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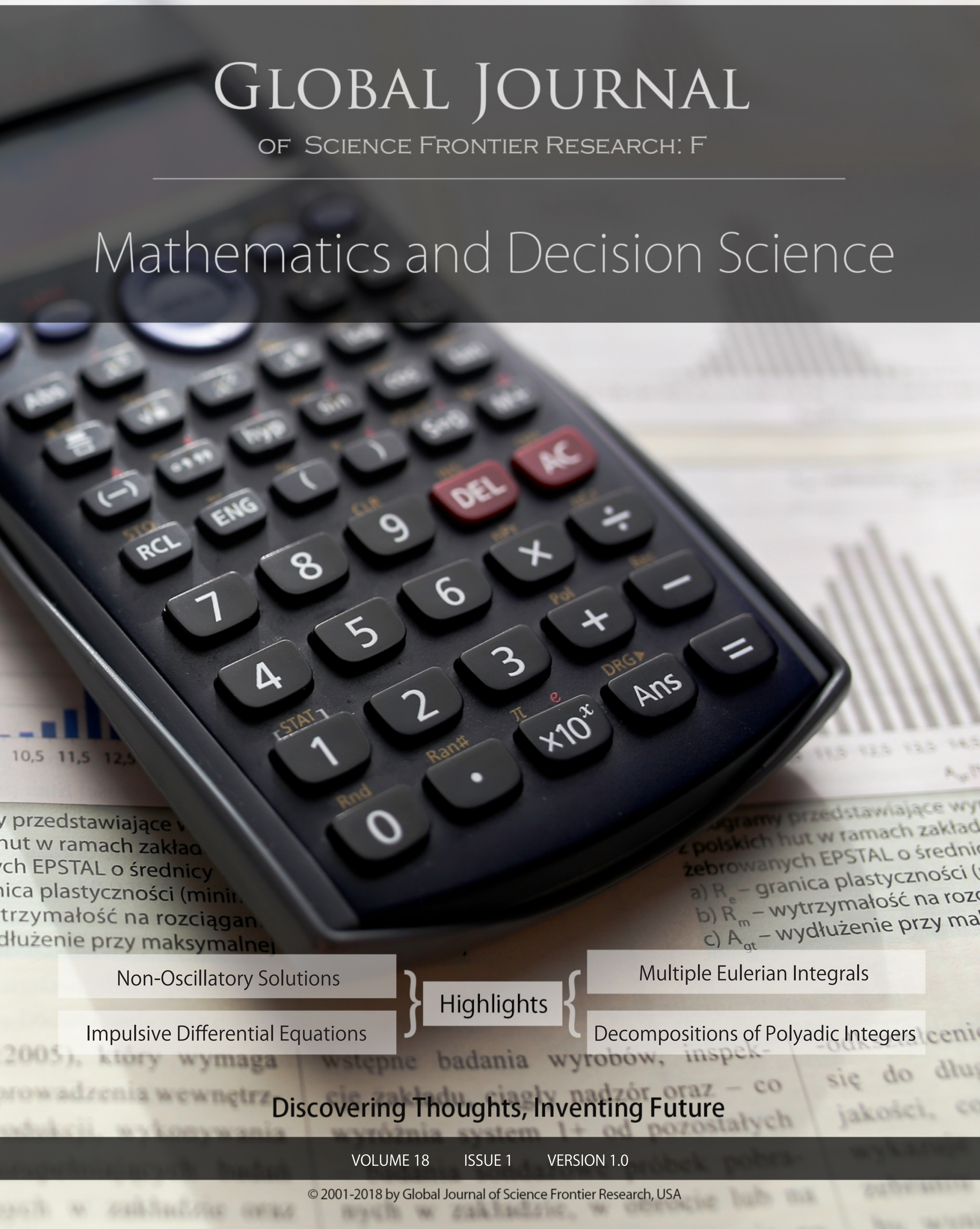

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Mathematics and Decision Science

Highlights

- Non-Oscillatory Solutions
- Impulsive Differential Equations
- Multiple Eulerian Integrals
- Decompositions of Polyadic Integers

Discovering Thoughts, Inventing Future

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On a General Class of Multiple Eulerian Integrals with Multivariable I-Functions

By Frederic Ayant

Abstract- Recently, Raina and Srivastava and Srivastava and Hussain have provided closed-form expressions for a number of a Eulerian integral involving multivariable H-functions. Motivated by these recent works, we aim at evaluating a general class of multiple Eulerian integrals concerning the product of two multivariable I-functions defined by Prathima et al. [6], a class of multivariable polynomials and the spheroidal function. These integrals will serve as a capital formula from which one can deduce numerous integrals.

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2. A. Bhargava, A. Srivastava and O. Mukherjee, On a General Class of Multiple Eulerian Integrals. International Journal of Latest Technology in Engineering, Management & Applied Science (IJLTEMAS), 3(8) (2014), 57-64.

On a General Class of Multiple Eulerian Integrals with Multivariable I-Functions

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Keywords: multivariable I-function, multiple eulerian integrals, class of polynomials, spheroidal functions, I-function of two variables, I-function of one variable.

I. Introduction and Prerequisites

The well-known Eulerian Beta integral [5]

$$\int_a^b (z-a)^{\alpha-1}(b-t)^{\beta-1}\mathrm{d}t = (b-a)^{\alpha+\beta-1}B(\alpha,\beta)(Re(\alpha)>0, Re(\beta)>0, b>a) \tag{1.1}$$

Is a basic result of evaluation of numerous other potentially useful integrals involving various special functions and polynomials. The mathematicians Raina and Srivastava [7], Saigo and Saxena [9], Srivastava and Hussain [14], Srivastava and Garg [13] et cetera have established a number of Eulerian integrals involving the various general class of polynomials, Meijer's G-function and Fox's H-function of one and more variables with general arguments. Recently, several Author study some multiple Eulerian integrals, see Bhargava [2], Goyal and Mathur [4] and others. The aim of this paper is to obtain general multiple Eulerian integrals of the product of two multivariable I-functions defined by Prathima et al [7], a general class of multivariable polynomials [12] and the spheroidal functions.

The spheroidal function $\psi_{\alpha n'}(c,\eta)$ of general order $\alpha > -1$ can be expanded as ([10], [18]).

$$\psi_{\alpha n''}(c,\eta) = \frac{i^{n''}\sqrt{2\pi}}{V_{\alpha n''}(c)} \sum_{k=0,or1}^{\infty *} a_k(c|\alpha n'')(c\eta)^{-\alpha-\frac{1}{2}} J_{k+\alpha+\frac{1}{2}}(c\eta) \tag{1.2}$$

Which represents the expression uniformly on (∞,∞), where the coefficients $a_k(c|\alpha n'')$ satisfy the recursion formula and the asterisk over the summation sign indicate that the sum is taken over only even or odd values of according as n'' is even or odd. As $c\to 0, a_k(c|\alpha n'')\to 0, k\neq n''$

The class of multivariable polynomials defined by Srivastava [12], is given in the following manner:

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$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1,K_1;\cdots;N_v,K_v] y_1^{K_1}\cdots y_v^{K_v} \tag{1.3}$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary real or complex constants.

We shall note $a'_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1,K_1;\cdots;N_v,K_v]$

The I-function of several variables is a generalization of the multivariable H-function studied by Srivastava et Panda [16,17]. The multiple Mellin-Barnes integrals occurring in this paper will be referred to as the multivariables I-function of r-variables throughout our present study and will refer and represented as follows:

$$\bar{I}(z_1,\cdots,z_r) = \bar{I}_{p,q:p_1,q_1;\cdots;p_r,q_r}^{0,n:m_1,n_1;\cdots;m_r,n_r}\left(\begin{array}{c|l} \mathrm{z}_1 & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p} : \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q} : \end{array}\right.$$

$$\left.\begin{array}{l} (\mathrm{c}_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,n_1},(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{n_1+1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,n_r},(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{n_r+1,p_r} \\ (\mathrm{d}_j^{(1)},\delta_j^{(1)};1)_{1,m_1},(d_j^{(1)},\delta_j^{(1)};D_1)_{m_1+1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};1)_{1,m_r},(d_j^{(r)},\delta_j^{(r)};D_r)_{m_r+1,q_r} \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.4}$$

Where $\psi(s_1,\cdots,s_r),\theta_i(s_i), i=1,\cdots,r$ are given by :

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)}$$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}-\delta_j^{(i)}s_i\right)}$$

For more details, see Prathima et al. [6].
Following the result of Braaksma [3] the I-function of r variables is analytic if :

$$U_i = \sum_{j=1}^{p}A_j\alpha_j^{(i)} - \sum_{j=1}^{q}B_j\beta_j^{(i)} + \sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)} \leqslant 0; i=1,\cdots,r \tag{1.5}$$

The integral (1.4) converges absolutely if

$|arg(z_k)| < \frac{1}{2}\Delta_k\pi, k=1,\cdots,r$ where

$$\Delta_k = -\sum_{j=n+1}^{p}A_j\alpha_j^{(k)} - \sum_{j=1}^{q}B_j\beta_j^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)} > 0 \tag{1.6}$$

Ref

3. B. L. J. Braaksma, "Asymptotic expansions and analytic continuations for a class of Barnes integrals," Composition Mathematical, vol. 15, pp. 239–341, 1964.

and if all the poles of (1.6) are simple ,then the integral (1.4) can be evaluated with the help of the residue theorem to give

$$\bar{I}(z_1,\cdots,z_r) = \sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!} \tag{1.7}$$

Where ϕ and ϕ_i are defined by

$$\phi = \frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}S_k\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}S_k\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}S_k\right)}$$

and

$$\phi_i = \frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}S_k\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}S_k\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}S_k\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}S_k\right)}, i=1,\cdots,r$$

where

$$S_k = \eta_{G_k,g_k} = \frac{d_{g_k}^{(k)}+G_k}{\delta_{g_k}^{(k)}} \text{ for } k=1,\cdots,r$$

which is valid under the following conditions: $\epsilon_{M_k}^{(k)}[p_j^{(k)}+p'_k] \neq \epsilon_j^{(k)}[p_{M_k}+g_k]$
Consider the second multivariable I-function.

$$I(z'_1,\cdots,z'_s) = I_{p',q':p'_1,q'_1;\cdots;p'_s,q'_s}^{0,n':m'_1,n'_1;\cdots;m'_s,n'_s}\left(\begin{array}{c|c} z'_1 & (\text{a}'_j;\alpha'^{(1)}_j,\cdots,\alpha'^{(s)}_j;A'_j)_{1,p'}: \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (\text{b}'_j;\beta'^{(1)}_j,\cdots,\beta'^{(s)}_j;B'_j)_{1,q'}: \end{array}\right.$$

$$\left.\begin{array}{l}(\text{c}'^{(1)}_j,\gamma'^{(1)}_j;C'^{(1)}_j)_{1,p'_1};\cdots;(c'^{(s)}_j,\gamma'^{(s)}_j;C'^{(s)}_j)_{1,p'_s} \\ \\ (\text{d}'^{(1)}_j,\delta'^{(1)}_j;D'^{(1)}_j)_{1,q'_1};\cdots;(d'^{(s)}_j,\delta'^{(s)}_j;D'^{(s)}_j)_{1,q'_s}\end{array}\right) = \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\zeta(t_1,\cdots,t_s)\prod_{k=1}^{s}\phi_k(t_k)z'^{t_k}_k\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.8}$$

where $\zeta(t_1,\cdots,t_s)$, $\phi_i(s_i)$, $i=1,\cdots,s$ are given by :

$$\zeta(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{n'}\Gamma^{A'_j}\left(1-a'_j+\sum_{i=1}^{s}\alpha'^{(i)}_j t_j\right)}{\prod_{j=n'+1}^{p'}\Gamma^{A'_j}\left(a'_j-\sum_{i=1}^{s}\alpha'^{(i)}_j t_j\right)\prod_{j=1}^{q'}\Gamma^{B'_j}\left(1-b'_j+\sum_{i=1}^{s}\beta'^{(i)}_j t_j\right)}$$

$$\phi_i(s_i) = \frac{\prod_{j=1}^{n'_i}\Gamma^{C'^{(i)}_j}\left(1-c'^{(i)}_j+\gamma'^{(i)}_j t_i\right)\prod_{j=1}^{m'_i}\Gamma^{D'^{(i)}_j}\left(d'^{(i)}_j-\delta'^{(i)}_j t_i\right)}{\prod_{j=n'_i+1}^{p'_i}\Gamma^{C'^{(i)}_j}\left(c'^{(i)}_j-\gamma'^{(i)}_j t_i\right)\prod_{j=m'_i+1}^{q'_i}\Gamma^{D'^{(i)}_j}\left(1-d'^{(i)}_j-\delta'^{(i)}_j t_i\right)}$$

For more details, see Prathima et al. [6].
Following the result of Braaksma [3], the I-function of r variables is analytic if:

Ref

6. J. Prathima, V. Nambisan and S.K.Kurumujji, A Study of I-function of Several Complex Variables, International Journal of Engineering Mathematics Vol (2014), 2014, 1-12.

$$U'_i = \sum_{j=1}^{p'} A'_j \alpha'^{(i)}_j - \sum_{j=1}^{q'} B'_j \beta'^{(i)}_j + \sum_{j=1}^{p'_i} C'^{(i)}_j \gamma'^{(i)}_j - \sum_{j=1}^{q'_i} D'^{(i)}_j \delta'^{(i)}_j \leqslant 0; i = 1, \cdots, s \tag{1.9}$$

The integral (1.13) converges absolutely if

$|arg(z'_k)| < \frac{1}{2}\Delta'_k \pi; k = 1, \cdots, s$. Where

$$\Delta'_k = -\sum_{j=n'+1}^{p'} A'_j \alpha'^{(k)}_j - \sum_{j=1}^{q'} B'_j \beta'^{(k)}_j + \sum_{j=1}^{m'_k} D'^{(k)}_j \delta'^{(k)}_j - \sum_{j=m'_k+1}^{q'_k} D'^{(k)}_j \delta'^{(k)}_j + \sum_{j=1}^{n'_k} C'^{(k)}_j \gamma'^{(k)}_j - \sum_{j=n'_k+1}^{p'_k} C'^{(k)}_j \gamma'^{(k)}_j > 0 \tag{1.10}$$

II. Integral Representation of Generalized Hypergeometric Function

The following generalized hypergeometric function regarding multiple integrals contour is also required [15, page 39 eq .30]

$$\frac{\prod_{j=1}^{P}\Gamma(A_j)}{\prod_{j=1}^{Q}\Gamma(B_j)} {}_PF_Q\left[(A_P);(B_Q);-(x_1+\cdots+x_r)\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \frac{\prod_{j=1}^{P}\Gamma(A_j+s_1+\cdots+s_r)}{\prod_{j=1}^{Q}\Gamma(B_j+s_1+\cdots+s_r)}\Gamma(-s_1)\cdots\Gamma(-s_r)x_1^{s_1}\cdots x_r^{s_r}\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{2.1}$$

where the contours are of Barnes type with indentations, if necessary, to ensure that the poles of $\Gamma(A_j+s_1+\cdots+s_r)$ are separated from those of $\Gamma(-s_j), j=1,\cdots,r$. The above result (2.1) is easily established by an appeal to the calculus of residues by calculating the residues at the poles of $\Gamma(-s_j), j=1,\cdots,r$

The equivalent form of Eulerian beta integral is given by (1.1):

III. Main Integral

We shall note:

$$X = m'_1, n'_1; \cdots; m'_s, n'_s; 1,0; \cdots; 1,0; 1,0; \cdots; 1,0$$

$$Y = p'_1, q'_1; \cdots; p'_s, q'_s; 0,1; \cdots; 0,1; 0,1; \cdots; 0,1$$

$$\mathbb{A} = [1+\sigma_i^{(1)} - \sum_{k'=1}^{v} K_{k'}\rho_i''^{(1,k')} - \sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(1,k)} - \theta_i^{(1)}(2R+k'); \rho_i'^{(1,1)}, \cdots, \rho_i'^{(1,s)}, \tau_i^{(1,1)}, \cdots, \tau_i^{(1,l)}, 1, 0, \cdots, 0; 1]_{1,s}$$

$$, \cdots, [1+\sigma_i^{(T)} - \sum_{k'=1}^{v} K_{k'}\rho_i''^{(T,k')} - \sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(T,k)} - \theta_i^{(T)}(2R+k'); \rho_i'^{(T,1)}, \cdots, \rho_i'^{(T,s)}, \tau_i^{(T,1)}, \cdots, \tau_i^{(T,l)}, 1, 0, \cdots, 0; 1]_{1,s},$$

$$[1-A_j; 0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0; 1]_{1,P},$$

$$[1-\alpha_i - \sum_{k'=1}^{v} K_{k'}\delta_i''^{(k')} - \sum_{k=1}^{r}\eta_{G_k,g_k}\delta_i^{(k)} - (2R+k')\zeta_i; \delta_i'^{(1)}, \cdots, \delta_i'^{(s)}, \mu_i^{(1)}, \cdots, \mu_i^{(l)}, 1, \cdots, 1, 0, \cdots, 0; 1]_{1,s},$$

Ref

15. H.M. Srivastava and P.W. Karlsson, Multiple Gaussian Hypergeometric series. Ellis. Horwood. Limited. New-York, Chichester. Brisbane. Toronto, 1985.

$$[1-\beta_i-\sum_{k'=1}^{v}K_{k'}\eta_i''^{(k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\eta_i^{(k)}-(2R+k')\lambda_i;\eta_i'^{(1)},\cdots,\eta_i'^{(s)},\theta_i^{(1)},\cdots,\theta_i^{(l)},1,\cdots,1,0,\cdots,0;1]_{1,s}$$

$$\mathbf{A}=(a_j';\alpha_j'^{(1)},\cdots,\alpha_j'^{(s)},0,\cdots,0,0,\cdots,0;A_j')_{1,p'}:(c_j'^{(1)},\gamma_j'^{(1)};C_j'^{(1)})_{1,p_1'};\cdots;(c_j'^{(r)},\gamma_j^{(s)};C_j'^{(s)})_{1,p_s'};$$

Notes

$$(1,0;1);\cdots;(1,0;1);(1,0;1);\cdots;(1,0;1)$$

$$\mathbb{B}=[1+\sigma_i^{(1)}-\sum_{k'=1}^{v}K_{k'}\rho_i''^{(1,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(1,k)}-\theta_i^{(1)}(2R+k');\rho_i'^{(1,1)},\cdots,\rho_i'^{(1,s)},\tau_i^{(1,1)},\cdots,\tau_i^{(1,l)},0,\cdots,0;1]_{1,s}$$

$$,\cdots,[1+\sigma_i^{(T)}-\sum_{k'=1}^{v}K_{k'}\rho_i''^{(T,k')}-\sum_{k=1}^{r}\eta_{G_k,g_k}\rho_i^{(T,k)}-\theta_i^{(T)}(2R+k');\rho_i'^{(T,1)},\cdots,\rho_i'^{(T,s)},\tau_i^{(T,1)},\cdots,\tau_i^{(T,l)},0,\cdots,0;1]_{1,s},$$

$$[1-B_j;0,\cdots,0,1,\cdots,1,0,\cdots,0]_{1,Q},$$

$$[1-\alpha_i-\beta_i-\sum_{k'=1}^{v}(\delta_i''^{(k')}+\eta_i''^{(k')})K_{k'}-\sum_{k=1}^{r}(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}-(\zeta_i+\lambda_i)(2R+k');$$

$$(\delta_i'^{(1)}+\eta_i'^{(1)}),\cdots,(\delta_i'^{(s)}+\eta_i'^{(s)}),(\mu_i^{(1)}+\theta_i^{(1)}),\cdots,(\mu_i^{(l)}+\theta_i^{(l)}),1,\cdots,1;1]_{1,s}$$

$$\mathbf{B}=(b_j';\beta_j'^{(1)},\cdots,\beta_j'^{(s)},0,\cdots,0,0,\cdots,0;B_j')_{1,q'}:(\mathrm{d}_j'^{(1)},\delta_j'^{(1)};D_j'^{(1)})_{1,q_1'};\cdots;(d_j'^{(s)},\delta_j'^{(s)};D_j'^{(s)})_{1,q_s'};$$

$$(0,1;1);\cdots;(0,1;1);(0,1;1);\cdots;(0,1;1)$$

We have the following multiple Eulerian integrals, and we obtain the I-function of variables.

Theorem

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$I\begin{pmatrix}\mathrm{z}_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(r)}}(v_i-x_i)^{\eta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,r)}}}\right]\end{pmatrix} I\begin{pmatrix}\mathrm{z}_1'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_s'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(s)}}(v_i-x_i)^{\eta_i'^{(s)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,s)}}}\right]\end{pmatrix}$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} z_1'' \prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ z_v'' \prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right] \end{array}\right)\psi_{\alpha n''}\left[c^\sigma, \prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l} g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{k''=0,or1}^{\infty_*}\sum_{R=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty}\phi\frac{\prod_{k=1}^{r}\phi_k z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{r}g_k}}{\prod_{k=1}^{r}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{r}g_k!}a_v' z_1''^{K_1}\cdots z_v''^{K_v}\frac{(-)^R a_{k''}(c^\sigma|\alpha n'')}{R!\Gamma(R+k''+\alpha+\frac{3}{2})}E_{ij}$$

$$I_{sT+P+2s+p',sT+Q+s+q':Y}^{0,sT+P+2s+n':X}\left(\begin{array}{c|c} z_1'w_1 & \mathbb{A},\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_1'w_s & \cdot \\ g_1W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ g_lW_l & \cdot \\ G_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ G_T & \mathbb{B},\ \mathbf{B} \end{array}\right) \tag{3.1}$$

Where

$$E_{ij}=\frac{\frac{i^{n''}\sqrt{2\pi}}{V_{\alpha n''}(c)}}{\prod_{i=1}^{t}\prod_{j=1}^{W}(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^{v}\rho_i''^{(j,k')}K_{k'}+\sum_{k=1}^{r}\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}(2R+k'')}}$$

$$\times\frac{\prod_{i=1}^{t}(v_i-u_i)^{\sum_{k'=1}^{v}(\delta_i''^{(k')}+\eta_i''^{(k')})K_{k'}+\sum_{k=1}^{r}(\delta_i^{(k)}+\eta_i^{(k)})\eta_{G_k,g_k}+(\zeta_i+\lambda_i)(2R+k'')}}{\prod_{i=1}^{t}\prod_{j=W+1}^{T}(u_iU_i^{(j)}+V_i^{(j)})^{\sum_{k'=1}^{v}\rho_i''^{(j,k')}K_{k'}+\sum_{k=1}^{r}\rho_i^{(j,k)}\eta_{G_k,g_k}+\theta_i^{(j)}(2R+k'')}}$$

$$w_m=\prod_{i=1}^{t}\left[(v_i-u_i)^{\delta_i'^{(m)}+\eta_i'^{(m)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\rho_i'^{(j,m)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\rho_i'^{(j,m)}}\right],m=1,\cdots,s$$

Notes

$$W_k = \prod_{i=1}^{t}\left[(v_i-u_i)^{\mu_i^{(k)}+\theta_i^{(k)}}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{-\tau_i^{(j,k)}}\right], k=1,\cdots,l$$

$$G_j = \prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=1,\cdots,W$$

Notes

$$G_j = -\prod_{i=1}^{t}\left[\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right], j=W+1,\cdots,T$$

$$\sum_{G_k=1}^{m_k}\sum_{g_k=0}^{\infty} = \sum_{G_1,\cdots,G_r=1}^{m_1,\cdots,m_r}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

Provided that:

(A) $W\in[0,T]; u_i,v_i\in\mathbb{R}; i=1,\cdots,t$

(B) $min\{\delta_i^{(g)},\eta_i^{(g)},\delta_i'^{(h)},\eta_i'^{(h)},\delta_i''^{(k)},\eta_i''^{(k)},\zeta_i,\eta_i\}\geqslant 0; g=1,\cdots,r; i=1,\cdots,t; h=1,\cdots,s; k=1,\cdots,v$

$min\{\rho_i^{(j,g)},\rho_i'^{(j,h)},\rho_i''^{(j,k')},\theta_i^{(j)},\tau_i^{(j,k)}\}\geqslant 0; j=1,\cdots,T; i=1,\cdots,t; g=1,\cdots,r; h=1,\cdots,s; k'=1,\cdots,v, k=1,\cdots,l$

(C) $\sigma_i^{(j)}\in\mathbb{R}, U_i^{(j)}, V_i^{(j)}\in\mathbb{C}, z_{i'}, z_{j'}, z''_{k'}, g_k, G_j\in\mathbb{C}; i=1,\cdots,t; j=1,\cdots,T; i'=1,\cdots,r;$

$j'=1,\cdots,s; k'=1,\cdots,v; k=1,\cdots,l$

(D) $max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right]<1, i=1,\cdots,s; j=1,\cdots,W$ and

$$max\left[\left|\frac{(v_i-u_i)U_i^{(j)}}{u_iU_i^{(j)}+V_i^{(j)}}\right|\right]<1, i=1,\cdots,s; j=W+1,\cdots,T$$

(E) $\left|arg\left(z_i\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i^{(j,k)}}\right)\right|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,r,$ where

$$\Delta_k = -\sum_{j=n+1}^{p}A_j\alpha_j^{(k)} - \sum_{j=1}^{q}B_j\beta_j^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)} + \sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)} - \sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}$$

$$-\delta_i^{(k)} - \eta_i^{(k)} - \sum_{j=1}^{T}\rho_i^{(j,k)} > 0$$

$\left|arg\left(z_i'\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\rho_i'^{(j,k)}}\right)\right|\frac{1}{2}\Delta_k'\pi, k=1,\cdots,s,$ where

$$\Delta_k' = -\sum_{j=n'+1}^{p'}A_j'\alpha_j'^{(k)} - \sum_{j=1}^{q'}B_j'\beta_j'^{(k)} + \sum_{j=1}^{m_k'}D_j'^{(k)}\delta_j'^{(k)} - \sum_{j=m_k'+1}^{q_k'}D_j'^{(k)}\delta_j'^{(k)} + \sum_{j=1}^{n_k'}C_j'^{(k)}\gamma_j'^{(k)} - \sum_{j=n_k'+1}^{p_k'}C_j'^{(k)}\gamma_j'^{(k)}$$

$$-\delta_i'^{(k)} - \eta_i'^{(k)} - \sum_{j=1}^{T}\rho_i'^{(j,k)} > 0$$

(F) $Re\left(\alpha_i+\zeta_i(2R+k'')+\sum_{j=1}^{r}\delta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\delta_i'^{(k)}\min_{1\leqslant j\leqslant m_k'}Re\left(\frac{d_j'^{(k)}}{\delta_j'^{(k)}}\right)>0$ and

$$Re\left(\beta_i+\lambda_i(2R+k'')+\sum_{j=1}^{r}\eta_i^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}\eta_i'^{(k)}\min_{1\leqslant j\leqslant m_k'}Re\left(\frac{d_j'^{(k)}}{\delta_j'^{(k)}}\right)>0 \text{ for } i=1,\cdots,t$$

(G) $P\leqslant Q+1$. The equality holds, when , in addition,

$$\text{either } P>Q \text{ and } \sum_{k=1}^{l}\left|g_k\left(\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\tau_i^{(j,k)}}\right)\right|^{\frac{1}{Q-P}}<1 \qquad (u_i\leqslant x_i\leqslant v_i;i=1,\cdots,t)$$

$$\text{or } P\leqslant Q \text{ and } \max_{1\leqslant k\leqslant l}\left[\left|\left(g_k\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{-\tau_i^{(j,k)}}\right)\right|\right]<1 \qquad (u_i\leqslant x_i\leqslant v_i;i=1,\cdots,t)$$

Proof

To establish the formula (4.7), we first express the spheroidal function, the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ and the multivariable I-function $I(z_1,\cdots,z_r)$ in series with the help of (1,2), (1,3) and (1.7) respectively, use integral contour representation with the help of (1.8) for the multivariable I-function $I(z_1',\cdots,z_s')$ occurring in its left-hand side and use the integral contour representation with the help of (2.1) for the Generalized hypergeometric function ${}_PF_Q(.)$. Changing the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and the summations involved in the process). Now we write:

$$\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}} \tag{3.2}$$

where $K_i^{(j)}=\upsilon_i^{(j)}-\theta_i^{(j)}(2R+k'')-\sum_{l=1}^{r}\rho_i^{(j,l)}\eta_{G_l,g_l}-\sum_{l=1}^{s}\rho_i'^{(j,l)}\psi_l-\sum_{l=1}^{v}\rho_i''^{(j,v)}K_l$ where $i=1,\cdots,t;j=1,\cdots,T$

and express the factors occurring in R.H.S. Of (3.1) in terms of following Mellin-Barnes integrals contour, we obtain:

$$\prod_{j=1}^{W}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}u_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^W}\int_{L_1'}\cdots\int_{L_W'}\prod_{j=1}^{W}\left[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')\right.$$

$$\left.\prod_{j=1}^{W}\left[\frac{(U_i^{(j)}(x_i-u_i)}{(u_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\right]\mathrm{d}\zeta_1'\cdots\mathrm{d}\zeta_W' \tag{3.3}$$

$$\prod_{j=W+1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{K_i^{(j)}}=\prod_{j=W+1}^{T}\left[\frac{(U_i^{(j)}v_i+V_i^{(j)})^{K_i^{(j)}}}{\Gamma(-K_i^{(j)})}\right]\frac{1}{(2\pi\omega)^{T-W}}\int_{L_{W+1}'}\cdots\int_{L_T'}\prod_{j=W+1}^{T}\left[\Gamma(-\zeta_j')\Gamma(-K_i^{(j)}+\zeta_j')\right.$$

$$\left.\prod_{j=W+1}^{T}\left[-\frac{(U_i^{(j)}(v_i-x_i)}{(v_iU_i^{(j)}+V_i^{(j)})}\right]^{\zeta_j'}\right]\mathrm{d}\zeta_{W+1}'\cdots\mathrm{d}\zeta_T' \tag{3.4}$$

We apply the Fubini's theorem for multiple integrals. Finally evaluating the innermost **x**-integral with the help of (1.1) and reinterpreting the multiple Mellin-Barnes integrals contour in terms of multivariable I-function of $(r+l+T)$ variables, we obtain the formula (3.1).

IV. Particular Cases

a) I-functions of two variables

Corollary 1

If $r=s=2$, then the multivariable I-functions reduce to I-functions of two variables defined by Rathie et al. [9]. We have.

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$\bar{I}\left(\begin{array}{c} \mathrm{z}_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_2\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(2)}}(v_i-x_i)^{\eta_i^{(2)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,2)}}}\right]\end{array}\right) I\left(\begin{array}{c} \mathrm{z}'_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_2\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(2)}}(v_i-x_i)^{\eta_i'^{(2)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,2)}}}\right]\end{array}\right)$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}''_v\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right]\end{array}\right)\psi_{\alpha n''}\left[c^{\sigma},\prod_{j=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{j=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{k''=0,or1}^{\infty_*}\sum_{R=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{G_2=1}^{m_2}\sum_{g_1,g_2=0}^{\infty}\phi_2\frac{\prod_{k=1}^{2}\phi_{2k}z_k^{\eta_{G_k,g_k}}(-)^{\sum_{k=1}^{2}g_k}}{\prod_{k=1}^{2}\delta_{G^{(k)}}^{(k)}\prod_{k=1}^{2}g_k!}a'_v z_1''^{K_1}\cdots z_v''^{K_v}\frac{(-)^R a_{k''}(c^{\sigma}|\alpha n'')}{R!\Gamma(R+k''+\alpha+\frac{3}{2})}$$

Ref

9. A.K. Rathie, K.S. Kumari and T.M. Vasudevan Nambisan, A study of I-functions of two variables, Le Matematiche 69(1) (2014), 285-305.

$$E_{ij} I^{0,sT+P+2s+n':X_2}_{sT+P+2s+p',sT+Q+s+q':Y_2}\left(\begin{array}{c|c} \mathrm{z}'_1 w_1 & \mathbb{A}_2,\ \mathbf{A}_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}'_2 w_2 & \cdot \\ \mathrm{g}_1 W_1 & ; \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{g}_l W_l & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{G}_T & \mathbb{B}_2,\ \mathbf{B}_2 \end{array}\right) \tag{4.1}$$

The validity conditions are the same that (3.1) with $r=s=2$. The quantities $\phi_2, \phi_{2k}, V_2, W_2, \mathbb{A}_2, \mathbb{B}_2, \mathbf{A}_2, \mathbf{B}_2$ are equal to $\phi, \phi_k, V, W, \mathbb{A}, \mathbb{B}, \mathbf{A}, \mathbf{B}$ respectively for $r=s=2$.

b) I-function of one variable

Corollary 2

If $r=s=1$, the multivariable I-functions reduce to I-functions of one variable defined by Rathie [8]. We have

$$\int_{u_1}^{v_1}\cdots\int_{u_t}^{v_t}\prod_{i=1}^{t}\left[(x_i-u_i)^{\alpha_i-1}(v_i-x_i)^{\beta_i-1}\prod_{j=1}^{T}(U_i^{(j)}x_i+V_i^{(j)})^{\sigma_i^{(j)}}\right]$$

$$I\left(\mathrm{z}\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i^{(1)}}(v_i-x_i)^{\eta_i^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i^{(j,1)}}}\right]\right) I\left(\mathrm{z}'\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i'^{(1)}}(v_i-x_i)^{\eta_i'^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i'^{(j,1)}}}\right]\right)$$

$$S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}_{N_1,\cdots,N_v}\left(\begin{array}{c} \mathrm{z}''_1\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(1)}}(v_i-x_i)^{\eta_i''^{(1)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,1)}}}\right] \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}''_v\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\delta_i''^{(v)}}(v_i-x_i)^{\eta_i''^{(v)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\rho_i''^{(j,v)}}}\right] \end{array}\right)\psi_{\alpha n''}\left[\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{\zeta_i}(v_i-x_i)^{\lambda_i}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\theta_i^{(j)}}}\right]\right]$$

$${}_PF_Q\left[(A_P);(B_Q);-\sum_{k=1}^{l}g_k\prod_{i=1}^{t}\left[\frac{(x_i-u_i)^{u_i^{(k)}}(v_i-x_i)^{\theta_i^{(r)}}}{\prod_{j=1}^{T}\left(U_i^{(j)}x_i+V_i^{(j)}\right)^{\tau_i^{(j,k)}}}\right]\right]\mathrm{d}x_1\cdots\mathrm{d}x_s$$

$$=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}\prod_{i=1}^{t}\left[(v_i-u_i)^{\alpha_i+\beta_i-1}\prod_{j=1}^{W}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\prod_{j=W+1}^{T}\left(u_iU_i^{(j)}+V_i^{(j)}\right)^{\sigma_i^{(j)}}\right]$$

$$\sum_{k''=0,or1}^{\infty_*}\sum_{R=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_1=1}^{m_1}\sum_{g_1=0}^{\infty}\phi_1\frac{z_k^{\eta_{G_1,g_1}}(-)^{g_1}}{\delta^{(1)}_{G^{(1)}}g_1!}a'_v z_1''^{K_1}\cdots z_v''^{K_v}\frac{(-)^R a_{k''}(c^\sigma|\alpha n'')}{R!\Gamma(R+k''+\alpha+\frac{3}{2})}E_{ij}$$

Ref
8. A.K. Rathie, A new generalization of generalized hypergeometric functions, Le Matematiche,52(2) (1997), 297-310.

$$I^{0,sT+P+2s+n':X_1}_{sT+P+2s+p',sT+Q+s+q':Y_1}\left(\begin{array}{c|c} \text{z'w}_1 & \mathbb{A}_1,\ \mathbf{A}_1 \\ \mathrm{g}_1 W_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{g}_l W_l & \cdot \\ \mathrm{G}_1 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{G}_T & \mathbb{B}_1,\ \mathbf{B}_1 \end{array}\right) \tag{4.2}$$

The validity conditions are the same that (3.1) with $r = s = 1$. The quantities $\phi_1, V_1, W_1, \mathbb{A}_1, \mathbb{B}_1, \mathbf{A}_1, \mathbf{B}_1$ are equal to $\phi_k, V, W, \mathbb{A}, \mathbb{B}, \mathbf{A}, \mathbf{B}$ respectively for $r = s = 1$.

Remark: By the similar procedure, the results of this document can be extended to the product of any finite number of multivariable I-functions and class of multivariable polynomials defined by Srivastava [12].

V. Conclusion

Our main integral formula is unified in nature and possesses manifold generality. It acts a fundamental expression and using various particular cases of the multivariable I-function, the class of multivariable polynomials and a general spheroidal functions, one can obtain a large number of other integrals involving simpler special functions and polynomials of one and several variables.

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Matrix Expression of a Complete Flag of Riemannian Extensions on a Manifold

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GJSFR-F Classification: *MSC 2010: 53C12, 58A30*

MATRIXEXPRESSIONOFACOMPLETEFLAGOFRIEMANNIANEXTENSIONSONAMANIFOLD

Strictly as per the compliance and regulations of:

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I. Introduction

We note above all about that the idea of this paper came to us from the proof of structure theorem of complete Riemannian flags of H. Diallo being in [7].

Let $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$, ..., $(M_q, \mathcal{F}_q)$ be q codimension 1 transversaly orientable Riemannian foliations, let $M = M_1 \times M_2 \times ... \times M_q$, let $\widetilde{\mathcal{F}}_k = \mathcal{F}_1 \times \mathcal{F}_2 \times ... \times \mathcal{F}_k \times M_{k+1} \times M_2 \times ... \times M_q$, let $(U_i^k, f_i^k, T^k, \gamma_{ij}^k)_{i \in I^k}$ be a foliated cocycle defining $(M_k, \mathcal{F}_k)$, $\widetilde{T}^k = T^1 \times T^2 \times ... \times T^k$, let p_s be the projection of M on M_s, let $\overline{p}_s$ be the projection of $\widetilde{T}^q$ on T^s, let $\widetilde{U}_i^k = U_i^1 \times U_i^2 \times ... \times U_i^k \times M_{k+1} \times M_2 \times ... \times M_q$, let $\widetilde{f}_i^k = \left(f_i^1 \circ p_1, f_i^2 \circ p_2, ..., f_i^k \circ p_k\right)$ and let $\widetilde{\gamma}_{ij}^k = \left(\gamma_{ij}^1 \circ \overline{p}_1, \gamma_{ij}^2 \circ \overline{p}_2, ..., \gamma_{ij}^k \circ \overline{p}_k\right)$.

We easily verify that $\left(\widetilde{U}_i^k, \widetilde{f}_i^k, \widetilde{T}^k, \widetilde{\gamma}_{ij}^k\right)$ is a foliated cocycle defining the codimension k Riemannian foliation $\widetilde{\mathcal{F}}_k$.

We have $\widetilde{\mathcal{F}}_q \subset \widetilde{\mathcal{F}}_{q-1} \subset ... \subset \widetilde{\mathcal{F}}_1$. We say that the sequence $\mathcal{D}_{\widetilde{\mathcal{F}}_q} = \left(\widetilde{\mathcal{F}}_{q-1}, ..., \widetilde{\mathcal{F}}_1\right)$ is a completed flag of Riemannian extension of Riemannian foliation $\widetilde{\mathcal{F}}_q$ on M.

Specifically, being given a codimension q foliation $\mathcal{F}_q$ on a manifold M, a flag of extensions of a foliation $\mathcal{F}_q$ is a sequence $\mathcal{D}_{\mathcal{F}_q}^k = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_k)$ of foliations on M such as $\mathcal{F}_q \subset \mathcal{F}_{q-1} \subset \mathcal{F}_{q-2} \subset ... \subset \mathcal{F}_k$ and each foliation $\mathcal{F}_s$ is a codimension s foliation.

For $k = 1$, the flag of extensions $\mathcal{D}_{\mathcal{F}_q}^k$ will be called complete and will be noted $\mathcal{D}_{\mathcal{F}_q}$.

If each foliation $\mathcal{F}_s$ is Riemannian, the flag of extensions $\mathcal{D}_{\mathcal{F}_q}^k$ will be called flag of Riemannian extensions of $\mathcal{F}_q$.

That said, is denoted by X_k the unitary field of T^k orienting T^k and $(X_k)^{h_k}$ the lifted of X_k on the tangent bundle $T\widetilde{T}^q$ of $\widetilde{T}^q$.

One checks easily [10] that $\left[(X_k)^{h_k}, (X_s)^{h_s}\right] = 0$ for $k \neq s$. There is thus obtained a coordinates system $(x_1, ..., x_q)$ on $\widetilde{T}^q$ such as $\frac{\partial}{\partial x_k} = (X_k)^{h_k}$. As each γ_{ij}^k is an local isometry of T^k then relative to the coordinates system $(x_1, ..., x_q)$ the Jacobian matrix $J_{\widetilde{\gamma}_{ij}^k}$ of $\widetilde{\gamma}_{ij}^k$ checked:

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Ref

10. R. Nasri and M. Djaa, 2006. *"Sur la courbure des variétés riemanniennes produits"*, Laboratoire de géométrie.Centre universitaire de Saida, Algérie. M.S.C.2000:53C50-53C42. Sciences et Technologie A-N°24; Decembre, pp.15-20.

$$J_{\gamma_{ij}^k}=\begin{pmatrix} \varepsilon_{ij}^1 & 0 & \dots & \dots & \dots & 0 \\ 0 & \varepsilon_{ij}^2 & 0 & \dots & \dots & 0 \\ 0 & 0 & \varepsilon_{ij}^3 & 0 & \dots & \vdots \\ \vdots & \vdots & 0 & . & . & \vdots \\ \vdots & \vdots & \vdots & . & . & 0 \\ 0 & 0 & 0 & \dots. & 0 & \varepsilon_{ij}^k \end{pmatrix} \text{ where } \varepsilon_{ij}^r=\pm 1.$$

From the foregoing the foliation $\widetilde{\mathcal{F}}_k$ will be said Riemannian transversaly diagonal.

Specifically a foliation $\mathcal{F}$ on a manifold N is said transversely diagonal if and only if it is defined by a foliated cocycle $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ such as the opens U_i are $\mathcal{F}$ -distinguished and on each open $f_i(U_i)$ it exists a local $\mathcal{F}$-transverse coordinates system $(y_q^i, y_{2,\dots,}^i y_1^i)$ such as relatively to local $\mathcal{F}$-transverse coordinates systems $\left((y_q^i, y_{2,\dots,}^i y_1^i)\right)_{i\in I}$ on the opens $f_i(U_i)$,

the Jacobian matrix $J_{\gamma_{ij}}$ of γ_{ij} is diagonal. In the case where there exists a metric h_T on the transverse manifold T such as the γ_{ij}^s are local isometries for this transverse metric, we say that $\mathcal{F}$ is Riemannian transversaly diagonal.

The primary purpose of this paper is to show that the existense of a transversaly diagonal foliation $\mathcal{F}$ on a manifold implies the existence of a complete flag of extensions of $\mathcal{F}$. The second purpose of this paper is to prove the existence of Riemannian transversely diagonal foliation <u>*nontrivial*</u>. Indeed, we show that if $\mathcal{F}_q$ is a codimension q Riemannian foliation having a complete flag of Riemannian extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ \dots,\ \mathcal{F}_1)$ on a connected manifold N and if there exists a metric h that is bundlelike for any foliation $\mathcal{F}_s$ of this flag then each foliation $\mathcal{F}_s$ is Riemannian transversely diagonal.

In all that follows, the manifolds considered are supposed connected and differentiability is C^∞.

II. Reminders

In this paragraph, we reformulate in the direction that is helpful to us some definitions and theorems that are in ([2],[3], [5], [7], [8], [10]).

Definition 2.1 Let M be a manifold.

An extension of a codimension q foliation $(M,\mathcal{F})$ is a codimension q' foliation $(M,\mathcal{F}')$ such that $0 < q' < q$ and $(M,\mathcal{F}')$ leaves are $(M,\mathcal{F})$ leaves meetings (it is noted $\mathcal{F}\subset\mathcal{F}'$).

We show ([2], [6]) that if $(M,\mathcal{F}')$ is a simple extension of a simple foliation $(M,\mathcal{F})$ and if $(M,\mathcal{F})$ and $(M,\mathcal{F}')$ are defined respectively by submersions $\pi : M \to T$ and $\pi' : M \to T'$, then there exists a submersion $\theta : T \to T'$ such that $\pi' = \theta\circ\pi$.

We say that the submersion θ is a bond between the foliation $(M,\mathcal{F})$ and its extension foliation $(M,\mathcal{F}')$.

It is shown in [3] that if the foliation $(M,\mathcal{F})$ and its extension $(M,\mathcal{F}')$ are defined respectively by the cocycles $(U_i, f_i, T, \gamma_{ij})_{i\in I}$ and $(U_i, f_i', T', \gamma_{ij}')_{i\in I}$ then we have

$$f_i' = \theta_i\circ f_i \ \text{ and } \gamma_{ij}'\circ\theta_j = \theta_i\circ\gamma_{ij}$$

where θ_s is a bond between the foliation $(U_s,\mathcal{F})$ and its extension foliation $(U_s,\mathcal{F}')$.

Proposition 2.2 Given a foliation $(M,\mathcal{F})$ having T for model transverse foliation, let T' be a dimension $q'>0$ manifold. If the local diffeomorphisms of transition of $\mathcal{F}$ preserve the fibers of a submersion of T on T', then the foliation $\mathcal{F}$ admits a codimension q' extension having T' for model transverse manifold.

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages".* Thèse unique, Université de Cocody, Abidjan.

The following theorem is demonstrated in the same way that the structure theorem of complete Riemannian flags being in [7] :

Theorem 2.3 *Let $(M;h)$ be a connected Riemannian manifold not necessarily compact and let $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$ be a complete flag of riemannian extensions of a codimension q Riemannian foliation $\mathcal{F}_q$ having T^q for transverse manifold and having h for bundlelike metric.*

If the metric h is bundlelike for any foliation of $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$ then:

1) Each foliation $\mathcal{F}_k$ is transversaly parallelizable. The vector fields of parallelism $\mathcal{F}_k-$transverse $\left(\overline{Y}_s\right)_{0\le s\le k-1}$ are orthogonal. For $s\neq 0$, each vector field $\overline{Y}_s$ is an unitary section of $(T\mathcal{F}_{s+1})^{\perp}\cap(T\mathcal{F}_s)$ and $\overline{Y}_0$ is an unitary section of $(T\mathcal{F}_1)^{\perp}$ where $(T\mathcal{F}_k)^{\perp}$ is the orthogonal bundle of $T\mathcal{F}_k$. Additionally each vector field $\overline{Y}_s$ directs the flow it generates.

2) The induced parallelism $(Y_s)_{0\le s\le q-1}$ of T^q by $\left(\overline{Y}_s\right)_{0\le s\le q-1}$ satisfies the equality $[Y_s,Y_r]=k_{sr}Y_s$ for $q-1\ge s>r\ge 0$. Functions k_{sr} are called structure functions of $\mathcal{D}_{\mathcal{F}_q}$.

Note that the parallelism $(Y_s)_{0\le s\le q-1}$ of T^q will be said parallelism $\mathcal{F}_q-$transverse of Diallo associated to $\mathcal{D}_{\mathcal{F}_q}$.

We note also, relatively to the transverse induced metric h_T by h on T^q, that vectors fields Y_s are unitary and orthogonal two by two.

We end these reminders by the following proposition being in [10]. It will allow us to construct local coordinate systems in the proof of the theorem 3.2 which is the main theorem of this paper in the following paragraph.

Proposition 2.4 *Let $M\times N$ be the product of two manifolds M and N, let $X_i\in\mathcal{X}(M)$ and let $Y_j\in\mathcal{X}(N)$ then*

$$\left[X_1^{h_1},X_2^{h_1}\right]=[X_1,X_2]^{h_1},\ \left[Y_1^{h_2},Y_2^{h_2}\right]=[Y_1,Y_2]^{h_2}\ \ and\ \left[X_1^{h_1},Y_2^{h_2}\right]=0$$

where

$$\begin{array}{cccc} R^{h_1}: & T_xM & \rightarrow & T_{(x,y)}M\times N \\ & u & \mapsto & u^{h_1}=(u,0) \end{array}\quad and \quad \begin{array}{cccc} R^{h_2}: & T_xN & \rightarrow & T_{(x,y)}M\times N \\ & w & \mapsto & w^{h_2}=(0,w) \end{array}.$$

III. Matrix Expression of a Complete Flag of Riemannian Extensions on a Manifold

There is a link between transversaly diagonal foliations and complete flags of extensions.

Specifically we have:

Proposition 3.1 *Let $\mathcal{F}_q$ be a codimension q transversaly diagonal foliation on a manifold M.*

Then $\mathcal{F}_q$ admits a complete flag of extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1},\ \mathcal{F}_{q-2},\ ...,\ \mathcal{F}_1)$.

Proof. Let $(U_i^q, f_i^q, T^q, \gamma_{ji}^q)_{i\in I}$ be a foliated cocyle defining the transversaly diagonal foliation $\mathcal{F}_q$ and let $\left((y_q^i, y_{q-1,...,}^i y_1^i)\right)_{i\in I}$ be $\mathcal{F}_q-$transverse coordinates systems on opens $f_i^q\left(U_i^q\right)$ such as the Jacobian matrix $J_{\gamma_{ij}^q}$ is diagonal. Let also

$$P^{qi}(x)=\overset{q-1}{\underset{k=1}{\oplus}} < \frac{\partial}{\partial y_k^i}(x) >$$

be the integrable differential system on $f_i^q(U_i^q)$ and let $\mathcal{U}_i^{q-1}$ be a leaf of this differential system.

It is clear that the foliation defined by the integrable differential system $x\rightarrow P^{qi}(x)$ is transverse for the flow $\mathcal{F}_{\partial y_q^i}$ of $\frac{\partial}{\partial y_q^i}$.

Ref

10. R. Nasri and M. Djaa, 2006. *"Sur la courbure des variétés riemanniennes produits"*, Laboratoire de géométrie.Centre universitaire de Saida, Algérie. M.S.C.2000:53C50-53C42. Sciences et Technologie A-N°24; Decembre, pp.15-20.

That said, quits to reduce the "size" of opens U_i^q, it may be considered an open recovering $(U_i^q)_{i\in I}$ of M such as in each $f_i^q(U_i^q)$ the flow $\mathcal{F}_{\partial y_q^i}$ of $\frac{\partial}{\partial y_q^i}$ and the integrable differential system $x \to P^{qi}(x)$ define simple foliations so that $\mathcal{U}_i^{q-1}$ is diffeomorphic to quotient manifold of simple foliation $\mathcal{F}_{\partial y_q^i}$.

Let $\theta_i^q : f_i^q(U_i^q) \to \mathcal{U}_i^{q-1}$ be the projection on $\mathcal{U}_i^{q-1}$ following the flow $\mathcal{F}_{\partial y_q^i}$ of $\frac{\partial}{\partial y_q^i}$.

The manifold T^q can be regarded as a disjoint union of $f_i^q(U_i^q)$. Therefore we can say that submersions θ_i^q defines a submersion θ^q on T^q whose restriction to each $f_i^q(U_i^q)$ is θ_i^q.

Note that $J_{\gamma_{ji}^q} = \left(\lambda_{jirs}^q\right)_{rs}$ being a invertible and diagonal matrix has all its diagonal elements non-zero.

As

$$\left(\gamma_{ji}^q\right)_*\left(\frac{\partial}{\partial y_q^i}\right) = \left(\lambda_{jirs}^q\right)_{rs}\left(\frac{\partial}{\partial y_q^i}\right) = \lambda_{ji11}^q . \frac{\partial}{\partial y_q^j} \quad \text{and } \lambda_{ji11}^q \neq 0$$

then the γ_{ji}^q preserve the fibers of the submersion θ^q.

It follows from this ([3], [5])(*cf.prop.2.2*) that the codimension q foliation $\mathcal{F}_q$ have an codimension $q-1$ extension $\mathcal{F}_{q-1}$.

We set

$$f_i^{q-1} = \theta_i^q \circ f_i^q \text{ and } T^{q-1} = \underset{i\in I}{\cup} \mathcal{U}_i^{q-1} \text{ and } f_r^{q-1}(U_r^q \cap U_s^q) = \mathcal{V}_{rs}^{q-1}$$

for $U_r^q \cap U_s^q \neq \emptyset$.

As γ_{ji}^q preserves the fibers of the submersion θ^q then γ_{ji}^q induces a local diffeomorphism $\gamma_{ji}^{q-1} : \mathcal{V}_{ij}^{q-1} \to \mathcal{V}_{ji}^{q-1}$ and this diffeomorphism checks ([3], [5]) (*cf def.2.1*) the equality

$$\gamma_{ji}^{q-1} \circ \theta_i^q = \theta_j^q \circ \gamma_{ji}^q .$$

We easily verify that $(U_i^q, f_i^{q-1}, T^{q-1}, \gamma_{ji}^{q-1})_{i\in I}$ is a foliated cocycle defining the extension $\mathcal{F}_{q-1}$ of $\mathcal{F}_q$.

We now show that the foliation $\mathcal{F}_{q-1}$ is transversaly diagonal.

We have $\left[\frac{\partial}{\partial y_q^i}, \frac{\partial}{\partial y_k^i}\right] = 0$ for all k. Hence the $q-1$ vectors fields $\frac{\partial}{\partial y_{q-1}^i}, \frac{\partial}{\partial y_{q-2}^i}, ..., \frac{\partial}{\partial y_1^i}$ are foliated for the flow of $\frac{\partial}{\partial y_q^i}$. Therefore $(\theta^q)_*\left(\frac{\partial}{\partial y_k^i}\right)$ is a vectors field on $\mathcal{U}_i^{q-1}$ for all $k \leq q-1$.

But the $q-1$ vector fields $\frac{\partial}{\partial y_{q-1}^i}, \frac{\partial}{\partial y_{q-2}^i}, ..., \frac{\partial}{\partial y_1^i}$ are tangent to $\mathcal{U}_i^{q-1}$ at any point in $\mathcal{U}_i^{q-1}$ so for $k \neq q$ and $a \in f_i^q(U_i^q)$ we have $(\theta_i^q)_{*a}\left(\frac{\partial}{\partial y_k^i}\right) = \frac{\partial}{\partial y_k^i}(\theta_i^q(a))$.

Therefore the $q-1$ vector fields $\frac{\partial}{\partial y_{q-1}^i}, \frac{\partial}{\partial y_{q-2}^i}, ..., \frac{\partial}{\partial y_1^i}$ define a coordinates system $\mathcal{F}_{\partial y_q^i}$-transverse on $\mathcal{U}_i^{q-1}$ and this coordinates system is the restriction to $\mathcal{U}_i^{q-1}$ of $\left(y_q^i, y_{q-1}^i, ..., y_1^i\right)$. So it will be noted yet $\left(y_{q-1}^i, ..., y_1^i\right)$.

For clarity in the presentation we note for following $(\theta_i^q)_*\left(\frac{\partial}{\partial y_k^i}\right) = \frac{\partial}{\partial y_k^i}_{/\mathcal{U}_i^{q-1}}$.

Using equality $\gamma_{ji}^{q-1} \circ \theta_i^q = \theta_j^q \circ \gamma_{ji}^q$ we obtain for $k \neq q$,

$$\begin{aligned}\left(\gamma_{ji}^{q-1}\right)_* \left.\frac{\partial}{\partial y_k^i}_{/\mathcal{U}_i^{q-1}}\right) &= \left(\gamma_{ji}^{q-1}\right)_* \circ (\theta_i^q)_* \left(\frac{\partial}{\partial y_k^i}\right) \\ &= (\theta_j^q)_* \circ \left(\gamma_{ji}^q\right)_* \left(\frac{\partial}{\partial y_k^i}\right)\end{aligned}$$

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages"*. Thèse unique, Université de Cocody, Abidjan.

$$= \left(\theta_j^q\right)_* \left(\lambda^q_{ij(q-k+1)(q-k+1)} \cdot \frac{\partial}{\partial y_k^j}\right)$$

$$= \lambda^q_{ij(q-k+1)(q-k+1)} \left(\theta_j^q\right)_* \left(\frac{\partial}{\partial y_k^j}\right)$$

$$= \lambda^q_{ij(q-k+1)(q-k+1)} \cdot \frac{\partial}{\partial y_k^j}_{/\mathcal{U}_j^{q-1}} .$$

Notes

Thus the foliation $\mathcal{F}_{q-1}$ is transversaly diagonal.

Before closing we note that equality

$$\gamma_{ji}^{q-1} \circ \theta_i^q = \theta_j^q \circ \gamma_{ji}^q \text{ show that } J_{\gamma_{ji}^{q-1}} . J_{\theta_i^q} = J_{\theta_j^q} . J_{\gamma_{ji}^q}$$

where $J_{\theta_s^q}$ is the Jacobian matrix of θ_s^q. But

$$J_{\theta_s^q} = \begin{pmatrix} 0 & 0 & ... & ... & ... & 0 \\ 0 & 1 & 0 & ... & ... & 0 \\ 0 & 0 & 1 & 0 & ... & : \\ : & : & 0 & . & . & : \\ : & : & : & . & . & 0 \\ 0 & 0 & 0 & .... & 0 & 1 \end{pmatrix}$$

so $J_{\gamma_{ij}^{q-1}}$ is obtained by removing the first line and the first collonne of $J_{\gamma_{ij}^q}$.

We are constructing $\mathcal{F}_{q-2}$, $\mathcal{F}_{q-3}$, ... , $\mathcal{F}_1$ using the same technical of construction of $\mathcal{F}_{q-1}$.

We note that each foliation $\mathcal{F}_k$ will be defined by $(U_i^q, f_i^k, T^k, \gamma_{ji}^k)_{i\in I}$ a foliated cocycle where θ_i^k is defined in the same way as θ_i^q, $f_i^{k-1} = \theta_i^k \circ f_i^k$, $T^k = \bigcup_{i\in I} \mathcal{U}_i^k$ with $\mathcal{U}_i^k$ defined in the same way as $\mathcal{U}_i^{q-1}$ and $\gamma_{ji}^{k-1} \circ \theta_i^k = \theta_j^k \circ \gamma_{ji}^k$.

We note that if the transversaly diagonal foliation $\mathcal{F}_q$ is Riemannian relatively to a transverse metric h_T and if local vector fields $\frac{\partial}{\partial y_k^i}$ are Killing vector fields for h_T for all i and all k then the foliations of complete flag of extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$ are Riemannian transversaly diagonal and there existe a common metric bundlelike for any foliation $\mathcal{F}_k$. Indeed in this case the submersions θ_i^k are Riemannian submersions for all i and all k. And, the equality $\gamma_{ji}^{k-1} \circ \theta_i^k = \theta_j^k \circ \gamma_{ji}^k$ implies that γ_{ji}^{k-1} is a local isometry once γ_{ji}^k is a local isometry.

Under certain conditions specified in Theorem 3.2, the previous proposal admits a reciprocal.

In the proof of the following result is given matrix expression of a complete flag of Riemannian extensions of a Riemannian foliation .

Theorem 3.2 Let $\mathcal{F}_q$ be a codimension q Riemannian foliation on a connected manifold M and h a metric $\mathcal{F}_q-$bundlelike on M.

If $\mathcal{F}_q$ admits a complete flag of Riemannian extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, ..., \mathcal{F}_1)$ such as the métric h is bundlelike for any Riemannian foliation $\mathcal{F}_k$ then each foliation $\mathcal{F}_k$ is Riemannian transversaly diagonal.

Proof. We assume that the foliation $\mathcal{F}_q$ admits a complete flag of Riemannian extensions $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$ such as the metric h is bundlelike for any Riemannian foliation $\mathcal{F}_k$.

We denote by T^q a transverse manifold of $\mathcal{F}_q$.

According to the theorem 2.3 each foliation $\mathcal{F}_k$ is transversaly parallelizable. The vector fields of $\mathcal{F}_k-$transverse parallelism $\left(\overline{Y}_s\right)_{0\leq s\leq k-1}$ are orthogonal. For s$\neq$ 0, each vector fields $\overline{Y}_s$ is an unitary section of $(T\mathcal{F}_{s+1})^{\perp}\cap(T\mathcal{F}_s)$ and $\overline{Y}_0$ is an unitary section of $(T\mathcal{F}_1)^{\perp}$ where $(T\mathcal{F}_1)^{\perp}$ is the orthogonal bundle of $T\mathcal{F}_1$. Additionally each vector field $\overline{Y}_s$ directs the flow it generates and the induced parallelism $(Y_s)_{0\leq s\leq q-1}$ of T^q by $\left(\overline{Y}_s\right)_{0\leq s\leq q-1}$ satisfies the equality $[Y_s, Y_r] = k_{sr}Y_s$ for $q-1\geq s>r\geq 0$.

We have $T\left(T^q\right) = \underset{s=0}{\overset{q-1}{\oplus}} < Y_s >$ where $T\left(T^q\right)$ is the tangent bundle of T^q and $< Y_s >$ is the tangent bundle of flow of Y_s.

The differential system

$$S^k\left(x\right) = \underset{\substack{s=0\\ s\neq k}}{\overset{q-1}{\oplus}} < Y_s\left(x\right) >$$

is integrable because $[Y_s, Y_r] = k_{sr}Y_s$ for $s>r$. It then defines a foliation $\mathcal{S}_k$.

For $k\neq r$, one checks easily that $\mathcal{S}_r$ is an extension of flow $\mathcal{F}_{Y_k}$ of unitary vector field Y_k. On the other side $\mathcal{F}_{Y_k}$ is transverse for the foliation $\mathcal{S}_k$ because for all $x\in T^q$, $T_x\left(T^q\right) = T_x\mathcal{S}_k\oplus < Y_k\left(x\right) >$.

That said, for all $x_i\in T^q$ there exists an open V_i of T^q containing x_i, distinguished for each foliation $\mathcal{S}_k$ and for each flow $\mathcal{F}_{Y_k}$.

Let s_k^i be a submersion defining $\mathcal{S}_k$ on V_i .

As $\mathcal{F}_{Y_k}$ is transverse to the foliation $\mathcal{S}_k$ and $\dim\left(\mathcal{F}_{Y_k}\right) = co\dim\left(\mathcal{S}_k\right)$ then the open V_i can be chosen such that for all maximal plaques (for the natural relation of inclusion) $L^i_{Y_k}$ and $P^i_{Y_k}$ for $\mathcal{F}_{Y_k}$ contained in V_i we have $s_k^i\left(L^i_{Y_k}\right) = s_k^i\left(V_i\right) = s_k^i\left(P^i_{Y_k}\right)$.

Thus, we can assume that $s_k^i\left(V_i\right) = L^i_{Y_k}$ because $s_k^i\left(L^i_{Y_k}\right)$ is diffeomorphic to $L^i_{Y_k}$.

It follows from the foregoing that the application $\mu^i : V_i\to L^i_{Y_{q-1}}\times ...\times L^i_{Y_0}$ such as for all $x\in V_i$, $\mu^i\left(x\right) = \left(s^i_{q-1}\left(x\right), s^i_{q-1}\left(x\right), ..., s^i_0\left(x\right)\right)$ is a diffeomorphism.

It is easy to verify that the leaves of $\left(\left(\mu^i\right)^{-1}\right)^*\left(\mathcal{F}_{Y_k/V_i}\right)$ are of the form $\{p\}\times L^i_{Y_k}\times\{p'\}$ where $p\in L^i_{Y_{q-1}}\times L^i_{Y_{q-2}}\times ...\times L^i_{Y_{k+1}}$ and $p'\in L^i_{Y_{k-1}}\times L^i_{Y_{k-2}}\times ...\times L^i_{Y_0}$.

This being, is considered in what follows a recovery $(U_i)_{i\in I}$ of opens of the manifold M such as each open U_i is $\mathcal{F}_k-$distinguished for each k and if $(U_i, f_i^q, T^q, \gamma^q_{ji})_{i\in I}$ is a foliated cocycle defining $\mathcal{F}_q$ then there exists a diffeomorphism $\mu^i_q : f^q_i\left(U_i\right)\to L^i_{Y_{q-1}}\times L^i_{Y_{q-2}}\times ...\times L^i_{Y_0}$ such as the leaves of $\left(\left(\mu^i_q\right)^{-1}\right)^*\left(\mathcal{F}_{Y_k/V_i}\right)$ are of the form $\{p\}\times L^i_{Y_k}\times\{p'\}$ where $p\in L^i_{Y_{q-1}}\times L^i_{Y_{q-2}}\times ...\times L^i_{Y_{k+1}}$ and $p'\in L^i_{Y_{k-1}}\times L^i_{Y_{k-2}}\times ...\times L^i_{Y_0}$.

Let h_T be the metric $\mathcal{F}_q$-transverse associated to the metric $\mathcal{F}_q-$bundlelike h and let Y^i_{Lk} be the unitary vector field tangent to the leaf $L^i_{Y_k}$ induced by the unitary vector field Y_k of $\mathcal{F}_q-$transverse parallelism $(Y_s)_{0\leq s\leq q-1}$ of Diallo of T^q.

We note in passing that the vector fields Y^i_{Lk} are orthogonal two by two relatively to the metric h_T.

We now consider $a = \left(a^i_{q-1}, ..., a^i_0\right)\in L^i_{Y_{q-1}}\times ...\times L^i_{Y_0}$.

We set

$$\begin{array}{cccc} R^{h_k}_a : & T_{a^i_k}L^i_{Y_k} & \to & T_{a^i_{q-1}}L^i_{Y_{q-1}}\times ...\times T_{a^i_0}L^i_{Y_0} \\ & X^i_{a^i_k} & \mapsto & \left(X^i_{a^i_k}\right)^{h_k}_{a^{qi}} = \left(0_{a^i_{q-1}}, ..., 0_{a^i_{k+1}}, X^i_{a^i_k}, 0_{a^i_{k-1}}, ..., 0_{a^i_0}\right) \end{array}$$

Notes

where $0_{a_t^i}$ is the null vector of $T_{a_t^i}L^i_{Y_t}$.

Let $\mathcal{X}\left(L^i_{Y_k}\right)$ be the Lie algebra of vector fields tangent to $L^i_{Y_k}$ and let $X_k^i \in \mathcal{X}\left(L^i_{Y_k}\right)$.

By varying $a_k^i \in L^i_{Y_k}$, the lifted $R_a^{h_k}\left(\left(X_k^i\right)_{a_k^i}\right)$ of $\left(X_k^i\right)_{a_k^i}$ in $TL^i_{Y_{q-1}} \times ... \times TL^i_{Y_0}$ define a vector field on $L^i_{Y_{q-1}} \times ... \times L^i_{Y_0}$. We will note it by $\left(X_k^i\right)^{h_k}$. So, we have $\left(X_k^i\right)_a^{h_k} = \left(X_k^i\right)^{h_k}(a) = R_a^{h_k}\left(\left(X_k^i\right)_{a_k^i}\right)$.

Is shown in [10] $(cf.prop.2.4)$ that for $X_k^i \in \mathcal{X}\left(L^i_{Y_k}\right)$ and $X_s^i \in \mathcal{X}\left(L^i_{Y_s}\right)$ we have $\left[\left(Y_k^i\right)^{h_k}, \left(Y_s^i\right)^{h_s}\right] = 0$ for $k \neq s$. It follows from this, Y_{Lr}^i being the unitary vector field on $L^i_{Y_r}$ induced by Y_r of $\mathcal{F}_q-$transverse parallelism $(Y_s)_{0\leq s\leq q-1}$ of Diallo, that $\left[\left(Y_{Lk}^i\right)^{h_k}, \left(Y_{Ls}^i\right)^{h_s}\right] = 0$ for $k \neq s$.

Thus, $\mu_q^i : f_i^q(U_i) \to L^i_{Y_{q-1}} \times L^i_{Y_{q-2}} \times ... \times L^i_{Y_0}$ being a diffeomorphism,the vector fields $\left(\left(\mu_q^i\right)^{-1}\right)^*\left(\left(Y_{Ls}^i\right)^{h_s}\right)$ define on $f_i^q(U_i)$ a coordinated system $\mathcal{F}_q-$transverse.

In the following $\left(\left(\mu_q^i\right)^{-1}\right)^*\left(\left(Y_{Ls}^i\right)^{h_s}\right)$ is noted $\frac{\partial}{\partial y_s^i}$.

We note in passing that if we denote by Y_s^i the restriction of Y_s at $f_i^q(U_i)$ and $p_k^i : L^i_{Y_{q-1}} \times L^i_{Y_{q-2}} \times ... \times L^i_{Y_0} \to L^i_{Y_k}$ the projection on $L^i_{Y_k}$ then:

1) We have $\left(p_k^i\right)_*\left(\left(Y_k^i\right)^{h_k}\right) = Y_{Lk}^i$ and $\left(p_k^i\right)_*\left(\left(Y_s^i\right)^{h_s}\right) = 0$ for $s \neq k$.

2) The projections $\left(p_s^i\right)_*$ not being necessarily Riemannian submersions. Therefore the vector fields $\left(Y_s^i\right)^{h_s}$ are not necessarily unitary (relatively to the metric $\left(\left(\mu_q^i\right)^{-1}\right)^* h_T$ on the product $L^i_{Y_{q-1}} \times ... \times L^i_{Y_0}$) despite the fact that Y_{Ls}^i is unitary on $L^i_{Y_s}$.

3) For all $p \in L^i_{Y_{q-1}} \times ... \times L^i_{Y_{s+1}}$ and $p' \in L^i_{Y_{s-1}} \times ... \times L^i_{Y_0}$ we have $\left(\mu_q^i\right)_*\left(Y_s^i\right)$ which is tangent to $\{p\} \times L^i_{Y_s} \times \{p'\}$. Hence the vector fields $\left(Y_{Ls}^i\right)^{h_s}$ and $\left(\mu_q^i\right)_*\left(Y_s^i\right)$ are collinear. And that implies that $\frac{\partial}{\partial y^i_s}$ and Y_s^i are also collinear.

4) Y_s^i coincides with $\frac{\partial}{\partial y_s^i}$ on $L^i_{Y_s}$ however, the values of these two fields on $V_i^q \backslash L^i_{Y_s}$ are not necessarily identical where $f_i^q(U_i) = V_i^q$.

One checks easily that the vector fields $\frac{\partial}{\partial y_s^i}$ on $f_i^q(U_i) = V_i^q$ are orthogonal two by two.

Let $\mathcal{F}_k^i = \mathcal{F}_{k/U_i}$ and $\mathcal{D}_{\mathcal{F}_q^i} = \left(\mathcal{F}_{q-1}^i, \mathcal{F}_{q-2}^i, ..., \mathcal{F}_1^i\right)$ the restriction of $\mathcal{D}_{\mathcal{F}_q} = (\mathcal{F}_{q-1}, \mathcal{F}_{q-2}, ..., \mathcal{F}_1)$ at the open U_i.

The flag $\mathcal{D}_{\mathcal{F}_q^i} = \left(\mathcal{F}_{q-1}^i, \mathcal{F}_{q-2}^i, ..., \mathcal{F}_1^i\right)$ is projected on $f_i^q(U_i) = V_i^q$ following the leaves of $\mathcal{F}_q^i$ in a complete Riemannian flag $\overline{\mathcal{D}}_i = \left(\overline{\mathcal{F}}_1^i, \overline{\mathcal{F}}_2^i, ..., \overline{\mathcal{F}}_{q-1}^i\right)$ where $\overline{\mathcal{F}}_k^i = f_i^q\left(\mathcal{F}_{q-k}^i\right)$.

We have $T\overline{\mathcal{F}}_k^i = \overset{q-1}{\underset{s=k}{\oplus}} < Y_s^i > = \overset{q-1}{\underset{s=k}{\oplus}} < \frac{\partial}{\partial y_s^i} >$. Thus on each open U_i the submersion $p_{k-1}^{iq} \circ \mu_q^i \circ f_i^q$ defines the foliation $\mathcal{F}_k^i$ where p_{k-1}^{iq} is the projection of $L^i_{Y_{q-1}} \times ... \times L^i_{Y_0}$ on $L^i_{Y_{k-1}} \times L^i_{Y_{k-2}} \times ... \times L^i_{Y_0}$.

Let θ_i^{qk} be the Riemannian bond between the Riemannian foliations $\mathcal{F}_q^i$ and $\mathcal{F}_k^i$, let θ_i^{k-1} be the Riemannian bond between the Riemannian foliations $\mathcal{F}_k^i$ and $\mathcal{F}_{k-1}^i$ and let $p_{k-2}^{i(k-1)}$ be the projection of $L^i_{Y_{k-1}} \times L^i_{Y_{k-2}} \times ... \times L^i_{Y_0}$ on $L^i_{Y_{k-2}} \times ... \times L^i_{Y_0}$.

10. R. Nasri and M. Djaa, 2006. *"Sur la courbure des variétés riemanniennes produits"*, Laboratoire de géométrie.Centre universitaire de Saida, Algérie. M.S.C.2000:53C50-53C42. Sciences et Technologie A-N°24; Decembre, pp.15-20.

We note [3] that fibers of θ_i^{qk} are leaves of foliation $\overline{\mathcal{F}}_k$ and fibers of θ_i^{k-1} are leaves of flow $\mathcal{F}_{\partial y_{k-1}^i}$ of $\frac{\partial}{\partial y_{k-1}^i}$.

For all s, it follows from the above that there exists a diffeomorphism μ_s^i : $f_i^s(U_i) \to L_{Y_{s-1}}^i \times L_{Y_{s-2}}^i \times ... \times L_{Y_0}^i$ making the following diagram commutative

$$\begin{array}{ccccc} U_i & \overset{f_i^q}{\to} & f_i^q(U_i) & \overset{\mu_q^i}{\to} & L_{Y_{q-1}}^i \times ... \times L_{Y_0}^i \\ Id\downarrow & & \downarrow \theta_i^{qk} & & \downarrow p_{k-1}^{iq} \\ U_i & \overset{f_i^k}{\to} & f_i^k(U_i) & \overset{\mu_k^i}{\to} & L_{Y_{k-1}}^i \times ... \times L_{Y_0}^i \\ Id\downarrow & & \downarrow \theta_i^{k-1} & & \downarrow p_{k-2}^{i(k-1)} \\ U_i & \overset{f_i^{k-1}}{\to} & f_i^{k-1}(U_i) & \overset{\mu_{k-1}^i}{\to} & L_{Y_{k-2}}^i \times ... \times L_{Y_0}^i \end{array} \quad (*) .$$

Let $a_k^i \in L_{Y_{q-1}}^i \times ... \times L_{Y_k}^i$ and $\tau_{a_k^i}$ the immersion of $L_{Y_{k-1}}^i \times ... \times L_{Y_0}^i$ in $L_{Y_{q-1}}^i \times ... \times L_{Y_0}^i$ such as $\tau_{a_k^i}(b) = a_k^i \times b$ for all $b \in L_{Y_{k-1}}^i \times ... \times L_{Y_0}^i$.

We have $\left(\mu_q^i\right)^{-1} \circ \tau_{a_k^i} \circ \mu_k^i$ which is an immersion of $f_i^k(U_i)$ in $f_i^q(U_i)$.

Thus, quits to replace f_i^k by $\left(\mu_q^i\right)^{-1} \circ \tau_{a_k^i} \circ \mu_k^i \circ f_i^k$, we can suppose that $f_i^k(U_i)$ is an immersed submanifold of $f_i^q(U_i)$ and this submanifold is a leaf of differential integrable system $P^k(x) = \overset{k-1}{\underset{s=0}{\oplus}} < Y_s^i(x) >= \overset{k-1}{\underset{s=0}{\oplus}} < \frac{\partial}{\partial y_s^i}(x) >$ on $f_i^q(U_i)$.

Using the same arguments one can assume that $f_i^1(U_i) \subset f_i^2(U_i) \subset .... \subset f_i^q(U_i)$.

That said, it is assumed in what follows that $f_i^1(U_i) \subset f_i^2(U_i) \subset .... \subset f_i^q(U_i)$ and $f_i^k(U_i)$ is a leaf of differential integrable system $x \to P^k(x)$ on $f_i^q(U_i)$.

We have $\left[\frac{\partial}{\partial y_s^i}, \frac{\partial}{\partial y_k^i}\right] = 0$ for all (k,s) and $T\overline{\mathcal{F}}_k^i = \overset{q-1}{\underset{s=k}{\oplus}} < \frac{\partial}{\partial y_s^i} >$. Hence the vector fields $\frac{\partial}{\partial y_s^i}$ are foliate for the foliation $\overline{\mathcal{F}}_k^i$ for $s < k$. Therefore $\left(\theta_i^{qk}\right)_* \left(\frac{\partial}{\partial y_s^i}\right)$ is a vector field on $f_i^k(U_i)$ for all $s < k$ because θ_i^{qk} is the projection of $f_i^q(U_i)$ on $f_i^k(U_i)$ following the leaves of $\overline{\mathcal{F}}_k^i$.

But for $s < k$ the vector fields $\frac{\partial}{\partial y_s^i}$ are tangent to $f_i^k(U_i)$ at any point of $f_i^k(U_i)$ so for $s < k$ and $a \in f_i^q\left(U_i^q\right)$ we have $\left(\theta_i^{qk}\right)_{*a}\left(\frac{\partial}{\partial y_s^i}\right) = \frac{\partial}{\partial y_s^i}\left(\theta_i^{qk}(a)\right)$.

Therefore for $s < k$ the vector fields $\frac{\partial}{\partial y_s^i}$ define a coordinate system $\overline{\mathcal{F}}_k^i$-transverse on $f_i^k(U_i)$ and this coordinates system is the restriction of $\left(y_{q-1}^i, ..., y_0^i\right)$ to $f_i^k(U_i)$. It will be noted therefore $\left(y_{k-1}^i, ..., y_0^i\right)$.

We can write that $\left(\theta_i^{qk}\right)\left(y_{q-1}^i, ..., y_0^i\right) = \left(y_{k-1}^i, ..., y_0^i\right)$.

Using the diagram $(*)$ it is easy to verify that $\theta_i^{q(k-1)} = \theta_i^{k-1} \circ \theta_i^{qk}$. Which causes that

$$\begin{aligned} \theta_i^{k-1}\left(y_{k-1}^i, y_{k-2}^i, ..., y_0^i\right) &= \left(\theta_i^{k-1} \circ \theta_i^{qk}\right)\left(y_{q-1}^i, ..., y_0^i\right) \\ &= \theta_i^{q(k-1)}\left(y_{q-1}^i, ..., y_0^i\right) \\ &= \left(y_{k-2}^i, ..., y_0^i\right) . \end{aligned}$$

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages"*. Thèse unique, Université de Cocody, Abidjan.

For $U_i \cap U_j \neq \emptyset$, we set:

$$\gamma^k_{_{ij}}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^{ki}_{k-1}, \gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right).$$

We have [3] the equality,

$$f_i^{k-1} = \theta_i^{k-1} \circ f_i^k \quad \text{and } \gamma^{k-1}_{_{ij}} \circ \theta_j^{k-1} = \theta_i^{k-1} \circ \gamma^k_{_{ij}}$$

that is to say that the following diagram is commutative

$$\begin{array}{ccccc} U_i \cap U_j & \overset{f_j^k}{\rightarrow} & f_j^k(U_i \cap U_j) & \overset{\theta_j^{k-1}}{\rightarrow} & f_j^{k-1}(U_i \cap U_j) \\ {}^{Id_{U_i \cap U_j}} \downarrow & & \downarrow \gamma^k_{_{ij}} & & \downarrow \gamma^{k-1}_{_{ij}} \\ U_i \cap U_j & \overset{f_i^k}{\rightarrow} & f_i^k(U_i \cap U_j) & \overset{\theta_i^{k-1}}{\rightarrow} & f_i^{k-1}(U_i \cap U_j) \end{array}.$$

From where the equality $\theta_i^{k-1}\left(y^i_{k-1}, y^i_{k-2}, ..., y^i_0\right) = \left(y^i_{k-2}, ..., y^i_0\right)$ causes that:

$$\begin{aligned} \gamma^{k-1}_{_{ij}}\left(y^j_{k-2}, ..., y^j_0\right) &= \gamma^{k-1}_{_{ij}} \circ \theta_j^{k-1}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) \\ &= \theta_i^{k-1} \circ \gamma^k_{_{ij}}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) \\ &= \theta_i^{k-1}\left(\gamma^{ki}_{k-1}, \gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right) \\ &= \left(\gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right) \end{aligned}$$

but

$$\gamma^{k-1}_{_{ij}}\left(y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^{(k-1)i}_{k-2}, ..., \gamma^{(k-1)i}_0\right) \text{ so } \left(\gamma^{(k-1)i}_{k-2}, ..., \gamma^{(k-1)i}_0\right) = \left(\gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right).$$

Which implies $\gamma^{si}_r = \gamma^{ti}_r$ for all r, s and t.

We can set

$$\gamma^k_{_{ij}}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^i_{k-1}, \gamma^i_{k-2}, ..., \gamma^i_0\right).$$

The equality

$$\gamma^{k-1}_{_{ij}}\left(y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^{ki}_{k-2}, ..., \gamma^{ki}_0\right) = \left(\gamma^i_{k-1}, \gamma^i_{k-2}, ..., \gamma^i_0\right)$$

show that

$$\gamma^k_{_{ij}}\left(y^j_{k-1}, y^j_{k-2}, ..., y^j_0\right) = \left(\gamma^i_{k-1}, \gamma^i_{k-2}, ..., \gamma^i_0\right)$$

with

$$\left(\gamma^i_{k-s-1}, \gamma^i_{k-s-2}, ..., \gamma^i_0\right) = \gamma^{k-s}_{_{ij}}\left(y^j_{k-s-1}, ..., y^j_0\right) \text{ for } s \in \{1, ..., k-1\}.$$

It follows from the foregoing that the Jacobian matrix $J_{\gamma^k_{ij}}$ of $\gamma^k_{_{ij}}$ satisfies the equality:

$$J_{\gamma^k_{ij}} = \begin{pmatrix} \frac{\partial \gamma^i_{k-1}}{\partial y^j_{k-1}} & \frac{\partial \gamma^i_{k-1}}{\partial y^j_{k-2}} & ... & ... & ... & \frac{\partial \gamma^i_{k-1}}{\partial y^j_0} \\ 0 & \frac{\partial \gamma^i_{k-2}}{\partial y^j_{k-2}} & \frac{\partial \gamma^i_{k-2}}{\partial y^j_{k-3}} & ... & ... & \frac{\partial \gamma^i_{k-2}}{\partial y^j_0} \\ 0 & 0 & \frac{\partial \gamma^i_{k-3}}{\partial y^j_{k-3}} & ... & ... & \frac{\partial \gamma^i_{k-3}}{\partial y^j_0} \\ \vdots & \vdots & 0 & . & & \vdots \\ \vdots & \vdots & \vdots & . & . & \vdots \\ 0 & 0 & 0 & .... & 0 & \frac{\partial \gamma^i_0}{\partial y^j_0} \end{pmatrix}.$$

Therefore

Ref

3. C. Dadi, 2008. *"Sur les extensions des feuilletages"*. Thèse unique, Université de Cocody, Abidjan.

$$\left(\gamma_{ij}^k\right)_* \left(\frac{\partial}{\partial y_{k-r}^j}\right) = \sum_{t=1}^{r} \frac{\partial \gamma_{k-t}^i}{\partial y_{k-r}^j} . \frac{\partial}{\partial y_{k-t}^i}.$$

The matrix $J_{\gamma_{ij}^k}$ is invertible and triangular. From where all diagonal element of $J_{\gamma_{ij}^k}$ is nonzero that is to say that $\frac{\partial \gamma_{k-r}^i}{\partial y_{k-r}^j} \neq 0$ for all r.

Notes

As γ_{ij}^k is an isometric for the transverse metric h_T and as the vector fields $\frac{\partial}{\partial y_{k-1}^j}$ are orthogonal two by two then

$$\begin{aligned} 0 &= h_T\left(\frac{\partial}{\partial y_{k-1}^j}, \frac{\partial}{\partial y_{k-2}^j}\right) \\ &= h_T\left(\left(\gamma_{ij}^k\right)_* \frac{\partial}{\partial y_{k-1}^j}\right), \left(\gamma_{ij}^k\right)_* \frac{\partial}{\partial y_{k-2}^j}\right)\right) \\ &= \frac{\partial \gamma_{k-1}^i}{\partial y_{k-2}^j} . \frac{\partial \gamma_{k-1}^i}{\partial y_{k-1}^j} h_T\left(\frac{\partial}{\partial y_{k-1}^i}, \frac{\partial}{\partial y_{k-1}^i}\right). \end{aligned}$$

But

$$\frac{\partial \gamma_{k-1}^i}{\partial y_{k-1}^j} \neq 0 \text{ and } h_T\left(\frac{\partial}{\partial y_{k-1}^i}, \frac{\partial}{\partial y_{k-1}^i}\right) \neq 0 \text{ so } \frac{\partial \gamma_{k-1}^i}{\partial y_{k-2}^j} = 0.$$

Let $r_0 \in \{1, 2, ..., k\}$. Suppose by recurrence that for all $r \leq r_0$ and all $s < r$, $\frac{\partial \gamma_{k-s}^i}{\partial y_{k-r}^j} = 0$.

We have for all $s < r_0 + 1$

$$\begin{aligned} 0 &= h_T\left(\frac{\partial}{\partial y_{k-s}^j}, \frac{\partial}{\partial y_{k-(r_0+1)}^j}\right) \\ &= h_T\left(\left(\gamma_{ij}^k\right)_* \frac{\partial}{\partial y_{k-s}^j}\right), \left(\gamma_{ij}^k\right)_* \left(\frac{\partial}{\partial y_{k-(r_0+1)}^j}\right)\right) \\ &= h_T\left(\frac{\partial \gamma_{k-s}^i}{\partial y_{k-s}^j} . \frac{\partial}{\partial y_{k-s}^i}, \sum_{t=1}^{r_0+1} \frac{\partial \gamma_{k-t}^i}{\partial y_{k-(r_0+1)}^j} . \frac{\partial}{\partial y_{k-t}^i}\right) \\ &= \frac{\partial \gamma_{k-s}^i}{\partial y_{k-(r_0+1)}^j} . \frac{\partial \gamma_{k-s}^i}{\partial y_{k-s}^j} . h\left(\frac{\partial}{\partial y_{k-s}^i}, \frac{\partial}{\partial y_{k-s}^i}\right). \end{aligned}$$

But

$$\frac{\partial \gamma_{k-s}^i}{\partial y_{k-s}^j} \neq 0 \text{ and } h_T\left(\frac{\partial}{\partial y_{k-s}^i}, \frac{\partial}{\partial y_{k-s}^i}\right) \neq 0$$

so

$$\frac{\partial \gamma_{k-s}^i}{\partial y_{k-(r_0+1)}^j} = 0 \text{ for all } s < r_0 + 1.$$

In conclusion

$$\frac{\partial\gamma^i_{k-s}}{\partial y^j_{k-r}} = 0 \text{ for all } s < r \text{ and } \frac{\partial\gamma^i_{k-r}}{\partial y^j_{k-r}} \neq 0 \text{ for all } r.$$

It follows from the foregoing that

$$J_{\gamma^k_{ij}} = \begin{pmatrix} \frac{\partial\gamma^i_{k-1}}{\partial y^j_{k-1}} & 0 & \dots & \dots & \dots & 0 \\ 0 & \frac{\partial\gamma^i_{k-2}}{\partial y^j_{k-2}} & 0 & \dots & \dots & 0 \\ 0 & 0 & \frac{\partial\gamma^i_{k-3}}{\partial y^j_{k-3}} & 0 & \dots & : \\ : & : & 0 & . & . & : \\ : & : & : & . & . & 0 \\ 0 & 0 & 0 & \dots. & 0 & \frac{\partial\gamma^i_0}{\partial y^j_0} \end{pmatrix} \text{ with } \frac{\partial\gamma^i_{k-r}}{\partial y^j_{k-r}} \neq 0 \text{ for all } r.$$

Thus $\mathcal{F}_k$ is Riemannian transversaly diagonal.

We will say that the covering of opens $(U_i)_{i\in I}$ of the manifold M and the local $\mathcal{F}_k-$transverse coordinates system $\left(\left(y^i_{k-1}, ..., y^i_0\right)\right)_{i\in I}$ are compatible with the flag $\mathcal{D}_{\mathcal{F}_q}$.

We show in [4] that if $\mathcal{F}_q$ is a foliation with dense leaves then the local compatible $\mathcal{F}_k-$transverse coordinates system with $\mathcal{D}_{\mathcal{F}_q}$ is a global $\mathcal{F}_k-$transverse coordinates system on any $\mathcal{F}_k-$transverse manifold. And, in this case it is not necessary to specify in the previous theorem the fact that the $\mathcal{F}_q-$bundlelike metric h is bundlelike for each foliation $\mathcal{F}_k$ of $\mathcal{D}_{\mathcal{F}_q}$ for $k < q$.

We note that for any $\mathcal{F}_k-$transverse manifold T^k there exists k foliations $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_k$ and $\dim(\mathcal{H}_r) = 1$ for each r (just write $\mathcal{H}_{r+1} = \mathcal{F}_{Y_r}$ where $\mathcal{F}_{Y_r}$ is the flow of unitary field Y_r) such as:

$i)$ $T\left(T^k\right) = \overset{k}{\underset{r=0}{\oplus}} T\mathcal{H}_r$ and each foliation $\mathcal{H}_r$ is invariant by the changers γ^k_{ij} of $\mathcal{F}_k-$transverse coordinates,

$ii)$ the differential system $S^{(k,t)}(x) = \overset{k}{\underset{\substack{r=1\\ r\neq t}}{\oplus}} T_x\mathcal{H}_r$ is integrable.

We say that a n dimension manifold N is *almost produces p-type multi-foliate* if and only if there exists p foliations $\mathcal{H}_1, \mathcal{H}_2, ..., \mathcal{H}_p$ where $p \leq n$ such as $TN = \overset{k}{\underset{r=0}{\oplus}} T\mathcal{H}_r$ and the differential system $S^{(k,t)}(x) = \overset{k}{\underset{\substack{r=1\\ r\neq t}}{\oplus}} T_x\mathcal{H}_r$ is integrable.

A such manifold is locally diffeomorphic to $L_1 \times L_2 \times ... \times L_p$ where L_r is a plaque of $\mathcal{H}_r$ (*Cf. proof of theorem* 3.2).

That said, we now consider a codimension q foliation $\mathcal{F}$ having N^q for $\mathcal{F}-$transverse manifold.

The proofs of proposition 3.1 and of theorem 3.2 allow us to see that $\mathcal{F}$ is transversaly diagonal if and only if N^q is *almost produces n-type multi-foliate* and the foliations $\mathcal{H}_r$ allowing the decomposition of TN^q are invariant by the changers γ_{ij} of $\mathcal{F}-$transverse coordinates.

In the case where the $\mathcal{F}-$transverse manifold N^q is *almost produces p-type multi-foliate* with $p < n$, if the foliations $\mathcal{H}_r$ allowing the decomposition of TN^q are invariant by the changers γ_{ij} of $\mathcal{F}-$transverse coordinates then using Proposition 2.4 we can construct as in the proof of Theorem 3.2 a family of local $\mathcal{F}-$transverse coordinates system $\left(\left(y^i_n, ..., y^i_1\right)\right)_{i\in I}$ on N^q and following this family we have

Ref

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$$J_{\gamma_{ij}} = \begin{pmatrix} J_{ij}^1 & 0 & \dots & \dots & \dots & 0 \\ 0 & J_{ij}^2 & 0 & \dots & \dots & 0 \\ 0 & 0 & . & 0 & \dots & : \\ : & : & 0 & . & . & : \\ : & : & : & . & . & 0 \\ 0 & 0 & 0 & \dots. & 0 & J_{ij}^p \end{pmatrix}$$

where $J_{\gamma_{ij}}$ is the Jacobian matrix of γ_{ij} and J_{ij}^r is a square matrix of order n_r where $n_r = \dim(\mathcal{H}_r)$.

We say in this case that the foliation $\mathcal{F}$ is *transversaly diagonal by block.*

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Oscillations of Second order Impulsive Differential Equations with Advanced Arguments

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Abstract- A comparison theorem providing sufficient conditions for the oscillation of all solutions of a class of second order linear impulsive differential equations with advanced argument is formulated. A relation between the oscillation (non-oscillation) of second order impulsive differential equations with advanced arguments and the oscillation (non-oscillation) of the corresponding impulsive ordinary differential equations is established by means of the Lebesgue dominated convergence theorem. Obtained comparison principle essentially simplifies the examination of the studied equations.

Keywords: impulsive differential equations, comparison theorem, advanced arguments, second order, oscillation.

GJSFR-F Classification: *MSC 2010: 31B35*

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Oscillations of Second order Impulsive Differential Equations with Advanced Arguments

Ubon Akpan Abasiekwere [α], Ita Micah Esuabana [σ], Idongesit Okon Isaac [ρ] & Zsolt Lipcsey [ω]

Abstract- A comparison theorem providing sufficient conditions for the oscillation of all solutions of a class of second order linear impulsive differential equations with advanced argument is formulated. A relation between the oscillation (non-oscillation) of second order impulsive differential equations with advanced arguments and the oscillation (non-oscillation) of the corresponding impulsive ordinary differential equations is established by means of the Lebesgue dominated convergence theorem. Obtained comparison principle essentially simplifies the examination of the studied equations.

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I. Introduction

Impulsive differential equations with deviating arguments are adequate mathematical models of numerous processes studied in physics, biology, electronics, etc. In spite of the great possibilities for applications, the theory of these equations is developing rather slowly due to the obstacles of technical and theoretical character arising in the investigation of impulsive differential equations.

Oscillation theory is one of the directions which initiated the investigations of the qualitative properties of differential equations. This theory started with the classical works of Sturm and Kneser, and still attracts the attention of many mathematicians as much for the interesting results obtained as for their various applications.

In recent few decades, the number of investigations of the oscillatory properties of the solutions of functional differential equations have been constantly growing (Ladde *et al.*, 1987, Samoilenko and Perestyuk, 1987, Gyori and Ladas, 1991, Erbe *et al.*, 1995,). In 1989 the paper of Gopalsamy and Zhang was published, where the first investigation on oscillatory properties of impulsive differential equations

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with deviating arguments was carried out (Gopalsamy and Zhang, 1989). However, not much has been done in the study of oscillation theory of second order impulsive differential equations with advanced arguments. This paper therefore is targeted at filling this gap by examining sufficient conditions for oscillation of the solutions of a class of second order linear impulsive differential equations with advanced arguments.

Before the formulation of the problem considered in this study, we present some basic definitions and concepts that will be useful throughout our discussions.

Usually, the solution $y(t)$ for $t\in[t_0,T)$ of a given impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k\in[t_0,T)$, $t_k\neq t$. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r\in N$, $D:=[T,\infty)\subset R$ and let $S:=\{t_k\}_{k\in E}$,where E is our subscript set which can be the set of natural numbers N or the set of integers Z, be fixed. Throughout this discussion,we will assume that the elements of the sequence $S:=\{t_k\}_{k\in E}$ are the moments of impulsive effects and satisfy the following properties:

C1.1: If $\{t_k\}$ is defined for all $k\in N$, then $0<t_1<t_2<\cdots$ and $\lim\limits_{k\to\infty} t_k=+\infty$.

C1.2: If $\{t_k\}$ is defined for all $k\in Z$, then $t_0\leq 0<t_1$, $t_k<t_{k+1}$ for $k\in Z$, $k\neq 0$ and $\lim\limits_{k\to\pm\infty} t_k=\pm\infty$.

We denote by $PC(D,R)$ the set of all values $\psi:D\to R$ which is continuous for all $t\in D$, $t\notin S$. They are functions from the left and have discontinuity of the first kind at the points for $t\in S$. By $PC^r(D,R)$, we denote the set of functions $\psi:D\to R$ having derivative $\dfrac{d^j\psi}{dt^j}\in PC(D,R)$, $0\leq j\leq r$ (Lashmikantham *et al.*, 1989, Bainov and Simeonov, 1998). To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S)$, $r\in N$ (Isaac and Lipcsey, 2009, Isaac and Lipcsey, 2010a, Isaac and Lipcsey, 2010b).

Now, let (P) be some property of the solution of $y(t)$ of an impulsive differential equation, which can be fulfilled for some $t\in R$. Hereafter, we shall say that the function $y(t)$ enjoys the property (P) *finally*, if there exists $T\in R$ such that $y(t)$ enjoys the property (P) for all $t\geq T$ (Bainov and Simeonov, 1998).

Definition 1.1: The solution $y(t)$ of an impulsive differential equation is said to be

i) Finally positive (finally negative) if there exist $T\geq 0$ such that $y(t)$ is defined and is strictlypositive(negative) for $t\geq T$ (Isaac *et al.*, 2011);

ii) Non-oscillatory, if it is either finally positive or finally negative; and

iii) Oscillatory, if it is neither finally positive nor finally negative (Bainov and Simeonov, 1998, Isaac and Lipcsey, 2010a).

II. Statement of the Problem

In this work, we seek sufficient conditions for the oscillation of the solutions of the linear impulsive differential equation with advanced argument of the form

Notes

$$\begin{cases}\left(r(t)y'(t)\right)'+q(t)y\left(\tau(t)\right)=0,\ t\notin S\\ \Delta\left(r(t_k)y'(t_k)\right)+q_k y\left(\tau(t_k)\right)=0,\ \forall\, t_k\in S,\end{cases} \tag{2.1}$$

Notes

where $0\le t_0<t_1<\cdots<t_k<\cdots$ with $\lim_{k\to+\infty} t_k=+\infty$, $\Delta y^{(i)}(t_k)=y^{(i)}(t_k^+)-y^{(i)}(t_k^-)$, $i=0,1$ and $y(t_k^-)$, $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t=t_k$, respectively. For the sake of definiteness, we shall suppose that the functions $y(t)$ and $y'(t)$ are continuous from the left at the points t_k such that $y'(t_k^-)=y'(t_k)$, $y(t_k^-)=y(t_k)$ and $\Delta(r(t_k)y'(t_k))=r(t_k^+)y'(t_k^+)-r(t_k)y'(t_k)$.

Without further mentioning, we will assume throughout this paper the following conditions:

C2.1: $\tau\in C([t_0,\infty),R)$, τ is a non-decreasing function in R_+, $\tau(t)\ge t$ for $t\in R_+$ and $\lim_{t\to\infty}\tau(t)=+\infty$;

C2.2: $r\in PC^1\left(\left[t_0,\infty\right),R_+\right)$ and $r(t)>0$, $r\left(t_k^+\right)>0$, for $t,t_k\in R_+$;

C2.3: $q\in PC\left(\left[t_0,\infty\right),R_+\right)$ and $q_k\ge 0$, $k\in N$;

C2.4: $\displaystyle\int_0^\infty \frac{dt}{r(t)}=\infty.$.

Using the method of steps, the real valued function $y(t)$ is said to be the solution of equation (2.1) if there exists a number $t_0\in R$ such that $y(t)\in PC([t_0-\tau(t_0),\infty),R)$, the function $r(t)y'(t)$ is continuously differentiable for $t\ge t_0-\tau(t_0)$, $t\ne t_k$, $k\in N$ and $y(t)$ satisfies equation (2.1) for all $t\ge t_0-\tau(t_0)$. We remark that every solution $y(t)$ of equation (2.1) that is under consideration here, is continuable to the right and is nontrivial. That is, $y(t)$ is defined on some half-line $[T_y,\infty)$ and $\sup\left\{|y(t)|:\ t\ge T\right\}>0$ for all $T\ge T_y$. Such a solution is called a regular solution of equation (2.1). Equation (2.1) is said to be oscillatory if all is solutions are oscillatory.

Before proceeding, we establish the following lemmas which will be useful in proving the main results. The lemmas are extensions of Erbe 's work on pages 284-287 of his monograph (Erbe *et al.*, 1995).

Lemma 2.1: Assume that

$$\int_{t_0}^{\infty} q(t)dt+\sum_{t_0\le t_k<\infty} q_k<\infty. \tag{2.2}$$

Let $y(t)>0$, $t\ge t_1$, be a solution of equation (2.1). Set

$$\omega(t)=\frac{r(t)y'(t)}{y(t)}. \tag{2.3}$$

Then $\omega(t)>0,\ \lim_{t\to\infty}\omega(t)=0,$

$$\int_{t_1}^{\infty}\frac{\omega^2(t)}{r(t)}dt+\sum_{t_1\le t_k<\infty}\frac{\omega^2(t_k)}{r(t_k)}\le\infty \tag{2.4}$$

and

$$\omega(t)=\int_t^{\infty}\frac{\omega^2(s)}{r(s)}ds+\sum_{t\le t_k<\infty}\frac{\omega^2(t_u)}{r(t_k)}+\int_t^{\infty}q(s)exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)ds+$$

$$+\sum_{t\le t_k<\infty}q_k\,exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right). \tag{2.5}$$

Proof: From equation (2.1), bearing condition (2.2) in mind, we obtain

$$\begin{cases}\left[y(t)\omega(t)\right]'+q(t)y(\tau(t))=0, & t\notin S\\ \Delta\left[y(t_k)\omega(t_k)\right]+q_ky(\tau(t_k))=0, & \forall\ t_k\in S,\end{cases}$$

since

$$\frac{y(\tau(t))}{y(t)}=exp\left(\int_t^{\tau(t)}\frac{\omega(s)}{r(s)}ds+\sum_{t\le t_k<\tau(t)}\frac{\omega(t_k)}{r(t_k)}\right),$$

and

$$\omega'(t)+\frac{\omega^2(t)}{r(t)}+q(t)exp\left(\int_t^{\tau(t)}\frac{\omega(s)}{r(s)}ds+\sum_{t\le t_k<\tau(t)}\frac{\omega(t_k)}{r(t_k)}\right)=0. \tag{2.6}$$

Integrating equation (2.6) from t to T for $T\ge t\ge t_1$, we have

$$\omega(T)-\omega(t)+\int_t^{T}\frac{\omega^2(s)}{r(s)}ds+\sum_{t\le t_k<T}\frac{\omega^2(t_k)}{r(t_k)}+\int_t^{T}q(s)exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)ds+$$

$$+\sum_{t\le t_k<T}q_k\,exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)=0. \tag{2.7}$$

Because $r(t)y'(t)>0$, so $\omega(t)>0$. We shall show that $\lim_{t\to\infty}\omega(t)=0.$ In fact, if $\lim_{t\to\infty}r(t)y'(t)=c>0$, then there exists a $t_2\ge t_1$ such that for $t\ge t_2$,

Notes

$$y(t)\geq\left[y(t_2)+\int_{t_2}^{t}\frac{c}{2r(s)}ds+\sum_{t_2\leq t_k<\tau}\frac{c}{2r(t_k)}\right]\to\infty,\quad t\to\infty,$$

and hence, $\lim_{t\to\infty}\omega(t)=0$. If $\lim_{t\to\infty}r(t)y'(t)=0$, then $\lim_{t\to\infty}\omega(t)=0$ also. Letting $T\to\infty$, in equation(2.7), we obtain condition (2.5). This completes the proof of Lemma 2.1.

Notes

Lemma 2.2: Equation (2.1) has a non-oscillatory solution if and only if there exists a positive differential function $\phi(t)$ such that

$$\phi'(t)+\frac{\phi^2(t)}{r(t)}\leq-q(t)exp\left(\int_{t}^{\tau(t)}\frac{\omega(s)}{r(s)}ds+\sum_{t\leq t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right),\quad t\geq t_2. \tag{2.8}$$

Proof: The necessity follows from Lemma 2.1. Now we assume that inequality (2.8) holds. Then, $\phi'(t)<0$ and hence $\lim_{t\to\infty}\phi(t)=-\infty$, a contradiction. Therefore, $\lim_{t\to\infty}\phi(t)=0$. Integrating inequality (2.8) from t to ∞, we obtain

$$\int_{t}^{\infty}\frac{\phi^2(s)}{r(s)}ds+\sum_{t\leq t_k<\infty}\frac{\phi^2(t_k)}{r(t_k)}+\int_{t}^{\infty}q(s)exp\left(\int_{s}^{\tau(s)}\frac{\phi(u)}{r(u)}du+\sum_{s\leq t_k<\tau(s)}\frac{\phi(t_k)}{r(t_k)}\right)+$$

$$+\sum_{t\leq t_k<\infty}q_k\,exp\left(\int_{s}^{\tau(s)}\frac{\phi(u)}{r(u)}ds+\sum_{s\leq t_k<\tau(s)}\frac{\phi(t_k)}{r(t_k)}\right)\leq\phi(t),\ t\geq t_2 \tag{2.9}$$

which implies that

$$\int_{t}^{\infty}\frac{\phi^2(s)}{r(s)}ds+\sum_{t\leq t_k<\infty}\frac{\phi^2(t_k)}{r(t_k)}<\infty$$

and

$$\int_{t}^{\infty}q(s)exp\left(\int_{s}^{\tau(s)}\frac{\phi(u)}{r(u)}du+\sum_{s\leq t_k<\tau(s)}\frac{\phi(t_k)}{r(t_k)}\right)ds+\sum_{t\leq t_k<\infty}q_k\,exp\left(\int_{s}^{\tau(s)}\frac{\phi(u)}{r(u)}du+\sum_{s\leq t_k<\tau(s)}\frac{\phi(t_k)}{r(t_k)}\right)<\infty.$$

For all functions $x(t)$ satisfying $0\leq x(t)\leq\phi(t),\ t\geq t_2$, define a mapping J by

$$(Jx)(t)=\int_{t}^{\infty}\frac{x^2(s)}{r(s)}ds+\sum_{t\leq t_k<\infty}\frac{x^2(t_k)}{r(t_k)}+\int_{t}^{\infty}q(s)exp\left(\int_{s}^{\tau(s)}\frac{x(u)}{r(u)}du+\sum_{s\leq t_k<\tau(s)}\frac{x(t_k)}{r(t_k)}\right)ds+$$

$$+\sum_{t\leq t_k<\infty}q_k\,exp\left(\int_{s}^{\tau(s)}\frac{x(u)}{r(u)}du+\sum_{s\leq t_k<\tau(s)}\frac{x(t_k)}{r(t_k)}\right),\quad t\geq t_2.$$

If is easy to see that $0\le x_1(t)\le x_2(t),\ \ t\ge t_2$, implies $(Jx_1)(t)\le(Jx_2)(t),\ t\ge t_2$.

Define $y_0(t)\equiv 0$ and $y_n(t)=(Jy_{n-1})(t),\ n=1,2,\cdots$. Then $y_{n-1}(t)\le y_n(t)\le\phi(t),\ n=1,2,\cdots$, and $\lim_{n\to\infty} y_n(t)=\omega(t)\le\phi(t)$. By the Lebesgue dominated convergence theorem, we have

$$\omega(t)=\int_t^\infty \frac{\omega^2(s)}{r(s)}ds+\sum_{t\le t_k<\infty}\frac{\omega^2(t_k)}{r(t_k)}+\int_t^\infty q(s)\exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right)ds+$$

$$+\sum_{t\le t_k<\infty} q_k\exp\left(\int_s^{\tau(s)}\frac{\omega(u)}{r(u)}du+\sum_{s\le t_k<\tau(s)}\frac{\omega(t_k)}{r(t_k)}\right),\ t\ge t_2.$$

Set

$$y(t)=\exp\left(\int_{t_2}^{t}\frac{\omega(u)}{r(u)}du+\sum_{t_2<t_k<t}\frac{\omega(t_k)}{r(t_k)}\right),\ t\ge t_2.$$

Then

$$\omega(t)=\frac{r(t)y'(t)}{y(t)}$$

and

$$\begin{cases}(r(t)y'(t))'+q(t)y(\tau(t))=0, & t\ge t_2,\ t\notin S\\ \Delta(r(t_k)y'(t_k))+q_k y(\tau(t_k))=0, & t_k\ge t_2,\ \forall\, t_k\in S,\end{cases}$$

that is, $y(t)$ is a non-oscillatory solution of equation (2.1). This completes the proof of Lemma 2.2.

In what follows we try to deduce the oscillatory conditions on equation taking advantage of the above lemmas.

III. Results

The following theorems are extensions of Theorem 4.9.1 and Theorem 4.9.2 as identified on pages 284 and 287 of the monograph by Erbe (Erbe *et al.*, 1995).Without loss of generality, we will deal only with the positive solutions of equation (2.1) where applicable.

Theorem 3.1: Assume that

$$\int_{t_0}^{\infty} q(t)dt+\sum_{t_0\le t_k<\infty} q_k=\infty. \tag{3.1}$$

Then every solution of equation (2.1) is oscillatory.

Proof: Let us assume, by contradiction, that $y(t)$ is a finally positive solution of equation (2.1). Onecan see that $r(t)y'(t)>0$ for $t\ge T\ge t_0$. Then

Notes

$$\int_{T}^{\infty} q(t)y(\tau(t))dt + \sum_{T \le t_k < \infty} q_k y(\tau(t_k)) < \infty \tag{3.2}$$

which contradicts equation (3.1). Therefore, every solution of equation (2.1) is oscillatory. This completes the proof of Theorem 3.1.

Notes

Theorem 3.2: If equation (2.1) has a non-oscillatory solution, then the second order linear impulsive differential equation

$$\begin{cases} (r(t)y'(t))' + q(t)y(t) = 0, & t \notin S \\ \Delta(r(t_k)y'(t_k)) + q_k y(t_k) = 0, & \forall\, t_k \in S \end{cases} \tag{3.3}$$

is non-oscillatory. Conversely, if equation (3.3) is oscillatory, then every solution of equation (2.1) is oscillatory.

Proof: Assume that equation (2.1) has a non-oscillatory solution. By Lemma 2.2, there exists a positive differential function $\phi(t)$ such that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \le -q(t)\exp\left(\int_{t}^{\tau(t)} \frac{\phi(u)}{r(u)}du + \sum_{t \le t_k < \tau(t)} \frac{\phi(t_k)}{r(t_k)}\right), \; t \ge t_2, \tag{3.4}$$

which implies that

$$\phi'(t) + \frac{\phi^2(t)}{r(t)} \le -q(t). \tag{3.5}$$

Taking advantage of Lemma 2.2 for the case in which $h(t) \equiv t$, equation (3.3) is non-oscillatory.

Consequently, the second part of the theorem is immediately obtained. This completes the proof of Theorem 3.2.

IV. Conclusion

It has become imperative in recent times to determine the properties of the solutions of certain mathematical equations from the knowledge of associated equations. In this work, we established a comparison theorem which compares theimpulsive differential equation with advanced argument (2.1) with the impulsive ordinary differential equation (3.3), in the sense that the oscillation (non-oscillation) of the impulsive ordinary differential equation (3.3) guarantees the oscillation (non-oscillation) of the impulsive differential equation with advanced argument (2.1). The formulated comparison theorem essentially simplifies the examination of the oscillatory properties of equation (2.1) and enables us also to eliminate some conditions imposed on the given problem.

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On Liouville Decompositions of Polyadic Integers

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Keywords: *polyadic integers, transcendental numbers, Liouville decompositions.*

GJSFR-F Classification: *MSC 2010: 35B53*

ONLIOUVILLEDECOMPOSITIONSOFPOLYADICINTEGERS

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On Liouville Decompositions of Polyadic Integers

V. G. Chirskii [α] & E. S. Krupitsin [σ]

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I. Introduction

In 1844 J. Liouville showed that if for any positive integer m there exists a rational number $\frac{p}{q}$, $(p,q)=1$ such that $\left|\alpha-\frac{p}{q}\right|<\frac{1}{q^m}$, then α is a transcendental number (a so-called Liouville number). In 1962 P. Erdős [1], in particular, gave a simple proof, that any real number can be represented as a sum of two Liouville numbers. In 1996 his result was essentially generalized by E.D. Burger [2]. He also mentioned that one can prove the results on Liouville decompositions for direct products of local fields. In this note, we show, following the lines from [1], how any polyadic integer can be explicitly expressed as a sum of two polyadic Liouville numbers.

We first supply a brief introduction to the theory of polyadic integers. Let K be a commutative ring. A mapping v of K into non-negative real numbers is called a non-archimedean pseudo-valuation of K if it has the following properties:

1. $v(a)\geqslant 0$ for all $a\in K$, $v(a)=0$ if $a=0\in K$.
2. $v(ab)\leqslant v(a)v(b)$ for all $a,b\in K$.
3. $v(a\pm b)\leqslant \max(v(a),v(b))$ for all $a,b\in K$.

If for all $a,b\in K$ the stronger condition $v(ab)=v(a)v(b)$ holds, then v is called a valuation. For a prime p, the p-adic valuation $|\alpha|_p$ of an element $\alpha\in\mathbb{Q}$ is defined as follows. If $a\in\mathbb{Z}$ is divisible by p^f and is not divisible by p^{f+1}, then $|a|_p=p^{-f}$. For $\alpha=\frac{a}{b}$, $a\in\mathbb{Z}, b\in\mathbb{N}$ we have $|\alpha|_p=\frac{|a|_p}{|b|_p}$. As usual, $\mathbb{Q}_p$ denotes the corresponding completion of $\mathbb{Q}$. It is the field of p-adic numbers, and $\mathbb{Z}_p$ denotes the ring of p-adic integers which satisfy the inequality $|\alpha|_p\leqslant 1$. For $g=p_1^{r_1}\cdot\ldots\cdot p_s^{r_s}$, where p_i are primes and r_i are positive integers we can consider the g-adic pseudo-valuation which is defined in a similar way (cf. [3]). The corresponding completion is denoted by $\mathbb{Q}_g$. A well-known theorem by K. Hensel (cf. [3]) asserts that $\mathbb{Q}_g$ is a direct sum

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2. E.D. Burger. On Liouville decomposition in local fields. Proc.Amer.Math.Soc., v.124, Num.11, (1966),3305-3310.

of the fields $\mathbb{Q}_{p_1}, \ldots, \mathbb{Q}_{p_s}$, so any element $A \in \mathbb{Q}_g$ can be expressed as $A = (A_1, \ldots, A_s)$ with $A_i \in \mathbb{Q}_{p_i}$ and for any polynomial $P(x) \in \mathbb{Z}[x]$ one has $P(A) = (P(A_1), \ldots, P(A_s))$. Recall that $\alpha \in K$ is called *algebraic* (over $\mathbb{Q}$) if there exists a nonzero polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\alpha) = 0$. Otherwise it is called *transcendental*. Therefore $A \in \mathbb{Q}_g$ is algebraic if $A_i \in \mathbb{Q}_{p_i}$, $i = 1, \ldots, s$ are algebraic.

We introduce a topology in the ring $\mathbb{Z}$ by considering the set of all ideals (m) as the system of vicinities of zero. Addition and multiplication are continuous with respect to this topology. The completion of this topological ring is called the ring of *polyadic integers*. (The detailed descriptions of the construction of polyadic numbers are presented in [4]). The elements of this ring have canonical representations of the form

$$\alpha = \sum_{n=0}^{\infty} a_n \cdot n!, \quad a_n \in \{0, 1, \ldots, n\}. \tag{1}$$

A series of this form converges in any $\mathbb{Q}_p$ and, for example, in any $\mathbb{Q}_p$ we have $\sum\limits_{n=1}^{\infty} n \cdot n! = -1$. One can prove that the ring of polyadic integers is a prime product of the rings $\mathbb{Z}_p$ over all primes p. Therefore any polyadic integer can be expressed as $\alpha = (\alpha_1, \ldots, \alpha_s, \ldots)$ where the components α_s belong to $\mathbb{Z}_{p_s}$, e.g. $\sum_{n=1}^{\infty} n \cdot n! = (-1, \ldots, -1, \ldots)$. This remark allows us to give the following definition: a polyadic integer α is called *algebraic*, if there exists a polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\alpha) = 0$ (where $0 = (0, \ldots, 0, \ldots)$), in other words, if for any s one has $P(\alpha_s) = 0$ in $\mathbb{Z}_{p_s}$. In terms of [5], [6] it means that α satisfies a global relation.

We call the polyadic integer α *transcendental*, if for any nonzero polynomial $P(x) \in \mathbb{Z}[x]$ there exists at least one prime p such that $P(\alpha_p) \neq 0$ in $\mathbb{Q}_p$. We call the polyadic integer α *infinitely transcendental*, if for any nonzero polynomial $P(x) \in \mathbb{Z}[x]$ there exist infinitely many primes p such that $P(\alpha_p) \neq 0$ in $\mathbb{Q}_p$. At last, a polyadic integer is *globally transcendental*, if for any nonzero polynomial $P(x) \in \mathbb{Z}[x]$ and for all primes p the inequality $P(\alpha_p) \neq 0$ holds in $\mathbb{Q}_p$. Of course, globally transcendental polyadic integers form a subset of infinitely transcendental polyadic integers which, in turn, form a subset of transcendental polyadic integers. Hensel's theorem mentioned above implies that there exist transcendental polyadic integers, which are not infinitely transcendental, and there exist infinitely transcendental polyadic integers, which are not globally transcendental.

The arithmetic properties of polyadic integers are studied in [5]-[12].

We call a polyadic integer α a *polyadic Liouville number*, if for any positive P, D and any prime p, $p \leqslant P$ there exists a positive integer A such that

$$|\alpha - A|_p < A^{-D}. \tag{2}$$

It is easy that for fixed positive P, D there exist infinitely many positive integers A satisfying this inequality. One can easily prove that the Liouville number is globally transcendental.

Let $\tau(k)$ be any integer-valued function defined on non-negative integers and satisfying

$$\frac{\tau(k+1)}{\tau(k)\ln\tau(k)} \to +\infty, \quad \text{as} \quad k \to +\infty. \tag{3}$$

We define the functions $l_1(n) = 1, l_2(n) = 0$ for all n with $\tau(k) \leqslant n < \tau(k+1)$ for $k = 0, 2, 4, \ldots$. We also put $l_1(n) = 0, l_2(n) = 1$ for all n with $\tau(k) \leqslant n < \tau(k+1)$ when $k = 1, 3, 5, \ldots$

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4. H. Prüfer, Neue Begründung der algebraischen Zahlentheorie, Math. Ann. V.94, 3-4 (1925), 198-243.

Theorem. For any polyadic integer (1) *we have* $\alpha = L_1 + L_2$, *where*

$$L_1 = \sum_{n=0}^{\infty} l_1(n) \cdot a_n \cdot n!, \quad L_2 = \sum_{n=0}^{\infty} l_2(n) \cdot a_n \cdot n! \tag{4}$$

are the two polyadic Liouville numbers.

Notes

Proof. The equation $\alpha = L_1 + L_2$ is evident. We now prove that L_1 is a polyadic Liouville number. The number L_2 can be treated in analogy. Let's denote

$$A_{1,k} = \sum_{n=0}^{\tau(k)} l_1(n) \cdot a_n \cdot n!. \tag{5}$$

Since $a_n \in \{0, 1, \ldots, n\}$, we have, using (5) and a rough estimate for factorial, the inequality

$$A_{1,k} < (\tau(k)+1)! \leqslant e^{(\tau(k)+1)\ln((\tau(k)+1))} \leqslant e^{2\tau(k)\ln\tau(k)} \tag{6}$$

From (4) and (5) we get

$$L_1 - A_{1,k} = \sum_{n=\tau(k+1)}^{\infty} l_1 \cdot a_n \cdot n!$$

so, for any prime p, $p \leqslant P$ we obtain

$$|L_1 - A_{1,k}|_p \leqslant |\tau(k+1)!|_p = e^{-\frac{\ln p}{p-1}\left(\tau(k+1) - S_{\tau(k+1)}\right)} \leqslant e^{-\frac{\frac{1}{2}\ln p}{p-1}(\tau(k+1))} \tag{7}$$

for sufficiently large k. Here for any positive integer N the symbol S_N denotes the sum of digits in the p-adic representation of N and it's evident that $S_N \leqslant (p-1)(\log_p N + 1)$.

For any fixed P, D and for any prime p, $p \leqslant P$ from (3), (6), (7) we get that if k is sufficiently large, then

$$|L_1 - A_{1,k}|_p \leqslant e^{-\frac{\frac{1}{2}\ln p}{p-1}(\tau(k+1))} \leqslant \left(e^{2\tau(k)\ln\tau(k)}\right)^{-D} \leqslant (A_{1,k})^{-D}.$$

This means that (2) holds and that the theorem is proved.

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Notes

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Variational Inequalities for Systems of Strongly Nonlinear Elliptic Operators of Infinite Order

By A. T. El-dessouky
Helwan University

Abstract- We are concerned with the existence of weak solutions of strongly nonlinear variational inequalities for systems of infinite order elliptic operators of the form:

$$A^r(u)(x) + Br(u)(x),\ \boldsymbol{x} \in \Omega,$$

where

$Ar(u)(x) = \Sigma(-\mathbf{1})|\boldsymbol{\alpha}|\infty|\boldsymbol{\alpha}|=\mathbf{0}\boldsymbol{D\alpha A\alpha r}(x,\boldsymbol{D\gamma u}(\boldsymbol{x}))$,

$(\mathbf{u})(\mathbf{x})\ \Sigma(-\mathbf{1})|\boldsymbol{\alpha}|\boldsymbol{D\alpha B\alpha r}|\boldsymbol{\alpha}|\leq \boldsymbol{Mr}(x,\ \boldsymbol{D\alpha u}(\boldsymbol{x})),\ \boldsymbol{Mr} \in \mathbb{N}$ *fixed,*

$(\boldsymbol{x})|\boldsymbol{\partial}\Omega=\mathbf{0}, |\boldsymbol{\omega}|=0,1,2,\ldots,$

Ω is a bounded domain in $\mathbf{R^N}$, $|\boldsymbol{\gamma}| \leq |\boldsymbol{\alpha}|$ *and* $\mathbf{r}=\mathbf{1}, 2, \ldots \mathbf{m}$.

We require that the coefficients A_a^r satisfy only some growth and coerciveness conditions and B_α^r obey a sign condition.

Keywords: *systems of strongly nonlinear elliptic operators of infinite order-variational inequalities.*

GJSFR-F Classification: *MSC 2010: 11J89*

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$$(x)|\partial\Omega = 0, |\omega| = 0,1,2,\ldots,$$

Ω is a bounded domain in $\mathbf{R}^N$, $|\gamma| \leq |\alpha|$ *and* $\mathbf{r} = \mathbf{1}, 2, \ldots \mathbf{m}$.

We require that the coefficients $\mathbf{A}_\alpha^r$ satisfy only some growth and coerciveness conditions and $\mathbf{B}_\alpha^r$ obey a sign condition.

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I. Introduction

In a recent paper, Benkirane, Chrif and El-Manouni[1]considerd the existence of solutions for strongly nonlinear elliptic equations of the form

$$\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|} D^\alpha A_\alpha (x, D^\gamma u(x)) + g(x,u) = f(x), x \in \Omega, |\gamma| \leq |\alpha|$$

where $\mathbf{A}_\alpha$ are assumed to satisfy polynomial growth and coerciveness

Conditions and g is strongly nonlinear in the sense that no growth condition is imposed but only a sign condition and $f \in L^1(\Omega)$. They relaxed the monotonicity condition, but we cann't see this.

In this paper, we extend the result of[1] to the corresponding class of variational inequalities of the above system without assuming this condition.

II. Function Spaces

Let Ω be a bounded domain in $\mathbf{R}$(N$\geq$**1**)having a locally Lipschitz property.

Let V be a closed linear subspace of $[W(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m$ such that

$$[W_0^k(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m \subseteq V \subseteq [W^k(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m$$

Ref

1. A. Benkirane, M. Chrif and S. El Manouni, Existence results for strongly nonlinear elliptic equations of infinite order, Z.Anal. Anwend.(J.Anal. Appl.)26, 2007, 303-313.

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$where [W^k(a_{\alpha,}\mathrm{p}_\alpha)(\Omega)]^m = \Pi_{r=1}^m [W^k(a_{\alpha,r,}\mathrm{p}_\alpha)(\Omega)]$, equipped with the norm

$$\|u\|_{k,\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^k a_{\alpha,r} \|D^\alpha u^r\|_{\mathrm{p}_\alpha}^{\mathrm{p}_\alpha}$$

and $[W_0^k(a_{\alpha,}\mathrm{p}_\alpha)(\Omega)]^m = \overline{[C_0^\infty(\Omega)]^m}^{\|\ \|_{k,\mathrm{p}_\alpha}}$

where $\{a_{\alpha,r}\}$ is an arbitrary sequence of nonnegative numbers and $\mathrm{p}_\alpha > 1$.

Denote by $\mathrm{p}'_\alpha = \frac{\mathrm{p}_\alpha}{\mathrm{p}_\alpha - 1}, |\alpha| \le k$.Put W=V$\cap \left[W^{k+1}(a_{\alpha,}s_\alpha)(\Omega)\right]^m$, $s_\alpha > \max\{N,, \mathrm{p}_\alpha\}$.

W is furnished with the norm

$$\|.\|_W = \max\{\|.\|_V\ ,\ \|.\|_{k+1,s_\alpha}\}$$

By the Sobolev embedding theorem

$$\left[W^{k+1}(a_{\alpha,}s_\alpha)(\Omega)\right]^m \to [C^k(\bar{\Omega})]^m \tag{1}$$

Consider the m-product of Sobolev spaces of infinite order:

$[W_0^\infty(a_{\alpha,}\mathrm{p}_\alpha)(\Omega)]^m = \Pi_{r=1}^m W_0^\infty(a_{\alpha,r,}\mathrm{p}_\alpha)(\Omega)$

$\{u \in [C_0^\infty(\Omega)]^m : \|u\|_{\infty,\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty a_{\alpha,r}\|D^\alpha u\|_{\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} < \infty\}$,

$[W^\infty(a_{\alpha,}\mathrm{p}_\alpha)(\Omega)]^m = \{u \in [C^\infty(\Omega)]^m : \|u\|_{\infty,\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty a_{\alpha,r}\|D^\alpha u\|_{\mathrm{p}_\alpha}^{\mathrm{p}_\alpha} < \infty\}$

$[W^{-\infty}(a_{\alpha,}\mathrm{p}'_\alpha)(\Omega)]^m = \{h : h = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty (-1)^{|\alpha|} a_{\alpha,r} D^\alpha h_\alpha^r;\ h_\alpha^r \in L^{\mathrm{p}'_\alpha}(\Omega)\}$,

$\|u\|_{-\infty,\mathrm{p}'_\alpha}^{\mathrm{p}'_\alpha} = \sum_{r=1}^m \sum_{|\alpha|=0}^\infty a_{\alpha,r}\|h_\alpha^r\|_{\mathrm{p}'_\alpha}^{\mathrm{p}'_\alpha} < \infty$

The nontriviality of these spaces are discussed by Dubinskii in[5]. So we choose $a_\alpha = (a_{\alpha,r})_{r=1}^m$ such that the nontriviality of these spaces holds.

III. Strongly Nonlinear Variational Inequalities of Finite Order

We start with the existence of weak solutions of strongly nonlinear variational inequalities for systems of the finite order elliptic operators:

$$\sum_{|\alpha|=0}^k (-1)^{|\alpha|} D^\alpha A_\alpha^r(\mathrm{x}, D^\gamma u(x)) + \sum_{|\alpha|\le M_r} (-1)^{|\alpha|} D^\alpha B_\alpha^r(\mathrm{x}, D^\alpha u(x)),\ x \in \Omega, \tag{2}$$

To define the system(2)more precisely we introduce the following hypotheses:

A₁) $A_\alpha^r(\mathrm{x}, \xi_\gamma) : \Omega \times \mathbb{R}^{N_0^1} \times \ldots \times \mathbb{R}^{N_0^m} \to \mathbb{R}$ are carathéodory functions.

There exist a constant $c_0 > 0$, independent of k and a function $K_1^r \in L^{\mathrm{p}'_\alpha}(\Omega)$ such that

$$|A_\alpha^r(\mathrm{x}, \xi_\gamma)| \le c_0 a_{\alpha,r} |\xi_\gamma{}^r|^{\mathrm{p}_\alpha - 1} + K_1^r(\mathrm{x}) \quad \forall\, \mathrm{x} \in \Omega, all\ \xi_\gamma{}^r, all\ r = 1, 2, .. m, |\gamma| \le |\alpha|$$

5. Yu. A. Dubinskii, Higher order parabolic differential equations, Translated from Itogi Nauki i Tekhniki, Seriya Sovremennye Problemy Matematiki, Noveishie Dostizheniya, Vol.37,1990,89-166.

where $a_\alpha>0, p_\alpha > 1$ are real numbers.

A_2) There exists a constant c_1, independent of k and a function $K_2 \in [L^1(\Omega)]^m$ Such that

$$\sum_{r=1}^{m}\sum_{|\alpha|=0}^{k} A_\alpha^r(x,\xi_\gamma)\,\xi_\alpha{}^r \geq c_1\sum_{r=1}^{m}\sum_{|\alpha|=0}^{k} a_{\alpha,r}\,|\,\xi_\alpha{}^r|^{p_\alpha} + K_2(x),$$

$$\forall\, \xi_\gamma{}^r, \xi_\alpha{}^r \in \mathbb{R}^{N_0^r}, |\gamma| \leq |\alpha|$$

B) $B_\alpha^r(x,\eta)$ are carathéodory functions defined for all $x \in \Omega$, $all\ \eta_\alpha{}^r \in \mathbb{R}^{N_1^r}$, each $r = 1, 2, \dots m$ and α with $|\alpha| \leq M_r < k$ such that $B_\alpha^r(x,\eta)\eta_\alpha{}^r \geq 0$ and

$$\sup_{|\eta|\leq a}|B_\alpha^r(x,\eta)| \leq h_\alpha^r(x) \in L^1(\Omega)$$

Consider the nonlinear form

$$a(u,v)= \int_\Omega \sum_{r=1}^{m}[\sum_{|\alpha|=0}^{k} A_\alpha^r(x, D^\gamma u(x))D^\alpha v^r(x)dx +$$

$$\sum_{|\alpha|\leq M_r} B_\alpha^r(x, D^\alpha u(x))\, D^\alpha v^r(x)dx]$$

which by A_1) and B) gives rise to a nonlinear mapping $S{:}K\cap W \to W^*$ such that

$$a(u,v)=(S(u),v) \quad (v \in K \cap W)$$

Theorem 1. Let the hypotheses A_1) - A_2) and B)be satisfied. Let K be a closed convex subset of V with $0\in K$. Let $f\in V^*$ be given. Suppose that for some $R>0$,

$$(S(v)-f,v)>0 \text{ for all } v \in K \cap W, \|v\|_V=R,$$

Then there exists $u \in K\cap W, \|u\|_V \leq R$, such that

$$(S(u),v-u)\geq (f,v-u) \text{for all } v \in K \cap W$$

Outline of proof.

Let Λ be the family of all finite dimensional linear subspaces F of W, which is a directed set under inclusion, and let F be provided with the norm $\|v\|_F = \|v\|_V$.

For each $F \in \Lambda$ let J_F be the injection mapping of F into W and $J_F^*{:}W^* \to F^*$ its adjoint.

In view of the compactness of the embedding(1) is easy to see that the the restriction of S to W is demicontinuous and moreover

$$(S_F(v)-f,v)>0 \quad \text{for all } v \in F \cap K \quad \text{with } \|v\|_F= R.$$

Therefore by lemma 2 of [2] there exists $u_F\in F$ with $\|u_F\|_F \leq R$ such that

$$(S_F(u_F),v- u_F)-(J_F^* f\ ,v- u_F)\geq 0 \text{ for all } v \in F \cap K \tag{3}$$

For any $F'\in\Lambda$,let $U_{F'}=\{ u_F: F\in \Lambda, F'\subset F, u_F$ as above$\}$. The family $\{(U_{F'}):F\in\Lambda\}$ has the finite intersection property and by the reflexivity of V, there exists

$$u\in \cap_{F'\in\Lambda}\{weak\ cl._V\,(U_{F'})\}$$

with $\|\mathbf{u}\|_V \le$ R. Since u$\in \{weak\ cl._V\ (\mathbf{U_{F'}})\}$, then for each $\mathbf{F_0} \in \mathbf{\Lambda}$ there exists a sequence (Fn)$\subset \Lambda$, whose union is dense in W, with $\mathbf{F_0} \subset \mathbf{F_1} \subset \ldots$, and for each $\boldsymbol{n} \in \mathbb{N}$ an elemaent $\mathbf{u_n} \in \mathbf{F_n}$ such that $\mathbf{u_n} \to \mathbf{u}$ weakly in V[proposition 11 of[3]. Therefore for each $\mathbf{n} \in \mathbb{N}$ we have from(3)

$$(\mathrm{S}(\mathrm{u_n}),\mathrm{v}-\mathrm{u_n})-(\mathrm{f},\mathrm{v}-\mathrm{u_n}) \ge 0 \quad \text{for all } \mathrm{v} \in \mathrm{F_n} \cap \mathrm{K} \tag{4}$$

Setting v=0 in (4) we conclude the uniform boundedness from above of the numerical sequence $\{(\mathbf{S(u_n)}, \mathbf{u_n})\}_{\mathbf{n}\in\mathbb{N}}$. From the compactness of the embedding (1), we get

$$\mathbf{D^\alpha u_n(x)} \to \mathbf{D^\alpha u(x)} \text{ uniformly on } \bar{\Omega} \text{ for all } \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| \le \mathbf{k}, \tag{5}$$

From A_1) and A_2), we obtain

$$\|\mathbf{u_n}\|_{k,\mathbf{p}_\alpha}^{\mathbf{p}_\alpha} \le c_2, \qquad \int_\Omega \left|A_\alpha^r(\mathbf{x}, \mathbf{D}^\gamma \mathbf{u_n}(\mathbf{x}))\right|^{\mathbf{p}'_\alpha} \le c_3.$$

From the inequality

$$|B_\alpha^r(x,\eta)| \le \sup_{|\eta|\le\delta^{-1}} |B_\alpha^r(x,\eta)| + \delta B_\alpha^r(\mathbf{x}, \mathbf{D}\mathbf{u_n}(\mathbf{x}))\mathbf{D}^\alpha \mathbf{u_n}(\mathbf{x})$$

which is always true for each $\boldsymbol{\delta} > 0, r = 1,2, \ldots m$ *and* all $\boldsymbol{\alpha}$ with $|\boldsymbol{\alpha}| \le \boldsymbol{k_r}$.
For any measurable subset A of Ω, we get from B)

$$\int_A |B_\alpha^r(x,\eta)| dx \le c_4 \qquad (c_2 - c_4 \text{ are constants})$$

Now, allowing n→∞ in (4), taking these estimates into cosideration as well as Vitali's and dominated convergence theorems and fatou's lemma, the proof follows.

IV. Strongly Nonlinear Variational Inequalities of Infinite Order

Now we consider the existence of weak solutions of strongly nonlinear variational inequalities for systems of the infinite order elliptic operators:

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha^r(\mathbf{x}, D^\gamma u(x)) + \sum_{|\alpha|\le M_r} (-1)^{|\alpha|} D^\alpha B_\alpha^r(\mathbf{x}, D^\alpha u(x)), \ x \in \Omega, \tag{6}$$

Theorem 2. Let the hypotheses A_1)- A_2) and B)be satisfied. Let K be a closed convex Subset of $[\mathbf{W_0^\infty}(\mathbf{a_\alpha}, \mathbf{p_\alpha})(\Omega)]^{\mathbf{m}}$ with $\mathbf{0} \in \mathbf{K}$. Let $\mathbf{f} \in [\mathbf{W}^{-\infty}(\mathbf{a_\alpha}, \mathbf{p'_\alpha})(\Omega)]^{\mathbf{m}}$ be given. Then there exists at least one solution u$\in$**K**, such that

$$(\mathrm{A^r(u)} + \mathrm{B^r(u)}, \mathrm{v^r} - \mathrm{u^r}) \ge (\mathbf{f}^r, \mathrm{v^r} - \mathrm{u^r}), \mathrm{r} \in \{1, 2, \ldots, m\} \tag{7}$$

Proof. We adopt the ideas of [6].Consider the auxiliary Dirchlet problem of order 2k, which may be thought as the partial sum of the series (6):

$$(\mathrm{A}_{2k}^{\mathrm{r}}(u_k), \mathrm{v^r} - \mathrm{u}_k^{\mathrm{r}}) + (\mathrm{B^r}(u_k), \mathrm{v^r} - \mathrm{u}_k^{\mathrm{r}}) \ge (\mathrm{f}_k^{\mathrm{r}}, \mathrm{v^r} - \mathrm{u}_k^{\mathrm{r}}), \ \mathrm{v} \in \mathrm{K} \cap W, \ \mathrm{r} \in \{1, 2, \ldots, m\} \tag{8}$$

where

$$\mathrm{A}_{2k}^{\mathrm{r}}(u_k)(x) = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} D^\alpha A_\alpha^r(\mathbf{x}, D^\gamma u_k(x)),$$

$$\mathrm{B^r}(u_k)(x) = \sum_{|\alpha|\le M_r < k} (-1)^{|\alpha|} D^\alpha B_\alpha^r(\mathbf{x}, D^\alpha(u_k)(x))$$

Ref

6. A.T.El-dessouky, Variational inequalities of strongly nonlinear elliptic operators of infinite order, Publications de l'institut mathematique, Nouvelle serie tome(67),1993,81-87.

and

$$\mathbf{f}_k^{\mathrm{r}} = \sum_{|\alpha|=0}^{k} (-1)^{|\alpha|} a_{\alpha,r} D^{\alpha} f_{\alpha}^{r} \in W^{-k}(a_{\alpha,r}, \mathbf{p}'_{\alpha})(\Omega)$$

The solvability of (8) in view of the hypotheses A_1)- A_2) and B) is a consequence of theorem1. Thus there exists $u_k \in \mathbf{K} \cap \mathbf{W}$ solving (8).

One of the fundamental roles in finding the solution of (8)is played by the so called a priori estimates. By A_2) and B), we get

$$\|\mathbf{u_k}\|_{k,\mathbf{p}_\alpha}^{\mathbf{p}_\alpha} \leq c_2$$

Since $\mathbf{u_k} \in [W^k(\mathrm{a}_\alpha, \mathbf{p}_\alpha)(\Omega)]^{\mathbf{m}}$ Implies $\mathbf{u_k} \in [W^1(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m$ we get from the compactness of $[W^1(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m \to [C(\overline{\Omega})]^{\mathbf{m}}$, the uniform convergence of $\mathbf{u_k}(\mathbf{x}) \to u(x)$ *on* $\overline{\Omega}$ *as* $k \to \infty$.

Similarly, by the compactness of $[W^k(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m \to [C^{k-\ell}(\overline{\Omega})]^m$, for large enough k and $\ell \in \mathbb{N}$, we have $\mathbf{D}^\alpha \mathbf{u_k}(\mathbf{x}) \to \mathbf{D}^\alpha \mathbf{u}(\mathbf{x})$ uniformly on $\overline{\Omega}$ *as* $k \to \infty$.

By the definition of $[W_0^\infty(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m$, *we get* $u \in [W_0^\infty(a_\alpha, \mathbf{p}_\alpha)(\Omega)]^m$ and by closedness of **K**, $u \in K$. It remains to slow that u is a solution of (**7**).
For this aim it suffices to prove the following assertions:

$$\lim_{k\to\infty} (\mathrm{A}_{2k}^{\mathrm{r}}(u_k), z^r) = (\mathrm{A}^{\mathrm{r}}(\mathbf{u}), z^r) \tag{9}$$

$$\lim_{k\to\infty} (\mathrm{B}^{\mathrm{r}}(u_k), z^r) = (\mathrm{B}^{\mathrm{r}}(\mathbf{u}), z^r) \tag{10}$$

$$\liminf_{k\to\infty} (\mathrm{A}_{2k}^{\mathrm{r}}(u_k), u_k{}^r) \geq (\mathrm{A}^{\mathrm{r}}(\mathbf{u}), u^r) \tag{11}$$

$$\liminf_{k\to\infty} (\mathrm{B}^{\mathrm{r}}(u_k), u_k{}^r) \geq (\mathrm{B}^{\mathrm{r}}(\mathbf{u}), u^r \tag{12}$$

for all $\mathbf{z} \in K$, $r = 1, 2, \dots, m$.

As above, (9)and(10) are consequence of the uniform boundedness of

$$\{\sum_{r=1}^{m} \sum_{|\alpha|=0}^{k} \mathrm{A}_\alpha^{\mathrm{r}}(\mathbf{x}, \mathbf{D}^\gamma u_k)\, D^\alpha u_k\}_{k\in\mathbb{N}}, \ \{\sum_{r=1}^{m} \sum_{|\alpha|=0}^{M_r} \mathrm{B}^{\mathrm{r}}(\mathbf{x}, D^\alpha u_k)\, D^\alpha u_k\}_{k\in\mathbb{N}} \text{ and}$$

uniform equi-integrability of

$$\{\sum_{r=1}^{m} \sum_{|\alpha|=0}^{k} \mathrm{A}_\alpha^{\mathrm{r}}(\mathbf{x}, \mathbf{D}^\gamma u_k)\}, \ \{\sum_{r=1}^{m} \sum_{|\alpha|=0}^{M_r} \mathrm{B}^{\mathrm{r}}(\mathbf{x}, D^\alpha u_k)\} \text{ in } [L^1(\Omega)]^{\mathbf{m}}$$

in view of Vitali's and dominated convergence theorems as well as (5). Assertions(11) and(12)are direct consequences of Fatou's lemma and(5).

Example. As a particular example which can be handled by our result but fails outside the Scope of [4], we consider the nonlinear system

Ref

4. Yu. A. Dubinskii, Sobolev spaces of infinite order and the behavior of solutions of some boundary-value problems with unbounded increase of the order of the equation, Math. USSR Sbornik 72, 1972, 143-162.

$$\begin{cases} \sum_{j=0}^{\infty} \sum_{|\alpha|=j} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha,1} |D^{\alpha} u_1|^{p_\alpha - 2} D^{\alpha} u_1) + h_1(x) |u_2| e^{|u_2|} \\ \sum_{j=0}^{\infty} \sum_{|\alpha|=j} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha,2} |D^{\alpha} u_2|^{p_\alpha - 2} D^{\alpha} u_2) + h_2(x) |u_1| e^{|u_1|} \end{cases}$$

$(h_i(x))_{i=1}^{2}$ are arbitrary nonnegative $L^1(x)-$functions, $u = (u_1, u_2)$.

Notes

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A Volume Preserving Map from Cube to Octahedron

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Abstract- Using simple geometric reasoning we deduce a volume preserving map from the cube to the octahedron.

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AVOLUMEPRESERVINGMAPFROMCUBETOOCTAHEDRON

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Abstract- Using simple geometric reasoning we deduce a volume preserving map from the cube to the octahedron.

I. Preliminaries

Consider the cube $\mathbb{C} = [-1,1]^3$ centered at the origin O and the regular octahedron $\mathbb{K}$ of the same volume, centered at O and with vertices on the coordinate axes

$$\mathbb{K} = \left\{(x,y,z) \in \mathbb{R}^3, |x| + |y| + |z| \le a\right\}.$$

Let L denote the edge of $\mathbb{K}$. Since the volume of the octahedron $\mathbb{K}$ is $\sqrt{2}L^3/3$, and this is equal to the volume of the cube $\mathbb{C}$, we have $8 = \sqrt{2}L^3/3$. Then, the distance from the origin to each vertex of $\mathbb{K}$ is

$$a = L/\sqrt{2} = \sqrt[3]{6}. \tag{1}$$

We will construct a map $\mathcal{U}\colon \mathbb{C} \to \mathbb{K}$ which preserves the volume, i.e.

$$Volume(D) = Volume(\mathcal{U}(D)), \qquad \text{for all } D \subseteq \mathbb{C},$$

where $Volume(D)$ denotes the volume of a domain D. For an arbitrary point $(x,y,z) \in \mathbb{C}$ we denote

$$(X,Y,Z) = \mathcal{U}(x,y,z) \in \mathbb{K}.$$

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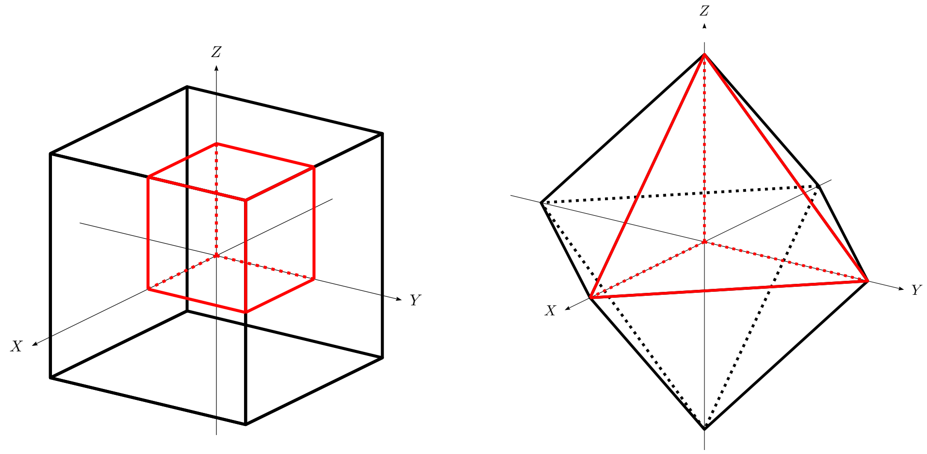

Figure 1: In the left, the cube $\mathbb{C}$ in black and the little cube $\mathbb{C}^1$ from the positive octant in red. In the right, the octahedron $\mathbb{K}$ in black and the tetrahedron K^1 in red

For the construction of $\mathcal{U}$, we split the cube into eight congruent cubes separated by the coordinate planes XOY, YOZ and ZOX, and thus, the construction of $\mathcal{U}$ can be reduced to the construction of its restriction to one of these cubes. We will denote $\mathbb{C}^1$ the eight part of $\mathbb{C}$ situated in the positive octant. We will denote by $\mathbb{K}^1$ the part of $\mathbb{K}$ situated in the positive octant. The map $\mathcal{U}$ will be constructed in such a way that $\mathbb{C}^1$ will be mapped in $\mathbb{K}^1$ and all the other seven cubes of $\mathbb{C}$ will be mapped to the corresponding tetrahedrons of $\mathbb{K}$.

II. CONSTRUCTION OF THE VOLUME PRESERVING MAP $\mathcal{U}$

We focus on the region $\mathbb{C}^1$ of $\mathbb{C}$ situated in the positive octant

$$I_0^+ = \{(x, y, z) \in \mathbb{R}^3,\ x \geq 0,\ y \geq 0,\ z \geq 0\},$$

and we denote the vertices of the cube $\mathbb{C}^1$ as follows: $A = (1, 0, 0)$, $B = (1, 1, 0)$, $C = (0, 1, 0)$, $D = (0, 1, 1)$, $E = (0, 0, 1)$, $F = (1, 0, 1)$ and $G = (1, 1, 1)$, see Figure 2 (left). We also consider the following points in $\mathbb{K}^1 = \mathbb{K} \cap I_0$: $A' = (a, 0, 0)$, $B' = (a/2, a/2, 0)$, $C' = (0, a, 0)$, $D' = (0, a/2, a/2)$,

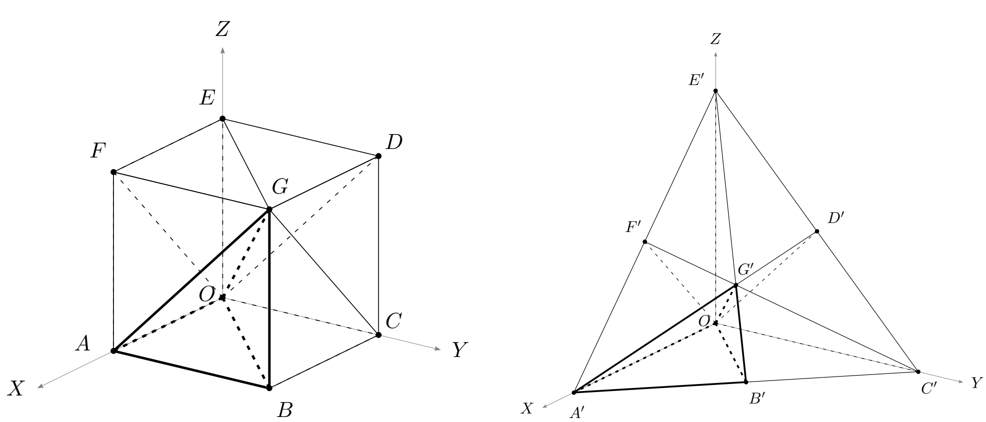

Figure 2: The cubical region $\mathbb{C}^1$ and its image $\mathbb{K}^1$

Notes

$E' = (0, 0, a)$, $F = (a/2, 0, a/2)$ and $G' = (a/3, a/3, a/3)$, see Figure 2 (right). We split the region $\mathbb{C}^1$ into six tetrahedrons of equal volume:

$$\begin{aligned} OABG &= \{(x, y, z) \in I_0,\ 1 \geq x \geq y \geq z \geq 0\}, \\ OBCG &= \{(x, y, z) \in I_0,\ 1 \geq y \geq x \geq z \geq 0\}, \\ OCDG &= \{(x, y, z) \in I_0,\ 1 \geq y \geq z \geq x \geq 0\}, \\ ODEG &= \{(x, y, z) \in I_0,\ 1 \geq z \geq y \geq x \geq 0\}, \\ OEFG &= \{(x, y, z) \in I_0,\ 1 \geq z \geq x \geq y \geq 0\}, \\ OFAG &= \{(x, y, z) \in I_0,\ 1 \geq x \geq z \geq y \geq 0\}. \end{aligned}$$

Notes

They will be mapped onto the tetrahedrons $OA'B'G'$, $OB'C'G'$, $OC'D'G'$, $OD'E'G'$, $OE'F'G'$ and $OF'A'G'$, respectively.

We focus on the region $OABG$ and the corresponding region $OA'B'G'$

$$OA'B'G' = \{(X, Y, Z) \in \mathbb{R}^3 \,|\, 0 \leq Z \leq Y \leq X,\ X + Y + Z \leq a\}.$$

Consider a point $M(x, y, z)$ in $OABG$ and the corresponding point $M'(X, Y, Z)$ through the map $\mathcal{U}$. Consider the plane π_1 through M paralel with the plane ABG and P the point of intersection of π_1 with OA. Suppose this plane is mapped in the plane π_1' through M'paralel with $A'B'G'$ and P' is the corresponding point on OA', the intersection of the plane π_1' with OA'.

The ratio of the volumes of the two pyramids with the same vertex O and the bases on the planes π_1 and ABG must be equal with the ratio of the volumes of the two pyramids with the same vertex O and the bases on the planes π_1' and $A'B'G'$ and equal with the cube of the ratio OP/OA and equal with the cube of the ratio OP'/OA'. We obtain

$$OP' = OA' \cdot OP,$$

which is equivalent with

$$X + Y + Z = ax. \tag{2}$$

Consider the plane π_2 through M paralel with the plane OBG and Q the point of intersection of π_2 with OA. Suppose this plane is mapped in the plane π_2' through M'paralel with $OB'G'$ and Q' is the corresponding point on OA', the intersection of the plane π_2' with OA'.

The ratio of the volumes of the two pyramids with the same vertex A and the bases on the planes π_2 and OBG must be equal with the ratio of the volumes of the two pyramids with the same vertex A' and the bases on the planes π_2' and $OB'G'$ and equal with the cube of the ratio AQ/AO and equal with the cube of the ratio $A'Q'/A'O$. We obtain

$$A'Q' = OA' \cdot AQ,$$

which is equivalent with $OQ' = a \cdot OQ$. We obtain

$$X - Y = a(x - y). \tag{3}$$

Consider the plane π_3 through M paralel with the plane OAG and R the point of intersection of π_3 with AB. Suppose this plane is mapped in the plane π_3' through M'paralel with $OA'G'$ and R' is the corresponding point on $A'B'$, the intersection of the plane π_3' with $A'B'$.

The ratio of the volumes of the two pyramids with the same vertex B and the bases on the planes π_3 and OAG must be equal with the ratio of the volumes of the two pyramids with the same vertex B' and the bases on the planes π_3' and $OA'G'$ and equal with the cube of the ratio BR/BA and equal with the cube of the ratio $B'R'/B'A'$. We obtain

$$B'R' = A'B' \cdot BR,$$

which is equivalent with $A'R' = \frac{a\sqrt{2}}{2} \cdot AR$. We obtain

$$Y - Z = \frac{a\sqrt{2}}{2} \cdot \frac{(y-z)\sqrt{2}}{2} = \frac{a}{2}(y-z). \tag{4}$$

Solving the system of the three equations (2), (3) and (4), we obtain

$$\begin{aligned} X &= ax - \frac{a}{2}y - \frac{a}{6}z \\ Y &= \frac{a}{2}y - \frac{a}{6}z \\ Z &= \frac{a}{3}z, \end{aligned}$$

where the value of a is specified by (1).

More general, the equations for all eight octants can be obtain in the following way: let

$$\begin{aligned} M &= \max(|x|,|y|,|z|) & M' &= \max(|X|,|Y|,|Z|) \\ m &= \min(|x|,|y|,|z|) & m' &= \min(|X|,|Y|,|Z|) \\ c &= |x|+|y|+|z| - M - m & c' &= |X|+|Y|+|Z| - M' - m'. \end{aligned}$$

Using the same reasoning we obtain the following system of equations

$$\begin{aligned} |X|+|Y|+|Z| &= a \cdot M \\ M' - c' &= a \cdot (M - c) \\ c' - m' &= \frac{a}{2} \cdot (c - m) \end{aligned}$$

with the solution

$$\begin{aligned} M' &= aM - \frac{a}{2}c - \frac{a}{6}m \\ c' &= \frac{a}{2}c - \frac{a}{6}m \\ m' &= \frac{a}{3}m. \end{aligned}$$

Notes

III. Applications

A volume preserving map can be useful in some statistics applications and for a decomposition of a 3D solid into smaller elements of equal volume.

For example, if we start with a uniform distribution of points in the solid cube we can obtain a uniform distribution of points in the solid octahedron (see [1] and the references therein, for a similar application in the case of planar domains).

For a cube is easy to obtain a uniform grid, a grid with all the elements having the same volume. By applying the map $\mathcal{U}$ we get a uniform grid for the octahedron. Using the map constructed in [2] we obtain a uniform grid for the ball. This method to obtain a uniform grid in the ball is different than the method described in [3].

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Classification of Non-Oscillatory Solutions of Nonlinear Neutral Delay Impulsive Differential Equations

By U. A. Abasiekwere, I. M. Esuabana, I. O. Isaac & Z. Lipscey
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Abstract- In this paper, a general class of second order nonlinear neutral delay impulsive differential equation of the form

$$\begin{cases} \left[y(t)-\sum_{i=1}^{m}p_i(t)y(t-\tau_i)\right]'' + \sum_{j=1}^{n} f_j\left(t, y\left(g_{jl}(t)\right),\cdots,y\left(g_{jl}\right)\right)=0, \ t\geq t_0\in R_+, t\neq t_k \\ \Delta\left[y(t_k)-\sum_{i=1}^{m}p_{ik}y(t_k-\tau_i)\right]' + \sum_{j=1}^{n} f_{jk}\left(t_k, y\left(g_{jl}(t_k)\right),\cdots,y\left(g_{jl}(t_k)\right)\right)=0, \ t_k\geq t_0\in R_+, t=t_k \end{cases}$$

is considered. We classifyits non-oscillatory solutions into four types of solution sets, namely $\Lambda^{(0,0,0)}, \Lambda^{(b,a,0)}, \Lambda^{(\infty,\infty,0)}$ and $\Lambda^{(\infty,\infty,d)}$ and establish necessary and sufficient conditions for the existence of these non-oscillatory solutions by means of Schauder-Tychonoff fixed point theorem and Lebesgue's Monotone Convergence Theorem. Some examples are given to illustrate the obtained results.

GJSFR-F Classification: *MSC 2010: 35R12*

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Abstract- In this paper, a general class of second order nonlinear neutral delay impulsive differential equation of the form

$$\begin{cases} \left[y(t)-\sum_{i=1}^{m} p_i(t)\,y(t-\tau_i)\right]'' + \sum_{j=1}^{n} f_j\Big(t, y\big(g_{j1}(t)\big),\cdots,y\big(g_{jl}\big)\Big)=0, & t\ge t_0\in R_+, t\ne t_k \\ \Delta\left[y(t_k)-\sum_{i=1}^{m} p_{ik}\,y(t_k-\tau_i)\right]' + \sum_{j=1}^{n} f_{jk}\Big(t_k, y\big(g_{j1}(t_k)\big),\cdots,y\big(g_{jl}(t_k)\big)\Big)=0, & t_k\ge t_0\in R_+, t=t_k \end{cases}$$

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I. Introduction

A survey of recent studies in neutral impulsive differential equations reveal that most of such works revolve around the quest for oscillatory conditions for impulsive differential equations, with or without delay, linear or nonlinear ([2], [3], [5], [6], [7], [8], [12], [13], [14]). The development of oscillatory and non-oscillatory criteria for nonlinear impulsive differential equations has so far attracted very little attention. In fact, the concept of non-oscillation for nonlinear neutral impulsive equations presently suffers almost complete neglect.

In this study, we attempt to classify the non-oscillatory solutions of a general class of second order nonlinear neutral delay impulsive differential equations into different solution sets and make conscious efforts to provide conditions for the existence of these solutions.

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In what follows, we recall some of the basic notions and definitions that will be of importance as we advance through the article.

Usually, the solution $y(t)$ for $t \in [t_0, T)$ of a given impulsive differential equation or its first derivative $y'(t)$ is a piece-wise continuous function with points of discontinuity $t_k \in [t_0, T)$, $t_k \neq t$. Therefore, in order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows:

Let $r \in N$, $D := [T, \infty) \subset R$ and let $S := \{t_k\}_{k \in E}$, where E is our subscript set which can be the set of natural numbers N or the set of integers Z, be fixed. Throughout this discussion, we will assume that the elements of the sequence $S := \{t_k\}_{k \in E}$ are the moments of impulsive effects and satisfy the following properties:

C1.1: If $\{t_k\}$ is defined for all $k \in N$, then $0 < t_1 < t_2 < \cdots$ and $\lim_{k \to \infty} t_k = +\infty$.

C1.2: If $\{t_k\}$ is defined for all $k \in Z$, then $t_0 \leq 0 < t_1, t_k < t_{k+1}$ for $k \in Z$, $k \neq 0$ and $\lim_{k \to \pm\infty} t_k = \pm\infty$.

We denote by $PC(D,R)$ the set of all values $\psi : D \to R$ which is continuous for all $t \in D$, $t \notin S$. They are functions from the left and have discontinuity of the first kind at the points for $t \in S$. By $PC^r(D,R)$, we denote the set of functions $\psi : D \to R$ having derivative $\frac{d^j \psi}{dt^j} \in PC(D,R)$, $0 \leq j \leq r$ ([1], [4]).

To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes use the symbols $PC(D,R;S)$ and $PC^r(D,R;S)$, $r \in N$.

The solution $y(t)$ of an impulsive differential equation is said to be

1. Finally positive (finally negative) if there exist $T \geq 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t \geq T$ ([9]);
2. Oscillatory, if it is neither finally positive nor finally negative; and
3. Non-oscillatory, if it is neither finally positive nor finally negative ([1], [10]).

II. Statement of the Problem

Here, we are considering the second order nonlinear neutral impulsive differential equation of the form

$$\begin{cases} \left[y(t) - \sum_{i=1}^{m} p_i(t) y(t-\tau_i) \right]'' + \sum_{j=1}^{n} f_j\left(t, y\left(g_{j1}(t), \cdots, y\left(g_{jl}\right)\right)\right) = 0, \ t \geq t_0 \in R_+, t \notin S \\ \Delta\left[y(t_k) - \sum_{i=1}^{m} p_{ik} y(t_k - \tau_i) \right]' + \sum_{j=1}^{n} f_{jk}\left(t_k, y\left(g_{j1}(t_k), \cdots, y\left(g_{jl}(t_k)\right)\right)\right) = 0, \ t_k \geq t_0 \in R_+, \forall\, t_k \in S. \end{cases} \tag{2.1}$$

We introduce the following conditions:

C2.1: $\tau_i > 0$, $p_{ik} \geq 0$, $p_i \in PC^1([t_0, \infty), R_+)$, $i = 1, 2, \cdots, m$ and there exists $\delta \in (0,1]$ such that

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$$\sum_{i=1}^{m} p_i(t) + \sum_{j=1}^{n} p_j \le 1-\delta, \quad t, \ge t_0 \in R_+;$$

C2.2: $g_{js} \in C([t_0,\infty), R), \ \lim_{t\to\infty} g_{js}(t) = \infty, \ j=1,2,\cdots,n, \ s=1,2,\cdots,\ell;$

C2.3: $f_j \in PC([t_0,\infty)\times R^\ell, R), \ x_1 f_j(t, x_1,\cdots,x_\ell) > 0;$

$x_1 f_{jk}(t_k, x_1,\cdots,x_l) > 0$ for $x_1 \, x_i > 0, \ i=1,2,\cdots,\ell, \ j=1,2,\cdots,n$. Moreover,

$$\begin{cases} \left|f_j(t, y_1,\cdots,y_\ell)\right| \ge \left|f_j(t,x_1,\cdots,x_\ell)\right| \\ \left|f_{jk}(t_k, y_1 \cdots, y_\ell)\right| \ge \left|f_{jk}(t_k, x_1,\cdots,x_\ell)\right| \end{cases}$$

whenever

$$|x_i| \le |y_i| \text{ and } y_i x_i > 0, \ i=1,2,\cdots,\ell, \ j=1,2,\cdots,n;$$

C2.4: Set

$$x(t) = y(t) - \sum_{i=1}^{m} p_i(t) y(t-\tau_i). \tag{2.2}$$

Our aim in this paper is to give the classification of non-oscillatory solutions of equation (2.1). But first, we define some concepts and establish the following lemmas which will be useful in the discussion of the main results.

Theorem 2.1: (Schauder-Tychonoff fixed point theorem) Let X be a locally convex linear space, S a compact convex subset of X, and let $T: S \to S$ be a continuous mapping with T(S) compact. Then T has a fixed point in S.

Theorem 2.2: (Lebesgue's Monotone Convergence Theorem) Let (A, Σ, μ) be a measure space and $f_1, f_2, f_3, \cdots$ a pointwise non-decreasing sequence of $[0,\infty)-$valued $\Sigma-$measurable functions. Let $\lim_{n\to\infty} f_n(t) := f(t)$ for all $t \in A$, then f is $\Sigma-$measurable and

$$\lim_{n\to\infty} \int_A f_n d\mu = \int_A f \, d\mu.$$

Lemma 2.1 and 2.2 are extensions of Lemma 4.5.1 and 4.5.2 on pages 242 and 243 respectively of the monograph by Erbe et al [11]

Lemma 2.1: Let $y(t)$ be a finally positive (or negative) solution of equation (2.1). If $\lim_{t\to\infty} y(t) = 0$, then $x(t)$ is finally negative (or positive) and $\lim_{t\to\infty} x(t) = 0$. Otherwise, $x(t)$ is finally positive (or negative).

Proof: Let $y(t)$ be a finally positive solution of equation (2.1). From the same equation (2.1), $x''(t), \Delta x'(t_k) > 0$ or $x'(t), \Delta x(t_k) < 0$ finally. Also, $x(t) > 0$ or $x(t) < 0$ finally. If $\lim_{t\to\infty} y(t) = 0$, from equation (2.2), it follows that $\lim_{t\to\infty} x(t) = 0$. Since x(t) is monotonic, so $\lim_{t\to\infty} x'(t) = 0$, $\lim_{t_k\to\infty} \Delta x(t_k) = 0$ which implies that $x'(t) > 0, \ \Delta x(t_k) > 0$.

Ref

11. L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Dekker, New York, 1995.

Therefore, $x(t)<0$ finally. If $\lim_{t\to\infty} y(t)\neq 0$, then $\limsup_{t\to\infty} y(t)>0$. We show that $x(t)>0$ finally. If not, then $x(t)<0$ finally. If $y(t)$ is unbounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n=\infty$, $y(t_n)=\max_{t_0\le t<t_n} y(t)$ and $\lim_{n\to\infty} y(t_n)=\infty$. From equation (2.2), we obtain

$$x(t_n)=y(t_n)-\sum_{i=1}^{m} p_i(t_n)y(t_n-\tau_i)\ge y(t_n)\left(1-\sum_{i=1}^{m} p_i(t_n)\right). \tag{2.3}$$

Thus, $\lim_{n\to\infty} x(t_n)=\infty$, which is a contradiction. If $y(t)$ is bounded, then there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n=\infty$ and $\lim_{n\to\infty} y(t_n)=\limsup_{t\to\infty} y(t)$. Since the sequences $\{p_i(t_n)\}$ and $\{y(t_n-\tau_i)\}$ are bounded, there exists convergent subsequences. Without loss of generality, we may assume that $\lim_{n\to\infty} y(t_n-\tau_i)$ and $\lim_{n\to\infty} p_i(t_n)$, $i=1,2\cdots,m$, exist. Hence

$$0\ge\lim_{n\to\infty} x(t_n)=\lim_{n\to\infty}\left(y(t_n)-\sum_{i=1}^{m} p_i(t_n)y(t_n-\tau_i)\right)\ge\limsup_{t\to\infty} y(t)\left(1-\sum_{i=1}^{m} p_i(t_n)\right)>0,$$

which, again, is a contradiction. Therefore, $x(t)>0$ finally. A similar proof can be repeated if $y(t)<0$ finally.

Lemma 2.2: Assume that $\lim_{t\to\infty}\sum_{i=1}^{m} p_i(t)=P\in(0,1]$, and $y(t)$ is a finally positive (or negative) solution of equation (2.1). If $\lim_{t\to\infty} x(t)=a\in R$, then $\lim_{t\to\infty} y(t)=\frac{a}{1-p}$. If $\lim_{t\to\infty} x(t)=\infty$ (or $-\infty$), then $\lim_{t\to\infty} y(t)=\infty$ (or $-\infty$).

Proof: Let $y(t)$ be a finally positive solution of equation (2.1), then $y(t)\ge x(t)$ finally. If $\lim_{t\to\infty} x(t)=\infty$, then $\lim_{t\to\infty} y(t)=\infty$. Now we consider the case that $\lim_{t\to\infty} x(t)=a\in R$. Thus, $x(t)$ is bounded which implies, by equation (2.3), that $y(t)$ is bounded. Therefore, there exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n=\infty$ and $\lim_{n\to\infty} y(t_n)=\limsup_{t\to\infty} y(t)$. As before, without loss of generality, we may assume that $\lim_{n\to\infty} p_i(t_n)$ and $\lim_{n\to\infty} y(t_n-\tau_i)$, $i=1,2,\cdots,n$ exist. Hence

$$a=\lim_{n\to\infty} x(t_n)=\lim_{n\to\infty} y(t_n)-\sum_{i=1}^{m}\lim_{n\to\infty} p_i(t_n)\lim_{n\to\infty} y(t_n-\tau_i)\ge\limsup_{t\to\infty} y(t)(1-p),$$

that is,

$$\frac{a}{1-p}\ge\limsup_{t\to\infty} y(t). \tag{2.4}$$

Notes

On the other hand, there exists $\{t'_n\}$ such that $\lim_{n\to\infty} y(t'_n) = \liminf_{t\to\infty} y(t)$. Without loss of generality, we assume that $\lim_{n\to\infty} p_i(t'_n)$ and $\lim_{n\to\infty} y(t'_n - \tau_i)$, $i = 1, 2, \cdots, m$ exist. Hence

$$a = \lim_{n\to\infty} x(t'_n) = \lim_{n\to\infty} y(t'_n) - \sum_{i=1}^{m} \lim_{n\to\infty} p_i(t'_n) \lim_{n\to\infty} y(t'_n - \tau_i) \le \liminf_{t\to\infty} y(t)(1-p)$$

or

$$\frac{a}{1-p} \le \liminf_{t\to\infty} y(t). \tag{2.5}$$

Combining inequalities (2.4) and (2.5), we obtain $\lim_{t\to\infty} y(t) = \frac{a}{1-p}$. A similar argument can be repeated if $y(t) < 0$.

We are now ready to prove the following results.

III. Main Results

Here, Theorem 3.1, 3.2, 3.3, 3.4, 3.5 are extensions of Theorem 4.5.1, 4.5.2, 4.5.3, 4.5.4, 4.5.5 found on pages 244, 245, 249, 251, 251, respectively, being their neutral delay versions as identified in the work by Erbe et al ([11]).

Theorem 3.1: Assume that $\lim_{t\to\infty} \sum_{i=1}^{m} p_i(t) = p \in [0,1)$. Let $y(t)$ be a non-oscillatory solution of equation (2.1). Let Λ denote the set of all non-oscillatory solutions of equation (2.1), and define

$$\Lambda^{(0,0,0)} = \left\{ y \in \Lambda : \lim_{t\to\infty} y(t) = 0,\ \lim_{t\to\infty} x(t) = 0,\ \lim_{t,t_k\to\infty} \left(x'(t), \Delta x(t_k)\right) = 0 \right\},$$

$$\Lambda^{(b,a,0)} = \left\{ y \in \Lambda :\ \lim_{t\to\infty} y(t) = b := \frac{a}{1-p},\ \lim_{t\to\infty} x(t) = a,\ \lim_{t,t_k\to\infty} \left(x'(t), \Delta x(t_k)\right) = 0 \right\},$$

$$\Lambda^{(\infty,\infty,0)} = \left\{ y \in \Lambda : \lim_{t\to\infty} y(t) = \infty,\ \lim_{t\to\infty} x(t) = \infty,\ \lim_{t,t_k\to\infty} \left(x'(t), \Delta x(t_k)\right) = 0 \right\},$$

$$\Lambda^{(\infty,\infty,d)} = \left\{ y \in \Lambda : \lim_{t\to\infty} y(t) = \infty,\ \lim_{t\to\infty} x(t) = \infty,\ \lim_{t,t_k\to\infty} \left(x'(t), \Delta x(t_k)\right) = d \ne 0 \right\}.$$

Then

$$\Lambda = \Lambda^{(0,0,0)} \cup \Lambda^{(b,a,0)} \cup \Lambda^{(\infty,\infty,0)} \cup \Lambda^{(\infty,\infty,d)}.$$

Proof: Without loss of generality, let $y(t)$ be a finally positive solution of equation (2.1). If $\lim_{t\to\infty} y(t) = 0$, then by Lemma 2.1, $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t,t_k\to\infty} \left(x'(t), \Delta x(t_k)\right) = 0$, that is, $y \in \Lambda^{(0,0,0)}$. If $\lim_{t\to\infty} y(t) \ne 0$, then by Lemma 2.1, $x(t) > 0$ finally and it therefore implies that

Ref

11. L. H. Erbe, Q. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Dekker, New York, 1995.

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$x'(t), \Delta x(t_k) > 0$ and $x''(t), \Delta x'(t_k) < 0$ finally. If $\lim_{t\to\infty} x(t) = a > 0$ exists, then $\lim_{t,t_k\to\infty}(x'(t), \Delta x(t_k)) = 0$. By Lemma 2.2, we have $\lim_{t\to\infty} y(t) = \frac{a}{1-p} = b$, that is, $y \in \Lambda^{(b,a,0)}$. If $\lim_{t\to\infty} x(t) = \infty$, then by Lemma 2.2, $\lim_{t\to\infty} y(t) = \infty$. Since $x''(t), \Delta x'(t_k) < 0$ and $x'(t), \Delta x(t_k) > 0$, we obtain $\lim_{t,t_k\to\infty}(x'(t), \Delta x(t_k)) = d$, where $d = 0$ or $d > 0$. Then either $y \in \Lambda^{(\infty,\infty,0)}$ or $y \in \Lambda^{(\infty,\infty,d)}$.

This completes the proof of Theorem 3.1.

In what follows, we shall show some existence results for each kind of non-oscillatory solution of equation (2.1).

Theorem 3.2: Assume that there exist two constants $h_1 > h_2 > 0$ such that

$$|p_i(t_2) - p_i(t_1)| \le h_1|t_2 - t_1|, |p_i(t_{2k}) - p_i(t_{1k})| \le h_1|t_{2k} - t_{1k}|, \ i = 1, 2, \cdots, m,$$

$$\sum_{i=1}^{m} p_i(t)\exp(h_1\tau_i) + \exp(h_1 t)\sum_{i=1}^{m} p_{ik}\exp(-h_1(t_k - \tau_i)) > 1$$

$$\ge \sum_{i=1}^{m} p_i(t)\exp(h_2\tau_i) + \exp(h_2 t)\sum_{i=1}^{m} p_{ik}\exp(-h_2(t_k - \tau_i)) \tag{3.1}$$

and

$$\left(\sum_{i=1}^{m} p_i(t)\exp(h_1\tau_i) + \exp(h_1 t)\sum_{i=1}^{m} p_{ik}\exp(-h_1(t_k - \tau_i)) - 1\right)\exp(-h_1 t)$$

$$\ge \int_t^{x_n} (u-t)\sum_{j=1}^{m} f_j\left(u, \exp(-h_2 g_{j1}(u)), \dots, \exp(-h_2 g_{jl}(u))\right)du +$$

$$+ \sum_{t \le t_k < \infty} (t_k - t)\sum_{j=1}^{n} f_{jk}\left(t_k, \exp(-h_2 g_{j1}(t_k))\right), \cdots, \exp(-h_2 g_{jl}(t_k)) \tag{3.2}$$

finally. Then equation (2.1) has a solution $y \in \Lambda^{(0,0,0)}$.

Proof: Let us denote by B_p the space of all bounded piece-wise continuous functions in $PC([t_0, \infty))$ and define the sup norm in B_p as follows:

$$\|y\| := \sup_{t \ge t_0} |y(t)|.$$

Set

$$\Omega = \begin{cases} y \in B_p : \exp(-h_1 t) \le y(t) \le \exp(-h_2 t) \\ |y(t_2) - y(t_1)| \le L|t_2 - t_1|, |y(t_{2k}) - y(t_{1k})| \le L|t_{2k} - t_{1k}|, \end{cases}$$

for $t_1, t_2 \ge t_0$, $\forall k : t_{1k}, t_{2k} \ge t_0$ and for $L \ge h_1$. Then Ω is a nonempty, closed convex bounded set in B_p.

Notes

For the sake of convenience, denote

$$\begin{cases} f\left(u,y\left(g\left(u\right)\right)\right)=\sum_{j=1}^{n} f_j\left(u,y\left(g_{j1}\left(u\right)\right),\cdots,y\left(g_{jl}\left(u\right)\right)\right) \\ f_k\left(t_k,y\left(g\left(t_k\right)\right)\right)=\sum_{j=1}^{n} f_{jk}\left(t_k,y\left(g_{j1}\left(t_k\right)\right),\cdots,y\left(g_{jl}\left(t_k\right)\right)\right), \end{cases} \tag{3.3}$$

Notes

$$\begin{cases} f\left(u,\exp\left(-h_2 g\left(u\right)\right)\right)=\sum_{j=1}^{n} f_j\left(u,\exp\left(-h_2 t_{j1}\left(u\right)\right),\cdots,\exp\left(-h_2 g_{jl}\left(u\right)\right)\right) \\ f_k\left(t_k,\exp\left(-h_2 g\left(t_k\right)\right)\right)=\sum_{j=1}^{n} f_{jk}\left(t_k,\exp\left(-h_2 g_{j1}\left(t_k\right)\right),\cdots,\exp\left(-h_2 g_{jl}\left(t_k\right)\right)\right). \end{cases} \tag{3.4}$$

Define a mapping J on Ω as follows:

$$(Jy)(t)=\begin{cases} \sum_{i=1}^{m} p_i\left(t\right)y\left(t-\tau_i\right)+\sum_{i=1}^{m} p_{ik}\,y\left(t_k-\tau_i\right)-\int_t^{\infty}\left(u-t\right)f\left(u,y\left(g\left(u\right)\right)\right)du- \\ \qquad -\sum_{t\le t_k<\infty}\left(t_k-t\right)+f_k\left(t_k,y\left(g\left(t_k\right)\right)\right),\ t,t_k\ge T \\ \exp\left(-K(y)t\right)+\exp\left(-K(y)t_k\right),\ t_0\le t,t_k<T, \end{cases} \tag{3.5}$$

where

$$K\left(y\right)=-\frac{\ln(Jy)(T)}{T},$$

T is sufficiently large such that $t-\tau_i\ge t_0$; $t_k-\tau_i\ge t_0$; $g_{js}\left(t_k\right)\ge t_0$; $i=1,2,\cdots,m$; $j=1,2,\cdots,n$; $s=1,2,\cdots,\ell$, for $t,\ t_k\ge T$.
Now, we see that condition (3.2) implies that

$$\int_T^{\infty} f\left(u,\exp\left(-h_2\, g\left(u\right)\right)\right)du+\sum_{T\le t_k<\infty} f_k\left(t_k,\exp\left(-h_2\, g\left(t_k\right)\right)\right)<\infty,$$

while from condition C2.1, it follows that for a given $\alpha\in\left(1-\delta,1\right)$,

$$\begin{cases} \left(\alpha-\sum_{i=1}^{m} p_i\left(t\right)\right)L\ge\left[\alpha-\left(1-\delta\right)\right]L>0 \\ \left(\alpha-\sum_{i=1}^{m} p_{ik}\right)\ge\left[\alpha-\left(t-\delta\right)\right]L. \end{cases} \tag{3.6}$$

Therefore, T can be chosen so large that for $t,\ t_k\ge T$,

$$\begin{cases} \int_T^{\infty} f\left(u,\exp\left(-h_2\, g\left(u\right)\right)\right)du\le\left(\alpha-\sum_{i=1}^{m} p_i\left(t\right)\right)L \\ \sum_{T\le t_k<\infty} f_k\left(t_k\right)\exp\left(-h_2\, g\left(t_k\right)\right)\le\left(\alpha-\sum_{i=1}^{m} p_{ik}\right)L, \end{cases} \tag{3.7}$$

and

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$$\begin{cases} \alpha+\sum_{i=1}^{m}\exp\left(-h_2\left(t-\tau_i\right)\right)\le\frac{1}{2} \\ \left(\alpha+\sum_{i=1}^{m}\exp\left(-h_2\left(t_k-\tau_i\right)\right)\right)\le\frac{1}{2}\frac{\left|t_2-t_1\right|}{\left|t_{2k}-t_{1k}\right|}. \end{cases}$$

Hence from inequalities (3.1) and (3.2), it follows that

$$(Jy)(t)\le\sum_{i=1}^{m}p_i(t)y(t-\tau_i)+\sum_{i=1}^{m}p_{ik}\,y(t_k-\tau_i)$$

$$\le\sum_{i=1}^{m}p_i(t)\exp\left(-h_2\left(t_k-\tau_i\right)\right)+\sum_{i=1}^{m}p_{ik}\exp\left(-h_2\left(t_k-\tau_i\right)\right)$$

$$\le\exp\left(-h_2\,t\right)\left[\sum_{i=1}^{m}p_i(t)\exp\left(h_2\,\tau_i\right)+\exp\left(h_2\,t\right)\sum_{i=1}^{m}p_{ik}\exp\left(-h_2\left(t_k-\tau_i\right)\right)\right]$$

$$\le\exp\left(-h_2\,t\right)\ \text{for}\ t,t_k\ge T,$$

and

$$(Jy)(t)\ge\sum_{i=1}^{m}p_i(t)\exp\left(-h_1\left(t-\tau_i\right)\right)+\sum_{i=1}^{m}p_{ik}\exp\left(-h_1\left(t_k-\tau_i\right)\right)-$$

$$-\int_t^{\infty}(u-t)f\left(u,\exp\left(-h_2g(u)\right)\right)du-\sum_{it\le t_k<\infty}\left(t_k-t\right)f_k\left(t_k,\exp\left(-h_2g\left(t_k\right)\right)\right)$$

$$=\exp\left(-h_1t\right)+\exp\left(-h_1t\right)\left(\sum_{i=1}^{m}p_i(t)\exp\left(h_1\tau_i\right)+\exp\left(h_1t\right)\sum_{i=1}^{m}p_{ik}\exp\left(-h_1\left(t_k-\tau_i\right)\right)\right)-$$

$$-\int_t^{\infty}(u-t)f\left(u,\exp\left(-h_2g(u)\right)\right)du-\sum_{t\le t_k<\infty}\left(t_k-t\right)f_k\left(t_k,\exp\left(-h_2g\left(t_k\right)\right)\right)$$

$$\ge\exp\left(-h_1t\right)\ \text{for}\ t,t_k\ge T.$$

That is,

$$\exp\left(-h_1t\right)\le(Jy)(t)\le\exp\left(-h_2t\right),\ t\ge T,$$

$$\exp\left(-h_1\left(t_k\right)\right)\le(Jy)\left(t_k\right)\le\exp\left(-h_2t_k\right),\ t_k\ge T.$$

By the definition of $K(y)$ and the statement

$$\exp\left(-h_1T\right)\le(Jy)(T)\le\exp\left(-h_2T\right),$$

It is clear that $h_2\le K(y)\le h_1$. Hence

$$\exp\left(-h_1t\right)\le(Jy)(t)\le\exp\left(-h_2t\right),\ \ t_0\le t,t_k<T.$$

Next, we show that

Notes

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|\le L\left|t_2-t_1\right|, \tag{3.8}$$

for $t_1,\ t_2\in[t_0,\infty)$ and $k:\ t_{1k},\ t_{2k}\in[t_0,\infty)$. Without loss of generality, we assume that $t_2\ge t_1\ge t_0$ and $\forall\ k: t_{2k}\ge t_{1k}\ge t_0$. Indeed, for $t_2\ge t_1\ge T$ and $\forall\ k: t_{2k}\ge t_{1k}\ge T$, using condition (3.7) and inequality (3.8), we have that

Notes

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|=\left|(Jy)(t_2)+(Jy)(t_{2k})-(Jy)(t_1)-(Jy)(t_{1k})\right|$$

$$\le\sum_{i=1}^{m}\left|p_i(t_1)y(t_1-\tau_i)+p_i(t_{1k})y(t_{1k}-\tau_i)-p_i(t_2)y(t_2-\tau_i)-p_i(t_{2k})y(t_{2k}-\tau_i)\right|+$$

$$+\int_{t_1}^{\infty}(u-t_1)f\big(u,y(g(u))\big)du+\sum_{t_1\le t_{1k}<\infty}(t_{1k}-t_1)f_k\big(t_{1k},y(g(t_{1k}))\big)-\int_{t_2}^{\infty}(u-t_2)f\big(u,y(g(u))\big)du-$$

$$-\sum_{t_2\le t_{2k}<\infty}(t_{2k}-t_2)f_k\big(t_{2k},y(g(t_{2k}))\big)$$

$$\le\sum_{i=1}^{m}\left|p_i(t_1)y(t_1-\tau_i)-p_i(t_2)y(t_2-\tau_i)\right|+\sum_{i=1}^{m}\left|p_i(t_{1k})y(t_{1k}-\tau_i)-p_i(t_{2k})y(t_{2k}-\tau_i)\right|+$$

$$+\left|\int_{t_1}^{\infty}(u-t_1)f\big(u,y(g(u))\big)du-\int_{t_2}^{\infty}(u-t_2)f\big(u,y(g(u))\big)du\right|+$$

$$+\left|\sum_{t_1\le t_{1k}<\infty}(t_{1k}-t_1)f_k\big(t_{1k},y(g(t_{1k}))\big)-\sum_{t_2\le t_{2k}<\infty}(t_{2k}-t_2)f_k\big(t_{2k},y(g(t_{2k}))\big)\right|$$

$$\le\sum_{i=1}^{m}p_i(t_2)\left|y(t_2-\tau_i)-y(t_1-\tau_i)\right|+\sum_{i=1}^{m}\left|p_i(t_2)-p_i(t_1)\right|y(t_1-\tau_i)+$$

$$+\left|\int_{t_1}^{t_2}(u-t_2)f\big(u,y(g(u))\big)du+\int_{t_2}^{\infty}(t_2-t_1)f\big(u,y(g(u))\big)du\right|+$$

$$+\sum_{i=1}^{m}p_i(t_{2k})\left|y(t_{2k}-\tau_i)-y(t_{1k}-\tau_i)\right|+\sum_{i=1}^{m}\left|p_i(t_{2k})-p_i(t_{1k})\right|y(t_{1k}-\tau_i)+$$

$$+\left|\sum_{t_1\le t_k\le t_2}(t_k-t_{1k})f\big(t_k,y(g(t_k))\big)+\sum_{t_2\le t_k<\infty}(t_{2k}-t_{1k})f_k\big(t_k,y(g(t_k))\big)\right|$$

$$\le\left[\sum_{i=1}^{m}\Big(p_i(t_2)+\exp\big(-h_2(t_1-\tau_i)\big)\Big)L+\int_{t_1}^{\infty}f\big(u,\exp(-h_2g(u))\big)du\right]\left|t_2-t_1\right|+$$

$$+\left[\sum_{i=1}^{m}\Big(p_i(t_{2k})+\exp\big(-h_2(t_{1k}-\tau_i)\big)\Big)L+\sum_{t_1\le t_k<\infty}f_k\big(t_k,\exp(-h_2g(t_k))\big)\right]\left|t_{2k}-t_{1k}\right|$$

$$\le\left\{\left[\sum_{i=1}^{m}p_i(t_2)+\sum_{i=1}^{m}\exp\big(-h_2(t_1-\tau_i)\big)\right]+\left(\alpha-\sum_{i=1}^{m}p_i(t_2)\right)\right\}L\left|t_2-t_1\right|+$$

$$+\left\{\left[\sum_{i=1}^{m}p_i\left(t_{2k}\right)+\sum_{i=1}^{m}\exp\left(-h_2\right)\left(t_{1k}-\tau_i\right)\right]+\left(\alpha-\sum_{i=1}^{m}p_i\left(t_{2k}\right)\right)\right\}L\left|t_{2k}-t_{1k}\right|$$

$$=\left[\sum_{i=1}^{m}\exp\left(-h_2\left(t_1-\tau_i\right)\right)+\alpha\right]L\left|t_2-t_1\right|+\left[\sum_{i=1}^{m}\exp\left(-h_2\left(t_{1k}-\tau_i\right)\right)+\alpha\right]L\left|t_{2k}-t_{1k}\right|$$

$$\le\frac{L}{2}\left|t_2-t_1\right|+\frac{L}{2}\frac{\left|t_{2k}-t_{1k}\right|}{t}\cdot\frac{\left|t_2-t_1\right|}{\left|t_{2k}-t_{1k}\right|}$$

$$\le L\left|t_2-t_1\right|.$$

For $t_0\le t_1\le t_2\le T$ and $\forall\, k: t_0\le t_{1k}\le t_{2k}\le T$, we have

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|=\left|(Jy)(t_2)+(Jy)(t_{2k})-(Jy)(t_1)-(Jy)(t_{1k})\right|$$

$$=\left|\exp\left(-K(y)(t_2)\right)+\exp\left(-K(y)(t_{2k})\right)-\exp\left(-K(y)(t_1)\right)-\exp\left(-K(y)(t_{1k})\right)\right|$$

$$\le\left|\exp\left(-K(y)(t_2)\right)-\exp\left(-K(y)(t_1)\right)\right|+\left|\exp\left(-K(y)(t_{2k})\right)-\exp\left(-K(y)(t_{1k})\right)\right|$$

$$\le\frac{L}{2}\left|t_2-t_1\right|+\frac{L\left|t_{2k}-t_{1k}\right|}{2}\cdot\frac{\left|t_2-t_1\right|}{\left|t_{2k}-t_{1k}\right|}=L\left|t_2-t_1\right|.$$

For $t_0<t_1\le T\le t_2$ and $\forall\, k: t_0<t_{1k}\le T\le t_{2k}$, we obtain

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|\le\left|(Jy)(t_2)-(Jy)(t_1)\right|+\left|(Jy)(t_{2k})-(Jy)(t_{1k})\right|$$

$$\le\left|(Jy)(t_2)-(Jy)(T)\right|+\left|(Jy)(T)-(Jy)(t_1)\right|+\left|(Jy)(t_{2k})-(Jy)(T)\right|+\left|(Jy)(T)-(Jy)(t_{1k})\right|$$

$$\le\frac{L}{2}\left|t_2-T\right|+\frac{L}{2}\left|T-t_1\right|+\frac{L\left|t_2-t_1\right|}{2\left|t_{2k}-t_{1k}\right|}\left|t_{2k}-T\right|+\frac{L\left|t_2-t_1\right|}{2\left|t_{2k}-t_{1k}\right|}\left|T-t_{1k}\right|$$

$$=\frac{L}{2}\left|t_2-t_1\right|+\frac{L}{2}\left|t_2-t_1\right|=L\left|t_2-t_1\right|.$$

We have proved that inequality (3.8) holds for all $t_0\le t_1\le t_2$ and $\forall\, k: t_0\le t_{1k}\le t_{2k}$. Therefore, $J\Omega\subseteq\Omega$. Hence, J is piece-wise continuous. Since $J\Omega\subseteq\Omega$, $J\Omega$ is uniformly bounded.

Set $y\in\Omega$. It immediately implies that

$$\left|(Jy)(t)\right|\le b_0,$$

where $b_0>0$ and

$$\left|(Jy)(t_2)-(Jy)(t_1)\right|\le L\left|t_2-t_1\right|$$

for $t_2\ge t_1\ge t_0$ and $k: t_{2k}\ge t_0$. Without loss of generality, we set

Notes

$$b_0 = \exp(-h_2 t),\ t, t_k \ge t_0.$$

Hence, for any arbitrarily pre-assigned small positive number ε, there exists a sufficiently large $T' > t_0$ such that whenever $\exp(-h_2 t) < \frac{\varepsilon}{2}$,

$$\left|(Jy)(t_2) - (Jy)(t_1)\right| \le \exp(-h_2 t_2) + \exp(-h_2 t_1) \le \varepsilon \tag{3.9}$$

Notes

for $t, t_k \ge T', t_2 \ge t_1 \ge T'$ and $k: t_{2k} \ge t_{1k} \ge T'$.

On the other hand, if we set $\lambda = \frac{\varepsilon}{L}$ and assume that $|t_2 - t_1| < \lambda$, then for all $t_0 \le t_1 \le t_2 \le T'$ and $k: t_0 \le t_{1k} \le t_{2k} \le T'$, it becomes clear that

$$\left|(Jy)(t_2) - (Jy)(t_1)\right| \le \varepsilon \tag{3.10}$$

Thus, from inequalities (3.9) and (3.10), we can affirm that $J\Omega$ is quasi-equicontinuous. Therefore, $J\Omega$ is relatively compact. By virtue of Schauder-Tychonoff fixed point theorem, the mapping J has a fixed point $y^* \in J$ such that $y^* = Jy^*$. Then y^* is a positive solution of equation (2.1) and $y^* \in \Lambda^{(0,0,0)}$. This completes the proof of Theorem 3.2.

Theorem 3.3: Assume that $\lim\limits_{t\to\infty}\sum\limits_{i=1}^{m} p_i(t) + \lim\limits_{t_k\to\infty}\sum\limits_{i=1}^{m} p_{ik} = p \in [0,1)$. Then equation (2.1) has a non-oscillatory solution $y \in \Lambda^{(b,a,0)}$ $(b, a \ne 0)$ if and only if

$$\int_{t_0}^{\infty} u\left|\sum_{j=1}^{n} f_j(u, b_1, \ldots, b_1)\right| du + \sum_{t_0 \le t_k < \infty} t_k \left|\sum_{j=1}^{n} f_{jk}(t_k, b_1, \ldots, b_1)\right| < \infty \tag{3.11}$$

for $b_1 \ne 0$.

Proof

i) *Necessity:* Without loss of generality, let $y(t) \in \Lambda^{(b,a,0)}$ be a finally positive solution of equation (2.1). From Theorem 3.1, we know that $b > 0$ and $a > 0$. Using notations in equations (3.3) and (3.4), we obtain from equations (2.1) and (2.2),

$$\begin{cases} x''(t) = -f(t, y(g(t))) \\ \Delta x'(t_k) = f_k(t_k, y(g(t_k))). \end{cases}$$

Integrating it from s to ∞ for $s \ge t_0$, we have

$$x'(s) = \int_s^{\infty} f(u, y(g(u)))du + \sum_{s \le t_k < \infty} f_k(t_k, y(g(t_k))) . \tag{3.12}$$

Again, integrating equation (3.12) from T to t, where T is sufficiently large, we obtain

$$x(t)=x(T)+\int_T^t (u-T)f\left(u,y\left(g(u)\right)\right)du+\int_t^\infty (t-T)f\left(u,y\left(g(u)\right)\right)du+$$

$$+\sum_{T\le t_k\le t}(t_k-T)f_k\left(t_k,y\left(g(t_k)\right)\right)+\sum_{t\le t_k<\infty}(t-T)f_k\left(t_k,y\left(g(t_k)\right)\right). \tag{3.13}$$

Since $\lim_{u\to\infty} y\left(g_{jh}(u)\right)=b>0$ and $\lim_{t_k\to\infty} y\left(g_{jh}(t_k)\right)=b>0$, $j=1,2,\cdots,n$, $h=1,2,\cdots,\ell$, there exists a $T\ge t_0$ such that $y\left(g_{jh}(u)\right)\ge\frac{b}{2}$ for $u\ge T$ and $y\left(g_{jh}(t_k)\right)\ge\frac{b}{2}$ for $k:t_k\ge T$. Hence from equation (3.13) we have

$$\int_T^t (u-T)\left|\sum_{j=1}^n f_j\left(u,\frac{b}{2},\cdots,\frac{b}{2}\right)\right|du+\sum_{T\le t_k\le t}(t_k-T)\left|\sum_{j=1}^n f_{jk}\left(t_k,\frac{b}{2},\cdots,\frac{b}{2}\right)\right|<x(t)-x(T)$$

which implies that condition (3.11) holds.

ii) *Sufficiency:* Set $b_1>0$ and $A>0$ so that $A<(1-p)b_1$. From condition (3.11) there exists a sufficiently large T so that for $t,t_k\ge T$ we have $t-\tau_i\ge t_0$, $t_k-\tau_i\ge t_0$, $i=1,2,\cdots,m$, and $g_{jh}(t)\ge t_o$, $g_{jh}(t_k)\ge t_0$, $j=1,2,\cdots,n$, $h=1,2,\cdots,\ell$ and

$$\frac{A}{b_1}+\sum_{i=1}^m\left(p_i(t)+p_{ik}\right)+\frac{1}{b_1}\int_T^\infty u\sum_{j=1}^n f_j(u,b_1,\dots,b_1)du+\frac{1}{b}\sum_{T\le t_k<\infty}t_k\sum_{j=1}^n f_{jk}(t_k,b_1,\dots,b_1)\le 1. \tag{3.14}$$

Let Ω be the set of all piece-wise continuous functions $y(t)\in[t_0,\infty)$ such that $0\le y(t)\le b_1$, $t,t_k\ge t_0$. Define a mapping J in Ω as follows:

$$(Jy)(t)=\begin{cases} A+\sum_{i=1}^m p_i(t)y(t-\tau_i)+\sum_{i=1}^m p_{ik}y(t_k-\tau_i)+\int_T^t uf\left(u,y\left(g(u)\right)\right)du+ \\ \quad +\int_t^\infty tf\left(u,y\left(g(u)\right)\right)du+\sum_{T\le t_k\le t}t_kf_k\left(t_k,y\left(g(t_k)\right)\right)+\sum_{t\le t_k<\infty}tf_k\left(t_k,y\left(g(t_k)\right)\right), \\ \quad t,t_k\ge T \\ (Jy)(T),\quad t_0\le t,t_k<T. \end{cases} \tag{3.15}$$

Set

$$y_0(t)=0,\ t\ge t_0;$$

$$y_\ell(t)=\left(Ty_{\ell-1}\right)(t),\ t\ge t_0,\ \ell=1,2,\cdots. \tag{3.16}$$

It immediately follows that $y_0(t)<y_1(t)=A\le b_1$, $t\ge t_0$. By induction, we obtain

$$A\le y_\ell(t)\le y_{\ell+1}(t)\le b_1,\ t\ge t_0,\ \ell=1,2,\cdots.$$

Notes

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Thus, $\lim_{\ell\to\infty} y_\ell(t) \le y(t)$ exists and $A \le y(t) \le b_1,\ t\in[t_0,\infty)$. By Lebesgue's monotone convergence theorem, we obtain from equation (3.16) the result

$$y(t)=\begin{cases} A+\sum_{i=1}^{m} p_i(t)y(t-\tau_i)+\sum_{i=1}^{m} p_{ik}y(t_k-\tau_i)+\int_T^t uf(u,y(g(u)))du+\int_t^\infty tf(u,y(g(u)))du \\ \qquad +\sum_{T\le t_k\le t} t_k f_k(t_k,y(g(t_k)))+\sum_{t\le t_k<\infty} tf_k(t_k,y(g(u))),\quad t,t_k\ge T \\ y(T),\quad t_0\le t,t_k<T. \end{cases}$$

Hence, $y(t)$ is a positive solution of equation (2.1). Since $0<A\le y(t)<b_1$, from Theorem 3.1, $y\in\Lambda^{(b,a,0)}$. This completes the proof of Theorem 3.3.

Using reasoning analogous to that given in the proof of Theorem 3.3 above, we can verify the following results.

Theorem 3.4: Assume that $\lim_{t\to\infty}\sum_{i=1}^{m} p_i(t)+\lim_{t_k\to\infty}\sum_{i=1}^{m} p_{ik}=p\in[0,1)$. Then equation (2.1) has a non-oscillatory solution $y\in\Lambda^{(\infty,\infty,d)}$, $(d\ne 0)$ if and only if

$$\int_{t_0}^{\infty}\left|\sum_{j=1}^{n} f_j(u,d_1g_{j1}(u),\dots,d_1g_{j\ell}(u))\right|du+\sum_{t_0\le t_k<\infty}\left|\sum_{j=1}^{n} f_{jk}(t_k,d_1g_{ji}(t_k),\dots,d_1g_{j\ell}(t_k))\right|<\infty, \tag{3.17}$$

for some $d_1\ne 0$.

Theorem 3.5: Assume that $\lim_{t\to\infty}\sum_{i=1}^{m} p_i(t)+\lim_{t_k\to\infty}\sum_{i=1}^{m} p_{ik}=p\in[0,1)$. Further assume that

$$\int_{t_0}^{\infty}\left|\sum_{j=1}^{n} f_j(u,d_1g_{ji}(u),\dots,d_1g_{j\ell}(u))\right|du+\sum_{t_0\le t_k<\infty}\left|\sum_{j=1}^{n} f_{jk}(t_k,d_1g_{j1}(t_k),\dots,d_1g_{j\ell}(t_k))\right|<\infty \tag{3.18}$$

for some $d_1\ne 0$ and

$$\int_{t_0}^{\infty} u\left|\sum_{j=1}^{n} f_j(u,b_1,\dots,b_1)\right|du+\sum_{t_0\le t_k\infty} t_k\left|\sum_{j=1}^{n} f_{jk}(t_k,b_1,\dots,b_1)\right|=\infty \tag{3.19}$$

for some $d_1\ne 0$, where $b_1d_1>0$. Then equation (2.1) has a non-oscillatory solution $y\in\Lambda^{(\infty,\infty,0)}$.

We examine the following to help illustrate the obtained results.

Example 3.1: Consider

$$\begin{cases} \left[y(t)-\frac{1}{2}y(t-1)\right]''+\frac{2(t-1)^3-t^3}{(t-1)^6}y^3(t)=0 \\ \Delta\left[y(t_k)-\frac{1}{2}y(t_k-1)\right]'+\frac{1(t_k-1)^3-t_k^3}{(t_k-1)^6}y^3(t_k)=0, \end{cases} \tag{3.20}$$

where

$$q(t)=\frac{2(t-1)^3-t^3}{(t-1)^6} \text{ and } q_k=\frac{2(t_k-1)^3-t_k^3}{(t_k-1)^6}.$$

It is obvious that inequality (3.11) holds. Therefore, equation (3.20) has a non-oscillatory solution $y\in\Lambda^{(b,a,0)}, b\neq 0, a\neq 0$. In fact, $y(t)=1-\frac{1}{t}$ is such a solution, where $a=\frac{1}{2}$ and $b=1$.

Example 3.2: Consider

$$\begin{cases} \left[y(t)-\frac{1}{2}y(t-1)\right]'' + q(t)y^{\frac{1}{3}}(t)=0 \\ \Delta\left[y(t_k)-\frac{1}{2}y(t_k-1)\right]' + q_k y^{\frac{1}{3}}(t_k)=0, \end{cases} \tag{3.21}$$

where

$$q(t)=\frac{1}{4}\left(t^{3/2}-\tfrac{1}{2}(t-1)^{-3/2}\right)t^{-1/6},\quad q_k=\frac{1}{4}\left(t_k^{-3/2}-\tfrac{1}{2}(t_k-1)^{-3/2}\right)t_k^{-1/6}.$$

For large t and t_k, $q(t)\sim Mt^{-5/3}$ and $q_k\sim Mt_k^{-5/3}$. It is obvious that inequalities (3.18) and (3.19) are satisfied. From Theorem 3.5, equation (3.21) has a solution $y\in\Lambda^{(\infty,\infty,0)}$. In fact, $y(t)=\sqrt{t}$ is such a solution of equation (3.21).

Remark 3.1: The above arguments can be applied to the equation

$$\begin{cases} \left[y(t)-\sum_{i=1}^{m}p_i(t)y(t-\tau_i)\right]'' = \sum_{j=1}^{n}f_j\left(t,y\left(g_{j1}(t)\right),\cdots,y\left(g_{j\ell}(t)\right)\right),\ t\geq t_0,\ t\notin S \\ \Delta\left[y(t_k)-\sum_{i=1}^{m}p_{ik}y(t_k-\tau_i)\right]' = \sum_{j=1}^{n}f_{jk}\left(t_k,y\left(g_{j1}(t_k)\right),\cdots,y\left(g_{j\ell}(t_k)\right)\right),\ t_k\geq t_0,\ \forall\, t_k\in S. \end{cases} \tag{3.22}$$

For instance, under the assumptions of Theorem 3.1, we have

$$\Lambda=\Lambda^{(0,0,0)}\cup\Lambda^{(b,a,0)}\cup\Lambda^{(\infty,\infty,\alpha)}\cup\Lambda^{(\infty,\infty,\infty)}.$$

Therefore, Theorems 3.3 and 3.4 hold for equation (3.22). Furthermore, equation (3.22) has a non-oscillatory solution $y(t)\in\Lambda^{(\infty,\infty,\infty)}$ if

$$\int_{t_0}^{\infty}\left|\sum_{j=1}^{n}f_j\left(t,d_1g_{j1}(t),\dots,d_1g_{j\ell}(t)\right)\right|dt+\sum_{t_0\leq t_k<\infty}\left|\sum_{j=1}^{n}f_{jk}\left(t_k,d_1g_{j1}(t_k),\dots,d_1g_{j\ell}(t_k)\right)\right|<\infty \tag{3.23}$$

for some $d_1\neq 0$.

Notes

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Own Waves in a Cylindrical Shell in Contact With a Viscous Liquid

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Abstract- This article focuses on the dynamic behavior of a cylindrical shell (elastic or visco-elastic) contacting with ideal (or viscous) liquid. The problem of wave propagation in a cylindrical shell filled or submerged liquid has great practical importance. The phenomenon of wave-like motion of the fluid in the elastic cylindrical shells attracted the attention of many researchers [1, 2, 3, 4, 5, 6]. In these works devoted to wave processes in the elastic cylindrical shell - ideal liquid, used and refined classical equations of shells, consider the influence of the radial and longitudinal inertial forces, considered the average density of the flow of liquid or gas. In works [7, 8, 9] analyzes the laws of wave processes in an elastic shell with viscous fluid in the model of the linear equations of hydrodynamics of a viscous compressible fluid. Unlike other systems are cylindrical shell (elastic or viscoelastic) and liquid (ideal or viscous) is regarded as inhomogeneous dissipative mechanical system [10, 11, 12].

Keywords: the cylindrical shell, viscous barotropic liquid, wave process, dissipative non-uniform, wavy motion.

GJSFR-F Classification: MSC 2010: 46T12

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Own Waves in a Cylindrical Shell in Contact with a Viscous Liquid

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Annotation- This article focuses on the dynamic behavior of a cylindrical shell (elastic or visco-elastic) contacting with ideal (or viscous) liquid. The problem of wave propagation in a cylindrical shell filled or submerged liquid has great practical importance. The phenomenon of wave-like motion of the fluid in the elastic cylindrical shells attracted the attention of many researchers [1, 2, 3, 4, 5, 6]. In these works devoted to wave processes in the elastic cylindrical shell - ideal liquid, used and refined classical equations of shells, consider the influence of the radial and longitudinal inertial forces, considered the average density of the flow of liquid or gas. In works [7, 8, 9] analyzes the laws of wave processes in an elastic shell with viscous fluid in the model of the linear equations of hydrodynamics of a viscous compressible fluid. Unlike other systems are cylindrical shell (elastic or viscoelastic) and liquid (ideal or viscous) is regarded as inhomogeneous dissipative mechanical system [10, 11, 12].

Keywords: the cylindrical shell, viscous barotropic liquid, wave process, dissipative non-uniform, wavy motion.

I. Statement of the Problem

An infinite length of deformable (viscoelastic) cylindrical shell of radius R with constant thickness h_0, density ρ_0, Poisson's ratio ν_0, filled with a viscous fluid with density at equilibrium. Fluctuations of a shell under a load, the density of which is denoted p_1 , p_2 , p_n respectively, can be described by following [1, 2, 4], equations:

$$L\left[1-\Gamma^{C}(\omega_R)-i\Gamma^{S}(\omega_R)\right]\vec{u}=\frac{\left(1-\nu_0^2\right)}{E_0 h_0}\vec{p}+\rho_0\frac{\left(1-\nu_0^2\right)}{E_0}\left(\frac{\partial^2\vec{u}}{\partial t^2}\right), \tag{1}$$

Here $\vec{u}=\vec{u}(u_r, u_\theta, u_z)$- displacement vector points of the middle surface of the shell and membranes for Kirchhoff - Love it has a dimension equal to three $(u_r=u;\quad u_\theta=v;\quad u_z=w)$, and to membranes such as the dimension of Timoshenko $\vec{u}$ is five. Here, in addition to the axial, circumferential and normal movements added more angles of rotation normal to the middle surface in the axial and circumferential directions [12]; $\{u\quad v\quad w\}^T$ - the displacement vector with axial, radial and circumferential components, respectively ("+" sign in front of p_n and the sign "-" before the last component of the inertial member says that is considered positive motion towards the

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center of curvature); $R_E(t-\tau)$– the core of relaxation; E_0– instantaneous modulus of elasticity.

The amplitudes of the oscillations are considered small, which allows you to record the basic relations in the framework of the linear theory. The system of linear equations of motion of a viscous barotropic liquid can be written as [12]:

$$\frac{\partial\vec{\vartheta}}{\partial t}-\nu*\Delta\vec{\vartheta}+\frac{1}{\rho_0^*}grad\,P-\frac{\nu*}{3}grad\;div\,\vec{\vartheta}=0$$

$$\frac{1}{\rho_0^*}\frac{\partial\rho*}{\partial t}+div\,\vec{\vartheta}=0;\qquad \frac{\partial P}{\partial\rho^*}=a_0^2\,,a_0=const.$$

$$\dot{u}_z=\vartheta_z\,,\dot{u}_r=\vartheta_r\,,\,\dot{u}_\theta=\vartheta_\theta\,,$$

$$q_z=-p_{rz}\,,q_r=-p_r\,,\,q_\theta=-p_{r\theta}. \tag{2}$$

$$p_{rz}=\mu^*\left(\frac{\partial\vartheta_z}{\partial r}+\frac{\partial\vartheta_r}{\partial z}\right);$$

$$p_{rr}=-p+\lambda^*\left(\frac{\partial\vartheta_r}{\partial r}+\frac{\partial\vartheta_z}{\partial z}+\frac{\vartheta_r}{r}\right)+2\mu^*\frac{\partial\vartheta_r}{\partial r}\,; \tag{6}$$

$$p_{r\theta}=\mu^*\left(\frac{1}{r}\frac{\partial\vartheta_z}{\partial\theta}+\frac{\partial\vartheta_\theta}{\partial r}-\frac{\vartheta_\theta}{r}\right).$$

Here, in the equations (2) $\vec{\vartheta}=\ \vec{\vartheta}(\vartheta_r,\vartheta_\theta,\vartheta_z)$ - the velocity vector of fluid particles; ρ^* and P- disturbance density and fluid pressure; ρ_0^* and a_0 – density and sound velocity in the fluid at rest; ν^*, μ^* - kinematic and dynamic viscosity; for the second viscosity coefficient λ^* accepted ratio $\lambda^*=-\frac{2}{3}\mu^*$; $p_{rz},p_{rr},p_{r\theta}$ - components of the stress tensor in the fluid. Equation (1), respectively, kinematic and dynamic boundary conditions, which, because of the thin-walled shell, we will meet on the middle surface *(r=R)*. Equations (1) and (2) is a closed system of relations hydro visco elastic cylindrical shell for containing a viscous compressible fluid. This for shell obeying Kirchhoff-Love hypotheses. Be investigated joint shell and liquid fluctuations, harmonic of the axial coordinate *z* and decay exponentially over time, or time-harmonic and damped with respect to *z*.

II. Method of Solution

We accept the integral terms in (1) small, then the function $\vec{u}(\vec{r},t)=\psi(\vec{r},t)e^{-i\omega_R t}$, where $\psi(\vec{r},t)$- slowly varying function of time, ω_R- real constant. Next, using the procedure of freezing [18], then the integral-differential equation (1) takes the form

$$L\left[1-\Gamma^C(\omega_R)-i\Gamma^S(\omega_R)\right]\vec{u}=\frac{(1-\nu_0^2)}{E_0h_0}\vec{p}+\rho_0\frac{(1-\nu_0^2)}{E_0}\left(\frac{\partial^2\vec{u}}{\partial t^2}\right), \tag{3}$$

where, for shell Kirchhoff - Love

Ref

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$$L=\begin{pmatrix} \frac{\partial^2}{\partial x^2}+\frac{1-\nu}{2R^2}\frac{\partial^2}{\partial\varphi^2} & \frac{1+\nu}{2R}\frac{\partial^2}{\partial x\partial\varphi} & \frac{\nu}{R}\frac{\partial}{\partial x} \\ \frac{1+\nu}{2R}\frac{\partial^2}{\partial x\partial\varphi} & \frac{1+\nu}{2}(1+4a)\frac{\partial^2}{\partial x^2}+(1+a)\frac{\partial^2}{\partial\varphi^2} & \frac{1}{R^2}\frac{\partial}{\partial\varphi}-a(2-\nu)\frac{\partial^3}{\partial x^2\partial\varphi}-\frac{a}{R^2}\frac{\partial^3}{\partial\varphi^3} \\ \frac{\nu}{R}\frac{\partial}{\partial x} & \frac{1}{R^2}\frac{\partial}{\partial\varphi}-a(2-\nu)\frac{\partial^3}{\partial x^2\partial\varphi}-\frac{a}{R^2}\frac{\partial^3}{\partial\varphi^3} & \frac{1}{R^2}+a\left(\frac{\partial^2}{\partial x^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}\right)^2 \end{pmatrix},$$

Notes

$\Gamma^C(\omega_R)=\int_0^\infty R(\tau)\cos\omega_R\tau\,d\tau$, $\Gamma^S(\omega_R)=\int_0^\infty R(\tau)\sin\omega_R\tau\,d\tau$ - respectively, cosine and sine Fourier transforms relaxation kernel material. As an example, the viscoelastic material take three parametric kernel relaxation $R(t)=Ae^{-\beta t}/t^{1-\alpha}$, ρ – material density shell; E – Young's modulus; ν – Poisson's ratio, $a=h^2/12R^2$. Let's move on to the dimensionless axial coordinate $\xi=x/R$ and multiply by R^2 system (3). The matrix of the resulting system will take the form

$$L=\begin{pmatrix} \frac{\partial^2}{\partial \xi^2}+\frac{1-\nu}{2}\frac{\partial^2}{\partial\varphi^2} & \frac{1+\nu}{2}\frac{\partial^2}{\partial \xi\partial\varphi} & \nu\frac{\partial}{\partial \xi} \\ \frac{1+\nu}{2}\frac{\partial^2}{\partial \xi\partial\varphi} & \frac{1-\nu}{2}(1+4a)\frac{\partial^2}{\partial \xi^2}+(1+a)\frac{\partial^2}{\partial\varphi^2} & \frac{\partial}{\partial\varphi}-a(2-\nu)\frac{\partial^3}{\partial \xi^2\partial\varphi}-a\frac{\partial^3}{\partial\varphi^3} \\ \nu\frac{\partial}{\partial \xi} & \frac{\partial}{\partial\varphi}-a(2-\nu)\frac{\partial^3}{\partial \xi^2\partial\varphi}-\frac{a}{R^2}\frac{\partial^3}{\partial\varphi^3} & \frac{1}{R^2}+a\left(\frac{\partial^2}{\partial \xi^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\varphi^2}\right)^2 \end{pmatrix}. \quad (4)$$

Expanding equation (2) and (3) in coordinate form, it is easy to see that the relations (2) - (3) break up into independent boundary value problems:

- *Torsional vibrations:*

$$\begin{gathered} \frac{\partial p_{r\theta}}{\partial r}+\frac{2p_{r\theta}}{r}+\frac{\partial p_{\theta z}}{\partial z}=\rho_0^*\ddot{\vartheta}_\theta, \\ p_{r\theta}=\mu^*\left(\frac{\partial\vartheta_\theta}{\partial r}-\frac{\vartheta_\theta}{r}\right),\quad p_{\theta z}=\mu^*\frac{\partial\vartheta_\theta}{\partial r}, \\ r=R_1:\ \bar{G}h_0\frac{\partial^2 u_\theta}{\partial z^2}-(\rho_0 h\ddot{u}_\theta\pm\sigma_{\varphi r})=0, \\ \bar{G}=\frac{\bar{E}}{2(1+\nu_0)}, \\ r=0:\quad p_{r\theta}=0,\ \ \bar{E}=E_0\left(1-\Gamma^C(\omega_R)-i\Gamma^S(\omega_R)\right) \end{gathered} \quad (5)$$

- *Longitudinal transverse vibrations:*

$$\frac{\partial p_{rr}}{\partial r}+\frac{p_{rr}-p_{\theta\theta}}{r}+\frac{\partial p_{rz}}{\partial z}=\rho_0^*\frac{\partial\vartheta_r}{\partial t}$$

$$\frac{\partial p_{rz}}{\partial r}+\frac{p_{rz}}{r}+\frac{\partial p_{zz}}{\partial z}=\rho_0^*\frac{\partial\vartheta_z}{\partial t}$$

$$p_{rr} = -p + \lambda^* k_\eta div\,\vec{\vartheta} + 2\mu^* \frac{\partial \vartheta_r}{\partial r},$$

$$p_{\theta\theta} = -p + \lambda^* div\,\vec{\vartheta} + 2\mu^* \frac{\vartheta_r}{r} \qquad (6)$$

$$p_{zz} = -p + \lambda^* div\,\vec{\vartheta} + 2\mu^* \frac{\partial \vartheta_z}{\partial z}$$

$$p_{rz} = \mu^*\left(\frac{\partial \vartheta_z}{\partial r} + \frac{\partial \vartheta_r}{\partial z}\right), \quad p_{r\theta} = \mu^*\left(\frac{1}{r}\frac{\partial \vartheta_z}{\partial \theta} + \frac{\partial \vartheta_\theta}{\partial r} + \frac{\vartheta_r}{r}\right)$$

$$\frac{\partial \rho^*}{\partial t} + \rho_0 div\vec{\vartheta} = 0, \quad div\,\vec{\vartheta} = \frac{\partial \vartheta_r}{\partial r} + \frac{\vartheta_r}{r} + \frac{\partial \vartheta_z}{\partial z}, \qquad \frac{\partial p}{\partial \rho} = a_0^2$$

$r = R_1$:

$$\overline{D}\frac{\partial^4 u_r}{\partial z^4} + \frac{\overline{C}}{R_1}\left(\frac{u_r}{R_1} + \nu_0 \frac{\partial u_z}{\partial z}\right) + p_{rr} + \rho_0 h_0 \frac{\partial^2 u_r}{\partial t^2} = 0,$$

$$\overline{C}\left(\frac{\partial^2 u_r}{\partial z^2} + \frac{\nu_0}{R}\frac{\partial u_r}{\partial z}\right) - (p_{rz} \pm \rho_0 h_0 \frac{\partial^2 u_z}{\partial t^2}) = 0,$$

Let the wave process is periodic in z and fades over time, then is given a real wave number k, and the complex frequency is the desired characteristic value. Solution of (2) - (6) for the major unknowns satisfying constraints imposed above the dependence on time and coordinates z, should be sought in the form [14]

$$(p_{rr}, p_{rz}, p_{r\theta}, \vec{u}, \vec{\vartheta})^T = \sum_m (\sigma_{rm}(\xi,\varphi,t), \tau_{zm}(\xi,\varphi,t), \tau_{\varphi m}(\xi,\varphi,t), \vec{u}_m(\xi,\varphi,t), \vec{\vartheta}_m(\xi,\varphi,t))^T, \qquad (7)$$

where $\vec{u}_m(\xi,\varphi,t) = \vec{u}_m\{U_m, V_m, W_m\}^T, \qquad \vec{\vartheta}(\xi,\varphi,t) = \vec{\vartheta}_m\{\vartheta_{rm}, \vartheta_{\theta m}, \vartheta_{zm}\}^T,$.

Expressions (7) in the form

$$(\sigma_{rm}(\xi,\varphi,t), \tau_{zm}(\xi,\varphi,t), \tau_{\varphi m}(\xi,\varphi,t))^T = (\sigma_r \cos(m\varphi), \tau_z \cos(m\varphi), \tau_\varphi \sin(m\varphi))^T\ e^{i\kappa z - i\omega t},$$

$$(\vec{u}_m(\xi,\varphi,t), \vec{\vartheta}_m(\xi,\varphi,t))^T =$$

$$= (U_m \cos(m\varphi), V_m \sin(m\varphi), W_m \cos(m\varphi), \vartheta_r \cos(m\varphi), \vartheta_\theta \cos(m\varphi), \vartheta_z \cos(m\varphi))^T\ e^{i\kappa z - i\omega t},$$

where $\sigma_r, \tau_z, \tau_\varphi$, $U_m, V_m, W_m, \vartheta_r, \vartheta_\theta, \vartheta_z$ - Amplitude integrated vector - function; κ– wavy number; C - phase velocity; ω – complex frequency; m – circumferential wave number (the number of district-wave), takes values $m = 1, 2, 3$...... When $m = 0$, happening Ax symmetrical vibrations. This approach allows you to seek a solution for every fixed value of the wave number of the district m independently.

In this way C, k, ω it is well-known real and complex spectral parameters of the type of problem.

To elucidate their physical meaning consider two cases:

1) $\kappa = \kappa_R$; C = $C_R + iC_i$, Then the solution of (5) has the form of a sine wave ***x***, whose amplitude decays over time;

Ref
14. Grinchenko V.T, V.V. Myaleshka. Harmonic Waves in elastic bodies. - Kiev: 1981, p. 284.

2) $\kappa = \kappa_R + i\kappa_I$; C = C_R, Then at each point x fluctuations established, but $\boldsymbol{x}$ attenuate. In the case of axially symmetric on the axis r = 0 conditions must be satisfied conditions $p_{r\theta} = p_{rz} = 0$, $\vartheta_r = 0$. If the outer surface $z = R$ assumed stationary, then $u_r = u_z = u_\varphi = 0$. The superposition of the solutions (8) forms an exponentially decaying over time the standing wave that describes the natural oscillations of a liquid and a cylindrical shell of finite length with boundary conditions. With infinite length sheath similarly specified type of movement (8) will be called *private or free* fluctuations. In the case of steady-state over time and fading coordinate the process, in contrast, is a well-known real rate of ω, as desired be a complex wave number *k*. In contrast to their own, these fluctuations will be called the established. Actual values of the ω in the first case, and k, second frequency have the physical meaning of the process in time and the coordinate, respectively. Imaginary part - the rate of decay of wave processes in time and *z*, respectively [13]. The value of *1/Imk* sometimes defined as the interval damped wave propagation. In the extreme case, the elastic range spread endless. The degree of attenuation of wave process in the time period is characterized by the logarithmic decrement

$$\delta_c = 2\pi |\mathrm{Im}\,\omega| / \mathrm{Re}\,\omega \tag{8}$$

Decrement is similar to the spatial

$$\delta_y = 2\pi |\mathrm{Im}\,k| / \mathrm{Re}\,k\,.$$

You can also introduce the concept of phase velocity of its own and steady motions

$$c_c = \frac{\mathrm{Re}\,\omega}{R}, c_y = \frac{\omega}{\mathrm{Re}\,k}$$

The values C_c and C_y have physical sense speeds of zero state at its own and steady oscillations, respectively, and, in contrast to the elastic (real) case, do not coincide with each other at the same frequencies. Two types of oscillations (and set their own), you can put two different formulations of the problem. And in the non-stationary case, namely the Cauchy problem for an infinite shell and boundary value problem for the semi-infinite interval changes Z. In either case, the solution is using the integral transformation of the decisions of the respective steady-state problems. For example, in the case of the Cauchy problem, the main vector of unknowns $\overline{Y}^c$. It can be in a superposition of waves

$$\overline{Y}^c = (r,z,t) = \sum_n \int_{-\infty}^{\infty} Y_n^c(r,k) \exp[t(kz - \varpi_n(k)t)]dk, \tag{9}$$

where vectors $\overline{Y}_n^c$ are their own form of the problem of natural oscillations, normalized so that the spatial Fourier spectrum of the initial disturbance $\overline{f}(r,z) = \overline{Y}^c(r,z,0)$ forms a linear combination

$$\overline{f}(r,z) = \int_{-\infty}^{\infty} F(r,k)e^{ikz}\,dk, \qquad \overline{f}(r,k) = \sum_n \overline{Y}_n^c(r,k)\,. \tag{10}$$

Ref

13. Kayumov S.S, Safarov I.I. Propagation and diffraction of waves in dissipative - inhomogeneous cylindrical deformable mechanical systems. Tashkent: 2002, 214p.

Similarly, the main vector of unknowns $\overline{Y}^{y}$ boundary value problem is calculated according to the expression

$$\overline{Y^{y}}(r,z,t)=\sum_{n}\int_{-\varpi}^{\varpi}\overline{Y}^{y}_{k}(r,\omega)\exp[ik(\omega)z-\omega t]d\omega \tag{11}$$

where $\overline{Y}^{y}_{k}$ - forms steady-state oscillation, the linear combination of which should form a Fourier spectrum given boundary perturbation

$$\overline{q}(r,t)=\overline{Y}^{y}(r,0,t),\overline{q}(r,t)=\int_{-\infty}^{\infty}q(r,\omega)e^{-i\omega t}d\omega,\ \ \overline{q}(r,\omega)=\sum_{n}\int_{-\infty}^{\infty}Y_{n}^{y}(r,\omega)$$

Obviously, the solutions (8) and (9) have a meaning only when there are (10) and (11). So, there are four possible variants of steady motions, which are discussed below, and established their own systems fluctuations shell - fluid inside and outside the sheath liquid [15]. Substituting the solution (7) in the system of differential equations (2) - (6) we obtain a system of ordinary differential equations with complex coefficients, which is solved by Godunov's orthogonal sweep method with a combination of method of Muller [18] in the complex arithmetic.

III. Torsional Vibrations

After performing in (5) the change of variables (7) permitting relations describing stationary torsional vibrations of the shell liquid, formulated in the form of the spectral boundary value problem for a system of two ordinary differential equations

$$\begin{aligned}
&\frac{d\tau_{\varphi}}{dr}=-(\rho_{0}^{*}\ \omega^{2}-i\mu^{*2}\xi^{2}\omega)\vartheta_{\theta}-\frac{2\tau_{\varphi}}{r}\\
&\frac{dv}{dr}=\frac{\vartheta_{\theta}}{r}+\frac{i}{\omega\mu^{*}}\tau_{\varphi}\\
&r=R_{1}:h_{0}(\overline{G}\xi^{2}-\xi\rho_{0}\omega^{2})\vartheta_{\theta}\pm\tau_{\varphi}=0\\
&r=0:\tau_{\varphi}=0
\end{aligned} \tag{12}$$

First investigate fluctuations of fluid in the walls. Equations (12) can be converted to a single equation for the displacement v

$$\frac{d^{2}\vartheta_{\theta}}{dr^{2}}+\frac{d\vartheta_{\theta}}{rdr}+(-\xi^{2}+i\frac{\omega}{v^{*}}-\frac{1}{r^{2}})\vartheta_{\theta}=0;\quad v^{*}=\frac{\mu^{*}}{\rho_{0}^{*}} \tag{13}$$

The solution of equation (13) is limited at $r=0$ has the form

$$v=A_{1}J_{1}(r\sqrt{-k^{2}+i\frac{\omega}{v^{*}}})=0\cdot \tag{14}$$

where J_1- Bessel function of the first order, and A is an arbitrary constant. Given the immobility of the shell, we obtain the dispersion equation

$$J_{1}(R_{1}\sqrt{-k^{2}+i\frac{\omega}{v^{*}}})=0 \tag{15}$$

from whence

Ref

15. Frolov K.V., Antonov A.N. Vibrations of a shell of liquid-Moscow: Science, 1983. 365p.

$$\omega_n = -i(\nu^* k^2 + \Gamma_m^2) \tag{16}$$

in the case of natural oscillations and

$$k_n = \sqrt{-\Gamma^2 n + i\frac{\omega}{\nu^*}} \tag{17}$$

Notes

in the case of steady-state oscillations. Here, through the Γ_n marked the roots of Bessel functions assigned to *R*. As it can be seen from (15), (16) own motion aperiodicity always on time, with the anchor points are fixed (the phase velocity C_o=0), while the steady motion are oscillatory in nature, as the nodal point move at the speed of $C_{V.}$ a monotonically increasing from zero to indefinitely with an increase in viscosity or ν^*. These characteristic features of the motion of a viscous medium will appear in the following more complex example.

Let us now consider the relation (12) in the case of the internal arrangement of the liquid. This problem can be solved in the same way using special features and have a dispersion equation

$$-k^2 + \frac{\omega^2}{a^2} + \frac{\omega\nu^*}{a^3\tilde{p}\tilde{h}R^2} + (z\frac{J_0(z)}{J_1(z)} - 2) = 0 \tag{18}$$

which was first obtained in *A*. Guz [7]. Here we have introduced new designations

$$\tilde{p} = \frac{\rho^*}{\rho_0}; \tilde{h} = \frac{h}{R_1}; z = R_1\sqrt{-k^2 + i\frac{\omega}{\nu^*}};\ a = \sqrt{\frac{G}{\rho_0}}$$

shear wave velocity shell: J_o- Bessel function of zero order.

The direct solution of the equation (18) comes up against certain difficulties caused by the need to calculate the Bessel functions of complex argument. Therefore we examine (18) by means of asymptotic representations of these functions at small and large arguments *z*. The smallness of *z* occurs in the low-frequency vibrations. According to the known expansion J_0 and J_1 power series

$$J_0 = 1 - \frac{z^2}{4} = ...; J_1(z) = \frac{z}{2}(1 - \frac{z^2}{8} + ...); \tag{19}$$

Hold the expansions (19) only the first term, we obtain

$$-k^2 + \frac{\omega}{a^2} = 0$$

dispersion equation of torsional vibrations or dry shell filled with an ideal liquid, keeping in (19) on the first two terms, we have the equation

$$-k^2 + \frac{\omega^2}{a^2} + i\frac{\omega\nu^*}{4a^2\tilde{p}\tilde{h}}(k^2 - i\frac{\omega}{\nu^*}) = 0 \tag{20}$$

the root of which, for example, in the case of steady-state oscillations is given by

$$k = \frac{\omega}{a}\left[(1 + \frac{1}{4\tilde{p}\tilde{h}})/(1 - \frac{\omega\nu^*}{4a^2\tilde{p}\tilde{h}})\right]^{1/2}. \tag{21}$$

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The physical interpretation of (18) is provided below. Consider now the situation when z is large enough, which corresponds to a high-frequency vibrations and low viscosity. In this case the asymptotic formulas for the Bessel functions have the form

$$J_0(z) \cong (\frac{2}{\pi z})^{1/2}\cos(z-\frac{\pi}{4}),\ J_1(z) \cong (\frac{2}{\pi z})^{1/2}\sin(z-\frac{\pi}{4})$$

On the basis of (20) and (21) it is easy to show that for sufficiently large positive imaginary part z: $J_0(z)/J_1(z) \cong -i.$ Substituting (1) and further assuming smallness ν^* in comparison with the $\frac{\omega}{k^2}$, to obtain an approximate dispersion equation, which is also contained in the [7]

$$-k^2+\frac{\omega^2}{a^3}(1+\sqrt{\frac{\nu^*}{\omega}}\frac{\tilde{p}}{\tilde{h}R}\frac{l+i}{1.41})=0 \tag{22}$$

Where, in the pursuit of the viscosity ν^* to zero (and also tends ω to infinity), we have a trivial result $\frac{\omega}{k}\to 0$, which was obtained at low ω from equation (20). Equation (22) when an unacceptably high viscosities. In this case, the phase velocity C unlimited increases with ω. This example shows inconsistencies of various asymptotic estimates in the mid-frequency vibrations. Thus, the analysis of wave processes asymptotic methods in the first approximation is not possible to establish the limits of applicability of formulas and calculations to estimate the error. In this paper for solving spectral problems using a direct numerical integration of permitting relations of the type (12) by the method of orthogonal shooting in complex arithmetic. This approach avoids the above difficulties associated with the calculation of Bessel functions of complex argument. Another advantage is due to the specificity of the orthogonal sweep method, which is due to the procedure orthonormality can solve highly rigid system with a boundary layer. As a result of a numerical study has found that the problem of natural oscillations (12) admits no more than one complex value ω, corresponding vibrations of the shell together with the adjacent liquid layers to it. The rest found the Eigen values appeared purely imaginary. They correspond to the a periodic motion of a fluid with almost stationary shell. Proper form corresponding complex values also are complex, that is, the phase of joint oscillations of the shell and liquid is not the same along the radius. In the case of steady-state oscillations all the calculated Eigen values k and their own forms be complex.

Notes

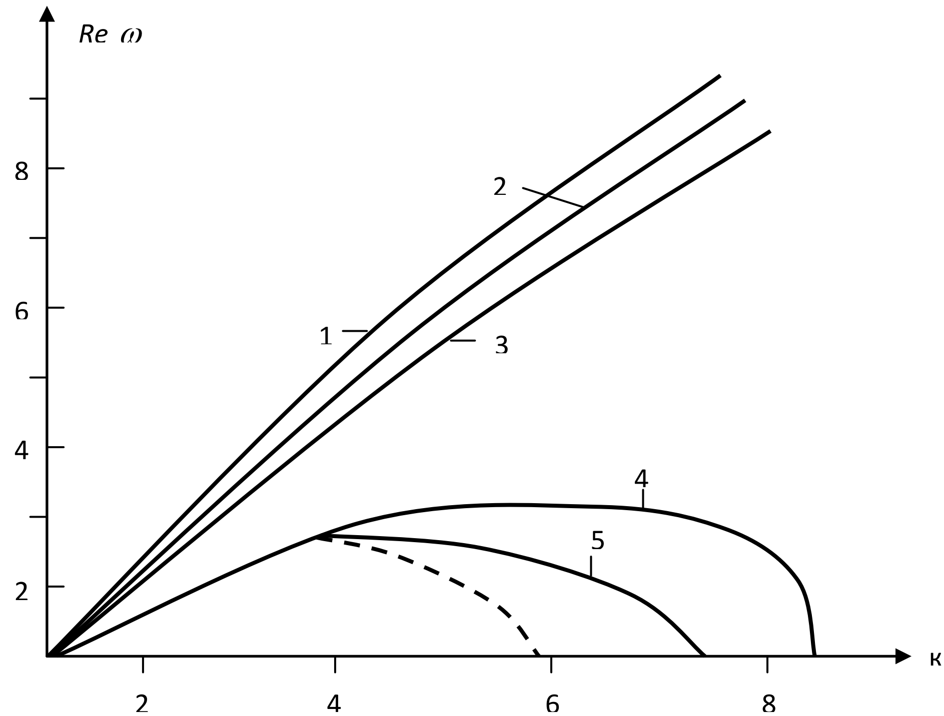

Figure 1: Dependence of the real part of the complex frequencies ($\mathrm{Re}\,\omega$) to wave numbers (k) for different values of η. 1-0.0009; 2-0.0018; 3-0.18, 4-0.19, 5-according to the formula (20); By the formula (22)

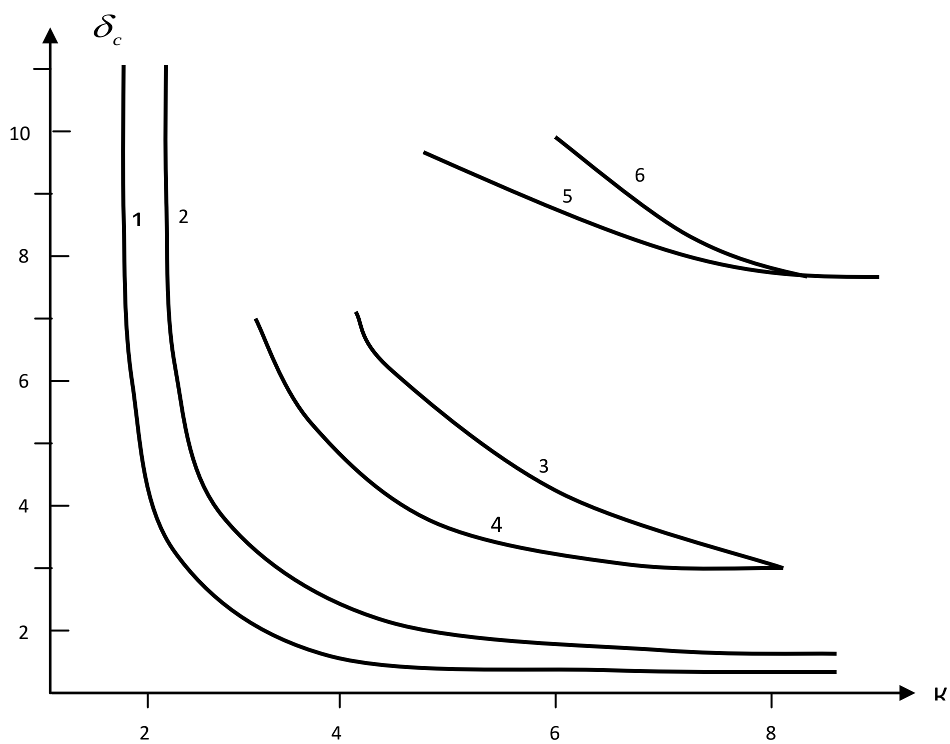

Figure 2: Dependence of the logarithmic decrement (δ_c) on the wave numbers (k) for different values of η. 1-0.0009; 2-0.0018; 3-0.18, 4-0.19, 5-0.20; 6-0.22

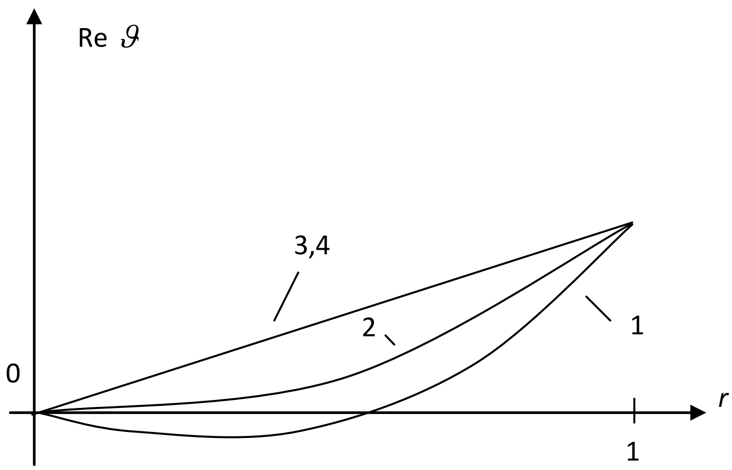

Figure 3: Dependence of ϑ on the wave number r, for different values of the viscosity of a liquid 1-0.0009; 2-0.0018; 3-0.18;4-0.19

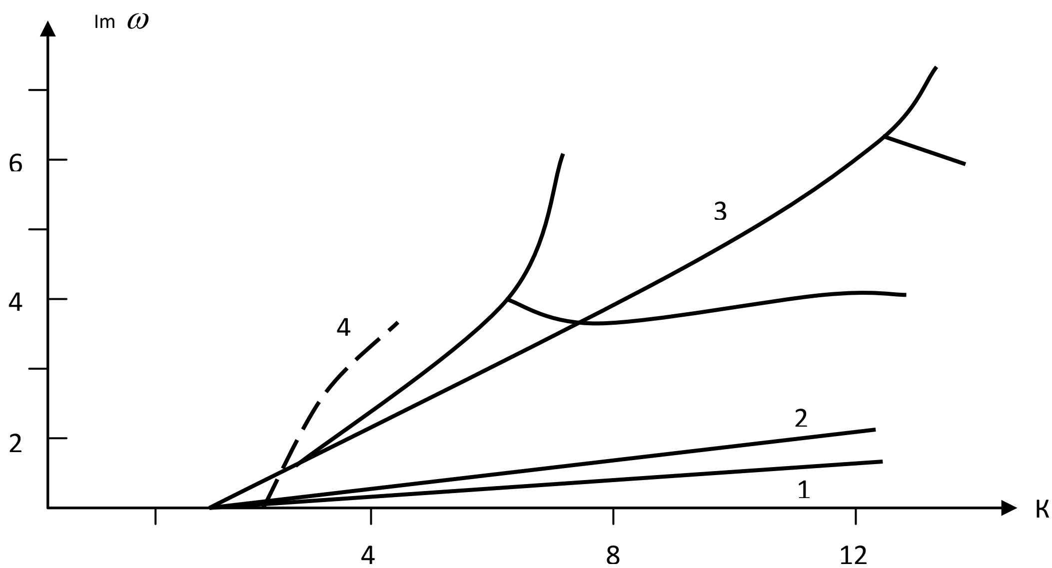

Figure 4: Dependence of the imaginary part of the complex frequencies (Im) to wave numbers (k) for different values of η: 1-0.0009; 2-0.0018; 3-0.18; 4-0.19;--- by the formula (22)

Notes

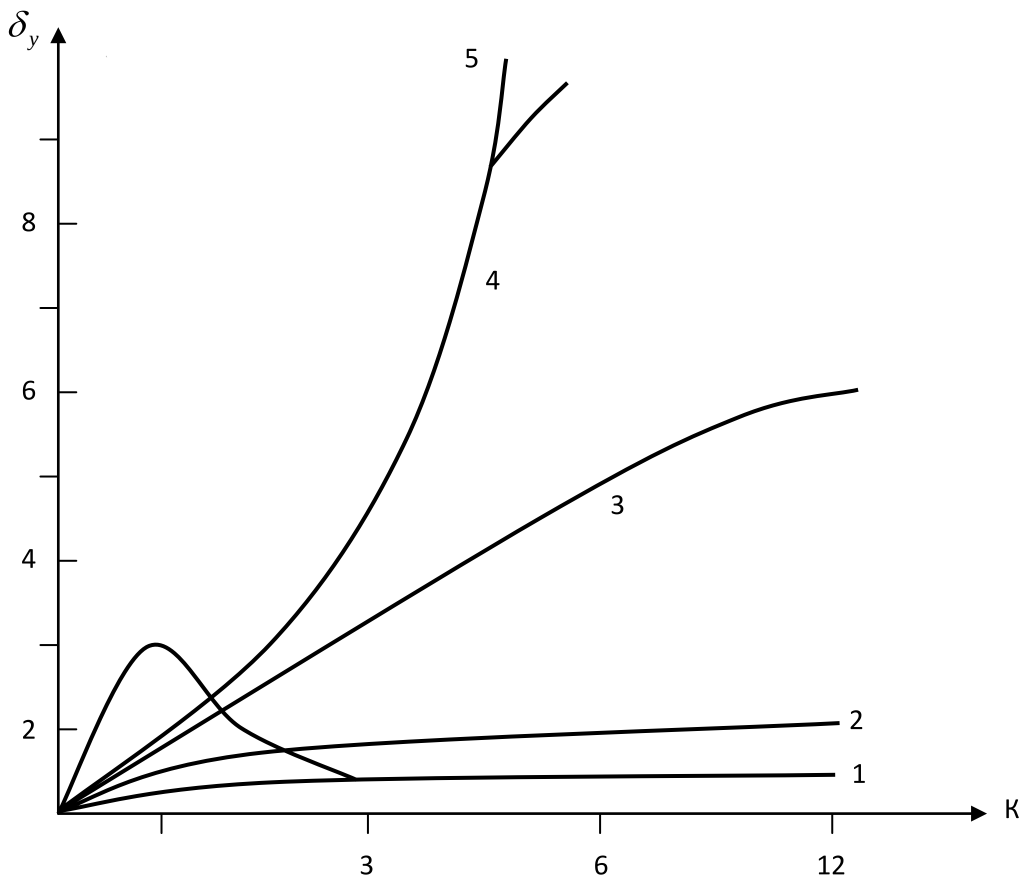

Figure 5: Dependence of the spatial decrement on the wave number k for different values of η: 1-0.0009; 2-0.0018; 3-0.18; 4-0.19

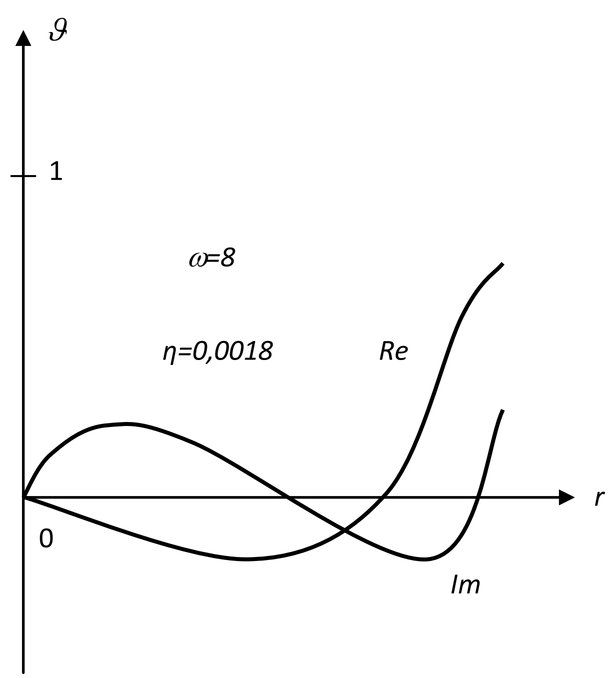

Figure 6: Dependence of ϑ on the wave number r. When $\omega = 8, \eta = 0{,}0018$

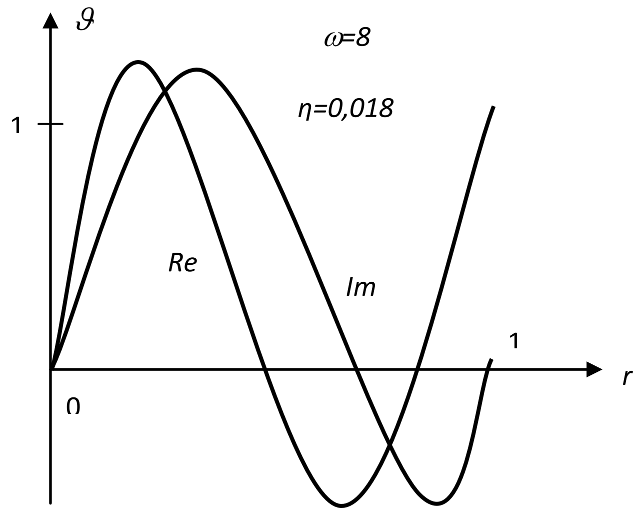

Figure 7: Dependence of ϑ on the wave number. When $\omega = 8, \eta = 0{,}018$

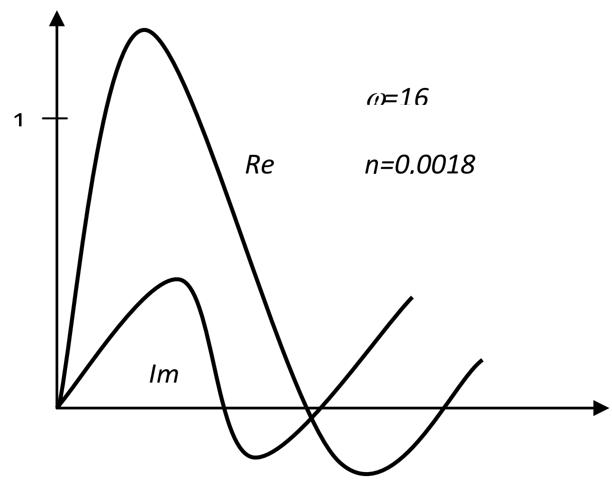

Figure 8: Dependence of ϑ on the wave number r. When $\omega = 16, \eta = 0{,}0018$

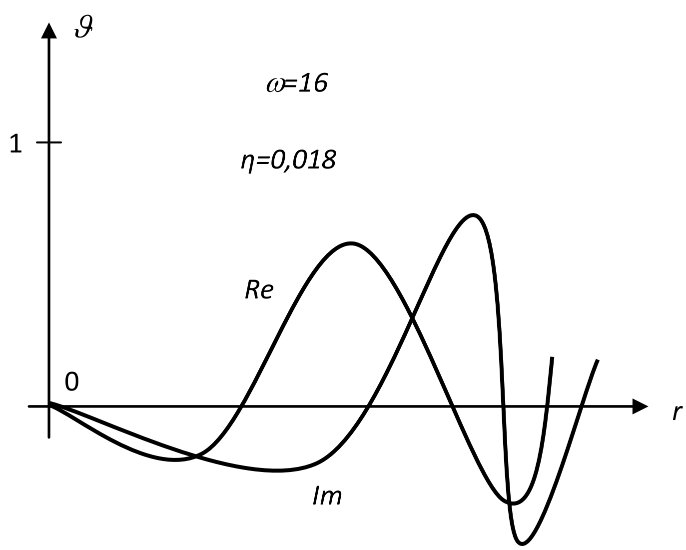

Figure 9: Dependence of ϑ on the wave number r. When $\omega = 16, \eta = 0{,}018$

Notes

IV. Numerical Results

Notes

Consider the case of natural oscillations, when the shell is filled with liquid. In Figure 1,2,3,4,6,7,8 c and 1,2,4,5 show, respectively, depending on the dispersion curves Re ω, Im ω, δ the wave number k- the first mode, in which the damping coefficients of the smallest, and the Eigen values are complex Bat. In accordance with the numbering of graphs asked four different values of the coefficient η 1) 0.0009: 2) 0.0018 3)0.15 4)0.018 ($a=$ 0.6199; $\tilde{\rho}=0.0529; \tilde{h}=0.0101; R=1; \nu_0=0.25$.) for the remaining parameters according to (1) In Fig. 3,6,7,8,9 to show their own forms *Rev* for values k equal to 1 and 8, respectively. It is easy to notice the difference in the behavior characteristic of the dispersion curves 1.2 and 3.4. In the last two cases, there is a wave number since a variable with only takes purely imaginary values corresponding to a periodic motion of the system. For curves 1.2 with less viscosity real part of the Eigen values $\mathrm{Re}\,\omega$ nonzero at all wave numbers and the damping rate has a finite limit at infinity. The greater the viscosity, the earlier start a periodic traffic (curves 3,4) and the higher limit of the damping rate (curves 1,2). It follows that where is a minimum critical viscosity η_k, above which a zone of high wave numbers of the first mode, there are a periodic wave number. As a result of numerical experiment, it was found that the critical values of the coefficient of viscosity η_k, is in the range$[0.0120 \quad 0.0125]$. Analyzing the dependence of energy dissipation on the wave number, two opposite tendencies should be noted. As the wave number increases, at a fixed amplitude, tangential stresses linearly increase according to (6): c another, as shown in Fig. 3, localization of the fluid motion amplitudes near the shell simultaneously results, which leads to a decrease in the mass of fluid involved in the motion, as well as tangential stresses. The difference in the behavior of curves 1,2 and 3,4 is due to which of the two tendencies prevails. At small wave numbers, a linear dependence of the eigenfunction v on the radius is observed, that is, the entire mass of the liquid is involved in the motion. As k increases, the central part of the liquid begins to "not keep up" with the vibrations of the shell, which leads to the localization of the amplitudes. The rate of localization depends on the viscosity of the liquid. If the localization occurs slowly, then starting from some k (owing to the growth of stresses), the self-motions become aperiodic (curves 3,4). If, on the other hand, the average amplitude of the fluid oscillation decreases rapidly enough, the motions will always remain oscillatory (curves 1,2). In this case, large voltage wave numbers prevail over voltages, and increase with increasing localization. In view of the latter circumstance, the damping coefficient always increases with increasing k. The linear dependence of the shape on the radius at small k also indicates the fulfillment of the flat-section hypothesis on which the elementary theory of viscoelastic rods is based. Using the Ritz method one can find the parameters of the Feucht core model and determine the limits of applicability of this model in the framework of the hydrodynamic theory, but for a narrower class of straight rods of circular cross section. Variational equation of the principle of possible displacements, equivalent to the relations

$$\int_v h\left(\frac{\partial u_\varphi}{\partial z}\delta\frac{\partial u_\varphi}{\partial z}+\rho_1\frac{\partial u_\varphi}{\partial z}\delta u_\varphi\right)R_1 d\varphi dz-\int_v\left(\sigma_{r\varphi}\delta\varepsilon_{r\varphi}+\sigma_{z\varphi}\delta\varepsilon_{z\varphi}+\rho_0\frac{\partial^2 u_\varphi}{\partial z^2}\delta u_\varphi\right)r d\varphi dr dz=0 \qquad (23)$$

has the form. Choosing a linear function as the basis

$$u_\varphi(r,z,t)=\varphi(z,t)r, \tag{24}$$

and after substituting (24) into (23) and the standard procedure, we obtain where the parameters β and a_0 are expressed in terms of the polar moments of inertia of the shell I_1 and liquid I_0 as follows

$$\beta=\frac{\eta I_0}{GI_1};a_0=a/(1+\frac{I_0}{\tilde{\rho}I_1})^{\frac{1}{2}}$$

Equation (23) describes the torsional vibrations of a viscoelastic Feucht rod according to the relations

$$\left(1+\beta\frac{\partial}{\partial t}\right)\frac{\partial^2\varphi}{\partial z^2}=\frac{1}{a_0^2}\frac{\partial^2\varphi}{\partial t^2}, \tag{25}$$

The solution of (25) is represented in the form

$$\varphi(z,t)=\varphi_0\exp(i(kz-\omega t)) \ .$$

Where the following relations satisfy

$$a_0^2\ \kappa^2(1-i\omega\beta)-\omega^2=0. \tag{26}$$

Taking into account the relation $I_1 / I_0 = 4h$, it is easy to see that equation (26) coincides with equation (22), which was obtained for the asymptotic solution of problem (16) for small oscillation frequencies. In Fig.1,4 The dotted lines show the dispersion curves of natural oscillations found from Eq. (22). As follows from the figures, a satisfactory coincidence of dotted and continuous lines is observed in the region of small wave numbers whose upper bound exceeds unity in this case and increases with increasing viscosity of the liquid. In the short-wavelength range, there is a discrepancy due to the localization of the oscillation amplitudes near the shell. Small wave numbers correspond to the natural vibrations of long finite tubes. We now turn to an analysis of the steady-state oscillations of a shell filled with a liquid. Figure 1-9 shows the dispersion curves and waveforms for two values of the viscosity coefficient (below and above the critical value) 1) 0.0018, 2) 0.018 and the same values of the remaining parameters as in (22). In the first case of relatively low viscosity, the results of the calculation are in good agreement with the asymptotic solutions of the Goose equation (18) at high frequencies.

V. Longitudinal - Transverse Vibrations

This section analyzes the stationary longitudinal-transverse vibrations of a shell filled with fluid, which according to (6) can be described by a system of four ordinary differential equations

$$\frac{d\vartheta_r}{dr}=-\frac{\vartheta_r}{r}-ik\vartheta_z-p$$

$$\frac{d\vartheta_z}{dr}=ik\vartheta_r+\frac{1}{\eta\omega}\tau_{\hat{y}}$$

Notes

$$\frac{d\sigma_r}{dr} = -\rho_0\omega^2\vartheta_z + 2i\eta\omega(\frac{d\vartheta_r}{dr} - \frac{\vartheta_r}{r}) = ik\tau_z \tag{27}$$

$$\frac{d\tau_z}{dr} = -\rho_0\omega^2\vartheta_z + 2\eta\omega k(\frac{d\vartheta_r}{dr} - ik\vartheta_z) - ik\sigma_r - \frac{\tau_z}{r}$$

With the boundary conditions

$$r = 0: \vartheta_r = 0, \tau_z = 0;$$

$$r = R: D\nabla^4 u + \frac{C}{R}(\frac{u}{R} + iv_0 kw) + \sigma_r - \rho_1 h\omega^2 u = 0; \tag{28}$$

$$C(iv_0 k\frac{u}{R} - \nabla^2 u) - \tau_z + \rho_1 h\omega^2 w = 0;\ C = \frac{Eh_0}{1 - {v_0}^2}.$$

The value of p in the first equation of system (27) is defined through the main unknowns according to the expression

$$p = \frac{-\sigma_r + 2i\eta\omega\left(iku + \frac{\vartheta_r}{r}\right)}{\rho_0 C_0^2 - i\omega(k + 2\eta)} \tag{29}$$

The spectral problems (27), (28), as in the case of longitudinal - transverse vibrations were solved by orthogonal shooting. To find the roots of the characteristic equation method were used Mueller.

VI. Numerical Results

The results of numerical study of natural oscillations. Figure 10 shows the dispersion curves $\mathrm{Re}\,\omega$ the wave number k - for the case of an incompressible ($C_0 = \infty$ - dot-dash line) and compressible (C_0=0, 1 - solid line) of the liquid. Shell parameters and coefficients of viscosity taken following: $h_0 = 0, 05$; p=1,8; $v_0 = 0,25$; h=6, 011*10 (-4); κ=-2 η/3. Here and henceforth given dimensionless quantities for which the units of length and mass density are $R, R\left(\frac{\rho_0}{E}\right)^{\frac{1}{2}}, \frac{1}{\rho_0}$. For an incompressible fluid, there are two modes, corresponding mainly longitudinal (curve 1) and preferably a cross (curve 2) fluctuations in the shell, with complex Eigen values. All other traffic have their own imaginary Eigen values, that is a periodic in time. The dashed lines in Figure 10 are designated the dispersion curves corresponding to the vibrations of a shell with an ideal incompressible fluid. The solution of the latter problem is given below. It should be noted that, unlike the dry shell joint oscillations transverse vibrations of said sheath fluid density p_1, It takes place on a smaller compared to the frequency of longitudinal vibrations in the entire range of the wave number. When administered viscosity oscillation frequency of the first mode decreases, apparently due to the involvement of additional masses in movement of fluid in the boundary layer and in the second mode appears critical wave number restricting oscillatory motions bottom region. In [15], who investigated the steady oscillations, noted the desire for zero phase velocity of the lowest mode with decreasing frequency. Proper motion of the shell and the viscous

Ref

15. Frolov K.V., Antonov A.N. Vibrations of a shell of liquid-Moscow: Science, 1983. 365p.

compressible fluid has an infinite number of modes. The paper S. Vasin et al. [16] using asymptotic methods of solving, the latter effect could not be found. Fig.11 shows the dispersion curves for the first four events with a minimum of vibration frequencies (curves 3,4,5,6) in ascending order of magnitude $\mathrm{Re}\,\omega$. Comparing curves 1.2 and 3.4 together, we can see that the second worse than the first few vibration modes of the shell - compressible fluid to the selected parameters are satisfactorily described by a model of an incompressible fluid in the region of wave numbers $k < 1$. This gives grounds for the study of the said system in the first approximation neglect the compressibility of the fluid. System elastic shell - is a viscous liquid dissipations-inhomogeneous viscoelastic body at a radial coordinate. Moreover, in contrast to the earlier torsional vibrations here for an incompressible fluid, there are two, and compressible - unlimited number of vibration modes. It is interesting to find out how this system can be shown a synergistic effect. Figure 11 shows the dispersion curves (2) for the following parameters of the shell and liquid:

$$h = 0{,}05; \rho_3 = 80; v = 0{,}25t; \eta = 7{,}071 \cdot 10^{-4}; C_0 = \infty$$

Dash-dotted lines correspond to fluctuations in the dry shell. The dashed lines show the frequency dependence for the case of an ideal fluid v = 0. In contrast with the previously discussed embodiment, the density p = 8, in this case partial frequency (v = 0) of the longitudinal and transverse vibrations of the shell with a perfect fluid intersect. It is natural to expect that the v near the intersection of partial frequencies will be a strong connectedness of both modes, leading to increased energy, resulting in a synergistic effect. Indeed, the presence of events demonstrates the effect of the conversion of Vina- longitudinal mode in transverse and longitudinal cross-section in a change of the wave number in the vicinity of the intersection of partial frequencies. Violation of the monotony of growth and synergies. Compared to the previous description of this effect there are two features. Firstly, the effect is far from the place of approximation curves of two modes, secondly, damping factor curves do not intersect. Yu. Novichkov in [17] investigated the coherence of joint oscillations of ideal compressible gas and the shell with the help of diagrams wines. As he examined the frequency of partial oscillations of gas in rigid walls and an empty shell.

Returning to Figure 11, we note a similar manifestation of the effect of wines in places of convergence curves 4.5 and 5.6. In these areas in Fig. 3 there is a synergistic effect for the curves. It is interesting to trace the influence of fluid viscosity on connectivity modes. 3.4 Curves in Figure 4 correspond to the value of the viscosity coefficient η=0,11 at constant other parameters. In this case, fashion predominantly transverse vibrations are defined on a finite interval of the wave of change, and the effect of guilt is not observed, indicating a loose coupling modes. Another large increase in viscosity (η=0,13, curve 5) leads to the fact that fashion is everywhere transverse vibrations becomes a periodic and *y* longitudinal oscillations appear critical wave numbers, limiting the scope of the vibration motions of the top. The physical nature of the observed effect is revealed when analyzing the vibrations of a shell filled with a perfect fluid. The equations of harmonic oscillations of an ideal liquid is easy to deduce from (27), formally putting viscosity coefficients equal to zero.

$$\frac{d\vartheta_r}{dr} = -\frac{\vartheta_r}{r} - ik\vartheta_z - \frac{\sigma}{\rho_0 C_0^2}, \; \frac{d\sigma}{dr} = -\rho_0\omega^2\vartheta_r, \; \sigma = i\rho_0\omega^2\vartheta_z \qquad (30)$$

General solution of (26) satisfying the finiteness condition unknown at zero, has the form

Ref

16. S.V. Vasin, V.V. Mikolyuk. Free oscillations tolerable cylindrical shells separated by a viscous fluid Hydro Aeromechanics. And the theory of elasticity. 1983. №3. p.108-116.

$$\vartheta_z = AJ_0(qr); \sigma = i\rho_0 \frac{\omega^2}{k} AI_0(qr)$$
$$\vartheta_z = i\frac{q}{R} AI_1(qr); q^2 = \frac{\omega^2}{C_0^2} - k_0^2 \quad (31)$$

Notes

where A arbitrary constant:J_0, J_1,- Bessel functions of zero and first order, respectively. The boundary conditions at the r=R similarly written conditions (28)

$$D\nabla^4 u + \frac{C}{R}(\frac{u}{R} + iv_0 kw) + \sigma_r - \rho_1 h\omega^2 u = 0;$$
$$C(iv_0 k\frac{u}{R} - \nabla^2 u) + \rho_1 h\omega^2 w = 0; \quad (32)$$

where w - axial movement of the shell, which is not now coincides with the axial movement of the liquid. After substitution of the solutions (22) of (23) there is a system of homogeneous linear algebraic equations in the unknown A and U_1. The roots of the determinant of this system are the desired Eigen values, and its decision to define the relation between A and U_1 .

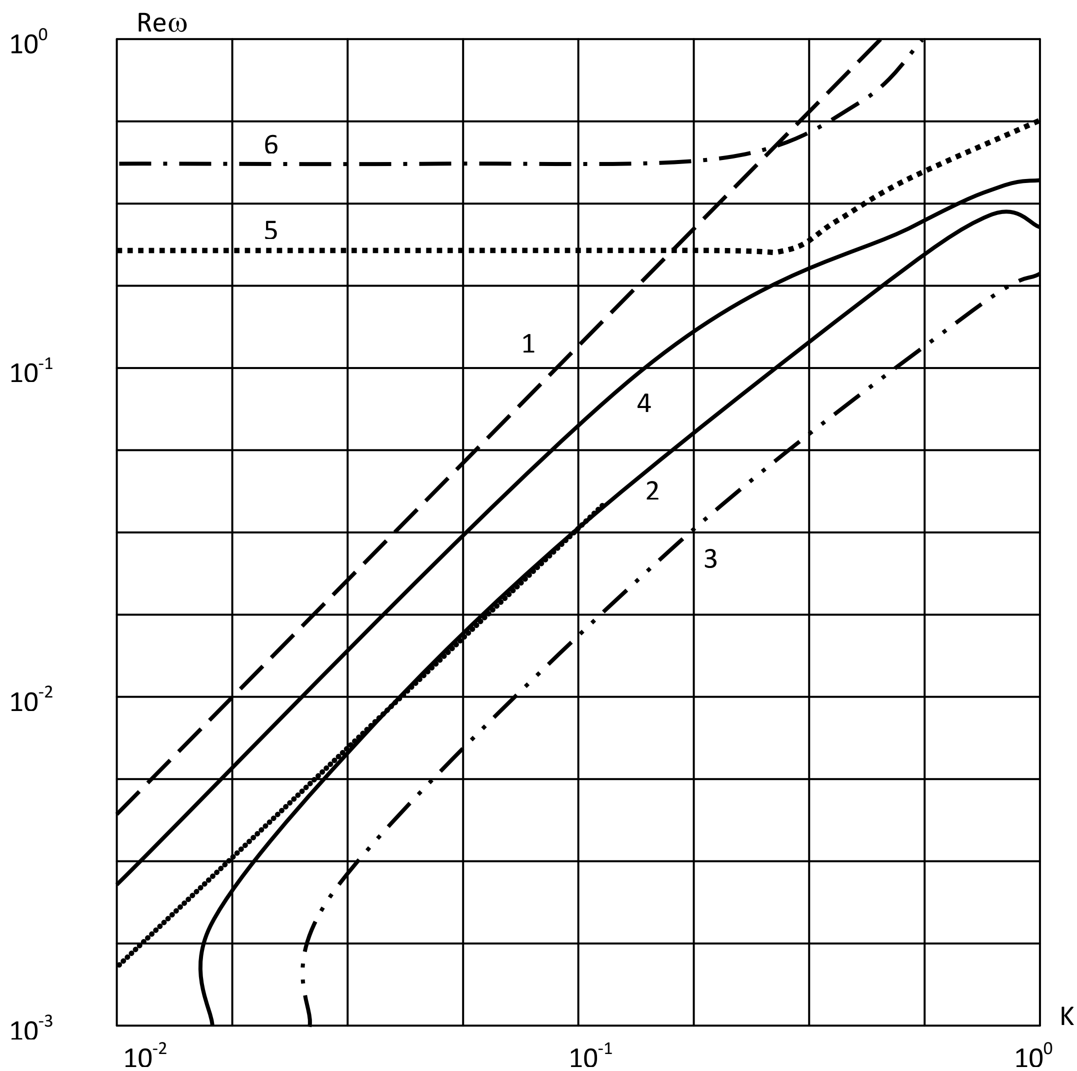

Figure 10: Addiction Reω with the wave number *k* in the case of an incompressible fluid

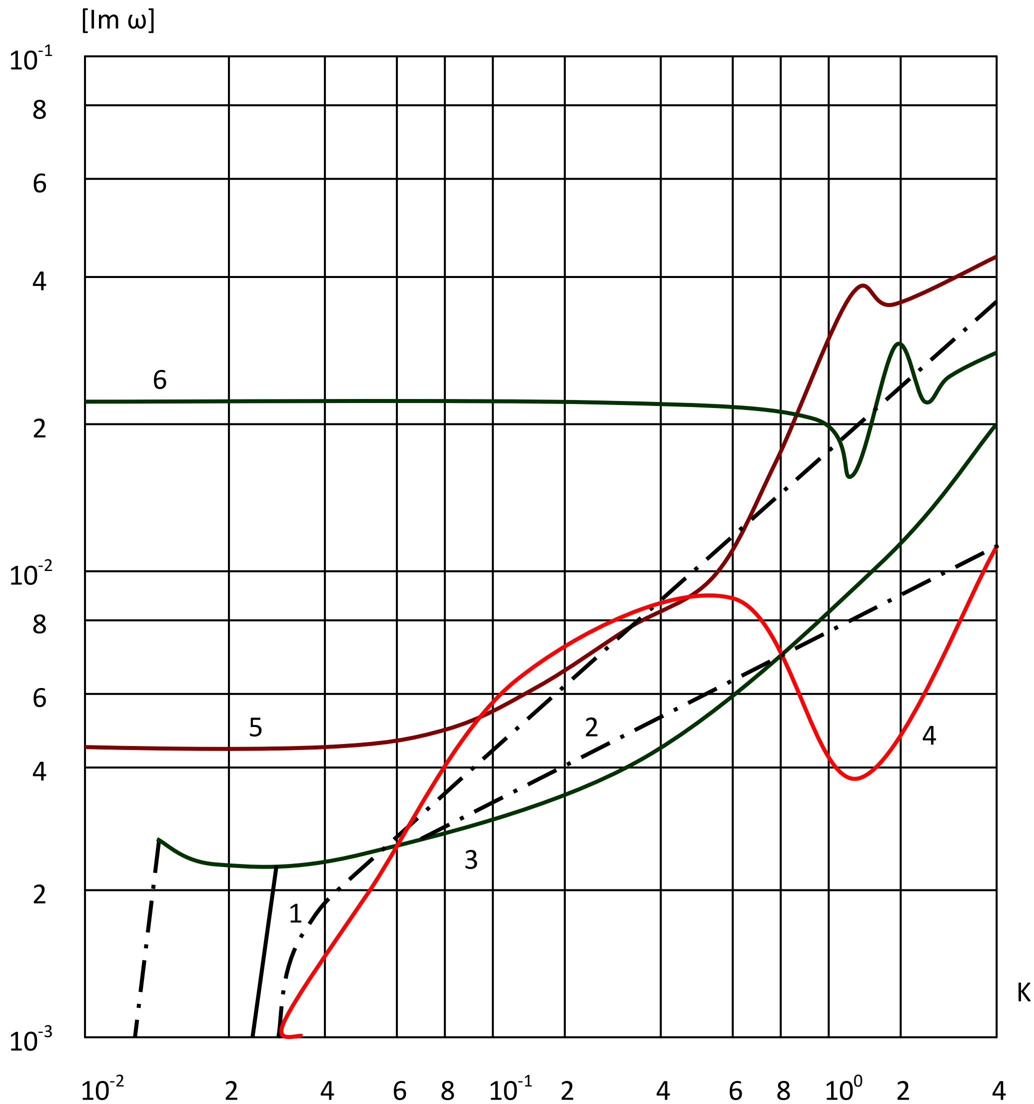

Figure 11: Addiction *Im a* the wave number in the case of a compressible fluid

For an incompressible fluid, there are two real own Bessel functions I_0 and I_1

$$\omega_1 = R\left(\frac{E}{\rho_1}\right)^{\frac{1}{2}}; \omega^2 = \left[\frac{E}{R_1\rho_1}\left(1+h^2R_1\kappa^4\right)/\left(1+\frac{\rho_n I(kR_1)}{h\rho_1 I_1(kR_1)\kappa}\right)\right]^{\frac{1}{2}} \quad (33)$$

Unlike dry shell here second frequency locking is absent and the phase speed at low k equal to the

$$C_R = \left(\frac{Eh}{2\rho_0 R_1}\right)^{\frac{1}{2}} \quad (34)$$

which coincides with the speed of the wave Rezalya (see. the review at the beginning of this chapter). In the case of a compressible fluid $\nu = 0$ and limiting the phase velocity of the transverse mode oscillation in the shell k —> 0 is the velocity of waves Korteweg Zhukovsky.

Notes

$$C_k = \frac{C_0 C_R}{(C_0^2 + C_R^2)^{\frac{1}{2}}} \tag{35}$$

Notes

Numerical study showed that the critical value C_k does not depend on the viscosity of the liquid, but with increasing η weakening the dependence of oscillations of Poisson's ratio, so that the ratio $(\max im\varpi)/(\min im\varpi) \to 1$ and own form U It becomes flat. As follows from the above results, generally within the engineering problem statement, we can not adequately describe the longitudinal vibrations of the cylindrical shell filled with a viscous fluid via rod theory.

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Notes

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First Order Reactant of Dusty Fluid MHD Turbulence Prior to the Ultimate Phase of Decay for Four-Point Correlation in A Rotating System

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Abstract- In this paper, first order chemical reactant will be analyzed to study the effect of the fluctuation of MHD turbulence before the ultimate phase of decay in present of dust particles in a rotating system at four point correlation. Following Deissler's approach two, three and four point correlation equations have been obtained, and the set of correlation equations is made determinate by neglecting the quintuple correlations in comparison to the third and fourth order correlation terms. By taking Fourier-transforms, the correlation equations are converted to spectral form. Finally, integrating the energy spectrum over all wave numbers. The energy decay of MHD turbulence for first-order reactant in the presence of dust particles in a rotating system is obtained, and the result is shown graphically in the text.

Keywords: first order reactant, 4-point correlations, deissler's method, dusty fluid, mhd turbulence, fourier-transforms, rotating system.

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Abstract- In this paper, first order chemical reactant will be analyzed to study the effect of the fluctuation of MHD turbulence before the ultimate phase of decay in present of dust particles in a rotating system at four point correlation. Following Deissler's approach two, three and four point correlation equations have been obtained, and the set of correlation equations is made determinate by neglecting the quintuple correlations in comparison to the third and fourth order correlation terms. By taking Fourier-transforms, the correlation equations are converted to spectral form. Finally, integrating the energy spectrum over all wave numbers. The energy decay of MHD turbulence for first-order reactant in the presence of dust particles in a rotating system is obtained, and the result is shown graphically in the text.

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I. Introduction

Chemical reactions occur in the gas phase, in solution in a variety of solvents, at gas-solid and other interfaces, in the liquid state, and in the solid state. It is sometimes convenient to work with amounts of substances instead of with concentrations. The essential characteristic of turbulent flows is that chaotic fluctuations are random. Chemical reaction as used in chemistry, chemical engineering, physics, fluid mechanics, heat, and mass transport. The behavior of dust particles in a turbulent flow depends on the concentrations of the particles and the size of the particles concerning the scale of the disordered fluid.

Deissler (Deissler 1958) developed a theory on the decay of homogeneous turbulence before the final period. Deissler (Deissler 1960) further described a assumption of decaying homogeneous turbulence. By considering Deissler's hypothesis, Sarker and Kishore (Sarker and Kishore 1991) studied the decay of MHD insecurity before the final period. Ahmed (Ahmed 2013) discussed the turbulent energy of dusty fluid in a rotating system. Sarker and Ahmed (Sarker and Ahmed 2011) studied the fiber motion in unclean fluid turbulent flow with two-point correlation. Kulandaivel *et* al. (Kulandaivel *et al.*2009) considered the chemical reaction on moving vertical plate with constant mass flux in the presence of thermal radiation.

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Islam and Sarker (Islam and Sarker 2001) calculated the first order reactant in MHD turbulence before the final period of decay for the case of multi-point and multi-time. Loeffler and Deissler (Loeffler and Deissler 1961) studied the decompose of temperature fluctuations in homogeneous turbulence before the final period. Chandrasekhar (Chandrasekhar 1951a) studied the invariant theory of isotropic turbulence in magneto-hydrodynamics. Corrsin (Corrsin 1951b) considered the spectrum of isotropic temperature fluctuations in isotropic turbulence. Sarker *et al.* (Sarker *et al.* 2010) studied the effect of Coriolis force on dusty viscous fluid between two parallel plates in the MHD flow. Azad *et al.* (Azad *et al.* 2011) also studied the statistical theory of some distribution functions in MHD turbulent flow for velocity and concentration undergoing a first order reaction in a rotating system. Azad *et al.* (Azad *et al.* 2015) also discussed 3-point distribution functions in the statistical theory in MHD turbulent flow for velocity magnetic temperature and concentration undergoing a first- order reaction. Bkar Pk *et al.*(Bkar Pk *et al.* 2012) discussed the decay of energy of MHD turbulence for four-point correlation. Bkar Pk *et al.* (Bkar Pk *et al.* 2013) also calculated the decay of MHD turbulence before the ultimate phase in the presence of dust particle for four-point correlation. Bkar Pk (Bkar Pk *et al.* 2016) further studied the effect of first-order chemical reaction for Coriolis force and dust particles for small Reynolds number in the atmospheric over territory.

In this paper, we have studied the first order reactant of MHD turbulence before the ultimate phase of decay in the presence of dust particles for the case of four-point correlation in a rotating system. The energy decay of MHD turbulence depends on the variation of the magnitude of dust particle parameters in the magnetic field and causes a vital role between pure and non-pure system. The effects due to dust particle in magnetic field fluctuation of MHD turbulence have been graphically discussed. It is observed that energy decays increases with the increases of the values in M (dust particle) and minimums at the point where the dust particle is absence. The energy decay of MHD fluid turbulence for four-point correlation in the presence of dust particle is more rapidly than the energy decay of MHD turbulence for clean fluid.

II. Four Point Correlation and Equation

For first order reactant in four-point correlation in the present of dust particle, we take the momentum equation of MHD turbulence at the point p and the induction equation of magnetic field fluctuation at $p', p'' and\ p'''$ as

$$\frac{\partial u_l}{\partial t} + u_k \frac{\partial u_l}{\partial x_k} - h_k \frac{\partial h_l}{\partial x_k} = -\frac{\partial w}{\partial x_l} + \nu \frac{\partial^2 u_l}{\partial x_k \partial x_k} - 2\varepsilon_{\text{pkl}} \Omega_p u_i + f(u_l - v_l) - Ru_l \tag{1}$$

$$\frac{\partial h_i'}{\partial t} + u_k' \frac{\partial h_i'}{\partial x_k'} - h_k' \frac{\partial u_i'}{\partial x_k'} = \frac{\nu}{p_M} \frac{\partial^2 h_i'}{\partial x_k' \partial x_k'} \tag{2}$$

$$\frac{\partial h_j''}{\partial t} + u'' \frac{\partial h_j''}{\partial x_k''} - h'' \frac{\partial u_j''}{\partial x''_{k\ k}} = \frac{\nu}{p_M} \frac{\partial^2 h_j''}{\partial x_k'' \partial x_k''} \tag{3}$$

$$\frac{\partial h_m'''}{\partial t} + u_k''' \frac{\partial h_m'''}{\partial x_k'''} - h_k''' \frac{\partial u_m'''}{\partial x_k'''} = \frac{\nu}{p_M} \frac{\partial^2 h_m'''}{\partial x_k'''} \tag{4}$$

Notes

Multiplying equations (1) by $h_i'h_j''h_m'''$ (2) by $u_l h_j'' h_m'''$ (3) by and $u_l h_i' h_m'''$ (4) by and $u_l h_i' h_j''$ and adding the four equations, we then taking the space or time averages or ensemble average, both of the values give the same representation. Space or time averages denoted by $\overline{(......)}$ and $\langle..........\rangle$ respectively. We get,

Notes

$$\frac{\partial}{\partial t}(\overline{u_l h_i' h_j'' h_m'''}) + \frac{\partial}{\partial x_k}(\overline{u_l u_k h_i' h_j'' h_m'''}) - \frac{\partial}{\partial x_k}(\overline{h_k h_l h_i' h_j'' h_m'''}) + \frac{\partial}{\partial x_k'}(\overline{u_l u_k h_i' h_j'' h_m'''}) -$$

$$\frac{\partial}{\partial x_k'}(\overline{u_l u_i' h_k' h_j'' h_m'''}) + \frac{\partial}{\partial x_k''}(\overline{u_l u_k'' h_i' h_j'' h_m'''}) - \frac{\partial}{\partial x_k''}(\overline{u_l u_j'' h_i' h_k'' h_m'''}) + \frac{\partial}{\partial x_k'''}(\overline{u_l u_k'' h_i' h_j'' h_m'''}) -$$

$$\frac{\partial}{\partial x_k'''}(\overline{u_l u_j'' h_i' h_j'' h_m'''}) = -\frac{\partial}{\partial x_l}(\overline{w h_i' h_j'' h_m'''}) + \frac{\partial^2}{\partial x_k \partial x_k}(\overline{u_l h_i' h_j'' h_m'''}) + \frac{\nu}{p_M}[\frac{\partial^2}{\partial x_k' \partial x_k'}(\overline{u_l h_i' h_j'' h_m'''}) +$$

$$\frac{\partial^2}{\partial x_k'' \partial x_k''}(\overline{u_l h_i' h_j'' h_m'''}) + \frac{\partial^2}{\partial x_k''' \partial x_k'''}(\overline{u_l u_k h_i' h_j'' h_m'''})]$$

$$-2\varepsilon_{\text{pkl}}\Omega_p\left(\overline{u_l h_i' h_j'' h_m'''}\right) + f\left[\left(\overline{u_l h_i' h_j'' h_m'''}\right) - \left(\overline{v_l h_i' h_j'' h_m'''}\right)\right] - \left(\overline{R u_l h_i' h_j'' h_m'''}\right) \tag{5}$$

By using $\frac{\partial}{\partial x_k''} = \frac{\partial}{\partial r_k'}, \frac{\partial}{\partial x_k'} = \frac{\partial}{\partial r_k'}, \frac{\partial}{\partial x_k} = -(\frac{\partial}{\partial r_k'} + \frac{\partial}{\partial r_k'} + \frac{\partial}{\partial r_k''})$ into equation (5) and then following nine dimensional Fourier transforms as [Equations (6)-(13) in [Bkar Pk *et al.* 2013]] and interchange of point's p' and p etc. in the subscripts with the facts

$$\overline{u_l u_k''' h_i' h_j'' h_m'''} = \overline{u_l u_k' h_i' h_j'' h_m'''};\ \overline{u_l u_k'' h_i' h_j'' h_m'''} = \overline{u_l u_k' h_i'' h_j'' h_m'''};$$

$$\overline{u_l u_m''' h_i' h_j'' h_m'''} = \overline{u_l u_i' h_i' h_k' h_j'' h_m'''};\ \overline{u_l u_j'' h_i' h_k'' h_m'''} = \overline{u_l u_i' h_i' h_k' h_j'' h_m'''};$$

then taking contraction for indices i and j of the resulting equation we obtain,

$$\frac{\partial}{\partial t}(\overline{\phi_l \gamma_i' \gamma_i'' \gamma_m'''}) + \frac{\nu}{p_M}[(1+p_M)(k^2 + k'^2 + k''^2) + 2p_M(kk' + k'k'' + kk'') - 2\varepsilon_{\text{pkl}}\Omega_p p_M/\nu + Rp_M/\nu](\overline{\phi_l \gamma_i' \gamma_i'' \gamma_m'''})$$

$$-f\left(\overline{\phi_l \gamma_i' \gamma_i'' \gamma_m'''}\right) + f\left(\overline{\eta_l \gamma_i' \gamma_i'' \gamma_m'''}\right) = i(k_k + k_k' + k_k'')(\overline{\phi_l \phi_k \gamma_i' \gamma_i'' \gamma_m'''})$$

$$-i(k_k + k_k' + k_k'')(\overline{\phi_l \gamma_k \gamma_i' \gamma_i'' \gamma_m'''}) - i(k_k + k_k' + k_k'')(\overline{\phi_l \gamma_k' \gamma_i' \gamma_i'' \gamma_m'''})$$

$$+i(k_k + k_k' + k_k'')(\overline{\phi_l \phi_i' \gamma_k' \gamma_i'' \gamma_m'''}) + i(k_k + k_k' + k_k'')(\overline{\delta \gamma_i' \gamma_i'' \gamma_m'''}) \tag{6}$$

If we take the derivative concerning x_l of equation (1) at p, we have,

$$-\frac{\partial^2 w}{\partial x_l \partial x_l} = \frac{\partial^2}{\partial x_l \partial x_l}(u_l u_k - h_l h_k) \tag{7}$$

Multiplying (7) by $h_i' h_j'' h_m'''$, using time averages and writing the equation regarding the independent variables, $\vec{r}, \vec{r}', \vec{r}''$ we have,

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$$-(\overline{\delta\gamma_i'\gamma_j''\gamma_m'''}) = \frac{(K_lK_k + K_lK_k' + K_lK_k'' + K_l'K_k + K_l'K_k' + K_l'K_k'' + K_l''K_k + K_l''K_k + K_l''K_k' + K_l''K_k'')}{K_lK_l + K_l'K_l' + K_l''K_l'' + 2K_lK_l' + 2K_l'K_l'' + 2K_lK_l''}$$

$$\overline{(\phi_l\phi_k\gamma_i'\gamma_j''\gamma_m'''} - \overline{\gamma_l\gamma_k\gamma_i'\gamma_j''\gamma_m'''}) \tag{8}$$

Equation (8) can be used to eliminate from $\left(\overline{\delta\gamma_i'\gamma_j''\gamma_m'''}\right)$ equation (6). Equation (6) and (8) are the spectral equations corresponding to the four-point correlation equation.

where $\omega = \frac{P}{\rho} + \frac{1}{2}\left|\overline{h}\right|^2$ is the total MHD pressure,

$R =$ First order chemical reactant

$p(x,t) =$ Hydrodynamic pressure,

$\rho =$ Fluid density,

$P_M = \frac{\nu}{\lambda}$ Magnetic Prandtl number,

$\Omega_s =$ angular velocity components,

ε_{skm} is the alternating tensor,

$\nu =$ Kinematics viscosity,

$\lambda =$ Magnetic diffusivity,

$h_i(x,t) =$ Magnetic field fluctuation,

$u_k(x,t) =$ Turbulent velocity,

$m_i = \frac{4}{3}\pi R^3{}_i\rho_i$, is the mass of a single spherical dust particle of the radius and R_i and ρ_i a constant density of the material in the dust particles.

$f = \frac{KN}{\rho}$, is the dimensions of frequency, K is the Stock's drug resistance, N is a constant number density of dust particle,

$v_l =$ Dust velocity component, t is the time,

$x_k =$ Space co-ordinate and repeated subscripts are summed, from 1 to 3.

III. Three Point Correlation and Equation

The spectral equations corresponding to the three-point correlation equations by contraction of the indices i and j are

$$\frac{\partial}{\partial t}\overline{(\phi_l\beta_i'\beta_i'')} + \frac{\nu}{p_M}[(1+P_M)(K^2+K'^2) + 2p_MKK']\,\overline{(\phi_l\beta_i'\beta_i'')} = i\,(K_k+K_k')\,\overline{(\phi_l\phi_k\beta_i'\beta_i'')} -$$

$$i\,(K_k+K_k')\,\overline{(\beta_l\beta_k\beta_i'\beta_i'')} - i(K_k+K_k')\,\overline{(\phi_l\phi_k'\beta_i'\beta_i'')} + i(K_k+K_k')\,\overline{(\phi_l\phi_i'\beta_i'\beta_i'')} + i(k_l+k_l')\overline{\gamma\beta_i'\beta_i''} \tag{9}$$

and

$$-(\gamma\,\overline{\beta_i'\beta_j''}) = \frac{(K_lK_k + K_l'K_k + K_lk_k' + K_l'K_k')}{(K_l{}^2 + K_l'^2 + 2K_lK_l')}\,\overline{(\phi_l\phi_k\beta_i'\beta_i''} - \overline{\beta_l\beta_k\beta_i'\beta_j''}) \tag{10}$$

where the spectral tensors are defined by [equations (19)-(24) in [Bkar Pk *et al.* 2013]]

A relation between $\phi_l\phi'_k\beta'_i\beta''_j$ and $\phi_l\gamma'_i\gamma''_j\gamma'''_m$ can be obtained, by letting $\vec{r''}=0$ in equation.

$$\left\langle u_l h'_i(\bar{r}) h''_j(\bar{r}') h'''_m(\bar{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle \phi_l\gamma'_i(\bar{k})\gamma''_j(\bar{k}')\gamma'''_m(\bar{k}'')\right\rangle \exp\left[i(\bar{k}.\bar{r}+\bar{k}'.\bar{r}'+\bar{k}''.\bar{r}''\right]d\bar{k}d\bar{k}'d\bar{k}''$$

and comparing the result with the equation

$$\left\langle u_l u'_k(\bar{r}) h'_i(\bar{r}) h''_j(\bar{r}')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle \phi_l\phi'_k(\bar{k})\beta'_i(\bar{k})\beta''_j(\bar{k}')\right\rangle \exp\left[i(\bar{k}.\bar{r}+\bar{k}'.\bar{r}'\right]d\bar{k}d\bar{k}',$$

we get

$$\left\langle \phi_l\phi'_k(\bar{k})\beta'_i(\bar{k})\beta''_j(\bar{k}')\right\rangle = \int_{-\infty}^{\infty}\left\langle \phi_l\gamma'_i(\bar{k})\gamma''_j(\bar{k}')\gamma'''_m(\bar{k}'')\right\rangle \exp\left[i(\bar{k}.\bar{r}+\bar{k}'.\bar{r}'+\bar{k}''.\bar{r}''\right]d\bar{k}d\bar{k}'d\bar{k}'' \quad (11)$$

IV. Two Point Correlation and Equation

The spectral equation corresponding to the two-point correlation equation taking contraction of the indices is

$$\frac{\partial}{\partial t}\left\langle \varphi_i\varphi'_i(\bar{k})\right\rangle + \frac{2\nu}{p_M}k^2\left\langle \varphi_i\varphi'_i(\bar{k})\right\rangle = 2ik_k[\left\langle \alpha_i\varphi_k\varphi'_i(\bar{k})\right\rangle - \left\langle \alpha_k\varphi_i\varphi'_i(-\bar{k})\right\rangle] \quad (12)$$

where $\varphi_i\varphi'_i$ and $\alpha_i\phi_k\phi'_i$ are defined by

$$\left\langle h_i h'_i(\bar{r})\right\rangle = \int_{-\infty}^{\infty}\left\langle \varphi_i\varphi'_i(\bar{k})\right\rangle \exp(i\bar{k}.\bar{r})dk \quad (13)$$

and

$$\left\langle h_i h_k h'_i(\bar{r})\right\rangle = \int_{-\infty}^{\infty}\left\langle \alpha_i\varphi_k\varphi'_i(\bar{k})\right\rangle \exp(i\bar{k}.\bar{r})dk \quad (14)$$

The relation between $\alpha_i\varphi_k\varphi'_j(\vec{k})$ and $\varphi_l\beta'_i\beta''_j$ is obtained by letting $\vec{r'}=0$ in equation

$$\left\langle u_l h'_i(\bar{r}) h''_j(\bar{r}')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle \phi_l\beta'_i(\bar{k})\beta''_j(\bar{k}')\right\rangle \exp\left[i(\bar{k}.\bar{r}+\bar{k}'.\bar{r}'\right]d\bar{k}d\bar{k}'$$

and comparing the result with equation

$$\left\langle w h'_i(\bar{r}) h''_j(\bar{r}')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\langle \gamma\beta'_i(\bar{k})\beta''_j(\bar{k}')\right\rangle \exp\left[i(\bar{k}.\bar{r}+\bar{k}'.\bar{r}'\right]d\bar{k}d\bar{k}'$$

then

$$\left\langle \alpha_i\varphi_k\varphi'_i(\bar{k})\right\rangle = \int_{-\infty}^{\infty}\phi_l\beta'_i(\bar{k})\beta''_i(\bar{k}')dk' \quad (15)$$

V. Solutions Neglecting Quintuple Correlation

Using $f\left(\overline{\eta_l\gamma_i'\gamma_i''\gamma_m'''}\right)=L\left(\overline{\phi_l\gamma_i'\gamma_i''\gamma_m'''}\right)$ and 1- $L=s$ in equation (6) and after simplifying, we neglect all the terms on the right hand side of the resulting equation and then integrating between t_1 to t, we obtained

$$\langle\phi_l\gamma_i'\gamma_j''\gamma_m'''\rangle=\langle\phi_l\gamma_i'\gamma_j''\gamma_m'''\rangle_1$$

$$\exp[\left\{-\frac{\nu}{p_M}(1+p_M)\left(k^2+k'^2+k''^2+2kk'+2k'k''+2kk'\right)-2\varepsilon_{\text{pkl}}\Omega_p u_l+fs-R\right\}(t-t_1)] \tag{16}$$

Where is $\langle\phi_l\gamma_i'\gamma_j''\gamma_m'''\rangle_1$ the value of at $\langle\phi_l\gamma_i'\gamma_j''\gamma_m'''\rangle$ at $t=t_1$ that is stationary value for small value of k, k' and k'' when the quintuple correlations are negligible? Substituting of equations (10), (11), (16) in equation (9) we get,

$$\frac{\partial}{\partial t}(\overline{k_k\phi_l\beta_i'\beta_i''})+\frac{\nu}{p_M}[(1+p_M)(k^2+k'^2)+2p_M kk'](\overline{k_k\phi_l\beta_i'\beta_i''})$$

$$[a]_1\int_{-\infty}^{\infty}\exp[\frac{-\nu}{p_M}(t-t_1)\{(1+p_M)(k^2+k'^2+k''^2)+2p_M(kk'+k'k''+kk')\}]$$

$$\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)]\,dk''$$

$$+[b]_1\int_{-\infty}^{\infty}\exp[\frac{-\nu}{p_M}(t-t_1)\{(1+p_M)(k^2+k'^2+k''^2)+2p_M(kk'-kk'')\}]\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)]\,dk''$$

$$+[c]_1\int_{-\infty}^{\infty}\exp[\frac{-\nu}{p_M}(t-t_1)\{(1+p_M)(k^2+k'^2+k''^2)+2p_M(kk'-k'k'')\}]\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)]dk'' \tag{17}$$

At, t_1 $\gamma's$ have been assumed independent of $\vec{k''}$; that assumption is not, made for other times. This is one of several ideas made concerning the initial conditions, although continuity equation satisfied the circumstances. The complete specification of initial turbulence is complicated; the ideas for the initial situation made here in are partially by simplicity. Substituting $dk''=dk_1''dk_2''dk_3''$ and integrating concerning k_1'', k_2'' and k_3'', we get,

$$\frac{\partial}{\partial t}(\overline{k_k\phi_l\beta_i'\beta_i''})+\frac{\nu}{p_M}\left[(1+p_M)\left(k^2+k'^2\right)+2p_M kk'\right](\overline{k_k\phi_l\beta_i'\beta_i''})$$

$$=\frac{\sqrt{\pi.p_M}}{\sqrt{[\nu(t-t_1)(1+p_M)}}[a_1]\exp\left[-\frac{\nu(t-t_1)(1+p_M)}{p_M}\left\{\frac{(1+2p_M)(k^2+k'^2)}{(1+p_M)^2}+\frac{2p_M kk'}{(1+p_M)^2}\right\}\right]\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)]$$

$$+\frac{\sqrt{\pi.p_M}}{\sqrt{[\nu(t-t_1)(1+p_M)}}[b_1]\exp\left[-\frac{\nu(t-t_1)(1+p_M)}{p_M}\left\{\frac{(1+2p_M)(k^2)}{(1+p_M)^2}+\frac{2p_M kk'}{(1+p_M)}+k'^2\right\}\right]\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)]$$

$$+\frac{\sqrt{\pi.p_M}}{\sqrt{[\nu(t-t_1)(1+p_M)}}[c_1]\exp\left[-\frac{\nu(t-t_1)(1+p_M)}{p_M}\left\{k^2+\frac{(1+2p_M)(k'^2)}{(1+p_M)^2}+\frac{2p_M kk'}{(1+p_M))}\right\}\right]\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)] \tag{18}$$

Notes

Integration of equation (18) with respect to time, and in order to simplify calculations, we will assume that $[a]_1 = 0$;That is we assume that a function sufficiently general to represent the initial conditions can obtain by considering only the terms involving $[b]_1$ and $[c]_1$,then substituting of equation (15) in equation (12) and setting $H = 2\pi k^2 \varphi_i \varphi_i'$ result in

Notes

$$\frac{\partial H}{\partial t} + \frac{2\nu k^2}{p_M} H = G \tag{19}$$

Where,

$$G = k^2 \int_{-\infty}^{\infty} 2\pi i [\, \overline{k_k \phi_l \beta_i' \beta_i''(\vec{k}, \vec{k'})}\ \overline{k_k \phi_i \beta_i' \beta_i''(-\vec{k}, -\vec{k'})}\,]_0 .$$

$$\exp[-\frac{\nu}{p_M}(t - t_0)\{(1 + p_M)(k^2 + k'^2) + 2 p_M k k'\}] dk'$$

$$+ k^2 \int_{-\infty}^{\infty} \frac{2\pi^{\frac{5}{2}} i}{\nu} [b(\vec{k}.\vec{k'}) - b(-\vec{k}.-\vec{k'})]_1 \exp[(fs - 2\varepsilon_{\text{pkl}} \Omega_p - R)(t - t_1)] .$$

$$-\!\!-\ \omega^{-1} \exp[-\omega^2 \left(\frac{(1 + 2 p_M) k^2}{(1 + p_M)^2} + \frac{2 P_m k k'}{1 + p_M} + k'^2 \right)]$$

$$+ k \exp\, [-\omega^2 (\quad (1 + p_M)(k^2 + k'^2) + 2 p_M k k'] \int_0^{\frac{\omega k}{2}} \exp(x^2) dx \} dk'$$

$$+ k^2 \int_{-\infty}^{\infty} \frac{2\pi^{\frac{5}{2}} i}{\nu} [c(\vec{k}.\vec{k'}) - c(-\vec{k}.-\vec{k'})]_1 \exp[(fs - 2\varepsilon_{\text{pkl}} \Omega_p - R)(t - t_1)]$$

$$-\!\!-\ \omega^{-1} \exp[-\omega^2 \left(k^2 + \frac{2 P_m k k'}{1 + p_M} + \frac{(1 + 2 p_M) k'^2}{(1 + p_M)^2} \right)]$$

$$+ k' \exp\, [-\omega^2 (\ (1 + p_M)(k^2 + k'^2) + 2 p_M k k')] . \int_0^{\frac{\omega k'}{2}} \exp(x^2) dx \} dk' \tag{20}$$

where H is the magnetic energy spectrum function, which represents contributions from various wave number (or eddy sizes) to the energy and G is the energy transfer function, which is responsible for the transfer of energy between wave number. In order to make further calculations, an assumption must be made for the forms of the bracketed quantities with the subscripts 0 and 1 in equation (20) which depends on the initial conditions.

$$(2\pi)^2 [\, \overline{k_k \phi_l \beta_i' \beta_i''(\vec{k}, \vec{k'})} - \overline{k_k \phi_i \beta_i' \beta_i''(-\vec{k}, -\vec{k'})}\,]_0 = -\xi_0 (k^2 k'^4 - k^4 k'^2) \tag{21}$$

Where ξ_0 is a constant depending on the initial conditions for the other bracketed quantities in equation (20), we get,

$$\frac{4\pi^{\frac{7}{2}}i}{\nu}[b(\vec{k}.\vec{k'})-b(-\vec{k}.-\vec{k'})]_1=\frac{4\pi^{\frac{7}{2}}i}{\nu}[c(\vec{k}.\vec{k'})-c(-\vec{k}.-\vec{k'})]_1=-2\,\xi_1(k^4k'^6-k^6k'^4) \tag{22}$$

Remembering, $d\vec{k'}=-2\pi\overrightarrow{k'^2}d(\cos\theta)dk'$, $kk'=kk'\cos\theta$, and carrying out the integration with respect to θ we get,

$$G=\int_0^\infty[\frac{\xi_0(k^2k'^4-k^4k'^2)kk'}{\nu(t-t_0)}\{\exp[-\frac{\nu}{p_M}(t-t_0)\{(1+p_M)(k^2+k'^2)-2p_Mkk'\}]$$

$$-\exp[-\frac{\nu}{p_M}(t-t_0)\{(1+p_M)(k^2+k'^2)+2p_Mkk'\}]$$

$$+\frac{\xi_1(k^4k'^6-k^6k'^4)kk'}{\nu(t-t_0)}\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)](\omega^{-1}\exp[-\omega^2\left(\frac{(1+2p_M)k^2}{(1+p_M)^2}-\frac{2p_Mkk'}{(1+p_M)}+k'^2\right)$$

$$-\omega^{-1}\exp[-\omega^2\left(\frac{(1+2p_M)k^2}{(1+p_M)^2}+\frac{2p_Mkk'}{(1+p_M)}+k'^2\right)]$$

$$+\omega^{-1}\exp[-\omega^2\left(k^2-\frac{2p_Mkk'}{(1+p_M)}+\frac{(1+2p_M)k'^2}{(1+p_M)^2}\right)]$$

$$-\omega^{-1}\exp[-\omega^2\left(k^2+\frac{2p_Mkk'}{(1+p_M)}+\frac{(1+2p_M)k'^2}{(1+p_M)^2}\right)]$$

$$+\{k\exp[-\omega^2((1+p_M)(k^2+k'^2)-2p_Mkk')]$$

$$-k\exp[-\omega^2((1+p_M)(k^2+k'^2)+2p_Mkk')]\}\int_0^{\frac{\omega k}{2}}\exp(x^2)dx$$

$$+\{k'\exp[-\omega^2((1+p_M)(k^2+k'^2)-2p_Mkk')]$$

$$-k'\exp[-\omega^2((1+p_M)(k^2+k'^2)+2p_Mkk')]\}\int_0^{\frac{\omega k'}{2}}\exp(x^2)dx)]dk' \tag{23}$$

where, $\omega=\left(\frac{\nu(t-t_1)(1+p_M)}{p_M}\right)^{\frac{1}{2}}$.

Integrating equation (23) with respect to k' we have,

$$\text{G}=G_\beta+G_\gamma\exp[(fs-2\varepsilon_{\text{pkl}}\Omega_p-R)(t-t_1)] \tag{24}$$

The integral expression in equation (24), the quantity G_β represents the transfer function arising due to consideration of magnetic field at three-point correlation

Notes

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equation; G_γ arises from consideration of the four-point equation. Integration of equation (24) over all wave numbers shows that

$$\int_0^\infty G.d\vec{k} = 0 \tag{25}$$

Indicating that the expression for G satisfies the conditions of continuity and homogeneity, physically, it was to be expected, since G is a measure of transfer of energy and the numbers must be zero. From equation (19),

we get, $H = \exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right]\int G\exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right]dt + J(k)\exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right]$,

$J(k) = N_0 k^2/\pi$, is a constant of integration and can be obtained as by Corrsion [Corrsion 1951b]. Therefore we obtained,

$$H = \frac{N_0 k^2}{\pi}\exp\left[\frac{-2\nu k^2(t-t_0)}{p_M}\right] + \exp\left[\frac{-2\nu k^2(t-t_0)}{p_M}\right]\int (G_\beta + G_{\gamma_1} + G_{\gamma_2} + G_{\gamma_3} + G_{\gamma_4})$$

$$\exp\left[\frac{-2\nu k^2(t-t_0)}{p_M}\right]dt \tag{26}$$

where,
$$G = G_\beta + G_{\gamma_1} + G_{\gamma_2} + G_{\gamma_3} + G_{\gamma_4} \tag{27}$$

After integration equation (26) becomes

$$H = \frac{N_0 k^2}{\pi}\exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right] + H_\beta + [H_{\gamma_1} + H_{\gamma_2} + H_{\gamma_3} + H_{\gamma_4}]\exp[(fs - 2\varepsilon_{\text{pkl}}\Omega_p - R)(t-t_1)] \tag{28}$$

From equation (28) we get,

$$H = H_1 + H_2\exp[(fs - 2\varepsilon_{\text{pkl}}\Omega_p - R)(t-t_1)] \tag{29}$$

$$H_1 = \frac{N_0 k^2}{\pi}\exp\left[-\frac{2\nu k^2(t-t_0)}{p_M}\right] + H_\beta,\; H_2 = \left(H_{\gamma_1} + H_{\gamma_2} + H_{\gamma_3} + H_{\gamma_4}\right);$$

In equation (29) H_1 and H_2 magnetic energy spectrum arising from consideration of the three and four-point correlation equations respectively. Equation (29) can be integrated over all wave numbers to give the total magnetic turbulent energy. That is

$$\frac{\overline{h_i h_i'}}{2} = \int_0^\infty H dk \tag{30}$$

Now,

$$\int_0^\infty H_1 dk = \frac{N_0 p^{3/2}{}_M \nu^{-3/2}(t-t_0)^{-3/2}}{8\sqrt{2\pi}} + \xi_0 Q\nu^{-6}(t-t_0)^{-5}$$

$$\int_0^\infty H_2 dk = \xi_1[L_1\nu^{-17/2}(t-t_1)^{-15/2} + L_2\nu^{-19/2}(t-t_1)^{-17/2}].\exp[(fs - 2\varepsilon_{\text{pkl}}\Omega_p - R)(t-t_1)]$$

where

$$L_1 = Q_2 + Q_4 + Q_6 + Q_7,\ L_2 = Q_1 + Q_3 + Q_5$$

and $G_\beta,\ G_\gamma\ \ H_\beta, H_{\gamma_1}, H_{\gamma_2}, H_{\gamma_3}, H_{\gamma_4}, Q, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7$ are defined in [Bkar Pk *et al.* 2013].

Therefore, from equation (30)

$$\frac{\overline{h_i h_i'}}{2} = \frac{N_0 p^{3/2} {}_M \nu^{-3/2}(t-t_0)^{-3/2}}{8\sqrt{2\pi}} + \xi_0 Q\nu^{-6}(t-t_0)^{-5}$$

$$+\xi_1[L_1\nu^{-17/2}(t-t_1)^{-15/2} + L_2\nu^{-19/2}(t-t_1)^{-17/2}]\exp[(fs - 2\varepsilon_{\text{pkl}}\Omega_p - R)(t-t_1)] \tag{31}$$

This is the first order reactant of MHD turbulence in presence of dust particles for four-point correlation. It can be written as

$$\langle h^2\rangle = A(t-t_0)^{-3/2} + B(t-t_0)^{-5} + [C(t-t_1)^{-15/2} + D(t-t_1)^{-17/2}]\exp[(M - 2\varepsilon_{\text{pkl}}\Omega_p - R)(t-t_1)] \tag{32}$$

If R=0 in equation (32) then it becomes

$$\langle h^2\rangle = A(t-t_0)^{-3/2} + B(t-t_0)^{-5} + [C(t-t_1)^{-15/2} + D(t-t_1)^{-17/2}]\exp[(M - 2\varepsilon_{\text{pkl}}\Omega_p)(t-t_1)] \tag{33}$$

Where, $M=fs$ is the dust particle parameter. This was obtained earlier by Bkar Pk et al. [Bkar PK et al. 2013].

In the absent of M, $2\varepsilon_{\text{pkl}}\Omega_p$ and R equation (32) reduces to the form

$$\langle h^2\rangle = A(t-t_0)^{-3/2} + B(t-t_0)^{-5} + C(t-t_1)^{-15/2} + D(t-t_1)^{-17/2}] \tag{34}$$

This is the decay of energy of MHD turbulence for four-point correlation. It is fully same with Bkar PK et al. [Bkar PK et al. 2012].

If ξ_1=0, then equation (34) becomes

$$\langle h^2\rangle = A(t-t_0)^{-3/2} + B(t-t_0)^{-5}$$

This is the decay of MHD turbulence for three-point correlation. This is totally same with the result obtained by Sarker and Kishore [Sarker and Kishore 1991].

VI. Results and Discussion

This study shows that the terms associated with the higher-order correlations die out faster than those associated with the lower-order ones. If the quadruple and quintuple correlations were not neglected, then more conditions the negative higher power of $(t-t_1)$ would be added to the equation(32), and for large times the last terms in the equations (32), becomes negligible, leaving the -3/2 power decay law for the final period.

Notes

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$h1$, $h2$, $h3$, $h4$ and $h5$ are energy decay curves of equation (32) at 0.5, 1, 1.5, 2 and 2.5 respectively. For different values of M, $2\varepsilon_{\text{pkl}}\Omega_p$ and R we have seen the energy decay curves in the figures bellow. In the presence of dust particles energy decay increases, and more increases either rotating force or chemical reaction is absent. In the absence of dust particles and Coriolis force (or chemical reaction) energy decay curves increases due to decreases of the chemical reaction (or Coriolis force) and maximum at the point where chemical reaction (or Coriolis force) is equal to zero. Energy decay increases with the decreases of chemical reaction and maximum at zero.

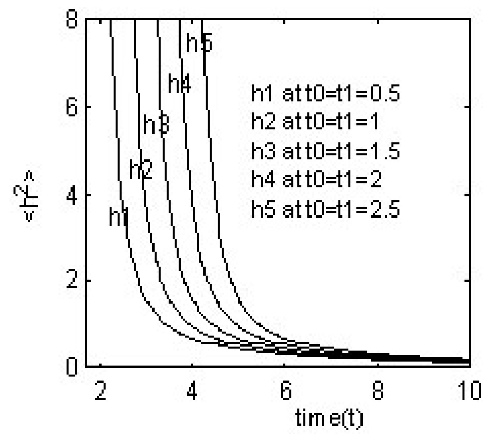

Figure 1: Energy curves for M=3, $2\varepsilon_{\text{pkl}}\Omega_p = 0.50$ R=0.50

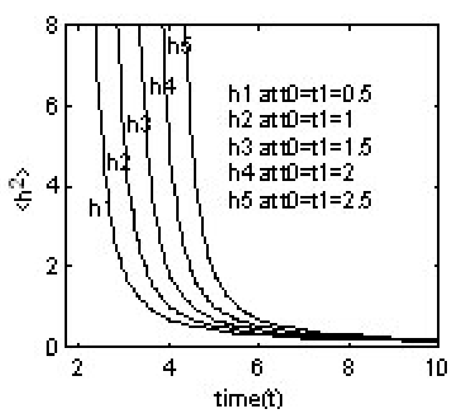

Figure 2: Energy decay curves of equation (32) if M=3, $2\varepsilon_{\text{pkl}}\Omega_p = 0$, R=0.50

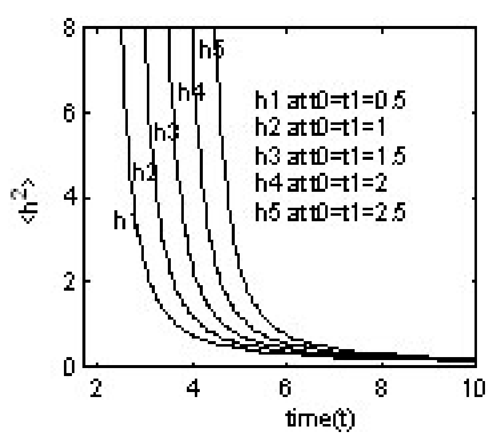

Figure 3: Energy decay curves of equation (32) if M=3, $2\varepsilon_{\text{pkl}}\Omega_p = 0$, R=0

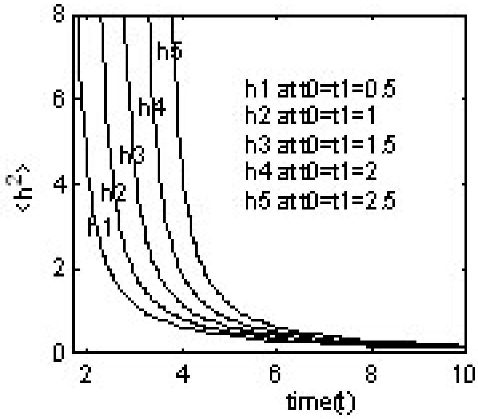

Figure 4: Energy decay curves of equation (32) if M=0, $2\varepsilon_{\mathrm{pkl}}\Omega_p = 0.50$, R=0.50

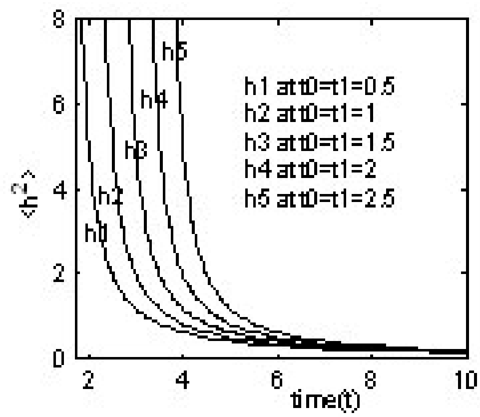

Figure 5: Energy decay curves of equation (32) if M=0, $2\varepsilon_{\mathrm{pkl}}\Omega_p = 0$, R=0.50

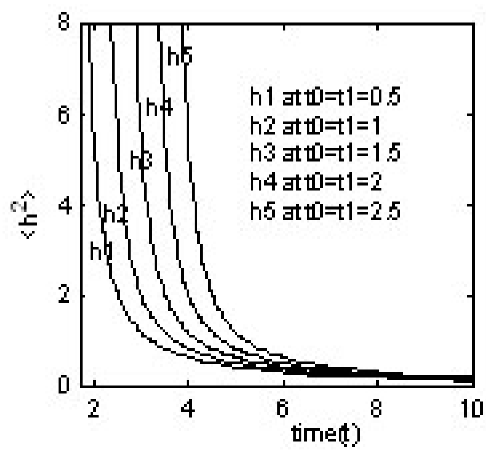

Figure 6: Energy decay curves of equation (32) if M=0, $2\varepsilon_{\mathrm{pkl}}\Omega_p = 0$, R=0.25

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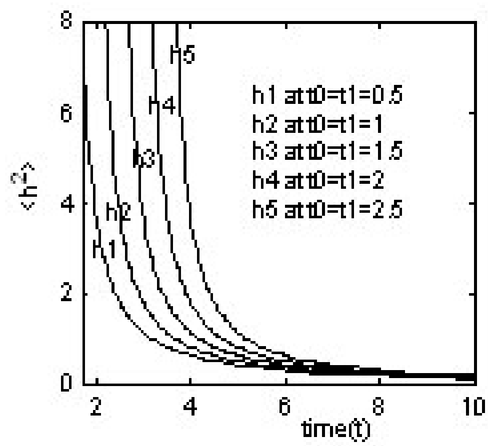

Figure 7: Energy decay curves of equation (32) if $M=0, 2\varepsilon_{\mathrm{pkl}}\Omega_p = 0, R=5$

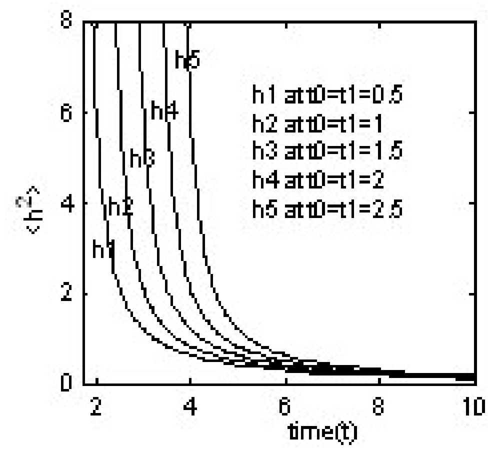

Figure 8: Energy decay curves of equation (34) if $M=0, 2\varepsilon_{\mathrm{pkl}}\Omega_p = 0, R=0$

From the above figures and discussion, we conclude that if $(M - 2\varepsilon_{\mathrm{pkl}}\Omega_p - R) \geq 0$ then energy decay increases rapidly and if $(M - 2\varepsilon_{\mathrm{pkl}}\Omega_p - R) < 0$ in the chemical reaction energy decay of MHD fluid turbulence for four-point correlation energy decay increases more slowly.

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Quasi-Hadamard Product of Certain Starlike and Convex P-Valent Functions

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Abstract- In this paper, we obtained some results using the quasi-Hadamard product for two classes of *p*- valent functions related to starlike and convex with respect to symmetric points.

Keywords: *starlike function; convex function with respect to symmetric points; quasi-hadamard product.*

GJSFR-F Classification: *MSC 2010: 15B34*

QUASIHADAMARDPRODUCTOFCERTAINSTARLIKEANDCONVEXPVALENTFUNCTIONS

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Quasi-Hadamard Product of Certain Starlike and Convex P-Valent Functions

R. M. El-Ashwah [α], A. Y. Lashin [σ] & A. E. El-Shirbiny [ρ]

Abstract- In this paper, we obtained some results using the quasi-Hadamard product for two classes of p- valent functions related to starlike and convex with respect to symmetric points.

Keywords: starlike function; convex function with respect to symmetric points; quasi-hadamard product.

I. Introduction

let $T(p)$ denote of the class of functions of the form:

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p}, \qquad (p \in \mathbb{N} = \{1, 2, ...\}, a_1 \geq 0, a_{n+p} \geq 0), \tag{1}$$

$$f_r(z) = a_{p,r} z^p - \sum_{n=1}^{\infty} a_{n+p,r} z^{n+p} \qquad (r \in \mathbb{N}, a_{p,r} > 0, a_{n+p,r} \geq 0), \tag{2}$$

$$g(z) = b_p z^p - \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \qquad (p \in \mathbb{N} = \{1, 2, ...\}, b_p \geq 0, b_{n+p} \geq 0), \tag{3}$$

and

$$g_j(z) = b_{p,j} z^p - \sum_{n=1}^{\infty} b_{n+p,j} z^{n+p} \qquad (j \in \mathbb{N}, b_p \geq 0, b_{n+p,j} \geq 0), \tag{4}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C}, \ |z| < 1\}$. We write $T(1) = T$, the class of analytic functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_1 \geq 0, a_n \geq 0), \tag{5}$$

which are analytic in the open unit disc $U = \{z : z \in \mathbb{C}, \ |z| < 1\}$.

Let S^* be the subclass of functions T consisting of starlike functions in U. It is well known that $f \in S^*$ if and only if

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$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>0, \qquad (z\in U).$$

and C be the subclass of functions T consisting of convex functions in U. It is well known that $f\in C$ if and only if

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0, \qquad (z\in U).$$

Let S_S^* be the subclass of T consisting of functions of the form (5) satisfying

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0, \qquad (z\in U).$$

These functions are called starlike with respect to symmetric points and introduced by Sakaguchi [13] (see also Robertson [12], Stankiewics [14], Wu [16] and Owa et al. [8]).

Khairnar and Rajas (see [4], with $\delta=0$) introduced the class $S_s^*(p,\alpha,\beta)$ consisting of functions of the form (1) and satisfying the following condition

$$\left|\frac{zf'(z)}{f(z)-f(-z)}-p\right|<\beta\left|\frac{\alpha zf'(z)}{f(z)-f(-z)}+p\right|, 0\le\alpha<1, 0<\beta\le1.$$

Let $S_c^*(p,\alpha,\beta)$ denote the class of functions of the form (1) for which $zf'(z)\in S_s^*(p,\alpha,\beta)$.

We note that $S_s^*(1,\alpha,\beta)=S_s^*(\alpha,\beta)$ (see [15] and [1]) and $S_c^*(1,\alpha,\beta)=S_c^*(\alpha,\beta)$ (see [3]).

By using the technique of Khairnar and Rajas (see [4], with $\delta=0$), we get the following theorem.

Theorem 1. Let the function $f(z)$ *defined by* (1) *then*

(i) $f(z)\in S_s^*(p,\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty}\left[\left(\frac{n+p}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p}\le(\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p)\,a_p$$

(ii) $f(z)\in S_c^*(p,\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)\left[\left(\frac{n+p}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p}\le(\beta[\alpha+(1-(-1)^p)]+(-1)^p)\,a_p$$

(iii) $f(z)\in S_{s,h}^*(p,\alpha,\beta)$ if and only if

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^h\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{p+n}-1\right)\right]a_{n+p}$$
$$\le\quad(\beta[\alpha+(1-(-1)^p)]+(-1)^p)\,a_p;$$

where $0\le\alpha<1$, $0<\beta<1$, $0<\frac{2(1-\beta)}{1+\alpha\beta}<1$, $z\in U$ and h is a nonnegative real number.

Ref
13. K. Sakaguchi, On certain univalent mapping, J. Math. Soc. Japen., 11 (1959), 72-75.

We note that for a nonnegative real number h the class $S_{s,h}^{*}(p,\alpha,\beta)$ is nonempty as the function of the form

$$f(z)=a_pz^p-\sum_{n=1}^{\infty}\frac{\beta\left[\alpha+(1-(-1)^p)\right]+(-1)^p}{\left(\frac{n+p}{p}\right)^h\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]}a_p\lambda_{n+p}z^{n+p}, \qquad (6)$$

where $a_p>0,\lambda_{n+p}\geq 0$ and $\sum_{n=1}^{\infty}\lambda_{n+p}\leq 1$ satisfy the inequality (6). It is evident that $S_{s,1}^{*}(p,\alpha,\beta)=S_c^{*}(p,\alpha,\beta)$ and for $h=0$, $S_{s,0}^{*}(p,\alpha,\beta)$ is identical to $S_s^{*}(p,\alpha,\beta)$. Further more $S_{s,h}^{*}(p,\alpha,\beta)\subset S_{s,m}^{*}(p,\alpha,\beta)$ for $h>m$, the containment being proper. Hence for any positive integer h, the inclusion relation

$$S_{s,h}^{*}(p,\alpha,\beta)\subset S_{s,h-1}^{*}(p,\alpha,\beta)\subset\ldots\subset S_{s,m}^{*}(p,\alpha,\beta)\subset\ldots\subset S_{s,2}^{*}(p,\alpha,\beta)\subset S_c^{*}(p,\alpha,\beta)\subset S_s^{*}(p,\alpha,\beta).$$

The quasi-Hadamard product of two or more functions has recently defined by Darwich et al.[2], Kumar[5, 6, 7], Owa [9, 10, 11], and others. Accordingly, the quasi-Hadmard product of two functions $f(z)$ and $g(z)$

$$f(z)*g(z)=a_pb_pz^p-\sum_{n=1}^{\infty}a_{n+p}b_{n+p}z^{n+p}.$$

In this paper, we obtained some results concerning the quasi-Hadamard product for two classes of $p-$ valent functions related to starlike and convex with respect to symmetric points.

II. Main Results

Theorem 2 *A functions* $f_r(z)$ *defined by* (2) *in the* $S_c^{*}(p,\alpha,\beta)$ *for each* $r=1,2,\ldots,u$. *then we get quasi-Hadamard prouduct*

$$f_1(z)*f_2(z)*\ldots*f_u(z)\in S_{s,2(u-1)+1}^{*}(p,\alpha,\beta).$$

Proof. To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)^{2(u-1)+1}\left\{\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right\}\prod_{r=1}^{u}a_{n+p,r}$$

$$\leq\left(\beta[\alpha+(1-(-1)^p)]+(-1)^p\right)\prod_{r=1}^{u}a_{p,r}.$$

Since $f_r(z)\in S_c^{*}(p,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}\left(\frac{n+p}{p}\right)\left\{\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right\}a_{n+p,r}\leq[\beta[\alpha+(1-(-1)^p)]+(-1)^p]a_{p,r} \qquad (7)$$

2. H. E. Darwish, A. Y. Lashin and A. N. Alnayyef, Quasi Hadmeard product of certain starlike and convex functions, Global Journal of Science Frontier Research, 15 (10) (2015), no.1, 1-7.

for each $r = 1, ..., u$. Therefore

$$\binom{n+p}{p}\left[\binom{p+n}{p}(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p,r}\leq[\beta[\alpha+(1-(-1)^{p})]+(-1)^{p}]a_{p,r}$$

or

$$a_{n+p,r}\leq\left\{\frac{\beta\left[\alpha+(1-(-1)^{p})\right]+(-1)^{p}}{\binom{n+p}{p}\left[\binom{p+n}{p}(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]}\right\}a_{p,r}$$

for each $r = 1, 2, ..., u$. The right hand expression of this last inequality is not greater than $\binom{n+p}{p}^{-2}a_{p,r}$, hence

$$a_{n+p,r}\leq\binom{n+p}{p}^{-2}a_{p,r}\qquad r=1,...,u. \tag{8}$$

By (8) for each $r = 1, 2, ..., u-1$ and (7) for $r = u$, we get

$$\sum_{n=1}^{\infty}\binom{n+p}{p}^{2(u-1)+1}\left[\binom{p+n}{p}(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\prod_{r=1}^{u}a_{n+p,r}\leq$$

$$\sum_{n=1}^{\infty}\left\{\left[\binom{n+p}{p}^{2(u-1)+1}\left[\binom{p+n}{p}(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\right]\binom{n+p}{p}^{-2(u-1)}\prod_{r=1}^{u-1}a_{p,r}\right\}a_{n+p,u}$$

$$=\left[\prod_{r=1}^{u-1}a_{p,r}\right]\sum_{n=1}^{\infty}\binom{n+p}{p}\left[\binom{p+n}{p}(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p,u}$$

$$\leq\left(\beta\left[\alpha+(1-(-1)^{p})\right]+(-1)^{p}\right)\prod_{r=1}^{u}a_{p,r}$$

hence $f_1 * f_2 * ... * f_u \in S^*_{2(u-1)+1}(p,\alpha,\beta)$. This completes the proof of Theorem 2.

Theorem 3 A functions $f_r(z)$ defined by (2)in the class $S^*_s(p,\alpha,\beta)$ for each $r = 1, 2, ..., u$. Then, we get the quasi-Hadamard product

$$f_1(z)*f_2(z)*...*f_u(z)\in S^*_{s,(u-1)}(p,\alpha,\beta).$$

Proof. Using $f_r(z)\in S^*_s(p,\alpha,\beta)$, we have

$$\sum_{n=1}^{\infty}\left[\binom{p+n}{p}(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]a_{n+p,r}\leq\beta\left[\alpha+(1-(-1)^{p})\right]+(-1)^{p}a_{p,r} \tag{9}$$

for each $r = 1, 2, ..., u$, therefore,

$$a_{n+p,r}\leq\left(\frac{\beta[\alpha+(1-(-1)^{p})]+(-1)^{p}}{\binom{p+n}{p}(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)}\right)a_{p,r}$$

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and hence

$$a_{n+p,r} \leq \left(\frac{n+p}{p}\right)^{-1} a_{p,r}, \qquad r = 1,2,...,u. \tag{10}$$

By (10) *for* $r = 1,2,...,(u-1)$ *and* (9) *for* $r = u$, *we get*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{(u-1)}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]\right]\prod_{r=1}^{u}a_{n+p,r}\right\} \leq$$

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{(u-1)}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\left[\left(\frac{n+p}{p}\right)^{-(u-1)}\prod_{r=1}^{u-1}a_{p,r}\right]\right\}a_{n+p,u}$$

$$= \left[\prod_{r=1}^{u-1}a_{p,r}\right]\sum_{n=1}^{\infty}\left\{\left(\frac{p+n}{p}\right)(1+\alpha\beta)+\left[(\beta-1)\left(1-(-1)^{n+p}\right)\right]a_{n+p,u}\right\}$$

$$\leq \left(\beta[\alpha+(1-(-1)^{p}]+(-1)^{p}\right)\prod_{r=1}^{u}a_{p,r}$$

Hence $f_1(z)*f_2(z)*...*f_u(z) \in S^*_{s,u-1}(p,\alpha,\beta)$. *This completes the proof of Theorem 3.*

Theorem 4 A functions $f_r(z)$ *defined by* (2) *in the class* $S^*_c(p,\alpha,\beta)$ *for each* $r = 1,2,...,u$ *and the functions* $g_j(z)$ *defined by* (1.4) *in the class* $S^*_s(p,\alpha,\beta)$ *for* $j = 1,2,...,q$. *Then, we get the quasi-Hadamard product;*

$$f_1(z)*f_2(z)*...*f_u(z)*g_1(z)*g_2(z)*...*g_q(z) \in S^*_{s,2u+q-1}(p,\alpha,\beta).$$

Proof. We denote the quasi-Hadamard produc $f_1(z)*f_2(z)*...*f_u(z)*g_1(z)*g_2(z)*...*g_q(z)$ *by function* $h(z)$, *for the sake of the convenience. Clearly*

$$h(z) = \left[\prod_{r=1}^{u}a_{p,r}\cdot\prod_{j=1}^{q}b_{p,j}\right]z^p - \sum_{n=1}^{\infty}\left[\prod_{r=1}^{u}a_{n+p,r}\cdot\prod_{j=1}^{q}b_{n+p,j}\right]z^{n+p}.$$

To prove the theorem, we need to show that

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+q-1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\left[\prod_{r=1}^{u}a_{n+p,r}\cdot\prod_{j=1}^{q}b_{n+p,j}\right]\right\}$$

$$\leq \left(\beta[\alpha+(1-(-1)^{p})]+(-1)^{p}\right)\left[\prod_{r=1}^{u}a_{p,r}\cdot\prod_{j=1}^{q}b_{p,j}\right] \tag{11}$$

Since $f_r(z) \in S^*_c(p,\alpha,\beta)$, *the inequalities* (7) *and* (8) *hold for every* $r = 1,2,...,u$. *Further, since* $g_j(z) \in S^*_s(p,\alpha,\beta)$, *the inequality* (10) *holds for each* $j = 1,2,...,q$. *By* (8) *for* $r = 1,2,...,u$, (10) *for* $j = 1,2,...,q-1$ *and* (9) *for* $j = q$, *we get*

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+q-1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]\left[\prod_{r=1}^{u}a_{n+p,r}\cdot\prod_{j=1}^{q}b_{n+p,j}\right]\right]\right\}$$

$$\leq \sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+q-1}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left((-1)^{n+p}-1\right)\right]\right\}\times$$

$$\left(\frac{n+p}{p}\right)^{-2u}\quad\prod_{r=1}^{u}a_{p,r}.\prod_{j=1}^{q}b_{n+p,j}\Bigg)$$

$$\leq \sum_{n=1}^{\infty}\left\{\left(\frac{n+p}{p}\right)^{2u+(q-1)}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]\right]\right\}\times$$

$$\left[\prod_{r=1}^{u}a_{p,r}.\prod_{j=1}^{q-1}b_{p,j}\left(\frac{n+p}{p}\right)^{-2u}.\left(\frac{n+p}{p}\right)^{-(q-1)}\right]b_{n+p,q}$$

$$=\left[\prod_{r=1}^{u}a_{p,r}.\prod_{j=1}^{q-1}b_{p,j}\right]\sum_{n=1}^{\infty}\left[\left(\frac{p+n}{p}\right)(1+\alpha\beta)+(1-\beta)\left[(-1)^{n+p}-1\right]b_{n+p,q}\right]$$

$$\leq (\beta[\alpha+(1-(-1)^{p})]+(-1)^{p})\left[\prod_{r=1}^{u}a_{p,r}\prod_{j=1}^{q}b_{p,j}\right].$$

Hence, $h(z)\in S^{*}_{s,2u+q-1}(p,\alpha,\beta)$. *This completes the proof of Theorem 4.* ■

Remark 5 Putting $p=1$ *in the above results, we obtain the results obtained by Darwish et al. [2].*

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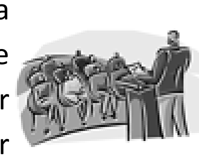

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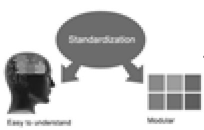


The FARSS will be eligible for a free application of standardization of their researches. Standardization of research will be subject to acceptability within stipulated norms as the next step after publishing in a journal. We shall depute a team of specialized research professionals who will render their services for elevating your researches to next higher level, which is worldwide open standardization.

The FARSS member can apply for grading and certification of standards of their educational and Institutional Degrees to Open Association of Research, Society U.S.A. Once you are designated as FARSS, you may send us a scanned copy of all of your credentials. OARS will verify, grade and certify them. This will be based on your academic records, quality of research papers published by you, and some more criteria. After certification of all your credentials by OARS, they will be published on your Fellow Profile link on website https://associationofresearch.org which will be helpful to upgrade the dignity.

The FARSS members can avail the benefits of free research podcasting in Global Research Radio with their research documents. After publishing the work, (including published elsewhere worldwide with proper authorization) you can upload your research paper with your recorded voice or you can utilize chargeable services of our professional RJs to record your paper in their voice on request.

The FARSS member also entitled to get the benefits of free research podcasting of their research documents through video clips. We can also streamline your conference videos and display your slides/ online slides and online research video clips at reasonable charges, on request.

The FARSS is eligible to earn from sales proceeds of his/her researches/reference/review Books or literature, while publishing with Global Journals. The FARSS can decide whether he/she would like to publish his/her research in a closed manner. In this case, whenever readers purchase that individual research paper for reading, maximum 60% of its profit earned as royalty by Global Journals, will be credited to his/her bank account. The entire entitled amount will be credited to his/her bank account exceeding limit of minimum fixed balance. There is no minimum time limit for collection. The FARSS member can decide its price and we can help in making the right decision.

The FARSS member is eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get remuneration of 15% of author fees, taken from the author of a respective paper. After reviewing 5 or more papers you can request to transfer the amount to your bank account.

MEMBER OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (MARSS)

The ' MARSS ' title is accorded to a selected professional after the approval of the Editor-in-Chief / Editorial Board Members/Dean.

The “MARSS” is a dignified ornament which is accorded to a person’s name viz. Dr. John E. Hall, Ph.D., MARSS or William Walldroff, M.S., MARSS.

MARSS accrediting is an honor. It authenticates your research activities. After becoming MARSS, you can add 'MARSS' title with your name as you use this recognition as additional suffix to your status. This will definitely enhance and add more value and repute to your name. You may use it on your professional Counseling Materials such as CV, Resume, Visiting Card and Name Plate etc.

The following benefitscan be availed by you only for next three years from the date of certification.

MARSS designated members are entitled to avail a 25% discount while publishing their research papers (of a single author) in Global Journals Inc., if the same is accepted by our Editorial Board and Peer Reviewers. If you are a main author or co-author of a group of authors, you will get discount of 10%.

As MARSS, you will be given a renowned, secure and free professional email address with 30 GB of space e.g. johnhall@globaljournals.org. This will include Webmail, Spam Assassin, Email Forwarders,Auto-Responders, Email Delivery Route tracing, etc.

We shall provide you intimation regarding launching of e-version of journal of your stream time to time.This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.

The MARSS member can apply for approval, grading and certification of standards of their educational and Institutional Degrees to Open Association of Research, Society U.S.A.

Once you are designated as MARSS, you may send us a scanned copy of all of your credentials. OARS will verify, grade and certify them. This will be based on your academic records, quality of research papers published by you, and some more criteria.

It is mandatory to read all terms and conditions carefully.

Institutional Fellow of Global Journals Incorporation (USA)-OARS (USA)

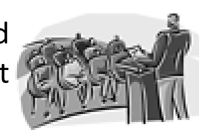
Global Journals Incorporation (USA) is accredited by Open Association of Research Society, U.S.A (OARS) and in turn, affiliates research institutions as "Institutional Fellow of Open Association of Research Society" (IFOARS).

The "FARSC" is a dignified title which is accorded to a person's name viz. Dr. John E. Hall, Ph.D., FARSC or William Walldroff, M.S., FARSC.

The IFOARS institution is entitled to form a Board comprised of one Chairperson and three to five board members preferably from different streams. The Board will be recognized as "Institutional Board of Open Association of Research Society"-(IBOARS).

The Institute will be entitled to following benefits:

The IBOARS can initially review research papers of their institute and recommend them to publish with respective journal of Global Journals. It can also review the papers of other institutions after obtaining our consent. The second review will be done by peer reviewer of Global Journals Incorporation (USA) The Board is at liberty to appoint a peer reviewer with the approval of chairperson after consulting us.
The author fees of such paper may be waived off up to 40%.



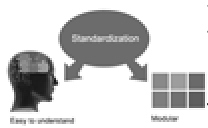
The Global Journals Incorporation (USA) at its discretion can also refer double blind peer reviewed paper at their end to the board for the verification and to get recommendation for final stage of acceptance of publication.

The IBOARS can organize symposium/seminar/conference in their country on behalf of Global Journals Incorporation (USA)-OARS (USA). The terms and conditions can be discussed separately.

The Board can also play vital role by exploring and giving valuable suggestions regarding the Standards of "Open Association of Research Society, U.S.A (OARS)" so that proper amendment can take place for the benefit of entire research community. We shall provide details of particular standard only on receipt of request from the Board.

The board members can also join us as Individual Fellow with 40% discount on total fees applicable to Individual Fellow. They will be entitled to avail all the benefits as declared. Please visit Individual Fellow-sub menu of GlobalJournals.org to have more relevant details.

We shall provide you intimation regarding launching of e-version of journal of your stream time to time. This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.



After nomination of your institution as “Institutional Fellow” and constantly functioning successfully for one year, we can consider giving recognition to your institute to function as Regional/Zonal office on our behalf.
The board can also take up the additional allied activities for betterment after our consultation.

The following entitlements are applicable to individual Fellows:

Open Association of Research Society, U.S.A (OARS) By-laws states that an individual Fellow may use the designations as applicable, or the corresponding initials. The Credentials of individual Fellow and Associate designations signify that the individual has gained knowledge of the fundamental concepts. One is magnanimous and proficient in an expertise course covering the professional code of conduct, and follows recognized standards of practice.

Open Association of Research Society (US)/ Global Journals Incorporation (USA), as described in Corporate Statements, are educational, research publishing and professional membership organizations. Achieving our individual Fellow or Associate status is based mainly on meeting stated educational research requirements.

Disbursement of 40% Royalty earned through Global Journals : Researcher = 50%, Peer Reviewer = 37.50%, Institution = 12.50% E.g. Out of 40%, the 20% benefit should be passed on to researcher, 15 % benefit towards remuneration should be given to a reviewer and remaining 5% is to be retained by the institution.

We shall provide print version of 12 issues of any three journals [as per your requirement] out of our 38 journals worth $ 2376 USD.

Other:

The individual Fellow and Associate designations accredited by Open Association of Research Society (US) credentials signify guarantees following achievements:

- The professional accredited with Fellow honor, is entitled to various benefits viz. name, fame, honor, regular flow of income, secured bright future, social status etc.

- In addition to above, if one is single author, then entitled to 40% discount on publishing research paper and can get 10%discount if one is co-author or main author among group of authors.
- The Fellow can organize symposium/seminar/conference on behalf of Global Journals Incorporation (USA) and he/she can also attend the same organized by other institutes on behalf of Global Journals.
- The Fellow can become member of Editorial Board Member after completing 3yrs.
- The Fellow can earn 60% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.
- Fellow can also join as paid peer reviewer and earn 15% remuneration of author charges and can also get an opportunity to join as member of the Editorial Board of Global Journals Incorporation (USA)
- • This individual has learned the basic methods of applying those concepts and techniques to common challenging situations. This individual has further demonstrated an in–depth understanding of the application of suitable techniques to a particular area of research practice.

Note :

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- In future, if the board feels the necessity to change any board member, the same can be done with the consent of the chairperson along with anyone board member without our approval.
- In case, the chairperson needs to be replaced then consent of 2/3rd board members are required and they are also required to jointly pass the resolution copy of which should be sent to us. In such case, it will be compulsory to obtain our approval before replacement.
- In case of "Difference of Opinion [if any]" among the Board members, our decision will be final and binding to everyone.

"

Preferred Author Guidelines

We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template.

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

Before and during Submission

Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

1. Authors must go through the complete author guideline and understand and *agree to Global Journals' ethics and code of conduct,* along with author responsibilities.
2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
5. Authors should submit paper in a ZIP archive if any supplementary files are required along with the paper.
6. Proper permissions must be acquired for the use of any copyrighted material.
7. Manuscript submitted *must not have been submitted or published elsewhere* and all authors must be aware of the submission.

Declaration of Conflicts of Interest

It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

Policy on Plagiarism

Plagiarism is not acceptable in Global Journals submissions at all.

Plagiarized content will not be considered for publication. We reserve the right to inform authors' institutions about plagiarism detected either before or after publication. If plagiarism is identified, we will follow COPE guidelines:

Authors are solely responsible for all the plagiarism that is found. The author must not fabricate, falsify or plagiarize existing research data. The following, if copied, will be considered plagiarism:

- Words (language)
- Ideas
- Findings
- Writings
- Diagrams
- Graphs
- Illustrations
- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

Copyright

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Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- o Explain the value (significance) of the study.
- o Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- o Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- o Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- o Report the method and not the particulars of each process that engaged the same methodology.
- o Describe the method entirely.
- o To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- o Simplify—detail how procedures were completed, not how they were performed on a particular day.
- o If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- o Resources and methods are not a set of information.
- o Skip all descriptive information and surroundings—save it for the argument.
- o Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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