956i, 0.6754 + 0.9595i, -1.3077 + 0.4956i, -0.6754 - 0.9595i which are indicated by red dots. The particle begins its journey from x=8 and enters into left-half of the complex-x plane crossing imaginary axis x=0. Then spirals around the left pair of turning points, and then it returns to right-half of the complex-x plane. So, in this case, we have two times tunneling through imaginary axis x=0. The trajectory is not periodic because it is open.

*g) Classical Trajectory of a Particle of Energy* $-1+2.5i$

Let us investigate what happens if we take a classical particle having complex conjugate of energy $\mathrm{E}=-1-2.5\mathrm{i}$, i.e.$-1+2.5\mathrm{i}$in the double-well potential. A single classical trajectory of a particle with energy $\mathrm{E}=-1+2.5\mathrm{i}$ in the double-well potential in the complex-x plane presented in figure 9. The four turning points associated with energy $\mathrm{E}=-1+2.5\mathrm{i}$ are 1.3077 + 0.4956i, 0.6754 - 0.9595i, -1.3077 - 0.4956i, -0.6754 + 0.9595i which are indicated by red dots. The particle begins its journey from x=8 and enters into left-half of the complex-x plane crossing imaginary axis x=0. Then spirals around left pair of turning points, and then it returns to right-half of the complex-x plane. So, in this case, we have two times tunneling through imaginary axis x=0. Although we take energy $\mathrm{E}=-1+2.5\mathrm{i}$, the complex conjugate of $\mathrm{E}=-1-2.5\mathrm{i}$, we get same result i.e., tunneling. The trajectory is not periodic because it is open.

## IV. Results and Discussion

We investigated the motion of a classical particle in the classical system using potential $V(x)=x^4-x^2$, in the complex domain by numerically. We found that the trajectories of the particle in the potential, $x^4-x^2$ are always confined to either right-half or left-half of complex-x plane for a negative real value of energy. Thus, the potential acts like a double-well potential, one well is left-half and another well is right-half of complex-x plane, separated by the imaginary axis x=0, leads to no effect analogous to quantum tunneling. Butthe energy of a deterministic particle being complex in the double-well potential, $x^4-x^2$, the corresponding system presents an effect analogous to quantum tunneling. We examine the analog to the quantum tunneling in the complex classical system for different four energies $-1-2i, -1+2i, -1+2.5i$ and $-1-2.5i$. The classical 'tunneling' process is less abstract and hence easier to understand than quantum-mechanical analog. During quantum tunneling the particle disappears from one classical region and reappears almost immediately in another region giving no idea about the path. For a classical particle, it is clear how the particle travels from one classically allowed region to the other i.e., it follows a well-defined path in the complex-x plane.

## V. Conclusion

Investigation of the trajectories of particle having real but negative energy in the potential $x^4-x^2$ gives $x^4-x^2$ acts like double-well potential. Particle exhibits periodic motion for real energy and confined to either right-half or left-half of the complex-x plane separated by imaginary axis $x=0$ for negative energy. The open classical trajectories that result from complex energy are particularly interesting because of their behavior reminiscent of the phenomenon of quantum tunneling-a negative energy quantum particle in such potential tunnels back and forth from one-well to another-well.

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## Figures

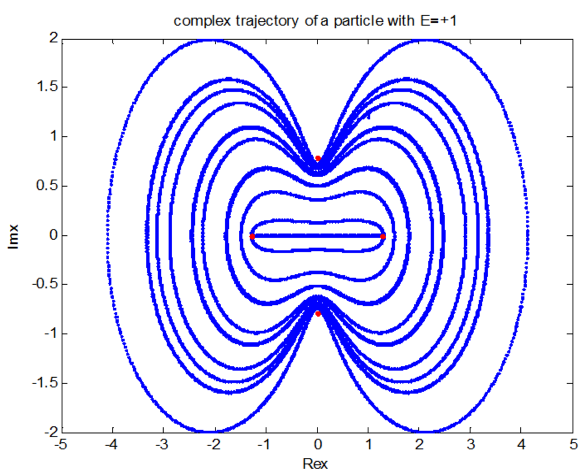


Figure 1

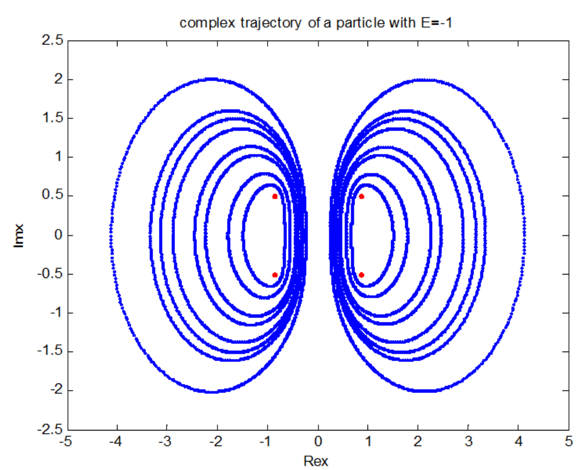


Figure 2

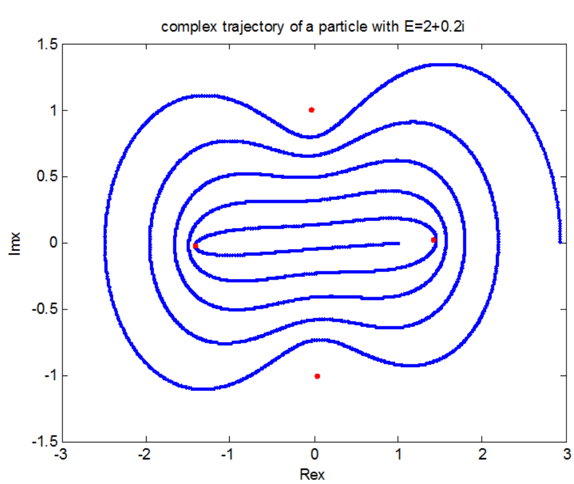


Figure 3

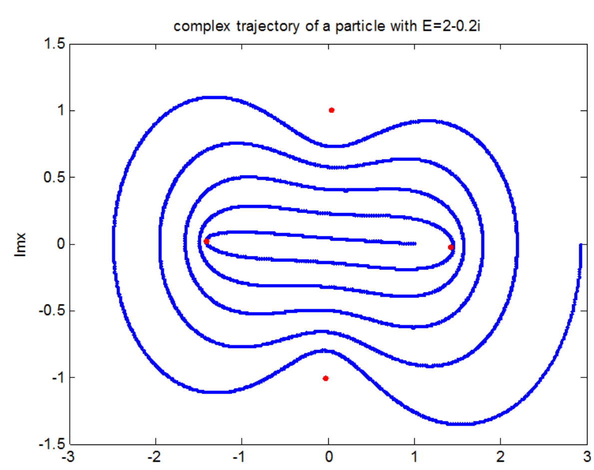


Figure 4

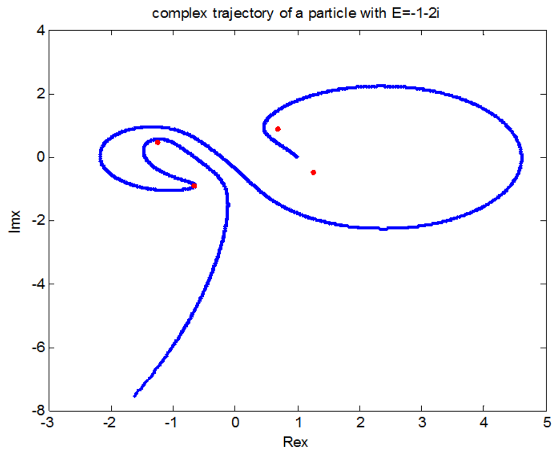


Figure 5

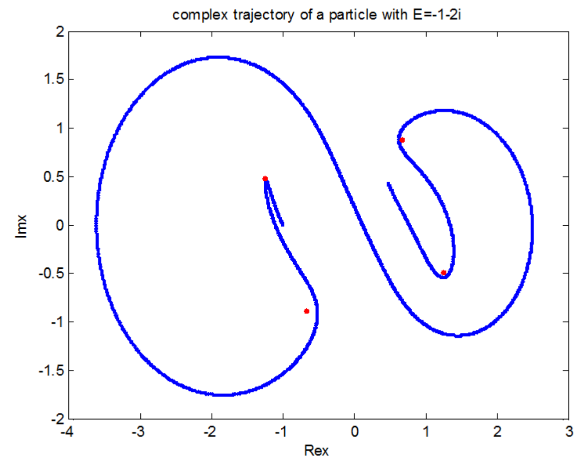


Figure 6

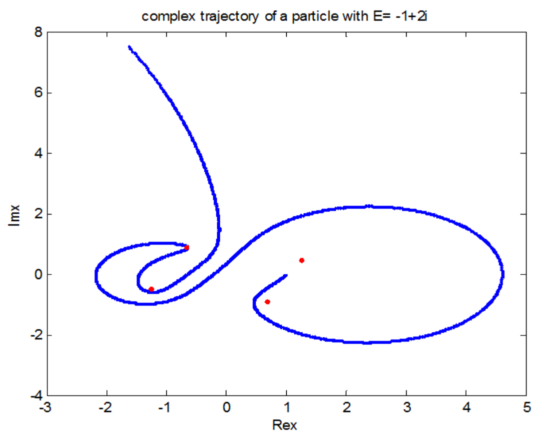


Figure 7

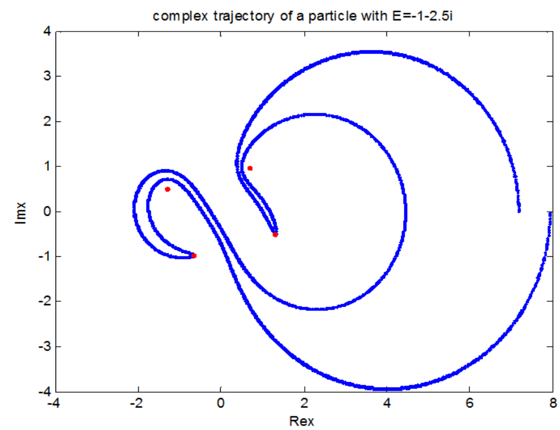


Figure 8

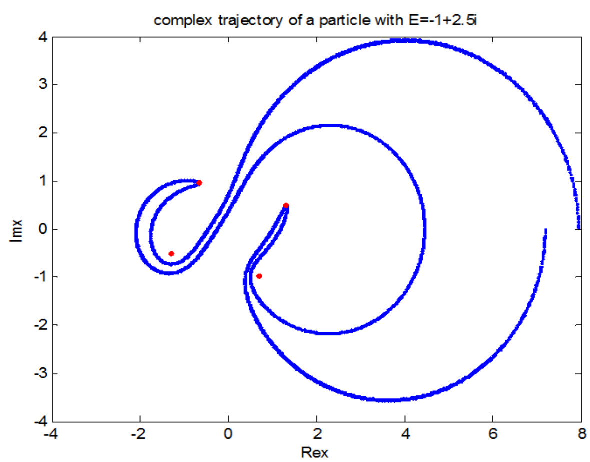


Figure 9