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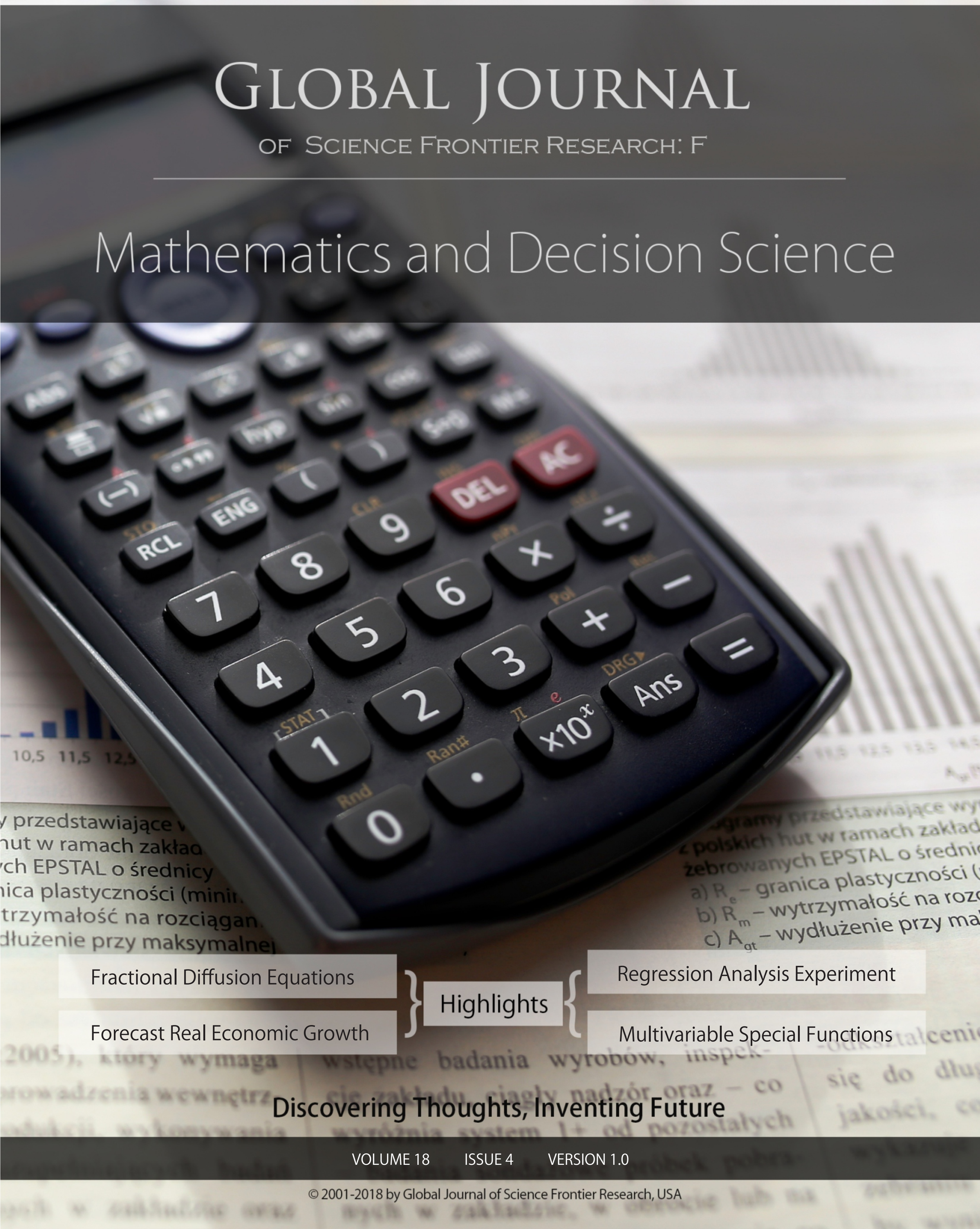

Highlights

- Fractional Diffusion Equations
- Forecast Real Economic Growth
- Regression Analysis Experiment
- Multivariable Special Functions

Discovering Thoughts, Inventing Future

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# Selberg Integral Involving the Product of Multivariable Special Functions

By FY. AY. Ant

*Abstract-* The Selberg integral was an integral first evaluated by Selberg in 1944. The aim of the present paper is to estimate generalized Selberg integral. It involves the product of the general class of multivariable polynomials, multivariable I-function and modified multivariable H-function. The result is believed to be new and is capable of giving a large number of integrals involving a variety of functions and polynomials as its cases. We shall see several corollaries and particular cases at the end.

*Keywords: modified multivariable H-function, selberg integral, multivariable I-function, class of multivariable polynomials, h-function.*

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# Selberg Integral Involving the Product of Multivariable Special Functions

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*Keywords: modified multivariable H-function, selberg integral, multivariable I-function, class of multivariable polynomials, h-function.*

## I. Introduction and Prerequisites.

The Selberg integral is the following integral first evaluated by Selberg [6] in 1944 :

$$S_n(a,b,c)=\int_0^1\cdots\int_0^1\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n$$

$$=\prod_{j=0}^{n-1}\frac{\Gamma(a+jc)\Gamma(b+jc)\Gamma(1+(j+1)c)}{\Gamma(a+b+(n-1+j)c)\Gamma(1+c)} \tag{1.1}$$

where n is a positive integer, $a$, $b$ and $c$ are the complex number such that

$$Re(a)>0, Re(b)>0, Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$$

We refer the reader to Forrester and Warnaar's exposition [2] for the history and importance of the Selberg integral.

In this document, we evaluate a generalized Selberg integral involving the product of the multivariable I-function defined by Prasad [4], modified multivariable H-function defined by Prasad and Singh [5] and class of multivariable polynomials defined by Srivastava [7].

The generalized multivariable polynomials defined by Srivastava [7], is given in the following manner:

$$S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]=\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\frac{(-N_1)_{\mathfrak{M}_1K_1}}{K_1!}\cdots\frac{(-N_v)_{\mathfrak{M}_vK_v}}{K_v!}A[N_1,K_1;\cdots;N_v,K_v]y_1^{K_1}\cdots y_v^{K_v} \tag{1.2}$$

where $\mathfrak{M}_1,\cdots,\mathfrak{M}_v$ are arbitrary positive integers and the coefficients $A[N_1,K_1;\cdots;N_v,K_v]$ are constants Real or complex. On suitably specializing the quantities, $A[N_1,K_1;\cdots;N_v,K_v]$, $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[y_1,\cdots,y_v]$ yields a Number of known polynomials, the

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Ref

4. Y. N. Prasad, Multivariable I-function, Vijnana Parishad Anusandhan Patrika 29 (1986), 231-237.

Laguerre polynomials, the Jacobi polynomials, and several other ([10], page. 158-161]. We shall note.

$$a_v = \frac{(-N_1)_{\mathfrak{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathfrak{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots ; N_v, K_v] \tag{1.3}$$

The multivariable I-function of r-variables is defined in term of multiple Mellin-Barnes types integral:

$$I(z_1, z_2, \cdots, z_r) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \left( \begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_{2j}; \alpha'_{2j}, \alpha''_{2j})_{1,p_2}; \cdots ; \\ \cdot & \\ \cdot & \\ \cdot & \\ \mathrm{z}_r & (\mathrm{b}_{2j}; \beta'_{2j}, \beta''_{2j})_{1,q_2}; \cdots ; \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{a}_{rj}; \alpha^{(1)}_{rj}, \cdots, \alpha^{(r)}_{rj})_{1,p_r} : (a^{(1)}_j, \alpha^{(1)}_j)_{1,p^{(1)}}; \cdots ; (a^{(r)}_j, \alpha^{(r)}_j)_{1,p^{(r)}} \\ \\ (\mathrm{b}_{rj}; \beta^{(1)}_{rj}, \cdots, \beta^{(r)}_{rj})_{1,q_r} : (b^{(1)}_j, \beta^{(1)}_j)_{1,q^{(1)}}; \cdots ; (b^{(r)}_j, \beta^{(r)}_j)_{1,q^{(r)}} \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi(s_1, \cdots, s_r) \prod_{i=1}^{r} \phi_i(s_i) z_i^{s_i} \mathrm{d}s_1 \cdots \mathrm{d}s_r \tag{1.4}$$

where

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}} \Gamma(b^{(i)}_j - \beta^{(i)}_j s_i) \prod_{j=1}^{n^{(i)}} \Gamma(1 - a^{(i)}_j + \alpha^{(i)}_j s_i)}{\prod_{j=m^{(i)}+1}^{q^{(i)}} \Gamma(1 - b^{(i)}_j + \beta^{(i)}_j s_i) \prod_{j=n^{(i)}+1}^{p^{(i)}} \Gamma(a^{(i)}_j - \alpha^{(i)}_j s_i)}, i = 1, \cdots, r \tag{1.5}$$

and

$$\phi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{n_2} \Gamma(1 - a_{2j} + \sum_{i=1}^{2} \alpha^{(i)}_{2j} s_i) \prod_{j=1}^{n_3} \Gamma(1 - a_{3j} + \sum_{i=1}^{3} \alpha^{(i)}_{3j} s_i) \cdots}{\prod_{j=n_2+1}^{p_2} \Gamma(a_{2j} - \sum_{i=1}^{2} \alpha^{(i)}_{2j} s_i) \prod_{j=n_3+1}^{p_3} \Gamma(a_{3j} - \sum_{i=1}^{3} \alpha^{(i)}_{3j} s_i) \cdots}$$

$$\frac{\cdots \prod_{j=1}^{n_r} \Gamma(1 - a_{rj} + \sum_{i=1}^{r} \alpha^{(i)}_{rj} s_i)}{\cdots \prod_{j=n_r+1}^{p_r} \Gamma(a_{rj} - \sum_{i=1}^{r} \alpha^{(i)}_{rj} s_i) \prod_{j=1}^{q_2} \Gamma(1 - b_{2j} - \sum_{i=1}^{2} \beta^{(i)}_{2j} s_i)}$$

$$\times \frac{1}{\prod_{j=1}^{q_3} \Gamma(1 - b_{3j} + \sum_{i=1}^{3} \beta^{(i)}_{3j} s_i) \cdots \prod_{j=1}^{q_r} \Gamma(1 - b_{rj} - \sum_{i=1}^{r} \beta^{(i)}_{rj} s_i)} \tag{1.6}$$

About the above integrals and these existence and convergence conditions, see Prasad [4] for more details. Throughout the present document, we assume that the existence and convergence conditions of the multivariable I-function. We have:

$|arg z_i| < \frac{1}{2}\Omega_i \pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha^{(i)}_k - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha^{(i)}_k + \sum_{k=1}^{m^{(i)}} \beta^{(i)}_k - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta^{(i)}_k + \left( \sum_{k=1}^{n_2} \alpha^{(i)}_{2k} - \sum_{k=n_2+1}^{p_2} \alpha^{(i)}_{2k} \right) + \cdots +$$

Ref

4. Y. N. Prasad, Multivariable I-function, Vijnana Parishad Anusandhan Patrika 29 (1986), 231-237.

$$\left(\sum_{k=1}^{n_r}\alpha_{rk}^{(i)}-\sum_{k=n_r+1}^{p_r}\alpha_{rk}^{(i)}\right)-\left(\sum_{k=1}^{q_2}\beta_{2k}^{(i)}+\sum_{k=1}^{q_3}\beta_{3k}^{(i)}+\cdots+\sum_{k=1}^{q_r}\beta_{rk}^{(i)}\right) \tag{1.7}$$

where $i=1,\cdots,r$

The complex numbers $z_i$ are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable I-function. We may establish the asymptotic expansion in the following convenient form:

$$I(z_1,\cdots,z_r)=0(\,|z_1|^{\alpha_1},\cdots,|z_r|^{\alpha_r}\,),max(\,|z_1|,\cdots,|z_r|\,)\rightarrow 0$$

$$I(z_1,\cdots,z_r)\;=0(\,|z_1|^{\beta_1},\cdots,|z_r|^{\beta_r}\,),min(\,|z_1|,\cdots,|z_r|\,)\rightarrow\infty$$

where $k=1,\cdots,r:\alpha_k'=min[Re(b_j^{(k)}/\beta_j^{(k)})],j=1,\cdots,m^{(k)}$ and

$$\beta_k'=max[Re((a_j^{(k)}-1)/\alpha_j^{(k)})],j=1,\cdots,n^{(k)}$$

If all the poles of (1.7) are simples, then the integral (1.6) can be evaluated with the help of the residue theorem to give

$$I(z_1,\cdots,z_r)=\sum_{G_i=1}^{m^{(i)}}\sum_{g_i=0}^{\infty}\phi\frac{\prod_{i=1}^r\phi_i z_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r\delta_{G^{(i)}}^{(i)}\prod_{i=1}^r g_i!} \tag{1.8}$$

where

$$\phi=\phi(\eta_{G_1,g_1},\cdots,\eta_{G_r,g_r})\,,\phi_i=\phi_i(\eta_{G_i,g_i}),i=1,\cdots,r \tag{1.9}$$

$$\eta_{G_i,g_i}=\frac{d_{g_i}^{(i)}+G_i}{\delta_{g_i}^{(i)}}\quad\text{for } i=1,\cdots,r\quad\text{and}\quad\sum_{G_i=1}^{m^{(i)}}\sum_{g_i=0}^{\infty}=\sum_{G_1,\cdots,G_r=1}^{m^{(1)},\cdots,m^{(r)}}\sum_{g_1,\cdots,g_r=0}^{\infty}$$

which is valid under the following conditions: $\epsilon_{M_i}^{(i)}[p_j^{(i)}+p_i']\neq\epsilon_j^{(i)}[p_{M_i}+g_i]$. $\phi_i$ and $\phi$ are are given by (1.5) and (1.6) respectively.

The modified H-function studied by Prasad and Singh [5] generalizes the multivariable H-function defined by Srivastava and Panda [8,9]. It is defined in term of multiple Mellin-Barnes types integral:

$$H(z_1',\cdots,z_s')=H_{\mathbf{p},\mathbf{q}:|R:p_1,q_1;\cdots;p_s,q_s}^{\mathbf{m},\mathbf{n}:|\mathbf{R}:m_1,n_1;\cdots;m_s,n_s}\left(\begin{array}{c|c}\mathrm{z}_1' & (\mathrm{a}_j;\alpha_j^{(1)},\cdots,\alpha_j^{(s)})_{1,\mathbf{p}}:\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \\ \mathrm{z}_s' & (\mathrm{b}_j;\beta_j^{(1)},\cdots,\beta_j^{(s)})_{1,\mathbf{q}}:\end{array}\right.$$

$$\left.\begin{array}{c}(\mathrm{e}_j;u_j^{(1)}g_j^{(1)},\cdots,u_j^{(s)}g_j^{(s)})_{1,\mathbf{R}}:(c_j^{(1)};\gamma_j^{(1)})_{1,p_1},\cdots,(c_j^{(s)};\gamma_j^{(s)})_{1,p_s}\\ \cdot\\ \cdot\\ (\mathrm{l}_j;U_j^{(1)}f_j^{(1)},\cdots,U_j^{(s)}f_j^{(r)})_{1,R}:(d_j^{(1)};\delta_j^{(1)})_{1,q_1},\cdots,(d_j^{(s)};\delta_j^{(s)})_{1,q_s}\end{array}\right) \tag{1.10}$$

$$=\frac{1}{(2\pi\omega)^s}\int_{L_1'}\cdots\int_{L_s'}\theta(t_1,\cdots,t_s)\prod_{i=1}^s\theta_i(t_i)z_i'^{t_i}\mathrm{d}t_1\cdots\mathrm{d}t_s \tag{1.11}$$

Ref

5. Y. N. Prasad and A. K. Singh, Basic properties of the transform involving and H-function of r-variables as kernel, Indian Acad Math, (2) (1982), 109-115.

where $\theta(t_1,\cdots,t_s),\theta_i(t_i), i=1,\cdots,\ s$ are given by:

$$\theta(t_1,\cdots,t_s)=\frac{\prod_{j=1}^{\mathbf{m}}\Gamma(b_j-\sum_{i=1}^{s}\beta_j^{(i)}t_i)\prod_{j=1}^{\mathbf{n}}\Gamma\left(1-a_j+\sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)}{\prod_{j=\mathbf{n}+1}^{\mathbf{p}}\Gamma\left(a_j-\sum_{i=1}^{s}\alpha_j^{(i)}t_j\right)\prod_{j=\mathbf{m}+1}^{\mathbf{q}}\Gamma\left(1-b_j+\sum_{i=1}^{s}\beta_j^{(i)}t_j\right)}$$

$$\frac{\prod_{j=1}^{\mathbf{R}}\Gamma(e_j+\sum_{i=1}^{s}u_j^{(i)}g_j^{(i)}t_i)}{\prod_{j=1}^{R}\Gamma(l_j+\sum_{i=1}^{s}U_j^{(i)}f_j^{(i)}t_i)} \tag{1.12}$$

$$\theta_i(t_i)=\frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\gamma_j^{(i)}t_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}t_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}t_i\right)\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}t_i\right)} \tag{1.13}$$

The integrals (1.14) converges absolutely if

$$|argz_i'|<\frac{1}{2}U_i\pi\quad(i=1,\cdots,s) \tag{1.14}$$

with $$U_i=\sum_{j=1}^{\mathbf{m}}\beta_j^{(i)}-\sum_{j=\mathbf{m}+1}^{\mathbf{q}}\beta_j^{(i)}+\sum_{j=1}^{\mathbf{n}}\alpha_j^{(i)}-\sum_{j=\mathbf{n}+1}^{\mathbf{p}}\alpha_j^{(i)}+\sum_{j=1}^{m_i}\delta_j^{(i)}-\sum_{j=1+m_i}^{q_i}\delta_j^{(i)}$$

$$+\sum_{j=1}^{n_i}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}\gamma_j^{(i)}+\sum_{j=1}^{\mathbf{R}}g_j^{(i)}-\sum_{j=1}^{R}f_j^{(i)}>0(i=1,\cdots,s) \tag{1.15}$$

For more details, see Prasad and Singh [5].

## II. Required Integral

We have the following integrals, see Andrew and R. Askey for more details ([1], p. 402).
*Lemma.*

$$\int_0^1\cdots\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,\mathrm{d}x_1\cdots\mathrm{d}x_n=\prod_{i=1}^{k}\frac{(a+(n-i)c)}{(a+b+(2n-i-1)c)}S_n(a,b,c) \tag{2.1}$$

Where $Re(a)>0,Re(b)>0,Re(c)>Max\left\{-\frac{1}{n},-\frac{Re(a)}{n-1},-\frac{Re(b)}{n-1}\right\}$ and $k\leqslant n.$ $S_n(a,b,c)$ is defined by (1.1).

## III. Main Integral

Let

$$X_{u,v,w}(x_1,\cdots,x_n)=\prod_{i=1}^{n}x_i^u(1-x_i)^v\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2w} \tag{3.1}$$

$$X=m_1,n_1;\cdots;m_s,n_s \tag{3.2}$$

$$Y=p_1,q_1;\cdots;p_s,q_s \tag{3.3}$$

Ref

1. G. G Andrew and R. Askey, Special function. Cambridge. University. Press 1999.

Notes

$$\mathbb{A}=(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(s)})_{1,\mathbf{p}};(\mathrm{e}_j;u_j^{(1)}g_j^{(1)},\cdots,u_j^{(s)}g_j^{(s)})_{1,\mathbf{R}}:\mathrm{C}=(\mathrm{c}_j^{(1)};\gamma_j^{(1)})_{1,p_1};\cdots,(c_j^{(s)};\gamma_j^{(s)})_{1,p_s} \tag{3.4}$$

$$\mathbb{B}=(b_j;\beta_j^{(1)},\cdots,\beta_j^{(s)})_{1,\mathbf{q}};\quad(\mathrm{l}_j;U_j^{(1)}f_j^{(1)},\cdots,U_j^{(s)}f_j^{(s)})_{1,R}:D=(d_j^{(1)};\delta_j^{(1)})_{1,q_1};\cdots;(d_j^{(s)};\delta_j^{(s)})_{1,q_s} \tag{3.5}$$

In this section, we establish the general Selberg integral about the product of a class of multivariable polynomials, multivariable I-function and modified H-function of several variables.

*Theorem.*

$$\int_0^1\cdots\int_0^1\prod_{i=1}^k x_i\prod_{i=1}^n x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}\begin{pmatrix}\mathrm{y}_1X_{\alpha_1,\beta_1,\gamma_1}(x_1,\cdots,x_n)\\ \cdot\\ \cdot\\ \mathrm{y}_vX_{\alpha_v,\beta_v,\gamma_v}(x_1,\cdots,x_n)\end{pmatrix}$$

$$I\begin{pmatrix}\mathrm{z}_1X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n)\\ \cdot\\ \cdot\\ \mathrm{z}_rX_{\delta_r,\psi_r,\phi_r}(x_1,\cdots,x_n)\end{pmatrix}H_{\mathbf{p},\mathbf{q}:|R:Y}^{\mathbf{m},\mathbf{n}:|\mathbf{R}:X}\begin{pmatrix}\mathrm{z}'_1X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n)\\ \cdot\\ \cdot\\ \mathrm{z}'_sX_{\epsilon_s,\eta_s,\zeta_s}(x_1,\cdots,x_n)\end{pmatrix}\mathrm{d}x_1\cdots\mathrm{d}x_n=$$

$$\sum_{K_1=0}^{[N_1/\mathfrak{M}_1]}\cdots\sum_{K_v=0}^{[N_v/\mathfrak{M}_v]}\sum_{G_i=1}^{m^{(i)}}\sum_{g_i=0}^{\infty}\phi\frac{\prod_{i=1}^r\phi_iz_i^{\eta_{G_i,g_i}}(-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r\delta_{G^{(i)}}^{(i)}\prod_{i=1}^r g_i!}\quad a_vy_1^{R_1}\cdots y_v^{R_v}$$

$$H_{\mathbf{p}+3n+2k,\mathbf{q}+2n+2k:|R:Y}^{\mathbf{m},\mathbf{n}+3n+2k:|\mathbf{R}:X}\left(\begin{array}{c|c}\mathrm{z}'_1 & \mathrm{A}_1,A_2,A_3,A_4,A_5,\mathbb{A}:C\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \\ \mathrm{z}'_s & \mathbb{B},\mathrm{B}_1,B_2,B_3,B_4:D\end{array}\right) \tag{3.6}$$

where

$$A_1=\left[1-a-\sum_{i=1}^vK_i\alpha_i-\sum_{i=1}^r\eta_{G_i,g_i}\delta_i-j(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\epsilon_1+j\zeta_1,\cdots,\epsilon_s+j\zeta_s\right]_{0,n-1} \tag{3.7}$$

$$A_2=\left[1-b-\sum_{i=1}^vK_i\beta_i-\sum_{i=1}^r\eta_{G_i,g_i}\psi_i-j(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\eta_1+j\zeta_1,\cdots,\eta_s+j\zeta_s\right]_{0,n-1} \tag{3.8}$$

$$A_3=\left[-j(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});j\zeta_1,\cdots,j\zeta_s)\right]_{1,n} \tag{3.9}$$

$$A_4=\left[-a-\sum_{i=1}^vK_i\alpha_i-\sum_{i=1}^r\eta_{G_i,g_i}\delta_i-(n-j)(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\right.$$

$$\left.\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\right]_{1,k} \tag{3.10}$$

$$A_5=\left[1-a-b-\sum_{i=1}^vK_i(\alpha_i+\beta_i)-\sum_{j=1}^r\eta_{G_j,g_j}(\delta_j+\psi_j)-(2n-j-1)(c+\sum_{i=1}^v\gamma_iK_i+\sum_{i=1}^r\phi_i\eta_{G_i,g_i});\right.$$

$$\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots\epsilon_s+\eta_s+(2n-j-1)\zeta_s\Big]_{1,k} \tag{3.11}$$

$$B_1=\left(-c-\sum_{i=1}^{v}K_i\gamma_i-\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s\right),\cdots,\left(-c-\sum_{i=1}^{v}\gamma_iR_i-\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i};\zeta_1,\cdots,\zeta_s\right) \tag{3.12}$$

$$B_2=\Bigg[1-a-\sum_{i=1}^{v}K_i\alpha_i-\sum_{i=1}^{r}\eta_{G_i,g_i}\delta_i-(n-j)(c+\sum_{i=1}^{v}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\Big]_{1,k} \tag{3.13}$$

$$B_3=\Bigg[-a-b-\sum_{i=1}^{v}K_i(\alpha_i+\beta_i)-\sum_{i=1}^{r}\eta_{G_i,g_i}(\delta_i+\psi_i)-(2n-j-1)(c+\sum_{i=1}^{v}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots\epsilon_s+\eta_s+(2n-j-1)\zeta_s\Big]_{1,k} \tag{3.14}$$

$$B_4=\Bigg[1-a-b-\sum_{i=1}^{v}K_i(\alpha_i+\beta_i)-\sum_{i=1}^{r}\eta_{G_i,g_i}(\delta_i+\psi_j)-(n+j-1)(c+\sum_{i=1}^{v}\gamma_iK_i+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i});$$

$$\epsilon_1+\eta_1+(n+j-1)\zeta_1,\cdots\epsilon_s+\eta_s+(n+j-1)\zeta_s\Big]_{0,n-1} \tag{3.15}$$

Provided

$min\{\alpha_i,\beta_i,\gamma_i,\delta_j,\psi_j,\phi_j,\epsilon_l,\eta_l,\zeta_l\}>0, i=1,\cdots,v, j=1,\cdots,r, l=1,\cdots,s\,; a,b,c\in\mathbb{C}$

$$A=Re\left(a+\sum_{i=1}^{r}\delta_i\eta_{G_i,g_i}\right)+\sum_{i=1}^{s}\epsilon_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

$$B=Re\left(b+\sum_{i=1}^{r}\psi_i\eta_{G_i,g_i}\right)+\sum_{i=1}^{s}\eta_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0$$

$$C=Re\left(c+\sum_{i=1}^{r}\phi_i\eta_{G_i,g_i}\right)+\sum_{i=1}^{s}\zeta_i\min_{1\leqslant j\leqslant m_i}Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>Max\left\{-\frac{1}{n},-\frac{A}{n-1},-\frac{B}{n-1}\right\}\text{ and }k\leqslant n.$$

$|arg(z_iX_{\delta_i,\psi_i,\phi_i}(x_1,\cdots,x_n))|<\frac{1}{2}\Omega_i\pi$, where $\Omega_i$ is defined by (1.7) for $i=1,\cdots,r$

$|arg(z_i'X_{\epsilon_i,\eta_i,\zeta_i}(x_1,\cdots,x_n))|<\frac{1}{2}U_i\pi$, where $U_i$ is defined by (1.15) for $i=1,\cdots,s$

the multiple series on the left-hand side of (3.6) converges absolutely.

*Proof*

To evaluate the integrals (3.6), first, we replace the class of multivariable polynomials $S_{N_1,\cdots,N_v}^{\mathfrak{M}_1,\cdots,\mathfrak{M}_v}[.]$ and multivariable I-function occurring on the left-hand side of your integral in term of series with the help of (1.2) and (1.8) Respectively. Now we express the modified multivariable H-function in Mellin-Barnes contour integrals by using (1.11). Nest, we change the order of the $(t_1,\cdots,t_s)$-integrals and $(x_1,\cdots,x_n)$-integrals, (which is justified under the conditions stated), we obtain the following result (say L.H.S.):

Notes

$$L.H.S = \sum_{K_1=0}^{[N_1/\mathfrak{M}_1]} \cdots \sum_{K_v=0}^{[N_v/\mathfrak{M}_v]} \sum_{G_i=1}^{m^{(i)}} \sum_{g_i=0}^{\infty} \phi \frac{\prod_{i=1}^r \phi_i z_i^{\eta_{G_i,g_i}} (-)^{\sum_{i=1}^r g_i}}{\prod_{i=1}^r \delta_{G^{(i)}}^{(i)} \prod_{i=1}^r g_i!}$$

$$a_v y_1^{K_1} \cdots y_v^{K_v} \frac{1}{(2\pi\omega)^s} \int_{L_1'} \cdots \int_{L_s'} \theta(t_1, \cdots, t_s) \prod_{l'=1}^{s} \theta_{l'}(t_{l'}) z_{l'}^{\prime t_{l'}} \left[ \int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} x_i \right.$$

$$\prod_{i=1}^{n} x_i^{a-1+\sum_{l=1}^v K_l\alpha_l + \sum_{l=1}^r \eta_{G_l,g_l}\delta_l + \sum_{l=1}^s \epsilon_l t_l} (1-x_i)^{b-1+\sum_{l=1}^v K_l\beta_l + \sum_{l=1}^r \eta_{G_l,g_l}\psi_l + \sum_{l=1}^s \eta_l t_l}$$

$$\left. \prod_{1\leqslant j<k\leqslant n} |x_j - x_k|^{2(c+\sum_{l=1}^v K_l\gamma_l + \sum_{l=1}^r \eta_{G_l,g_l}\phi_l + \sum_{l=1}^s \zeta_l t_l)} \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \right] \mathrm{d}t_1 \cdots \mathrm{d}t_s \tag{3.16}$$

Now we evaluate the inner $(x_1, \cdots, x_n)$ - integrals with the help of the lemma and reinterpreting the Mellin-Barnes contour integrals thus obtained regarding modified H-function of $s$-variables, we arrive at the required result after algebraic manipulations.

## IV. Special Cases

The multivariable polynomial vanishes and the multivariable I-function reduces to H-function defined by Fox [3], we obtain:

*Corollary 1.*

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{k} x_i \prod_{i=1}^{n} x_i^{a-1}(1-x_i)^{b-1} \prod_{1\leqslant j<k\leqslant n} |x_j - x_k|^{2c} H\left( \ z_1 X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n) \ \right)$$

$$H_{\mathbf{p},\mathbf{q}:|R:Y}^{\mathbf{m},\mathbf{n}:|\mathbf{R}:X} \left( \begin{array}{c} \mathrm{z}_1' X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n) \\ \cdot \\ \cdot \\ \mathrm{z}_s' X_{\epsilon_s,\eta_s,\zeta_s}(x_1,\cdots,x_n) \end{array} \right) \mathrm{d}x_1 \cdots \mathrm{d}x_n = \sum_{G=1}^{m^{(1)}} \sum_{g=0}^{\infty} \phi_1 \frac{z_1^{\eta_{G,g}} (-)^g}{\delta_G g!}$$

$$H_{\mathbf{p}+3n+2k,\mathbf{q}+2n+2k:|R:Y}^{\mathbf{m},\mathbf{n}+3n+2k:|\mathbf{R}:X} \left( \begin{array}{c|c} \mathrm{z}_1' & \mathrm{A}_1', A_2', A_3', A_4', A_5', \mathbb{A} : C \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \\ \mathrm{z}_s' & \mathbb{B}, \mathrm{B}_1', B_2', B_3', B_4' : D \end{array} \right) \tag{4.1}$$

where

$$A_1' = [1 - a - \eta_{G,g}\delta_1 - j(c + \phi_1\eta_{G,g}); \epsilon_1 + j\zeta_1, \cdots, \epsilon_s + j\zeta_s]_{0,n-1} \tag{4.2}$$

$$A_2' = [1 - b - \eta_{G,g}\psi_1 - j(c + \phi_1\eta_{G,g}); \eta_1 + j\zeta_1, \cdots, \eta_s + j\zeta_s]_{0,n-1} \tag{4.3}$$

$$A_3' = [-j(c + \phi_1\eta_{G,g}); j\zeta_1, \cdots, j\zeta_s)]_{1,n} \tag{4.4}$$

$$A_4' = \big[ -a - \eta_{G,g}\delta_1 - (n-j)(c + \phi_1\eta_{G,g}); \epsilon_1 + (n-j)\zeta_1, \cdots, \epsilon_s + (n-j)\zeta_s \big]_{1,k} \tag{4.5}$$

Ref

3. C. Fox, The G and H-functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961), 395-429.

$$A'_5 = \left[1-a-b-\eta_{G,g}(\delta_1+\psi_1) - (2n-j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots\right.$$

$$\left.\epsilon_s+\eta_s+(2n-j-1)\zeta_s\right]_{1,k} \tag{4.6}$$

$$B'_1 = (-c-\phi_1\eta_{G,g};\zeta_1,\cdots,\zeta_s),\cdots,(-c-\phi_1\eta_{G,g};\zeta_1,\cdots,\zeta_s) \tag{4.7}$$

$$B'_2 = \left[1-a-\eta_{G,g}\delta_1-(n-j)(c+\phi_1\eta_{G,g});\epsilon_1+(n-j)\zeta_1,\cdots,\epsilon_s+(n-j)\zeta_s\right]_{1,k} \tag{4.8}$$

$$B'_3 = \left[-a-b-\eta_{G,g}(\delta_1+\psi_1) - (2n-j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(2n-j-1)\zeta_1,\cdots\right.$$

$$\left.\epsilon_s+\eta_s+(2n-j-1)\zeta_s\right]_{1,k} \tag{4.9}$$

$$B'_4 = \left[1-a-b-\eta_{G,g}(\delta_1+\psi_1) - (n+j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(n+j-1)\zeta_1,\cdots\right.$$

$$\left.\epsilon_s+\eta_s+(n+j-1)\zeta_s\right]_{0,n-1} \tag{4.10}$$

Provided

$$min\{\delta_1,\psi_1,\phi_1,\epsilon_l,\eta_l,\zeta_l\}>0; l=1,\cdots,s; a,b,c\in\mathbb{C}$$

$$A' = Re\,(a+\delta_1\eta_{G,g}) + \sum_{i=1}^{s}\epsilon_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

$$B' = Re\,(b+\psi_1\eta_{G,g}) + \sum_{i=1}^{s}\eta_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0$$

$$C' = Re\,(c+\phi_1\eta_{G,g}) + \sum_{i=1}^{s}\zeta_i \min_{1\leqslant j\leqslant m_i} Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > Max\left\{-\frac{1}{n},-\frac{A'}{n-1},-\frac{B'}{n-1}\right\} \text{ and } k\leqslant n.$$

$$|arg(z_1X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n))| < \frac{1}{2}\Omega_1\pi\,, \text{ where } \Omega_1=\sum_{k=1}^{n^{(1)}}\alpha_k^{(1)} - \sum_{k=n^{(1)}+1}^{p^{(1)}}\alpha_k^{(1)} + \sum_{k=1}^{m^{(1)}}\beta_k^{(1)} - \sum_{k=m^{(1)}+1}^{q^{(1)}}\beta_k^{(1)}$$

$|arg(z'_iX_{\epsilon_i,\eta_i,\zeta_i}(x_1,\cdots,x_n))| < \frac{1}{2}U_i\pi$, where $U_i$ is defined by (1.15) for $i=1,\cdots,s$

the series on the left-hand side of (4.1) converges absolutely.

Consider the above corollary, now the modified multivariable H-function reduces to H-function of one variable defined by Fox [3], we have.

*Corollary 2.*

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{k}x_i\prod_{i=1}^{n}x_i^{a-1}(1-x_i)^{b-1}\prod_{1\leqslant j<k\leqslant n}|x_j-x_k|^{2c}\,H\left(\ z_1X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n)\ \right)$$

Ref

3. C. Fox, The G and H-functions as symmetrical Fourier Kernels, Trans. Amer. Math. Soc. 98 (1961), 395-429.

$$H_{p_1,q_1}^{m_1,n_1}\left( z_1' X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n) \right)\mathrm{d}x_1\cdots\mathrm{d}x_n = \sum_{G=1}^{m^{(1)}}\sum_{g=0}^{\infty}\phi_1\frac{z_1^{\eta_{G,g}}(-)^g}{\delta_G g!}$$

$$H_{p_1+3n+2k,q_1+2n+2k}^{m_1,n_1+3n+2k}\left( z_1' \left| \begin{array}{c} A_1'',A_2'',A_3'',A_4'',A_5'',(c_j^{(1)},\gamma_j^{(1)})_{1,p_1} \\ \cdot \\ \cdot \\ (d_j^{(1)},\delta_j^{(1)})_{1,q_1},B_1'',B_2'',B_3'',B_4'' \end{array} \right. \right) \tag{4.11}$$

Notes

where

$$A_1'' = [1-a-\eta_{G,g}\delta_1 - j(c+\phi_1\eta_{G,g});\epsilon_1+j\zeta_1]_{0,n-1} \tag{4.12}$$

$$A_2'' = [1-b-\eta_{G,g}\psi_1 - j(c+\phi_1\eta_{G,g});\eta_1+j\zeta_1]_{0,n-1} \tag{4.13}$$

$$A_3'' = [-j(c+\phi_1\eta_{G,g});j\zeta_1)]_{1,n} \tag{4.14}$$

$$A_4'' = \left[-a-\eta_{G,g}\delta_1-(n-j)(c+\phi_1\eta_{G,g});\epsilon_1+(n-j)\zeta_1\right]_{1,k} \tag{4.15}$$

$$A_5'' = \left[1-a-b-\eta_{G,g}(\delta_1+\psi_1)-(2n-j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(2n-j-1)\zeta_1\right]_{1,k} \tag{4.16}$$

$$B_1'' = (-c-\phi_1\eta_{G,g};\zeta_1),\cdots,(-c-\phi_1\eta_{G,g};\zeta_1) \tag{4.17}$$

$$B_2'' = \left[1-a-\eta_{G,g}\delta_1-(n-j)(c+\phi_1\eta_{G,g});\epsilon_1+(n-j)\zeta_1\right]_{1,k} \tag{4.18}$$

$$B_3'' = \left[-a-b-\eta_{G,g}(\delta_1+\psi_1)-(2n-j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(2n-j-1)\zeta_1\right]_{1,k} \tag{4.19}$$

$$B_4'' = \left[1-a-b-\eta_{G,g}(\delta_1+\psi_1)-(n+j-1)(c+\phi_1\eta_{G,g});\epsilon_1+\eta_1+(n+j-1)\zeta_1\right]_{0,n-1} \tag{4.20}$$

Provided

$$min\{\delta_1,\psi_1,\phi_1,\epsilon_1,\eta_1,\zeta_1\}>0;a,b,c\in\mathbb{C}$$

$$A'' = Re\,(a+\delta_1\eta_{G,g})+\epsilon_1\min_{1\leqslant j\leqslant m_1} Re\left(\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)>0$$

$$B'' = Re\,(b+\psi_1\eta_{G,g})+\eta_1\min_{1\leqslant j\leqslant m_1} Re\left(\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)>0$$

$$C'' = Re\,(c+\phi_1\eta_{G,g})+\zeta_1\min_{1\leqslant j\leqslant m_1} Re\left(\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)>Max\left\{-\frac{1}{n},-\frac{A''}{n-1},-\frac{B''}{n-1}\right\}\ \text{and } k\leqslant n.$$

$$|arg(z_1 X_{\delta_1,\psi_1,\phi_1}(x_1,\cdots,x_n))| < \frac{1}{2}\Omega_1\pi\,, \text{ where } \Omega_1 = \sum_{k=1}^{n^{(1)}} \alpha_k^{(1)} - \sum_{k=n^{(1)}+1}^{p^{(1)}} \alpha_k^{(1)} + \sum_{k=1}^{m^{(1)}} \beta_k^{(1)} - \sum_{k=m^{(1)}+1}^{q^{(1)}} \beta_k^{(1)}$$

$$|arg(z_1' X_{\epsilon_1,\eta_1,\zeta_1}(x_1,\cdots,x_n))| < \frac{1}{2}U_1\pi, \text{where } \quad U_1 = \sum_{j=1}^{m_1} \delta_j^{(1)} - \sum_{j=1+m_1}^{q_1} \delta_j^{(1)} + \sum_{j=1}^{n_1} \gamma_j^{(1)} - \sum_{j=n_1+1}^{p_1} \gamma_j^{(1)}$$

the series on the left-hand side of (4.11) converges absolutely.

## V. Conclusion

In this paper, we have evaluated a general Selberg integral involving the product of an expansion of multivariable I function defined by Prasad [5], modified multivariable H-function defined by Prasad and Singh [6] and class of multivariable polynomials defined by Srivastava [9] with general arguments. The formulae evaluated in this paper are very general nature. Thus, the results established in this research work would serve as a formula from which, upon specializing the parameters, as many as desired results involving the special functions of one and several variables can be obtained.

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# The Extended Q- Image (basic analogue) of Mittag-Leffler Function and it's Simply Properties

By Alok Jain, Farooq Ahmad, D. K. Jain & Renu Jain

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*Keywords: extended q-beta function, fractional q-operator; basic analogue of the extended mittag-leffler function.*

*GJSFR-F Classification: FOR Code: MSC 2010: 33E12*

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# The Extended Q- Image (basic analogue) of Mittag-Leffler Function and it's Simply Properties

Alok Jain [α], Farooq Ahmad [σ], D. K. Jain [ρ] & Renu Jain [ω]

*Abstract-* In the present paper, the authors derived the basic analogue of extended Mittag-Leffler function by using the extended q-beta function and obtain some important results of Mittag-Leffler function in terms of generalized Wright function. Special cases are also discussed at the end section of paper.

*Keywords: extended q-beta function, fractional q-operator; basic analogue of the extended mittag-leffler function.*

## I. Introduction and Mathematical Preliminaries

The purpose of this paper is to increase the accessibility of different dimensions of q-fractional calculus and the q-analogue of the extended Mittag-Leffler function to the real world problems of engineering science.

Let us start with giving the historical background of the ML-function. The function

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad \ldots(1)$$

was defined and studied by ML-function in the year 1903 in [1-3].

It is direct generalization of the exponential series, since for $\alpha = 1$, we have to exponential function.

The function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)},$$

gives a generalization of equation (1) this generalization was studied by Wiman in 1905 [4,5]. Agarwal in 1953, and Humbert and Agarwal [6, 7] in 1953. Afterward, Prabhakar [8] introduced the generalized ML-function.

$$E^{\delta}{}_{\beta,\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(\beta k+\gamma)} \frac{z^k}{k!}, \text{where } \beta,\gamma,\delta \in C, \text{ with } \mathrm{Re}(\beta) > 0.$$

Recently, Mehmet Ali etal [9] extended ML-function as follows

$$E^{\gamma;c}{}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\beta(\gamma+k,c-\gamma)}{\beta(\gamma,c-\gamma)} \frac{(c)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \text{ where } \beta,\gamma,\delta \in C, \text{ with } \mathrm{Re}(\beta) > 0.$$

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*Riemann-Liouville fractional integrals:* As defined in [10, 11], the Riemann-Liouville fractional integral is given by:

$$(I_{a+}^{\alpha} \mathrm{f})x \;=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(\mathrm{t})\,\mathrm{dt},\ (x>a\,, a\epsilon R).$$

Thus, in general the Riemann-Liouville fractional integrals of arbitrary order for a function f(t), is a natural consequence of the well-knownformula (Cauchy-Dirichlets?) that reduces the calculation of the n-fold primitive of a function f (t) to a single integral of convolution type.

Recently, Yadav et. al. [12] introduces a new q-extension of the Leibnitz rule for the derivatives of a product of two basic functions in terms of a finite q-series involving Weyl type q-derivatives of the functions in the following manner.

$$\underset{z,\ \infty,q}{\overset{\alpha}{D}}[U(v)V(z)]=\sum_{r=0}^{\infty}\frac{(-1)^r q^{r(r+1)/2}(q^{-\alpha};q)_r}{(q;q)_r}\underset{z\ \ \infty,q}{\overset{\alpha-r}{D}}[U(z)]\underset{z\ \ \infty,q}{\overset{\alpha}{D}}\left[V(zq^{\alpha-r})\right]$$

Where U(z) and V(z) are two regular functions and the functional q-differential operator $\underset{z,\ \infty,q}{\overset{\alpha}{D}}(.)$ of Weyl type is given by

$$\underset{z,\ \infty,q}{\overset{\alpha}{D}}[f(z)]=\frac{q^{\frac{-\alpha(\alpha+1)}{2}}}{\Gamma_q(-\alpha)}\int_z^{\infty}(t-z)_{-q-1}f(tq^{1+\alpha})d_q(t)$$

In particular for f(z) = $z^{-p}$ the equation becomes:

$$\underset{z,\ \infty,q}{\overset{\alpha}{D}}\left[z^{-p}\right]=\frac{\Gamma_q(p+\alpha)q^{-\alpha p+\frac{\alpha(1-\alpha)}{2}}z^{-p-\alpha}(1-q)}{\Gamma_q(p)}$$

## II. Main Results

In this section of paper, we defined the q-analogue of Mehmet Ali etal [1] extended ML-function as follows.

We defined basic analogue of extended ML-function by using the fact that

$$\frac{(\gamma:q)_k}{(c:q)_k}=\frac{B_q(\gamma+k,c-\gamma)}{B_q(\gamma,c-\gamma)}$$

$$E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\;=\;\sum_{k=0}^{\infty}\frac{B_q(\gamma+k,c-\gamma)\quad(c:q)_k\qquad z^k}{B_q(\gamma,c-\gamma)\quad\Gamma_q(\alpha k+\beta)\qquad(q;q)_k}.$$

The function $E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})$ converges under convergence of basic analogue of H-function which are as follows. The integral converges if $Re[slog(z)-\log sin\pi s]<0$, on the contour C,

*Theorem I:* (Integral representation) For the extended Mittag-Leffler function, we have

$$I_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z{:}\,q})\}=\ I_q^{\mu}\left[\sum_{k=0}^{\infty}\frac{B_q(\gamma+k,c-\gamma)\quad(c:q)_k\qquad z^k}{B_q(\gamma,c-\gamma)\quad\Gamma_q(\alpha k+\beta)\qquad(q;q)_k}\right]$$

Or

Ref

12. Yadav, R. K., & Purohit, S. D. (2006). On fractional q-derivatives and transformations of the generalized basic hypergeometric functions. *J. Indian Acad. Math, 28*(2), 321-326.

$$I_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\} = \left\{\frac{B_q(\gamma+k,c-\gamma)\quad (c;q)_k \qquad 1}{B_q(\gamma,c-\gamma)\ \ \Gamma_q(\alpha k+\beta)\quad (q;q)_k}\right\} I_q^{\mu}\ (z^k)$$

Or

$$I_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\} = \frac{\Gamma_q(\gamma+k)\Gamma_q(c-\gamma)\Gamma_q(c)\Gamma_q(c+k)\Gamma_q(\mu+k)}{\Gamma_q(c+k)\ \Gamma_q(c-\gamma)\Gamma_q(\gamma)\Gamma_q(\mu+1+k)\Gamma_q(\beta+\alpha k)} \frac{z^{k+\mu}}{(q;q)_k}$$

Notes

Thus,

$$I_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\} = \frac{z^{\mu}}{\Gamma_q(\gamma)}\ {}^{2}\Psi_2\left[\begin{matrix}(\gamma,1)(\mu,1);\ z\\(\beta+\alpha,1)(\mu+1{,}1)\end{matrix}\right].$$

This is the proof of the theorem.

*Theorem II:* For the extended Mittag-Leffler function, we have

$$D_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\} = D_q^{\mu}\left[\sum_{k=0}^{\infty}\frac{B_q(\gamma+k,c-\gamma)\quad (c;q)_k \qquad z^k}{B_q(\gamma,c-\gamma)\ \ \Gamma_q(\alpha k+\beta)\quad (q;q)_k}\right]$$

Or

$$D_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\} = \left\{\frac{B_q(\gamma+k,c-\gamma)\quad (c;q)_k \qquad 1}{B_q(\gamma,c-\gamma)\ \ \Gamma_q(\alpha k+\beta)\quad (q;q)_k}\right\} D_q^{\mu}\ (z^k)$$

Or

$$D_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\} = \frac{\Gamma_q(\gamma+k)\Gamma_q(c-\gamma)\Gamma_q(c)\Gamma_q(c+k)\Gamma_q(\mu+k)}{\Gamma_q(c+k)\ \Gamma_q(c-\gamma)\Gamma_q(\gamma)\Gamma_q(\mu-1+k)\Gamma_q(\beta+\alpha k)} \frac{z^{k+\mu}}{(q;q)_k}$$

Thus,

$$D_q^{\mu}\ \{E^{\gamma;c}{}_{\alpha,\beta}(\mathrm{z:q})\} = \frac{z^{\mu}}{\Gamma_q(\gamma)}\ {}^{2}\Psi_2\left[\begin{matrix}(\gamma,1)(\mu,1);\ z\\(\beta+\alpha,1)(\mu-1{,}1)\end{matrix}\right].$$

This is the proof of the theorem.

## III. Conclusion

In this paper, we have explored the possibility for derivation of some expansions of basic analogue ML-function. The results thus derived are general in character and likely to find certain applications in the theory of hyper geometric functions. Finally we conclude with the remark that the results and the operators proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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# Regression Analysis of an Experiment with Treatment as a Qualitative Predictor

By Eze, Chinonso Michael, Asogwa, Oluchukwu Chukwuemeka & Onwuamaeze, Uchenna Charity
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*Abstract-* This research work explores the indept use of multiple regression analysis in modelling the optimum weight gain of Broiler chicken as a function of initial weight and feed combination (treatment). The statistical relationship between the weight gain, initial weight and different feed combination was established using a regression model. The measurement on the initial weights was kept silent, allowing the use of analysis of variance technique to determine the error inherent in the study. Application of the model was done on the result of the experiment carried out on agritted breed of Broilers. The best feed combination (among the different feed combination considered) was determined by looking out for the combination that yields the optimum weight gain when the initial weight is known.

*Keywords:* *multiple regression; qualitative predictor.*

*GJSFR-F Classification:* *FOR Code: MSC 2010: 47N30*

REGRESSIONANALYSISOFANEXPERIMENTWITHTREATMENTASAQUALITATIVEPREDICTOR

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# Regression Analysis of an Experiment with Treatment as a Qualitative Predictor

Eze, Chinonso Michael [α], Asogwa, Oluchukwu Chukwuemeka [σ] & Onwuamaeze, Uchenna Charity [ρ]

*Abstract-* This research work explores the indept use of multiple regression analysis in modelling the optimum weight gain of Broiler chicken as a function of initial weight and feed combination(treatment). The statistical relationship between the weight gain, initial weight and different feed combination was established using a regression model. The measurement on the initial weights was kept silent, allowing the use of analysis of variance technique to determine the error inherent in the study. Application of the model was done on the result of the experiment carried out on agritted breed of Broilers. The best feed combination (among the different feed combination considered) was determined by looking out for the combination that yields the optimum weight gain when the initial weight is known.

*Keywords: multiple regression; qualitative predictor.*

## I. Introduction

In design and analysis of experiment, interest is centred on getting a design structure that will facilitate the collection of appropriate data for proper analysis. In an experimental design, some necessary considerations are made pertaining the scope of the study, the factors(controllable and uncontrollable) to be studied, suitable layout(design) and the experimental units(plots). These considerations are made prior to the conduct of the experiment and they avail the experimenter the opportunity to understand the aims of the experiment and possibly identify the appropriate technique for the actualization of the set aside aims.

Basically, the purpose of every experiment conducted is to ascertain the effects and the relationship between the dependent variable and some other variables usually known as the predictor variables (Udom, 2015). The ability of a particular analytical technique to actualize the purpose of ascertaining the effects of identified factors depends on the structure provided for the study. In other words, if the proper structure is not provided, valid effects may not be ascertained. When several treatments or treatment combinations are randomly applied to the experimental units, analysis of variance (ANOVA) technique emerges as one of the tools which can analyze the linear model(s) that represent the experiment situation. The ANOVA procedure attempts to analyze the variation in a set of responses and assign portions of this variation to each variable in a set of independent variables(Wackerly et, al, 2008). With the concept of pairwise comparison, the significant treatment effects among all the treatments under study can be determined.

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On the other hand, the analysis of variance layout for the experiment can be re-arranged such that the observations can be analyzed using multiple linear regression model where the treatments or treatment combinations form part of the independent variables. Multiple linear regression is an extension of simple linear regression to allow for more than one independent variable (Mendenhall, 2003). So, we combine the treatment and the initial weight to form the two factors that affect the weight gain. Oftentimes, The treatments in experiments are not quantitative, thus, it becomes appropriate for them to be premised on an indicator variable like

$$z_i = \begin{cases} 1 \ if \ i^{th} \ treatment \ is \ applied \ to \ the \ j^{th} replicate \\ \qquad\qquad 0 \ if \ otherwise \end{cases} \tag{1}$$

so that they can form part of the information matrix. The model appropriate for such setting can be expressed in compact form as $y = X\beta + \varepsilon$, where $y = y_{ij}$ is the observed response of $j^{th}$ replicate receiving $i^{th}$ treatment, $\beta$ is a vector that contains the treatment effects, $X$ is the information matrix containing the indicator variables and $\varepsilon$ is error column matrix. The task of predictive analysis is to obtain $\beta$ such that the predicted value is close to the observed value as much as possible (Motoyama, 1978). However, the effects of the treatments cannot be estimated using $\beta = (X'X)^{-1}X'y$ since the first column (the coefficient of the constant in the model) in $X$ is the sum of the second and subsequent columns, which are linearly independent. Also, $(X'X)^{-1}$ does not exist since $X$ is not of full rank. However, if another one or more quantitative measurements are made on the experimental units and added to the model as an additional independent variable different from the indicator variables such that the first column in $X$ is not the sum of the second and subsequent columns, $(X'X)^{-1}$ will exist and $(X'X)^{-1}X'y$ will be able to estimate the parameters $\beta$. Inferences on $\beta$ can then be made to determine which effect is higher than the other.

In the area of weight gain analysis as it applies to other arears of life, meta-analysis has often been used. This involves statistical procedure that combines the results of multiple scientific studies. Nianogo, et, al (2017) adopted meta-analysis in studying the weight gain of prisoners during incarceration. A meta-analysis of eight studies combined showed an average weight gain of 0.43 (0.14, 0.72) lb/week. In all the studies they consulted, a high proportion (43% to 73%) of participants reported weight gain during incarceration. Thus, they recommended the incorporation of initiatives aimed at combating unhealthy weight developments in health promotion activities within prisons.

However, our interest is to first, analyze the result of the experiment in a regression format and make use of the ANOVA precedure to analyze the variation in the experiment and assign portions of this variation to each variable in a set of independent variables while relaxing the initial weight. Effort is made also to determine the feed combination that yields optimum weight gain so as to solve the problem of unimaginable increase in the cost of poultry feed stuff pointed out by Adene (2004) as the greatest source of dilemma in poultry industry. For other problems associated with Broiler feeding (see Berepubo et al (1995), Offiong and olumu (1980), and Akpodiete (2008)).

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## II. Design of the Experiment

The experiment was designed in a complete randomized design with one factor which is a combination of PKC based feed and bioactive yeast having four and three levels respectively.

The treatments in the experiment are represented as follows:

$T_1$ = 0.4g of bioactive yeast/kg X 15kg of PKC based feed
$T_2$ = 0.8g of bioactive yeast/kg X 15kg of PKC based feed
$T_3$ = 1.2g of bioactive yeast/kg X 15kg of PKC based feed
$T_4$ = 0.4g of bioactive yeast/kg X 20kg of PKC based feed
$T_5$ = 0.8g of bioactive yeast/kg X 20kg of PKC based feed
$T_6$ = 1.2g of bioactive yeast/kg X 20kg of PKC based feed
$T_7$ = 0.4g of bioactive yeast/kg X 25kg of PKC based feed
$T_8$ = 0.8g of bioactive yeast/kg X 25kg of PKC based feed
$T_9$ = 1.2g of bioactive yeast/kg X 25kg of PKC based feed
$T_{10}$ = 0.4g of bioactive yeast/kg X 30kg of PKC based feed
$T_{11}$ = 0.8g of bioactive yeast/kg X 30kg of PKC based feed
$T_{12}$ = 1.2g of bioactive yeast/kg X 30kg of PKC based feed

Since interest is on the growth of broilers, measurements were taken on

1. The initial weight gain
2. The weight gain of the broilers

*Table 1:* Layout of the Experiment

| | **Replication** | | | | | | |
|---|---|---|---|---|---|---|---|
| TREATMENT | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| T1 | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{14}$ | $X_{15}$ | $X_{16}$ | $X_{17}$ |
| T2 | $X_{21}$ | $X_{22}$ | $X_{23}$ | $X_{24}$ | $X_{25}$ | $X_{26}$ | $X_{27}$ |
| T3 | $X_{31}$ | $X_{32}$ | $X_{33}$ | $X_{34}$ | $X_{35}$ | $X_{36}$ | $X_{37}$ |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |
| . | . | . | . | . | . | . | . |
| T12 | $X_{121}$ | $X_{122}$ | $X_{123}$ | $X_{124}$ | $X_{125}$ | $X_{126}$ | $X_{127}$ |

## III. Methodology

Having stated earlier that one of the aims of this work is to develop an adequate model for predicting the weight gain of broilers given the initial weight and a specific feed combination, it is appropriate to analyze the data with multiple regression analysis technique.

Multiple regression analysis is a statistical evaluation of the relationship between one dependent variable and two or more independent variables. This technique provides an adequate mathematical model that explains the relationship between the variables under consideration. The general multiple regression model in a compact form can be written as

$$Y = \beta X + \varepsilon \tag{2}$$

$Y$ is a column vector containing the weight gain of broilers considered in the experiment

$\beta$ is a vector of the partial slopes or partial regression coefficients (the intercept or general constant inclusive) (see Mendenhall et, al, 2003)

$X$ is n X k matrix of the independent (predictor) variables which in this case are, the feed combination and initial weight.

$\varepsilon$ is the random error associated with the dependent variable $Y$

However, interest also is on examining the effects of different feed combination, thus, the analysis is done using an appropriate analysis of variance model. Here, we ignore the measurements on the initial weights and concentrate only on the weight gain and the feed combination effects. Since we are considering only one factor in this problem, which is the feed combination, we consider a one-way analysis of variance model given as

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \tag{3}$$

$Y_{ij}$ is the weight gain of jth broiler fed with ith feed combination

$\mu$ is the grand mean

$\alpha_i$ is the effect of the ith feed combination

$\varepsilon_{ij}$ is the error associated with the weight gain of jth broiler fed with ith feed combination

With eqt (3) above, we can determine the significant effect of the different feed combination on the weight gain of broilers considered in the experiment.

Thirdly, a proper adjustment on the initial weight (covariate) can be made so as to ascertain the true effect of the different feed combination. This can be achieved through analysis of covariance

## IV. Analysis and Results

As we know, statistical analysis- modeling and inference form the basis for objective generalizations from the observed data (Jammalamadaka & SenGupta, 2001). Here, we shall apply the methodologies given in the previous section on the weight gain data.

Consider rewritting the general multiple regression model in (2) above in a more explicit form.

$$Y_{ij} = \mu + \sum_{i=1}^{12} \alpha_i z_{ij} + \beta X_{ij} + \varepsilon_{ij} \tag{4}$$

In contrast to quantitative predictor variables, quality predictor variables can be entered into a regression model through dummy or indicator variables (Mendenhall et, al, 2003).

Observing the fact that $z_{ij}$ is a dummy (indicator) variable defined as eqt (1), eqt (4) transforms to

$$Y_{ij} = \mu + \alpha_i + \beta X_{ij} + \varepsilon_{ij}\,;\, i = 1,2,\dots,12; j = 1,2,\dots,7 \tag{5}$$

The definition of the model components remains the same as in (2), with $\alpha_i$ being the effect on weight gain due to $i^{th}$ feed combination.

Thus, the model can be expressed as (weight gain)$_{ij}$ = general mean + (feed combination)$_i$ + slope(initial weight)$_{ij}$ + (error)$_{ij}$.

Notes

Notes

*a) Assumptions of the Model*

In order to carry out a linear regression analysis on a set of data, it is reasonable to assume that the variables under consideration satisfy the following assumptions:

- The dependent variables as well as the error terms are normally distributed with mean zero and variance $\delta^2$. That is $Y_{ij} \sim N(0, \delta^2)$.
- The variance of the dependent variables as well as the variance of the error terms are the same for all the populations under study (Homoscedasticity).
- The independent variables are fixed (they can be controlled by the experimenter).

These assumptions can satisfactorily be justified by the normality and constant variance test performed in the following section.

i. *Normality Test*

Maximum weight gain = 1.1

Minimum weight gain = -0.41

Range (R) = maximum – minimum = 1.1 – (-0.41) = 1.51

By the application of Sturge's rule, we obtain the number of classes (C) as

C = $1+3.322\text{Log}_{10}N$; N is the total number of observations which in this case is 84.

Therefore, C = $1+3.322\text{Log}_{10}84$ = 7.3925 (approximately 7)

The class size (S) is obtained by the ratio of the range to the approximate number of classes. That is $S = \frac{R}{C} = \frac{1.51}{7} = 0.22$

The mean and standard deviation of the observations are 0.335 and 0.2780 respectively.

*Table 2:* Frequency Table of the Weight Gain

| C. interval | FREQ. | X | C. boundary | $z = \frac{x-\bar{x}}{s}$ | $P_i = P(z)$ | $Fe = NP_i$ |
|---|---|---|---|---|---|---|
| -0.42 - -0.19 | 2 | -0.305 | $-\infty - -0.185$ | $\leq -2.30$ | 0.0107 | 0.8988 |
| -0.18 – 0.05 | 4 | -0.065 | -0.185 - -0.045 | -2.30 - -1.37 | 0.0746 | 6.2664 |
| 0.06 – 0.29 | 39 | 0.175 | -0.045 - -0.295 | -1.37 - -0.14 | 0.359 | 30.156 |
| 0.3 – 0.53 | 24 | 0.415 | -0.295 – 0.535 | -0.14 – 0.72 | 0.3199 | 26.8716 |
| 0.54 – 0.77 | 7 | 0.655 | 0.535 – 0.775 | 0.72 – 1.58 | 0.1787 | 15.0108 |
| 0.78 – 1.01 | 6 | 0.895 | 0.775 – 1.015 | 1.58 – 2.45 | 0.05 | 4.2 |
| 1.02 – 1.25 | 2 | 1.135 | 1.015 - $+\infty$ | $\geq 2.45$ | 0.0071 | 0.5964 |
| TOTAL | 84 | | | | | |

With chi-square test statistic, the test was conducted and it yielded test value of 4.2568. At $\alpha = 0.01$ and (4-2-1=1) degree of freedom, the tabulated value is 6.63490. Based on the values above, it was concluded the weight gain observations are normally distributed.

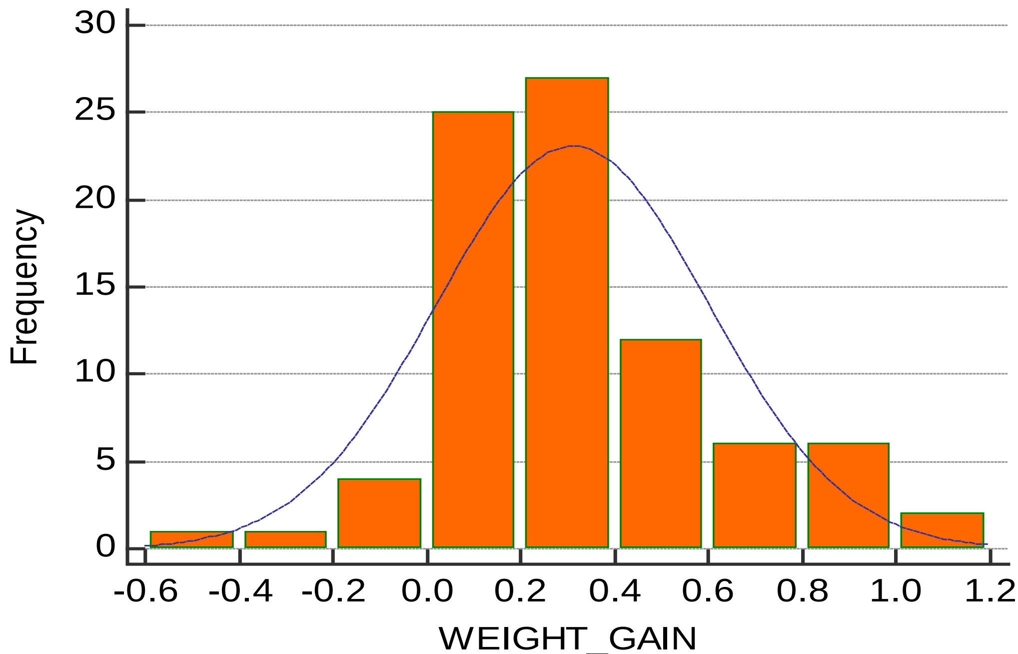


*Figure 1:* Normal Graph for the Weight Gain

ii. *Test For Constant Variance (Homoscedasticity)*

Another important assumption for the use of the model specified in section (3) is the assumption of constant variance. This assumption is important in that it makes the least square estimates of the model parameters to be linear unbiased estimates. In testing for constant variance, the most widely used procedure is the Bartlett's test. The procedure involves computing a statistic whose sampling distribution is closely approximated by the chi-square distribution with r-1 degree of freedom when random samples are from independent normal population.

Our interest is to show that the variances are the same across the groups. That is $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_{12}^2$. we made use of the chi-square distributed Test statistic below for the test

$$x^2 = 2.3026\frac{q}{c} \sim x_{r-1}^2$$

$$q = (N-r)Log_{10}S_p^2 - \sum_{i=1}^{r}(n_i - 1)Log_{10}S_i^2$$

$$c = 1 + \frac{1}{3(r-1)}[\sum_{i=1}^{r}(n_i-1)^{-1} - (N-r)^{-1}]$$

$$S_p^2 = \frac{\sum_{i=1}^{r}(n_i-1)S_i^2}{N-r}$$

Where N is the total number of observations, r is the number of populations, $S_i^2$ is the sample variance of the i[th] population, $S_p^2$ is the weighted average of the sample variance and $n_i$ is the number of observations in i[th] population.

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*Table 3:* Table of Sample Variances

| | | |
|---|---|---|
| $S_1^2 = 0.0092$ | $S_5^2 = 0.1076$ | $S_9^2 = 0.0635$ |
| $S_2^2 = 0.0643$ | $S_6^2 = 0.0369$ | $S_{10}^2 = 0.0745$ |
| $S_3^2 = 0.01123$ | $S_7^2 = 0.0253$ | $S_{11}^2 = 0.0369$ |
| $S_4^2 = 0.0358$ | $S_8^2 = 0.0679$ | $S_{12}^2 = 0.1739$ |

Notes

$n_i = 7$

$$S_p^2 = (7-1)\frac{0.0092 + 0.0643 + \ldots + 0.1739}{84-12} = 0.0673$$

$$q = (84-12)Log_{10}(0.0673) - 6[Log_{10}(0.0092) + Log_{10}(0.0643) + \ldots + Log_{10}(0.1739)] = 7.463$$

$$C = 1 + \frac{1}{3(12-1)}\left[\frac{1}{6} + \frac{1}{6} + \ldots + \frac{1}{6} - \frac{1}{72}\right] = 1.0602$$

The test statistic value therefore, becomes

$$x_{cal}^2 = 2.3026\frac{7.463}{1.0602} = 16.2085$$

At 0.05 level of significance and r-1=11 degree of freedom, the tabulated value, $x_{11}^2; 0.05$ =19.675.

Since the Bartlett's test statistic value (16.2085) is less than the chi-square tabulated value (19.675), we conclude that all the twelve population variances are the same at 0.05 level of significant.

Let eqt (5) be presented in terms of general linear regression model as in eqt (2),

$$Y = XB + e$$

Where $\boldsymbol{Y}$ is an $n \times 1$ vector of the dependent variable (the weight gain), $\boldsymbol{X}$ is an $n \times m$ matrix of the independent variables which contains information about the initial weight and feed combination considered, $\boldsymbol{B}$ is an $m \times 1$ vector of the parameters (the general constant inclusive) and $\boldsymbol{e}$ is an $n \times 1$ vector of the model error. With ordinary least square approach, we estimate the parameters of the model such that the error function is minimized. To get an optimum set of parameters for the model, we consider the sum of square of the error component and differentiate with respect to the parameter set as follows:

Consider

$Y = XB + e$, the sum of square of the error term implies

$$\mathrm{SSE} = \mathrm{e}'\mathrm{e} = (\mathrm{Y} - \mathrm{X}\beta)'(\mathrm{Y} - \mathrm{X}\beta)$$

Opening the bracket, we have that

$$\mathrm{SSE} = \mathrm{Y}'\mathrm{Y} - \mathrm{Y}'\mathrm{X}\beta - \mathrm{X}'\beta'\mathrm{Y} + \mathrm{X}'\beta'\mathrm{X}\beta$$

But $\mathrm{Y}'\mathrm{X}\beta = \mathrm{X}'\beta'\mathrm{Y}$

Therefore,

$$\mathrm{SSE} = \mathrm{Y}'\mathrm{Y} - 2\mathrm{X}'\beta'\mathrm{Y} + \mathrm{X}'\beta'\mathrm{X}\beta$$

Differentiating partially the sum of square of the error with respect to the elements of the parameter vector implies

$$\frac{\partial SSE}{\partial \beta} = -2X'Y + 2X'X\hat{\beta} = 0$$

$$\rightarrow X'X\hat{\beta} = X'Y$$

Pre-multiply both sides by $(X'X)^{-1}$ to have $\hat{\beta} = (X'X)^{-1}X'Y$.

$X'X$ is an information matrix that contains the relationship (covariance) existing among the independent variables. Fixing the data considered in this research work into the formula derived above, the following estimated values of the model parameters are generated as presented in the table below

Notes

*Table 4:* Parameter Estimates

| Parameter | Estimate | Parameter | Estimate |
|---|---|---|---|
| constant | 0.0309 | $\alpha_7$ | 0.2420 |
| $\alpha_1$ | 0.1236 | $\alpha_8$ | 0.3685 |
| $\alpha_2$ | 0.0823 | $\alpha_9$ | 0.5871 |
| $\alpha_3$ | 0.3667 | $\alpha_{10}$ | 0.6592 |
| $\alpha_4$ | 0.2782 | $\alpha_{11}$ | 0.1157 |
| $\alpha_5$ | 0.3824 | $\alpha_{12}$ | 0.4183 |
| $\alpha_6$ | 0.3222 | $\beta$ | -0.0301 |

Therefore, the fitted model is

$Y_{ij} = 0.0309 + \alpha_i - 0.0301X_{ij}\, ;\, \alpha_i's$ are presented in the table above.

*b) Goodness of Fit Test*

Having fitted a regression model to the set of the observations, it is appropriate to assess the robustness (goodness) of the fitted model. This test aims at knowing whether the true statistical relationship between the weight gain, the treatment effect (feed combination) and the covariate variable (initial weight) is reflected in the fitted model. Here, we carry out the test under the null hypothesis that each of the parameters $B = 0$ (the parameters are all equal to zero) against the alternative that at least one of the parameters is not equal to zero. Consider the analysis of variance table of the model given below.

*Table 5:* Analysis of Variance Table 1

| Source of variation | DF | SS | MS | F | P |
|---|---|---|---|---|---|
| Feed comb. And initial weight | 2 | 9.9420 | 4.97102 | 78.35 | 0.000 |
| Error | 81 | 5.139 | 0.06344 | | |
| Total | 83 | 15.0811 | | | |

The p-value (0.000) of the test is very small (smaller than any significance level one can imagine). This means that the predictor variables add significant information to the prediction of weight gain. Therefore, we conclude that since none of the model parameters is statistically equivalent to zero, the true linear relationship among the variables is reflected and the model optimally mimic the observations. In addition to the test above, the coefficient of determination $(R^2)$ was also considered to ascertain the proportion of the total variation in the weight gain that is explained by the initial weight and feed combination. The metric that captures the coefficient of determination is expressed as

$CD = \frac{SS_{feed\ comb\ .and\ initial\ weig\ ht}}{Total\ sum\ sum\ of\ square}$. From the Analysis of variance table 1,

$CD = \frac{9.9420}{15.0811} = 0.66$. This implies that 66% of the entire variation in the weight gain experiment is accounted for by the initial weight and feed combination, and the remaining 34% is attributed to the uncontrollable factors.

Notes

However, it is often necessary in a multiple regression model to analyse the seperate effect of the independent variables. This will give a proper ground for comparison between multiple regression model and classical analysis of variance model. In testing for the significance of fitting the feed combination effects after allowing for initial weight effect, we try to find the sum of square due to the "scanty" model

$$Y_{ij} = \mu + \beta X_{ij} + e_{ij} \tag{6}$$

Here, the relationship between the weight gain and only the covariate (initial weight) is examined.
The information matrix from the observations is given as

$$X'X = \begin{bmatrix} 84 & 138.55 \\ 138.55 & 402.47 \end{bmatrix} \text{ and } X'Y = \begin{bmatrix} 26.05 \\ 49.88 \end{bmatrix}$$

From the above, $(X'X)^{-1} = \begin{bmatrix} 0.0275 & -0.0095 \\ -0.0095 & 0.0057 \end{bmatrix}$

Thus, the estimated parameter $\hat{\beta} = (X'X)^{-1}X'Y = \begin{bmatrix} 0.2425 \\ 0.0368 \end{bmatrix}$.

The sum of square for regression (based only on the initial weight) from the "scanty" model above is expressed as $SS_{IW\,(only\,)} = \beta' Y' X = 8.1527$. Therefore, the sum of square for feed combination only (after allowing for feed combination) is the difference between the sum of square feed combination and initial weight generated earlier and the sum of square for regression (based only on the initial weight) generated from the scanty model. That is

$$SS_{FC\,(only\,)} = SS_{feed\ comb\ .and\ initial\ weig\ ht} - SS_{IW\,(only\,)}$$

Therefore, $SS_{FC\,(only\,)} = 9.9420 - 8.1527 = 1.7893$.

Now, we have succeded in breaking the total variation in the entire experiment into components that cause them. These information are presented in the comprehensive ANOVA table below which will be considered for further analysis.

*Table 6:* Analysis of Variance Table 2

| Source of variation | DF | SS | MS | F-ratio | P-value |
|---|---|---|---|---|---|
| Initial weight | 1 | 8.1527 | 8.1527 | 128.5915 | 0.000 |
| Feed combination after allowing for initial weight | 1 | 1.7894 | 1.7894 | 28.224 | 0.000 |
| Error | 81 | 5.139 | 0.0634 | | |
| Total | 83 | 15.0811 | | | |

*c) Test For Individual Effect of the Independent Variables*

Presenting feed combination as a factor, we intend to examine the significance of the factor in the entire experiment. Let the feed combination be denoted as $\gamma_i; i = 1,2,\ldots,12$. We test the null hypothesis $(H_0: \gamma_i = 0)$ against the alternative that atleast one of the levels of the feed combination is significant. The test statistic which is the ratio of the mean square feed combination after allowing for initial weight to the mean square error, follows F-distribution with $k-1$ numerator degree of freedom and $n-k-1$ denominator degree of freedom. The test supports the rejection of the null hypothesis at 0.05 level of significance if $F_{cal} > F_{tab}$.

From the analysis of variance table above, $F_{cal} = 28.22$. At 0.05 level of significance and specified degrees of freedom, the P-value of the test is 0.000. This suggests that the feed combinations have significant effects on the weight gain since the $P - valu(0.000) < 0.05$. Similarly, the initial weights have significant effect on the weight gain since it is obvious that the $P - value(0.000) < 0.05$. This gives the ground for subsequent section where interest is to analyze the true effects of the feed combinations after making adjustment for or relaxing the initial weight.

It is necessary to go on to break further, the coefficient of determination in section () into coefficient of determination for initial weight and coefficient of determination for feed combination. This gives expilicitly, the proportion of variation accounted for by each of the factors.

$$CD_{initial\ weig\,ht} = \frac{SS_{initial\ weig\,ht}}{SS_{Total}} 100\% = 54\%$$

Similarly,

$$CD_{feed\ combination} = \frac{SS_{feed\ combin\ ation}}{SS_{Total}} = 12\%$$

This implies that out of the 66% of the total variation explained by the two regressors, 12% is attributed to the feed and 54% is attributed to the initial weight. This results justify the belief that the weight gain of broiler chicken strongly, is a function of the initial weight.

*d) Prediction*

Here, we adopt the fitted model $Y = 0.0309 + \alpha_i - 0.0301X$ to predict the weight gain of broilers if their initial weights $(X)$ are known and they are fed with a particular feed combination $\alpha_i$. The essence is to ascertain which feed combination among all under consideration yields the maximum weight gain. Suppose that two initial weights $(X = 0.24\ and\ 1.32)$ are randomly sellected, with the estimated effects of the feed combination, predictions are made.

*Table 7:* Prediction

| Feed combination | Effect $(\alpha_i)$ | Initial weight | Weight gain |
|---|---|---|---|
| T1 | 0.1236 | 0.24 | 0.147276 |
| | | 1.32 | 0.114768 |
| T2 | 0.0823 | 0.24 | 0.105976 |
| | | 1.32 | 0.073468 |
| T3 | 0.3667 | 0.24 | 0.390376 |

Notes

Notes

| | | | |
|---|---|---|---|
| | | 1.32 | 0.357868 |
| T4 | 0.2782 | 0.24 | 0.301876 |
| | | 1.32 | 0.269368 |
| T5 | 0.3824 | 0.24 | 0.406076 |
| | | 1.32 | 0.373568 |
| T6 | 0.3222 | 0.24 | 0.345876 |
| | | 1.32 | 0.313368 |
| T7 | 0.2420 | 0.24 | 0.265676 |
| | | 1.32 | 0.233168 |
| T8 | 0.3685 | 0.24 | 0.392176 |
| | | 1.32 | 0.359668 |
| T9 | 0.5871 | 0.24 | 0.610776 |
| | | 1.32 | 0.578268 |
| T10 | 0.6592 | 0.24 | 0.682876 |
| | | 1.32 | 0.650368 |
| T11 | 0.1157 | 0.24 | 0.139376 |
| | | 1.32 | 0.106868 |
| T12 | 0.4183 | 0.24 | 0.441976 |
| | | 1.32 | 0.409468 |

In further analysis to determine the most significant among the feed combination, we keep the initial weight constant and concentrate on the feed combination. We tabulate the feed combination (yeast and PKC based feed) according to the level specified in the experiment, in a complete randomized design in table (1) and study the observations with the model

$$\begin{cases} Y_{ij} = \emptyset + \pi_i + \varepsilon_{ij};\ \varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2) \\ \sum_{i=1}^{12} \pi_i = 0 \end{cases} \tag{7}$$

First, the estimated marginal mean of the weight gain due to the feed combination only is tabulated below

*Table 8:* Estimated Marginal Mean

| Feed combination | Mean | Variance | Std. Error | 95% Confidence interval |
|---|---|---|---|---|
| T1 | -0.1006 | 0.1903 | 0.1649 | -0.4295 to 0.2282 |
| T2 | -0.1228 | 0.1675 | 0.1547 | -0.4314 to 0.1857 |
| T3 | 0.1747 | 0.1533 | 0.1480 | -0.1204 to 0.4698 |
| T4 | 0.1363 | 0.1087 | 0.1246 | -0.1121 to 0.3847 |
| T5 | 0.2962 | 0.0773 | 0.1051 | 0.08672 to 0.5058 |
| T6 | 0.3182 | 0.0717 | 0.1012 | 0.1164 to 0.5200 |
| T7 | 0.2352 | 0.0661 | 0.09717 | 0.04149 to 0.4290 |
| T8 | 0.3898 | 0.0715 | 0.1011 | 0.1882 to 0.5914 |
| T9 | 0.6293 | 0.0806 | 0.1073 | 0.4153 to 0.8433 |
| T10 | 0.7957 | 0.1454 | 0.1441 | 0.5084 to 1.0831 |
| T11 | 0.3309 | 0.2432 | 0.1864 | -0.04077 to 0.7026 |
| T12 | 0.6384 | 0.2506 | 0.1892 | 0.2611 to 1.0156 |

With model (7), the total variation in the entire experiment was partitioned into the components that caused them. Eventhough the initial weight is a factor in the experiment, it was kept silent so as to examine the through effects of the feed combination.

*Table 9:* Analysis of Variance Table 3

| Source of variation | Sum of Squares | DF | Mean Square | F-ratio | P-value |
|---|---|---|---|---|---|
| Feed combination | 2.1532 | 11 | 0.1957 | 2.906 | 0.003 |
| Error(other fluctuations) | 4.8493 | 72 | 0.0674 | | |
| Total | 7.0025 | 83 | | | |

Notes

The p-value ($p=0.003<0.05$) suggests that the effects of the different feed combinations are not the same. To determine which is different from the other, we embark on multiple comparison on the estimated marginal means due to the different feed combinations, using Turkey Honestly significant difference (Turkey HSD) test. This approach of multiple comparison is based on the studentized t-distribution. It has the ability of controlling type one error by taking into account the number of means that are being compared.

The test statistic for Turkey HSD test is expressed as

$$Q = \frac{|T_i - T_j|}{\left(MSE/m\right)^{1/2}} \sim t_{n-m}; \alpha \tag{8}$$

$T_i$ is the estimated marginal mean of the i$^{th}$ group and $T_j$ is the estimated marginal mean of the j$^{th}$ group. $m = 7$ is the number of observations in a group while $n = 84(7 \times 12)$ is the total number of observations across the group.

Here, we estimate the MSE as the mean of the estimated group marginal variances. That is $\frac{\sum_{i=1}^{12} s_i^2}{12} = 0.136$.

Using (8) above, the comparison test was conducted for different possible pairs and the P-value of each of the pairs are presented in table (10).

*Table 10:* The P-Values of the Comparison Test

| | T1 | T2 | T3 | T4 | T5 | T6 | T7 | T8 | T9 | T10 | T11 | T12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| T1 | - | 1.000 | 0.959 | 0.987 | 0.679 | 0.603 | 0.858 | 0.359 | 0.019 | 0.001 | 0.558 | 0.016 |
| T2 | | - | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| T3 | | | - | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| T4 | | | | - | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| T5 | | | | | - | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| T6 | | | | | | - | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| T7 | | | | | | | - | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| T8 | | | | | | | | - | 1.000 | 1.000 | 1.000 | 1.000 |
| T9 | | | | | | | | | - | 1.000 | 1.000 | 1.000 |
| T10 | | | | | | | | | | - | 1.000 | 1.000 |
| T11 | | | | | | | | | | | - | 1.000 |
| T12 | | | | | | | | | | | | - |

## V. Summary, Conclussion and Recommendation

In this work, we have been able to model the weight gain data with a linear multiple regression equation, taking the treatment in the experiment as a qualitative predictor. Also, we have been able to adopt the analysis of variance technique in assigning portions of variation in the entire experiment to each variable in a set of independent variables. The mean square error in the regression approach (0.0634) is smaller than that of one way analysis of variance approach (0.0674). This can be attributed to the fact that the presence of the initial weight which was carried along in the former analysis helped in reducing the totality of the error in the entire experiment. When the initial weight is ignored, information is lost. Consequently, the mean square

Notes

error shuts up with a difference of 0.004 (6.31% increment). This implies that the initial weight contributes 0.4% accuracy to the analysis.

The adjustment on the model succeded in making the true mean effects of T1 and T2 to be negative, with T10 having the highest effect. This result is in synch with the effects estimated earlier in terms of regression estimates where it was shown that T1 and T2 still have the most least effect on the subjects.

In table(10), it is revealed that significant difference only exist between the marginal means T1 and T9, T1and T10, and T1 and T12. In every other comparison, there is no significant difference. This does not mean that the effect of these non significant feed combinations are realy the same. Rather it means that there is no enough (convincing) evidence to justify that they are different. A cursory look at the pairs that are significantly different, reveals that the difference between T1 and T10 is the most significant. This goes on to show that T10 (0.4g of bioactive yeast/kg X 30kg of PKC based feed) has highest treatment effect since it has higher estimated marginal mean of the weight gain than every other treatment. This inference is in line with the postulation made earlier under the section for multiple linear regression where it was noted that T10 has higher effect since the ordinary least square estimate of it is higher than every other parameter within the effect space.

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# Analytic Solutions of (N+1) Dimensional Time Fractional Diffusion Equations by Iterative Fractional Laplace Transform Method

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*Abstract-* In this paper, the author has been examined how to obtain solutions of $(n+1)$ dimensional time fractional diffusion equations with initial conditions in the form of infinite fractional power series, in terms of Mittage Lefler function of one parameter and exact form by the use of iterative fractional Laplace transform method (IFLTM). The basic idea of the IFLTM was developed successfully. To see its effectiveness and applicability, three test examples were presented. The closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter, which rapidly converge to exact solutions, were successfully derived analytically by the use of iterative fractional Laplace transform method (IFLTM). Thus, the results show that the iterative fractional Laplace transform method works successfully in solving $(n+1)$ dimensional time fractional diffusion equations in a direct way without using linearization, perturbation, discretization or restrictive assumptions, and hence it can be extended to other fractional differential equations.

*Keywords:* *$(n+1)$ dimensional time fractional diffusion equations with initial conditions, caputo fractional derivatives, mittag-leffler function, iterative fractional laplace transform method.*

*GJSFR-F Classification:* *FOR Code: MSC 2010: 35J05*

Notes



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*Abstract-* In this paper, the author has been examined how to obtain solutions of $(n+1)$ dimensional time fractional diffusion equations with initial conditions in the form of infinite fractional power series, in terms of Mittage Lefler function of one parameter and exact form by the use of iterative fractional Laplace transform method (IFLTM). The basic idea of the IFLTM was developed successfully. To see its effectiveness and applicability, three test examples were presented. The closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter, which rapidly converge to exact solutions, were successfully derived analytically by the use of iterative fractional Laplace transform method (IFLTM). Thus, the results show that the iterative fractional Laplace transform method works successfully in solving $(n+1)$ dimensional time fractional diffusion equations in a direct way without using linearization, perturbation, discretization or restrictive assumptions, and hence it can be extended to other fractional differential equations.

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## I. Introduction

Fractional calculus theory is a mathematical analysis tool to the study of integrals and derivatives of arbitrary order, which unify and generalize the notations of integer-order differentiation and $n-$fold integration (El-Ajou, Arqub, Al-Zhour, & Momani, 2013; Millar & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1999).

The L'Hopital's letter raised the question "What does $\frac{\partial mf(x)}{\partial x^m}$ mean if $m=\frac{1}{2}$?" to Leibniz in 1695 is considered to be where the idea of fractional calculus started(Diethelm, 2010; Hilfer, 2000; Lazarevic, et al., 2014; Millar & Ross, 1993; Kumar & Saxena, 2016). Since then, much works on this question and other related questions have done up to the middle of the 19$^{th}$ century by many famous mathematicians such as Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnkov, Levy, Marchaud, Erdelyi and Reiszand these works sum up leads to contributions creating the field which is known today as fractional calculus(Oldham & Spanier, 1974).

Even though fractional calculus is nearly as old as the standard calculus, it was only in recent few decades that its theory and applications have rapidly developed. It

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was Ross who organized the first international conference on fractional calculus and its applications at the University of new Haven in June 1974, and edited the proceedings(Ross, 1975). Oldham and Spanier (1974) published the first monograph on fractional calculus in 1974. Next, because of the fact that fractional derivatives and integrals are non-local operators and then this property make them a powerful instrument for the description of memory and hereditary properties of different substances(Podlubny, 1999),theory and applications of fractional calculus have attracted much interest and become a pulsating research area.

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Due to this, fractional calculus has got important applications in different fields of science, engineering and finance. For instance, Shanantu Das(2011) discussed that fractional calculusis applicable to problems in: fractance circuits, electrochemistry, capacitor theory, feedback control system, vibration damping system, diffusion process, electrical science, and material creep. Podlubny(1999)discussed that fractional calculus is applicable to problems in fitting experimental data, electric circuits, electro-analytical chemistry, fractional multi-poles, neurons and biology(Podlubny, 1999).Fractional calculus is also applicable to problems in: polymer science, polymer physics, biophysics, rheology, and thermodynamics(Hilfer, 2000). In addition, it is applicable to problems in: electrochemical process(Millar & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1999), control theory(David, Linarese, & Pallone, 2011; Podlubny, 1999), physics(Sabatier, Agrawal, & Machado, 2007), science and engineering(Kumar & Saxena, 2016), transport in semi-infinite medium(Oldham & Spanier, 1974), signal processing(Sheng, Chen, & Qiu, 2011), food science (Rahimy, 2010), food gums (David & Katayama, 2013), fractional dynamics(Tarsov, 2011; zaslavsky, 2005), modeling Cardiac tissue electrode interface(Magin, 2008), food engineering and econophysics (David, Linarese, & Pallone, 2011), complex dynamics in biological tissues(Margin, 2010), viscoelasticity(Dalir & Bashour, 2010; Mainardi, 2010; Podlubny, 1999; Rahimy, 2010; Sabatier, Agrawal, & Machado, 2007), modeling oscillation systems(Gomez-Aguilar, Yepez-Martinez, Calderon-Ramon, Cruz-Orduna, Escobar-Jimenez, & Olivares-Peregrino, 2015). Some of these mentioned applications were tried to be touched as follows.

In the area of science and engineering, different applications of fractional calculus have been developed in the last two decades. For instance, fractional calculus was used in image processing, mortgage, biosciences, robotics, motion of fractional oscillator and analytical science(Kumar & Saxena, 2016). It was also used to generalize traditional classical inventory model to fractional inventory model (Das & Roy, 2014).

In the area of electrochemical process, for example half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models(Millar & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1999).

In the area of viscoelasticity, the use of fractional calculus for modeling viscoelastic materials is well known. For viscoelastic materials the stress-strain constitutive relation can be more accurately described by introducing the fractional derivative(Carpinteri, Cornetti, & sapora, 2014; Dalir & Bashour, 2010; Duan, 2016; Koeller, 1984; Mainardi, 2010; Podlubny, 1999).

Fractional derivatives, which are the one part of fractional calculus, are used to name derivatives of an arbitrary order(Podlubny, 1999). Recently, fractional derivatives have been successfully applied to describe (model) real world problems.

In the area of physics, fractional kinetic equations of the diffusion, diffusion-advection and Focker-Plank type are presented as a useful approach for the description of transport dynamics in complex systems that are governed by anomalous diffusion and non-exponential relaxation patterns (Metzler & Klafter, 2000). Metzler and Klafter(2000)derived these fractional equations asymptotically from basic random walk models, and from a generalized master equation. They presented an integral transformation between the Brownian solution and its fractional counterparts. Moreover, a phase space model was presented to explain the genesis of fractional dynamics in trapping systems. These issues make the fractional equation approach powerful. Their work demonstrates that the fractional equations have come of age as a complementary tool in the description of anomalous transport processes. L.R. Da Silva, Tateishi, M.K. Lenzi, Lenzi and Da silva(2009)were also discussed that solutions for a system governed by a non-Markovian Fokker Planck equation and subjected to a Comb structure were investigated by using the Green function approach. This structure consists of the axis of structure as the backbone and fingers which are attached perpendicular to the axis, and for this system, an arbitrary initial condition in the presence of time dependent diffusion coefficients and spatial fractional derivatives was considered and the connection to the anomalous diffusion was analyzed (L.R. Da Silva *et al.*, 2009).

In addition to these, the following are also other applications of fractional derivatives. Fractional derivatives in the sense of Caputo fractional derivatives were used in generalizing some theorems of classical power series to fractional power series (El-Ajou *et al.*, 2013). Fractional derivative in the Caputo sense was used to introduce a general form of the generalized Taylor's formula by generalizing some theorems related to the classical power series into fractional power series sense (El-Ajou, Abu Arqub, & Al-S, 2015). A definition of Caputo fractional derivative proposed on a finite interval in the fractional Sobolev spaces was investigated from the operator theoretic viewpoint(Gorenflo, Luchko, & Yamamoto, 2015). Particularly, some important equivalence of the norms related to the fractional integration and differentiation operators in the fractional Sobolev spaces are given and then applied for proving the maximal regularity of the solutions to some initial-boundary-value problems for the time-fractional diffusion equation with the Caputo derivative in the fractional Sobolev spaces(Gorenflo, Luchko, & Yamamoto, 2015).With the help of Caputo time-fractional derivative and Riesz space-fractional derivative, the $\alpha$-fractional diffusion equation, which is a special model for the two-dimensional anomalous diffusion, is deduced from the basic continuous time random walk equations in terms of a time- and space-fractional partial differential equation with the Caputo time-fractional derivative of order $\frac{\alpha}{2}$ and the Riesz space-fractional derivative of order $\alpha$ (Luchko, 2016). Fractional derivatives were also used to describe HIV infection of $CD4^{+}T$ with therapy effect (Zeid, Yousefi, & Kamyad, 2016).

In the area of modeling oscillating systems, caputo and Caputo-Fabrizio fractional derivatives were used to present fractional differential equations which are generalization of the classical mass-spring-damper model, and these fractional differential equations are used to describe variety of systems which had not been addressed by the classical mass-spring-damper model due to the limitations of the classical calculus (Gomez-Aguilar *et al.*, 2015).

Podlubny(1999)stated that fractional differential equations are equations which contain fractional derivatives. These equations can be divided into two categories such as fractional ordinary differential equations and fractional partial differential equations. Fractional partial differential equations (PDES) are a type of differential equations

(DEs) that involving multivariable function and their fractional or fractional-integer partial derivatives with respect to those variables(Abu Arqub, El-Ajou, & Momani, 2015). There are different examples of fractional partial differential equations. Some of them are: the time-fractional Boussines q-type equation, the time-fractional $B(2,1,1)$-type equation and the time-fractional Klein-Gordon-type equation stated in Abu Arqub *et al.*(2015), and time fractional diffusion equation stated in A. Kumar, kumar and Yan (2017), Cetinkaya and Kiymaz (2013), Kumar, Yildirim, Khan and Wei(2012) and so on.

Notes

Recently, fractional differential equations have been successfully applied to describe (model) real world problems. For instance, the generalized wave equation, which contains fractional derivatives with respect to time in addition to the second-order temporal and spatial derivatives, was used to model the viscoelastic case and the pure elastic case in a single equation(Wang, 2016).The time fractional Boussines q-type equations can be used to describe small oscillations of nonlinear beams, long waves over an even slope, shallow-water waves, shallow fluid layers, and nonlinear atomic chains; the time-fractional $B(2,1,1)$-type equations can be used to study optical solitons in the two cycle regime, density waves in traffic flow of two kinds of vehicles, and surface acoustic soliton in a system supporting love waves; the time fractional Klein-Gordon-type equations can be applied to study complex group velocity and energy transport in absorbing media, short waves in nonlinear dispersive models, propagation of dislocations within crystals(As cited in Abu Arqub *et al.*, 2015). As cited in Abu Arqub(2017), the time-fractional heat equation, which is derived from Fourier's law and conservation of energy, is used in describing the distribution of heat or variation in temperature in a given region over time; the time-fractional cable equation, which is derived from the cable equation for electro diffusion in smooth homogeneous cylinders and occurred due to anomalous diffusion, is used in modeling the ion electro diffusion at the neurons; the time-fractional modified anomalous sub diffusion equation, which is derived from the neural cell adhesion molecules, is used for describing processes that become less anomalous as time progresses by the inclusion of a second fractional time derivative acting on the diffusion term; the time fractional reaction sub diffusion equation is used to describe many different areas of chemical reactions, such as exciton quenching, recombination of charge carriers or radiation defects in solids, and predator pray relationships in ecology; the time-fractional Fokker–Planck equation is used to describe many phenomena in plasma and polymer physics, population dynamics, neurosciences, nonlinear hydrodynamics, pattern formation, and psychology; the time-fractional Fisher's equation is used to describe the population growth models, whilst, the time fractional Newell–Whitehead equation is used to describe fluid dynamics model and capillary–gravity waves. The fractional differential equations, generalization of the classical mass-spring-damper models, are useful to understand the behavior of dynamical complex systems, mechanical vibrations, control theory, relaxation phenomena, viscoelasticity, viscoelastic damping and oscillatory processes (Gomez-Aguilar *et al.*, 2015).The space-time fractional diffusion equations on two time intervals was used in finance to model option pricing and the model was shown to be useful for option pricing during some temporally abnormal periods (Korbel & Luchko, 2016). The $\alpha$-fractional diffusion equation for $0<\alpha<2$ describes the so called Levy flights that correspond to the continuous time random walk model, where both the mean waiting time and the jump length variance of the diffusing Particles are divergent(Luchko, 2016). Time fractional diffusion equations in the Caputo sense with initial conditions are used to model cancer tumor(Iyiola & Zaman, 2014).

Nonlinear diffusion equations play a great role to describe the density dynamics in a material undergoing diffusion in a dynamic system which includes different branches of science and technology. The classical and simplest diffusion equation which is used to model the free motion of the particle is:

Notes

$$\frac{\partial u(x,t)}{\partial t} = A\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}\, F(x)u(x,t),\ \mathrm{A}>0\,, \tag{1.1}$$

where $u(x,t)$ is the probability density function of finding a particle at the point $x$ in time instant $t$, $F(x)$ is the external force, and $A$ is a positive constant which depends on the temperature, the friction coefficient, the universal gas constant and the Avogadro number (A.Kumar*et al.*, 2017).

Recently, the fractional differential equations have gained much attention of researchers due to the fact that they generate fractional Brownian motion which is generalization of Brownian motion(Podlubny, 1999). Das, Visha, Gupta and Saha Ray(2011) stated that time fractional diffusion equation, which is one of the fractional differential equations, is obtained from the classical diffusion equation in mathematical physics by replacing the first order time derivative by a fractional derivative of order $\alpha$ where $0<\alpha<1$. Time fractional diffusion equation is an evolution equation that generates the fractional Brownian motion (FBM) which is a generalization of Brownian motion (Das, *et al.*, 2011; Podlubny, 1999). Due to the fact that fractional derivative provides an excellent tool for describing memory and hereditary properties for various materials and processes(Caputo & Mainardi, 1971), the time fractional diffusion equations(A. Kumar *et al.*, 2017; Cetinkaya & Kiymaz, 2013;Das, 2009; Kebede, 2018; Kumar *et al.*, 2012 ) were extended to the form

$$\begin{cases} \dfrac{\partial^\beta u}{\partial t^\beta} = D\displaystyle\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[F(x)u\right];\ 0<\beta<1,\ D>0,\ t>0,\ x=x_1,x_2,\cdots,x_n, x_1>0, x_2>0,\cdots,x_n>0 & (1.2a) \\ subject\ to\ \ I.C:\ u^k(x,0)=f_k(x);\ k=0,\ 1,\ 2,\cdots,\ q-1;\ x=x_1,x_2,\cdots,x_n;\ x_1>0, x_2>0,\ \cdots, x_n>0 & (1.2b) \end{cases},$$

which is generalization of equation $(1.1)$, was considered in this study. Here,

$$D_t^\beta u(x,t) = J_t^{1-\beta}\left[\frac{\partial}{\partial t}u(x_1,x_2,\cdots x_n,t)\right],\ \text{and}\ u=u\left(x_1,x_2,\ \cdots,\ x_n,t\right).$$

The fractional derivative $D_t^\beta$ is considered in the Caputo sense which has the main advantage that the initial conditions for fractional differential equations with Caputo derivative take on the same form as for integer order differential equations (Caputo, 1967). Due to this, considerable works on fractional diffusion equations have already been done by different authors to obtain exact, approximate analytic and pure numerical solutions by using various developed methods.

Recently, Adomian Decomposition Methodby Saha Ray and Bera in 2006 (As cited in Cetinkaya & Kiymaz, 2013; Kumar *et al.*, 2012; Das, 2009), variational iteration method (Das, 2009), Homotopy Analysis Method(Das, *et al.*, 2011), Laplace Transform Method (Kumar*et al.*, 2012), Generalized Differential Transform Method(Cetinkaya & Kiymaz, 2013)and Residue fractional power series method (Kumar*et. al.*, 2017), fractional reduced differential transform method (kebede, 2018)which have their own inbuilt deficiencies: the complexity and difficulty of solution procedure for calculation of adomain polynomials, the restrictions on the order of the nonlinearity term or even the form of the boundary conditions and uncontrollability of non-zero end conditions, unrestricted freedom to choose base function to approximate the linear and nonlinear problems, and complex computations respectively, were used to obtain solutions of ($(1+1)$dimensional time fractional diffusion equations with initial conditions. To overcome these deficiencies, the iterative fractional Laplace transform method(IFLTM)was preferably taken in this paper to solve $(n+1)$ dimensional time fractional diffusion equations with initial conditions of the form $(1.2a)$ given that $(1.2b)$ analytically.

The iterative method was firstly introduced by Daftardar-Gejji and Jafari(2006) to solve numerically the nonlinear functional equations. By now, the iterative method has been used to solve many integer and fractional boundary value problem ( Daftardar-Gejji & Bhalekar, 2010). Jafari et al.(2013) firstly solved the fractional partial differential equations by the use of iterative Laplace transform method (ILTM). More recently, Yan (2013), Sharma and Bairwa (2015), and Sharma and Bairwa(2014)were used ILTM for solving Fractional Fokker-Planck equations, generalized time-fractional biological population model, and Fractional Heat and Wave-Like Equations respectively.

In this paper, the author has been examined how to obtain the solutions of $(n+1)$ dimensional time fractional diffusion equations with initial conditions in the form infinite fractional power series, in terms of Mittage Lefler function of one parameter and exact form by the use of iterative fractional Laplace transform method (IFLTM). The basic idea of IFLTM was developed successfully. To see its effectiveness and applicability, three test examples were presented. Their closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter, which rapidly converge to exact solutions, were successfully derived by the use iterative fractional Laplace transform method (IFLTM). The results show that the iterative fractional Laplace transform method works successfully in solving $(n+1)$ dimensional time fractional diffusion equations in a direct way without using linearization, perturbation, discretization or restrictive assumptions, and hence it can be extended to other fractional differential equations.

This paper is organized as follows: in the next sections which is the methodology, which is the way the study was designed to go through, was discussed. In section 3, results and discussion which include: some definitions, properties and theorems of fractional calculus theory, the results which are is the basic idea of fractional Laplace transform method, application models and discussion of application of the results obtained were presented. Finally, conclusions are presented in Section 4.

## II. Methodology

In this paper, it was designed to set the theoretical foundation of the study to come to its objective. Next, it was designed to consider time fractional differential equations under initial conditions, which are$(n+1)$ dimensional time fractional diffusion equations with initial conditions of the form: $(1.2a)$ given that $(1.2b)$ and then use analytical design to solve the manalytically by using iterative fractional Laplace transform method by following the next five procedures sequentially. First, it was designed to revisit some basic definitions and properties of fractional calculus and Laplace transform. Secondly, it was designed to develop basic idea of iterative fractional Laplace transform method for$(3.10a)$given that$(3.10b)$and then obtain a remark 3.2.2.1.

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Thirdly, it was designed to obtain closed solutions of $(1.2a)$ given that $(1.2b)$ in the form of infinite fractional power series by using the remark 3.2.2.1. Fourthly, it was designed to determine closed solutions equations of the form of $(1.2a)$ given that $(1.2b)$ in terms of Mittag-Leffler functions in one parameter from these infinite fractional power series. Lastly, it was designed to obtain exact solutions of $(1.2a)$ given that $(1.2b)$ for the special case $\alpha = 1$.

Notes

## III. Results and Discussion

### a) *Preliminaries and Notations*

#### i. *Fractional Calculus*

Here, some basic definitions and properties of fractional calculus and Laplace transform were revisited as follows to use them in this paper; see (Kilbas,Srivastava, & Trujillo, 2006; Mainardi, 2010; Podlubny, 1999; Millar & Ross, 1993).

*Definition 3.1.1.* A real valued function $u(x,t), x \in IR, t > 0$, is said to be in the space $C_\mu, \mu \in IR$, if there exists a real number $q > \mu$ such that $u(x) = t^q u_1(x,t)$, where $u_1(x,t) \in C(IR \times [0,+\infty))$ and it is said to be in the space $C_\mu^m$ if $u^{(m)}(x,t) \in C_\mu, n \in IN$.

*Definition 3.1.2.* The Riemann-Liouville fractional integral operator of order $\beta \geq 0$ of a function $u(x,t) \in C_\mu, \mu > \text{-}1$ is defined as

$$J_t^\beta u(x,t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (x-\xi)^{\beta-1} u(x,\xi) d\xi, \ 0 < \xi < \mathrm{t}, \beta > 0 \\ u(x,t), \beta = 0 \end{cases} \tag{3.1}$$

Consequently, for $\alpha, \beta \geq 0, C \in IR, u(x,t) \in C_\mu^m, u(x,t) \in C_\mu, \mu > \text{-}1$, the operator $J_t^\beta$ has the following properties:

i. $J_t^\alpha J_t^\beta u(x,t) = J_t^{\alpha+\beta} u(x,t) = J_t^\beta J_t^\alpha u(x,t)$

ii. $J_t^\alpha c = \left(\dfrac{c}{\Gamma(\alpha+1)}\right) t^\alpha$.

iii. $J_t^\alpha t^\gamma = \left(\dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\right) t^{\gamma+\alpha}$.

The Riemann Liouville derivative has the disadvantage that it does not allow the utilization of initial and boundary conditions involving integer order derivatives when trying to model real world problems with fractional differential equations. To beat this disadvantage of Riemann Liouville derivative (Millar & Ross, 1993; Podlubny, 1999), Caputo proposed a modified fractional differentiation operator $D_a^\beta$ (Caputo & Mainardi, 1971)to illustrate the theory of viscoelasticity as follows:

$$D_a^\beta f(x) = J_a^{m-\beta} D^m f(x) = \frac{1}{\Gamma(m-\beta)} \int_a^x (x-\xi)^{m-\beta-1} f^{(m)}(\xi) d\xi, \beta \geq 0 \tag{3.2}$$

where $m-1<\beta<m, x>a$ and $f\in C_{-1}^{m}$.

This Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations of the real situations.

*Definition 3.1.3.* For the smallest integer that exceeds $\beta$, the Caputo time fractional derivative order $\beta>0$ of a function $u(x,t)$ is defined as:

$$D_t^{\beta}u(x,t)=\begin{cases}\dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_0^t(t-\xi)^{m-\beta-1}\dfrac{\partial^m u(x,\xi)}{\partial\xi^m}d\xi \\ \dfrac{\partial^m u(x,t)}{\partial t^m},\ \beta=m\end{cases}=J^{m-\beta}\frac{d^m}{dt^m}u(x,t),\ 0\le m-1<\beta<m \tag{3.3}$$

*Theorem 3.1.1.* If $m-1<\beta\le m, \forall \mathrm{m}\in \mathrm{IN}, u(x,t)\in C_{\gamma}^{m}, \gamma\ge -1$ then

i. $D_t^{\beta}J_t^{\beta}u(x,t)=u(x,t)$.

ii. $J^{\beta}D^{\beta}u(x,t)=u(x,t)-\displaystyle\sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u(x,0^+)\frac{t^k}{k!},\ t>0$.

The reader is kindly requested to go through (Kilbas, Srivastava, & Trujillo, 2006; Mainardi, 2010) in order to know more details about the mathematical properties of fractional derivatives and fractional integrals, including their types and history, their motivation for use, their characteristics, and their applications.

*Definition 3.1.4.* According to Millar and Ross(1993), Podlubny(1999), and Sontakke and Shaikh(2015), the Mittag-Leffler function, which is a one parameter generalization of exponential function, is defined as

$$E_{\alpha}(z)=\sum_{q=0}^{\infty}\frac{z^q}{\Gamma(q\alpha+1)},\alpha\in C,\mathrm{Re}(\alpha)>0 \tag{3.4}$$

*Definition 3.1.5.* (Kilbas, Srivastava, & Trujillo, 2006) Laplace transform of $\phi(t),t>0$ is

$$L[f(t)]=F(s)=\int_0^{\infty}e^{-st}f(t)dt \tag{3.5}$$

*Definition 3.1.6.* (Kilbas, Srivastava, & Trujillo, 2006)Laplace transform of $D_t^{\beta}u(x,t)$ is

$$L[D_t^{\beta}u(x,t)]=L[u(x,t)]-\sum_{k=1}^{q-1}u^k(x,0)s^{\beta-k-1},q-1<\beta\le q,q\in IN \tag{3.6}$$

### *b) Main Results*

i. *Some basic definitions of fractional calculus and Laplace Transform*

Here, some definitions of fractional calculus and Laplace transform, one theorem and basic idea of iterative fractional Laplace transform method were developed and introduced.

*Definition 3.2.1:* A real valued $(n+1)$ dimensional function $u(x_1,x_2,\cdots x_n,t)$, where $(x_1,x_2,\cdots x_n)\in IR^n, t>0$ is said to be in the space $C_{\mu},\mu\in IR$, if there exists a real number

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$p > \mu$ such that $u(x) = t^p u_1(x_1, x_2, \cdots x_n, t)$, where $u_1(x_1, x_2, \cdots x_n, t) \in C(IR^n \times [0, +\infty))$ and it is said to be in the space $C_\mu^m$ if $u^{(m)}(x_1, x_2, \cdots x_n, t) \in C_\mu, n \in IN$.

*Definition 3.2.2.* The Riemann-Liouville fractional integral operator of order $\beta \geq 0$ of $(n+1)$dimensional function $u(x_1, x_2, \cdots x_n, t) \in C_\mu, \mu > \text{-}1$ is defined as

$$J_t^\beta u(x_1, x_2, \cdots x_n, t) = \begin{cases} \frac{1}{\Gamma(\beta)} \int_0^t (x - \xi)^{\beta - 1} u(x_1, x_2, \cdots x_n, \xi) d\xi, \ 0 < \xi < \mathrm{t}, \beta > 0 \\ u(x_1, x_2, \cdots x_n, t), \beta = 0 \end{cases} \tag{3.7}$$

*Lemma 3.2.1.* For $\alpha, \beta \geq 0, C \in IR, u(x_1, x_2, \cdots x_n, t) \in C_\mu^m, u(x_1, x_2, \cdots x_n, t) \in C_\mu, \mu > \text{-}1,$ the operator $J_t^\beta$ has the property:

$$J_t^\alpha J_t^\beta u(x_1, x_2, \cdots x_n, t) = J_t^{\alpha+\beta} u(x_1, x_2, \cdots x_n, t) = J_t^\beta J_t^\alpha u(x_1, x_2, \cdots x_n, t)$$

*Definition 3.2.3.* For the smallest integer that exceeds $\beta$, the Caputo time fractional derivative order $\beta > 0$ of $(n+1)$dimensional function, $u(x_1, x_2, \cdots x_n, t)$ is defined as:

$$D_t^\beta u(x_1, x_2, \cdots x_n, t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\xi)^{m-\beta-1} \frac{\partial^m u(x_1, x_2, \cdots x_n, \xi)}{\partial \xi^m} d\xi & = J^{m-\beta} \frac{d^m}{dt^m} u(x_1, x_2, \cdots x_n, t), \ 0 \leq m-1 < \beta < m \\ \frac{\partial^m u(x_1, x_2, \cdots x_n, t)}{\partial t^m}, \beta = m \end{cases} \tag{3.8}$$

*Theorem 3.2.1.* If $m-1 < \beta \leq m, \forall \mathrm{m} \in \mathrm{IN}$, $(n+1)$ dimensional function $u(x_1, x_2, \cdots x_n, t) \in C_\gamma^m, \gamma \geq -1,$ then

i. $D_t^\beta J_t^\beta u(x_1, x_2, \cdots x_n, t) = u(x_1, x_2, \cdots x_n, t)$.

ii. $J^\beta D^\beta u(x_1, x_2, \cdots x_n, t) = u(x_1, x_2, \cdots x_n, t) - \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x_1, x_2, \cdots x_n, 0^+) \frac{t^k}{k!}, \ t > 0$.

*Definition 3.2.4.* Laplace transform of $D_t^\beta u(x,t)$ is

$$L[D_t^\beta u(x_1, x_2, \cdots x_n, t)] = L[u(x_1, x_2, \cdots x_n, t)] - \sum_{r=0}^{p-1} u^r(x_1, x_2, \cdots x_n, 0) s^{\beta - r - 1}, p-1 < \beta \leq p, p \in IN, \tag{3.9}$$

where $u(x_1, x_2, \cdots x_n, t)$ is $(n+1)$ dimensional function and $u(x_1, x_2, \cdots x_n, 0)$ is the $r$ order derivative of $u(x_1, x_2, \cdots x_n, t)$ at $t = 0$.

ii. *Basic idea of Iterative fractional Laplace transform method*

The basic idea of this method is illustrated as follows.

*Step 1.* Consider a general $(n+1)$ dimensional time fractional non-linear non homogeneous partial differential equation with initial conditions of the form:

$$\begin{cases} D_t^\beta u + Lu + Nu = f; \ p-1 < \beta \leq p & (3.10a) \\ u_0^r = g_r; r = 0, 1, 2, \cdots, p-1 & (3.10b) \end{cases}$$

where $u = u(x_1, x_2, \cdots, x_n, t)$, $u_0^r = g_r(x_1, x_2, \cdots, x_n, 0)$, $D_t^\beta u(x_1, x_2, \cdots, x_n, t)$ is the Caputo fractional derivative of the function, $L$ is the linear operator, $N$ is general nonlinear operator and $f(x_1, x_2, \cdots, x_n, t)$ is the source term respectively.

*Step2.* Now apply fractional Laplace transform method to $(3.10a)$ given that $(3.10b)$. as follows.

i. Applying the Laplace transform denoted by $L$ in equation $(3.10a)$, we obtain:

$$L[D_t^\beta u] + L[Ru + Nu] = L[f] \tag{3.11}$$

ii. By using equation $(3.9)$, we get:

$$L[u] = \frac{1}{s^\beta}\sum_{r=0}^{p-1} u^r s^{\beta-r-1} + \frac{1}{s^\beta}L[f] - \frac{1}{S^\beta}L[Ru + Nu] \tag{3.12}$$

iii. Taking inverse Laplace transform of equation $(3.12)$ we get:

$$u = L^{-1}\left[\frac{1}{S^\beta}\left[\sum_{r=0}^{r-1} u^r s^{\beta-r-1} + L[f]\right]\right] - L^{-1}\left[\frac{1}{S^\beta}L[Ru + Nu]\right] \tag{3.13}$$

*Step3.* Now we apply the iterative method to $(3.13)$ as follows.

i. Let $u$ be the solution of $(3.10a)$ and has the infinite series form

$$u = \sum_{i=0}^{\infty} u_i \tag{3.14}$$

ii. Since, $R$ is the linear operator, using equation $(3.14)$,

$$Ru = R\sum_{i=0}^{\infty} u_i = \sum_{i=0}^{\infty} Ru_i \tag{3.15}$$

iii. Since $N$ is the non-linear operator, by using equation $(3.14)$, $N$ is decomposed as:

$$Nu = N\left(\sum_{i=1}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left(N\left(\sum_{r=0}^{i} u_r\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right) \tag{3.16}$$

iv. By substituting Equations $(3.14)$, $(3.15)$ and $(3.16)$ in Equation $(3.13)$, we get

$$\sum_{i=1}^{\infty} u_i = L^{-1}\left[\frac{1}{S^\beta}\left[\sum_{r=0}^{p-1} u^r s^{\beta-r-1} + L[f]\right]\right] - L^{-1}\left[\frac{1}{S^\beta}L\left[\sum_{i=0}^{\infty} Ru_i + N(u_0) + \sum_{i=1}^{\infty}\left(N\left(\sum_{r=0}^{i} u_j\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right)\right]\right] \tag{3.17}$$

v. Now from Equation $(3.17)$, we define recurrence relations as follows:

$$u_0 = L^{-1}\left[\frac{1}{S^\beta}\left[\sum_{r=0}^{p-1} u^r s^{\beta-r-1} + L[f]\right]\right] \tag{3.18}$$

$$u_1 = -L^{-1}\left[\frac{1}{S^\beta}L[Ru_0 + N(u_0)]\right] = -L^{-1}\left[\frac{1}{S^\beta}L[R(u_0) + N(u_0)]\right] \tag{3.19}$$

$$u_2 = u_{1+1} = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\,u_1 + N\left(u_0 + u_1\right) - N\left(u_0\right)\right]\right] = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\left[\sum_{i=0}^{1} u_i - u_0\right] + N\left(\sum_{i=0}^{1} u_i\right) - N\left(u_0\right)\right]\right] \quad (3.20)$$

$$u_3 = u_{2+1} = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\,u_2 + N\left(u_0 + u_1 + u_2\right) - N\left(u_0 + u_1\right)\right]\right] = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\left(\sum_{i=0}^{2} u_i - \sum_{i=0}^{1} u_i\right) + N\left(\sum_{i=0}^{2} u_i\right) - N\left(\sum_{i=0}^{1} u_i\right)\right]\right] \quad (3.21)$$

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Continuing with this procedure, we get

$$u_i = u_{p+1} = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right) + N\left(\sum_{i=0}^{p} u_i\right) - N\left(\sum_{i=0}^{p-1} u_i\right)\right]\right];\ p \in IN, p \geq 1, i = 0, 1, 2, \cdots, p+1 \quad (3.22)$$

Therefore the $i^{th}$ term approximate solution of Equation $(3.10a)$ given that $(3.10b)$ in series form is given by

$$\tilde{u}_i \cong u_0 + u_1 + u_2 + \cdots + u_{p+1};\ p = 1, 2, 3, \cdots \quad (3.23)$$

*Step4.* The infinite power series form solution of $(3.10a)$ given that $(3.10b)$ as $p \in IN$ approaches $\infty$, is obtained from Equation $(3.23)$ and it is given as Equation $(3.14)$.

*Step5.* The solution of $(3.10a)$ given that $(3.10b)$ in term of Mittag Leffler function of one parameter is obtained from step5.

*Remark 3.2.2.1:* If $Lu = -\left(D \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[F(x)u\right]\right)$, $Nu = 0$ and $f = 0$, then Equation $(3.10a)$ given that $(3.10b)$ becomes Equation $(1.2a)$ given that $(1.2b)$ and

i. $u_0$ which is given by Equation $(3.18)$ becomes

$$u_0 = L^{-1}\left[\frac{1}{S^{\beta}}\left[\sum_{r=0}^{p-1} u^{r} s^{\beta-r-1}\right]\right] \quad (3.24)$$

ii. $u_1$ which is given by Equation $(3.19)$ becomes

$$u_1 = L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(D \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} u_0 - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[F(x)u_0\right]\right)\right]\right] \quad (3.25)$$

iii. $u_i = u_{p+1}$ which is given by Equation $(3.20)$ becomes

$$u_i = u_{p+1} = L^{-1}\left[\frac{1}{S^{\beta}} L\left[D \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F(x)\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right)\right]\right];\ p \in IN, p \geq 1, i = 0, 1, 2, \cdots, p+1 \quad (3.26)$$

iii. *Applications*

To validate (show) the simplicity, effectiveness and applicability of iterative fractional Laplace transform method (IFLTM) for determining closed solutions of $(n+1)$ dimensional time fractional diffusion equations of the form $(1.2a)$ given that $(1.2b)$ in infinite fractional power series form, in terms of Mittag-Leffler functions in one parameter and exact form, three application examples were considered and solved as follows.

*Example 3.2.1.1.* Taking $F(x_1)=-x_1, \lambda=1$ in $(1.2a)$ and choosing $f(x_1)=1$ in $(1.2b)$ (A.Kumar et al., 2017; Cetinkaya & Kiymaz, 2013; Kebede, 2018; Kumar et al., 2012),consider the initial value problem:

$$\begin{cases} \dfrac{\partial^\beta u}{\partial t^\beta}=\dfrac{\partial^2 u}{\partial {x_1}^2}+\dfrac{\partial}{\partial x_1}(x_1 u),\ x_1>0,\ t>0,\ 0<\beta\le 1 & (3.27a) \\ \text{Subject to initial condition}: u(x_1,0,\cdots,0,0)=1 & (3.27b) \end{cases}$$

Since $F(x_1)=-x_1, \lambda=1$ and $f(x_1)=1$,

By Equation $(3.24)$:

$$u_0(x_1,\ t)=L^{-1}\left[\frac{1}{S^\beta}\left[u^{\,0}(x_1,0)s^{\beta-0-1}\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\left[u^{\,0}(x_1,0)s^{\beta-0-1}\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\times 1\times s^{\beta-1}\right]=1$$

$$u_0(x_1,\ t)=u_0(x_1,\ t)=1 \qquad (3.28b)$$

By Equation $(3.25)$:

$$u_1(x_1,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\frac{\partial^2}{\partial x_1^2}(1)-\frac{\partial}{\partial x_1}\left[(-x_1)\times 1\right]\right)\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\right]=\frac{t^\beta}{\Gamma(\beta+1)}$$

$$u_1(x_1,t)=\frac{t^\beta}{\Gamma(\beta+1)},\quad 0<\beta\le 1, x>0, t>0 \qquad (3.29b)$$

By Equation $(3.26)$:

For $p=1$, $u_2=L^{-1}\left[\frac{1}{S^\beta}L\left[D\frac{\partial^2}{\partial x_1^2}(u_0+u_1-u_0)-\frac{\partial}{\partial x_1}F(x_1)(u_0+u_1-u_0)\right]\right]$

$$u_2(x_1,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\frac{\partial^2}{\partial x_1^2}\left(1+\frac{t^\beta}{\Gamma(\beta+1)}-1\right)-\frac{\partial}{\partial x_1}\left[(-x_1)\times\left(1+\frac{t^\beta}{\Gamma(\beta+1)}-1\right)\right]\right)\right]\right]=L^{-1}\left[\frac{1}{S^\beta}L\left[\frac{t^\beta}{\Gamma(\beta+1)}\right]\right]=L^{-1}\left[\frac{1}{S^{2\beta}}\right]$$

$$u_2(x_1,t)=\frac{t^{2\beta}}{\Gamma(2\beta+1)},\quad 0<\beta\le 1, x>0, t>0 \qquad (3.30b)$$

For $p=2$, $u_3=L^{-1}\left[\frac{1}{S^\beta}L\left[D\frac{\partial^2}{\partial x_1^2}(u_0+u_1+u_2-(u_0+u_1))-\frac{\partial}{\partial x_1}F(x_1)(u_0+u_1+u_2-(u_0+u_1))\right]\right]$

$$u_3(x_1,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\frac{\partial^2}{\partial x_1^2}\left(1+\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\left(1+\frac{t^\beta}{\Gamma(\beta+1)}\right)\right)-\frac{\partial}{\partial x_1}\left[(-x_1)\times\left(1+\frac{t^\beta}{\Gamma(\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\left(1+\frac{t^\beta}{\Gamma(\beta+1)}\right)\right)\right]\right)\right]\right]$$

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$$u_3(x_1,t)=L^{-1}\left[\frac{1}{S^{\beta}}L\left[\frac{t^{2\beta}}{\Gamma(2\beta+1)}\right]\right]=L^{-1}\left[\frac{1}{S^{\beta}}\times\frac{1}{S^{2\beta}}\right]=L^{-1}\left[\frac{1}{S^{3\beta}}\right]=\frac{t^{3\beta}}{\Gamma(3\beta+1)}$$

$$u_3(x_1,t)=\frac{t^{3\beta}}{\Gamma(3\beta+1)},\quad 0<\beta\le 1, x>0, t>0 \tag{3.31}$$

Notes

Continuing with this process, we obtain that:

$$u_i=u_{p+1}=u_1(x_1,t)=\frac{t^{i\beta}}{\Gamma(i\beta+1)},\quad 0<\beta\le 1, x>0, t>0, i=1,\ 2,\ 3,\ \cdots,P+1, P\in IN \tag{3.32}$$

The $i^{th}$ order approximate solution of Equation $(3.27a)$ given that $(3.27b)$, denoted by $\tilde{u}_i(x_1,t)$ is given by:

$$\tilde{u}_i(x_1,t)=\sum_{i=0}^{p+1}\left(\frac{1}{\Gamma(i\beta+1)}\right)t^i\ ,0<\beta\le 1, x>0, t>0 \tag{3.33}$$

By letting $p\in IN$ to $\infty$ or taking limit of both sides of Equation $(3.33)$ as $p\in IN\to\infty$, the closed solution of Equation $(3.27\text{a})$ in the form of infinite fractional power series denoted by $u(x_1,t)$ is:

$$u(x_1,t)=\sum_{i=0}^{\infty}\frac{t^{i\beta}}{\Gamma(i\beta+1)},0<\beta\le 1, x>0, t>0 \tag{3.34}$$

Thus, by using Equation $(3.4)$ in Equation $(3.34)$, the closed solution of Equation $(3.23\text{a})$ in terms of Mittag-Leffler function of one parameter is given by:

$$u(x_1,t)=E_{\beta}\left(t^{\beta}\right), 0<\beta\le 1, x>0, t>0 \tag{3.35}$$

If $\beta=\frac{1}{2}$, then Equation $(3.35)$ becomes $u(x_1,t)=E_{\frac{1}{2}}\left(\sqrt{t}\right)$

Lastly, the exact solution of Equation $(3.27\text{a})$, $u_{exact}(x_1,t)$ can be obtained from Equation $(3.27)$ as $\beta$ approaches to 1from left and it is given by

$$u_{exact}(x_1,t)=e^t, \beta=1, x>0, t>0 \tag{3.36}$$

In order to guarantee the agreement between the exact solution, Equation $(3.36)$ and the $i^{th}$ order approximate solution, Equation $(3.33)$ of Equation $(3.27\text{a})$ given that Equation $(3.27\text{b})$, the absolute errors: $E_5(u)=\left|u_{exact}(x_1,t)-\tilde{u}_5(x_1,t)\right|$ and $E_6(u)=\left|u_{exact}(x_1,t)-\tilde{u}_6(x_1,t)\right|$ were computed as they were shown below by tables 3.1and 3.2 by considering the $5^{th}$ order approximate solution, $\tilde{u}_5(x_1,t)=\sum_{i=0}^{5}\frac{t^{i\beta}}{\Gamma(i\beta+1)}, x_1,t,\beta\in\{0.25, 0.5, 0.75, 1\}$ and the $6^{th}$ order approximate solutions, $\tilde{u}_6(x_1,t)=\sum_{k=0}^{6}\frac{t^{i\beta}}{\Gamma(i\beta+1)}, x_1,t,\beta\in\{0.25,0.5,0.75,1\}$ of Equation (3.27a) given that Equation (3.27b) without loss of generality.

*Table 3.1:* Absolute error of approximating the solution of Equation(3.27a) given that Equation(3.27b) to 5th order using IFLTM

| Variables | | Absolute error, $E_5(u)=\lvert u_{exact}(x_1,t)-\tilde{u}_5(x_1,t)\rvert$ | | | |
|---|---|---|---|---|---|
| $t$ | $x_1$ | $\alpha=0.25$ | $\alpha=0.50$ | $\alpha=0.75$ | $\alpha=1$ |
| 0.25 | 0.25 | 4.852591 | 0.827357 | 0.530995 | $3.515836\times10^{-7}$ |
| 0.25 | 0.50 | 4.852591 | 0.827357 | 0.530995 | $3.515836\times10^{-7}$ |
| 0.25 | 0.75 | 4.852591 | 0.827357 | 0.530995 | $3.515836\times10^{-7}$ |
| 0.25 | 1.00 | 4.852591 | 0.827357 | 0.530995 | $3.515836\times10^{-7}$ |
| 0.50 | 0.25 | 6.317463 | 1.770632 | 0.449889 | $2.335403\times10^{-5}$ |
| 0.50 | 0.50 | 6.317463 | 1.770632 | 0.449889 | $2.335403\times10^{-5}$ |
| 0.50 | 0.75 | 6.317463 | 1.770632 | 0.449889 | $2.335403\times10^{-5}$ |
| 0.50 | 1.00 | 6.317463 | 1.770632 | 0.449889 | $2.335403\times10^{-5}$ |
| 0.75 | 0.25 | 7.356235 | 2.253027 | 0.544978 | 0.000276 |
| 0.75 | 0.50 | 7.356235 | 2.253027 | 0.544978 | 0.000276 |
| 0.75 | 0.75 | 7.356235 | 2.253027 | 0.544978 | 0.000276 |
| 0.75 | 1.00 | 7.356235 | 2.253027 | 0.544978 | 0.000276 |
| 1.00 | 0.25 | 8.108369 | 2.660339 | 0.591061 | 0.001615 |
| 1.00 | 0.50 | 8.108369 | 2.660339 | 0.591061 | 0.001615 |
| 1.00 | 0.75 | 8.108369 | 2.660339 | 0.591061 | 0.001615 |
| 1.00 | 1.00 | 8.108369 | 2.660339 | 0.591061 | 0.001615 |

*Table 3.2:* Absolute error of approximating the solution of Equation(3.27a) given that Equation (3.27b) to 6th order using IFLTM

| Variables | | Absolute error, $E_6(u)=\lvert u_{exact}(x_1,t)-\tilde{u}_6(x_1,t)\rvert$ | | | |
|---|---|---|---|---|---|
| $t$ | $x_1$ | $\alpha=0.25$ | $\alpha=0.50$ | $\alpha=0.75$ | $\alpha=1$ |
| 0.25 | 0.25 | 4.915278 | 1.194657 | 0.047116 | $1.249937\times10^{-8}$ |
| 0.25 | 0.50 | 4.915278 | 1.194657 | 0.047116 | $1.249937\times10^{-8}$ |
| 0.25 | 0.75 | 4.915278 | 1.194657 | 0.047116 | $1.249937\times10^{-8}$ |
| 0.25 | 1.00 | 4.915278 | 1.194657 | 0.047116 | $1.249937\times10^{-8}$ |
| 0.50 | 0.25 | 6.494771 | 1.791466 | 0.087453 | $1.652645\times10^{-6}$ |
| 0.50 | 0.50 | 6.494771 | 1.791466 | 0.087453 | $1.652645\times10^{-6}$ |
| 0.50 | 0.75 | 6.494771 | 1.791466 | 0.087453 | $1.652645\times10^{-6}$ |
| 0.50 | 1.00 | 6.494771 | 1.791466 | 0.087453 | $1.652645\times10^{-6}$ |
| 0.75 | 0.25 | 7.681970 | 2.32334 | 0.126940 | $2.919142\times10^{-5}$ |
| 0.75 | 0.50 | 7.681970 | 2.32334 | 0.126940 | $2.919142\times10^{-5}$ |
| 0.75 | 0.75 | 7.681970 | 2.32334 | 0.126940 | $2.919142\times10^{-5}$ |
| 0.75 | 1.00 | 7.681970 | 2.32334 | 0.126940 | $2.919142\times10^{-5}$ |

Notes

| | | | | | |
|---|---|---|---|---|---|
| 1.00 | 0.25 | 8.609871 | 2.827005 | 0.595306 | 0.000226 |
| 1.00 | 0.50 | 8.609871 | 2.827005 | 0.595306 | 0.000226 |
| 1.00 | 0.75 | 8.609871 | 2.827005 | 0.595306 | 0.000226 |
| 1.00 | 1.00 | 8.609871 | 2.827005 | 0.595306 | 0.000226 |

Notes

*Example 3.2.3.2.* Taking $F(x_1,x_2)=-x_1-x_2, \lambda=1$, $u=u(x_1,x_2,\ t)$ and choosing $f(x_1,x_2)=x_1+x_2$, in Equation $(1.2a)$, consider the initial value problem:

$$\begin{cases}\dfrac{\partial^\beta u}{\partial t^\beta}=\dfrac{\partial^2 u}{\partial x_1{}^2}+\dfrac{\partial^2 u}{\partial x_2{}^2}+\left(\dfrac{\partial}{\partial x_1}+\dfrac{\partial}{\partial x_2}\right)((x_1+x_2)u),\ x_1>0,\ x_2>0, \mathrm{t}>0, 0<\beta\le 1 & (3.37a)\\ \text{Subject to initial condition } u(x_1,x_2,0)=x_1+x_2 & (3.37b)\end{cases}$$

Since $F(x_1,x_2)=-x_1-x_2, \lambda=1$ and $f(x_1,x_2)=x_1+x_2$

By Equation $(3.24)$:

$$u_0(x_1,x_2,\ t)=L^{-1}\left[\frac{1}{S^\beta}\left[u^{\ 0}(x_1,x_2,0)s^{\beta-0-1}\right]\right]=\ L^{-1}\left[\frac{1}{S^\beta}\left[(x_1+x_2)s^{\beta-0-1}\right]\right]\ =\ L^{-1}\left[\frac{1}{S}\times(x_1+x_2)\right]=x_1+x_2$$

$$u_0(x_1,x_2,\ t)=u_0(x_1,x_2,\ t)=x_1+x_2 \qquad (3.38)$$

By Equation $(3.25)$:

$$u_1((x_1,x_2,\ t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)(x_1+x_2)\ -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_1}\right)\left[(-x_1-x_2)\times(x_1+x_2)\right]\right)\right]\right]$$

$$u_1((x_1,x_2,\ t)=L^{-1}\left[\frac{1}{S^\beta}L[3x_1+3x_2\ ]\right]=L^{-1}\left[\frac{1}{S^\beta}\times\frac{1}{s}\times[3x_1+3x_2\ ]\right]=\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)}$$

$$u_1(x_1,x_2,t)=\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, t>0 \qquad (3.39)$$

By Equation $(3.26)$:

For $p=1$, $u_2\ =\ L^{-1}\left[\frac{1}{S^\beta}L\left[D\frac{\partial^2}{\partial x_1^2}(u_0+u_1-u_0)-\frac{\partial}{\partial x_1}F(x_1)(u_0+u_1-u_0)\right]\right]$

$$u_2(x_1,x_2,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\left(x_1+x_2+\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)}-(x_1+x_2)\right)-\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left[(-x_1-x_{21})\times\left(x_1+x_2+\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)}-(x_1+x_2)\right)\right]\right)\right]\right]$$

$$u_2(x_1,x_2,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\frac{3^2(x_1+x_2)t^\beta}{\Gamma(\beta+1)}\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\times\frac{3^2(x_1+x_2)}{s^{\beta+1}}\right]=\frac{3^2(x_1+x_2)t^{2\beta}}{\Gamma(2\beta+1)}$$

$$u_2(x_1,x_2,t)=\frac{3^2(x_1+x_2)t^{2\beta}}{\Gamma(2\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, t>0 \tag{3.40}$$

Continuing with this process, we obtain that:

$$u_i(x_1,x_2,t)=u_{p+1}(x_1,x_2,t)=\frac{3^i(x_1+x_2)t^{i\beta}}{\Gamma(i\beta+1)},\ 0<\beta\le 1,\ x_1>0,\ x_2>0,\ t>0, i=1,2,3,\cdots,P+1, P\in IN \tag{3.41}$$

The $i^{th}$ order approximate solution of Equation $(3.37a)$ given that $(3.37b)$, denoted by $\tilde{u}_i(x_1,x_2,t)$ is given by:

$$\tilde{u}_i(x_1,x_2,t)=\sum_{i=0}^{p+1}\frac{3^i(x_1+x_2)t^{i\beta}}{\Gamma(i\beta+1)},0<\beta\le 1, x_1>0, x_2>0, t>0 \tag{3.42}$$

By letting $p\in IN$ to $\infty$ or taking limit of both sides of Equation $(3.33)$ as $p\in IN\to\infty$, the closed solution of Equation $(3.27\text{a})$ in the form of infinite fractional power series denoted by $u(x_1,x_2,t)$ is:

$$u(x_1,x_2,t)=\sum_{i=0}^{\infty}\frac{3^i(x_1+x_2)t^{i\beta}}{\Gamma(i\beta+1)},0<\beta\le 1, x_1>0, x_2>0, t>0 \tag{3.43}$$

Thus, by using Equation $(3.4)$ in Equation $(3.43)$, the closed solution of Equation $(3.37\text{a})$ in terms of Mittag-Leffler function of one parameter is given by:

$$u(x_1,x_2,t)=(x_1+x_2)E_\beta\left(3t^\beta\right),0<\beta\le 1, x_1>0, x_2>0, t>0 \tag{3.44}$$

Lastly, the exact solution of equation $(3.37\text{a})$, $u_{exact}(x_1,x_2,t)$ can be obtained from Equation $(3.27)$ as $\beta$ approaches to 1from left and it is given by

$$u_{exact}(x_1,x_2,t)=(x_1+x_2)e^{3t},\beta=1, x_1>0, x_2>0, t>0 \tag{3.45}$$

In order to show the agreement between the exact solution, equation $(3.45)$ and the $i^{th}$ order approximate solution, equation $(3.42)$ of equation $(3.37\text{a})$ given that equation $(3.37b)$, the absolute errors: $E_4(u)=\left|u_{exact}(x_1,x_2,t)-\tilde{u}_4(x_1,x_2,t)\right|$ and $E_5(u)=\left|u_{exact}(x_1,x_2,t)-\tilde{u}_5(x_1,x_2,t)\right|$ were computed as they were shown below by tables 3.3 and 3.4 by considering the $4^{rd}$ order approximate solutions, $\tilde{u}_4(x_1,x_2,t)=\sum_{i=0}^{4}\frac{3^i(x_1+x_2)t^{i\beta}}{\Gamma(i\beta+1)},x_1,x_2,t,\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}$ and the $5^{th}$ order approximate solutions, $\tilde{u}_5(x_1,x_2,t)=\sum_{i=0}^{5}\frac{3^i(x_1+x_2)t^{i\beta}}{\Gamma(i\beta+1)},x_1,x_2,t,\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}$ of equation $(3.37\text{a})$ given that equation $(3.37\text{b})$ without loss of generality.

Notes

*Table 3.3:* Absolute error of approximating the solution of Equation(3.37a) given that Equation(3.37b) to4$^{th}$orderusing IFLTM

| Variables | | | Absolute error, $E_4(u)=\lvert u_{exact}(x_1,x_2,t)-\tilde{u}_4(x_1,x_2,t)\rvert$ | | |
|---|---|---|---|---|---|
| $t$ | $x_1$ | $x_2$ | $\alpha=\frac{1}{3}$ | $\alpha=\frac{2}{3}$ | $\alpha=1$ |
| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 20.084832 | 2.803182 | 0.031324 |
| $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 40.169667 | 5.606364 | 0.062648 |
| $\frac{1}{3}$ | 1 | 1 | 60.254500 | 8.409546 | 0.093972 |
| $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 41.182782 | 5.875512 | 0.654432 |
| $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 82.365564 | 11.751027 | 1.308864 |
| $\frac{2}{3}$ | 1 | 1 | 123.548346 | 17.626554 | 1.963296 |
| 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | 59.516178 | 16.990026 | 2.473692 |
| 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | 119.032359 | 33.980055 | 4.947384 |
| 1 | 1 | 1 | 178.548536 | 50.970084 | 7.421074 |

*Table 3.4:* Absolute error of approximating the solution of Equation(3.37a) given that Equation (3.37b) to 5$^{th}$ order using IFLTM

| Variables | | | Absolute error, $E_5(u)=\lvert u_{exact}(x_1,x_2,t)-\tilde{u}_5(x_1,x_2,t)\rvert$ | | |
|---|---|---|---|---|---|
| $t$ | $x_1$ | $x_2$ | $\alpha=\frac{1}{3}$ | $\alpha=\frac{2}{3}$ | $\alpha=1$ |
| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 36.532842 | 3.25242 | 0.025768 |
| $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 73.06569 | 6.504837 | 0.051536 |
| $\frac{1}{3}$ | 1 | 1 | 109.598534 | 9.757256 | 0.077304 |
| $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 97.535262 | 11.539632 | 0.476655 |
| $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 195.070521 | 23.079261 | 0.95331 |
| $\frac{2}{3}$ | 1 | 1 | 292.60578 | 34.618892 | 1.429965 |

Notes

| | | | | | |
|---|---|---|---|---|---|
| 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | 167.187744 | 34.483632 | 1.123692 |
| 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | 334.375491 | 68.967261 | 2.247384 |
| 1 | 1 | 1 | 501.563236 | 103.450892 | 3.371074 |

Notes

*Example 3.2.2.3.* Taking $F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$, $\lambda=1$, and $u=u(x_1,x_2,x_2,\ t)$ in $(1.2a)$ and choosing $f(x)=e^{x_1+x_2+x_3}$ in equation $(1.2b)$, we have the initial value problem:

$$\begin{cases}\dfrac{\partial^\beta u}{\partial t^\beta}=\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial}{\partial x}(e^{-x_1-x_2-x_3}u), 0<\beta\le 1,\ x>0, t>0 & (3.47\text{a})\\ u(x,0)=e^{x_1+x_2+x_3} & (3.47\text{b})\end{cases}$$

Since $F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$ and $f(x)=e^{x_1+x_2+x_3}$,

By Equation $(3.24)$:

$$u_0(x_1,x_2,x_3,\ t)=L^{-1}\left[\frac{1}{S^\beta}\left[u^{0}(x_1,x_2,x_3,0)s^{\beta-0-1}\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\left[\left(e^{x_1+x_2+x_3}\right)s^{\beta-0-1}\right]\right]=L^{-1}\left[\frac{1}{S}\times\left(e^{x_1+x_2+x_3}\right)\right]=e^{x_1+x_2+x_3}$$

$$u_0(x_1,x_2,x_3\ t)=e^{x_1+x_2+x_3} \qquad (3.48)$$

By Equation $(3.25)$:

$$u_1((x_1,x_2,x_3,\ t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)(e^{x_1+x_2+x_3})-\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\right)\left[\left(e^{-x_1-x_2-x_3}\right)\times\left(e^{x_1+x_2+x_3}\right)\right]\right)\right]\right]$$

$$u_1((x_1,x_2,x_2,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[e^{x_1+x_2+x_3}+e^{x_1+x_2+x_3}+e^{x_1+x_2+x_3}-0\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\times\frac{1}{s}\times\left[3e^{x_1+x_2+x_3}\right]\right]=\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)}$$

$$u_1(x_1,x_2,x_3,t)=\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, x_3>0, t>0 \qquad (3.49)$$

By Equation $(3.26)$:

For $p=1$, $u_2=L^{-1}\left[\frac{1}{S^\beta}L\left[D\frac{\partial^2}{\partial x_1^2}(u_0+u_1-u_0)-\frac{\partial}{\partial x_1}F(x_1)(u_0+u_1-u_0)\right]\right]$

$$u_2(x_1,x_2,x_3,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\left(e^{x_1+x_2+x_3}+\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)}-\left(e^{x_1+x_2+x_3}\right)\right)-\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left[\left(e^{-x_1-x_2-x_3}\right)\times\left(e^{x_1+x_2+x_3}+\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)}-\left(e^{x_1+x_2+x_3}\right)\right)\right]\right)\right]\right]$$

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$$u_2(x_1,x_2,x_3,t)=L^{-1}\left[\frac{1}{S^{\beta}}L\left[\frac{3^2\left(e^{x_1+x_2+x_3}\right)t^{\beta}}{\Gamma(\beta+1)}\right]\right]=L^{-1}\left[\frac{1}{S^{\beta}}\times\frac{3^2\times e^{x_1+x_2+x_3}}{s^{\beta+1}}\right]=\frac{3^2\times e^{x_1+x_2+x_3}t^{2\beta}}{\Gamma(2\beta+1)}$$

$$u_2(x_1,x_2,x_3,t)=\frac{3^2\times e^{x_1+x_2+x_3}t^{2\beta}}{\Gamma(2\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, x_3>0,\ t>0 \qquad (3.50)$$

Notes

Continuing with this process, we obtain that:

$$u_i(x_1,x_2,x_3,t)=u_{p+1}(x_1,x_2,x_3,t)=\frac{3^i\times e^{x_1+x_2+x_3}t^{i\beta}}{\Gamma(i\beta+1)},\quad 0<\beta\le 1,\ x_1>0,\ x_2>0,\ x_3>0,\ t>0,\ i=1,2,3,\cdots,P+1, P\in IN \qquad (3.51).$$

Then the $i^{th}$ order approximate solution of Equation $(3.47a)$ given that $(3.47b)$, denoted by $\tilde{u}_i(x_1,x_2,x_3,t)$ is given by:

$$\tilde{u}_i(x_1,x_2,x_3,t)=\sum_{i=0}^{p+1}\frac{3^i e^{x_1+x_2+x_3}t^{i\beta}}{\Gamma(i\beta+1)},0<\beta\le 1, x_1>0,\ x_2>0,\ x_3>0, t>0 \qquad (3.52)$$

By letting $p\in IN$ to $\infty$ or taking limit of both sides of Equation $(3.52)$ as $p\in IN\to\infty$, the closed solution of Equation $(3.47a)$ in the form of infinite fractional power series denoted by $u(x_1,x_2,x_3,t)$ is:

$$u(x_1,x_2,x_3,t)=\sum_{i=0}^{\infty}\frac{3^i e^{x_1+x_2+x_3}t^{i\beta}}{\Gamma(i\beta+1)},0<\beta\le 1, x_1>0,\ x_2>0,\ x_3>0, t>0 \qquad (3.53)$$

Thus, by using Equation $(3.4)$ in Equation $(3.53)$, the closed solution of Equation $(3.47a)$ in terms of Mittag-Leffler function of one parameter is given by:

$$u(x_1,x_2,x_3,t)=e^{x_1+x_2+x_3}E_{\beta}\left(3t^{\beta}\right),0<\beta\le 1, x_1>0,\ x_2>0,\ x_3>0, t>0 \qquad (3.54)$$

Lastly, the exact solution of Equation $(3.47a)$, $u_{exact}(x_1,x_2,x_3,t)$ can be obtained from Equation $(3.54)$ as $\beta$ approaches to 1from left and it is given by

$$u_{exact}(x_1,x_2,x_3,t)=e^{x_1+x_2+x_3}e^{3t},\beta=1, x_1>0,\ x_2>0,\ x_3>0, t>0 \qquad (3.55)$$

In order to show the agreement between the exact solution, Equation $(3.55)$ and the $i^{th}$ order approximate solution, Equation $(3.52)$ of Equation $(3.47a)$ given that $(3.47b)$, the absolute errors: $E_4(u)=\left|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_4(x_1,x_2,x_3,t)\right|$ and $E_5(u)=\left|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_5(x_1,x_2,x_3,t)\right|$ were computed as shown below by tables3.5 and3.6 by considering the $4^{th}$ order approximate solutions, $\tilde{u}_4(x_1,x_2,x_3,t)=\sum_{k=0}^{4}\frac{3^i e^{x_1+x_2+x_3}t^{i\beta}}{\Gamma(i\beta+1)},x_1,x_2,x_3,t,\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}$ and the $5^{th}$ order approximate solutions, $\tilde{u}_5(x_1,x_2,x_3,t)=\sum_{i=0}^{5}\frac{3^i e^{x_1+x_2+x_3}t^{k\beta}}{\Gamma(k\beta+1)},x_1,x_2,x_3,t,\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}$ of Equation $(3.47a)$ given that $(3.47b)$ without loss of generality.

*Table 3.5:* Absolute error of approximating the solution of Equation(3.47a) given that (3.47b) to 4th order using IFLTM

| Variables | | | | Absolute error, $E_4(u)=\left\|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_4(x_1,x_2,x_3,t)\right\|$ | | |
|---|---|---|---|---|---|---|
| $t$ | $x_1$ | $x_2$ | $x_3$ | $\alpha=\frac{1}{3}$ | $\alpha=\frac{2}{3}$ | $\alpha=1$ |
| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 81.894351 | 11429758 | 0.027033 |
| $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 222.611942 | 30.570542 | 0.073517 |
| $\frac{1}{3}$ | 1 | 1 | 1 | 605.121992 | 84.455123 | 0.199811 |
| $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 167.919612 | 23.956946 | 8.446234 |
| $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 456.45283 | 65.121748 | 11.479623 |
| $\frac{2}{3}$ | 1 | 1 | 1 | 1240.767433 | 177.01926 | 20.803233 |
| 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 242.672618 | 69.275518 | 10.086288 |
| 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 659.652584 | 188.310399 | 27.417373 |
| 1 | 1 | 1 | 1 | 1793.121606 | 511.880752 | 74.528128 |

*Table 3.6:* Absolute error of approximating the solution of Equation(3.47a)given that(3.47b) to 5thorder using IFLTM

| Variables | | | | Absolute error, $E_5(u)=\left\|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_5(x_1,x_2,x_3,t)\right\|$ | | |
|---|---|---|---|---|---|---|
| $t$ | $x_1$ | $x_2$ | $x_3$ | $\alpha=\frac{1}{3}$ | $\alpha=\frac{2}{3}$ | $\alpha=1$ |
| $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 148.959841 | 13.261491 | 0.165307 |
| $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 404.914862 | 36.048454 | 0.449334 |
| $\frac{1}{3}$ | 1 | 1 | 1 | 1100.672701 | 97.989863 | 1.221402 |
| $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 397.692495 | 47.051958 | 8.446234 |
| $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 1081.040267 | 127.900466 | 11.479623 |
| $\frac{2}{3}$ | 1 | 1 | 1 | 2938.572099 | 347.669517 | 20.803233 |

Notes

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 681.69511 | 140.604345 | 4.581767 |
| 1 | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 1853.039446 | 382.20222 | 12.454535 |
| 1 | 1 | 1 | 1 | 5037.083448 | 1038.933356 | 33.854916 |

Notes

*c) Disscusion*

Here, the results obtained from the three application examples considered above are discussed. Through the three examples above, the iterative fractional Laplace transform method (IFLTM)was successfully applied to the time fractional diffusion equations, that is, Equation $(1.2a)$ given that $(1.2b)$, for $F(x_1)=-x_1$ with initial conditions $f(x_1)=1$, $F(x_1,x_2)=-x_1-x_2$ with initial conditions $f(x_1,x_2)=x_1+x_2$, $F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$ with initial conditions $f(x_1,x_2,x_3)=e^{x_1+x_2+x_3}$, $\lambda=1$ and $0<\beta\le 1$.

As a result, through example one, the closed solutions of Equation $(1.2a)$ given that $(1.2b)$ in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as its exact solution were obtained and they are in complete agreement with the results obtained by Cetinkaya and Kiymaz(2013), kebede(2018), Kumar et al.(2012)and A. Kumar et al.(2017). For $\beta=\frac{1}{2}$ with $F(x_1)$, $\lambda$ and $f(x_1)$ specified in example one above, the closed solutions of equation $(1.2a)$ given that $(1.2b)$ in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as their exact solution, which were obtained by IFLTM, are in complete agreement with the results obtained by kebede(2018)and S. Das (2009).

From the application of IFLTM to Equation $(1.2a)$ given that $(1.2b)$ through the second and third examples above, where $F(x_1,x_2)=-x_1-x_2$ with initial condition $f(x_1,x_2)=x_1+x_2$, $F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$ with initial condition $f(x_1,x_2,x_3)=e^{x_1+x_2+x_3}$ and $0<\beta\le 1$, the closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as exact solution were obtained.

Without loss of generality the $5^{th}$ and $6^{th}$ order approximate solutions of Equation $(3.27a)$; $\forall(x_1,t)\in\{0.25,0.5,0.75,1\}\times\{0.25,0.5,0.75,1\}$, $\forall\beta\in\{0.25,0.5,0.75,1\}$, and the $4^{rd}$ and $5^{th}$order approximate solutions of Equations: $(3.37a)$ and $(3.47a)$ $\forall(x_1,x_2,t,)\in\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}$ and $\forall(x_1,x_2,x_3,t,)\in\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}$ respectively were considered to compute absolute errors in this paper. The validity, accuracy and convergence of the IFLTM was checked through the computed absolute errors:

$$\begin{cases} E_5(u)=\left|u_{exact}(x_1,t)-\tilde{u}_5(x_1,t)\right| \\ E_6(u)=\left|u_{exact}(x_1,t)-\tilde{u}_6(x_2,t)\right| \end{cases}; \forall\beta\in\{0.25,0.5,0.75,1\}\subseteq(0,1],$$

$$\begin{cases} E_4(u)=\left|u_{exact}(x_1,x_2,t)-\tilde{u}_4(x_1,x_2,t)\right| \\ E_5(u)=\left|u_{exact}(x_1,x_2,t)-\tilde{u}_5(x_1,x_2,t)\right| \end{cases}; \forall\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}\subseteq(0,1],$$

$$\begin{cases} E_4(u) = \left|u_{exact}(x_1,x_2,x_3,t) - \tilde{u}_4(x_1,x_2,x_3,t)\right| \\ E_5(u) = \left|u_{exact}(x_1,x_2,x_3,t) - \tilde{u}_5(x_1,x_2,x_3,t)\right| \end{cases}; \forall \beta \in \left\{\frac{1}{3},\frac{2}{3},1\right\} \subseteq (0,1],$$

where $u_5(x_1,t)$ is the 5th order approximate solutions, $u_6(x_1,t)$ is the 6th order approximate solutions and $u_{exact}(x_1,t)$ is the exact solutions of example one; $u_4(x_1,x_2,t)$ is the 4th order approximate solutions, $u_5(x_1,x_2,t)$ is the 5th order approximate solutions and $u_{exact}(x_1,x_2,t)$ is the exact solution of example two; $u_4(x_1,x_2,x_3,t)$ is the 4th order approximate solutions, $u_5(x_1,x_2,x_3,t)$ is the 5th order approximate solutions and $u_{exact}(x_1,x_2,x_3,t)$ is the exact solution of example three. From observation made through tables 3.1 to 3.6, the absolute errors: $E_5(u)$ and $E_6(u)$ decrease as $\beta \in \{0.25, 0.5, 0.75, 1\}$ increases from 0.25 to 1; $E_4(u)$ and $E_5(u)$ decrease as $\beta \in \left\{\frac{1}{3},\frac{2}{3},1\right\}$ increases from $\frac{1}{3}$ to 1. These imply that the 5th order approximate solutions and the 6th order approximate solutions of Equation (3.27a)converge to their exact solution as $\beta \in \{0.25, 0.5, 0.75, 1\}$ increases from 0.25 to 1; the 4th order approximate solutions and the 5th order approximate solutions of Equations(3.37a)and (3.47a)converge to their exact solutions as $\beta \in \left\{\frac{1}{3},\frac{2}{3},1\right\}$ increases from $\frac{1}{3}$ to 1. It was also observed that $E_5(u) > E_6(u)$ for each $(x_1,t) \in \{0.25, 0.5, 0.75, 1\} \times \{0.25, 0.5, 0.75, 1\}$ and for each $\beta \in \{0.25, 0.5, 0.75, 1\}$ throughout tables: 3.1 and 3.2; $E_4(u) > E_5(u)$ for each $(x_1,x_2,t,) \in \left\{\frac{1}{3},\frac{2}{3},1\right\} \times \left\{\frac{1}{3},\frac{2}{3},1\right\} \times \left\{\frac{1}{3},\frac{2}{3},1\right\}$ and for each $\beta \in \left\{\frac{1}{3},\frac{2}{3},1\right\}$ throughout tables: 3.3 and 3.4; $E_4(u) > E_5(u)$ for each $(x_1,x_2,x_3,t,) \in \left\{\frac{1}{3},\frac{2}{3},1\right\} \times \left\{\frac{1}{3},\frac{2}{3},1\right\} \times \left\{\frac{1}{3},\frac{2}{3},1\right\} \times \left\{\frac{1}{3},\frac{2}{3},1\right\}$ and for each $\beta \in \left\{\frac{1}{3},\frac{2}{3},1\right\}$ throughout tables: 3.5 and 3.6.

These show that the validity, accuracy and convergence of the fractional power series solutions of equations $(3.27a)$, $(3.37a)$ and $(3.47a)$ can be improved by calculating more term in the series solutions by using the present method, IFLTM.

## IV. Conclusion

In this study, basic idea of iterative fractional Laplace transform method (IFLTM) for solving $(n+1)$ dimensional time fractional diffusion equations with initial conditions of the form $(1.2a)$ given that $(1.2b)$ was developed. The IFLTM was applied to three $(n+1)$ dimensional time fractional diffusion equations with initial conditions to obtain their closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter which rapidly converge to exact solutions. The closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter, which rapidly converge to exact solutions, were successfully derived by the use of iterative fractional Laplace transform method (IFLTM). The results evaluated for the first time fractional diffusion equations is in a good agreement with the one already existing in the literature. Precisely, IFLTM works successfully in solving time fractional diffusion equations with initial conditions to obtain their closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler function in on parameter as well as exact solutions with a minimum size of calculations.

Notes

Thus, we can conclude that the IFLTM used in solving time fractional diffusion equations with initial conditions can be extended to solve other fractional partial differential equations with initial conditions which can arise in fields of sciences.

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Notes

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# Can A Decision Tree Forecast Real Economic Growth from Relative Depth of Financial Sector in Nigeria?

By Alabi Nurudeen Olawale & Bada Olatunbosun

*Abstract-* We employed a decision tree statistical learning method which is lately gaining wide usage in the field of econometrics to establish the relationships between real gross domestic products growth rate and financial depth indicators such as stock market turnover ratio, credit to private sector (CPS) and broad money supply (M2) relative to gross domestic product (GDP) in Nigeria between 1981 to 2016. The data was divided into training and test datasets. The former was used to train the decision tree while the later was used to test the performance of the fitted decision tree model. Recursive binary splitting produced a fitted tree with nine nodes (leaves). This tree was pruned using cost complexity pruning procedure which uses a tuning parameter α to control the tradeoff between the tree complexity and overfitting the data. Pruning produced a tree with four terminal nodes and improved predictability in terms of lower model MSE on test dataset and interpretability. Bagging and Random Forest procedure were employed to further improve the performance of the model by aggregating bootstrapped training samples in order to reduce the variance.

*Keywords: decision tree, recursive binary splitting, cost complexity pruning, bagging, random forest, financial depth, stock market liquidity.*

*GJSFR-F Classification: FOR Code: MSC 2010: 62P05*

Ref

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Alabi Nurudeen Olawale [α] & Bada Olatunbosun [σ]

*Abstract-* We employed a decision tree statistical learning method which is lately gaining wide usage in the field of econometrics to establish the relationships between real gross domestic products growth rate and financial depth indicators such as stock market turnover ratio, credit to private sector (CPS) and broad money supply (M2) relative to gross domestic product (GDP) in Nigeria between 1981 to 2016. The data was divided into training and test datasets. The former was used to train the decision tree while the later was used to test the performance of the fitted decision tree model. Recursive binary splitting produced a fitted tree with nine nodes (leaves). This tree was pruned using cost complexity pruning procedure which uses a tuning parameter $\alpha$ to control the tradeoff between the tree complexity and overfitting the data. Pruning produced a tree with four terminal nodes and improved predictability in terms of lower model MSE on test dataset and interpretability. Bagging and Random Forest procedure were employed to further improve the performance of the model by aggregating bootstrapped training samples in order to reduce the variance. These resulted in lower model MSEs on the test dataset. The regression tree model reveals that stock market turnover ratio and broad money supply relative to GDP are the most important financial depth measures in the real economic growth model. However, real GDP growth rate rises with stock market turnover ratio but dips with values of broad money supply relative to GDP between 10 per cent and 15 per cent. The real GDP growth rate stability threshold for stock market turnover ratio and broad money supply (M2) relative to gross domestic product are 10 per cent and 20 per cent respectively. R programming language was used throughout the paper.

*Keywords: decision tree, recursive binary splitting, cost complexity pruning, bagging, random forest, financial depth, stock market liquidity.*

## I. Introduction

A decision tree is a supervised learning method with associated learning algorithms that analyze data used for classification and regression analysis. According to [1], amongst the learning methods in the Data Mining field, decision tree is considered to be closest to being "off-the-shelf" methods in terms of data handling, robustness to outliers in the regressor space, insensitivity to monotone transformations of inputs, computational scalability, and ability to deal with unimportant regressors. However, despite all these good characteristics possessed by decision trees, they perform poorly in their ability to extract linear combinations of the regressor space and in terms of predictive power. Off-the-shelf learning methods do not require time costly tuning of the learning procedure and preprocessing of data [1]. Hence these methods are applied directly on our real GDP growth rate data. In this current work, we study the relevance of this increasingly accepted modeling technique to the forecast of real economic growth

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using three major measures of financial depth in Nigeria between 1981 to 2016. In addition, our interest in this learning method is directed at the interpretability of the relationships between the macroeconomic variables under study. These financial depth variables are functions of credit to private sector and broad money supply relative to gross domestic products (gdp) and Nigerian Stock market turnover ratio. Recursive Binary Splitting procedure was used to construct an unpruned tree. We then improve the interpretability and predictive power of this method by using the cost complexity pruning, bagging and random forest. Cost complexity pruning is quite efficient because it uses a nonnegative tuning parameter $\alpha$ to select a sequence of trees from a large number of subtrees resulting from splitting of the original tree. Bagging and Random Forest are better than cost complexity pruning in that they potentially increase the predictive power of our model. These concepts will be discussed later in the paper.

Notes

Financial development in Nigeria has witnessed series of reforms since 1986. Figure 1 shows five significant periods, which may be associated with periods the financial sector witnessed influx of institutions (including banking and nonbanking), changes in ownership and agency structure, depth and mode of operations. These structural reforms were envisioned to make the financial sector adapt to the fast changes in global financial system, more competitive, strong and reliable. Between 1986 and 2004, the Nigerian government through the Central Bank liberalized the banking sector by deregulating the sector, enhancing financial inclusion, influencing savings, investment and consumption through interest rates and credit control designed primarily for monetary and price stability. Furthermore, the state-owned banks were privatized. During this period, financial depth improved but far below the desired levels[1]. The banking crisis of 2004 and 2009 revealed weaknesses which brought about massive regulatory interventions and reinstatement of financial controls such as deleveraging of the banking sector, expansion of capital market, insurance market reform, creation of pension fund system, facilitation of bond issuance and establishment of long-term institutional investors. Institutional investors are pivotal to Nigeria financial system as their "*buy-and-hold*" strategies contribute in no small measure to low liquidity in the system[2]. The 2004 banking sector reform and consolidation led to significant rise in financial depth[3] with M2/GDP rising from 18.1 per cent of GDP in 2005 to 38 per cent in 2009, CPS/GDP rising from 12.6 per cent of GDP to 36.7 per cent of GDP, Stock market turnover ratio increasing by 0.68 per cent during the same period. A major increase in market liquidity was recorded in 2008 due to global financial crisis and Real economic growth rose marginally by 0.45 per cent during this period.



[1] M2/GDP, CPS/GDP, Stock market turnover ratio and real economic growth rate averaged geometrically 13.9 per cent, 9.01 per cent, 3.71 per cent and 3.99 per cent respectively.

[2] Whilst the total asset of pension funds accounted for about 5 per cent of GDP, the insurance companies accounted for less than 1 per cent of GDP

[3] Stock market has grown by more than 16 per cent of GDP since 2000, Liquidity rose marginally by 0.74 per cent and M2 averaged geometrically 20.81 per cent of GDP since 2000. Highest M2/GDP and CPS/GDP were recorded in 2009. In 2014, outstanding credit to private sector accounted for 22.4 per cent of GDP with emerging markets averaging over 50 per cent in 2014.

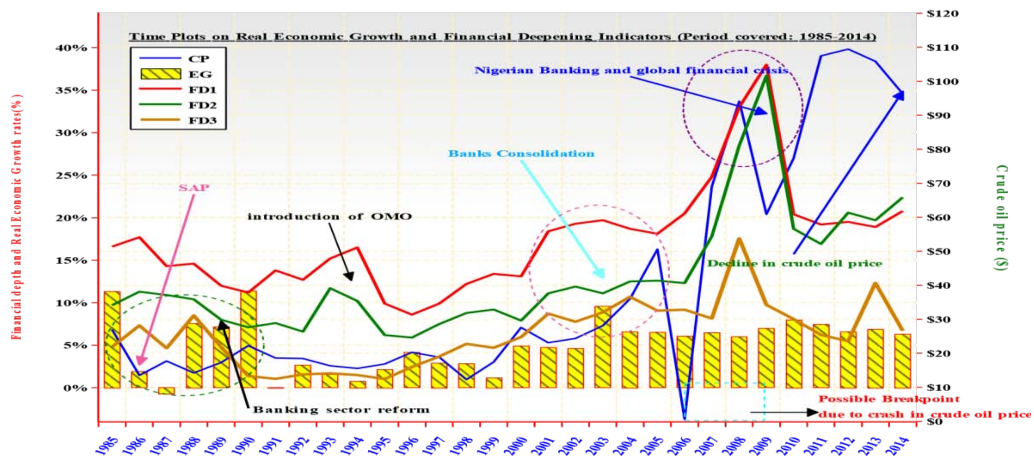


*Figure 1:* Time plot of Real Economic Growth rate (reg) versus M2/GDP ($fd_1$), CPS/GDP ($fd_2$) and Stock Market Turnover ratio ($fd_3$). This plot shows the five significant periods associated with periods the financial sector witnessed influx of institutions (including banking and nonbanking), changes in ownership and agency structure, depth and mode of operations

The banking sector remained dominant by the end of 2014, 21 major banks remain standing out of 89 major banks during pre-consolidation era[4]. Financial development has become an increasingly attractive topic to researchers globally in the last decade especially with the occurrence of the 2008 global financial crisis. The reasons for this recent interest in the topic stems from the believe that a well-structured financial development provides resilience and ensures economic growth needed to avert the effects of such crisis. By definition, financial development is a combination of depth, access and efficiency of financial system which provides opportunity for savings mobilization required for investment purposes, promotion of efficient information propagation, improvement of resource allocation and management of risks. Depth refers to size and liquidity of financial markets, access involves the ability of individuals to receive financial services and efficiency refers to ability of institutions to provide these services at reduced cost with persistent income and the level of activity in the capital market [2]. Depth of financial market gives rise to financial deepening, which may imply excessive financial development influenced by worsen regulatory framework. Although financial development enhances resilience and economic growth, concession between growth and stability due to financial deepening can emerge. A crucial question is "what are the limits to which financial development boosts long-term positive economic growth or limits of financial depth sustainable for enhancement of long-term positive economic growth in advanced and emerging economies?" [2] came up with different thresholds for stability and growth driven financial development for both the advanced and emerging economies. They based their work on newly developed financial development index, which combines the financial institution and financial market developments characteristics [2].

Prior to their work, a rich and diverse literature on the relationships between financial development and economic growth exists for both the advanced and emerging markets. The list is quite exhaustive but the earliest works were carried out by [3]; [4];

[4] Total assets of major banks accounted for more than 27 per cent of GDP in 2014. Branches have grown by more than 300 per cent since 1985, opening up access to well-developed branch networks, more than 7 branches per 100,000 adults and nearly 20 ATM per 100,000 adults.

Ref

3. King Robert, and Ross Levine. (1993), "Finance and Growth: Schumpeter Might Be Right." *The Quarterly Journal of Economics*, 108 (3): 717–37.

[5]. They applied various econometric techniques such as causality analysis amongst financial depth indicators like size of banking system relative to gross domestic products (GDP), stock market depth, and instrumental variables. They concluded that the past and current sizes of banking system relative to GDP could predict future values of economic growth by establishing unidirectional causality from these financial depth indicators to economic growth. [6], through Johansen vector error correction model (VECM) studied the long run causal relationships between financial depth indicators such as value added ratio and ratio of total private credit extension to GDP. They concluded that financial depth has significant effects on economic growth and credit rationing is prevalent in South Africa with firms with extensive reliance on internally generated income for operational requirements. Other works include [7] who employed panel data analysis in which financial depth indicators were instruments and controlling for other determinants of growth. [8] through correlation analysis found strong positive correlation between financial depth and real economic activity such as economic growth. Their work emphasized that countries with solid and well-established regulatory framework suffer less from shocks on inflation than countries with faulty regulatory structure. [9] established the existence of threshold cointegration between financial depth and economic growth in Taiwan and pointed out that there exists a positive and significant financial impact on economic growth. They concluded that financial depth could increase economic growth in Taiwan. Another study by [10] tried answering the question of how important financial development is to economic growth through a costly state verification model. Their resolve was that financial intermediation is crucial to economic development in the United States and a cross-country data. Furthermore, emphasis was that 29 per cent of US economic growth could be associated with technological improvement in financial intermediation. [11] analyzed the impact of financial deepening on economic growth in Turkey and found a strong negative relationship between financial deepening and economic growth. They summed up that financial development does not always lead to positive economic growth. In Nigeria, numerous monetary aggregates are calculated relative to GDP and used to measure the level of financial development/innovations. However, this current work focuses on three widely reported indicators; stock market liquidity/turnover ratio, M2/GDP and CPS/GDP. By the end of 2014, real economic growth in Nigeria stood at 6.30 per cent, broad money supply (M2) was 20.78 per cent of the GDP. Similarly, credit to private sector (CPS) relative to GDP stood at 22.40 per cent. However, the annualized economic growth rate in the last decade was 6.69 which made growth rather normally distributed. The country's financial development has gone through various phases of evolutions. Over time, several research works have emerged looking in-depth at the impact of financial deepening and economic growth in Nigeria. Prominent is the work of [12] which examined the relationship between real economic growth and three measures of development; M2/GDP, real interest rate and CPS/GDP. The study concluded that out of the financial depth indicators, only real interest rate is negatively related to real economic growth. The postulated model was statistically insignificant despite a high coefficient of determination. By implication the research failed to determine the data generating process (*d.g.p*, henceforth) underlying the time series, which is necessary to establish the order of integration of the series. Failure to do this may indicate that the estimated coefficients are mere contemporaneous correlations. Nevertheless, the study is important to this current work as it emphasized that financial development does not always result in positive economic growth in Nigeria, which is in line with conclusions made in previous studies by [11] in Turkey. [13]; [14] conducted Johansen test of

Ref

7. Beck, Thorsten and Ross Levine (2004), "Stock Markets, Banks and Growth: Panel Evidence." *Journal of Banking and Finance* 28 (3): 423–42.

cointegration and time varying granger causality respectively using data between 1960-2012 on poverty level, financial development and economic growth. The former found no evidence of long run co-movement between M2/GDP, per capita consumption and real GDP per capita. However, the later by introducing structural break in the granger causality model of M2/GDP and real economic growth between 1961-2012 found predictive powers from financial depth to real economic growth and vice versa during different time periods. Another research by [15] concluded that CPS is insignificant in promoting higher economic growth in Nigeria. This paper focuses on the effects of three major financial depth indicators; M2/GDP, Stock market liquidity and CPS/GDP on real economic growth in Nigeria. This will be achieved by recursive binary splitting, cost complexity pruning, bagging and random forest. The data employed are yearly observations from 1985 to 2014 of real economic growth (**reg**), broad money supply (M2) relative to GDP (**fd**$_1$), credit to private sector (CPS) relative to GDP (**fd**$_2$) and stock market turnover ratio (**fd**$_3$). The source of all time-series data is the Central Bank of Nigeria (CBN) database.

Ref

15. Oriavwote Victor E., Eshenake Samuel J., (2014), "An empirical assessment of financial sector development and economic growth in Nigeria", *International Review of Management and Business Research*, Vol. 3 (1).

## II. Methods, Empirical Analyses and Results

The four macroeconomic variables included in this paper are real GDP growth rate (**reg**), M2/GDP (**fd**$_1$), CPS/GDP (**fd**$_2$) as proxies for financial deepening and turnover ratio as proxy for stock market liquidity (**fd**$_3$). We divided the quarterly data between 1981 to 2016 into two. One part (training dataset) was used to grow our decision tree and the second half (test dataset) to test the performance of the decision tree model. As earlier mentioned in this paper, decision trees either for classification or regression is considered to be "*off-the-shelf*" statistical learning method which presently is gaining acceptance in the field of econometrics. This method derive motivation from a tree analogy grown with many branches and leaves. The leaves are referred to as the *terminal nodes* and the points along which the regressor space is split are called the *internal nodes*. The internal and terminal nodes are linked by *branches*. We draw our inspiration from the procedure outlined by [1] which involves dividing the regressor space $\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3]$ where $\boldsymbol{x}_1=\mathbf{fd}_1$, $\boldsymbol{x}_2=\mathbf{fd}_2$, $\boldsymbol{x}_3=\mathbf{fd}_3$, into $j$ distinct and disjoint regions $R_1, R_2,\dots.R_j$. The prediction is then done by determining the region $R_j$ in which an observation falls into and using the mean of the response values of the training observations in that region $R_j$ as the predicted value for that observation. The regions are determined such that the residual sum of squares given in equation 1.0 is minimized:

$$RSS = \sum_{j=1}^{J}\sum_{i\in Rj}(y_i - \bar{y}_{Rj})^2 \qquad 1.0$$

Where $\bar{y}_{Rj}$ is the mean response (real GDP growth rate, **reg**) for training observations in the $j^{th}$ region. Generally speaking, it may be numerically infeasible to generate every partition of the regression space into $J$ regions.

### a) Recursive Binary Splitting on the fitted Real GDP Regression tree model

We adopted the *recursive binary splitting* (RBS, hereon) which start with single region at the top of the tree and gradually perform splitting in an optimal fashion at each step of the tree construction. The RBS procedure select a regressor $\boldsymbol{x}_j$ in $\boldsymbol{X}$ and a cutoff $c$ such that the regressor space is split into two regions.

$$R_1 = \{X \mid x_j < c\} \text{ and } R_2 = \{X \mid x_j \geq c\}$$

With the ultimate aim of reducing the RSS in equation 1.0. this implies that we try to generate the value of *j* and *c*, which minimize equation 1.1

$$\sum_{i: x_i \in R_1(j,c)} (y_i - \bar{y}_{R_1})^2 + \sum_{i: x_i \in R_2(j,c)} (y_i - \bar{y}_{R_2})^2 \qquad 1.1$$

Where $\bar{y}_{R_1}$ is the mean response (real GDP growth rate, reg) for training observations in the $R_1(j, c)$ and $\bar{y}_{R_2}$ is the mean response for training observations in the $R_2(j, c)$ regions. This process was repeated in the subsequent steps minimizing RSS in each step. This resulted in the tree in Figure 2 with the value of *j* and *c* that minimizes the RSS in equation 1.1 are 3 and 7.585 per cent respectively. That is stock market turnover ratio ($fd_3$) is the regressor at the top of the tree used for the initial split such that

$$R_1 = \{X \mid fd_3 < 7.585\} \text{ and } R_2 = \{X \mid fd_3 \geq 7.585\}$$

The value of RSS = 362.9, MSE = 5.76 and the number of terminal nodes = 9. The splitting was terminated as soon as we have not more than 5 observations in each region Table 1. The unpruned tree used up about 88 per cent of the training observations.

*Table 1:* Analysis of Recursive Binary Splitting on Real GDP Growth Regression Tree Model

| Node | Split | Number of observation | RSS | Predicted Response |
|---|---|---|---|---|
| 1 | root | 72 | 1472 | 3.875 |
| 2 | $fd_3 < 7.585$ | 47 | 877.3 | 2.189 |
| 4 | $fd_2 < 8.770$ | 21 | 253.9 | 4.359 |
| 8 | $fd_1 < 12.420$ | 12 | 166.9 | 5.381 |
| 16 | $fd_1 < 10.795$ | 6 | 1.583 | 3.0380* |
| 17 | $fd_1 > 10.795$ | 6 | 99.480 | 7.7230* |
| 9 | $fd_1 > 12.420$ | 9 | 57.760 | 2.9970* |
| 5 | $fd_2 < 8.770$ | 26 | 444.6 | 0.4358 |
| 10 | $fd_1 < 17.290$ | 17 | 225.5 | -1.5580 |
| 20 | $fd_3 < 5.380$ | 12 | 10.120 | 0.3708* |
| 21 | $fd_3 > 5.380$ | 5 | 63.560 | 6.1880* |
| 11 | $fd_1 > 17.290$ | 9 | 23.82 | 4.2020* |
| 3 | $fd_3 > 7.585$ | 25 | 209.6 | 7.0460 |
| 6 | $fd_3 < 10.875$ | 15 | 89.940 | 8.3050 |
| 12 | $fd_1 < 18.560$ | 5 | 3.439 | 5.892* |
| 13 | $fd_1 > 18.560$ | 10 | 42.840 | 9.511* |
| 7 | $fd_3 > 10.875$ | 10 | 60.280 | 5.159* |

**Terminal node (leave)*　　*Source: Computations using **R** language*

Notes

*Recursive Binary Splitting of Regression Tree on Real GDP growth rate and Financial Depth Regression Space*

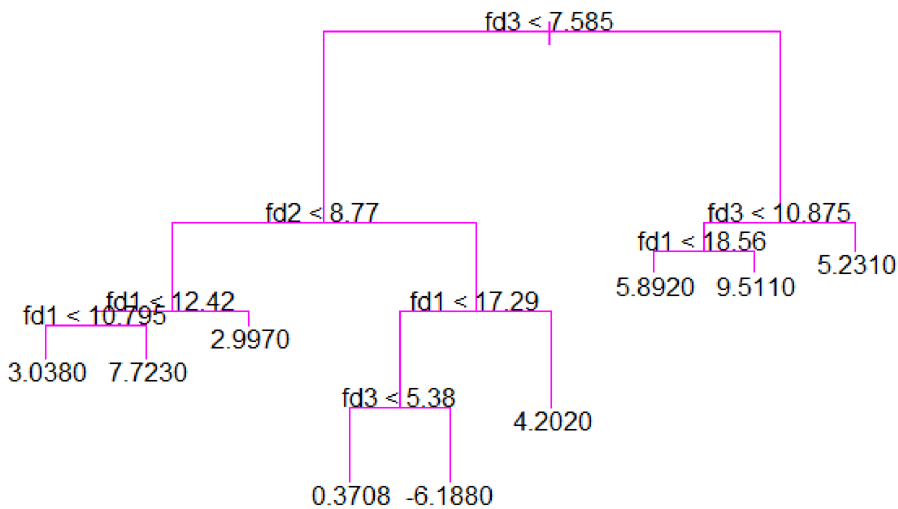


*Figure 2:* Unpruned Regression tree with 9 terminal nodes (leaves) and 8 internal nodes. This regression tree model shows that $\mathbf{fd}_3$ which is the measure of stock market liquidity is the most important regressor in the model since it minimizes the RSS in equation 1.0. Splitting was done in such a way that at each step, one of the regressor is selected. At a given internal node, the left-hand branch is represented by $X_j < q_k$ resulting from the split and $X_j > q_k$ indicates the right hand branch. At the top of the tree, the split resulted in two branches in which the left hand branch corresponds to $\mathbf{fd}_3 < 7.585$ per cent and the right hand branch corresponds to $\mathbf{fd}_3 \geq 7.585$ per cent. Splitting ensures simplicity and ease of interpretability of the regression tree model. **Source:** Computations using **R** language.

This procedure of RBS resulting in **Figure** 2 is quite simple which may have produced good predictions on the training dataset but eventually overfitting the real GDP growth and financial depth data. One snag is that this might lead to a very poor performance of the regression tree model on the test dataset. Consequently, in order to achieve a good test performance, we generated a smaller tree that contain fewer splits using *cost complexity pruning*. With this method, we are able to achieve a regression tree model with lower variance but slightly higher bias. The cost complexity pruning involves a nonnegative tuning parameter α such that for every value of this positive tuning parameter, there exists a subtree $T \subset T_0$ for which equation 1.2 is as small as possible.

$$\sum_{m=1}^{|T|} \sum_{i:x_i \in R_m} (y_i - \bar{y}_{R_m})^2 + \alpha |T| \quad — \quad 1.2$$

The quantity —$T$— is the number of terminal nodes or leaves of the regression tree $T$, $R_m$ is the regions relating to the $m^{th}$ terminal node and $\bar{y}_{Rm}$ is the mean of the training dataset's real GDP growth rate (**reg**) corresponding to the region $R_m$. Increasing the value of α from zero in equation 1.2 ensures that the tree branches get pruned and controls the tradeoff between the complexity of the regression subtree and

Notes

the goodness of fit. We employed the 10-fold cross validation to determine the value of the positive tuning parameter and a most complex tree. We computed a tuning parameter $\alpha = 151.829$ with an MSE value of 1302.169 using cross validation Table 2 shows the result of the cost complexity pruning for various values of $\alpha$ at each split. We set the value of the best number of terminal nodes to 3 which is equivalent to the number of regressors in the regressor space for the tree pruning. This pruned tree was generated from a large tree on the training dataset containing 72 observations and varying the nonnegative tuning parameter $\alpha$ in equation 1.2 which resulted in 7 subtrees.

Notes

*Table 2:* *Cost Complexity Pruning on Regression Tree Model*

| Node | Split | Number of observation | RSS | Predicted Response |
|---|---|---|---|---|
| 1 | root | 72 | 1472 | 3.875 |
| 2 | $fd_3 < 7.585$ | 47 | 877.3 | 2.189 |
| 4 | $fd_2 < 8.770$ | 21 | 253.9 | 4.359* |
| 5 | $fd_2 > 8.770$ | 26 | 444.6 | 0.4358 |
| 10 | $fd_1 < 17.29$ | 17 | 225.5 | -1.5580* |
| 11 | $fd_1 > 17.29$ | 9 | 23.82 | 4.2020* |
| 3 | $fd_3 < 7.585$ | 25 | 209.6 | 7.0460* |

*Terminal node (leave) *Source: Computations using R language*

*Cost Complexity Pruning of Regression Tree on Real GDP growth rate and Financial Depth Regression Space*

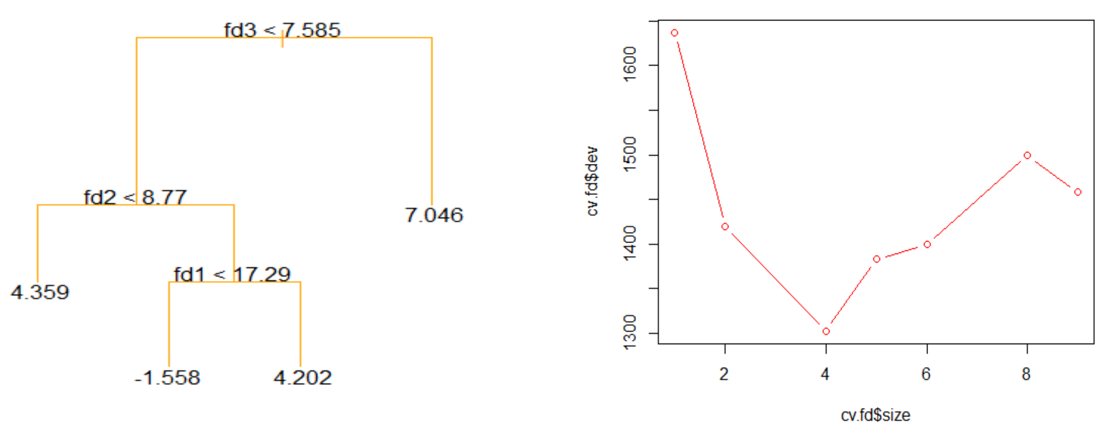


*Figure 3:* Analysis of cost complexity pruning on the fitted regression tree of Figure 2. Left panel: Pruned tree with 4 terminal nodes, 3 internal nodes. At a given internal the left-hand branch is represented by $X_j < q_k$ resulting from the split and $X_j > q_k$ indicates the right hand branch. At the top of the tree, the split resulted in two branches in which the left hand branch corresponds to $fd_3 < 7.585$ per cent and the right hand branch corresponds to $fd_3 \geq 7.585$ per cent. Right panel: The result of 10-fold cross validation showing the cross-validation error (cv.fd$dev) as a function of the terminal nodes (cv.fd$size). It indicates that the CV error dipped at terminal node value of 4 with MSE = 1,302.169. Source: Computations using R language

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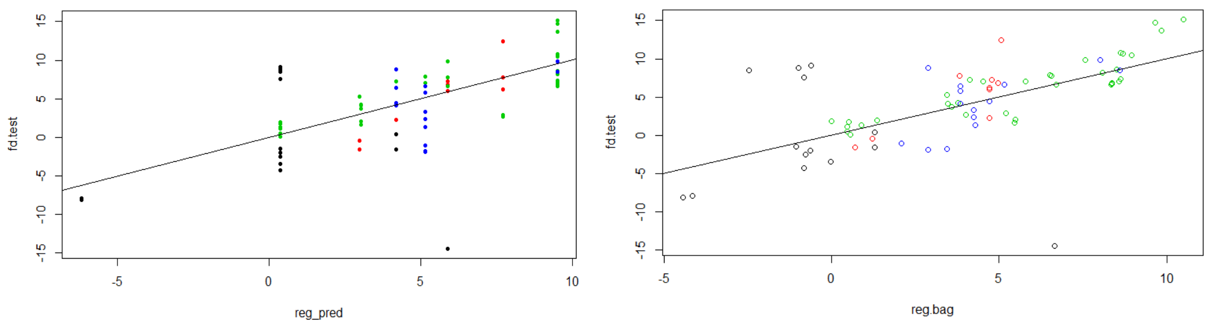


*Figure 4:* Left panel: The scatter plot on the residuals of the optimally-pruned regression tree using the test dataset. Right Panel: The scatter plot on the residuals of the aggregated bootstrapping on G = 200 regression trees using the test dataset (i.e. Bagging procedure). The straight line through the dotted points is the trend line. Source: Computation using R language

The pruned tree in Figure 3 reveals that out of the three regressors, stock market turnover ratio ($fd_3$) is the most important in predicting real economic growth. Using 25 training observations, given that $fd_3$ is greater or equal to 7.585 per cent, the mean of the observations on the real economic growth is 7.046 per cent which represent the first terminal node. The fitted regression was further split by credit to private sector ($fd_2$) given the stock market turnover ratio is less than 7.585 per cent. At this step, if $fd_2$ less than 8.77 per cent, a total of 21 training observations produced a mean response of 4.359 per cent. Otherwise another split was carried out which resulted in two terminal nodes. These two terminal nodes are -1.558 per cent if broad money supply relative to GDP ($fd_1$) is less than 17.29 per cent and 4.404 per cent otherwise. A total of 17 and 9 training observations were involved in calculating these two mean responses of the reg. using the result of this pruned tree, we predicted the real GDP growth rate for the 72 test observations to determine its performance. The mean square error (MSE) of the model using test dataset is 17.685 compared to the MSE of 34.785 using the train dataset.

*b) Bagging and Random Forest on the fitted Real GDP Regression tree model*

Decision trees are known to present some setbacks in terms of predictive accuracy (high variance) and lack of robustness to changes in data. Therefore, we employed bagging and random forests to improve the accuracy of the predictions generated by the regression tree model. Generally, bagging is a bootstrap aggregation procedure used in reducing the variance of a statistical learning method such as our fitted regression tree. It has been shown that averaging a set of observations reduces the variance. Bootstrapping involves taking single training dataset, generate $B = 400{,}000$ different bootstrapped regression trees and then trained our model on the $400{,}000^{th}$ bootstrapped regression tree. Using $G = 400{,}000$ subtrees, we computed

$$\hat{f}^{1}(x), \hat{f}^{2}(x)\ldots\ldots\ldots\ldots\ldots\hat{f}^{400{,}000}(x)$$

and an average prediction which gives

$$\hat{f}_{bag}(x) = \frac{1}{400{,}000}\sum_{g=1}^{400{,}000}\hat{f}^{*g}(x) \qquad 1.3$$

Because these trees are grown deep and unpruned, they possess higher variance but low bias. Hence averaging using equation 1.3 reduces the variance. The MSE of the bagging is 13.409 with 34.41 per cent explained variance. The performance of the regression tree model was verified using the test dataset. The MSE of the model using the test dataset is 18.292. In order to further reduce the test MSE of the regression model, we reduced the number of trees $G$ to 200 trees, the training MSE becomes 13.410 and test MSE equals 17.653 which is slightly better than the optimally-pruned single tree. The right panel of Figure 4 shows residual plot on the bagging procedure. Bagging removes any simple structure in the regression tree model due to extremely large number of trees to be bootstrapped. Thus, our bagged tree on reg model stops to be a tree. Another problem with bagging is that it gives priority to strong and moderately strong regressors at the top split during the bootstrapping and aggregation process.

Subsequently, we employed another powerful method called the *Random Forest*. Random Forest is an improvement over the bagging method in the sense that it reduces the extent of correlation between the sampled trees. The procedure on Random Forest like bagging uses regression trees as foundation. We constructed a large number of trees on bootstrapped training samples using a random sample of $m$ ¡ $p$ predictors at every split in the regression tree. Specifically, we set $m = p/3$ and since $p = 3$ regressors in our regression tree model, we use $m = 1$ regressor at every split in the tree. This procedure ensures that no strong or moderately strong regressor is given priority over weaker ones by considering only a subset of the regressors which eliminates high correlations. by this method, we found that two-third of the splits might not involve the strong predictors thereby giving the other regressors a chance of being involved in the tree building process. A total of $G = 500$ trees was used for the bootstrapping process. This regression tree model produced a training dataset MSE value of 11.629 and 43.12 per cent explained variation. Furthermore, the test dataset MSE was 16.41 which is an improvement over bagged regression model.

*Table 2:* Analysis of regressor importance

| Regressor | Accuracy (MSE) | Reduction in RSS |
|---|---|---|
| $fd_2$ | 9.180 | 425.052 |
| $fd_1$ | 16.615 | 438.143 |
| $fd_3$ | 10.347 | 450.300 |

*Source: Computations using R language*

Notes

Notes

*Plots on measures of regressor importance*

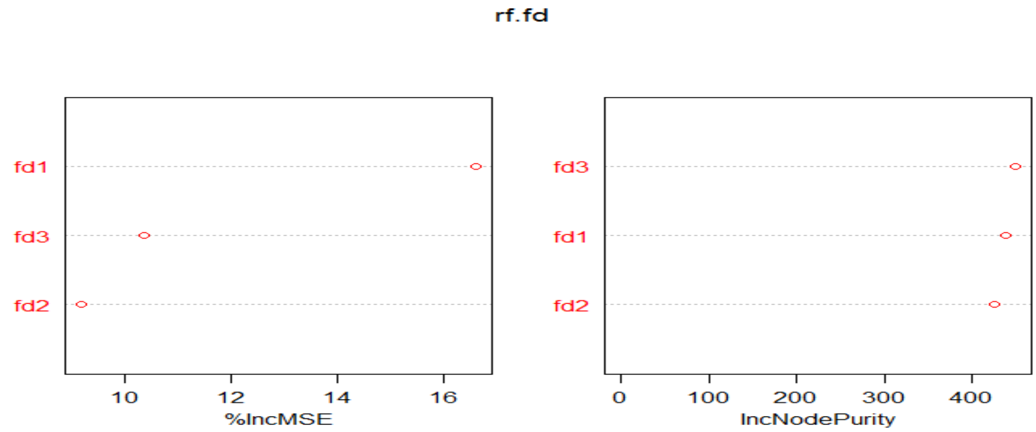


Figure 5: Left panel: Plot on the mean reduction in accuracy as measured by MSE in predictions on out of bag samples when a corresponding regressor is excluded from the model. In bagging, the bagged tree utilizes about two-third of observations. The remaining one-third of the observations not used up in the bagging procedure are referred to as out-of-bag (OOB) observations. These OOB observations are used to predict the response (reg) for the $i^{th}$ observations using each of the trees in which this observation is an OOB. Right panel: Plot on the decrease in node impurity that results from splits over that regressor averaged over all trees. Node impurity is measured by the RSS. This plot indicates that on 500 trees, the RSS decreased by 450.300 if the split is done over $fd_3$ at the top of the tree. Source: Computation using R language

*Partial Dependence Plots on Stock Market Turnover Ratio and Broad Money Supply Relative to GDP*

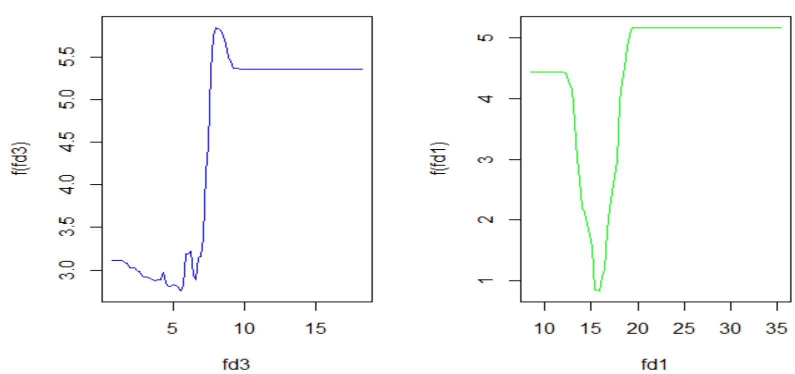


*Figure 6:* Plots showing the marginal effects of the $fd_3$ and $fd_1$ in the real GDP growth model. Left panel: This panel is the marginal effect of stock market turnover ratio ($fd_3$) on real GDP growth rate (reg). This plot shows that reg rises with increasing $fd_3$. reg rose sharply when $fd_3$ is about 6 per cent before stabilizing at values of $fd_3$ at 10 per cent or more. Right Panel: Marginal effect of the broad money supply relative to GDP ($fd_1$) on reg. Similar to the $fd_3$, this plot indicates a rising reg with increasing $fd_1$ and vice versa. Particularly, the reg dipped for values of $fd_1$ between 10 per cent and 15 per cent. The marginal effects of these regressors on reg were computed by integrating out other regressors. Source: Computations using R language

This procedure ensures that no strong or moderately strong regressor is given priority over weaker ones by considering only a subset of the regressors which eliminates high correlations. By this method, we found that two-third of the splits might not involve the strong predictors thereby giving the other regressors a chance of being involved in the tree building process. A total of $G = 500$ trees was used for the bootstrapping process. This regression tree model produced a training dataset MSE value of 11.629 and 43.12 per cent explained variation. Furthermore, the test dataset MSE was 16.41 which is an improvement over bagged regression model.

Notes

## III. Conclusion

Decision tree for regression has been used to study the relationship between real gross domestic growth rate and three major financial depth indicators such as turnover ratio as proxy for stock market liquidity, credit to private sector (CPS) and broad money supply (M2) relative to gross domestic product (GDP) in Nigeria between 1981 to 2016. The analysis of the tree was done using the recursive binary splitting (RBS), cost complexity pruning, bagging and random forest. Recursive Binary Splitting produced a tree with nine leaves or terminal nodes. Though this single large tree produced a low MSE for the model on the training dataset, its performance on the test dataset was weak. Hence, we applied the cost complexity pruning (CCP) to trim down the number of terminal nodes (leaves) and improve the predictive capacity alongside the interpretability of our model. This model resulted in lower model MSE for the test dataset and 4 terminal nodes. Stock market turnover ratio minimizes the residual sum of squares (RSS). Therefore, this indicator was used as the initial splitting variable at the top of the tree. The CCP procedure was based on varied values of a tuning parameter which was used to control the tradeoff between the complexity of the model and it fitting the data adequately. We employed bagging and random forest to further improve the predictive capacity of the postulated regression tree model. Bagging and Random Forest are procedure that involve growing large number of separate trees which made plotting a tree impossible. However, the results of these two procedures indicate improved performance of the regression tree model on the test data set in terms of lower MSE. Partial dependence analysis of the regressors in terms of node purity indicates that the real GDP growth rate rises with stock market turnover ratio. It dipped for some values of broad money supply relative to GDP during the period under review. The fitted regression tree shows that stock market liquidity and broad money supply relative to GDP are the most important financial depth variables imparting the growth of real GDP in Nigeria during the period under review.

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# Signifficance of Mathematical Equations in Electroanalytical Techniques

By Pinnamreddy Sreeharireddy & Sarvareddy Rajasekhar Reddy

*Abstract-* Importance of mathematical equations was discussed in this effort. Mathematical formulae calculated instrumental out puts and experimental out comes from various electro analytical techniques such as cyclic voltammetry, stripping voltammetry and controlled potential electrolysis are mentioned in this study. In addition to that statistical equations used in estimations and determinations carried by standard addition method and differential methods also discussed.

*Keywords:* *electronalytical techniques, standard addition method, differential methods.*

*GJSFR-F Classification:* *FOR Code: MSC 2010: 83C05*

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Ref

1. M.S. El-Shahawi and M.M. Kamal, Fresenius J. Anal. Chem., 313, 1982, 390.

# Signifficance of Mathematical Equations in Electroanalytical Techniques

Pinnamreddy Sreeharireddy [α] & Sarvareddy Rajasekhar Reddy [σ]

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## I. Introduction

Many modern analytical methods are widely employed for determining active ingredient in formulations, and minute quantities of pesticide residues in various environmental samples. Although each pesticide requires a specific procedure, the following techniques are widely used for the detection and determination of pesticides individually or sometimes combinely. These techniques are:

1. Spectrophotometry
2. Fluorescence spectrophotometry
3. Chromatography
4. Radiochemical Methods
5. Electrochemical Methods

In this article application of mathematical formulae in Electrochemical Methods was discussed in this article.

## II. Discussion

### a) *In Measurments*

In the standard addition method, the voltammogram of the unknown is first recorded after which a known volume of standard solution of the same electroactive species is added to the cell and second voltammogram is taken. From the magnitude of the peak height, the unknown concentration of species may be calculated using the equation[1].

$$Cu = \frac{C_s \, x \, V}{V_t \, x \, i_2} \, x \, i_1$$

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where,
$i_1$=the observed maximum current of voltammogram in microamperes of unknown solution.
$i_2$=the maximum current of the voltammogram after adding volume of V ml of unknown concentration.
$C_s$=concentration of the standard solution in mM
$C_u$=concentration of the unknown solution in mM
$V_t$=total volume of the solution (V + v)

Notes

## III. Cyclic Voltammetry

Cyclic voltammetry is one of the most exploited techniques in electrochemical studies. Its primary advantage comes from the fact that it gives insight into both the half-reactions taking place at the working electrode, providing at the same time information about the chemical or physical phenomena coupled to the studied electrochemical reaction. Hence cyclic voltammetry is often considered as electrochemical spectroscopy. Although its usage is relatively minimal in quantitative food analysis, it is important to elaborate the principles of cyclic voltammetry, since every electroanalytical study almost inevitably commences with this technique. In cyclic voltammetry, starting from an initial potential Ei, a staircase potential sweep (or linear sweep in older potentiostats) is applied to the working electrode. After reaching a switching potential Ef, the sweep is reversed and the potential returns to its initial value. The main instrumental parameter in the cyclic voltammetry is the scan rate, since it controls the timescale of the voltammetric experiment. The useful scan rates range from 1 to 1000 mV=s, although scan rates of over 10 V=s are technically achievable. The instrumental output in cyclic voltammetric techniques is a current–potential curve, a cyclic voltammogram . The main features of the cyclic voltammogram are the cathodic and anodic peak potentials, the cathodic and anodic peak currents, and the formal (or half-peak) potential. While the half-peak potential (defined simply as a median between the cathodic and the anodic peak potentials) provides mainly thermodynamics information, the magnitudes of the peak currents reveal the kinetics involved in the electrochemical reaction. The shape of the cyclic voltammogram gives information about the type of the electrode reaction, the number of electrons involved in the elementary step of electrochemical transformation, as well as about the additional phenomena coupled to the electrochemical reaction of interest, like those for coupled chemical reactions or adsorption and crystallization. If the electron transfer process is much faster than the kinetics of the mass transport processes (diffusion), then the electrode reaction is electrochemically reversible. In this case, the peak separation DEp is defined as follows:

$$DEp \quad = (Ep,c\text{-}Ep,a) = 2.303\ RT/nF$$

For example, in a simple reversible and diffusion-controlled electrochemical reaction, where one electron is exchanged in an elementary act, the peak separation should be about 59 mV (at 258C). Moreover, the peak potential separation should not vary by increasing the scan rate, while both cathodic and anodic peak currents should be a linear function of the square root of the scan rate. Every breach of these criteria means deflection of the electrochemical reversibility, caused either by the slow electron transfer (quasi-reversibility or irreversibility) or by additional involvement of the electroactive species in chemical reactions or adsorption phenomena. In this technique, the potential applied between the working electrode and the reference electrode is varied with time in known fashion in a triangular sweep mode. The forward or cathode potential sweep gives a reduction wave where as backward or anodic potential sweep gives an oxidation wave. Generally linear diffusion conditions are employed in cyclic voltammetry. Fast scan rates minimize diffusion problems.

This technique is based on varying the applied potential at working electrode in both forward and reverse directions (at some scan rate) while monitoring the current for example. The initial scan could be in the negative direction to the switching potential. At the point, the scan would be reversed and run in the positive direction. Depending on the analysis, one or more potential cycles can be performed, hence the term 'Cyclic Voltammetry'. The potential of working electrode is controlled verses a reference electrode such as saturated calomel electrode or silver electrode. The controlling potential, which is applied across these two electrodes, can be considered in excitation signal. The excitation signal for cyclic voltammetry is liner potential scan with triangular wave form. If only a single anodic/cathodic sweep is performed the technique usually called linear potential sweep voltammetry.

Matheson and Nichols[2] established this technique and later Sevcik[3] developed it. A series of papers by Nicholson and Shain[4] did present a detailed development of the theory of cyclic voltammetry. Randles[5] worked out the utility of the technique for reversible processes and Delahay[6] for irreversible processes. A description of triangular wave generator was given by Kaufman et al.[7]The peak current in a reversible process is quantitatively expressed by[8-9]

$$i_p = KACD^{1/2}\ V^{1/2}\ n^{3/2} \tag{1}$$

where,

$i_p$ = peak current in microamperes
K = Randles-Sevcik constant
A = area of working electrode in $cm^2$
C = concentration of the depolarizer in mM
D = diffusion coefficient of electroactive species in $cm^2s^{-1}$
V = scan rate in $mVs^{-1}$
n = number of electrons

Delahay[10] indicated 2.75 x $10^5$ as the reliable value for the 'K' in the case of reversible process.
For an irreversible processes[11]

$$i_p=3.01 \times 10^5\, n\ (\alpha\ na)^{1/2}\ AD^{1/2}\ CV^{1/2} \tag{2}$$

where,

$\alpha$ = transfer coefficient.
$n_a$ = number of electrons involved during the rate determining step and other terms have their usual significance.

The above equations (1 and 2) are used for evaluating diffusion coefficient values of reversible and irreversible electrode processes respectively.
The nature of the irreversible process is known from following equations[12]

(i) $E_{p/2} - E_p = 0.0565/n$ volts (3)

(ii) $(E_p)_{anodic} - (E_p)_{cathodic} = 0.058/n$ volts (4)

Ref

2. S.J. Reddy, P. Valenta and H.W. Nurnberg, Fresenius, J. Anal. Chem., 312, 1982, 390.

where, $E_p$ = peak potential in volts

$E_{p/2}$ = half-peak potential in volts

Any deviation from the above equations (3 and 4) leads to irreversible nature of the electrode process. Absence of anodic signal on the reverse scan indicates irreversible nature of the electrode process.

$\alpha_{na}$ values can be evaluated from the equation[13]

$$E_{p/2} - E_p = 0.048/\alpha_{na} \text{ volts} \tag{5}$$

The forward rate constant values for all irreversible processes can be evaluated using the equation

$$E_p = \frac{-1.14RT}{\alpha n_a F} + \frac{RT}{\alpha n_a F} \ln \frac{k^0_{f,h}}{D^{1/2}} - \frac{RT}{2\alpha n_a F} \ln(\alpha n_a v) \tag{6}$$

Where, all the terms have their usual significance.

The most useful aspects of cyclic voltammetry is, its application to the qualitative diagnostic of the electrode reactions which are coupled to the homogeneous chemical reactions. Cyclic voltammetry provides a particularly convenient means to study adsorption phenomena in detail. For adsorption controlled waves, the current function ($i_p$ / $Cv^{1/2}$) increased rapidly with an increase in scan rate. Wopschall and Shain[14] have studied the nature of effects of adsorption processes with the aid of cyclic voltammetry. If the product or reactant is strongly adsorbed on the surface of the electrode, a separate adsorption peak will appear prior to or after the normal peak respectively.[15-16] Cyclic voltammetric studies have been useful in deciding EC,CE and catalytic mechanisms. It has several applications in pesticide analysis.

## IV. Adsorptive Stripping Voltammetry

For quantitative determination of depolarizer using DPP, AdSV either standard method or calibration method can be employed. The conditions in adsorptive stripping voltammetry are rather complicated as many factors concerning adsorption of the species on the electrode surface are to be taken into account. The factors include the type of adsorption and the respective isotherm, diffusion conditions, type of electrode reaction of the adsorbed species.

For fast, diffusion controlled adsorption, which occurs must often in adsorptive accumulation, where during the accumulation period the mass transport is assumed under the limiting current condition. The following relationship was derived for the peak current for the reduction of the compound at working electrode assuming that $I_p$ is a function of the surface concentration:

$$I_p = kAT \tag{7}$$

$$I_p = KAC/(D/r)\ t_{acc} + 2\ (D/\Lambda)^{1/2}\ t_{acc}{}^{1/2} \tag{8}$$

where, k=proportionality constant

A=electrode area

I=surface concentration of the compound

C=concentration of the compound

D=diffusion coefficient

Ref

14. P. Zuman and M. Brezina "D.C. Polarography in Der. Medizein, Biochemic and Pharamazie" AkademicSche, Leipig, 1956.

r=radius of working electrode

$t_{acc}$=accumulation period

at large values of C and / or $t_{acc}$, $I_p$ approaches a limiting value, for which it is assumed that

$$I_{p\ max} = K A \Gamma m \tag{9}$$

for complete coverage eq. has derived by Koryta[17]

$$\Gamma m = 7.36 \times 10^{-4}\ C D^{1/2}\ t^{1/2} \tag{10}$$

where, 't' is the time required for complete electrode coverage. It has been found that $I_p$ increase linearly with $t_{acc}^{1/2}$ (assuming that there is no interaction between the adsorbed molecules). $I_p$ is proportional to the product of C and $t_{acc}^{1/2}$, when neither of these two values is too large. The linear dependence of $I_p$ on scan rate is predicted by the equation.

$$I_p = n^2\ F^2\ \Gamma v / 4RT \tag{11}$$

That was derived by Brown and Amson[18] this is a proof of adsorption, in contrast to the $v^{1/2}$ dependence valid for pure Faradic process. Concerning adsorption parameters, for the concentration dependence of the voltammetric signal in the region of small concentration levels, Novotny[19] derived the equation.

$$I = K_2\ \Gamma m\ \beta c - K_3\ \Gamma \beta c^2 \tag{12}$$

and thus, for the slope I/C

$$I/C = K_2 \beta\ \Gamma m - K_3\ \Gamma m\ \beta c \tag{13}$$

where, $K_2$ and $K_3$ are proportionality constants and Γm denotes the maximum coverage of the electrode. The slope I/C can be considered therefore as a function of the adsorption coefficient β,

$$I/C = F(\beta) \tag{14}$$

and the adsorptivity expressed by the slope as a parameter typical for the compound under study.

Wenrui Jin[20] has derived equations for the dependence of $I_p$ on parameters such as concentration of the analyte and accumulation time for application of micro electrode. These electrodes due to their large edge effect have a uniquely enhanced mass transport characteristic and exhibit lower iR drop. It has been found that the dependences of the peak current on voltage scan rate, accumulation time in quiescent solution, electrode radius (mercury ultramicro electrode) and bulk concentration of adsorbed substances were in agreement with theoretical prediction.

Thus, for a reversible process, equation was derived

$$I_p = (n^2\ F^2\ /\ 4RT)\ ADv\ t_{acc}\ c/r \tag{15}$$

and for an irreversible process

$$I_p = (n^2\ F^2 / e\ 4RT)\ \alpha\ ADv\ t_{acc}\ c/r \tag{16}$$

Where, e is the base of natural logarithms, r is the radius of the working electrode. V is the scan rate and α is the transfer coefficient.

Ref

18. T. Devries and T.L. Kroon, J. Amer. Chem. Soc., 75, 1953, 2484.

## V. Millicoulometry

There is a growing interest regarding the number of electrons 'n' involved per molecule during the reaction at the carbon nano tubes paste electrode.[21] The evaluation of number of electrons 'n' by the quantitative reduction of a known amount of electroactive species may be served by incomplete reduction of a voltammetric indicator electrode.

Devries and Kroon[21-22] employed this technique containing a known volume of a standard solution of a substance for which 'n' is known. When current is passed through two voltammetric cells in series, an equivalent amount of electrochemical reaction will occur in each cell. The change in concentration in each cell can be determined from the decrease in wave height and a comparison of these changes leads to determination of the number of electrons used in a unit of reaction for an unknown substance in terms of known 'n' in one of the cells. The amount of a material, which is reduced or oxidised, is given by

$$\Delta w = \left[1 - \frac{i_2}{i_1}\left[\frac{t_1}{t_2}\right]^{1/6}\right] N \tag{17}$$

by evaluating $\Delta w$ for the two cells under consideration and taking into account of the known 'n' of the unknown electrode process can easily be calculated. By using the equation (17) the number of electrons participated in the electrode process is evaluated.

## VI. Conclusion

This Work Is Very Useful for Instrumental Interpretations For one who Deals With Electro analytical Measurements based on oxidation and reduction process takes place at electrodes and electrodynamics.

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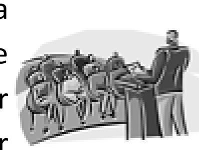

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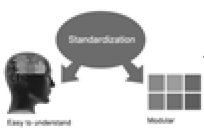


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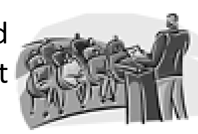

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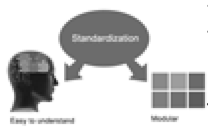


After nomination of your institution as “Institutional Fellow” and constantly functioning successfully for one year, we can consider giving recognition to your institute to function as Regional/Zonal office on our behalf.
The board can also take up the additional allied activities for betterment after our consultation.

**The following entitlements are applicable to individual Fellows:**

Open Association of Research Society, U.S.A (OARS) By-laws states that an individual Fellow may use the designations as applicable, or the corresponding initials. The Credentials of individual Fellow and Associate designations signify that the individual has gained knowledge of the fundamental concepts. One is magnanimous and proficient in an expertise course covering the professional code of conduct, and follows recognized standards of practice.


Open Association of Research Society (US)/ Global Journals Incorporation (USA), as described in Corporate Statements, are educational, research publishing and professional membership organizations. Achieving our individual Fellow or Associate status is based mainly on meeting stated educational research requirements.

Disbursement of 40% Royalty earned through Global Journals : Researcher = 50%, Peer Reviewer = 37.50%, Institution = 12.50% E.g. Out of 40%, the 20% benefit should be passed on to researcher, 15 % benefit towards remuneration should be given to a reviewer and remaining 5% is to be retained by the institution.

We shall provide print version of 12 issues of any three journals [as per your requirement] out of our 38 journals worth $ 2376 USD.

**Other:**

**The individual Fellow and Associate designations accredited by Open Association of Research Society (US) credentials signify guarantees following achievements:**

- The professional accredited with Fellow honor, is entitled to various benefits viz. name, fame, honor, regular flow of income, secured bright future, social status etc.

- In addition to above, if one is single author, then entitled to 40% discount on publishing research paper and can get 10%discount if one is co-author or main author among group of authors.
- The Fellow can organize symposium/seminar/conference on behalf of Global Journals Incorporation (USA) and he/she can also attend the same organized by other institutes on behalf of Global Journals.
- The Fellow can become member of Editorial Board Member after completing 3yrs.
- The Fellow can earn 60% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.
- Fellow can also join as paid peer reviewer and earn 15% remuneration of author charges and can also get an opportunity to join as member of the Editorial Board of Global Journals Incorporation (USA)
- • This individual has learned the basic methods of applying those concepts and techniques to common challenging situations. This individual has further demonstrated an in–depth understanding of the application of suitable techniques to a particular area of research practice.

**Note :**

"

- In future, if the board feels the necessity to change any board member, the same can be done with the consent of the chairperson along with anyone board member without our approval.
- In case, the chairperson needs to be replaced then consent of 2/3rd board members are required and they are also required to jointly pass the resolution copy of which should be sent to us. In such case, it will be compulsory to obtain our approval before replacement.
- In case of "Difference of Opinion [if any]" among the Board members, our decision will be final and binding to everyone.

"

# Preferred Author Guidelines

**We accept the manuscript submissions in any standard (generic) format.**

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

## Before and during Submission

Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

1. Authors must go through the complete author guideline and understand and *agree to Global Journals' ethics and code of conduct,* along with author responsibilities.
2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
5. Authors should submit paper in a ZIP archive if any supplementary files are required along with the paper.
6. Proper permissions must be acquired for the use of any copyrighted material.
7. Manuscript submitted *must not have been submitted or published elsewhere* and all authors must be aware of the submission.

**Declaration of Conflicts of Interest**

It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

## Policy on Plagiarism

Plagiarism is not acceptable in Global Journals submissions at all.

Plagiarized content will not be considered for publication. We reserve the right to inform authors' institutions about plagiarism detected either before or after publication. If plagiarism is identified, we will follow COPE guidelines:

Authors are solely responsible for all the plagiarism that is found. The author must not fabricate, falsify or plagiarize existing research data. The following, if copied, will be considered plagiarism:

- Words (language)
- Ideas
- Findings
- Writings
- Diagrams
- Graphs
- Illustrations
- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

## Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

**Changes in Authorship**

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

**Copyright**

During submission of the manuscript, the author is confirming an exclusive license agreement with Global Journals which gives Global Journals the authority to reproduce, reuse, and republish authors' research. We also believe in flexible copyright terms where copyright may remain with authors/employers/institutions as well. Contact your editor after acceptance to choose your copyright policy. You may follow this form for copyright transfers.

**Appealing Decisions**

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

**Acknowledgments**

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

**Declaration of funding sources**

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

## Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

***Manuscript Style Instruction (Optional)***

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

***Structure and Format of Manuscript***

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

## Format Structure

***It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.***

All manuscripts submitted to Global Journals should include:

**Title**

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

**Author details**

The full postal address of any related author(s) must be specified.

**Abstract**

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

**Keywords**

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

**Numerical Methods**

Numerical methods used should be transparent and, where appropriate, supported by references.

**Abbreviations**

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

**Formulas and equations**

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

**Tables, Figures, and Figure Legends**

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

**Figures**

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

## Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

## Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

***1. Choosing the topic:*** In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

***2. Think like evaluators:*** If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

***3. Ask your guides:*** If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

***4. Use of computer is recommended:*** As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

***5. Use the internet for help:*** An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

***6. Bookmarks are useful:*** When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

***7. Revise what you wrote:*** When you write anything, always read it, summarize it, and then finalize it.

***8. Make every effort:*** Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

***9. Produce good diagrams of your own:*** Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

***10. Use proper verb tense:*** Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

***11. Pick a good study spot:*** Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

***12. Know what you know:*** Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

***13. Use good grammar:*** Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

***14. Arrangement of information:*** Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

***15. Never start at the last minute:*** Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

***16. Multitasking in research is not good:*** Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

***17. Never copy others' work:*** Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

***18. Go to seminars:*** Attend seminars if the topic is relevant to your research area. Utilize all your resources.

***19. Refresh your mind after intervals:*** Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

***20. Think technically:*** Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

***21. Adding unnecessary information:*** Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

***22. Report concluded results:*** Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

***23. Upon conclusion:*** Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

## Informal Guidelines of Research Paper Writing

***Key points to remember:***

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

**Final points:**

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

*The introduction:* This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

**The discussion section:**

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

**General style:**

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

**To make a paper clear:** Adhere to recommended page limits.

*Mistakes to avoid:*

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

**Title page:**

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

**Abstract:** This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

*Reason for writing the article—theory, overall issue, purpose.*

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

**Approach:**

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

**Introduction:**

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

*The following approach can create a valuable beginning:*

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

**Approach:**

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

**Procedures (methods and materials):**

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

**Materials:**

*Materials may be reported in part of a section or else they may be recognized along with your measures.*

**Methods:**

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

**Approach:**

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

**What to keep away from:**

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

**Results:**

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

**Content:**

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

**What to stay away from:**

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

**Approach:**

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

**Figures and tables:**

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

**Discussion:**

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

**Approach:**

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

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| ***Methods and Procedures*** | Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads | Difficult to comprehend with embarrassed text, too much explanation but completed | Incorrect and unorganized structure with hazy meaning |
| ***Result*** | Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake | Complete and embarrassed text, difficult to comprehend | Irregular format with wrong facts and figures |
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| ***References*** | Complete and correct format, well organized | Beside the point, Incomplete | Wrong format and structuring |

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