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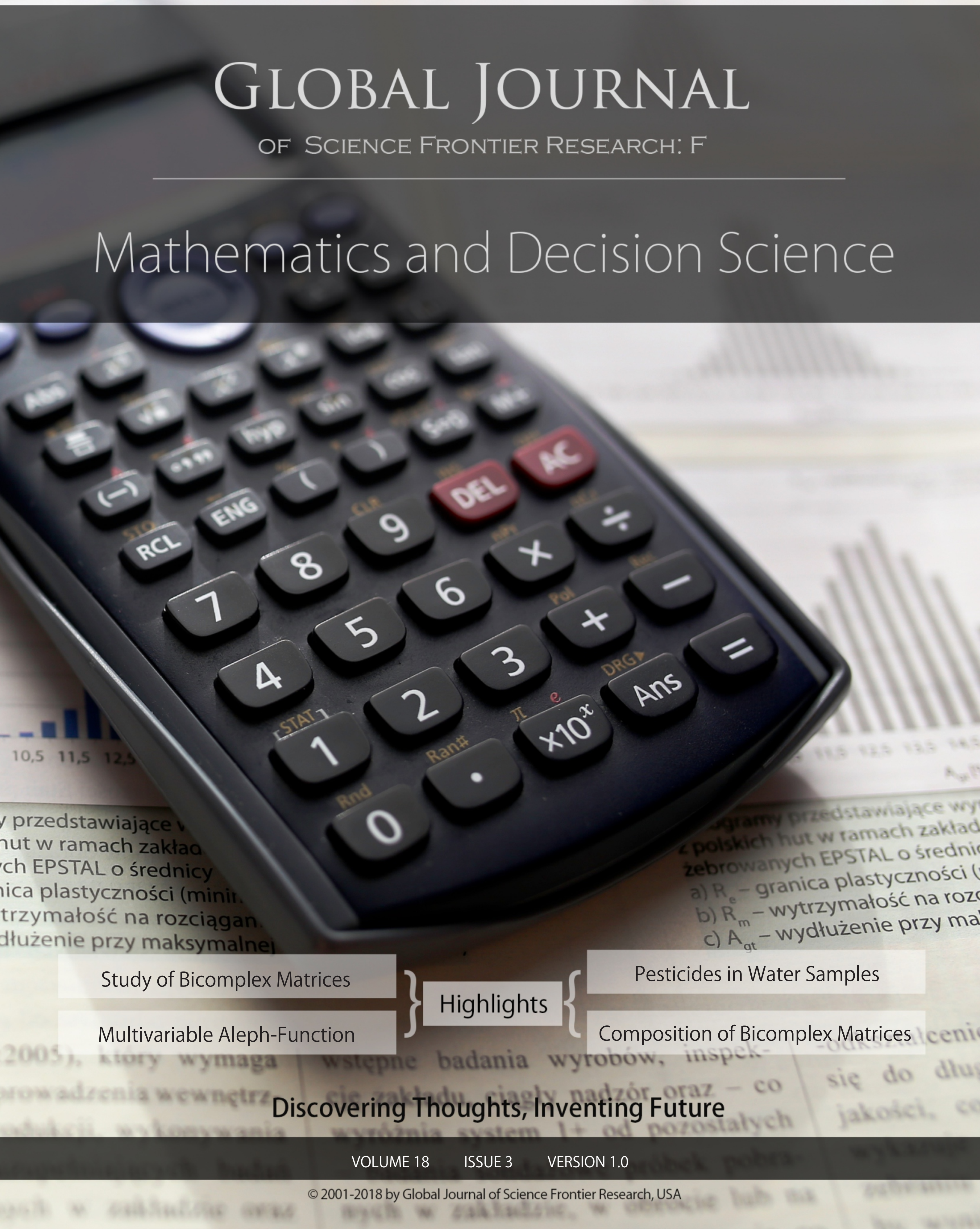

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Mathematics and Decision Science

Highlights

- Study of Bicomplex Matrices
- Multivariable Aleph-Function
- Pesticides in Water Samples
- Composition of Bicomplex Matrices

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Quantum Information Processing Via Hamiltonian Inverse Quantum Engineering

By Alan C. Santos
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I. Introduction

The heart of technologies of the future are based on our ability to control quantum system and designing very small quantum devices. Currently, controlling and protecting quantum systems against decoherence effects is the main challenging task for both theoretical and experimentalists. To protect a quantum system against decohering effects, for example, we can use protocols for speeding up quantum dynamics. In contrast, high speed quantum dynamics requests robust protocols against systematic errors, i.e., uncontrollable deviations in the fields parameters used to drive the system. For this reason, techniques for implementing robust and fast quantum dynamics has woke up interest in recent years.

For instance, we can consider shortcuts to adiabatic dynamics [1, 2, 3] and inverse quantum engineering [4] as two protocols for speeding up quantum tasks. Hamiltonian inverse quantum engineering (HIQE) is a useful technique to design Hamiltonians able to perform a desired dynamics. In particular, we could highlight the application o HIQE for implementing fast and robust quantum gates necessary for quantum information processing [5]. However, we can find many others interesting applications of both HIQE and shortcuts to adiabaticity techniques, for example in fast transfer/inversion population in nitrogen-vacancy systems [6], in Rydberg atoms [7] and trapped ions [8], as well as applications in two level systems coupled to decohering reservoirs [9, 10, 11, 12], quantum computation [13, 14, 15, 16], thermal machines [17, 18, 19, 20] and others [21, 22, 23, 24, 25].

In this paper we will use HIQE, where no shortcut to adiabaticity is performed, in order to obtain a large class of two-level system Hamiltonians able to drive a quantum system from input state $|\psi_{\text{inp}}\rangle$ to an output one $|\psi_{\text{our}}\rangle$, where $|\psi_{\text{our}}\rangle$ is output of some quantum algorithm (in our case, Deutsch and Grover's algorithm output state). In particular we design Hamiltonians associated to Deutsch

Ref

4. L.-A. Wu, J. Sun, and J.-Y. Zhong, Physics Letters A 183, 257 (1993).

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and Grover's algorithm. Remarkable we show how HIQE allow us to obtain feasible and time-independent Hamiltonians for implementing such algorithm.

II. Hamiltonian Inverse Quantum Engineering (HIQE)

When we start our studies on quantum mechanics, we learn that the dynamics of a quantum system is dictated by Schrdinger equation

$$i\hbar|\dot{\psi}(t)\rangle = H(t)|\psi(t)\rangle, \tag{1}$$

where $H(t)$ is the Hamiltonian of the system. From this equation, our aim is to solve it in order to find the evolved state $|\psi(t)\rangle$ of the system. Thus, given a Hamiltonian $H(t)$, the problem is to determinate how our system evolves. If we are interested to find a dynamics in particular, obviously we need to solve the above equation for many Hamiltonians until obtaining the desired dynamics. However, sometimes this can be a very hard task, so that we can use HIQE in order to solve this problem.

We can think about HIQE as a method for obtaining Hamiltonians able to drive a quantum system from a input state $|\psi(0)\rangle$ to a target state $|\psi(\tau)\rangle$ through a path $|\psi(t)\rangle$. So, given an evolved state $|\psi(t)\rangle$, we can use HIQE for finding the Hamiltonian $H(t)$ ables to perform this dynamics. In fact, let us write $|\psi(t)\rangle = U(t)|\psi(0)\rangle$, where $U(t)$ is a known unitary quantum operator called *evolution operator*, we can show that the Hamiltonian $H(t)$ associated with $U(t)$ is obtained from equation [4, 22, 23]

$$H(t) = i\hbar\dot{U}(t)U^{\dagger}(t). \tag{2}$$

The operator $U(t)$ has been considered in literature with different proposals. Furthermore, in this paper we are interested in a particular definition of the operator $U(t)$ as discussed in Ref. [5], where $U(t)$ is written as

$$U(t) = \sum\nolimits_{n} e^{i\varphi_n(t)}|\phi_n(t)\rangle\langle\phi_n(t)|, \tag{3}$$

where $|\phi_n(t)\rangle$ constitutes an orthonormal bases for the Hilbert space associated with the system and $\varphi_n(t)$ are real free parameters. We can see that $U(t)$ satisfies the unitarity condition $U(t)U^{\dagger}(t) = \mathbb{1}$, for any set of parameters $\varphi_n(t)$, and it satisfies the initial condition $U(0) = \mathbb{1}$ if we impose initial conditions for the parameters $\varphi_n(t)$ given by $\varphi_n(0) = 2m\pi$ for $m \in \mathbb{Z}$. As it was showed in Ref. [5], from the operator defined in Eq. (3) we can find Hamiltonians able to implement quantum gates.

It is important to highlight that we can implement quantum gates from others approaches of HIQE and definitions of the operator $U(t)$. But, as it was discussed in Ref. [5], these others protocols request physical system with dimension $d \geq 4$, two-qubit interaction and auxiliary qubits. For example, a good definition of the operator $U(t)$ has been considered in Ref. [7], where additional free parameters can be used for providing experimentally feasible Hamiltonians. However, if we use such method for implement single-quantum gates, for example, we need four-level system. On the other hand, by using the operator in Eq. (3), such gate can be performed in two-level systems. For this reason, we will consider the definition in Eq. (3) throughout this paper.

a) Implementing single-qubit quantum gates by HIQE

Let us consider an arbitrary input state $|\psi(0)\rangle = a|0\rangle + b|1\rangle$, where without less of generality we put $a \in \mathbb{R}$ and $b \in \mathbb{C}$. If we let the system evolves through the operator $U(t)$ from Eq. (3), with $\varphi_1(t) = 0$, $\varphi_2(t) = \varphi(t)$ and

Ref

5. A. C. Santos, J. Phys. B: At. Mol. Opt. Phys. 51, 015501 (2018).

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$$|\phi_1(t)\rangle = \cos[\theta(t)/2]|0\rangle + e^{i\Omega(t)}\sin[\theta(t)/2]|1\rangle, \tag{4a}$$

$$|\phi_2(t)\rangle = -\sin[\theta(t)/2]|0\rangle + e^{i\Omega(t)}\cos[\theta(t)/2]|1\rangle, \tag{4b}$$

with $\theta(t)$ and $\Omega(t)$ being real free parameters, at time $t > 0$ the evolved state $|\psi(t)\rangle$ will be given by

$$|\psi(t)\rangle = U_1(t)|\psi_{\rm inp}\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle, \tag{5}$$

where the coefficients $\alpha(t)$ and $\beta(t)$ are given, respectively by

$$\alpha(t) = \frac{a\sigma_+(t) - \sigma_-(t)\tilde{\alpha}(t)}{2} \quad , \quad \beta(t) = \frac{b\sigma_+(t) + \sigma_-(t)\tilde{\beta}(t)}{2}, \tag{6}$$

where we define $\sigma_\pm(t) = (e^{i\varphi(t)} \pm 1)$, $\tilde{\alpha}(t) = a\cos\theta(t) + be^{-i\phi(t)}\sin\theta(t)$ and $\tilde{\beta}(t) = b\cos\theta(t) - ae^{i\phi(t)}\sin\theta(t)$. Thus, we can associate the parameters $\theta(t)$, $\varphi(t)$ and $\Omega(t)$ with an arbitrary rotation of a single-qubit state in Bloch sphere [5], i.e., an arbitrary quantum gate.

The Hamiltonian that evolves the system as in Eq. (5) is obtained from Eq. (2) and it can be written as

$$H_1(t) = \frac{1}{2}\left[\omega_x(t)\sigma_x + \omega_y(t)\sigma_y + \omega_z(t)\sigma_z\right], \tag{7}$$

where

$$\omega_x(t) = (\cos\varphi - 1)\dot{\Omega}\cos\Omega\cos\theta\sin\theta + (\dot{\theta}\cos\theta\sin\varphi + \dot{\varphi}\sin\theta)\cos\Omega$$
$$+ [\dot{\Omega}\sin\theta\sin\varphi + (\cos\varphi - 1)\dot{\theta}]\sin\Omega, \tag{8a}$$

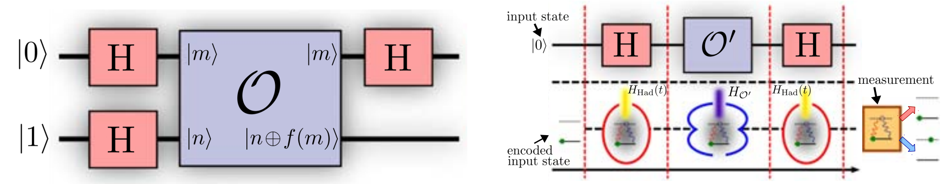
$$\omega_y(t) = (\cos\varphi - 1)\dot{\Omega}\sin\Omega\sin\theta\cos\theta + \sin\Omega(\dot{\theta}\cos\theta\sin\varphi + \dot{\varphi}\sin\theta)$$
$$+ [\dot{\Omega}\sin\theta\sin\varphi - (\cos\varphi - 1)\dot{\theta}]\cos\Omega, \tag{8b}$$

$$\omega_z(t) = -\dot{\theta}\sin\theta\sin\varphi - (\cos\varphi - 1)\dot{\Omega}\sin^2\theta + \dot{\varphi}\cos\theta. \tag{8c}$$

In general, there are systems in which the y-component of the Hamiltonian in Eq. (7) can be hard to implement. For instance, in systems composed by Bose-Einstein condensates in optical lattices [24] and some superconducting circuits [25, 26, 27, 28]. Remarkably, by using our approach we can choose the parameters $\theta(t)$, $\varphi(t)$ and $\Omega(t)$ so that $\omega_y(t) = 0$. In fact, without loss generality we can put $\Omega(t) = 0$ and $\theta(t) = \theta_0 = \text{cte}$, so that $\omega_x(t) = \sin(\theta_0)\dot{\varphi}(t)$, $\omega_y(t) = 0$ and $\omega_z(t) = \cos(\theta_0)\dot{\varphi}(t)$. In conclusion, an arbitrary single-qubit gate can be implemented without two qubit interaction and no additional resource. By using concrete examples (algorithm), in the next sections we will show how we can adequately choose these parameters.

Ref

24. M. G. Bason *et al.*, Nat. Phys. 8, 147 (2012).



(a) Deutsch's circuit (b) Alternative to Deutsch's circuit

Fig. 1: (a) Schematic representation of the Deutsch's circuit. (b) Circuit and schematic representation of two-level system associated to alternative approach presented in this paper

III. Deutsch's Algorithm with Inverse Quantum Engineering

The Deutsch's algorithm is a quantum algorithm used to solve the following problem: *Given a function $f(x) : \{0, 1\} \to \{0, 1\}$, where $f(x)$ is promised to be constant or balanced. How can we show if $f(x)$ is constant or balanced?* In 1980's, David Deutsch proposed an quantum algorithm to solve this problem [29], called *Deutsch's algorithm*. The Deutsch's algorithm can be implemented by using a quantum circuit composed by three (or four, optional) Hadamard gates and an oracle $\mathcal{O}$ that satisfies $\mathcal{O}|n\rangle|m\rangle = n\rangle|n \oplus f(m)\rangle$, as shown in Fig. 1a. In addition, we need two qubits: the *register* qubit, that will be read after circuit action, and an *auxiliary* qubit, that can be discarded.

As we said previous, we are interested to show how we can use HIQE for implementing the Deutsch's algorithm. Different from Ref. [5], here we will not provide Hamiltonians to simulate the quantum gates of the circuit in Fig. 1a. We are interested to consider a protocol in which the Deutsch's algorithm can be implemented through an alternative approach. As a first consequence of the our approach, as shown in Fig. 1b, our scheme is composed by a single-qubit instead two ones. We can think about others approach where we could implement the Deutsch's algorithm using a single-qubit, for example, adiabatic quantum Deutsch's algorithm [30]. Let us describe how our protocol works.

Without loss of generality, we consider that the qubit used in our scheme is initialized in state $0\rangle$ (eigenstate of the σ_z Pauli operator with eigenvalue $+1$). So, we implement an Hadamard gate for obtaining $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. In this step, the Hadamard gate is implemented by using the Hamiltonian in Eq. (7), where the simplest Hamiltonian for such operation is written as [5]

$$H_{\text{Had}}(t) = \frac{\dot{\varphi}(t)}{2\sqrt{2}}\sigma_z + \frac{\dot{\varphi}(t)}{2\sqrt{2}}\sigma_x, \tag{9}$$

where $\varphi(t)$ satisfies $\varphi(\tau) = \pi$. The above Hamiltonian is a Landau-Zener type Hamiltonian and it can be experimentally projected by using quantum dots [31], trapped ion [32] or nuclear magnetic resonance [33], for example.

Once we are using a different approach of the Deutsch's algorithm, here we need to define another oracle. In particular we will define the oracle as in Refs. [30, 34], where we have $\mathcal{O}'|n\rangle = (-1)^{f(n)}|n\rangle$. The evolution operator $U_{\mathcal{O}'}(t)$ used to provide the correct output associate to oracle $\mathcal{O}'$ is given by Eq. (3), with the vectors given by Eq. (4). The initial state of this second step of the protocol is $+\rangle$, so that the evolved state $\psi_2(t)\rangle = U_{\mathcal{O}'}(t)+\rangle$ will be

$$\psi_2(t)\rangle = \frac{1}{2\sqrt{2}}\left[e^{i\varphi_1} + e^{i\varphi_2} - \left(e^{i\varphi_1} - e^{i\varphi_2}\right)\left(\cos\theta + e^{i\Omega}\sin\theta\right)\right]|0\rangle$$

$$+ \frac{1}{2\sqrt{2}}\left[e^{i\varphi_1} + e^{i\varphi_2} + \left(e^{i\varphi_1} - e^{i\varphi_2}\right)\left(e^{i\Omega}\cos\theta - \sin\theta\right)\right]|1\rangle, \tag{10}$$

Ref

29. D. Deutsch, Proc. R. Soc. Lond. A 400, 97 (1985).

therefore, it is easy to show that if we choose the parameters $\Omega(t)$ and $\theta(t)$ so that $\Omega(\tau)=0$ and $\theta(\tau)=\pi$, the output can be written as

$$|\psi_2(\tau)\rangle = \frac{1}{\sqrt{2}}\left[e^{i\varphi_1(\tau)}|0\rangle + e^{i\varphi_2(\tau)}|1\rangle\right], \quad (11)$$

where we can use $\varphi_1(t)$ and $\varphi_2(t)$ to encode the function $f:\{0,1\}\rightarrow\{0,1\}$ as $\varphi_1(\tau)=\pi f(0)$ and $\varphi_2(\tau)=\pi f(1)$. Now, by using that $e^{i\pi f(n)}=(-1)^{f(n)}$, we can write

$$|\psi_2(\tau)\rangle = \frac{1}{\sqrt{2}}\left[(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right]. \quad (12)$$

We can note that $|\psi_2(\tau)\rangle$ is exactly $\mathcal{O}'|+\rangle$. Now, we can study the Hamiltonian that implements this dynamics. We note that the parameters $\Omega(t)$, $\theta(t)$, $\varphi_1(t)$ and $\varphi_2(t)$ should satisfy some boundary conditions, but we have not any condition about their time-dependence. Hence, as previous discussed, we can use this fact to provide feasible Hamiltonians. Firstly, we choose $\varphi_1(t)=\pi f(0)$ and $\varphi_2(t)=\pi f(1)$, and from Eq. (2) we get the oracle Hamiltonian $H_{\mathcal{O}'}(t)$ as in Eq. (7) where

$$\omega_x(t) = 2\sin^2\frac{F\pi}{2}\left[\dot{\Omega}(t)\cos\Omega(t)\sin\theta(t)\cos\theta(t) + \sin\Omega(t)\dot{\theta}(t)\right], \quad (13a)$$

$$\omega_y(t) = 2\sin^2\frac{F\pi}{2}\left[\cos\Omega(t)\dot{\theta}(t) - \dot{\Omega}(t)\sin\Omega(t)\sin\theta(t)\cos\theta(t)\right], \quad (13b)$$

$$\omega_z(t) = 2\dot{\Omega}(t)\sin^2\frac{F\pi}{2}\sin^2\theta(t), \quad (13c)$$

where $F=(-1)^{f(0)}-(-1)^{f(1)}$. Therefore, we can adjust the functions $\Omega(t)$ and $\theta(t)$ in order to obtain the simplest Hamiltonian. For example, because the $\Omega(t)$ and $\theta(t)$ needs to satisfy $\Omega(\tau)=0$ and $\theta(\tau)=\pi$, we can put $\Omega(t)=0$ and $\theta(t)=\pi t/\tau$. In this case we get the time-independent Hamiltonian

$$H_{\mathcal{O}'} = \frac{\hbar}{\tau}\sin^2\frac{F\pi}{2}\sigma_y\,. \quad (14)$$

It is important to highlight the role of F above. Note that if we have a constant function, so $F=0$, hence $H_{\mathcal{O}'}=0$. But this is not a problem of the theory, it is a trivial result of the protocol. In fact, since the input state of the second step is $|+\rangle$, an oracle associated with a constant f can be simulated without any dynamics. It is important to mention that the information about f should be encoded in the Hamiltonian. In addition, such result is not a particular characteristic of our approach, it is also present in adiabatic version of the Deutsch's algorithm [30].

To discuss about the last step of the protocol, we need to choose basis in which we will perform the measurement. If we want to measure the system in computational basis $\{|0\rangle, 1\rangle\}$, we need to apply a Hadamard gate. If we will measure the state in σ_x basis, $|\pm\rangle = (\ 0\rangle \pm |1\rangle)/\sqrt{2}$, no additional Hadamard gate need to be applied. In fact, let us consider the measurement in basis $|\pm\rangle$, by rewriting $\psi_2(\tau)\rangle$ in such basis, we get

$$\psi_2(\tau)\rangle = \frac{(-1)^{f(0)}+(-1)^{f(1)}}{2}|+\rangle + \frac{(-1)^{f(0)}-(-1)^{f(1)}}{2}|-\rangle. \quad (15)$$

where the result is $|+\rangle$ if f is constant, otherwise the result is $|-\rangle$.

Ref

30. R. Harris *et al.*, Phys. Rev. B 82, 024511 (2010).

IV. Search Algorithm with Inverse Quantum Engineering

To provide a more practical example of a quantum algorithm that can be implemented with this approach, in this section we are interested to provide Hamiltonians for implementing the search algorithm. This algorithm was devised by Lov Grover in 1990's [35, 36], where the problem solved was: *given an disordered database with N entires, one marked element $|m\rangle$ can be efficiently found (high probability) by using quantum mechanics*. In his paper, Grover considered an circuit composed by Hadamard gates and an oracle. Here we will make a different approach, where we will present Hamiltonians able to simulate such circuit. However, a detailed and good discussion about the original proposal of Grover's algorithm (search algorithm) can be found in Ref. [33].

In general, we can consider a input state for the Grover's algorithm as an n-qubit state $|0\rangle^{\otimes n} = |0\rangle_1|0\rangle_2 \cdots |0\rangle_n$. Thus, the first step is creating a *uniform* distribution of all element of the disordered list, where we apply the Hadamard gate to each qubit and we get $|\psi\rangle = |+\rangle^{\otimes n}$. It is common we represent $|\psi\rangle$ in decimal basis $|k\rangle = \{|0\rangle, |1\rangle, \cdots, |N-1\rangle\}$, where $N = 2^n$ and each state $|k\rangle$ represents $|0\rangle = |0\rangle^{\otimes n}$, $|1\rangle = |0\rangle^{\otimes n-1}|1\rangle$, $|2\rangle = |0\rangle^{\otimes n-2}|1\rangle|0\rangle$, $|3\rangle = |0\rangle^{\otimes n-2}|1\rangle|1\rangle$ and so on. Therefore, the state $|\psi\rangle$ is written as

$$|\psi_{\text{inp}}\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle. \tag{16}$$

From this representation, we can map our n-qubit system into a hypothetic single-qubit system. Such mapping provide us a simple way to treat our study and it is used in others situations [33, 37, 38]. Based on this representation, we can write the state in Eq. (16) as

$$|\psi_{\text{inp}}^{\text{Grov}}\rangle = \frac{\sqrt{N-1}}{\sqrt{N}}|m^{\perp}\rangle + \frac{1}{\sqrt{N}}|m\rangle. \tag{17}$$

where we define the marked state $|m\rangle$ and $|m^{\perp}\rangle$, with $|m^{\perp}\rangle$ being composed by a uniform combination of all unmarked state, i.e., $|m^{\perp}\rangle = (1/\sqrt{N-1}) \sum_{m\neq k} |k\rangle$. Thus, if we perform a measurement on the system, the probability p_m of obtaining $|m\rangle$ is $p_m = 1/N$, so that for $N \gg 1$, we have $p_m \ll 1$. To obtain an efficient protocol we need to drive $|\psi_{\text{inp}}^{\text{Grov}}\rangle$ to another state $|\psi_{\text{out}}\rangle$ in which $p_m^{\text{out}} \approx 1$.

To give a geometric representation of how our scheme works, consider the Fig. 2. We define the parameter α such that $\cos\alpha = \sqrt{(N-1)/N}$, in this case we get

$$|\psi_{\text{inp}}^{\text{Grov}}\rangle = \cos\alpha|m^{\perp}\rangle + \sin\alpha|m\rangle, \tag{18}$$

From Fig. 2 we can note that if we want to obtain an output state $|\psi_{\text{out}}^{\text{Grov}}\rangle$ with $p_m^{\text{out}} > p_m$, we should drive the system from $|\psi_{\text{inp}}^{\text{Grov}}\rangle$ to

$$|\psi_{\text{out}}^{\text{Grov}}\rangle = \cos\alpha^{\text{out}}|m^{\perp}\rangle + \sin\alpha^{\text{out}}|m\rangle, \tag{19}$$

where $\alpha^{\text{out}} > \alpha$. From definition of the parameter α in Eq. (18), we can see that $\alpha \approx 0$, therefore, for getting $p_m^{\text{out}} \approx 1$, we should be able to achieve $\alpha^{\text{out}} \approx \pi/2$.

We can show that our approach allow us to achieve this task by using the evolution operator $U(t)$ given in Eq. (3). In fact, by writing $U(t)$ in basis $\{|m\rangle, |m^{\perp}\rangle\}$ with

$$|\phi_1(t)\rangle = \cos[\theta(t)/2]|m^{\perp}\rangle + e^{i\Omega(t)}\sin[\theta(t)/2]|m\rangle, \tag{20a}$$

Ref

33. D. Deutsch, Proc. R. Soc. Lond. A 400, 97 (1985).

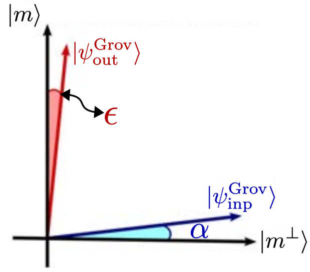


Fig. 2: Geometrical representation of the bi-dimensional Grover's algorithm mapping

$$|\phi_2(t)\rangle = -\sin[\theta(t)/2]|m^{\perp}\rangle + e^{i\Omega(t)}\cos[\theta(t)/2]|m\rangle, \tag{20b}$$

and by choosing $\varphi_1(t) = 0$, $\varphi_2(t) = \varphi(t)$ and $\Omega(t) = 0$, we get the evolved state

$$|\psi^{\text{Grov}}(t)\rangle = \frac{1}{2}\left[(1+e^{i\varphi(t)})\cos\alpha + (1-e^{i\varphi(t)})\cos[\alpha-\theta(t)]\right]|m^{\perp}\rangle$$
$$+\frac{1}{2}\left[(1+e^{i\varphi(t)})\sin\alpha - (1-e^{i\varphi(t)})\sin[\alpha-\theta(t)]\right]|m\rangle. \tag{21}$$

Remarkably, note that if we impose $|\psi^{\text{Grov}}(\tau)\rangle = |\psi_{\text{out}}^{\text{Grov}}\rangle$, the parameter $\varphi(t)$ in above equation could be picked so that $\varphi(\tau) = \pi$, and the final state $|\psi^{\text{Grov}}(\tau)\rangle$ is written as in Eq. (19), where $\alpha^{\text{out}} = \alpha - \theta(\tau)$. To end, by computing the probability p_m^{out} we find $p_m^{\text{out}} = \sin^2[\alpha - \theta(\tau)]$. Our result shows that there are infinity choices of $\theta(\tau)$ where $p_m^{\text{out}} \approx 1$. More specifically, by imposing $\sin^2[\alpha - \theta(\tau)] \approx 1$, we find

$$\theta(\tau) \approx (n+1/2)\pi + \alpha = \left(a + \frac{1}{2}\right)\pi + \arccos\left[\sqrt{(N-1)/N}\right], \tag{22}$$

for any integer a. Moreover, in limit $N \to \infty$ we have $\theta(\tau) \to (a+1/2)\pi$, where $\theta(\tau)$, as well as $\theta(t)$, is independent on the number of elements of the database. This result shows that we are able to implement the Grover algorithm with an arbitrary probability $1-\epsilon^2$ from a careful choice of the parameter $\theta(\tau)$. In fact, by taking p_m^{out} around $\alpha - \theta(\tau) \approx \pi/2$, we get $p_m^{\text{out}} = 1 - [\alpha - \theta(\tau) - \pi/2]^2$, where we can identify $\epsilon = \alpha - \theta(\tau) - \pi/2$.

To find the Hamiltonian, we start from Eq. (2). We can show that, in basis $\{|m\rangle, |m^{\perp}\rangle\}$, the Hamiltonian is written as in Eq. (7) with

$$\omega_x(t) = \dot{\varphi}(t)\sin\theta(t) - \dot{\theta}(t)\cos\theta(t)\sin\varphi(t), \tag{23a}$$

$$\omega_y(t) = 2\dot{\theta}(t)\sin^2[\varphi(t)/2], \tag{23b}$$

$$\omega_z(t) = \dot{\varphi}(t)\cos\theta(t) - \dot{\theta}(t)\sin\theta(t)\sin\varphi(t), \tag{23c}$$

where $\theta(t)$ needs to satisfy the Eq. (22) and $\varphi(t)$ should satisfy $\varphi(0) = 0$ and $\varphi(\tau) = \pi$. In particular, by putting $\theta(t) = \text{cte}$ we obtain $\omega_y(t) = 0$, but now we will not take into account any consideration.

V. Conclusion

In this paper we have considered the role of Hamiltonian inverse engineering when we wish to implement quantum algorithm. Since such approach is a robust protocol against systematic errors [5], such algorithm can be efficiently performed at finite

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5. A. C. Santos, J. Phys. B: At. Mol. Opt. Phys. 51, 015501 (2018).

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time. Remarkably, as we showed, the Grover algorithm can be effectively implemented with arbitrary probability though a single quantum evolution. In addition, as it can be obtained from others schemes of Grover algorithm [30, 34], no auxiliary qubits are required and we can use single qubit analysis (from two-dimensional Grover's algorithm version). Since the robustness of our protocol was carefully studied in the literature [5], we believe that our approach constitutes a robust scheme for providing high fidelity dynamics and successful implementations of the algorithm studied in this paper.

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Notes

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An Unified Study of Some Multiple Integrals

By FY. AY. Ant

Abstract- In this paper, we first evaluate unified finite multiple integrals whose integrand involves the product of the generalized hypergeometric function, ${}_PF_Q$ general class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$, the series expansion of multivariable A-function, a sequence of functions and the multivariable I-function. The arguments occurring in the integrand involve the product of factors of the form $z^{\rho-1}(a-x)^{\sigma}\{1+(bx)^l\}^{-\lambda}$ while that of ${}_PF_Q$, occurring herein involves a finite series of such coefficients. On account of the most general nature of the functions happening in the integrand of our integral, a large number of new and known integrals can be obtained from it merely by specializing the functions and parameters involved here. At the end, we shall see two corollaries.

Keywords: multivariable A-function, a sequence of functions, multiple integrals, multivariable I-function, class of multivariable polynomials, H-function, generalized hypergeometric function, A-function.

GJSFR-F Classification: *MSC 2010: 33C99, 33C60, 44A20*

ANUNIFIEDSTUDYOFSOMEMULTIPLEINTEGRALS

Strictly as per the compliance and regulations of:

5. K. C. Gupta and R.Jain, A unified study of some multiple integrals, Soochow Journal of Mathematics, 19(1) (1993), 73-81.

An Unified Study of Some Multiple Integrals

FY. AY. Ant

Abstract- In this paper, we first evaluate unified finite multiple integrals whose integrand involves the product of the generalized hypergeometric function, ${}_PF_Q$ general class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$, the series expansion of multivariable A-function, a sequence of functions and the multivariable I-function. The arguments occurring in the integrand involve the product of factors of the form $z^{\rho-1}(a-x)^{\sigma}\{1+(bx)^l\}^{-\lambda}$ while that of ${}_PF_Q$ occurring herein involves a finite series of such coefficients. On account of the most general nature of the functions happening in the integrand of our integral, a large number of new and known integrals can be obtained from it merely by specializing the functions and parameters involved here. At the end, we shall see two corollaries.

Keywords: multivariable A-function, a sequence of functions, multiple integrals, multivariable I-function, class of multivariable polynomials, H-function, generalized hypergeometric function, A-function.

I. Introduction and Preliminaries

Gupta and Jain [5] have studied unified multiple integrals involving the generalized hypergeometric function, class of multivariable polynomials [9] and multivariable H-function [13,14]. The aim of this paper is to establish a general finite multiple integrals about the generalized hypergeometric function, sequence of functions, general class of multivariable polynomials, the series expansion of the A-function [4] and multivariable I-function defined by Prasad [6].

For this study, we need the following series formula for the general sequence of functions introduced by Agrawal and Chaubey [1] and was established by Salim [7].

$$R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}]=\sum_{w,v',u,t',e,k_1,k_2}\psi(w,v',u,t',e,k_1,k_2)x^R \tag{1.1}$$

where $\psi(w,v',u,t',e,k_1,k_2)=\dfrac{(-)^{t'+w+k_2}(-v')_u(-t')_e(\alpha)_t l^n}{w!v'!u!t'!e!l'_n k_1!k_2!}\dfrac{\mathfrak{s}^{w+k_1}F^{\gamma n-t'}}{(1-\alpha-t')_e}(-\alpha-\gamma n)_e(-\beta-\delta n)_{v'}$

$$g^{v+k_2}h^{\delta n-v-k_2}\,(v'-\delta n)_{k_2}E^{t'}\left(\frac{pe+\tau w+\lambda+qu}{l}\right)_n \tag{1.2}$$

and $\displaystyle\sum_{w,v',u,t',e,k_1,k_2}=\sum_{w=0}^{\infty}\sum_{v'=0}^{n}\sum_{u=0}^{v'}\sum_{t'=0}^{n}\sum_{e=0}^{t'}\sum_{k_1,k_2=0}^{\infty}$

The infinite series on the right-hand side of (1.3) is convergent and $R=ln+qv+pt'+\tau w+\tau k_1+k_2q$
We shall note $R_n^{\alpha,\beta}[x;E,F,g,h;p,q;\gamma;\delta;e^{-\mathfrak{s}x^{\tau}}]=R_n^{\alpha,\beta}(x)$ (1.3)

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The generalized multivariablepolynomials defined by Srivastava [9], is given in the following manner :

$$S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_{\mathfrak{v}}}_{N_1,\cdots,N_v}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_{\mathfrak{v}}]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_{\mathfrak{v}}K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \qquad (1.4)$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_{\mathfrak{v}}$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are arbitrary constants Real or complex. On suitably specializing the coefficients, $A[N_1,K_1;\cdots;N_v,K_v]$, $S^{\mathfrak{M}_1,\cdots,\mathfrak{M}_{\mathfrak{v}}}_{N_1,\cdots,N_v}[y_1,\cdots,y_v]$ yields a Number of known polynomials, the Laguerre polynomials, the Jacobi polynomials, and several other ([15], page. 158-161]. We shall note

$$a_v=\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_{\mathfrak{v}}K_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v] \qquad (1.5)$$

The series representation of the multivariable A-function is given by Gautam [4] as

$$A[u_1,\cdots,u_r]=A^{0,\lambda:(\alpha',\beta');\cdots;(\alpha^{(r)},\beta^{(r)})}_{A,C:(M',N');\cdots;(M^{(r)},N^{(r)})}\left(\begin{array}{c|l} \mathrm{u}_1 & [(\mathrm{g}_j);\gamma',\cdots,\gamma^{(r)}]_{1,A}: \\ \cdot & \quad\cdots \\ \cdot & \\ \mathrm{u}_r & [(\mathrm{f}_j);\xi',\cdots,\xi^{(r)}]_{1,C}: \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{q}^{(1)},\eta^{(1)})_{1,M^{(1)}};\cdots;(q^{(r)},\eta^{(r)})_{1,M^{(r)}} \\ \cdots \\ \cdots \\ (\mathrm{p}^{(1)},\epsilon^{(1)})_{1,N^{(1)}};\cdots;(p^{(r)},\epsilon^{(r)})_{1,N^{(r)}} \end{array}\right)=\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi\frac{\prod_{i=1}^r\phi_i u_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r\epsilon^{(i)}_{G_i}g_i!} \qquad (1.6)$$

where

$$\phi=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-g_j+\sum_{i=1}^r\gamma_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\lambda'+1}^{A}\Gamma\left(g_j-\sum_{i=1}^r\gamma_j^{(i)}U_i\right)\prod_{j=1}^{C}\Gamma\left(1-f_j+\sum_{i=1}^r\xi_j^{(i)}\eta_{G_i,g_i}\right)} \qquad (1.7)$$

$$\phi_i=\frac{\prod_{j=1,j\neq m_i}^{\alpha^{(i)}}\Gamma\left(p_j^{(i)}-\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=1}^{\beta^{(i)}}\Gamma\left(1-q_j^{(i)}+\eta_j^{(i)}\eta_{G_i,g_i}\right)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}}\Gamma\left(1-p_j^{(i)}+\epsilon_j^{(i)}\eta_{G_i,g_i}\right)\prod_{j=\beta^{(i)}+1}^{M^{(i)}}\Gamma\left(q_j^{(i)}-\eta_j^{(i)}\eta_{G_i,g_i}\right)},i=1,\cdots,r \qquad (1.8)$$

and

$$\eta_{G_i,g_i}=\frac{p_{G_i}^{(i)}+g_i}{\epsilon_{G_i}^{(i)}},i=1,\cdots,r \qquad (1.9)$$

which is valid under the following conditions : $\epsilon_{m_i}^{(i)}[p_j^{(i)}+p_i']\neq\epsilon_j^{(i)}[p_{m_i}+g_i]$
and

$$u_i\neq 0,\sum_{j=1}^{A}\gamma_j^{(i)}-\sum_{j=1}^{C}\xi_j^{(i)}+\sum_{j=1}^{M^{(i)}}\eta_j^{(i)}-\sum_{j=1}^{N^{(i)}}\epsilon_j^{(i)}<0,i=1,\cdots,r \qquad (1.10)$$

Here $\lambda,A,C,\alpha_i,\beta_i,m_i,n_i\in\mathbb{N}^*;i=1,\cdots,r;f_j,g_j,p_j^{(i)},q_j^{(i)},\gamma_j^{(i)},\xi_j^{(i)},\eta_j^{(i)},\epsilon_j^{(i)}\in\mathbb{C}$

Ref

9. H. M. Srivastava, A multilinear generating function for the Konhauser set of biorthogonal polynomials suggested by Laguerre polynomial, Pacific. J. Math. 177(1985), 183-191.

The multivariable I-function of s-variables defined by Prasad [6] generalizes the multivariable H-function defined by Srivastava and Panda [13,14]. This representation of multiple Mellin-Barnes types integral is:

$$I(z'_1,\cdots,z'_s) = I^{0,n'_2;0,n'_3;\cdots;0,n'_s:m'^{(1)},n'^{(1)};\cdots;m'^{(s)},n'^{(s)}}_{p'_2,q'_2,p'_3,q'_3;\cdots;p'_s,q'_s:p'^{(1)},q'^{(1)};\cdots;p'^{(s)},q'^{(s)}}\left(\begin{array}{c|c} z'_1 & (\mathrm{a}'_{2j};\alpha'^{(1)}_{2j},\alpha'^{(2)}_{2j})_{1,p_2};\cdots; \\ \cdot & \\ \cdot & \\ \cdot & \\ z'_s & (\mathrm{b}'_{2j};\beta'^{(1)}_{2j},\beta'^{(2)}_{2j})_{1,q_2};\cdots; \end{array}\right.$$

$$\left.\begin{array}{c} (\mathrm{a}'_{sj};\alpha'^{(1)}_{sj},\cdots,\alpha'_{sj}{}^{(s)})_{1,p'_s}:(a'^{(1)}_j,\alpha'^{(1)}_j)_{1,p'^{(1)}};\cdots;(a'_j{}^{(s)},\alpha'_j{}^{(s)})_{1,p'^{(s)}} \\ \\ (\mathrm{b}'_{rj};\beta'^{(1)}_{sj},\cdots,\beta'_{sj}{}^{(s)})_{1,q_s}:(b'^{(1)}_j,\beta'^{(1)}_j)_{1,q'^{(1)}};\cdots;(b'_j{}^{(s)},\beta'_j{}^{(s)})_{1,q'^{(s)}} \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^s}\int_{L'_1}\cdots\int_{L'_s}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\phi_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.11}$$

where

$$\phi_i(t_i) = \frac{\prod_{j=1}^{m'^{(i)}}\Gamma(b'^{(i)}_j-\beta'^{(i)}_jt_i)\prod_{j=1}^{n'^{(i)}}\Gamma(1-a'^{(i)}_j+\alpha'^{(i)}_jt_i)}{\prod_{j=m'^{(i)}+1}^{q'^{(i)}}\Gamma(1-b'^{(i)}_j+\beta'^{(i)}_jt_i)\prod_{j=n'^{(i)}+1}^{p'^{(i)}}\Gamma(a'^{(i)}_j-\alpha'^{(i)}_jt_i)}\ ,i=1,\cdots,s \tag{1.12}$$

and

$$\phi(t_1,\cdots,t_s) = \frac{\prod_{j=1}^{n'_2}\Gamma(1-a'_{2j}+\sum_{i=1}^{2}\alpha'^{(i)}_{2j}t_i)\prod_{j=1}^{n'_3}\Gamma(1-a'_{3j}+\sum_{i=1}^{3}\alpha'^{(i)}_{3j}t_i)\cdots}{\prod_{j=n'_2+1}^{p_2}\Gamma(a'_{2j}-\sum_{i=1}^{2}\alpha'^{(i)}_{2j}t_i)\prod_{j=n'_3+1}^{p'_3}\Gamma(a'_{3j}-\sum_{i=1}^{3}\alpha'^{(i)}_{3j}t_i)\cdots}$$

$$\frac{\cdots\prod_{j=1}^{n'_s}\Gamma(1-a'_{sj}+\sum_{i=1}^{s}\alpha'^{(i)}_{sj}t_i)}{\cdots\prod_{j=n'_s+1}^{p'_s}\Gamma(a'_{sj}-\sum_{i=1}^{s}\alpha'^{(i)}_{sj}t_i)\prod_{j=1}^{q'_2}\Gamma(1-b'_{2j}-\sum_{i=1}^{2}\beta'^{(i)}_{2j}t_i)}$$

$$\times\frac{1}{\prod_{j=1}^{q'_3}\Gamma(1-b'_{3j}+\sum_{i=1}^{3}\beta'^{(i)}_{3j}t_i)\cdots\prod_{j=1}^{q'_s}\Gamma(1-b'_{sj}-\sum_{i=1}^{s}\beta'^{(i)}_{sj}t_i)} \tag{1.13}$$

About the above integrals and these existence and convergence conditions, see Prasad [4] for more details. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. We have:

$|argz'_i| < \frac{1}{2}\Omega'_i\pi$, where

$$\Omega'_i = \sum_{k=1}^{n'^{(i)}}\alpha'_k{}^{(i)} - \sum_{k=n'^{(i)}+1}^{p'^{(i)}}\alpha'_k{}^{(i)} + \sum_{k=1}^{m'^{(i)}}\beta'_k{}^{(i)} - \sum_{k=m'^{(i)}+1}^{q'^{(i)}}\beta'_k{}^{(i)} + \left(\sum_{k=1}^{n'_2}\alpha'_{2k}{}^{(i)} - \sum_{k=n'_2+1}^{p'_2}\alpha'_{2k}{}^{(i)}\right) +$$

$$+\cdots+\left(\sum_{k=1}^{n'_s}\alpha'^{(i)}_{sk} - \sum_{k=n'_s+1}^{p'_s}\alpha'^{(i)}_{sk}\right) - \left(\sum_{k=1}^{q'_2}\beta'^{(i)}_{2k} + \sum_{k=1}^{q'_3}\beta'^{(i)}_{3k} + \cdots + \sum_{k=1}^{q'_s}\beta'^{(i)}_{sk}\right) \tag{1.14}$$

Ref

6. Y. N. Prasad, Multivariable I-function, Vijnana Parishad Anusandhan Patrika 29 (1986), 231-237.

The complex numbers z_i are not zero.Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function.

We may establish the asymptotic expansion in the following convenient form :

$$I(z'_1,\cdots,z'_s)=0(\,|z'_1|^{\alpha'_1},\cdots,|z'_s|^{\alpha'_s}\,)\,,max(\,|z'_1|,\cdots,|z'_s|\,)\to 0$$

$$I(z'_1,\cdots,z'_s)\;=0(\,|z'_1|^{\beta'_1},\cdots,|z'_s|^{\beta'_s}\,)\,,min(\,|z'_1|,\cdots,|z'_s|\,)\to\infty$$

where: $k=1,\cdots,s:\alpha''_k=min[Re(b_j'^{(k)}/\beta_j'^{(k)})],j=1,\cdots,m'_k$ and

$$\beta''_k=max[Re((a_j'^{(k)}-1)/\alpha_j'^{(k)})],j=1,\cdots,n'_k$$

II. Main Integral

We have the following unified multiple integrals formula.

Theorem

$$\int_0^{a_1}\cdots\int_0^{a_t}\prod_{l=1}^{t}\left[x_l^{\rho_l-1}(a_l-x_l)^{\sigma_l}\left\{1+(b_lx_l)^{g_l}\right\}^{-\lambda_l}\right]R_n^{\alpha,\beta}\left[y\prod_{l=1}^{t}\left[x_l^{e_l}(a_l-x_l)^{f_l}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l}\right]\right]$$

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\left(\begin{array}{c} \mathrm{y}_1\left[\prod_{l=1}^{t}\left[x_l^{e_l''^{(1)}}(a_l-x_l)^{f_l''^{(1)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l''^{(1)}}\right]\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{y}_v\left[\prod_{l=1}^{t}\left[x_l^{e_l''^{(v)}}(a_l-x_l)^{f_l''^{(v)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l''^{(v)}}\right]\right]\end{array}\right)$$

$$A\left(\begin{array}{c} \mathrm{z}_1\left[\prod_{l=1}^{t}\left[x_l^{e_l'^{(1)}}(a_l-x_l)^{f_l'^{(1)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l'^{(1)}}\right]\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}_r\left[\prod_{l=1}^{t}\left[x_l^{e_l'^{(r)}}(a_l-x_l)^{f_l'^{(r)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l'^{(r)}}\right]\right]\end{array}\right)$$

$$I\left(\begin{array}{c} \mathrm{z}'_1\left[\prod_{l=1}^{t}\left[x_l^{e_l^{(1)}}(a_l-x_l)^{f_l^{(1)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l^{(1)}}\right]\right]\\ \cdot\\ \cdot\\ \cdot\\ \mathrm{z}'_s\left[\prod_{l=1}^{t}\left[x_l^{e_l^{(s)}}(a_l-x_l)^{f_l^{(s)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l^{(s)}}\right]\right]\end{array}\right)$$

Notes

$$ {}_PF_Q\left[(A_P);(B_Q);\sum_{l=1}^{t}B_l x_l^{\mu_l}(a_l-x_l)^{\upsilon_l}\left\{1+(b_lx_l)^{g_l}\right\}^{-\omega_l}\right]\mathrm{d}x_1\cdots\mathrm{d}x_t=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)} $$

$$ \prod_{l=1}^{t}a_l^{\rho_l+\sigma_l}\sum_{w,v',u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi_1\frac{\prod_{i=1}^{r}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{r}g_i}}{\prod_{i=1}^{r}\epsilon_{G_i}^{(i)}g_i!} $$

$$ a_v y_1^{K_1}\cdots y_v^{K_v}\psi(w,v',u,t',e,k_1,k_2)\,y^R\prod_{i=1}^{v}\prod_{l=1}^{t}a_l^{(e_l''^{(i)}+f_l''^{((i)})K_i}\prod_{j=1}^{r}\prod_{l=1}^{t}a^{(e_l'^{(j)}+f_l'^{((j)})\eta_{G_j,g_j}} $$

$$ I_{U:p_s'+3t+P,q_s'+2t+Q;Y}^{V;0,n_s'+3t+P;X}\left(\begin{array}{c|c} \mathrm{z}_1'\prod_{l=1}^{t}a_l^{(e_l^{(1)}+f^{(1)})} & \mathrm{A}\\ \cdots & .\\ \cdots & .\\ \mathrm{z}_s'\prod_{l=1}^{t}a_l^{(e_l^{(s)}+f^{(s)})} & .\\ \text{-}\mathrm{B}_1a_1^{\mu_1+\upsilon_1} & .\\ \cdots & .\\ \cdots & .\\ \text{-}\mathrm{B}_ta_t^{\mu_t+\upsilon_t} & .\\ (\mathrm{a}_1b_1)^{g_1} & .\\ \cdots & .\\ \cdots & \mathrm{B}\\ (\mathrm{a}_tb_t)^{g_t} & \end{array}\right) \qquad (2.1) $$

We obtain a Prasad's I-function of $(s+2t)$-variables.

Provided that

$min\{e_l''^{(i)},f_l''^{(i)},h_l''^{(i)},e_l'^{(j)},f_l'^{(j)},h_l'^{(j)},e_l^{(k)},f_l^{(k)},h_l^{(k)},e_l,f_l,h_l,\mu_l,\upsilon_l,\omega_l\}>0;(i=1,\cdots,u;j=1,\cdots,r;k=1,\cdots,s;l=1,\cdots,t)$

$Re(\lambda_l)>0,g_l>0;l=1,\cdots,t$

$$ Re\left(\rho_l+e_lR+\sum_{j=1}^{r}e_l'^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}e_l^{(k)}\min_{1\leqslant K\leqslant m'^{(k)}}Re\left(\frac{b_K'^{(k)}}{\beta_K'^{(k)}}\right)>0\,;(l=1,\cdots,t) $$

$$ Re\left(\sigma_l+f_lR+\sum_{j=1}^{r}f_l'^{(j)}\eta_{G_j,g_j}\right)+\sum_{k=1}^{s}f_l^{(k)}\min_{1\leqslant K\leqslant m'^{(k)}}Re\left(\frac{b_K'^{(k)}}{\beta_K'^{(k)}}\right)>0\,;(l=1,\cdots,t) $$

$$ z_i\neq 0,\sum_{j=1}^{A}\gamma_j^{(i)}-\sum_{j=1}^{C}\xi_j^{(i)}+\sum_{j=1}^{M^{(i)}}\eta_j^{(i)}-\sum_{j=1}^{N^{(i)}}\epsilon_j^{(i)}<0,i=1,\cdots,r $$

with $\lambda,A,C,\alpha_i,\beta_i,m_i,n_i\in\mathbb{N}^*;i=1,\cdots,r\,;f_j,g_j,p_j^{(i)},q_j^{(i)},\gamma_j^{(i)},\xi_j^{(i)},\eta_j^{(i)},\epsilon_j^{(i)}\in\mathbb{C}$

$$ \left|arg\left(z_k'\prod_{l=1}^{t}\left[(x_l^{e_l^{(k)}}(a_l-x_l)^{f_l^{(k)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l^{(k)}}\right)\right|<\frac{1}{2}\Omega_i''\pi\ \ (a_l\leqslant x_l\leqslant b_l;k=1,\cdots,s;l=1,\cdots,t) $$

Notes

where, $\Omega_i'' = \Omega_i' - (e_l^{(i)} + f_l^{(i)} + h_l^{(i)})$, Ω_i' is defined by (1.14). $P \leqslant Q+1$, and the multiple series on a left-hand side of (2.1) converges absolutely, where

$$U = p_2', q_2'; p_3', q_3'; \cdots ; p_{s-1}', q_{s-1}' \tag{2.2}$$

$$V = 0, n_2'; 0, n_3'; \cdots ; 0, n_{s-1}' \tag{2.3}$$

$$X = m'^{(1)}, n'^{(1)}; \cdots ; m'^{(s)}, n'^{(s)}; 1,0; \cdots ; 1,0; 1,0; \cdots ; 1,0 \tag{2.4}$$

$$Y = p'^{(1)}, q'^{(1)}; \cdots ; p'^{(s)}, q'^{(s)}; 0,1; \cdots ; 0,1; 0,1; \cdots ; 0,1 \tag{2.5}$$

$$A = (a_{2k}'; \alpha_{2k}'^{(1)}, \alpha_{2k}''^{(2)})_{1,p_2}; \cdots ; (a_{(s-1)k}'; \alpha_{(s-1)k}'^{(1)}, \alpha_{(s-1)k}'^{(2)}, \cdots , \alpha_{(s-1)k}'^{(s-1)})_{1,p_{s-1}'};$$

$$(1-\rho_1 - e_1 R - \sum_{i=1}^{v} e_1''^{(i)} K_i - \sum_{j=1}^{r} e_1'^{(j)} \eta_{G_j,g_j}; e_1^{(1)}, \cdots , e_1^{(s)}, \mu_1, \underbrace{0, \cdots , 0}_{t-1}, g_1, \underbrace{0, \cdots , 0}_{t-1}), \cdots ,$$

$$(1-\rho_t - e_t R - \sum_{i=1}^{v} e_t''^{(i)} K_i - \sum_{j=1}^{r} e_t'^{(j)} \eta_{G_j,g_j}; e_t^{(1)}, \cdots , e_t^{(s)}, \underbrace{0, \cdots , 0}_{t-1}, \mu_t, \underbrace{0, \cdots , 0}_{t-1}, g_t),$$

$$(-\sigma_1 - f_1 R - \sum_{i=1}^{v} f_1''^{(i)} K_i - \sum_{j=1}^{r} f_1'^{(j)} \eta_{G_j,g_j}; f_1^{(1)}, \cdots , f_1^{(s)}, \upsilon_1, \underbrace{0, \cdots , 0}_{2t-1}), \cdots ,$$

$$(-\sigma_t - f_t R - \sum_{i=1}^{v} f_t''^{(i)} K_i - \sum_{j=1}^{r} f_t'^{(j)} \eta_{G_j,g_j}; f_t^{(1)}, \cdots , f_t^{(s)}, \underbrace{0, \cdots , 0}_{t-1}, \upsilon_t, \underbrace{0, \cdots , 0}_{t}),$$

$$(1-\lambda_1 - h_1 R - \sum_{i=1}^{v} h_1''^{(i)} K_i - \sum_{j=1}^{r} h_1'^{(j)} \eta_{G_j,g_j}; h_1^{(1)}, \cdots , h_1^{(s)}, \omega_1, \underbrace{0, \cdots , 0}_{t-1}, 1, \underbrace{0, \cdots , 0}_{t-1}), \cdots ,$$

$$(1-\lambda_t - h_t R - \sum_{i=1}^{v} h_t''^{(i)} K_i - \sum_{j=1}^{r} h_t'^{(j)} \eta_{G_j,g_j}; h_t^{(1)}, \cdots , h_t^{(s)}, \underbrace{0, \cdots , 0}_{t-1}, \omega_t, \underbrace{0, \cdots , 0}_{t-1}, 1),$$

$$(1-E_j; \underbrace{0, \cdots , 0}_{s}, \underbrace{1, \cdots , 1}_{t}, \underbrace{0, \cdots , 0}_{t})_{1,P}, \ (a_{sk}'; \alpha_{sk}'^{(1)}, \alpha_{sk}'^{(2)}, \cdots , \alpha_{sk}'^{(s)}, \underbrace{0, \cdots , 0}_{2t})_{1,p_s'} :$$

$$(a_k'^{(1)}, \alpha_k'^{(1)})_{1,p^{(1)}}; \cdots ; (a_k'^{(s)}, \alpha_k'^{(s)})_{1,p'^{(s)}}; \ -; \cdots ; - \tag{2.7}$$

$$B = (b_{2k}'; \beta_{2k}'^{(1)}, \beta_{2k}'^{(2)})_{1,q_2'}; \cdots ; (b_{(s-1)k}'; \beta_{(s-1)k}'^{(1)}, \beta_{(s-1)k}'^{(2)}, \cdots , \beta_{(s-1)k}'^{(s-1)})_{1,q_{s-1}'}; (b_{sk}'; \beta_{sk}'^{(1)}, \beta_{sk}'^{(2)}, \cdots , \beta_{sk}'^{(s)}, \underbrace{0, \cdots , 0}_{2t})_{1,q_s'},$$

$$(\rho_1 - \sigma_1 - (e_1+f_1)R - \sum_{i=1}^{v} (e_1''^{(i)} + f_1''^{(i)}) K_i - \sum_{j=1}^{r} (e_1'^{(j)} + f_1'^{(j)}) \eta_{G_j,g_j}; e_1^{(1)} + f_1^{(1)}, \cdots , e_1^{(s)} + f_1^{(s)}, \mu_1 + \upsilon_1, \underbrace{0, \cdots , 0}_{t-1}, g_1, \underbrace{0, \cdots , 0}_{t-1}), \cdots ,$$

Notes

$$(\rho_t-\sigma_t-(e_t+f_t)R-\sum_{i=1}^{v}(e_t^{\prime\prime(i)}+f_t^{\prime\prime(i)})K_i-\sum_{j=1}^{r}(e_t^{\prime(j)}+f_t^{\prime(j)})\eta_{G_j,g_j};e_t^{(1)}+f_t^{(1)},\cdots,e_t^{(s)}+f_t^{(s)},\underbrace{0,\cdots,0}_{t-1},\mu_t+\upsilon_t,\underbrace{0,\cdots,0}_{t-1},g_t),$$

$$(1-\lambda_1-h_1R-\sum_{i=1}^{v}h_1^{\prime\prime(i)}K_i-\sum_{j=1}^{r}h_1^{\prime(j)}\eta_{G_j,g_j};h_1^{(1)},\cdots,h_1^{(s)},\omega_1,\underbrace{0,\cdots,0}_{2t-1}),\cdots,$$

$$(1-\lambda_t-h_tR-\sum_{i=1}^{v}h_t^{\prime\prime(i)}K_i-\sum_{j=1}^{r}h_t^{\prime(j)}\eta_{G_j,g_j};h_t^{(1)},\cdots,h_t^{(s)},\underbrace{0,\cdots,0}_{t-1},\omega_t,\underbrace{0,\cdots,0}_{t}),$$

$$(1-F_j;\underbrace{0,\cdots,0}_{s},\underbrace{1,\cdots,1}_{t},\underbrace{0,\cdots,0}_{t})_{1,Q}:(b_k^{\prime(1)},\beta_k^{\prime(1)})_{1,q^{\prime(1)}};\cdots;(b_k^{\prime(s)},\beta_k^{\prime(s)})_{1,q^{\prime(s)}};\underbrace{(0;1);\cdots;(0;1)}_{2t} \quad (2.8)$$

and

$$\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=0}^{\infty}=\sum_{G_1,\cdots,G_r=1}^{\alpha^{(1)},\cdots,(r)}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

Proof

To evaluate the multiple integrals (2.1), we first express the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ in series the multivariable A-function $A(z_1,\cdots,z_r)$ in serie, the sequence of functions $R_n^{(\alpha,\beta)}[.]$ in series with the help of equations (1.4), (1.6) and (1.1) respectively.Then we change the order of the multiple series and the $(x_1,\cdots,x_t)$-Integrals. Nest, we express the generalized hypergeometric function ${}_PF_Q[.]$ regarding a generalized Kampé de Fériet function of t-variables with the help of the formula ([11], page.39 Eq. (30)), and express this function of an H-function Of t variables with the help of the result ([12], page. 272, Eq. (4.7)). Next, we express the H-function of t-variables and The I-function of s-variables regarding their respective Mellin-Barnes integrals contour. Now we change the order of the $(t_1,\cdots,t_s),(\eta_1,\cdots,\eta_t)$ and $(x_1,\cdots,x_t)$-integrals which are permissible under the conditions stated with (2.1). Finally, on evaluating the $(x_1,\cdots,x_t)$-integrals thus got with the help of a case of the result ([10], page. 61, Eq. (5.2.1)) and we obtain the following result (say L.H.S.) :

$$\text{L.H.S}=\sum_{w,v',u,t',e,k_1,k_2}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_i=1}^{\alpha^{(i)}}\sum_{g_i=1}^{\infty}\phi\frac{\prod_{i=1}^{r}\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^{r}g_i}}{\prod_{i=1}^{r}\epsilon_{G_i}^{(i)}g_i!}a_v y_1^{K_1}\cdots y_v^{K_v}\psi(w,v',u,t',e,k_1,k_2)$$

$$\frac{1}{(2\pi\omega)^{s+t}}\int_{L_1}\cdots\int_{L_s}\int_{L_{s+1}}\cdots\int_{L_{s+t}}\phi(t_1,\cdots,t_s)\prod_{i=1}^{s}\phi_i(t_i)z_i^{\prime t_i}\frac{\prod_{j=1}^{P}\Gamma(E_j+\sum_{k=1}^{t}\eta_k)}{\prod_{j=1}^{Q}\Gamma(F_j+\sum_{k=1}^{t}\eta_k)}\left[\prod_{l=1}^{t}\{\Gamma(-\eta_l)(-B_l)^{\eta_l}\right.$$

$$a_k^{\rho_l+\sigma_l+(e_l+f_l)R+\sum_{i=1}^{v}(e_l^{\prime\prime(i)}+f_l^{\prime\prime(i)})K_i+\sum_{j=1}^{r}(e_l^{\prime(j)}+f_l^{\prime(j)})\eta_{G_j,g_j}+\sum_{k=1}^{s}(e_l^{(k)}+f_l^{(k)})t_k+(\mu_l+\upsilon_l)\eta_l}$$

$$\frac{\Gamma(1+f_lR+\sigma_l+\sum_{i=1}^{v}f_l^{\prime\prime(i)}K_i+\sum_{j=1}^{r}f_l^{\prime(j)}\eta_{G_j,g_j}+\sum_{k=1}^{s}f_l^{(k)}t_k+\upsilon_l\eta_l)}{\Gamma(\lambda_l+h_lR+\sum_{i=1}^{v}h_l^{\prime\prime(i)}K_i+\sum_{j=1}^{r}h_l^{\prime(j)}\eta_{G_j,g_j}+\sum_{k=1}^{s}h_l^{(k)}t_k+\omega_l\eta_l)}H_{2,2}^{1,2}\left[(\mathrm{b}_l a_l)^{g_j}\left|\begin{matrix}\mathrm{C,D}\\ .\\ \mathrm{E}\end{matrix}\right.\right]\Bigg]$$

$$\mathrm{d}t_1\cdots\mathrm{d}t_s\,\mathrm{d}\eta_1\cdots\mathrm{d}\eta_t$$

where

$$C=(1-\lambda_l-f_lR-\sum_{i=1}^{u}f_l^{\prime\prime(i)}K_i-\sum_{j=1}^{r}f_l^{\prime(j)}\eta_{G_j,g_j}-\sum_{k=1}^{s}f_l^{(k)}t_k-\omega_l\eta_l;1),$$

Ref

11. H. M. Srivastava and P. W. Karlson, Multiple Gaussian hypergeometric series, John, Wiley and Sons (Ellis Horwood Ltd.), New York, 1985.

$$D=(1-\rho_l-h_lR-\sum_{i=1}^{u}g_l^{\prime\prime(i)}K_i-\sum_{j=1}^{r}g_l^{\prime(j)}\eta_{G_j,g_j}-\sum_{k=1}^{s}g_l^{(k)}t_k-\mu_l\eta_l;g_l)\text{ and}$$

$$E=(0;1;1),(-\rho_l-\sigma_l-(f_l+h_l)R-\sum_{i=1}^{u}(f_l^{\prime\prime(i)}+g_l^{\prime\prime(i)})K_i-\sum_{j=1}^{r}(f_l^{\prime(j)}+g_l^{\prime(j)})\eta_{G_j,g_j}-\sum_{k=1}^{s}(f_l^{(k)}+g_l^{(k)})t_k-(\mu_k+\upsilon_k)\eta_l;g_l)$$

Now, if we express the product of the H-functions of one variable occurring in the above expression regarding their respective Mellin-Barnes integrals contour and reinterpreting the result thus obtained regarding the Prasad's I-function Of $(s+2t)$-variables, we arrive at the desired formula after algebraic manipulations.

III. Corollaries and Special Case

If the generalized multivariable polynomials, the multivariable A-function and multivariable I-function reduce respectively to a class of polynomials of one variable [8], A-function defined by Gautam and Asgar [3] and H-function defined by Fox [2], we get the following multiple integrals :

Corollary 1.

$$\int_0^{a_1}\cdots\int_0^{a_t}\prod_{l=1}^{t}\left[x_l^{\rho_l-1}(a_l-x_l)^{\sigma_l}\left\{1+(b_lx_l)^{g_l}\right\}^{-\lambda_l}\right]R_n^{\alpha,\beta}\left[y\prod_{l=1}^{t}\left[x_l^{e_l}(a_l-x_l)^{f_l}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l}\right]\right]$$

$$S_{N_1}^{\mathfrak{M}_1}\left(\ \mathrm{y}_1\left[\prod_{l=1}^{t}\left[x_l^{e_l^{\prime\prime(1)}}(a_l-x_l)^{f_l^{\prime\prime(1)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l^{\prime\prime(1)}}\right]\right]\ \right)$$

$$A\left(\ \mathrm{z}_1\left[\prod_{l=1}^{t}\left[x_l^{e_l^{\prime(1)}}(a_l-x_l)^{f_l^{\prime(1)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l^{\prime(1)}}\right]\right]\ \right)$$

$$H\left(\ \mathrm{z}_1'\left[\prod_{l=1}^{t}\left[x_l^{e_l^{(1)}}(a_l-x_l)^{f_l^{(1)}}\left\{1+(b_lx_l)^{g_l}\right\}^{-h_l^{(1)}}\right]\right]\ \right)$$

$${}_PF_Q\left[(A_P);(B_Q);\sum_{l=1}^{t}B_lx_l^{\mu_l}(a_l-x_l)^{\upsilon_l}\left\{1+(b_lx_l)^{g_l}\right\}^{-\omega_l}\right]\mathrm{d}x_1\cdots\mathrm{d}x_t=\frac{\prod_{j=1}^{Q}\Gamma(B_j)}{\prod_{j=1}^{P}\Gamma(A_j)}$$

$$\prod_{l=1}^{t}a_l^{\rho_l+\sigma_l}\sum_{w,v',u,t',e,k_1,k_2}\ \sum_{K=0}^{[N_1/\mathfrak{M}_1]}\ \sum_{G_1=1}^{\alpha^{(1)}}\ \sum_{g_1=1}^{\infty}\frac{\phi_1 z_1^{\eta_{G_1,g_1}}(-)^{g_1}}{\epsilon_{G_1}^{(1)}g_1!}\frac{(-\mathfrak{N})_{\mathfrak{M}K}A_{\mathfrak{N},K}}{K!}y_1^K$$

$$\psi(w,v',u,t',e,k_1,k_2)\,y^R\prod_{l=1}^{t}a_l^{(e_l^{\prime\prime(1)}+f_l^{\prime\prime((1)})K}\prod_{l=1}^{t}a^{(e_l^{\prime(1)}+f_l^{\prime((1)})\eta_{G_1,g_1}}$$

$$H^{m^{(1)},n^{(1)}+3t+P;X'}_{p^{(1)}+3t+P,q^{(1)}+2t+q;Y'}\left(\begin{array}{c|c}\mathrm{z}_1'\prod_{l=1}^{t}a_l^{(e_l^{(1)}+f^{(1)})} & \mathrm{A'}\\ \text{-}\mathrm{B}_1a_1^{\mu_1+\upsilon_1} & \cdot\\ \cdot\cdot\cdot & \cdot\\ \cdot\cdot\cdot & \cdot\\ \text{-}\mathrm{B}_ta_t^{\mu_t+\upsilon_t} & \cdot\\ (\mathrm{a}_1b_1)^{g_1} & \cdot\\ \cdot\cdot\cdot & \cdot\\ \cdot\cdot\cdot & \cdot\\ (\mathrm{a}_tb_t)^{g_t} & \mathrm{B'}\end{array}\right) \quad (3.1)$$

Ref

3. B. P. Gautam, A. S. Asgar and A. N. Goyal. The A-function. Revista. Mathematica. Tucuman (1980).

We obtain an H-function of to $(1+2t)$-variables.
Provided that

$$min\{e_l''^{(1)}, f_l''^{(1)}, h_l''^{(1)}, e_l'^{(1)}, f_l'^{(1)}, h_l'^{(1)}, e_l^{(1)}, f_l^{(1)}, h_l^{(1)}, e_l, f_l, h_l, \mu_l, \upsilon_l, \omega_l\} > 0; (l = 1, \cdots, t)$$

$$Re(\lambda_l) > 0, g_l > 0; l = 1, \cdots, t$$

Notes

$$Re\left(\rho_l + e_l R + e_l'^{(1)}\eta_{G_1,g_1}\right) + e_l^{(1)} \min_{1\leqslant K\leqslant m'^{(1)}} Re\left(\frac{b_K'^{(1)}}{\beta_K'^{(1)}}\right) > 0\,; (l = 1, \cdots, t)$$

$$Re\left(\sigma_l + f_l R + f_l'^{(1)}\eta_{G_1,g_1}\right) + f_l^{(1)} \min_{1\leqslant K\leqslant m'^{(1)}} Re\left(\frac{b_K'^{(1)}}{\beta_K'^{(1)}}\right) > 0\,; (l = 1, \cdots, t)$$

$$z_1 \neq 0, \sum_{j=1}^{A} \gamma_j^{(1)} - \sum_{j=1}^{C} \xi_j^{(1)} + \sum_{j=1}^{M^{(1)}} \eta_j^{(i)} - \sum_{j=1}^{N^{(1)}} \epsilon_j^{(i)} < 0$$

with $\lambda, A, C, \alpha_1, \beta_1, m_1, n_1 \in \mathbb{N}^*; ; f_j, g_j, p_j^{(1)}, q_j^{(1)}, \gamma_j^{(1)}, \xi_j^{(1)}, \eta_j^{(1)}, \epsilon_j^{(1)} \in \mathbb{C}$

$$\left|arg\left(z_1' \prod_{l=1}^{t}\left[(x_l^{e_l^{(1)}}(a_l - x_l)^{f_l^{(1)}}\{1 + (b_l x_l)^{g_l}\}^{-h_l^{(1)}}\right)\right| < \frac{1}{2}\Omega_i''\pi \;\; (a_l \leqslant x_l \leqslant b_l; l = 1, \cdots, t)$$

where $$\Omega_1'' = = \sum_{k=1}^{n'^{(1)}} \alpha_k'^{(1)} - \sum_{k=n'^{(1)}+1}^{p'^{(1)}} \alpha_k'^{(1)} + \sum_{k=1}^{m'^{(1)}} \beta_k'^{(1)} - \sum_{k=m'^{(1)}+1}^{q'^{(i)}} \beta_k'^{(1)} - (e_l^{(1)} + f_l^{(1)} + h_l^{(1)})$$

$P \leqslant Q+1$, and the multiple series on the left-hand side of (2.1) converges absolutely, where

$$X' = m'^{(1)}, n'^{(1)}; 1, 0; \cdots; 1, 0; 1, 0; \cdots; 1, 0 \tag{3.2}$$

$$Y' = p'^{(1)}, q'^{(1)}; 0, 1; \cdots; 0, 1; 0, 1; \cdots; 0, 1 \tag{3.3}$$

$$A' = (1 - \rho_1 - e_1 R - e_1''^{(i)} K - e_1'^{(1)}\eta_{G_1,g_1}; e_1^{(1)}, \mu_1, \underbrace{0, \cdots, 0}_{t-1}, g_1, \underbrace{0, \cdots, 0}_{t-1}), \cdots,$$

$$(1 - \rho_t - e_t R - e_t''^{(1)} K - e_t'^{(1)}\eta_{G_1,g_1}; e_t^{(1)}, \underbrace{0, \cdots, 0}_{t-1}, \mu_t, \underbrace{0, \cdots, 0}_{t-1}, g_t),$$

$$(-\sigma_1 - f_1 R - f_1''^{(1)} K - f_1'^{(1)}\eta_{G_1,g_1}; f_1^{(1)}, \upsilon_1, \underbrace{0, \cdots, 0}_{2t-1}), \cdots,$$

$$(-\sigma_t - f_t R - f_t''^{(1)} K - f_t'^{(1)}\eta_{G_1,g_1}; f_t^{(1)}, \underbrace{0, \cdots, 0}_{t-1}, \upsilon_t, \underbrace{0, \cdots, 0}_{t}),$$

$$(1 - \lambda_1 - h_1 R - h_1''^{(1)} K - h_1'^{(1)}\eta_{G_1,g_1}; h_1^{(1)}, \omega_1, \underbrace{0, \cdots, 0}_{t-1}, 1, \underbrace{0, \cdots, 0}_{t-1}), \cdots,$$

$$(1-\lambda_t-h_tR-h_t''^{(1)}K-h_t'^{(1)}\eta_{G_1,g_1};h_t^{(1)},\underbrace{0,\cdots,0}_{t-1},\omega_t,\underbrace{0,\cdots,0}_{t-1},1),$$

$$(1-E_j;\underbrace{0,\cdots,0}_{s},\underbrace{1,\cdots,1}_{t},\underbrace{0,\cdots,0}_{t})_{1,P},\ [(a_k'^{(1)},\alpha_k'^{(1)})_{1,p^{(1)}},\underbrace{0,\cdots,0}_{2t}];-;\cdots;- \tag{3.4}$$

$$B'=[(b_k'^{(1)},\beta_k'^{(1)})_{1,q'^{(1)}},\underbrace{0,\cdots,0}_{2t}],$$

$$(\rho_1-\sigma_1-(e_1+f_1)R-(e_1''^{(i)}+f_1''^{(1)})K-(e_1'^{(1)}+f_1'^{(1)})\eta_{G_1,g_1};e_1^{(1)}+f_1^{(1)},\mu_1+\upsilon_1,\underbrace{0,\cdots,0}_{t-1},g_1,\underbrace{0,\cdots,0}_{t-1})$$

$$,\cdots,(\rho_t-\sigma_t-(e_t+f_t)R-(e_t''^{(1)}+f_t''^{(1)})K-(e_t'^{(1)}+f_t'^{(1)})\eta_{G_1,g_1};e_t^{(1)}+f_t^{(1)},\underbrace{0,\cdots,0}_{t-1},\mu_t+\upsilon_t,\underbrace{0,\cdots,0}_{t-1},g_t),$$

$$(1-\lambda_1-h_1R-h_1''^{(1)}K_1-h_1'^{(1)}\eta_{G_1,g_1};h_1^{(1)},\omega_1,\underbrace{0,\cdots,0}_{2t-1}),\cdots,$$

$$(1-\lambda_t-h_tR-h_t''^{(1)}K-h_t'^{(1)}\eta_{G_1,g_1};h_t^{(1)},\underbrace{0,\cdots,0}_{t-1},\omega_t,\underbrace{0,\cdots,0}_{t}),$$

$$(1-F_j;\underbrace{0,\cdots,0}_{s},\underbrace{1,\cdots,1}_{t},\underbrace{0,\cdots,0}_{t})_{1,Q}:\underbrace{(0;1);\cdots;(0;1)}_{2t} \tag{3.5}$$

By applying our result given in (4.1) and (4.4) to the case the Laguerre polynomials ([16], page 101, eq.(15.1.6)) and ([15] ,page 159) and by setting

$$S_N^1(x)\to L_N^{\alpha'}(x)$$

In which case $\mathfrak{M}=1, A_{N,K}=\binom{N+\alpha'}{N}\frac{1}{(\alpha'+1)_K}$ we obtain the following multiple integrals.

Corollary 2.

$$\int_0^{a_1}\cdots\int_0^{a_t}\prod_{l=1}^t\left[x_l^{\rho_l-1}(a_l-x_l)^{\sigma_l}\{1+(b_lx_l)^{g_l}\}^{-\lambda_l}\right]R_n^{\alpha,\beta}\left[y\prod_{l=1}^t\left[x_l^{e_l}(a_l-x_l)^{f_l}\{1+(b_lx_l)^{g_l}\}^{-h_l}\right]\right]$$

$$L_N^{\alpha'}\Big(\ \mathrm{y}_1\left[\prod_{l=1}^t\left[x_l^{e_l''^{(1)}}(a_l-x_l)^{f_l''^{(1)}}\{1+(b_lx_l)^{g_l}\}^{-h_l''^{(1)}}\right]\right]\ \Big)$$

$$A\Big(\ \mathrm{z}_1\left[\prod_{l=1}^t\left[x_l^{e_l'^{(1)}}(a_l-x_l)^{f_l'^{(1)}}\{1+(b_lx_l)^{g_l}\}^{-h_l'^{(1)}}\right]\right]\ \Big)$$

$$H\Big(\ \mathrm{z}_1'\left[\prod_{l=1}^t\left[x_l^{e_l^{(1)}}(a_l-x_l)^{f_l^{(1)}}\{1+(b_lx_l)^{g_l}\}^{-h_l^{(1)}}\right]\right]\ \Big)$$

$${}_PF_Q\left[(A_P);(B_Q);\sum_{l=1}^tB_lx_l^{\mu_l}(a_l-x_l)^{\upsilon_l}\{1+(b_lx_l)^{g_l}\}^{-\omega_l}\right]\mathrm{d}x_1\cdots\mathrm{d}x_t=\frac{\prod_{j=1}^Q\Gamma(B_j)}{\prod_{j=1}^P\Gamma(A_j)}$$

Ref

16. C. Szego, (1975), Orthogonal polynomials. Amer. Math. Soc. Colloq. Publ. 23 fourth edition. Amer. Math. Soc. Providence. Rhodes Island, 1975.

$$\prod_{l=1}^{t} a_l^{\rho_l+\sigma_l} \sum_{w,v',u,t',e,k_1,k_2} \sum_{K=0}^{N} \sum_{G_1=1}^{\alpha^{(1)}} \sum_{g_1=1}^{\infty} \frac{\phi_1 z_1^{\eta_{G_1,g_1}} (-)^{g_1}}{\epsilon_{G_1}^{(1)} g_1!} \psi(w,v,u,t',e,k_1,k_2)$$

$$y^R \prod_{l=1}^{t} a_l^{(e_l''^{(1)}+f_l''^{((1)})K} \prod_{l=1}^{t} a^{(e_l'^{(1)}+f_l'^{((1)})\eta_{G_1,g_1}} \frac{(-N)_K}{K!} \binom{N+\alpha'}{N} \frac{1}{(\alpha'+1)_K} y_1^K$$

$$H_{p^{(1)}+3t+P,q^{(1)}+2t+q;Y'}^{m^{(1)},n^{(1)}+3t+P;X'} \left(\begin{array}{c|c} \mathrm{z}_1' \prod_{l=1}^t a_l^{(e_l^{(1)}+f^{(1)})} & \mathrm{A'} \\ -\mathrm{B}_1 a_1^{\mu_1+\upsilon_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ -\mathrm{B}_t a_t^{\mu_t+\upsilon_t} & \cdot \\ (\mathrm{a}_1 b_1)^{g_1} & \cdot \\ \cdots & \cdot \\ \cdots & \cdot \\ (\mathrm{a}_t b_t)^{g_t} & \mathrm{B'} \end{array} \right) \tag{3.6}$$

under the same notations and existence conditions that (3.1).

If, $s=t=2$, the general polynomial S_N^M reduces to the Jacobi polynomials $P_n^{(\alpha,\beta)}(1-2x)$, the H-function of two variables into Appell's function F_3 and the generalized hypergeometric function ${}_PF_Q$ into the Bessel's function J_υ with the help of results ([15], page.159, Eq. (1.6)), ([10], page. 89, Eq. (6.4.6) ,page.18 Eq. (2.6.3) (2.6.5)), respectively and the A-function and a sequence of functions vanish, we arrive at the following double integrals after simplifications (see Gupta and Jain [5] for more details):

$$\int_0^{a_1} \int_0^{a_2} \prod_{l=1}^{2} \left[x_l^{\rho_l-1} (a_l - x_l)^{\sigma_l} \left\{ 1 + (b_l x_l)^{h_l} \right\}^{-\lambda_l} \right] P_n^{(\alpha,\beta)} [1 - 2y x_1^{e_1} x_2^{e_2}] \left[2\sqrt{B_1 x_1^{\mu_1} + B_2 x_2^{\mu_2}} \right]^{-\frac{v}{2}}$$

$$F_3[k_1,k_2,h_1,h_2;L;z_1 x_1^{u_1}, z_2 x_2^{u_2}] J_\upsilon [2\sqrt{B_1 x_1^{\mu_1} + B_2 x_2^{\mu_2}}] \, \mathrm{d}x_1 \mathrm{d}x_2 =$$

$$\frac{\Gamma(L)\Gamma(1+\sigma_1)\Gamma(1+\sigma_2) a_1^{\rho_1+\sigma_1} a_2^{\rho_2+\sigma_2}}{\Gamma(k_1)\Gamma(k_2)\Gamma(h_1)\gamma(h_2)\Gamma(\lambda_1)\Gamma(\lambda_2)} \sum_{R=0}^{n} \frac{(-n)_R \binom{\alpha+n}{n} (\alpha+\beta+n+1)_R (y a_1^{e_1} a_2^{e_2})^R}{R!(\alpha+1)_R}$$

$$H_{2,4;2,1;2,1;0,1;0,1;1,1;1,1}^{0,2;1,2;1,2;1,0;1,0;1,1;1,1} \left(\begin{array}{c|c} -\mathrm{z}_1 a_1^{u_1} & \mathrm{A}_2 \\ -\mathrm{z}_2 a_2^{u_2} & \cdot \\ \mathrm{B}_1 a_1^{\mu_1} & \cdot \\ \mathrm{B}_2 a_2^{\mu_2} & \cdot \\ \mathrm{a}_1 b_1 & \mathrm{B}_2 \\ \mathrm{a}_2 b_2 & \end{array} \right) \tag{3.7}$$

with

$$A_2 = (1-\rho_1-e_1R; u_1, 0, \mu_1, 0, 1, 0), (1-\rho_2-e_2R; 0, u_2, 0, \mu_2, 0, 1) : (1-k_1;1), (1-k_2;1); (1-h_1;1), (1-h_2;1)$$

$$-;-;(1-\lambda_1;1);(1-\lambda_2;1) \tag{3.8}$$

$$B_2 = (-\upsilon; 0, 0, , 1, 1, 0, 0), (-\rho_1-\sigma_1-e_1R; u_1, 0, \mu_1, 0, 1, 0), (-\rho_2-\sigma_2-e_2R; 0, u_2, 0, \mu_2, 0, 1),$$

$$(1-L; 1, 1, 0, 0, 0, 0,) : (0,1); (0,1); (0,1); (0,1); (0,1); (0,1) \tag{3.9}$$

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IV. Conclusion

In this paper, we have evaluated unified multiple integrals involving the product of an expansion of the multivariable A-function, multivariable I-function defined by Prasad [6], a sequence of functions and class of multivariable polynomials defined by Srivastava [9] with general arguments. The formula established in this paper is very general nature. Thus, the results established in this research work would serve as a formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables, multiple integrals can be obtained.

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Statistical Estimation of the Persistence of Pesticides in Water Samples

By Pinnamreddy Sreehari Reddy & Thommandru Raveendranath Babu
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Abstract- This is a statistical loom in which to estimate the residual amounts of pesticides with various activities present in environmental samples by employing electro analytical techniques such as adsorptive stripping voltammetry. Average amounts for ten replicates found by applying statistical concepts such as standard deviation and correlation coefficient and in all the findings in this approach all the possible errors are minimised and accuracy is maximised. Water samples of various areas are collected and investigated for pesticide residues before and after the application of pesticides.

Keywords: *pesticides, voltammetry, water samples.*

GJSFR-F Classification: *MSC 2010: 91B82*

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Statistical Estimation of the Persistence of Pesticides in Water Samples

Pinnamreddy Sreehari Reddy α & Thommandru Raveendranath Babu σ

Abstract- This is a statistical loom in which to estimate the residual amounts of pesticides with various activities present in environmental samples by employing electro analytical techniques such as adsorptive stripping voltammetry. Average amounts for ten replicates found by applying statistical concepts such as standard deviation and correlation coefficient and in all the findings in this approach all the possible errors are minimised and accuracy is maximised. Water samples of various areas are collected and investigated for pesticide residues before and after the application of pesticides.

Keywords: *pesticides, voltammetry, water samples.*

I. Introduction

A pesticide is a substance used to kill feral animals, insects, fungi or plants. There are thousands of different pesticides in use today. Pesticides are used in houses, shops, offices, storerooms, sheds, gardens, farms, pastoral stations. Most of the pesticides used today are chemicals which have been developed in a laboratory by scientists and produced in factories. Some pesticides are quite hazardous, as they can be harmful to humans and other living things. They can contaminate land, the air, food crops, and water ways and seriously harm or kill native animals, pets and domestic animals. In addition to being hazardous to the user, pesticides can also cause great harm and sometimes death to a person or other living things nearby, if the instructions on the pesticide container is not followed carefully.

Pesticides come in three different forms: Solids, which come in powder form (like flour), or in crystal or granular form (like sugar) liquids, which look like milky water. Aerosols, which are sprayed out in a fine mist.

While pesticides are useful for the control of various pests, many of them are hazardous chemicals. They are hazardous because they can poison the land, the water and the air.

Some pesticides do not break down for a long time. These types of pesticides are often used when something must be protected from pest attack for a long period of time, for example, protecting houses from termite attack.

Pesticides which remain in the soil or on the treated surface are also often called residual chemicals.

When residual pesticides get into the environment they can remain poisonous and active for many years. If applied incorrectly or used in the wrong place, these chemicals may spread to other land areas and possibly to the water supply.

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Sometimes people do not know that the chemical is in the ground and may dig up the soil. They may then use it for a garden or some other purpose which brings other people, their pets and other animals into contact with it. As a result, many non-target animals can be affected by pesticides In this way.

Prior to 1996, some pesticides were non-biodegradable. Some of them, such as D.D.T and Dieldrin can still be found in the environment today, although they are no longer available and have not been used for many years.

Pesticides and the food chain: In nature, plants are eaten by animals. These animals are in turn eaten by other animals, which are eaten by other animals, and so on. This is called the food chain. Along the food chain there are many different ways pesticides can accidentally contaminate animals and plants which could then be eaten by humans. Pesticides can enter the food chain at different points.

After an insect pest has been killed by a pesticide the chemical may stay in its body and still be active. If another animal eats the insect's body the pesticide will be transferred to its body and it may also be harmed by the pesticide. The second animal may of course be eaten by a third animal and it too could be harmed by the pesticide and so on.

Fig. 1: An example of a food chain

There are good reasons (advantages) for using pesticides and there are reasons for not using them (disadvantages).

Advantages of using pesticides. Modern pesticides are very effective. This means that nearly all the target pests which come in contact with these pesticides are killed Results are quick. This means the pests are killed within a very short time.

Using pesticides can be an economical (cheap) way of controlling pests. Pesticides can be applied quickly and there is not the high labour cost which might apply to other methods of control, such as removing weeds by hand.

If pesticides are not used correctly, they can affect human health or cause serious injury or death to the pesticide operator, other people or household pets. Pesticides can also directly affect other non-target animals. For example, a gardener spraying his garden to kill caterpillars will probably also kill harmless lady bird beetles and praying mantises. If pesticides are used incorrectly or applied wrongly, they may find their way into places where they are not wanted, for example, they might be washed into rivers or into the soil. In this article an elstroanalytical method voltammetry supported by statistical findings was applied.

II. Instruments and Reagents

Voltammetric estimations coducted using a model meterohm Auto Lab 101 PG stat (Netherlands). CNTPE was used as working electrode for differential pulse adsorptive stripping voltammetry and cyclic voltammetry. pH measurements were carried out with an Eutech PC˙510 cyber scan. Meltzer Toledo (Japan) Xp26 delta

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range micro balancer were used to weigh the samples during the preparation of standard solutions. All the experiments were performed at 250C.

All reagents used are analytical reagent grade. Double distilled water was used throughout the analysis. In the present investigation universal buffers of pH 4.0 are used as supporting electrolytes and are prepared by using 0.2 M boric acid, 0.05M citric acid and 0.1M trisodium orthophosphate solutions.

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III. Measurements and Calculations

In this standard addition method, the voltammogram of the unknown is first recorded after which a known volume of standard solution of the same electro active species is added to the cell and second voltammogram is taken. From the magnitude of the peak height, the unknown concentration of species may be calculated using the following equations.

$$\text{C (un known)} = \frac{C_s \, x V}{V_t \, x i_2} \, x \, i_1$$

The values obtained is substituted in the following equations to find the amount of residue[1-13]

$$\bar{x} = \frac{1}{n} \sum x_i$$

$$s^2 = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2$$

$$\sigma^2 = \frac{1}{n} \sum_i (x_i - \mu)^2$$

$$\text{CL} = 100 \times (1 - \alpha) \, (\%)$$

IV. Analysis

Well resolvable and reproducible peak obtained for each sample is useful for the analysis of water samples. The optimum pH to get well defined peak for the detection is found to be 4.0. The peak current is found to vary linearly with the concentration of the pesticide over the range 1.0 x 10^{-5}M to 1.0 x 10^{-9}M. The lower detection was limit found to be 1.02 x 10^{-8}M. The correlation coefficient and relative standard deviation (for 10 replicates) obtained using the above procedure [14-20].

V. Analytical Procedure

A stock solution (1.0 x 10^{-3} M) of each sample is prepared in dimethyl formamide. In voltammetric cell, 1 mL of standard solution is taken and 9 mL of the supporting electrolyte (pH 4.0) is added to it. Then the solution is de aerated with nitrogen gas for 10 min. after obtaining the voltammogram, small additions of standard solution are added and the voltammograms are recorded under similar experimental conditions. The optimum conditions for analytical estimation at pH 4.0 are found to be pulse amplitude of 25 mV, applied potential of -0.35V and scan rate 40 mVs.$^{-1}$.

Water samples are collected from paddy fields which sprayed by the pesticides under investigation 48 hours after spraying the pesticides in swarnamukhi river belt, Vakadu, Nellore district, A.P., India. These samples were filtered through a Whatman No.41 filter paper and Aliquots of water samples were taken in a 25mL graduated tube, to it buffer solution was added and analyzed as described above [21-23]. The recoveries of samples obtained in water samples ranged from 41.00 to 47.00% and the results are summarized in Table 1.0.

Table 1.0: Recoveries of pesticides in water samples

Name of the pesticide	Amount added (mg/L)	Amount found (mg/L)	Recovery (%)	Standard deviation
Dinitramine(Herbicide)	5	2.11	42.20	0.07
Bromethalin(Rodenticide)	5	2.32	46.40	0.04
Isopropaline(Weedicide)	5	2.35	47.00	0.06
Benfluraline(Fungicide)	5	2.21	44.20	0.10
Trifluraline(Fungicide)	5	2.09	41.80	0.09
Metheocarb(Acharicide)	5	2.15	43.00	0.06
Cyometrinil(Herbicide)	5	2.10	42.00	0.11
Fluxafenim(Herbicide)	5	2.08	41.60	0.08
Fenomidone(fungicide)	5	2.07	41.40	0.07
Topramezone(Fungicide)	5	2.22	44.40	0.10

**Average of 10 replicates*

VI. Conclusions

In this approach statistical parameters for the determination of pesticide residues satisfactory applied to interpret the instrumental out puts without considerable errors. And during the estimations pollution arises due to heavy metal electrodes such as mercury electrodes is avoided by using carbon electrodes.

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A New Subclass of Univalent Functions

By Gagandeep Singh, Gurcharanjit Singh & Harjinder Singh
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Abstract- In this paper, a new subclass $\chi_t(A,B)$ of close-to-convex functions, defined by means of subordination is investigated. Some results such as coefficient estimates, inclusion relations, distortion theorems, radius of convexity and Fekete- Szegö problem for this class are derived. The results obtained here is extension of earlier known work.

Keywords: *subordination, univalent functions, analytic functions, convex functions, close-to-convex, coefficient estimates, fekete- szegö problem.*

GJSFR-F Classification: *MSC 2010: 30C45*

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A New Subclass of Univalent Functions

Gagandeep Singh [α], Gurcharanjit Singh [σ] & Harjinder Singh [ρ]

Abstract- In this paper, a new subclass $\chi_t(A,B)$ of close-to-convex functions, defined by means of subordination is investigated. Some results such as coefficient estimates, inclusion relations, distortion theorems, radius of convexity and Fekete- Szegö problem for this class are derived. The results obtained here is extension of earlier known work.

Keywords: subordination, univalent functions, analytic functions, convex functions, close-to-convex, coefficient estimates, fekete- szegö problem.

I. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic and univalent in the open unit disc $E=\{z:|z|<1\}$.

Let U be the class of bounded functions

$$w(z)=\sum_{n=1}^{\infty}c_nz^n, \tag{1.2}$$

which are regular in the unit disc and satisfying the conditions

$$w(0)=0 \quad \text{and} \quad |w(z)|<1 \text{ in } E.$$

For functions f and g analytic in E, we say that f is subordinate to g, denoted by $f \prec g$, if there exists a Schwarz function $w(z)\in U$, $w(z)$ analytic in E with $w(0)=0$ and $|w(z)|<1$ in E, such that $f(z)=g(w(z))$.

By S, S^* and C we denote subclass of A, consisting of functions which are univalent, starlike and convex in E.

Gao and Zhou [1] discussed the following subclass K_S of analytic functions, which is indeed a subclass of close-to-convex functions.

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Let K_S denote the class of functions $f(z)$ of the form (1.1) and satisfying the conditions

$$\mathrm{Re}\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > 0 \tag{1.3}$$

where $g(z) \in S^*\left(\frac{1}{2}\right)$.

Knwalczyk and Les-Bomba [5] extended the class K_S by introducing the following subclass of analytic functions.

A function $f \in A$ is said to be in the class $K_S(\gamma), 0 \le \gamma < 1$, if there exist a function $g(z) \in S^*\left(\frac{1}{2}\right)$, such that

$$\mathrm{Re}\left(-\frac{z^2 f'(z)}{g(z)g(-z)}\right) > \gamma.$$

Recently Prajapat [7] introduced the following subclass of analytic functions.

A function $f \in A$ is said to be in the class $\chi_t(\gamma)(|t| \le 1, t \ne 0, 0 \le \gamma < 1)$, if there exist a function $g(z) \in S^*\left(\frac{1}{2}\right)$, such that

$$\mathrm{Re}\left(\frac{tz^2 f'(z)}{g(z)g(tz)}\right) > \gamma.$$

Motivated by above defined classes, we introduce the following subclass of analytic functions.

Let $\chi_t(A,B)\,(|t| \le 1, t \ne 0)$, denote the class of functions $f(z)$ of the form (1.1) and satisfying the conditions

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, \quad z \in E \tag{1.4}$$

where $g(z) \in S^*\left(\frac{1}{2}\right)$.

In particular,

$$\chi_t(1-2\gamma, -1) \equiv \chi_t(\gamma).$$

$$\chi_{-1}(1-2\gamma, -1) \equiv K_S(\gamma).$$

$$\chi_{-1}(1, -1) \equiv K_S.$$

By definition of subordination it follows that $f(z) \in \chi_t(A,B)$ if and only if $f(z)$ can be represented in the form

Ref

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$$\frac{tz^2 f'(z)}{g(z)g(tz)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad w(z) \in U, \quad -1 \le B < A \le 1, \qquad z \in E. \tag{1.5}$$

In the present work, we obtained coefficient estimates, inclusion relation, distortion theorems, radius of convexity and Fekete- Szegö problem for functions in the functional class $\chi_t(A,B)$. Results obtained here extend the known results due to various authors.

Notes

Throughout our present discussion, to avoid repetition , we lay down once for all that $-1 \le B < A \le 1, 0 < |t| \le 1, t \ne 0$, $z \in E$.

II. Coefficient Estimates

Lemma 2.1 ([2]) Let

$$\frac{tz^2 f'(z)}{g(z)g(tz)} = P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

then $$|p_n| \le (A-B), \quad n \ge 1. \tag{2.2}$$

The bounds are sharp, being attained for the functions

$$P_n(z) = \frac{1+A\delta z^n}{1+B\delta z^n}, \quad |\delta| = 1.$$

Lemma 2.2 ([8]) As $g(z) \in S^*\left(\frac{1}{2}\right)$, then $G(z) = \frac{g(z)g(tz)}{tz} = z + \sum_{n=2}^{\infty} d_n z^n \in S^*$, so

$$|d_n| \le n. \tag{2.3}$$

Theorem 2.3 If $f(z) \in \chi_t(A,B)$, then

$$|a_n| \le 1 + \frac{(n-1)(A-B)}{2}. \tag{2.4}$$

Proof. As $f(z) \in \chi_t(A,B)$, therefore (1.5) can be expressed as

$$\frac{zf'(z)}{G(z)} = P(z). \tag{2.5}$$

Using (1.1), (2.1) and (2.3), (2.5) yields

$$1 + \sum_{n=2}^{\infty} n a_n z^{n-1} = \left(1 + \sum_{n=2}^{\infty} n d_n z^{n-1}\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right) \tag{2.6}$$

Equating the coefficients of z^{n-1} in (2.6), we have

$$n a_n = d_n + d_{n-1} p_1 + d_{n-2} p_2 + \ldots + d_2 p_{n-2} + p_{n-1}. \tag{2.7}$$

Therefore using (2.2) and (2.3), it gives

$$n|a_n| \le n + (A-B)[(n-1) + (n-2) + \ldots + 2 + 1]. \tag{2.8}$$

Hence from (2.8), we have

$$|a_n| \le 1 + \frac{(n-1)(A-B)}{2}.$$

On putting $A = 1-2\gamma, B = -1$ in Theorem 2.3, the following result due to Prajapat [7] is obvious:

Corollary 2.4 If $f(z) \in \chi_t(\gamma)$, then

$$|a_n| \le 1 + (n-1)(1-\gamma).$$

Again for $A = 1, B = -1, t = -1$, Theorem 2.3 gives the following result:

Corollary 2.5 If $f(z) \in K_S$, then $|a_n| \le n-1$.

III. Inclusion Relation

Lemma 3.1 ([8]) Let $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$. Then

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Theorem 3.2 Let $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$. Then

$$\chi_t(A_1, B_1) \subset \chi_t(A_2, B_2).$$

Proof. As $f(z) \in \chi_t(A_1, B_1)$, so

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+A_1 z}{1+B_1 z}.$$

Since $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, by Lemma 3.1, we have

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z},$$

it follows that $f(z) \in \chi_t(A_2, B_2)$ which proves the inclusion relation.

IV. Distortion Theorems

Theorem. 4.1 If $f(z) \in \chi_t(A, B)$, then for $|z| = r$, $0 < r < 1$, we have

$$\frac{(1-Ar)}{(1-Br)(1+r)^2} \le |f'(z)| \le \frac{(1+Ar)}{(1+Br)(1-r)^2} \tag{4.1}$$

and

$$\int_0^r \frac{(1-At)}{(1-Bt)(1+t)^2}\,dt \le |f(z)| \le \int_0^r \frac{(1+At)}{(1+Bt)(1-t)^2}\,dt. \tag{4.2}$$

Proof. From (2.5), we have

Notes

$$|f'(z)| = \frac{|G(z)|}{|z|}\left|\frac{1+Aw(z)}{1+Bw(z)}\right|, \quad w(z) \in U. \tag{4.3}$$

It is easy to show that the transformation

$$\frac{zf'(z)}{G(z)} = \frac{1+Aw(z)}{1+Bw(z)}$$

Notes

maps $|w(z)| \le r$ onto the circle

$$\left|\frac{zf'(z)}{G(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \le \frac{(A-B)r}{(1-B^2r^2)}, \quad |z| = r.$$

This implies that

$$\frac{1-Ar}{1-Br} \le \left|\frac{1+Aw(z)}{1+Bw(z)}\right| \le \frac{1+Ar}{1+Br}. \tag{4.4}$$

Since by Lemma 2.2, $G(z)$ is a starlike function. It is well known that,

$$\frac{r}{(1+r)^2} \le |G(z)| \le \frac{r}{(1-r)^2}. \tag{4.5}$$

(4.3) together with (4.4) and (4.5) yields (4.1). On integrating (4.1) from 0 to r, (4.2) follows.

For $A = 1-2\gamma, B = -1$, Theorem 4.1 gives the following result due to Prajapat [7]:

Corollary 4.2 If $f(z) \in \chi_t(\gamma)$, then

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \le |f'(z)| \le \frac{1+(1-2\gamma)r}{(1-r)^3}$$

and

$$\int_0^r \frac{1-(1-2\gamma)t}{(1+t)^3}dt \le |f(z)| \le \int_0^r \frac{1+(1-2\gamma)t}{(1-t)^3}dt.$$

V. Radius of Convexity

Theorem. 5.1. Let $f(z) \in \chi_t(A,B)$, then $f(z)$ is convex in $|z| < r_1$, where r_1 is the smallest positive root in (0,1) of

$$ABr^3 - A(B-2)r^2 - (2B-1)r - 1 = 0. \tag{5.1}$$

Proof. As $f(z) \in \chi_t(A,B)$, we have

$$zf'(z) = G(z)p(z). \tag{5.2}$$

After differentiating (5.2) logarithmically, we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)}. \tag{5.3}$$

Now for $G(z)\in S^*$, we have

$$\operatorname{Re}\left(\frac{zG'(z)}{G(z)}\right)\geq\frac{1-r}{1+r}.$$

Therefore (5.3) yields,

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\geq\frac{1-r}{1+r}-\left|\frac{zp'(z)}{p(z)}\right|$$

$$\geq\frac{1-r}{1+r}-\frac{r(A-B)}{(1+Ar)(1+Br)}$$

$$\geq\frac{-ABr^3+A(B-2)r^2+(2B-1)r+1}{(1+r)(1+Ar)(1+Br)}.$$

Hence $f(z)$ is convex in $|z|<r_1$, where r_1 is the smallest positive root in $(0,1)$ of

$$ABr^3-A(B-2)r^2-(2B-1)r-1=0.$$

Taking $A=1-2\gamma, B=-1$, Theorem 5.1 gives the following result due to Prajapat [7]:

Corollary 5.2 If $f(z)\in\chi_t(\gamma)$, then $f(z)$ is convex in $|z|<r_0=2-\sqrt{3}$.

VI. Fekete-Szegö Problem

Lemma 6.1 ([3],[6]) If $p(z)=1+p_1z+p_2z^2+p_3z^3+\ldots$ is a function with positive real part, then for any complex number μ,

$$|p_2-\mu p_1^2|\leq 2\max\{1,|2\mu-1|\}$$

and the result is sharp for the functions given by $p(z)=\dfrac{1+z^2}{1-z^2}$ and $p(z)=\dfrac{1+z}{1-z}$.

Lemma 6.2 ([4]) If $G(z)=z+\sum_{n=2}^{\infty}d_nz^n\in S^*$, then for any complex number λ,

$$|d_3-\lambda d_2^2|\leq\max\{1,|3-4\lambda|\}$$

and the result is sharp for the Koebe function k if $\left|\lambda-\frac{3}{4}\right|\geq\frac{1}{4}$ and for $\left(k\left(z^2\right)\right)^{\frac{1}{2}}=\dfrac{z}{1-z^2}$ if $\left|\lambda-\frac{3}{4}\right|\leq\frac{1}{4}$.

Theorem. 6.3 Let $f(z)\in\chi_t(A,B)$, then for $\mu\in C$,

$$|a_3-\mu a_2^2|\leq\frac{(A-B)}{3}\max\{1,|2\gamma_1-1|\}+\frac{1}{3}\max\{1,|3-4\mu_1|\}+2(A-B)\left|\frac{1}{3}-\frac{\mu}{2}\right|, \quad (6.1)$$

Notes

Notes

where $\gamma_1 = \frac{(1+B)}{2} + \frac{3(A-B)\mu}{8}$ and $\mu_1 = \frac{3\mu}{4}$.

Proof. As $f(z) \in \chi_t(A,B)$, from (1.5) we have

$$\frac{zf'(z)}{G(z)} = \frac{1+Aw(z)}{1+Bw(z)}.$$

Let $h(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots$, then $\operatorname{Re}(h(z)) > 0$ and $h(0) = 1$.

So
$$\frac{zf'(z)}{G(z)} = \frac{1-A+h(z)(1+A)}{1-B+h(z)(1+B)}. \tag{6.2}$$

On expanding (6.2), we have

$$1 + (2a_2 - d_2)z + (3a_3 - 2a_2 d_2 - d_3 + d_2^2)z^2 + \dots = 1 + \frac{p_1(A-B)z}{2} + \frac{(A-B)}{2}\left\{p_2 - p_1^2\left(\frac{1+B}{2}\right)\right\}z^2 + \dots \tag{6.3}$$

Equating coefficients of z and z^2 on both sides of (6.3), we get

$$a_2 = \frac{2d_2 + p_1(A-B)}{4}$$

and
$$a_3 = \frac{1}{3}\left\{d_3 + \frac{(A-B)}{2}\left(p_1 d_2 + p_2 - \frac{p_1^2(1+B)}{2}\right)\right\}.$$

Therefore, we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)}{6}\left|p_2 - \gamma_1 p_1^2\right| + \frac{1}{3}\left|d_3 - \mu_1 d_2^2\right| + \frac{(A-B)}{2}|d_2|\left[\frac{1}{3} - \frac{\mu}{2}\right]|p_1|.$$

Hence using Lemma 6.1 and Lemma 6.2, the desired result follows.

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Notes

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Unified Fractional Derivative Formulae for the Multivariable Aleph-Function

By FY. AY. Ant

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Keywords: general class of multivariable polynomial, saigo-maeda operator, multivariable aleph-function, multivariable H-function, alephfunction, fractional derivative formulae, generalized leibniz rule.

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Abstract- The object of this paper is to derive three unified fractional derivatives formulae for the Saigo-Maeda operators of fractional integration. The first formula deals with the product of a general class of multivariable polynomials and the multivariable Aleph-function. The second concerns the multivariable polynomials and two multivariable Aleph-functions with the help of the Leibniz rule for fractional derivatives. The last relation also implies the product of a class of multivariable polynomials and the multivariable Aleph-function but it is obtained by the application of the first formula twice and it implicates two independents variables instead of one. The polynomials and the functions have their arguments of the type $x^{\rho}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\sigma_i}$ are quite general nature. These formulae, besides being on very general character have been put in a compact form avoiding the occurrence of infinite series and thus making them put in applications. Our findings provide unifications and extensions of some (known and new) results. We shall give several corollaries and particular cases.

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I. Introduction and Preliminaries

The fractional integral operator involving various special functions has found significant importance and applications in mathematical analysis. Since last four decades, some workers like Love [14], McBride [18], Kalla [6,7], Kalla and Saxena [8,9], Saxena et al. [28], Saigo [21,22], Kilbas [10], Kilbas and Sebastian [11] have studied in depth the properties, applications and different extensions of various hypergeometric operators of Fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Samko et al. [25], Miller and Ross [19], Kiryakova [13,14], Kilbas, Srivastava and Trujillo [12] and Debnath and Bhatta [3]. A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [22,23], has been introduced by Marichev [15] (see details in Samko et al. [23] and also see Kilbas and Saigo [13]). The generalized fractional integral operator of arbitrary order, involving Appell function F_3 in the kernel defined and studied by Saigo and Maeda [24, p. 393, Eq (4.12) and (4.13)] in the following manner :

Let $\alpha,\alpha'\beta,\beta',\gamma$ be complex numbers. The fractional integral $(Re(\alpha)>0)$ and derivative $(Re(\alpha)<0)$ of a function $f(x)$ defined on $(0,\infty)$ is given by :

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}f(z)=\left[\begin{array}{l}\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x(x-t)^{\gamma-1}t^{-\alpha'}F_3\left[\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x};1-\frac{x}{t}\right]f(t)\mathrm{d}t,Re(\gamma)>0\\ \left(\frac{d}{dx}\right)^k\left(I_{0,x}^{\alpha,\alpha',\beta+k,\beta',\gamma+k}f\right)(x),Re(\gamma)\leqslant 0;k=[-Re(\gamma)]+1\end{array}\right. \quad (1.1)$$

and

$$I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}f(z)=\left[\begin{array}{l}\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty(t-x)^{\gamma-1}t^{-\alpha}F_3\left[\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t};1-\frac{t}{x}\right]f(t)\mathrm{d}t,Re(\gamma)>0\\ \left(-\frac{d}{dx}\right)^k\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k}f\right)(x),Re(\gamma)\leqslant 0;k=[-Re(\gamma)]+1\end{array}\right. \quad (1.2)$$

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The Appell hypergeometric function of the third type denoted F_3 is defined by :

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;z,t) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m t^n}{m!n!} \qquad |z|<1, |t|<1 \tag{1.3}$$

Recently, Agrawal [1], Soni and Singh [26], Ram and Suthar [20], Singh and Mandia [28] have studied several formulae about the fractional operator involving the product of a general class of polynomials of one variable defined by Srivastava [29] and multivariable H-functions introduced by Srivastava and Panda [34,35]. In this paper, we shall obtain three results that give the theorems of the product of two multivariable Aleph-functions and a general class of multivariable polynomials [30] in Saigo-Maeda operators.

The Aleph-function of several variables is an extension of the multivariable I-function defined by Sharma and Ahmad [25], itself is a generalization of G and H-functions of several variables defined by Srivastava et Panda [34,35]. The multiple Mellin-Barnes integral occurring in this paper will be referred to as the multivariable Aleph-function of $r-$ variables throughout our present study and will be defined and represented as follows (see Ayant [2]).

We have : $\aleph(z_1,\cdots,z_r) = \aleph^{0,\mathfrak{n}:m_1,n_1,\cdots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}} \left(\begin{array}{c} \mathrm{z}_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{z}_r \end{array} \middle| \begin{array}{c} [(\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)})_{1,\mathfrak{n}}], \\ \cdot \\ \cdots\cdots\cdots\cdots, \end{array} \right.$

$$[\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)})_{\mathfrak{n}+1,p_i}] : [(\mathrm{c}_j^{(1)}),\gamma_j^{(1)})_{1,n_1}], [\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_i^{(1)}}];\cdots;$$
$$[\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)})_{m+1,q_i}] : [(\mathrm{d}_j^{(1)}),\delta_j^{(1)})_{1,m_1}], [\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_i^{(1)}}];\cdots;$$

$$\left. \begin{array}{c} [(\mathrm{c}_j^{(r)}),\gamma_j^{(r)})_{1,n_r}], [\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_i^{(r)}}] \\ \cdot \\ [(\mathrm{d}_j^{(r)}),\delta_j^{(r)})_{1,m_r}], [\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_i^{(r)}}] \end{array} \right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\cdots,s_r)\prod_{k=1}^{r}\theta_k(s_k)z_k^{s_k}\,\mathrm{d}s_1\cdots\mathrm{d}s_r \tag{1.4}$$

with $\omega=\sqrt{-1}$

$$\psi(s_1,\cdots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}}\Gamma(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k)}{\sum_{i=1}^{R}[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k)]} \tag{1.5}$$

and

$$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma(d_j^{(k)}-\delta_j^{(k)}s_k)\prod_{j=1}^{n_k}\Gamma(1-c_j^{(k)}+\gamma_j^{(k)}s_k)}{\sum_{i^{(k)}=1}^{R^{(k)}}[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k)]} \tag{1.6}$$

For more details, see Ayant [2]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding Conditions for multivariable H-function given by as :

$|argz_k| < \frac{1}{2}A_i^{(k)}\pi,$ where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0 \tag{1.7}$$

Ref

26. C.K. Sharma and S.S. Ahmad, On the multivariable I-function. Acta ciencia Indica Math , 20(2) (1994), 113-116.

with, $k = 1, \cdots, r, i = 1, \cdots, R \; , i^{(k)} = 1, \cdots, R^{(k)}$

The complex numbers z_i are not zero.Throughout this document, we assume the existence and absolute convergence Conditions of the multivariable Aleph-function. We may establish the asymptotic expansion in the following convenient form :

Notes

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\alpha_1}, \cdots, |z_r|^{\alpha_r}), max(|z_1|, \cdots, |z_r|) \rightarrow 0$$

$$\aleph(z_1, \cdots, z_r) = 0(|z_1|^{\beta_1}, \cdots, |z_r|^{\beta_r}), min(|z_1|, \cdots, |z_r|) \rightarrow \infty$$

where: $k = 1, \cdots, r : \alpha_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = 1, \cdots, m_k$ and

$$\beta_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = 1, \cdots, n_k$$

We shall note: $\aleph(z_1, \cdots, z_r) = \aleph_1(z_1, \cdots, z_r)$.

We define the Aleph-function of s-variable in the following manner :

$$\aleph(z_{r+1}, \cdots, z_{r+s}) = \aleph^{0,N:m_{r+1},n_{r+1},\cdots,m_{r+s},n_{r+s}}_{P_i,Q_i,\iota_i;R':p_{i(r+1)},q_{i(r+1)},\tau_{i(r+1)};R'^{(r+1)};\cdots;p_{i(r+s)},q_{i(r+s)};\tau_{i(r+s)};R'^{(r+s)}} \left(\begin{array}{c} z_{r+1} \\ \cdot \\ \cdot \\ z_{r+s} \end{array} \right|$$

$$[(a'_j; \alpha_j^{(r+1)}, \cdots, \alpha_j^{(r+1)})_{1,N}], [\iota_i(a'_{ji}; \alpha_{ji}^{(r+1)}, \cdots, \alpha_{ji}^{(r+s)})_{N+1,P_i}] : [(c_j^{(r+1)}), \gamma_j^{(r+1)})_{1,n_{r+1}}], [\tau_{i^{(r+1)}}(c_{ji^{(r+1)}}^{(r+1)}, \gamma_{ji^{(r+1)}}^{(r+1)})_{n_{r+1}+1,p_i^{(r+1)}}]$$

$$\cdots\cdots\cdots , \quad [\iota_i(b'_{ji}; \beta_{ji}^{(r+1)}, \cdots, \beta_{ji}^{(r+s)})_{M+1,Q_i}] : [(d_j^{(r+1)}), \delta_j^{(r+1)})_{1,m_{r+1}}], [\tau_{i^{(r+1)}}(d_{ji^{(r+1)}}^{(r+1)}, \delta_{ji^{(r+1)}}^{(r+1)})_{m_{r+1}+1,q_i^{(r+1)}}]$$

$$;\cdots; [(c_j^{(r+s)}), \gamma_j^{(r+s)})_{1,n_{r+s}}], [\tau_{i^{(r+s)}}(c_{ji^{(r+s)}}^{(r+s)}, \gamma_{ji^{(r+s)}}^{(r+s)})_{n_{r+s}+1,p_i^{(r+s)}}]$$

$$;\cdots; [(d_j^{(r+s)}), \delta_j^{(r+s)})_{1,m_{r+s}}], [\tau_{i^{(r+s)}}(d_{ji^{(r+s)}}^{(r+s)}, \delta_{ji^{(r+s)}}^{(r+s)})_{m_{r+s+1,q_i^{(r+s)}}}] \right)$$

$$= \frac{1}{(2\pi\omega)^s} \int_{L_{r+1}} \cdots \int_{L_{r+s}} \psi(t_{r+1}, \cdots, t_{r+s}) \prod_{k=r+1}^{r+s} \phi_k(t_k) z_k^{t_k} \, \mathrm{d}t_{r+1} \cdots \mathrm{d}t_{r+s} \tag{1.8}$$

$$\psi(t_{r+1}, \cdots, t_{r+s}) = \frac{\prod_{j=1}^{N} \Gamma(1 - a'_j + \sum_{k=r}^{r+s} \alpha_j^{(k)} t_k)}{\sum_{i=1}^{R'} [\iota_i \prod_{j=N+1}^{P_i} \Gamma(a'_{ji} - \sum_{k=r+1}^{r+s} \alpha_{ji}^{(k)} t_k) \prod_{j=1}^{Q_i} \Gamma(1 - b'_{ji} + \sum_{k=r+1}^{r+s} \beta_{ji}^{(k)} t_k)]} \tag{1.9}$$

and

$$\theta_k(t_k) = \frac{\prod_{j=1}^{m_k} \Gamma(d_j^{(k)} - \delta_j^{(k)} t_k) \prod_{j=1}^{n_k} \Gamma(1 - c_j^{(k)} + \gamma_j^{(k)} t_k)}{\sum_{i^{(k)}=1}^{R'^{(k)}} [\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} t_k) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} t_k)]}, \; k = r+1, \cdots, r+s \tag{1.10}$$

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For more details, see Ayant [2]. $|argz_k| < \frac{1}{2}B_i^{(k)}\pi$, where

$$B_i^{(k)} = \sum_{j=1}^{N}\alpha_j^{(k)} - \iota_i \sum_{j=N+1}^{p_i}\alpha_{ji}^{(k)} - \iota_i \sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}} \sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)} + \sum_{j=1}^{m_k}\delta_j^{(k)} - \iota_{i^{(k)}} \sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0 \quad (1.11)$$

with $k = r+1, \cdots, r+s, i = 1, \cdots, R', i^{(k)} = 1, \cdots, R'^{(k)}$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence Conditions of the multivariable Aleph-function. We may establish the asymptotic expansion in the following convenient form :

$$\aleph(z_{r+1}, \cdots, z_{r+s}) = 0(\,|z_{r+1}|^{\alpha'_{r+1}}, \cdots, |z_{r+s}|^{\alpha'_{r+s}}\,), max(\,|z_{r+1}|, \cdots, |z_{r+s}|\,) \to 0$$

$$\aleph(z_{r+1}, \cdots, z_{r+s}) = 0(\,|z_{r+1}|^{\beta'_{r+1}}, \cdots, |z_{r+s}|^{\beta'_{r+s}}\,), min(\,|z_{r+1}|, \cdots, |z_{r+s}|\,) \to \infty$$

where: $k = r+1, \cdots, r+s : \alpha'_k = min[Re(d_j^{(k)}/\delta_j^{(k)})], j = m_{r+1}, \cdots, m_{r+s}$ and

$$\beta'_k = max[Re((c_j^{(k)} - 1)/\gamma_j^{(k)})], j = n_{r+1}, \cdots, n_{r+s}$$

We shall note: $\aleph(z_{r+1}, \cdots, z_{r+s}) = \aleph_2(z_{r+1}, \cdots, z_{r+s})$.

The generalized polynomials of multivariable defined by Srivastava [30], is given in the following manner :

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1, \cdots, y_v] = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots ; N_v, K_v] y_1^{K_1} \cdots y_v^{K_v} \quad (1.12)$$

where $\mathfrak{M}_1, \cdots, \mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots ; N_v, K_v]$ are arbitrary constants, real or complex.

We shall note $a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots ; N_v, K_v]$

II. Lemma

Lemma 1.

$$\left(I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\mu-1}\right)(x) = \frac{\Gamma(\mu)\Gamma(\mu+\gamma-\alpha-\alpha'-\beta)\Gamma(\mu+\beta'-\alpha')}{\Gamma(\mu-\alpha-\alpha'+\gamma)\Gamma(\mu-\alpha'-\beta+\gamma)\Gamma(\mu+\beta')} x^{\mu-\alpha-\alpha'+\gamma-1} \quad (2.1)$$

where $\alpha, \alpha'\beta, \beta', \gamma \in \mathbb{C}, Re(\gamma) > 0, Re(\mu) > max\{0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')\}$

Lemma 2.

$$\left(I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\mu-1}\right)(x) = \frac{\Gamma(1+\alpha+\alpha'-\gamma-\mu)\Gamma(1+\alpha+\beta'-\gamma-\mu)\Gamma(1-\beta-\mu)}{\Gamma(1-\mu)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\mu)\Gamma(1+\alpha-\beta-\mu)} x^{\mu-\alpha-\alpha'+\gamma-1} \quad (2.2)$$

where $\alpha, \alpha'\beta, \beta', \gamma \in \mathbb{C}, Re(\gamma) > 0, Re(\mu) < 1 + min\{Re(-\beta), Re(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)\}$

Notes

III. Main Results

We have the following results.

a) Fractional derivative formula 1.

Theorem 1.

Notes

$$\left\{I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left(x^{\rho}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\sigma_i}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}c_1x^{\lambda_1}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\eta_i^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ c_vx^{\lambda_v}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\eta_i^{(v)}}\end{pmatrix}\aleph_1\begin{pmatrix}z_1x^{\mu_1}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{\mu_r}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(r)}}\end{pmatrix}\right)\right\}$$

$$=\prod_{i=1}^{t}\alpha_i^{\sigma_i}x^{\rho-\alpha-\alpha'+\gamma}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}a_v\,c_1^{K_1}\cdots c_v^{K_v}x^{\sum_{j=1}^{v}\lambda_jK_j}\prod_{i=1}^{t}\alpha_i^{\sum_{j=1}^{v}K_j\eta_i^{(j)}}$$

$$\aleph_{p_i+t+3,q_i+t+3;\tau_i;R:W}^{0,\mathbf{n}+t+3;V}\left(\begin{array}{c|c}z_1x^{\mu_1}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(1)}} & \mathrm{A},\ \mathbf{A}:\ \mathrm{C}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ z_rx^{\mu_r}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(r)}} & \cdot\\ \alpha_1^{(-1)}x^{u_1} & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \alpha_t^{(-1)}x^{u_t} & \mathbf{B},\ \mathrm{B}:\ \mathrm{D}\end{array}\right) \tag{3.1}$$

where

$$V=m_1,n_1;\cdots;m_r,n_r:1,0;\cdots;1,0 \tag{3.2}$$

$$W=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\cdots;p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)};\underbrace{0,1;\cdots;0,1}_{t} \tag{3.3}$$

$$A=\Big(-\rho-\sum_{j=1}^{v}\lambda_jK_j;\mu_1,\cdots,\mu_r,u_1,\cdots,u_t\Big),\ \Big(\alpha'-\beta'-\rho-\sum_{j=1}^{v}\lambda_jK_j;\mu_1,\cdots,\mu_r,u_1,\cdots,u_t\Big),$$

$$\Big(-\rho-\gamma+\alpha'+\beta-\sum_{j=1}^{v}\lambda_jK_j;\mu_1,\cdots,\mu_r,u_1,\cdots,u_t\Big),\ \left(1+\sigma_1+\sum_{j=1}^{v}K_j\eta_1^{(j)};\upsilon_1^{(1)},\cdots,\upsilon_1^{(r)},1,\underbrace{0,\cdots,0}_{t-1}\right),\cdots,$$

$$\left(1+\sigma_t+\sum_{j=1}^{v}K_j\eta_t^{(j)};\upsilon_t^{(r)},\cdots,\upsilon_t^{(r)},\underbrace{0,\cdots,0}_{t-1},1\right) \tag{3.4}$$

$$\mathbf{A}=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},\underbrace{0,\cdots,0}_{t})\}_{1,n},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)},\underbrace{0,\cdots,0}_{t})_{n+1,p_i}\}$$

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\};-;\cdots;- \quad (3.5)$$

$$B=\Big(-\beta'-\rho-\sum_{j=1}^{v}\lambda_jK_j;\mu_1,\cdots,\mu_r,u_1,\cdots,u_t\Big),\ \Big(\alpha+\alpha'-\gamma-\rho-\sum_{j=1}^{v}\lambda_jK_j;\mu_1,\cdots,\mu_r,u_1,\cdots,u_t\Big),$$

$$\Big(-\rho-\gamma+\alpha'+\beta-\sum_{j=1}^{v}\lambda_jK_j;\mu_1,\cdots,\mu_r,u_1,\cdots,u_t\Big),\left(1+\sigma_i+\sum_{j=1}^{v}K_j\eta_i^{(j)};\upsilon_i^{(1)},\cdots,\upsilon_i^{(r)},\underbrace{0,\cdots,0}_{t}\right)_{1,t} \quad (3.6)$$

$$\mathbf{B}=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)},\underbrace{0,\cdots,0}_{t})_{m+1,q_i}\}:D=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\};\cdots;$$

$$\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\};\underbrace{(0;1),\cdots,(0;1)}_{t} \quad (3.7)$$

Provided that

$$Re(\gamma)>0;u_i,\lambda_j,\eta_i^{(j)},\mu_k,\upsilon_i^{(k)};i=1,\cdots,t;j=1,\cdots,v;k=1,\cdots,r.$$

$|argz_i|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (1.7).

$$Re(\rho)+\sum_{i=1}^{r}\mu_i\min_{1\leqslant l\leqslant m_i}Re\left(\frac{d_l^{(i)}}{\delta_l^{(i)}}\right)+1>\ max\{0,Re(\alpha+\alpha'+\beta-\gamma),Re(\alpha'-\beta')\}$$

Proof

To prove (3.1), we first express the general class of multivariable polynomials occurring on its left-hand side $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ in series with the help of (1.12), replace the multivariable Aleph-function by its Mellin-Barnes integrals contour with the help of (1.4), interchange the order of summations and $(s_1,\cdots,s_r)$-integrals and taking the fractional derivative Operator inside (which is permissible under the stated conditions) and make a little simplification. Next, we express the Following terms $(x^{u_1}+\alpha_1)^{\sigma_1+\sum_{j=1}^{v}\eta_1^{(j)}K_j-\sum_{k=1}^{r}\upsilon_1^{(k)}s_k},\cdots,(x^{u_t}+\alpha_t)^{\sigma_t+\sum_{j=1}^{v}\eta_t^{(j)}K_j-\sum_{k=1}^{r}\upsilon_t^{(k)}s_k}$ so obtained regarding Mellin-Barnes integrals contour ([33], p. 18, eq.(2.6.4); p.10, eq.(2.1.1)). Now, interchanging the order of $(v_1,\cdots,v_s)$ and $(s_1,\cdots,s_r)$-integrals (which is permissible under the stated conditions), and evaluating the x-integral with the help of the lemma 1 and reinterpreting the multiple Mellin-Barnes integrals contour so obtained regarding the Aleph-function Of $(r+t)$-variables, we get the desired formula (3.1) after algebraic manipulations.

b) Fractional derivative formula 2

Theorem 2.

$$\left\{I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left(x^{\rho}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\sigma_i}S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}c_1x^{\lambda_1}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\eta_i^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ c_vx^{\lambda_v}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\eta_i^{(v)}}\end{pmatrix}\aleph_1\begin{pmatrix}z_1x^{\mu_1}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(1)}}\\ \cdot\\ \cdot\\ \cdot\\ z_rx^{\mu_r}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(r)}}\end{pmatrix}\right.\right.$$

$$\left.\left.\aleph_2\begin{pmatrix}z_{r+1}x^{\mu_{r+1}}\prod_{i=1}^{t-1}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(r+1)}}\\ \cdot\\ \cdot\\ \cdot\\ z_{r+s}x^{\mu_{r+s}}\prod_{i=1}^{t-1}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(r+s)}}\end{pmatrix}\right)\right\}=\prod_{i=1}^{t}\alpha_i^{\sigma_i}x^{\rho-\beta}\sum_{l=0}^{\infty}\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\binom{-\beta}{l}a_vc_1^{K_1}\cdots c_v^{K_v}$$

Notes

$$x^{\sum_{j=1}^{v}\lambda_j K_j}\prod_{i=1}^{t}\alpha^{\sum_{j=1}^{v}K_j\eta_i^{(j)}}\aleph^{0,\mathbf{n}+N+2t+6;V}_{p_i+P_i+2t+6,q_i+Q_i+2t+6;\tau_i;\iota_i;R;R':W}\left(\begin{array}{c|c} \mathrm{z}_1 x^{\mu_1}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(1)}} & \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r x^{\mu_r}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(r)}} & \mathrm{A},\ \mathbf{A}\text{: C} \\ \alpha_1^{(-1)}x^{u_1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \alpha_t^{(-1)}x^{u_t} & \cdot \\ \mathrm{z}_{r+1}x^{\mu_{r+1}}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(r+1)}} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_{r+s}x^{\mu_{r+s}}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(r+s)}} & \cdot \\ \alpha_1^{(-1)}x^{u_1} & \mathbf{B},\ \text{B: D} \\ \cdot & \\ \cdot & \\ \cdot & \\ \alpha_{t-1}^{(-1)}x^{u_{t-1}} & \end{array}\right) \tag{3.8}$$

where

$$V = m_1, n_1; \cdots ; m_r, n_r : 1, 0; \cdots ; 1, 0; m_{r+1}; \cdots ; m_{r+s}; 1; 0; \cdots ; 1, 0 \tag{3.9}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \cdots ; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)}; \underbrace{0,1;\cdots;0,1}_{t}; p_{i^{(r+1)}}, q_{i^{(r+1)}}, \tau_{i^{(r+1)}}; R'^{(r+1)}; \cdots ;$$

$$p_{i^{(r+s)}}, q_{i^{(r+s)}}, \tau_{i^{(r+s)}}; R'^{(r+s)}; \underbrace{0,1;\cdots;0,1}_{t-1} \tag{3.10}$$

$$A = (-\sum_{j=1}^{v}\lambda_j K_j; \mu_1, \cdots, \mu_r, u_1, \cdots, u_t, \underbrace{0,\cdots,0}_{t+s-1}),\ (\alpha' - \beta' - \sum_{j=1}^{v}\lambda_j K_j; \mu_1, \cdots, \mu_r, u_1, \cdots, u_t, \underbrace{0,\cdots,0}_{t+s-1}),$$

$$(\alpha + \alpha' + l - \gamma - \sum_{j=1}^{v}\lambda_j K_j; \mu_1, \cdots, \mu_r, u_1, \cdots, u_t, \underbrace{0,\cdots,0}_{t+s-1}), (-\beta' - \sum_{j=1}^{v}\lambda_j K_j; \mu_1, \cdots, \mu_r, u_1, \cdots, u_t, \underbrace{0,\cdots,0}_{t+s-1}),$$

$$(\alpha' + l - \gamma - \sum_{j=1}^{v}\lambda_j K_j; \mu_1, \cdots, \mu_r, u_1, \cdots, u_t, \underbrace{0,\cdots,0}_{t+s-1}), (\alpha + \alpha' - \gamma - \sum_{j=1}^{v}\lambda_j K_j; \mu_1, \cdots, \mu_r, u_1, \cdots, u_t, \underbrace{0,\cdots,0}_{t+s-1}),$$

$$\left(1 + \sigma_1 + \sum_{j=1}^{v}K_i\eta_1^{(i)}; \upsilon_1^{(1)}, \cdots, \upsilon_1^{(r)}, 1, \underbrace{0,\cdots,0}_{s+2t-2}\right), \cdots, \left(1 + \sigma_t + \sum_{j=1}^{v}K_i\eta_t^{(i)}; \upsilon_t^{(r)}, \cdots, \upsilon_t^{(r)}, \underbrace{0,\cdots,0}_{t-1}, 1, \underbrace{0,\cdots,0}_{s+t-1}\right),$$

$$\left(1 + \sigma_i + \sum_{j=1}^{v}K_j\eta_i^{(j)}; \upsilon_i^{(1)}, \cdots, \upsilon_i^{(r)}, \underbrace{0,\cdots,0}_{s+2t-1}\right)_{1,t} \tag{3.11}$$

$$\mathbf{A} = \{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},\underbrace{0,\cdots,0}_{s+2t-1})_{1,\mathfrak{n}}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)},\underbrace{0,\cdots,0}_{s+2t-1})_{n+1,p_i}\},$$

$$\{(a'_j;\underbrace{0,\cdots,0}_{r+t},\alpha_j^{(r+1)},\cdots,\alpha_j^{(r+s)},\underbrace{0,\cdots,0}_{t-1})_{1,N}\},\{\iota_i(a'_{ji};\underbrace{0,\cdots,0}_{r+t},\alpha_{ji}^{(r+1)},\cdots,\alpha_{ji}^{(r+s)},\underbrace{0,\cdots,0}_{t-1})_{N+1,P_i}\},$$

$$C = \{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\};-;\cdots;-$$

Notes

$$;\cdots;\{(c_j^{(r+1)};\gamma_j^{(r+1)})_{1,n_{r+1}}\},\{\tau_{i^{(r+1)}}(c_{ji^{(r+1)}}^{(r+1)};\gamma_{ji^{(r+1)}}^{(r+1)})_{n_{r+1}+1,p_{i^{(r+1)}}}\}$$

$$\{(c_j^{(r+s)};\gamma_j^{(r+s)})_{1,n_{r+s}}\},\{\tau_{i^{(r+s)}}(c_{ji^{(r+s)}}^{(r+s)};\gamma_{ji^{(r+s)}}^{(r+s)})_{n_{r+s}+1,p_{i^{(r+s)}}}\};-;\cdots;- \tag{3.12}$$

$$B = \Big(-\rho-\sum_{j=1}^{v}\lambda_jK_j;\underbrace{0,\cdots,0}_{r+t},\mu_{r+1},\cdots,\mu_{r+s},u_1,\cdots,u_{t-1}\Big),\ \Big(\alpha+\alpha'-\gamma-\rho-\sum_{j=1}^{v}\lambda_jK_j;\underbrace{0,\cdots,0}_{r+s},\mu_{r+1},\cdots,\mu_{r+s},u_1,\cdots,u_{t-1}\Big),$$

$$\Big(\alpha'-\beta'-\rho-\sum_{j=1}^{v}\lambda_jK_j;\underbrace{0,\cdots,0}_{r+t},\mu_{r+1},\cdots,\mu_{r+s},u_1,\cdots,u_{t-1}\Big),\ \Big(\alpha+\alpha'-\beta-l-\rho-\sum_{j=1}^{v}\lambda_jK_j;\underbrace{0,\cdots,0}_{r+s},\mu_{r+1},\cdots,\mu_{r+t},u_1,\cdots,u_{t-1}\Big),$$

$$\Big(\alpha'+\beta-l-\rho-\gamma-\sum_{j=1}^{v}\lambda_jK_j;\underbrace{0,\cdots,0}_{r+t},\mu_{r+1},\cdots,\mu_{r+s},u_1,\cdots,u_{t-1}\Big),\Big(-\beta-\sum_{j=1}^{v}\lambda_jK_j;\underbrace{0,\cdots,0}_{r+t},\mu_{r+1},\cdots,\mu_{r+s},u_1,\cdots,u_{t-1}\Big),$$

$$\left(1+\sigma_{s-1};\underbrace{0,\cdots,0}_{r+t},\upsilon_{s-1}^{(r+1)},\cdots,\upsilon_{s-1}^{(r+s)},\underbrace{0,\cdots,0}_{t-2},1\right),\left(1+\sigma_{s-1};\underbrace{0,\cdots,0}_{r+t},\upsilon_{s-1}^{(r+1)},\cdots,\upsilon_{s-1}^{(r+s)},\underbrace{0,\cdots,0}_{t-2},1\right),\cdot,$$

$$\left(1+\sigma_i;\underbrace{0,\cdots,0}_{r+t},\upsilon_i^{(r+1)},\cdots,\upsilon_i^{(r+s)},\underbrace{0,\cdots,0}_{t-1}\right)_{1,t-1} \tag{3.13}$$

$$\mathbf{B} = \{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)},\underbrace{0,\cdots,0}_{s+2t-1})_{m+1,q_i}\},\{\iota_i(b'_{ji};\underbrace{0,\cdots,0}_{r+s},\beta_{ji}^{(r+1)},\cdots,\beta_{ji}^{(r+s)},\underbrace{0,\cdots,0}_{t-1})_{M+1,Q_i}\}:$$

$$D = \{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\};\cdots;\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\};\underbrace{(0;1);\cdots;(0;1)}_{t}$$

$$\{(d_j^{(r+s)};\delta_j^{(r+s)})_{1,m_{r+s}},\tau_{i^{(r+s)}}(\delta_{ji^{(r+s)}}^{(r+s)};\delta_{ji^{(r+s)}}^{(r+s)})_{m_{r+s}+1,q_{i^{(r+s)}}}\}\{(d_j^{(r+1)};\delta_j^{(r+1)})_{1,m_1},\{\tau_{i^{(r+1)}}(d_{ji^{(r+1)}}^{(r+1)};\beta_{ji^{(r+1)}}^{(r+1)})_{m_{r+1}+1,q_{i^{(r+1)}}}\}$$

$$;\cdots;\underbrace{(0;1);\cdots;(0;1)}_{t-1} \tag{3.14}$$

Ref

5. K.C. Gupta and R.C. Soni, A study of H-functions of one and several variables, J. Rajasthan. Acad. Phys. Sci. 1 (2002), 89-94.

Provided that

$Re(\gamma) > 0; u_i, \lambda_j, \eta_i^{(j)}, \mu_k, \upsilon_i^{(k)}; i = 1, \cdots, t; j = 1, \cdots, v; k = 1, \cdots, r+s.$

$|argz_i| < \frac{1}{2} A_i^{(k)} \pi$, where $A_i^{(k)}$ is defined by (1.7).

$|argz_k| < \frac{1}{2} B_i^{(k)} \pi$, where $B_i^{(k)}$ is defined by (1.11).

$$Re(\rho) + \sum_{i=1}^{r+s} \mu_i \min_{1 \leqslant l \leqslant m_i} Re\left(\frac{d_l^{(i)}}{\delta_l^{(i)}}\right) + 1 > max\{0, Re(\alpha + \alpha' + \beta - \gamma), Re(\alpha' - \beta')\}$$

and the multiple series on the left-hand side of (3.8) converges absolutely.

Proof

To prove the second theorem, we take

$$f(x) = x^{\rho} \prod_{i=1}^{t-1} (x^{u_i} + \alpha_i)^{\sigma_i} \aleph_2 \begin{pmatrix} z_{r+1} x^{\mu_{r+1}} \prod_{i=1}^{t-1} (x^{u_i} + \alpha_i)^{-\upsilon_i^{(r+1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_{r+s} x^{\mu_{r+s}} \prod_{i=1}^{t-1} (x^{u_i} + \alpha_i)^{-\upsilon_i^{(r+s)}} \end{pmatrix}$$

and

$$g(x) = (x^{u_t} + \alpha_t)^{\sigma_t} S_{N_1, \cdots, N_v}^{\mathfrak{M}_1, \cdots, \mathfrak{M}_v} \begin{pmatrix} c_1 x^{\lambda_1} \prod_{i=1}^{t} (x^{u_i} + \alpha_i)^{\eta_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ c_v x^{\lambda_v} \prod_{i=1}^{t} (x^{u_i} + \alpha_i)^{\eta_i^{(v)}} \end{pmatrix} \aleph_1 \begin{pmatrix} z_1 x^{\mu_1} \prod_{i=1}^{t} (x^{u_i} + \alpha_i)^{-\upsilon_i^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{\mu_r} \prod_{i=1}^{t} (x^{u_i} + \alpha_i)^{-\upsilon_i^{(r)}} \end{pmatrix}$$

in the left-hand side of the equation (3.8) and apply the following generalized Leibniz rule for the factional integrals

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma} \{f(x)g(x)\} = \sum_{l=0}^{\infty} \binom{-\beta}{l} I_{0,x}^{\alpha,\alpha',\beta-l,\beta',\gamma} \{f(x)\} \, I_{0,x}^{\alpha,\alpha',l,\beta',\gamma} \{g(x)\} \tag{3.15}$$

We obtain the second relation of fractional derivative after algebraic manipulations on making use of theorem 1 and the result ([5],p. 91, eq. (6)).

c) Fractional derivative formula 1.

Theorem 3.

$$I_{0,x}^{\alpha_1,\alpha_1',\beta_1,\beta_1',\gamma_1} I_{0,x}^{\alpha_2,\alpha_2',\beta_2,\beta_2',\gamma_2} \left\{ x^{\rho} y^{\rho'} \prod_{i=1}^{t} (x^{u_i} + \alpha_i)^{\sigma_i} \prod_{i=1}^{t} (y^{u_i'} + \beta_i)^{\sigma_i'} \right.$$

$$S_{N_1, \cdots, N_v}^{\mathfrak{M}_1, \cdots, \mathfrak{M}_v} \begin{pmatrix} c_1 x^{\lambda_1} y^{\zeta_1} \prod_{i=1}^{t} (x^{u_i} + \alpha_i)^{\eta_i^{(1)}} (y^{u_i'} + \beta_i)^{\eta_i'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ c_v x^{\lambda_v} y^{\zeta_v} \prod_{i=1}^{t} (x^{u_i} + \alpha_i)^{\eta_i^{(v)}} (y^{u_i'} + \beta_i)^{\eta_i'^{(v)}} \end{pmatrix}$$

$$\aleph_1\left(\begin{array}{c} z_1 x^{\mu_1} y^{\mu'_1}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(1)}}(x^{u'_i}+\alpha'_i)^{-\upsilon_i'^{(1)}} \\ \cdot \\ \cdot \\ \cdot \\ z_r x^{\mu_r} y^{\mu'_r}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{-\upsilon_i^{(r)}}(x^{u'_i}+\alpha'_i)^{-\upsilon_i'^{(r)}} \end{array}\right)\Bigg\} = \prod_{i=1}^{t}\alpha_i^{\sigma_i}\prod_{i=1}^{t}\beta_i^{\sigma'_i} x^{\rho-\alpha_1-\alpha'_1+\gamma_1} x^{\rho'-\alpha_2-\alpha'_2+\gamma_2}$$

$$\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} a_v\, c_1^{K_1}\cdots c_v^{K_v} x^{\sum_{j=1}^{v}\lambda_j K_j} y^{\sum_{j=1}^{v}\zeta_j K_j}\prod_{i=1}^{t}\alpha_i^{\sum_{j=1}^{v}K_j\eta_i^{(j)}}\beta_i^{\sum_{j=1}^{v}K_j\eta_i'^{(j)}}$$

$$\aleph^{0,\mathbf{n}+2t+6;V}_{p_i+2t+6,q_i+2t+6;\tau_i;R:W}\left(\begin{array}{c|c} z_1 x^{\mu_1} y^{\mu'_1}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(1)}}\beta_i^{-\upsilon_i'^{(1)}} & \mathrm{A},\ \mathbf{A}:\ \mathrm{C} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r x^{\mu_r} y^{\mu'_r}\prod_{i=1}^{t}\alpha_i^{-\upsilon_i^{(r)}}\beta_i^{-\upsilon_i'^{(r)}} & \cdot \\ \alpha_1^{(-1)}x^{u_1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \alpha_t^{(-1)}x^{u_t} & \cdot \\ \beta_1^{(-1)}y^{u'_1} & \cdot \\ \cdot & \mathbf{B},\ \mathrm{B}:\ \mathrm{D} \\ \cdot & \\ \cdot & \\ \alpha_t^{(-1)}y^{u'_t} & \end{array}\right) \tag{3.16}$$

where

$$V = m_1, n_1; \cdots; m_r, n_r : 1, 0; \cdots; 1, 0 \tag{3.17}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \cdots; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)}; \underbrace{0,1;\cdots;0,1}_{2t} \tag{3.18}$$

$$A = (-\rho-\sum_{j=1}^{v}\lambda_j K_j; \mu_1,\cdots,\mu_r,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),\ (-\alpha'_1-\beta'_1-\gamma_1-\rho-\sum_{j=1}^{v}\lambda_j K_j; \mu_1,\cdots,\mu_r,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),$$

$$(\alpha+\alpha'_1+\beta_1-\gamma_1-\rho-\sum_{j=1}^{v}\lambda_j K_j; \mu_1,\cdots,\mu_r,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),(-\beta'_1-\rho-\sum_{j=1}^{v}\lambda_j K_j; \mu_1,\cdots,\mu_r,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),$$

$$(\alpha_1+\alpha'_1-\gamma_1-\rho-\sum_{j=1}^{v}\lambda_j K_j; \mu_1,\cdots,\mu_r,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),(\beta_1+\alpha'_1-\gamma_1-\rho-\sum_{j=1}^{v}\lambda_j K_j; \mu_1,\cdots,\mu_r,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),$$

$$\left(1+\sigma_1+\sum_{j=1}^{v}K_j\eta_1^{(j)}; \upsilon_1^{(1)},\cdots,\upsilon_1^{(r)},\underbrace{0,\cdots,0}_{t},1,\underbrace{0,\cdots,0}_{t-1}\right),\cdots,\left(1+\sigma_t+\sum_{j=1}^{v}K_j\eta_t^{(j)}; \upsilon_t^{(r)},\cdots,\upsilon_t^{(r)},\underbrace{0,\cdots,0}_{2t-1},1\right)$$

Notes

$$\left(1+\sigma_i+\sum_{j=1}^{v}K_j\eta_i^{(j)};\upsilon_i^{(1)},\cdots,\upsilon_i^{(r)},\underbrace{0,\cdots,0}_{2t}\right)_{1,t} \quad (3.19)$$

$$\mathbf{A}=\{(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)},\underbrace{0,\cdots,0}_{2t})_{1,n}\},\{\tau_i(a_{ji};\alpha_{ji}^{(1)},\cdots,\alpha_{ji}^{(r)},\underbrace{0,\cdots,0}_{2t})_{n+1,p_i}\}$$

Notes

$$C=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};\cdots;\{(c_j^{(r)};\gamma_j^{(r)})_{1,n_r}\},\{\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)})_{n_r+1,p_{i^{(r)}}}\};-;\cdots;-$$

$$B=(-\rho'-\sum_{j=1}^{v}\zeta_jK_j;\mu_1',\cdots,\mu_r',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),\ (\alpha_2'-\beta_2'-\gamma_2-\rho'-\sum_{j=1}^{v}\zeta_jK_j;\mu_1',\cdots,\mu_r',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),$$

$$(\alpha_2'+\alpha_2+\beta_2-\gamma_2-\rho'-\sum_{j=1}^{v}\zeta_jK_j;\mu_1',\cdots,\mu_r',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),(-\rho'-\beta_2'-\sum_{j=1}^{v}\zeta_jK_j;\mu_1',\cdots,\mu_r',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),$$

$$(\alpha_2'+\alpha_2-\gamma_2-\rho'-\sum_{j=1}^{v}\zeta_jK_j;\mu_1',\cdots,\mu_r',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),(\alpha_2'-\beta_2-\gamma_2-\rho'-\sum_{j=1}^{v}\zeta_jK_j;\mu_1',\cdots,\mu_r',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),$$

$$\left(1+\sigma_1'+\sum_{j=1}^{v}K_j\eta_1'^{(j)};\upsilon_1'^{(1)},\cdots,\upsilon_1'^{(r)},1,\underbrace{0,\cdots,0}_{2t-1}\right),\cdots,\left(1+\sigma_t'+\sum_{j=1}^{v}K_j\eta_t'^{(j)};\upsilon_t'^{(1)},\cdots,\upsilon_t'^{(r)},\underbrace{0,\cdots,0}_{t},1,\underbrace{0,\cdots,0}_{t-1}\right)$$

$$,\left(1+\sigma_i'+\sum_{j=1}^{v}K_j\eta_i'^{(j)};\upsilon_i'^{(1)},\cdots,\upsilon_i'^{(r)},\underbrace{0,\cdots,0}_{2t}\right)_{1,t} \quad (3.20)$$

$$\mathbf{B}=\{\tau_i(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)},\underbrace{0,\cdots,0}_{2t})_{m+1,q_i}\}:D=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\};\cdots;$$

$$\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\},\{\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)})_{m_r+1,q_{i^{(r)}}}\};\underbrace{(0;1),\cdots,(0;1)}_{2t} \quad (3.21)$$

Provided that

$$Re(\gamma_1)>0,Re(\gamma_2)>0;u_i,u_i',\lambda_j,\zeta_j,\eta_i^{(j)},\eta_i'^{(j)},\mu_k,\mu_k',\upsilon_i^{(k)},\upsilon_i'^{(k)};i=1,\cdots,t;j=1,\cdots,v;k=1,\cdots,r.$$

$|argz_i|<\frac{1}{2}A_i^{(k)}\pi$, where Ω_i is defined by (1.7).

$$Re(\rho)+\sum_{i=1}^{r}\mu_i\min_{1\leqslant l\leqslant m_i}Re\left(\frac{d_l^{(i)}}{\delta_l^{(i)}}\right)+1>\ max\{0,Re(\alpha_1+\alpha_1'+\beta_1-\gamma_1),Re(\alpha_1'-\beta_1')\}$$

$$Re(\rho')+\sum_{i=1}^{r}\mu_i'\min_{1\leqslant l\leqslant m_i}Re\left(\frac{d_l^{(i)}}{\delta_l^{(i)}}\right)+1>\ max\{0,Re(\alpha_2+\alpha_2'+\beta_2-\gamma_2),Re(\alpha_2'-\beta_2')\}$$

Proof of (3.16).
To prove the theorem 3; we use the fractional derivative formula one twice, first concerning the variable y and then concerning the variable x; here x and y are independent variables.

IV. Special Cases and Applications

The fractional derivative formulae 1, 2 and three established here are unified in nature and act as main formulae. Thus a general class of polynomials involved in fractional derivative form 1, 2 and three reduces to a wide spectrum of polynomials listed by Srivastava and Singh ([36], pp. 158–161), and so from expressions 1, 2 and three we can further obtain various fractional derivative expressions involving some simpler polynomials. Again, the multivariable H -function occurring in these formulae can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of E; F; G, H, ℵ and I -functions of one, two or more variables. For example, if $N = P = Q = 0$ (or $N = P = Q = 1$), the multivariable H -function occurring in the left-hand side of these formulae would reduce immediately to the product of r (or τ) different Fox's H-functions [4]. Thus the table listing various particular cases of the H -function ([16], pp. 145–159) can be used to derive from these fractional derivative forms some other fractional derivative formula involving any of these simpler special functions. On reducing the operator to the Riemann–Liouville operator, we arrive at three fractional derivative formulae involving these operators, but we do not record them here explicitly. Again, our theorems 1, 2 and three will also give rise in essence to some other fractional derivative relation lying scattered in the literature (see [31], pp. 563–564, Eq. (2.1)–(2.3), [32], pp. 644–645, Eq. (2.1)–(2.3)) on making suitable substitutions.

We have the following result,(see Soni and Singh [28] for more details).

$$I_{0,x}^{\alpha,\beta,\gamma}\left\{x^{\rho+\sum_{l=1}^{r}+\frac{n_1}{2}}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\sigma_i}H_{n_1}\left(\frac{1}{2\sqrt{x}}\right)L_{n_2}^{(\theta)}(x)\prod_{l=1}^{r}e^{-\frac{z_l x}{2}}W_{\mu_l v_l}(z_l x)\right.$$

$$=\frac{\prod_{l=1}^{r}z_l^{-b_l}\alpha_1^{\sigma_1}\cdots\alpha_t^{\sigma_t}x^{\rho-\beta}}{\Gamma(-\sigma_1)\cdots\gamma(-\sigma_t)}\sum_{k_1=0}^{[n_1/2]}\sum_{k_2=0}^{[n_2]}\frac{(-n_1)_{2k_1}(-n_2)_k}{k_1!k_2!}(-)^k\binom{n_2+\theta}{n_2}\frac{x^{k_1+k_2}}{(\theta+1)_{k_2}}$$

$$H_{2,2:1,2;\cdots;1,2;1,1;\cdots;1,1}^{0,2:2,0;\cdots;2,0;1,1;\cdots;1,1}\left(\begin{array}{c|c} \mathrm{z}_1 & \mathrm{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_r x & \cdot \\ \alpha_1^{(-1)}x^{u_1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \mathrm{B} \\ \alpha_t^{(-1)}x^{u_t} & \end{array}\right) \tag{4.1}$$

where

$$A=(-\rho-k_1-k_2;1,\cdots,1,u_1,\cdots,u_t),(\beta-\gamma-\rho-k_1-k_2;1,\cdots,1,u_1,\cdots,u_t),$$

$$(b_i-\mu_i+1;1)_{1,r};(1+\sigma_i;1)_{1,t} \tag{4.2}$$

$$B=(\beta-\rho-k_1-k_2;1,\cdots,1,u_1,\cdots,u_t),(-\alpha-\gamma-\rho-k_1-k_2;1,\cdots,1,u_1,\cdots,u_t),$$

$$\left(b_i\pm v_i+\frac{1}{2};1\right)_{1,r};\underbrace{(0;1);\cdots;(0;1)}_{t} \tag{4.3}$$

Concerning the corollaries, the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ vanishes and the multivariable Aleph-function reduces to Aleph-function of one variable defined by Sudland [3,38]. We shall use respectively the theorem 1 and theorem 2.

Ref

28. R.C. Soni and D. Singh, Certain fractional derivative formulae involving the product of a general class of polynomials and the multivariable H-function, Proc. Indian Acad. Sci. (Math. Sci.), 112(4) (2002), 551-562.

Notes

Corollary 1.

$$I_{0,x}^{\alpha,\alpha',\beta,\beta',\gamma}\left\{x^{\rho}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{\sigma_i}\aleph\left(\ \mathrm{z}_1x^{\mu_1}\prod_{i=1}^{t}(x^{u_i}+\alpha_i)^{-v_i^{(1)}}\ \right)\right\}$$

$$=\prod_{i=1}^{t}\alpha_i^{\sigma_i}x^{\rho-\alpha-\alpha'+\gamma}\ \aleph_{t+3,t+3:W}^{0,t+3;V}\left(\begin{array}{c|c} \mathrm{z}_1x^{\mu_1}\prod_{i=1}^{t}\alpha_i^{-v_i^{(1)}} & \mathrm{A},\mathbf{A} \\ \alpha_1^{(-1)}x^{u_1} & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \alpha_t^{(-1)}x^{u_t} & \mathrm{B},\mathbf{B} \end{array}\right) \tag{4.4}$$

where

$$V=m_1,n_1:1,0;\cdots;1,0 \tag{4.5}$$

$$W=p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\underbrace{0,1;\cdots;0,1}_{t} \tag{4.6}$$

$$A=\big(-\rho;\mu_1,u_1,\cdots,u_t\big),\ \big(\alpha'-\beta'-\rho;\mu_1,u_1,\cdots,u_t\big),\big(-\rho-\gamma+\alpha'+\beta;\mu_1,u_1,\cdots,u_t\big)$$

$$\left(1+\sigma_1;v_1^{(1)},1,\underbrace{0,\cdots,0}_{t-1}\right),\cdots,\left(1+\sigma_t;v_t^{(r)},\underbrace{0,\cdots,0}_{t-1},1\right), \tag{4.7}$$

$$\mathbf{A}=\{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};-;\cdots;-$$

$$B=\big(-\beta'-\rho;\mu_1,u_1,\cdots,u_t\big),\ \big(\alpha+\alpha'-\gamma-\rho;\mu_1,u_1,\cdots,u_t\big),\big(-\rho-\gamma+\alpha'+\beta;\mu_1,u_1,\cdots,u_t\big)$$

$$\left(1+\sigma_i;v_i^{(1)},\underbrace{0,\cdots,0}_{t}\right)_{1,t}, \tag{4.8}$$

$$\mathbf{B}=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\};\underbrace{(0;1),\cdots,(0;1)}_{t} \tag{4.9}$$

Provided that

$$Re(\gamma)>0;u_i,\mu_1,v_i^{(1)};i=1,\cdots,t$$

$$|argz_1|<\frac{1}{2}\pi\left(\sum_{j=1}^{n_1}\gamma_j^{(1)}-\tau_{i^{(1)}}\sum_{j=n_1+1}^{p_{i^{(1)}}}\gamma_{ji^{(1)}}^{(1)}+\sum_{j=1}^{m_1}\delta_j^{(1)}-\tau_{i^{(1)}}\sum_{j=m_1+1}^{q_{i^{(1)}}}\delta_{ji^{(1)}}^{(1)}\right)$$

$$Re(\rho)+\mu_1\min_{1\leqslant l\leqslant m_1}Re\left(\frac{d_l^{(1)}}{\delta_l^{(1)}}\right)+1>\ max\{0,Re(\alpha+\alpha'+\beta-\gamma),Re(\alpha'-\beta')\}$$

Corollary 2.

$$I_{0,x}^{\alpha_1,\alpha_1',\beta_1,\beta_1',\gamma_1} I_{0,x}^{\alpha_2,\alpha_2',\beta_2,\beta_2',\gamma_2} \left\{ x^{\rho} y^{\rho'} \prod_{i=1}^{t} (x^{u_i}+\alpha_i)^{\sigma_i} \prod_{i=1}^{t} (y^{u_i'}+\beta_i)^{\sigma_i'} \right.$$

$$\left. \aleph_1\left(\mathrm{z}_1 x^{\mu_1} y^{\mu_1'} \prod_{i=1}^{t} (x^{u_i}+\alpha_i)^{-\upsilon_i^{(1)}} (x^{u_i'}+\alpha_i')^{-\upsilon_i'^{(1)}} \right) \right\} = \prod_{i=1}^{t} \alpha_i^{\sigma_i} \prod_{i=1}^{t} \beta_i^{\sigma_i'} x^{\rho-\alpha_1-\alpha_1'+\gamma_1} x^{\rho'-\alpha_2-\alpha_2'+\gamma_2}$$

$$\aleph_{2t+6,2t+6:W}^{0,2t+6;V} \left(\begin{array}{c|c} \mathrm{z}_1 x^{\mu_1} y^{\mu_1'} \prod_{i=1}^{t} \alpha_i^{-\upsilon_i^{(1)}} \beta_i^{-\upsilon_i'^{(1)}} & \\ \alpha_1^{(-1)} x^{u_1} & \mathrm{A}, \mathbf{A} \\ \vdots & \vdots \\ \alpha_t^{(-1)} x^{u_t} & \vdots \\ \beta_1^{(-1)} y^{u_1'} & \vdots \\ \vdots & \mathrm{B}, \mathbf{B} \\ \alpha_t^{(-1)} y^{u_t'} & \end{array} \right) \tag{4.10}$$

where

$$V = m_1, n_1; 1, 0; \cdots ; 1, 0 \tag{4.11}$$

$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)}; \underbrace{0,1;\cdots;0,1}_{2t} \tag{4.12}$$

$$A = \big(-\rho-;\mu_1, \underbrace{0,\cdots,0}_{t}, u_1,\cdots,u_t\big),\ \big(-\alpha_1'-\beta_1'-\gamma_1-\rho;\mu_1,,\underbrace{0,\cdots,0}_{t}, u_1,\cdots,u_t\big),$$

$$(\alpha+\alpha_1'+\beta_1-\gamma_1-\rho;\mu_1,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),(-\beta_1'-\rho;\mu_1,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),$$

$$(\beta_1+\alpha_1'-\gamma_1-\rho;\mu_1,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),(\alpha_1+\alpha_1'-\gamma_1-\rho;\mu_1,\underbrace{0,\cdots,0}_{t},u_1,\cdots,u_t),$$

$$\left(1+\sigma_1;\upsilon_1^{(1)},\underbrace{0,\cdots,0}_{t},1,\underbrace{0,\cdots,0}_{t-1}\right),\cdots,\left(1+\sigma_t;\upsilon_t^{(r)},\underbrace{0,\cdots,0}_{2t-1},1\right),\left(1+\sigma_i;\upsilon_i^{(1)},\underbrace{0,\cdots,0}_{2t}\right)_{1,t}$$

$$\mathbf{A} = \{(c_j^{(1)};\gamma_j^{(1)})_{1,n_1}\},\{\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)})_{n_1+1,p_{i^{(1)}}}\};-;\cdots;- \tag{4.13}$$

$$B = \big(-\rho';\mu_1',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}\big),\ \big(\alpha_2'-\beta_2'-\gamma_2-\rho';\mu_1',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}\big),$$

Notes

$$(\alpha_2'+\alpha_2+\beta_2-\gamma_2-\rho';\mu_1',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),(-\rho'-\beta_2'-\sum_{j=1}^{v}\zeta_jK_j;\mu_1',\cdots,\mu_r',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),$$

$$(\alpha_2'+\alpha_2-\gamma_2-\rho';\mu_1',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),(\alpha_2'-\beta_2-\gamma_2-\rho';\mu_1',u_1',\cdots,u_t',\underbrace{0,\cdots,0}_{t}),$$

Notes

$$\left(1+\sigma_1';\upsilon_1'^{(1)},1,\underbrace{0,\cdots,0}_{2t-1}\right),\cdots,\left(1+\sigma_t';\upsilon_t'^{(1)},\underbrace{0,\cdots,0}_{t-1},1,\underbrace{0,\cdots,0}_{t-1}\right),\left(1+\sigma_i';\upsilon_i'^{(1)},\underbrace{0,\cdots,0}_{2t}\right)_{1,t}$$

$$\left(1+\sigma_i';\upsilon_i'^{(1)},\underbrace{0,\cdots,0}_{2t}\right)_{1,t}, \tag{4.14}$$

$$\mathbf{B}=\{(d_j^{(1)};\delta_j^{(1)})_{1,m_1}\},\{\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)})_{m_1+1,q_{i^{(1)}}}\};\ \underbrace{(0;1),\cdots,(0;1)}_{2t} \tag{4.15}$$

Provided that

$$Re(\gamma_1)>0, Re(\gamma_2)>0; u_i,u_i',\mu_1,\mu_1',\upsilon_i^{(1)},\upsilon_i'^{(1)}; i=1,\cdots,t$$

$$|arg z_1| < \frac{1}{2}\pi\left(\sum_{j=1}^{n_1}\gamma_j^{(1)}-\tau_{i^{(1)}}\sum_{j=n_1+1}^{p_{i^{(1)}}}\gamma_{ji^{(1)}}^{(1)}+\sum_{j=1}^{m_1}\delta_j^{(1)}-\tau_{i^{(1)}}\sum_{j=m_1+1}^{q_{i^{(1)}}}\delta_{ji^{(1)}}^{(1)}\right)$$

$$Re(\rho)+\mu_1\min_{1\leqslant l\leqslant m_1}Re\left(\frac{d_l^{(1)}}{\delta_l^{(1)}}\right)+1> \ max\{0,Re(\alpha_1+\alpha_1'+\beta_1-\gamma_1),Re(\alpha_1'-\beta_1')\}$$

$$Re(\rho')+\mu_1'\min_{1\leqslant l\leqslant m_1}Re\left(\frac{d_l^{(1)}}{\delta_l^{(1)}}\right)+1> \ max\{0,Re(\alpha_2+\alpha_2'+\beta_2-\gamma_2),Re(\alpha_2'-\beta_2')\}$$

Remark: We can give the similar theorems by applying the operator $I_{x,\infty}^{\alpha,\alpha',\beta,\beta',\gamma}\{\}$.

V. Conclusion

In this paper, we have obtained the theorems about generalized fractional derivative operators given by Saigo-Maeda. The images have been developed regarding the product of one or two multivariable Aleph-functions and a general class of multivariable polynomials in a compact and elegant form with the help of Saigo-Maeda operators. Most of the results obtained in this paper are useful in deriving definite composition formulae involving Riemann–Liouville, Erdelyi–Kober fractional calculus operators and multivariable Aleph-functions. The findings of this paper provide an extension of the results given earlier by Kilbas, Kilbas and Saigo, Kilbas and Sebastian, Saxena et al. and Gupta et al. as mentioned before.

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Certain Study of Bicomplex Matrices and a New Composition of Bicomplex Matrices

By Prabhat Kumar & Akhil Prakash
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Certain Study of Bicomplex Matrices and a New Composition of Bicomplex Matrices

Prabhat Kumar [α] & Akhil Prakash [σ]

Abstract- In this paper, we have studied Orthogonal and Unitary matrix in C_2, some theorems and properties related to bicomplex matrix. We have defined the new concept over the bicomplex matrix, relation between bicomplex matrix and its complex component matrix, algebraic structure of bicomplex matrix in new system as well as the new definition of inverse of matrix in C_2 and some properties in new system. A similar relation between two bicomplex matrices is also defined in this paper.

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I. Introduction

In 1892, Corrado Segre (1860-1924) published a paper [9] in which he treated an infinite set of Algebras whose elements he called bicomplex numbers, tricomplex numbers,...., n-complex numbers. A number which can expressed in the form of $x_1+i_1x_2+i_2x_3+i_1i_2x_4$, $i_p^2 = -1$,for all p=1, 2 and $i_1i_2 = i_2i_1$ as well as $x_{1,}\ldots,x_4$ are real numbers, is called a bicomplex number. Segre showed that every bicomplex number $z_1+i_2z_2$ can be represented as the complex combination

$$(z_1-i_1z_2)\left[\frac{1+i_1i_2}{2}\right]+(z_1+i_1z_2)\left[\frac{1-i_1i_2}{2}\right]$$

Shrivastava [10] introduced the notations ${}^1\xi$ and ${}^2\xi$ for the idempotent components of the bicomplex number $\xi = z_1+i_2z_2$, so that

$$\xi = {}^1\xi.\frac{1+i_1i_2}{2}+{}^2\xi.\frac{1-i_1i_2}{2}$$

Michiji Futagawa seems to have been the first to consider the theory of functions of a bicomplex variable [2, 3] in 1928 and 1932.

The hyper complex system of Ringleb [8] is more general than the Algebras; he showed in 1933 that Futagawa system is a special case of his own.

In 1953 James D. Riley published a paper [7] entitled "Contributions to theory of functions of a bicomplex variable".

In the entire work, the symbols C_2, C_1 and C_0 denote the set of all bicomplex, complex and real numbers respectively.

In C_2-besides 0 and 1- there are exactly two non-trivial idempotent elements denoted as e_1 and e_2 and defined as

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$$e_1 = \frac{1+i_1 i_2}{2} \text{ and } e_2 = \frac{1-i_1 i_2}{2}$$

Obviously $(e_1)^n = e_1$, $(e_2)^n = e_2$

$$e_1 + e_2 = 1,\ e_1.e_2 = 0$$

Every bicomplex number ξ has unique idempotent representation as complex combination of e_1 and e_2 as follows

$$\xi = z_1+i_2 z_2 = (z_1-i_1 z_2)e_1 + (z_1+i_1 z_2)e_2$$

The complex numbers $(z_1-i_1z_2)$ and $(z_1+i_1z_2)$ are called idempotent component of ξ, and are denoted by ${}^1\xi$ and ${}^2\xi$ respectively (cf. Srivastava [10]).

Thus $\xi = {}^1\xi\ e_1 + {}^2\xi\ e_2$

The idempotent representation is perfectly consistent with the Algebraic structure of C_2 in the following sense

$$\xi \pm \eta = ({}^1\xi\ e_1 + {}^2\xi\ e_2) \pm ({}^1\eta\ e_1 + {}^2\eta\ e_2)$$
$$= ({}^1\xi \pm {}^1\eta)\ e_1 + ({}^2\xi \pm {}^2\eta)\ e_2$$

So that ${}^1(\xi \pm \eta) = {}^1\xi \pm {}^1\eta$ and ${}^2(\xi \pm \eta) = {}^2\xi \pm {}^2\eta$

$$a.\xi = a.\ ({}^1\xi\ e_1 + {}^2\xi\ e_2)$$
$$= (a.{}^1\xi)\ e_1 + (a.{}^2\xi)\ e_2$$

So that ${}^1(a.\xi) = a.{}^1\xi$ and ${}^2(a.\xi) = a.{}^2\xi$

$$\xi.\eta = ({}^1\xi\ e_1 + {}^2\xi\ e_2).\ ({}^1\eta\ e_1 + {}^2\eta\ e_2)$$
$$= ({}^1\xi.{}^1\eta)\ e_1 + ({}^2\xi.{}^2\eta)\ e_2$$

So that ${}^1(\xi.\eta) = {}^1\xi.{}^1\eta$ and ${}^2(\xi.\eta) = {}^2\xi.{}^2\eta$

$\xi / \eta = ({}^1\xi / {}^1\eta)\ e_1 + ({}^2\xi / {}^2\eta)\ e_2$; provided $\eta \notin O_2$

So that ${}^1(\xi / \eta) = {}^1\xi / {}^1\eta$ and ${}^2(\xi / \eta) = {}^2\xi / {}^2\eta$,

where O_2 = set of all singular element in C_2

a) Singular elements and Norm of a bicomplex number

There are infinite numbers of element in C_2 which do not possess multiplicative inverse. A bicomplex number $\xi = z_1 + i_2 z_2$ is singular iff $\left|z_1^2 + z_2^2\right| = 0$. Evidently a nonzero bicomplex number ξ is singular if and only if either ${}^1\xi = 0$ or ${}^2\xi = 0$. In fact C_2 is not a field while C_1 is a field.

The norm of a bicomplex number ξ is defined as

$$\|\xi\| = \|z_1 + i_2 z_2\|$$
$$= \left\{|z_1|^2 + |z_2|^2\right\}^{1/2}$$
$$= \left\{\frac{1}{2}\left(|{}^1\xi|^2 + |{}^2\xi|^2\right)\right\}^{1/2}$$

C_2 forms a modified Banach algebra. i.e. Banach algebra with modified consistency of the norm of product of two bicomplex number is less than or equal to $\sqrt{2}$ time of product of their individual norm i.e. $\|\xi\eta\| \le \sqrt{2}\|\xi\|\|\eta\|$

Ref

10. Srivastava, Rajiv K.: Bicomplex Numbers: Analysis and applications, Math. Student, 72 (1-4) 2003, 69-87.

Notes

b) Some special properties and subsets of bicomplex space

Every bicomplex number ξ possesses three types of conjugates called i_1-conjugate, i_2-conjugate and i_1i_2-conjugate corresponding to i_1, i_2 and i_1i_2 independent vectors respectively represented by $\bar{\xi}, \xi^{\sim}$ and $\xi^{\#}$. Thus

$$\bar{\xi} = (x_1 - i_1 x_2) + i_2 (x_3 - i_1 x_4) = \bar{z}_1 + i_2 \bar{z}_2 = ({}^2\bar{\xi})e_1 + ({}^1\bar{\xi})e_2$$

$$\xi^{\sim} = (x_1 + i_1 x_2) - i_2 (x_3 + i_1 x_4) = z_1 - i_2 z_2 = ({}^2\xi)e_1 + ({}^1\xi)e_2$$

$$\xi^{\#} = (x_1 - i_1 x_2) - i_2 (x_3 - i_1 x_4) = \bar{z}_1 - i_2 \bar{z}_2 = ({}^1\bar{\xi})e_1 + ({}^2\bar{\xi})e_2$$

We shall use specific notations for some special subset of C_2 that are given below.

$$C(i_1) = \{a + i_1 b : a, b \in C_0\}$$

$$C(i_2) = \{a + i_2 b : a, b \in C_0\}$$

$$H = \{a + i_1 i_2 b : a, b \in C_0\}$$

c) Representation of bicomplex matrix

A matrix 'A' whose entries are bicomplex numbers is called bicomplex matrix i.e.

$$A = \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn} \end{bmatrix}, \ \forall \xi_{pq} \text{ in } C_2,\ 1 \le p \le m \text{ and } 1 \le q \le n$$

According to three types of representation of a bicomplex number, there are three types of representation of a bicomplex matrix as real representation, complex representation and idempotent representation.

A square bicomplex matrix "A" is said to be non-singular if $|A| \notin O_2$ otherwise the matrix will be singular.

II. Certain Results on Bicomplex Matrices

a) Algebraic structure and Inversion of Bicomplex matrices[1]

2.1.1 Algebraic structure

Let M be the set of all square and non-singular bicomplex matrices of order n then the set M with operations addition "+" coordinate wise, multiplication "×" is term by term multiplication as well as scalar multiplication "." is also coordinate wise, forms an algebra over the field of complex number.

2.1.2 Determinant and Adjoint of a bicomplex matrix

Let A= $[\xi_{ij}]_{n\times n}$ be the bicomplex square matrix of order n where n is the positive integer. The determinant of A is defined by

$$|A| = |[\xi_{ij}]|,\ \xi_{ij} \in C_2$$

$$= \sum_{j^i=1}^{n} \pm \xi_{1j^1} \xi_{2j^2} \cdots \xi_{nj^n}$$

where $\pm$ sign is taken according to even and odd permutation of suffixes of ξ.

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Let A $= [\xi_{ij}]_{n\times n}$ be a bicomplex square matrix and $[\zeta_{ij}]_{n\times n}$ denote the co-factor matrix of A then the transpose of the matrix $[\zeta_{ij}]_{n\times n}$ is defined as Adjoint of A and denoted by Adj.A.

Some Results-

(a) $|A| = |{}^1A|\, e_1 + |{}^2A|\, e_2$

(b) If $|A| \notin O_2 \Leftrightarrow |{}^1A| \neq 0 \ \& \ |{}^2A| \neq 0$

(c) Adj.A = Adj.(^{1}A) e_1 + Adj.(^{2}A) e_2

2.1.3 Inversion of Bicomplex matrix by two techniques

Anjali [1] has developed two techniques to determine the inverse of bicomplex matrix.

a. *Adjoint technique*

Let A $= [\xi_{ij}]_{n\times n}$ be a square and non-singular matrix whose elements are in C_2 then Inverse of A is defined as

$$A^{-1} = \frac{A.(AdjA)}{|A|}$$

b. *Idempotent technique*

Suppose M = ^{1}M e_1 + ^{2}M $e_2 = [\xi_{ij}]_{n\times n}$ be a square and nonsingular bicomplex matrix of order n. Let $[z_{ij}]_{n\times n}$ and $[w_{ij}]_{n\times n}$ be the inverse of ^{1}M and ^{2}M respectively then Inverse of M is defined as

$$M^{-1} = [z_{ij}]_{n\times n}\, e_1 + [w_{ij}]_{n\times n}\, e_2 = [\eta_{ij}]_{n\times n} \ \text{(say)}$$

b) Hermitian and Skew-Hermitian matrix in C_2 [1]

2.2.1 Tranjugate of a bicomplex matrix

Analogous to three types of conjugate element in C_2 we have three types of conjugate of a matrix in C_2 viz. are i_1 conjugate matrix, i_2 conjugate matrix and i_1i_2 conjugate matrix. The transpose of the conjugate matrix is called tranjugate of the matrix. There are three types of tranjugates of a matrix in C_2.

a. *i_1 tranjugate of a bicomplex matrix*

Let A $= [a_{ij}]_{n\times n}$ be any bicomplex matrix and $\bar{A}$ denotes the i_1 conjugate of A obtained by taking i_1 conjugate of each entry of A. Transposing $\bar{A}$, we get the tranjugate of A. Simply denoted by $[\bar{A}]^T$ or A^{θ_1}

b. *i_2 tranjugate of a bicomplex matrix*

The i_2 conjugate of a bicomplex matrix A denoted by $\tilde{A}$ is the matrix obtained by taking i_2 conjugate of each entry of A. On taking transpose of $\tilde{A}$ then we obtain $[\tilde{A}]^T$ which is known as i_2 tranjugate of A and denoted by A^{θ_2}.

c. *i_1i_2 tranjugate of a bicomplex matrix*

The i_1i_2 conjugate of a bicomplex matrix A denoted by $A^\#$ is the matrix obtained from A by taking i_1i_2 conjugate of each entry of A. On taking transpose of $A^\#$, we obtain $[A^\#]^T$ which is known as i_1i_2 tranjugate of A and denoted by A^{θ_3}.

Properties of a bicomplex matrix [1]-

For all 'k' in C_2 and A, B of $C_2^{n\times n}$ then

(1) $\overline{[\bar{A}]} = A$

Ref

1. Anjali: Certain results on bicomplex matrices, M. Phil. Dissertation, Dr. B. R. Ambedker University, Agra (2011).

Notes

(2) $[\tilde{A}]^{\sim} = A$

(3) $[A^{\#}]^{\#} = A$

(4) $(\overline{A + B}) = \bar{A} + \bar{B}$

(5) $(A+B)^{\sim} = A^{\sim} + B^{\sim}$

(6) $(A+B)^{\#} = A^{\#} + B^{\#}$

(7) $\overline{kA} = \bar{k}\bar{A}$

(8) $[k\ A]^{\sim} = k^{\sim}.A^{\sim}$

(9) $[k\ A]^{\#} = k^{\#}.A^{\#}$

(10) $[\overline{[(\bar{A})^T]}]^T = A$

(11) $[[(\tilde{A})^T]^{\sim}]^T = A$

(12) $[[[A^{\#}]^T]^{\#}]^T = A$

(13) $(\overline{kA})^T = \bar{k}.[\bar{A}]^T$

(14) $[[kA]^{\sim}]^T = k^{\sim}.[A^{\sim}]^T$

(15) $[[kA]^{\#}]^T = k^{\#}.[A^{\#}]^T$

2.2.2 Symmetric and Skew-symmetric matrix in C_2 [1]

A square bicomplex matrix "A" is symmetric if $A^T = A$ or $a_{ij} = a_{ji}$ for all i, j and if $A^T = -A$ or $a_{ij} = -a_{ji}$ for all i, j then it is called a skew symmetric matrix.
In skew symmetric matrix all principal diagonal elements are zero.

2.2.3 Hermitian and Skew-Hermitian matrix in C_2

Since three types of conjugate elements exist in C_2 and each conjugate will introduce Hermitian matrix, so that in C_2, there will be three types of Hermitian matrices.

a. *i_1-Hermitian matrix*

A bicomplex square matrix A is said to be i_1-Hermitian matrix if $A = [\bar{A}]^T$.
The element of the principal diagonal of i_1-Hermitian matrix are the member of $C(i_2)$ i.e. i_2-complex number.

b. *i_2-Hermitian matrix*

A bicomplex square matrix A is said to be i_2-Hermitian matrix if $A = [\tilde{A}]^T$.

The element of the principal diagonal of i_2-Hermitian matrix are the member of $C(i_1)$ i.e. i_1-complex number.

c. *i_1i_2-Hermitian matrix*

A bicomplex square matrix A is said to be i_1i_2-Hermitian matrix if $A = [A^{\#}]^T$.

The elements of the principal diagonal of i_1i_2-Hermitian matrix are the member of H (set of hyperbolic numbers).
There are three types of skew Hermitian matrix in C_2.

- *i_1-Skew Hermitian matrix*

A bicomplex square matrix A is said to be i_1-skew Hermitian matrix If $A = -[\bar{A}]^T$.

The element of the principal diagonal of i_1-skew Hermitian matrix are the member of the type $i_1(s)$, $s \in C(i_2)$.

- *i_2-Skew Hermitian matrix*

A bicomplex square matrix A is said to be i_2-skew Hermitian matrix if $A = -[\tilde{A}]^T$.

The elements of the principal diagonal of i_2-skew Hermitian matrix are the member of the type $i_2(s)$, $s \in C(i_1)$.

- *i_1i_2-Skew Hermitian matrix*

A bicomplex square matrix A is said to be i_1i_2 - skew Hermitian matrix if $A = -[A^{\#}]^T$ or $(A^{\#})^T = -A$.

The elements of the principal diagonal of i_1i_2 - skew Hermitian matrix are the member of the type $i_1(s)$, $s \in H$

III. Study of Bicomplex Matrix under Traditional and New System

In this section we present the work which has been done by us. In this section we have studied Orthogonal and Unitary matrix in C_2 and defined the new concept over the bicomplex matrix. A similar relation between two bicomplex matrices is also defined in this section.

a) *Orthogonal and Unitary Bicomplex matrices*

3.1.1. Orthogonal Bicomplex matrix

Let A be any square and invertible bicomplex matrix then A is said to be orthogonal bicomplex matrix if

$$A^T A = I = A A^T$$

i.e .$A^{-1} = A^T$

where A^T is the transpose of A and I is the identity matrix.

3.1.2 Unitary bicomplex matrices

Corresponding to three types of tranjugate of any bicomplex matrix, there are three types of bicomplex Unitary matrix.

a. *i_1 Unitary matrix*

Let A be any square bicomplex matrix which is invertible and $\bar{A}$ denote the i_1 conjugate of A and $[\bar{A}]^T$ is the transpose of i_1 conjugate of A. We shall use A^{θ_1} in place of $[\bar{A}]^T$ in entire work.

The matrix A is called i_1Unitary matrix if $A^{\theta_1} A = I = A A^{\theta_1}$

i.e. $A^{-1} = A^{\theta_1}$

Thus, the matrix $[\zeta_{ij}]_{n\times n}$ is an i_1 Unitary matrix if

$[\overline{\zeta_{ji}}]_{n\times n} . [\zeta_{ij}]_{n\times n} = I_{n\times n}$,where $I_{n\times n}$is the identity matrix.

b. *i_2 Unitary matrix*

Let A be any square bicomplex matrix which is invertible and $A^{\sim}$ be the i_2 conjugate of A and $[A^{\sim}]^T$ be transpose of i_2 conjugate matrix of A and we shall use A^{θ_2} in place of $[A^{\sim}]^T$ in entire work.

If $A A^{\theta_2} = I = A^{\theta_2} A$

ie. $A^{-1} = A^{\theta_2}$

Then A is called i_2Unitary matrix.

c. *i_1i_2 Unitary matrix*

Let A be any square bicomplex matrix which is invertible and $A^{\#}$ be the i_1i_2 conjugate of A and $[A^{\#}]^T$ be transpose of i_1i_2 conjugate matrix of A. We shall use A^{θ_3} in place of $[A^{\#}]^T$in entire work. If $A A^{\theta_3} = I = A^{\theta_3} A$

Notes

i.e. $A^{-1} = A^{\theta_3}$

then A is called i_1i_2 Unitary matrix.

Remark: If A is any real Unitary matrix then it will obviously be an i_1 Unitary matrix, i_2 Unitary matrix and i_1i_2 Unitary matrix.

3.1.3 Theorem

If A and B are two i_1i_2 Unitary matrices of same order then AB will be i_1i_2 Unitary matrix similarly i_1AB, i_2AB, i_1i_2AB will also be i_1i_2 Unitary matrices.

Notes

Proof:

By the definition of i_1i_2Unitary matrix

$A^{\theta_3}.A = I$, similarly $B^{\theta_3}.B = I$

$$\begin{aligned}(AB)^{\theta_3}.AB &= \{^1(AB)e_1 + {}^2(AB)e_2\}^{\theta_3}.AB \\ &= \{^1(AB)^{\theta_3} e_1 + {}^2(AB)^{\theta_3} e_2\}.AB \\ &= \left[{}^1(B^{\theta_3} A^{\theta_3})e_1 + {}^2(B^{\theta_3} A^{\theta_3})e_2\right] AB \\ &= (B^{\theta_3} A^{\theta_3}).AB \\ &= B^{\theta_3} A^{\theta_3}.AB \\ &= B^{\theta_3}(A^{\theta_3}.A)B \\ &= B^{\theta_3}.I.B \qquad (\because A \text{ is } i_1i_2 \text{ unitary}) \\ &= B^{\theta_3}.B = I \qquad (\because B \text{ is } i_1i_2 \text{ unitary})\end{aligned}$$

Now we find out the nature of i_1AB, i_2AB, i_1i_2AB

Therefore

$$\left.\begin{aligned}(i_1AB)^{\theta_3}.i_1AB &= \bar{i}_1B^{\theta_3}A^{\theta_3}.i_1AB \\ &= B^{\theta_3}IB \\ &= I\end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned}(i_2AB)^{\theta_3}.i_2AB &= \bar{i}_2B^{\theta_3}A^{\theta_3}.i_2AB \\ &= B^{\theta_3}IB \\ &= I\end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned}(i_1i_2AB)^{\theta_3}.i_1i_2AB &= \bar{i}_1\bar{i}_2B^{\theta_3}A^{\theta_3}.i_1i_2AB \\ &= (-i_1)(-i_2),i_1i_2B^{\theta_3}IB \\ &= I\end{aligned}\right\} \qquad (3)$$

Proof of the theorem is complete.

Remark:

$$(AB)^{\theta_S} \neq B^{\theta_S}A^{\theta_S}, \quad where \quad S = 1,2$$

Therefore analogues of theorem 3.1.3 is not true for i_1Unitary and i_2Unitary matrices.

3.1.4 Theorem

The adjoint of i_1i_2 tranjugate of a bicomplex square matrix is equal to the i_1i_2 tranjugate of the adjoint of the matrix.

$$Adj(A^{\theta_3}) = (Adj\, A)^{\theta_3}$$

Proof:

$$(Adj\ A)^{\theta_3} = (Adj\ {}^1A\, e_1 + Adj\ {}^2A\, e_2)^{\theta_3} [by\ 2.1.2(c)]$$

$$= (Adj\ {}^1A)^{\theta_3} e_1 + (Adj\ {}^2A)^{\theta_3} e_2$$

$$= Adj\,({}^1A^{\theta_3})e_1 + Adj\,({}^2A^{\theta_3})e_2$$

$$\because (Adj\ A)^{\theta} = Adj\,(A^{\theta}),\ for\ A\ in\ C_1\ therefore$$

$$(Adj\ A)^{\theta_3} = Adj[({}^1A^{\theta_3})e_1 + ({}^2A^{\theta_3})e_2]$$

$$= Adj\,(A^{\theta_3})$$

Remark:

Since i_1 and i_2 conjugates of e_1 is e_2 and e_2 is e_1, we get $(Adj\ A)^{\theta_S} \neq Adj\,(A^{\theta_S})$ $where\ S = 1,2$

3.1.5 Theorem

Let A and B be two square bicomplex matrices of order n, such that $|A| \notin O_2$ and $|B| \notin O_2$, then their product (AB) will be invertible, and the inverse of AB will be $B^{-1}A^{-1}$.

Proof:

Since the bicomplex matrices A and B both are nonsingular i.e. $|A| \notin O_2$ and $|B| \notin O_2$, that means

$$|A| = |{}^1A|e_1 + |{}^2A|e_2 \notin O_2$$

$$\Leftrightarrow |{}^1A| \neq 0\ and\ |{}^2A| \neq 0$$

$$similarly\ |{}^1B| \neq 0\ and\ |{}^2B| \neq 0$$

$$\Rightarrow |{}^1A|.|{}^1B| \neq 0\ and\ |{}^2A|.|{}^2B| \neq 0$$

$$\Rightarrow \left(|{}^1A||{}^1B|\right)e_1 + \left(|{}^2A||{}^2B|\right)e_2 \notin O_2$$

$$\Rightarrow |{}^1(AB)|e_1 + |{}^2(AB)|e_2 \notin O_2$$

$$\Rightarrow |AB| \notin O_2$$

⇒ AB is invertible.

The inverse of the matrix (AB) is $(AB)^{-1}$therefore

Further

$$(AB)^{-1} = [{}^1(AB)e_1 + {}^2(AB)e_2]^{-1}$$

$$= {}^1(AB)^{-1}e_1 + {}^2(AB)^{-1}e_2$$

$$= ({}^1B^{-1}\,{}^1A^{-1})e_1 + ({}^2B^{-1}\,{}^2A^{-1})e_2 \text{ (since } (PQ)^{-1} = Q^{-1}P^{-1} \text{ in } C_1)$$

$$= [{}^1B^{-1}e_1 + {}^2B^{-1}e_2]\,[{}^1A^{-1}e_1 + {}^2A^{-1}e_2]$$

$$= B^{-1}A^{-1}$$

3.1.6 Theorem

Let A, B be two square bicomplex matrices then determinant of their product will be equal to product of their individual determinant.

Proof:

$$|AB| = |{}^1(AB)|e_1 + |{}^2(AB)|e_2$$

$$= \left(|{}^1A||{}^1B|\right)e_1 + \left(|{}^2A||{}^2B|\right)e_2$$

$$= \left(|{}^1A|e_1 + |{}^2A|e_2\right)\left(|{}^1B|e_1 + |{}^2B|e_2\right)$$

$$\Rightarrow |AB| = |A|.|B|$$

Notes

3.1.7 Some Properties of bicomplex matrices

(i) If A is any bicomplex square matrix of order n then det A and the det of transpose A are equal.

(ii) A and B are two bicomplex matrices of order n such that B is obtained from interchanging any two row /column only of A then $|A| = -|B|$.

(iii) If any one of the row/column in a square bicomplex matrix has each element in O_2 then matrix will be singular or non - invertible.

Notes

Proofs of these results are straight forward.

b) Study under new system

If A is any bicomplex matrix, then it can be written as

$$A = A_0 + i_2 A_1, where\ A_s \in C_1^{m\times n}, s = 0,1$$

The i_2 independent part and dependent part of bicomplex matrix A is denoted by A_0 and A_1 respectively. The matrices A_0 and A_1 are known as complex component of matrix A.

We define a new binary composition "ʘ" between two arbitrary square biocomplex matrices A and B as follow

$$\forall\ A, B \in C_2^{n\times n} \text{ then}$$

$$A ⊙ B = (A_0 + i_2 A_1) ⊙ (B_0 + i_2 B_1)$$

$$= (A_0 B_0 + i_2 A_1 B_1)$$

$$= (C_0 + i_2 C_1) \in C_2^{n\times n}, \text{ where } C_s \in C_1^{n\times n} \forall\ s = 0, 1$$

It is a new definition of product of two bicomplex matrices (specially) and the procedures of both, addition and scalar multiplication, will be the same as traditional system procedures.

Thus the three operations will be as follow

$$\forall A, B \in C_2^{n\times n}$$

$$\text{"+"} \rightarrow A+B = [A_0 + i_2 B] + [B_0 + i_2 B_1]$$

$$= [A_0 + B_0] + i_2 [A_1 + B_1]$$

$$\text{"⊙"} \rightarrow A ⊙ B = [A_0 + i_2 A_1] ⊙ [B_0 +_2 B_1]$$

$$= A_0 B_0 + i_2 A_1 B_1$$

$$\text{and "}\bullet\text{"} \rightarrow \alpha.A = \alpha [A_0 + i_2 A_1]$$

$$= \alpha A_0 + i_2 \alpha A_1$$

$$\xi.\ A = [\alpha + i_2 \beta].\ [A_0 + i_2 A_1]$$

$$= \alpha A_0 + i_2 \alpha A_1 - \beta A_1 + i_2 \beta A_0$$

$$= \alpha A_0 - \beta A_1 + i_2 [\alpha A_1 + \beta A_0]$$

3.2.1 Relation between the bicomplex matrix and its complex component matrices

We define addition on the set $C_1^{m\times n} \times C_1^{m\times n}$ as follow.

If (A_0, A_1) and (B_0, B_1) are two arbitrary element of $C_1^{m\times n} \times C_1^{m\times n}$ then $(A_0, A_1) + (B_0, B_1) = (A_0+B_0, A_1+B_1)$. Further (A_0, A_1) and (B_0, B_1) are said to be equal if and only if $A_0 = B_0$ and $A_1 = B_1$ The set $C_1^{m\times n} \times C_1^{m\times n}$ is an abelian group w.r.t. addition '+'.

We define a function $f : C_2^{m\times n} \rightarrow C_1^{m\times n} \times C_1^{m\times n}$

Such that $f(A) = (A_0, A_1)$

3.2.2 Theorem

If f: $C_2^{m\times n} \rightarrow C_1^{m\times n} \times C_1^{m\times n}$ is the function Such that $f(A) = (A_0, A_1)$ then f is an on to isomorphism i.e. $C_2^{m\times n} \cong C_1^{m\times n} \times C_1^{m\times n}$

Proof:

f is one-one:

$$\forall\ A, B \in C_2^{m\times n}$$

$$\text{Let}\quad f(A) = f(B)$$

$$\Rightarrow (A_0, A_1) = (B_0, B_1)$$

$$\Rightarrow A_0 = B_0 \text{ and } A_1 = B_1$$

$$\Rightarrow A = B$$

f is onto:

Let (A_0, A_1) be the arbitrary element of $C_1^{m\times n} \times C_1^{m\times n}$.

Corresponding to (A_0, A_1) there exist a bicomplex matrix $A = A_0+i_2A_1$ such that

$$f(A) = f(A_0+i_2A_1)$$

$$= (A_0, A_1)$$

Therefore A is the preimage of (A_0, A_1) in $C_2^{m\times n}$.

f is homomorphism:

$$\forall\ A, B \in C_2^{m\times n}$$

$$f(A+B) = f\,[(A_0+ B_0) + i_2\,(A_1+B_1)]$$

$$= (A_0+B_0,\ A_1+B_1)$$

$$= [(A_0, A_1) + (B_0, B_1)]$$

$$= f(A) + f(B)$$

Hence f is an on to isomorphism i.e. $C_2^{m\times n} \cong C_1^{m\times n} \times C_1^{m\times n}$

3.2.3 Theorem

Let M be the set of all square bicomplex matrix of order n. If we introduce the operation "⊙" with set M over new system and binary operation addition "+"taken coordinate wise and scalar multiplication "•" is term by term then the structure [M,"+","•","⊙"] forms an algebra with identity $(I + i_2 I)$.

Proof:

$$\forall\ A, B, C \in M^{n\times n}$$

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therefore $A= A_0+i_2A_1$, $B= B_0+ i_2B_1$, $C=C_0+i_2C_1$
(M,+) is an abelian group:
Closure:

$$A+B = [A_0+i_2A_1] + [B_0+ i_2B_1]$$
$$= [A_0+B_0] +i_2[A_1+B_1] \in M$$

Associativity:

$$A + (B + C) = (A + B) + C \text{ (Hold)}$$

Additive identity:

$$\forall\ A \in M,\ A = [A_0 + i_2A_1]\ \exists \text{an } (0 + i_20)$$
$$A + (0 + i_20) = [A_0 + i_2A_1] + [0 + i_20]$$
$$= [A_0 + 0] + i_2\ [A_1 + 0]$$
$$= A_0 + i_2A_1 = A$$

Hence $(0 + i_20)$ is the additive identity.

Inverse property:

$\forall\ A \in M,\ \exists\ -A\in M$, such that

$$[A_0+i_2A_1] - [A_0+ i_2A_1] = [A_0 - A_0] + i_2\ [A_1 - A_1] =[0 + i_20]$$

Commutativity:

$$A + B = B + A\ \ \forall\ A, B \in M$$

$[M, +, \odot]$ is ring structure:

Closure under new multiplication '$\odot$':

$$\forall\ A, B \in M$$
$$A \odot B = (A_0\ B_0) + i_2\,(A_1\ B_1) \in M$$

Associativity:

$$A \odot [B \odot C] = [A_0+i_2A_1] \odot [(B_0\ C_0) + i_2\,(B_1\ C_1)]$$
$$= A_0\ [B_0\ C_0] + i_2\ A_1\,[B_1\ C_1]$$

$\because$ Complex matrix are associative and A_s, B_S, C_S in $C_1^{n\times n}$, $\forall\ S = 0, 1$

$$A \odot [B \odot C] = [A_0\ B_0]\ C_0 + i_2\ [A_1\,B_1]\ C_1$$
$$= [(A_0+i_2A_1) \odot (B_0+i_2B_1)] \odot (C_0+i_2C_1)$$
$$= [A \odot B] \odot C$$

Distribution property:

$$\forall\ A, B, C \in M$$
$$A \odot [B + C] = [A_0+i_2A_1] \odot [(B_0 + C_0) + i_2\,(B_1 + C_1)]$$

$$= A_0\,(B_0 + C_0) + i_2\,A_1\,[B_1 + C_1)$$
$$= [A_0B_0 + A_0C_0] + i_2\,[A_1B_1 + A_1C_1]$$

(Distributive laws for complex matrices)

$$= [A_0\,B_0 + A_0\,C_0] + [i_2\,A_1\,B_1 + i_2\,A_1\,C_1]$$
$$= [A_0B_0 + i_2\,A_1B_1] + [A_0C_0 + i_2\,A_1C_1]$$
$$= \;A \odot B + A \odot C$$

Notes

Linear space:

Closed w.r.t scalar multiplication:

$$\forall\ \alpha\in C_0 \rightarrow \alpha.A = \alpha\,[A_0+i_2A_1]$$
$$= \alpha\,A_0+ \alpha\,i_2A_1$$

$$\alpha.A = \alpha\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} + i_2\,\alpha\begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha z_{11} & \alpha z_{12} & \cdots & \alpha z_{1n} \\ \alpha z_{21} & \alpha z_{22} & \cdots & \alpha z_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha z_{n1} & \alpha z_{n2} & \cdots & \alpha z_{nn} \end{bmatrix} + i_2\begin{bmatrix} \alpha w_{11} & \alpha w_{12} & \cdots & \alpha w_{1n} \\ \alpha w_{21} & \alpha w_{22} & \cdots & \alpha w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha w_{n1} & \alpha w_{n2} & \cdots & \alpha w_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha z_{11}+i_2\alpha w_{11} & \alpha z_{12}+i_2\alpha w_{12} & \cdots & \alpha z_{1n}+i_2\alpha w_{1n} \\ \alpha z_{21}+i_2\alpha w_{21} & \alpha z_{22}+i_2\alpha w_{22} & \cdots & \alpha z_{2n}+i_2\alpha w_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha z_{n1}+i_2\alpha w_{n1} & \alpha z_{n2}+i_2\alpha w_{n2} & \cdots & \alpha z_{nn}+i_2\alpha w_{nn} \end{bmatrix} \in M.$$

Again

$$\forall\ A \in M \text{ and } 1\in C_0,\ 1.A = 1.[\,A_0 + i_2A_1]$$
$$= 1.A_0 + i_2\,1.A_1$$
$$= A_0 + i_2A_1 = A$$

$$\forall\ \alpha,\beta\in F,\quad (\alpha + \beta)A = (\alpha + \beta)\ [\,A_0 + i_2A_1]$$
$$= (\alpha + \beta)A_0 + i_2(\alpha + \beta)A_1$$
$$= (\alpha A_0 + \beta A_0) + i_2(\alpha A_1 + \beta A_1)$$
$$= [\alpha\,A_0+i_2\,\alpha\,A_1] + [\beta\,A_0+i_2\,\beta\,A_1]$$
$$= \alpha\,A + \beta\,A$$

$$\forall\ A, B \in M, \text{ and } \alpha\in F$$

$$\alpha\,[\,A + B] = \alpha[(A_0+ B_0)+i_2\,(A_1+ B_1)]$$
$$= \alpha\,(A_0+ B_0) + i_2\,\alpha\,(A_1+ B_1)$$
$$= (\alpha\,A_0+ \alpha B_0) + i_2\,(\alpha A_1+ \alpha B_1)$$

Notes

$$= \alpha A + \alpha B$$

$$(\alpha \beta) A = \alpha \beta (A_0) + i_2 \alpha \beta (A_1)$$

$$= \alpha (\beta A_0) + i_2\alpha (\beta A_1)$$

$$= \alpha [\beta A_0 + i_2\beta A_1]$$

$$= \alpha. [\beta. A]$$

Consistency (compatibility) between ⊙ and •:

$$\forall A, B \in M, \text{ and } \alpha \in F$$

$$\alpha [A \odot B] = \alpha[(A_0 B_0)+i_2(A_1 B_1)]$$

$$= \alpha (A_0 B_0) + i_2 \alpha (A_1 B_1)$$

$$= (\alpha A_0) B_0 + i_2 (\alpha A_1) B_1$$

$$= (\alpha A_0 + i_2\alpha A_1) \odot (B_0 + i_2 B_1)$$

$$= (\alpha.A) \odot B$$

$$= (A_0\alpha) B_0 + i_2 (A_1\alpha) B_1$$

$$= A_0 (\alpha B_0) + i_2 A_1 (\alpha B_1)$$

$$= [A_0+i_2A_1] \odot [\alpha B_0+i_2 \alpha B_1]$$

$$= A \odot (\alpha B)$$

Hence M (set of all square bicomplex matrix of order n) is an algebra.
Identity:

$A_0 + i_2A_1 = A$, $\exists (I + i_2 I)$such that

$$[A_0 + i_2A_1] \odot [I + i_2 I] = A_0I + i_2A_1I = A = (I + i_2 I).A$$

$\Rightarrow (I + i_2I)$ will be the identity under new system
Moreover for all ξ in C_2

$$\xi. A = (z_1 + i_2z_2) (A_0 + i_2A_1)$$

$$= z_1 A_0 + i_2z_1 A_1 + i_2 z_2 A_0 - z_2 A_1$$

3.2.4 Definition: New inversion of a square bicomplex matrix

Let $A = A_0 + i_2 A_1 \in C_2^{n\times n}$ be given bicomplex matrix where A_0, A_1 are complex matrix of same order if the inverse of A_0 and A_1 both exist then bicomplex matrix A is said to be invertible and inverse of A is written as $A^{-} = A_0^{-} + i_2A_1^{-}$, where A_0^{-} and A_1^{-} are the inverse of A_0 and A_1 respectively as well as A^{-} is the inverse of A or reciprocal of A.

3.2.5 Theorem

Let A be any square bicomplex matrix which is invertible in new system, then the inverse of the bicomplex matrix A, will be $\left(\frac{adj\ A_0}{|A_0|}\right)+i_2\left(\frac{adj\ A_1}{|A_1|}\right)$

Proof:

$$A = A_0 + i_2\, A_1$$

Let A_0 and A_1 both has inverse A_0^- and A_1^-

Inverse of A = $A^- = A_0^- + i_2\, A_1^-$ (by definition)

Since A_0 and A_1 both are complex matrices therefore the inverse of A_0 and A_1 are

$$\left(\frac{adj\ A_0}{|A_0|}\right) \text{ and } \left(\frac{adj\ A_1}{|A_1|}\right) \text{ respectively}$$

Thus $$A^- = \left(\frac{adj\ A_0}{|A_0|\neq 0}\right)+i_2\left(\frac{adj\ A_1}{|A_1|\neq 0}\right) \quad (4)$$

Next from here we shall use M in place of $C_2^{n\times n}$

3.2.6 Some properties under new system

Property: 1

The multiplication is not commutative in general

$$A \odot B = (A_o + i_2\, A_1) \odot (B_0 + i_2 B_1)$$
$$= A_0 B_0 + i_2\, A_1 B_1$$
$$B \odot A = (B_o + i_2\, B_1) \odot (A_0 + i_2 A_1)$$
$$= B_0 A_0 + i_2\, B_1 A_1$$

Let $A \odot B = B \odot A$

$\Rightarrow A_0 B_0 + i_2\, A_1 B_1 = B_0 A_0 + i_2\, B_1 A_1$

$\Rightarrow A_0 B_0 = B_0 A_0$ and $A_1 B_1 = B_1 A_1$

$\Rightarrow$ Complex matrix is commutative which is contradicted.

$\Rightarrow A \odot B \neq B \odot A$

Counter example:

$$A=\begin{bmatrix}1 & i\\ -i & 2\end{bmatrix} \text{ and } B=\begin{bmatrix}i & 3\\ 2i & 5i\end{bmatrix} \text{ then } AB=\begin{bmatrix}i-2 & -2\\ 1+4i & -7i\end{bmatrix}$$

$$\text{but } BA=\begin{bmatrix}-2i & 5\\ 2i+5 & -2+10i\end{bmatrix} \Rightarrow AB\neq BA$$

Property: 2

$$(A \odot B)^T = (A_0 B_0 + i_2 A_1 B_1)^T$$
$$= (A_0 B_0)^T + i_2\,(A_1 B_1)^T$$
$$= (B_0^T A_0^T) + i_2\,(B_1^T A_1^T)$$

Notes

Since $(A\ B)^T = B^T A^T$ true in C_1

$$\text{Therefore } (A \odot B)^T = (B_0^T + i_2 B_1^T) \odot (A_0^T + i_2 A_1^T) = B^T \odot A^T$$

Property: 3

If A is any bicomplex square and invertible matrix whose inverse is A^- under new system then $(A^-)^- = A$

Notes

Proof:

$$\begin{aligned}(A^-)^- &= [(A_0 + i_2 A_1)^-]^- \\ &= (A_0^- + i_2 A_1^-)^- && (\because A^- = A_0^- + A_1^-, \text{ by definition}) \\ &= (C_0 + i_2 C_1)^- && (\text{say } C_0 = A_0^- \text{ and } C_1 = A_1^-) \\ &= (C_0^- + i_2 C_1^-) && (\text{by using again definition}) \\ &= (A_0^-)^- + i_2 (A_1^-)^- \\ &= (A_0 + i_2 A_1) \\ &= A\end{aligned}$$

Property: 4

$$\begin{aligned}(A^-)^k &= (A_0^- + A_1^-)^k \\ &= (A_0^- + i_2 A_1^-) \odot (A_0^- + i_2 A_1^-)^{k-1} \\ &= [(A_0^-)^2 + i_2 (A_1^-)^2] \odot (A_0^- + i_2 A_1^-)^{k-2} \\ &= [(A_0^-)^k + i_2 (A_1^-)^k] \qquad (5) \\ &= (A_0^- A_0^- \ldots\ldots k \text{ times}) + i_2 (A_1^- A_1^- \ldots\ldots k \text{ times}) \\ &= (A_0 A_0 \ldots\ldots k \text{ times})^- + i_2 (A_1 A_1 \ldots\ldots k \text{ times})^- \\ &= (A_0^k)^- + i_2 (A_1^k)^- \\ &= (A^k)^-\end{aligned}$$

Property: 5

The inverse of the product of two bicomplex matrices A and B is equal to product of their inverses in reverse order

Proof:

Let $\forall\ A, B \in M$ then

$$\begin{aligned}(A \odot B)^- &= [(A_0 + i_2 A_1) \odot (B_0 + i_2 B_1)]^- \\ &= [A_0 B_0 + i_2 A_1 B_1]^- \\ &= [A_0 B_0]^- + [A_1 B_1]^- \\ &= [B_0^- A_0^-] + i_2 [B_1^- A_1^-]\end{aligned}$$

(Since A_0, B_0, A_1 and B_1 are in $C_1^{n\times n}$ and $(A\ B)^{-} = B^{-}A^{-}$)

$$= [B^{-}_{0} + i_2 B^{-}_{1}] \odot [A^{-}_{0} + i_2\ A^{-}_{1}]$$

Therefore $[(A \odot B)]^{-} = B^{-} \odot A^{-}$

3.2.7 Theorem

If $A_1\ A_2 \ldots\ldots A_n$ are the invertible bicomplex matrix then the inverse of product of $A_1\ A_2 \ldots\ldots A_n$, will be equal to the individual product of their inverse in reverse order.

Proof:

Let $A_1\ A_2 \ldots\ldots A_n$ be the invertible bicomplex matrix then

$$(A_1 \odot A_2 \odot A_3 \odot \ldots\ldots\ldots \odot A_n)^{-}$$

$$= [(A_{10}+i_2\ A_{11}) \odot (A_{20}+i_2 A_{21}) \odot \ldots\ldots\ldots \odot (A_{n0}+i_2 A_{n1})]^{-}$$

$$= [(A_{10}\ A_{20} \ldots\ldots A_{n0}) + i_2 (A_{11}\ A_{21} \ldots\ldots A_{n1})]^{-}$$

$$= [(A_{10}\ A_{20} \ldots\ldots A_{n0})^{-} + i_2\ (A_{11}\ A_{21} \ldots\ldots\ A_{n1})^{-}]$$

$$= (A_{n0}^{-}\ A_{n-1\,0}^{-} \ldots\ldots A_{10}^{-}) + (A_{n1}^{-}\ A_{n-1\,1}^{-} \ldots\ldots A_{11}^{-})$$

$$= (A_{n0}^{-} + i_2\ A_{n1}^{-}) \odot (A_{n-10}^{-} + i_2\ A_{n-1\,1}^{-}) \odot \ldots\ldots (A_{10}^{-} + i_2\ A_{n1}^{-})$$

$$= (A_n^{-}) \odot (A_{n-1}^{-}) \ldots\ldots\ldots \odot (A_1^{-}).$$

Thus it is clear the inversion of the bicomplex matrix under new definition has the same fundamental properties as those under the traditional algebraic system.

c) Some definitions and theorems related to bicomplex matrix in both systems

3.3.1 Idempotent bicomplex matrix

Let A be a square bicomplex matrix and if $A^2 = A$, then A is called the idempotent bicomplex matrix, obviously identity matrices in C_2 will be idempotent matrix in their individual systems.

Remark:

The identity matrix is always idempotent bicomplex matrices in their own systems as well as the Null matrix is also idempotent matrix.

Example:

(1) $\begin{bmatrix} \frac{e_1}{2} & \frac{e_1}{2} \\ \frac{e_1}{2} & \frac{e_1}{2} \end{bmatrix}$ *is the example of idempotent matrix.*

(2) $\begin{bmatrix} \xi & {}^{2}\xi \\ {}^{1}\xi & {}^{1}\xi\, e_1 \end{bmatrix}$ $\forall\ \xi \neq 0 \in C_2$ is not idempotent matrix.

(3) $\begin{bmatrix} e_1 & 0 \\ 0 & e_1 \end{bmatrix}$ is also an idempotent matrix.

Notes

(4) $\left[\begin{bmatrix} 1 & 0 \\ i_1 & 0 \end{bmatrix} + i_2 \begin{bmatrix} 1 & 0 \\ 6i_1 & 0 \end{bmatrix}\right]$ is an idempotent matrix in new system.

3.3.2 Theorem

A is an idempotent bicomplex square matrix of order n if and only if both complex component matrixes A_0 and A_1 are idempotent complex matrix simultaneously.

Notes

Proof:

$\because$ A is an idempotent matrix i.e. by definition $A^2 = A$

$\Leftrightarrow (A_0 + i_2 A_1) \odot (A_0 + i_2 A_1) = A_0 + i_2 A_1$

$\Leftrightarrow A_0^2 + i_2 A_1^2 = A_0 + i_2 A_1$

$\Leftrightarrow A_0^2 = A_0$ and $A_1^2 = A_1$

$\Leftrightarrow$ Both A_0 and A_1 are idempotent complex matrix.

3.3.3 Involutory bicomplex matrix

Let A be any bicomplex square matrix if $A^2 = I$ matrix then A is known as involutory bicomplex matrix. i.e. the inverse of the given matrix A will be itself.

Clearly the identity matrices will be involutory bicomplex matrices in their own systems.

Remark:

All idempotent bicomplex matrices which are not identity matrix will not be involutory.

Example:

(1) $\begin{bmatrix} e_2 & e_1 \\ e_1 & e_2 \end{bmatrix}^2 = I$

(2) $\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}^2 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3.3.4 Theorem

A is an involutory bicomplex matrix if and only if both complex component matrix A_0 and A_1 (under the new system) are involutory.

Proof:

By new definition of product of bicomplex matrix

i.e. $A^2 = A_0^2 + i_2 A_1^2$, where $A_0, A_1 \in C_1$

$\because$ A is an involutory bicomplex matrix

$\therefore A^2 = I + i_2 I \rightarrow$ (identity under new system)

$\Leftrightarrow A_0^2 + i_2 A_1^2 = I + i_2 I$

$\Leftrightarrow A_0^2 = I$ and $A_1^2 = I$

$\Leftrightarrow$ Both A_0 and A_1 are involutory.

Remark:

Under new product definition $(I + i_2 I)$ is always an involutory matrix i.e.

$$(I + i_2 I) \odot (I + i_2 I) = (I + i_2 I)$$

3.3.5 Similar bicomplex matrix

Let A, B $\in$M if $\exists$ an invertible matrix P $\in$M such that

$A = P^{-1}BP$ then A and B are said to be similar bicomplex matrix and denoted by A˜ B

If A and B are similar in new system then

$$A = \bar{P} \odot B \odot P \quad (6)$$

And if A, B $\in C_1^{n \times n}$ then equation (6) will be equivalent to $A = \bar{P_0} B P_0$, where $P = P_0 + i_2 P_1$

3.3.6 Theorem

Let A and B be two square bicomplex matrices, and an invertible matrix P such that $A = P^{-1}BP$ or A ˜ B then

$$|A| = |B|$$

Proof:

$\because$ $A = P^{-1}BP$, where P is an invertible bicomplex matrix

Now $|A| = |P^{-1}B\,P|$

$$= |{}^1(P^{-1}BP)|e_1 + |{}^2(P^{-1}B\,P)|e_2$$

$$= |{}^1P^{-1}||{}^1B||{}^1P|\,e_1 + |{}^2P^{-1}||{}^2B||{}^2P|\,e_2$$

$$= |{}^1P^{-1}||{}^1P||{}^1B|\,e_1 + |{}^2P^{-1}||{}^2P||{}^2B|\,e_2$$

$$= |{}^1P^{-1}\,{}^1P||{}^1B|\,e_1 + |{}^2P^{-1}\,{}^2P||{}^2B|\,e_2$$

$$= |{}^1B|e_1 + |{}^2B|e_2$$

therefore $|A| = |B|$

3.3.7 Theorem

The $i_1 i_2$ tranjugate of inverse of a matrix A is equal to the inverse of $i_1 i_2$ tranjugate of A. i.e.

$$(A^{-1})^{\theta_3} = (A^{\theta_3})^{-1}$$

Proof:

$$(A^{-1}) = {}^1A^{-1}e_1 + {}^2A^{-1}e_2$$

$$(A^{-1})^{\theta_3} = \left[({}^1A^{-1}e_1 + {}^2A^{-1}e_2)^{\#}\right]^T$$

$$= \left(\overline{{}^1A^{-1}}\,\bar{e}_1 + \overline{{}^2A^{-1}}\,\bar{e}_2\right)^T$$

$$= \left(\overline{{}^1A^{-1}}\right)^T e_1 + \left(\overline{{}^2A^{-1}}\right)^T e_2$$

$$= conj\left({}^1A^T\right)^{-1} e_1 + conj\left({}^2A^T\right)^{-1} e_2$$

$$= \left(\overline{{}^1A^T}\right)^{-1} e_1 + \left(\overline{{}^2A^T}\right)^{-1} e_2$$

$$\text{i.e}\,(A^{-1})^{\theta_3} = (A^{\theta_3})^{-1}$$

Remark:

Since i_1 and i_2 conjugates of e_1 is e_2 and e_2 is e_1 therefore this result is not true for S = 1, 2 where $(A^{-1})^{\theta_s} = (A^{\theta_s})^{-1}$.

Notes

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3.3.8 Theorem

In new system, the $i_1$tranjugate of inverse of a matrix A is equal to the inverse of i_1 tranjugate of A. i.e. $\left(A^{-}\right)^{\theta_1}=\left(A^{\theta_1}\right)^{-}$

Proof:

$$\left(A^{-}\right)^{\theta_1}=\left[\overline{\left(A_0^{-}+i_2\,A_1^{-}\right)}\right]^T$$

$$\left(A^{-}\right)^{\theta_1}=\left[\overline{\left(A_0^{-}\right)}\right]^{'T}+i_2\left[\overline{\left(A_1^{-}\right)}\right]^T$$

$$\left(A^{-}\right)^{\theta_1}=\left(\overline{A}_0^{\,T}\right)^{-}+i_2\left(\overline{A}_1^{\,T}\right)^{-}$$

$$\left(A^{-}\right)^{\theta_1}=\left(A^{\theta_1}\right)^{-}$$

In this system this result is not valid for S = 2, 3 where $\left(A^{-}\right)^{\theta_S}=\left(A^{\theta_S}\right)^{-}$.

By above two theorems it is evident that the given bicomplex matrix has same property by taken different conjugate in both different system.

3.3.9 Theorem

A will be an orthogonal bicomplex matrix in new system if and only if the complex component matrix A_0 and A_1 are orthogonal complex matrices.

Proof:

Since A $\in$M is an orthogonal matrix.

Therefore $$A^T=A^{-} \qquad (7)$$

Since A can be express as the i_2 combination of two complex matrices A_0 and A_1 as follow

$$A = (A_0 + i_2A_1)$$

$A^T = (A_0 + i_2A_1)^T = A^{-}$(where A^{-} is the inverse of A)

And $A^{-} = A^{-}_{0} + i_2A^{-}_{1}$

From equation (7) we have

$$(A_0 + i_2A_1)^T = A^{-}_{0} + i_2A^{-}_{1}$$

$\Leftrightarrow A_0^{\,T} + i_2A_1^{\,T} = A^{-}_{0} + i_2\,A_1^{-}$

$\Leftrightarrow A_0^{\,T} = A_0^{-}$ and $A_1^{\,T} = A_1^{-}$

$\Leftrightarrow$ Both complex matrices A_0 and A_1 are orthogonal.

The proof of theorem 3.3.9 is complete.

If A is an orthogonal matrix in traditional system then

$$A^T= A^{-1}$$

$\Leftrightarrow {}^1A^T\,e_1 + {}^2A^T\,e_2 = {}^1A^{-1}\,e_1 + {}^2A^{-1}\,e_2$

$\Leftrightarrow {}^1A^T = {}^1A^{-1}$and ${}^2A^T = {}^2A^{-1}$

$\Leftrightarrow$ Both idempotent component matrices are orthogonal.

Hence A is an orthogonal matrix in traditional system if and only if both idempotent component matrices are orthogonal.

3.3.10 Theorem

A will be an i_1 Unitary bicomplex matrix if and only if the complex component matrix A_0 and A_1 of A are Unitary complex matrix but idempotent matrix ${}^1A$ and ${}^2A$ of A may or may not be Unitary.

Proof:

Part-1st

According to definition of i_1 Unitary bicomplex matrix

$$A^{\theta_1} = [(A_0 + i_2 A_1)^{-}]^{T} = [A_0^{-} + i_2 A_1^{-}]$$

$\Leftrightarrow \bar{A}_0^{T} + i_2 \bar{A}_1^{T} = A_0^{-} + i_2 A_1^{-}$

$\Leftrightarrow \bar{A}_0^{T} = A_0^{-}$ and $\bar{A}_1^{T} = A_1^{-}$

$\Leftrightarrow$ A_0 and A_1 are Unitary

Part-2nd

Note that

$$\left({}^1Ae_1 + {}^2Ae_2\right)^{\theta_1} = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$$

$\Leftrightarrow \left(\overline{{}^1A}\, e_2 + \overline{{}^2A}\, e_1\right)^{T} = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \left(\overline{{}^1A}^{T} e_2 + \overline{{}^2A}^{T} e_1\right) = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \overline{{}^2A}^{T} = {}^1A^{-1}$ and $\overline{{}^1A}^{T} = {}^2A^{-1}$

It is evident that if A is an i_1 Unitary bicomplex matrix then idempotent matrix ${}^1A$ and ${}^2A$ of A will be Unitary complex matrix only if A is a complex matrix.

It is clear from here that both component A_0 and A_1 as well as ${}^1A$ and ${}^2A$ be an unitary complex matrices then matrix A will be different type of unitary bicomplex matrix that means it has shown the different nature of representations of A.

3.3.11 Theorem

Let A be an i_1 Hermitian bicomplex matrix then the i_1 tranjugate of both ${}^1A$ and ${}^2A$ will be ${}^2A$ and ${}^1A$ respectively as well as A_0 and A_1 both will be Hermitian complex matrix.

Proof:

Part-1st

Since A is i_1 Hermitian $\Rightarrow$ $(\bar{A})^{T} = A$

$\Leftrightarrow \left(\overline{{}^1A^{-1} e_1 + {}^2A^{-1} e_2}\right)^{T} = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow \left(\overline{{}^1A}\, e_2 + \overline{{}^2A}\, e_1\right)^{T} = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow \overline{{}^2A}^{T} e_1 + \overline{{}^1A}^{T} e_2 = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow \left({}^2A^{\theta_1} = {}^1A\right)$ and $\left({}^1A^{\theta_1} = {}^2A\right)$

Notes

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Part-2nd

$$A = A_0 + i_2 A_1$$

$$\left(\overline{A_0 + i_2 A_1}\right)^T = (\overline{A_0})^T + i_2 (\overline{A_1})^T$$

Since A is i_1 Hermitian $\Leftrightarrow (\bar{A})^T = A$

$\Leftrightarrow (\overline{A_0})^T + i_2 (\overline{A_1})^T = A_0 + i_2 A_1$

$\Leftrightarrow (\overline{A_0})^T = A_0$ and $(\overline{A_1})^T = A_1$

$\Leftrightarrow$ Both complex component matrix A_0 and A_1 of A are Hermitian.

3.3.12 Theorem

Let A be an i_2 Hermitian bicomplex matrix if and only if the transpose of ^2A and ^1A are ^1A and ^2A respectively as well as A_0 and A_1 are symmetric and Skew – Symmetric bicomplex matrix respectively.

Proof:

Part-1st

By definition of i_2 Hermitian in C_2

$[(A)^{\sim}]^T = A^{\theta_2} = A$

$\Leftrightarrow \left({}^1Ae_1 + {}^2Ae_2\right)^{\theta_2} = \left({}^1Ae_1 + {}^2Ae_2\right)$

$\Leftrightarrow {}^1A^T e_2 + {}^2A^T e_1 = {}^1Ae_1 + {}^2Ae_2$

$\Leftrightarrow {}^2A^T = {}^1A$ and ${}^1A^T = {}^2A$

Part- 2nd

$$A = A_0 + i_2 A_1 \quad \forall\ A_S = C_1^{\ n\times n}, S = 0,1$$

$$A^{\theta_2} = (A_0 + i_2 A_1)^{\theta_2} = A_0^{\ T} - i_2 A_1^{\ T}$$

Since A is i_2 Hermitian $A^{\theta_2} = A$

$\Leftrightarrow A^{\theta_2} = A$

$\Leftrightarrow A_0^{\ T} - i_2 A_1^{\ T} = A_0 + i_2 A_1$

$\Leftrightarrow A_0^{\ T} = A_0$ and $A^T{}_1 = -A_1$

3.3.13 Theorem

A is a $i_1 i_2$ Hermitian matrix if and only if ^1A and ^2A both idempotent component matrix of A will be Hermitian as well as the complex component matrix A_0 and A_1 of A will be Hermitian and Skew – Hermitian respectively.

Proof:

Part-1st

$\because$ A is $i_1 i_2$ Hermitian

$$[A]^{\theta_3} = A$$

$\Leftrightarrow \left({}^1Ae_1 + {}^2Ae_2\right)^{\theta_3} = {}^1Ae_1 + {}^2Ae_2$

$\Leftrightarrow {}^1A^{\theta_3} = {}^1A$ *and* ${}^2A^{\theta_3} = {}^2A$

$\Leftrightarrow$ both ^1A *and* ^2A *are Hermitian*

Part- 2nd

$$A^{\theta_3} = (A_0 + i_2 A_1)^{\theta_3}$$

$\because$ A is Hermitian

$\Leftrightarrow (A_0 + i_2 A_1)^{\theta_3} = A_0 + i_2 A_1$

$\Leftrightarrow A_0^{\theta_3} - i_2 A_1^{\theta_3} = A_0 + i_2 A_1$

$\Leftrightarrow A_0^{\theta_3} = A_0$ and $A_1^{\theta_3} = -A_1$

$\Leftrightarrow$ A_0 is Hermitian and A_1 is Skew – Hermitian in C_1.

3.3.14 Theorem

Let A be an i_2 Unitary bicomplex matrix then idempotent component matrix ^1A and ^2A are not symmetric until $^1A^{-1} = {}^2A$ and $[A_0]^T = A_0^{-}$ and $[A_1]^T = -A_1^{-}$

Proof:

Part-1st

by definition of i_2 unitary bicomplex matrix $({}^1Ae_1 + {}^2Ae_2)^{\theta_2} = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \left[({}^1Ae_1 + {}^2Ae_2)^{\sim}\right]^T = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow [({}^1A)^{\sim}]^T e_1^{\sim} + [({}^2A)^{\sim}]^T e_2^{\sim} = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow [{}^1A]^T e_2 + [{}^2A]^T e_1 = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow [{}^2A]^T = {}^1A^{-1}$ and $[{}^1A]^T = {}^2A^{-1}$

Therefore it is clear that if $^1A^{-1} \neq {}^2A$ then ^1A and ^2A will never symmetric.

Part- 2nd

Since A is i_2 Unitary bicomplex matrix and

$A = (A_0 + i_2 A_1)$ *and* $A^{-} = (A_0^{-} + i_2 A_1^{-})$ *therefore*

$[A_0 + i_2 A_1]^{\theta_2} = A_0^{-} + i_2 A_1^{-}$

$\Leftrightarrow [(A_0 + i_2 A_1)^{\sim}]^T = A_0^{-} + i_2 A_1^{-}$

$\Leftrightarrow [A_0^{\sim}]^T - i_2 [A_1^{\sim}]^T = A_0^{-} + i_2 A_1^{-}$

$\Leftrightarrow [A_0]^T = A_0^{-}$ and $[A_1]^T = -A_1^{-}$

3.3.15 Theorem

Let A be an $i_1 i_2$ Unitary bicomplex matrix then the idempotent component matrix ^1A and ^2A both are Unitary simultaneously but complex component matrix A_0 and A_1 are not Unitary simultaneously. Moreover A_0 will be Unitary but A_1 will not be Unitary.

Proof:

$$A = (A_0 + i_2 A_1)$$

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$$A^{\theta_3} = [(A_0 + i_2 A_1)^{\#}]^T = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$$

$\Leftrightarrow \quad [\overline{{}^1A}]^T e_1 + [\overline{{}^2A}]^T e_2 = ({}^1A^{-1})e_1 + ({}^2A^{-1})e_2$

$\Leftrightarrow \quad [\overline{{}^1A}]^T = [{}^1A^{-1}]$ and $[\overline{{}^2A}]^T = [{}^2A^{-1}]$

$\Leftrightarrow$ both idempotent component ${}^1A$ and ${}^2A$ are Unitary

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and $A^{\theta_3} = [(A_0 + i_2 A_1)^{\#}]^T$

$$\because \; A^{\theta_3} = A^{-1} \;\Leftrightarrow\; [\overline{A_0}]^T - i_2[\overline{A_1}]^T = A_0^- + i_2 A_1^-$$

$\Leftrightarrow [\overline{A_0}]^T = A_0^-$ and $[\overline{A_1}]^T = -A_1^-$

Therefore A_0 is Unitary but A_1 is not Unitary.

3.3.16 Theorem

The similar relation ˜ between two arbitrary bicomplex square matrix A and B in new system i.e. A ˜ B then this relation will be an equivalence relation.

Proof:

Reflexive-

Since We know that $P = [I + i_2 I]$ is an invertible matrix such that $A = P^- \odot A \odot P$ i.e. every bicomplex square matrix is similar to itself.

Symmetric Relation-

Let A˜B then we have to prove B will be similar to A.

$\Rightarrow \quad A = P^- \odot B \odot P$ for some invertible $P \in M^{n \times n}$

$\Rightarrow \quad P\odot A = P\odot [P^- \odot B \odot P]$

$= [P\odot P^-] \odot [B \odot P]$

$= [I + i_2 I] \odot [B \odot P]$

$\Rightarrow \quad P\odot A = [B \odot P]$

i.e. $P\odot A \odot P^- = [B \odot P] \odot P^-$

$= B \odot [P \odot P^-]$

$= B \odot [I + i_2 I]$

$= [B_0 + i_2 B_1] \odot [I + i_2 I]$

$= B_0 + i_2 B_1$

$P\odot A \odot P^- = B$

$\Rightarrow \quad B \sim A$

Transitive-

If A ˜ B $\Rightarrow$ $\exists$ an invertible Bicomplex matrix P

Such that $A = P^- \odot B \odot P$

And B ˜ C $\Rightarrow$ $B = Q^- \odot C \odot Q$ for some $Q^- \in M^{n \times n}$

We have to show A ˜ C

$\because$ A = P^{-} ⊙ B ⊙ P and B = Q^{-} ⊙ C ⊙ Q

A = P^{-} ⊙ [Q^{-} ⊙ C ⊙ Q] ⊙ P

= [P^{-} ⊙ Q^{-}] ⊙ [C ⊙ Q] ⊙ P

= E^{-} ⊙ C ⊙ E, where E = Q⊙P

We have an invertible bicomplex matrix E
Such that A = E^{-} ⊙ C ⊙ E
therefore A is similar to C then Relation is transitive.
Hence the similar relation between bicomplex matrices is an equivalence relation.

Moreover the collection of bicomplex matrices similar to A forms a class denoted by [A] and is called the class of similar matrices of A.

Two classes [A] and [B] are either same or disjoint, in the sense that no matrix can belong to two different classes. Thus there exists a natural partition of the set of "All bicomplex square matrices".

The set of all square bicomplex matrices can also be viewed as the collection of all mutually disjoint equivalence classes with respect to a suitable defined equivalent relation.

Notes

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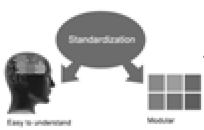

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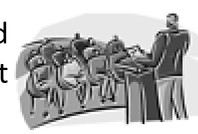

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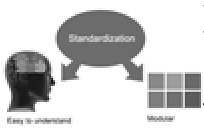
The IBOARS can organize symposium/seminar/conference in their country on behalf of Global Journals Incorporation (USA)-OARS (USA). The terms and conditions can be discussed separately.

The Board can also play vital role by exploring and giving valuable suggestions regarding the Standards of "Open Association of Research Society, U.S.A (OARS)" so that proper amendment can take place for the benefit of entire research community. We shall provide details of particular standard only on receipt of request from the Board.

The board members can also join us as Individual Fellow with 40% discount on total fees applicable to Individual Fellow. They will be entitled to avail all the benefits as declared. Please visit Individual Fellow-sub menu of GlobalJournals.org to have more relevant details.

We shall provide you intimation regarding launching of e-version of journal of your stream time to time. This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.



After nomination of your institution as “Institutional Fellow” and constantly functioning successfully for one year, we can consider giving recognition to your institute to function as Regional/Zonal office on our behalf.

The board can also take up the additional allied activities for betterment after our consultation.

The following entitlements are applicable to individual Fellows:

Open Association of Research Society, U.S.A (OARS) By-laws states that an individual Fellow may use the designations as applicable, or the corresponding initials. The Credentials of individual Fellow and Associate designations signify that the individual has gained knowledge of the fundamental concepts. One is magnanimous and proficient in an expertise course covering the professional code of conduct, and follows recognized standards of practice.

Open Association of Research Society (US)/ Global Journals Incorporation (USA), as described in Corporate Statements, are educational, research publishing and professional membership organizations. Achieving our individual Fellow or Associate status is based mainly on meeting stated educational research requirements.

Disbursement of 40% Royalty earned through Global Journals : Researcher = 50%, Peer Reviewer = 37.50%, Institution = 12.50% E.g. Out of 40%, the 20% benefit should be passed on to researcher, 15 % benefit towards remuneration should be given to a reviewer and remaining 5% is to be retained by the institution.

We shall provide print version of 12 issues of any three journals [as per your requirement] out of our 38 journals worth $ 2376 USD.

Other:

The individual Fellow and Associate designations accredited by Open Association of Research Society (US) credentials signify guarantees following achievements:

- The professional accredited with Fellow honor, is entitled to various benefits viz. name, fame, honor, regular flow of income, secured bright future, social status etc.

- In addition to above, if one is single author, then entitled to 40% discount on publishing research paper and can get 10%discount if one is co-author or main author among group of authors.
- The Fellow can organize symposium/seminar/conference on behalf of Global Journals Incorporation (USA) and he/she can also attend the same organized by other institutes on behalf of Global Journals.
- The Fellow can become member of Editorial Board Member after completing 3yrs.
- The Fellow can earn 60% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.
- Fellow can also join as paid peer reviewer and earn 15% remuneration of author charges and can also get an opportunity to join as member of the Editorial Board of Global Journals Incorporation (USA)
- • This individual has learned the basic methods of applying those concepts and techniques to common challenging situations. This individual has further demonstrated an in–depth understanding of the application of suitable techniques to a particular area of research practice.

Note :

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- In future, if the board feels the necessity to change any board member, the same can be done with the consent of the chairperson along with anyone board member without our approval.
- In case, the chairperson needs to be replaced then consent of 2/3rd board members are required and they are also required to jointly pass the resolution copy of which should be sent to us. In such case, it will be compulsory to obtain our approval before replacement.
- In case of "Difference of Opinion [if any]" among the Board members, our decision will be final and binding to everyone. "

Preferred Author Guidelines

We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

Before and during Submission

Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

1. Authors must go through the complete author guideline and understand and *agree to Global Journals' ethics and code of conduct,* along with author responsibilities.
2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
5. Authors should submit paper in a ZIP archive if any supplementary files are required along with the paper.
6. Proper permissions must be acquired for the use of any copyrighted material.
7. Manuscript submitted *must not have been submitted or published elsewhere* and all authors must be aware of the submission.

Declaration of Conflicts of Interest

It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

Policy on Plagiarism

Plagiarism is not acceptable in Global Journals submissions at all.

Plagiarized content will not be considered for publication. We reserve the right to inform authors' institutions about plagiarism detected either before or after publication. If plagiarism is identified, we will follow COPE guidelines:

Authors are solely responsible for all the plagiarism that is found. The author must not fabricate, falsify or plagiarize existing research data. The following, if copied, will be considered plagiarism:

- Words (language)
- Ideas
- Findings
- Writings
- Diagrams
- Graphs
- Illustrations
- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

Copyright

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Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

FORMAT STRUCTURE

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, “What words would a source have to include to be truly valuable in a research paper?” Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

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Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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