Online ISSN : 2249-4626
Print ISSN : 0975-5896
DOI : 10.17406/GJSFR

Global Journal of Science Frontier Research: F

Mathematics and Decision Science

Highlights

- Laplace Transform of Fourier
- Automata based on Lattice-Ordered
- Decomposition of Fuzzy Automata
- A Numerical Approach to the Solution

Discovering Thoughts, Inventing Future

VOLUME 18 ISSUE 8 VERSION 1.0

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences

Global Journal of Science Frontier Research: F
Mathematics & Decision Sciences
Volume 18 Issue 8 (Ver. 1.0)

Open Association of Research Society

Incorporation No.: 0423089
License No.: 42125/022010/1186
Registration No.: 430374
Import-Export Code: 1109007027
Employer Identification Number (EIN):
USA Tax ID: 98-0673427

Global Journals Inc.

*(A Delaware USA Incorporation with "Good Standing"; **Reg. Number: 0423089**)*

Sponsors: Open Association of Research Society
Open Scientific Standards

Publisher's Headquarters office

Global Journals® Headquarters
945th Concord Streets,
Framingham Massachusetts Pin: 01701,
United States of America
USA Toll Free: +001-888-839-7392
USA Toll Free Fax: +001-888-839-7392

Offset Typesetting

Global Journals Incorporated
2nd, Lansdowne, Lansdowne Rd., Croydon-Surrey,
Pin: CR9 2ER, United Kingdom

Packaging & Continental Dispatching

Global Journals Pvt Ltd
E-3130 Sudama Nagar, Near Gopur Square,
Indore, M.P., Pin:452009, India

Find a correspondence nodal officer near you

To find nodal officer of your country, please email us at *local@globaljournals.org*

eContacts

Press Inquiries: *press@globaljournals.org*
Investor Inquiries: *investors@globaljournals.org*
Technical Support: *technology@globaljournals.org*
Media & Releases: *media@globaljournals.org*

Pricing (Excluding Air Parcel Charges):

Yearly Subscription (Personal & Institutional)
250 USD (B/W) & 350 USD (Color)

Editorial Board

Global Journal of Science Frontier Research

Contents of the Issue

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Decomposition of Fuzzy Automata based on Lattice-Ordered Monoid

By Anupam K. Singh

Abstract- The purpose of the present work is to introduce decomposable concepts of fuzzy automaton having the membership values in the lattice-ordered monoids. Unlike to the usual fuzzy automata, we show that such concepts for fuzzy automata having the membership values in lattice-ordered monoids by the associated monoid structure.

Keywords: *lattice-ordered monoid; fuzzy automaton; primary; f-primary; decomposition.*

GJSFR-F Classification: *MSC 2010: 06D72*

DECOMPOSITIONOFFUZZYAUTOMATABASEDONLATTICEORDEREDMONOID

Strictly as per the compliance and regulations of:

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version 1 Year 2018

Decomposition of Fuzzy Automata based on Lattice-Ordered Monoid

Anupam K. Singh

Abstract- The purpose of the present work is to introduce decomposable concepts of fuzzy automaton having the membership values in the lattice-ordered monoids. Unlike to the usual fuzzy automata, we show that such concepts for fuzzy automata having the membership values in lattice-ordered monoids by the associated monoid structure.

Keywords: *lattice-ordered monoid; fuzzy automaton; primary; f-primary; decomposition.*

I. Introduction And Preliminaries

The study of fuzzy automata was initiated by [16] and [26] in 1960's after the introduction of fuzzy set theory by [27]. Much later, a considerably simpler notion of a fuzzy finite state machine (which is almost identical to a fuzzy automaton) was introduced by [10] (cf. [11], for more details). Somewhat different notions were introduced subsequently by [7, 8, 13]. Recently, Jun [4, 5, 6] generalized the concept of fuzzy finite state machine corresponding to one higher order fuzzy sets, viz., the intuitionistic fuzzy sets, and called it intuitionistic fuzzy finite state machine. In these studies, the membership values in the closed interval [0, 1] were considered. During the recent years, the researchers were initiated to work with fuzzy automata with membership values in complete residuated lattices, lattice ordered monoids and some other kind of lattices (cf., [3, 9, 14, 15]Anu5, Anu6).

In this paper, we study the decomposable properties of fuzzy automata with membership values in lattice ordered monoid via their primaries. We show that several results related to the decomposable properties of fuzzy automata introduced in [19, 20, 21, 22] may not hold well in the case of fuzzy automata with membership values in lattice ordered monoid. This paper is organised as follows: In section 1, we recall some notions related to lattice ordered monoid, monoid with and without zero divisors and some known results which are used in the paper. In Section 2, we introduce and study the concept of a primaries of a fuzzy automata and a compact fuzzy automata and their primary decomposition having the membership values in lattice ordered monoid. In Section 3, we study the primary decomposition and f-primary decomposition for an L-fuzzy automaton.

We recall the following from [2, 19, 23].

Definition 1.1 *An algebra* $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *is called a* **lattice-ordered monoid** *if*

1. $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ *is a lattice with the least element 0 and the greatest element 1,*
2. $(L, \bullet, e)$ *is a monoid with identity* $e \in L$ *such that for all* $a, b, c \in L$
   (i) $a \bullet 0 = 0 \bullet a = 0$,

Author: *Janta Koshi College, Biraul A Constituent Unit of Lalit Narayan Mithila University Darbhanga, Bihar-847203, India.*
e-mail: anupam09.bhu@gmail.com

Ref

10. D.S. Malik, J.N. Mordeson, M.K. Sen, Submachines of fuzzy finite state machine, *Journal of Fuzzy Mathematics*, 2 1994) 781-792.

(ii) $a \leq b \Rightarrow \forall x \in L, a \bullet x \leq b \bullet x$ *and* $x \bullet a \leq x \bullet b$,

(iii) $a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$ *and* $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

Definition 1.2 *A monoid* $(L, \bullet, e)$ *is called* **monoid without zero divisors** *if for all* $a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \bullet b \neq 0$.

Definition 1.3 *A monoid* $(L, \bullet, e)$ *is called* **monoid with zero divisors** *if for all* $a, b \in L, a \neq 0, b \neq 0 \Rightarrow a \bullet b = 0$.

Definition 1.4 *Let* L *be an lattice-ordered monoid. An* L**-fuzzy automaton** *is a triple* $M = (Q, X, \delta)$, *where* Q *is a nonempty set (of* **states** *of* M*),* X *is a monoid (the* **input monoid** *of* M*), whose identity shall be denoted as* e_X, *and* $\delta : Q \times X \times Q \to L$ *is a map, such that* $\forall q, p \in Q, \forall x, y \in X$,

$$\delta(q, e_X, p) = \begin{cases} e & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

$$\text{and} \quad \delta(q, xy, p) = \vee\{\delta(q, x, r) \bullet \delta(r, y, p) : r \in Q\}.$$

Definition 1.5 *Let* (Q, X, δ) *be an* L*-fuzzy automaton and* $A \subseteq Q$. *The* **source**, *the* **successor** *and the* **core** *of* A *are respectively the sets*

$$\sigma_Q(A) = \{q \in Q : \delta(q, x, p) > 0, \text{ for some } (x, p) \in X \times A\}, \text{ and}$$
$$s_Q(A) = \{p \in Q : \delta(q, x, p) > 0, \text{ for some } (x, q) \in X \times A\}.$$

We shall frequently write $\sigma_Q(A), s_Q(A)$ as just $\sigma(A), s(A)$and $\sigma(\{q\})$ and $s(\{q\})$ as just $\sigma(q)$ and $s(q)$.

Definition 1.6 *The* **core** *of any subset* R *of the state-set* Q *of an* L*-fuzzy automaton is the set*

$$\mu(R) = \{q \in Q : \sigma(q) \subseteq R\}.$$

We shall frequently write $\mu(\{q\})$ as just $\mu(q)$.

Proposition 1.1 *Let* $(L, \bullet, e)$ *be a monoid without zero divisors and* (Q, X, δ) *be an* L*-fuzzy automaton. Then for all* $A \subseteq Q, s(s(A)) = s(A)$ *and hence* $\sigma(\sigma(A)) = \sigma(A)$.

Remark 1.1 *Let* $M = (Q, X, \delta)$ *be an* L*-fuzzy automaton and* $p, q, r \in Q$. *Then* $p \in \sigma(q), q \in \sigma(r) \not\Rightarrow p \in \sigma(r)$.

Proposition 1.2 *Let* $(L, \bullet, e)$ *be a monoid without zero divisors and* (Q, X, δ) *be an* L*-fuzzy automaton. Then for all* $p, q, r \in Q$. *Then* $p \in \sigma(q), q \in \sigma(r) \Rightarrow p \in \sigma(r)$.

Definition 1.7 *Let* $M = (Q, X, \delta)$ *be an* L*-fuzzy automaton.* $R \in L^Q$ *is called an* L*-fuzzy subautomaton of* M *if* $s(R) \leq R$ *and* $\lambda = \delta|_{R \times X \times R}$.

A fuzzy automaton $M = (Q, X, \delta)$ is retrievable if $\forall p, q \in Q, q \in \sigma(p) \Rightarrow p \in \sigma(q)$ and strongly connected if $\forall q, p \in Q, q \in s(p)$.

II. Compact L-Fuzzy Automata And Their Primary Decomposition

In this section, we introduce the concept of a primary of an L-fuzzy automaton $M = (Q, X, \delta)$ and provide its topological interpretation through the concept of a *regular closed set* in topology, as introduced in [19, 20]. Also, we introduce the concept of a primary decomposition of an L-fuzzy automaton $M = (Q, X, \delta)$ and provide state-set topologies are compact (cf. [17] for a similar observation).

Definition 2.1 *A closed subset of topological space is called* **regular closed** *if it is equal to the closure of its interior*

Definition 2.2 *A subset* $R \subseteq Q$ *is called*

(i) **genetic** *if* $\sigma(R) \subseteq s(R)$,

(ii) **genetically closed** *if* $\exists P \subseteq R$ *such that* $\sigma(P) \subseteq s(P)$ *and* $s(P) = R$,

Ref

17. A.K. Srivastava, W. Shukla, A topology for automata , *Information and control* A note, 32 (1976) 163-168.

(iii) **Gen**(Q) *is defined as,* $\{R : R \subseteq Q$ *and* $s(R) = Q\}$,

(iv) *a* **primary** *subset of* Q *if* R *is a nonempty minimal genetically closed subset of* Q.

Proposition 2.1 *Let* (R, X, λ) *be a primary subautomaton of an L-fuzzy automaton* (Q, X, δ). *Then* $s(\sigma(p)) = R, \forall p \in \mu(R)$.

Proof : The proof is similar, as given in [23].

Proposition 2.2 *If* $(L, \bullet, e)$ *be a monoid without zero divisors. Then for each* $q \in Q, s(\mu(q))$ *is a primary of* Q, *if* $\mu(q) \neq \phi$.

Proof: The proof is similar, as given in [23].

Remark 2.1 *If* $(L, \bullet, e)$ *be a monoid with zero divisors. Then* $s(q)$ *is not a primary of* Q. *Hence for each* $q \in Q, s(\mu(q))$ *is not a primary of* Q.

Remark 2.2 *From the above proposition it is clear that whenever* $s(p)$ *is regular closed, it must be minimal regular closed.*

Proposition 2.3 *Let* $M = (Q, X, \delta)$ *be an L-fuzzy automaton and* $p \in Q$. *Then as* $(L, \bullet, e)$ *be a monoid without zero divisors. The following statements are equivalent.*

(i) $s(p)$ *is a primary of* Q:

(ii) $s(p)$ *is a regular closed subset of* Q:

(iii) $\{p\}$ *is not a nowhere dense subset of* Q:

(iv) $s(p)$ *is a minimal regular closed subset of* Q:

(v) $p \in \mu(s(p))$.

Proof: The proof is similar, as given in [19].

Proposition 2.4 *Let* $R \subseteq Q$ *and* $(L, \bullet, e)$ *be a monoid without zero divisors. Then* $s(\sigma(R))$ *is the union of all primaries of* Q *which contains at least one member of* R.

Proof: The proof is similar, as given in [12].

Lemma 2.1 *Let* $Q = s(q)$ *and* $(L, \bullet, e)$ *is a monoid without zero divisors. Then* $\sigma(q) = g_1(Q)$, *where* $g_1(Q) = \{p : \{p\} \in Gen(Q)\}$.

Proof: Let $p \in \sigma(q)$, then $q \in s(p)$ and $(L, \bullet, e)$ is a monoid without zero divisors. Then $Q = s(q) \subseteq s(s(p)) \subseteq s(p) \subseteq Q$. Hence $Q = s(p)$ and so $\{p\} \in Gen(Q)$. Thus $\sigma(q) \subseteq g_1(Q)$. On the other hand, if $p \in g_1(Q)$, then $q \in s(p)$. Hence $p \in \sigma(q)$. Thus $g_1(Q) \subseteq \sigma(q)$.

Remark 2.3 *Let* $Q = s(q)$ *and* $(L, \bullet, e)$ *be a monoid with zero divisors. Then* $\sigma(q) \neq g_1(Q)$, *where* $g_1(Q) = \{p : \{p\} \in Gen(Q)\}$, *as the following counter-example shows.*

counter-example 2.1 *For the lattice-ordered monoid* L, *consider the monoid* $(L, \bullet, e)$, *where* $L = [0, 1], a \bullet b = max(0, a + b - 1), \forall a, b \in L$ *and* $e = 1$. *Consider a L-fuzzy automaton* $M = (Q, X, \delta)$, *where* Q *is the set of integers,* $X = \{0, 1, 2, ...\}$, *and* $\delta : Q \times X \times Q \to L$ *is given by*

$$\delta(m, 0, n) = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n, \end{cases}$$

$\forall m, n \in Q$, *and* $\delta(m_0, x_0, n_0) = 1/3, \delta(n_0, x_0, k_0) = 1/3, \delta(k_0, x_0, l_0) = 1/3$, *for fixed* $m_0, n_0, k_0, l_0 \in Q$ *and for fixed* $x_0 \in X(x_0 \neq 0)$. *For other* $m, n \in Q$ *and* $x \in X, \delta(m, x, n) = 0$. *Here, Let* $s(\{m_0\}) = \{n_0\} = Q$ *then* $\sigma(\{m_0\}) = \phi$. *Now if* $\{m_0\} \in g_1(Q)$. *Then* $\{n_0\} \in s(\{m_0\}) \Rightarrow \sigma(\{n_0\}) = \{m_0\} \neq \phi$. *Thus* $\sigma(\{m_0\}) \neq g_1(Q)$.

Proposition 2.5 *Let* R *be a non-empty genetic subset of* Q *and* $(L, \bullet, e)$ *be a monoid without zero divisors. Then* $s(R)$ *is the union of those primaries of* Q *which contains at least one member of* R.

Proof: Since R is genetic, $\sigma(R) \subseteq s(R)$ and $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(\sigma(R)) \subseteq s(s(R)) = s(R)$. Since $R \subseteq \sigma(R), s(R) \subseteq s(\sigma(R))$. Hence $s(\sigma(R)) = s(R)$. The result follows from Proposition 2.4.

Remark 2.4 *Let R be a non-empty genetic subset of Q and $(L, \bullet, e)$ be a monoid with zero divisors. Then $s(R)$ is not the union of those primaries of Q which contains at least one member of R, as the following counter-examples shows.*

counter-example 2.2 *Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{n_o\}$. Then $s(A) = \{n_o, k_o\}, \sigma(A) = \{m_o, n_o\}$, $\sigma(\sigma(A)) = \{m_o, n_o\}$ and $s(\sigma(A)) = \{m_o, n_o, k_o\}$. Then $\sigma(\sigma(A)) \subseteq s(\sigma(A))$. Thus $\sigma(A)$ is genetic and hence A be genetic subset of Q. But $s(\sigma(A)) \neq s(A)$.*

Lemma 2.2 *Let (R, X, λ) be a primary subautomaton of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ be a monoid without zero divisors. Then for every finite subset T of R, $T \subseteq s(r)$, for some $r \in R$.*

Proof: We prove this Lemma by induction. Let $T = \{p_1, p_2.....p_n\}$ be any finite subset of R. Then the result is obvious for $n = 1$. Now assume the result is true for $n = k-1$; in particular for $T_{k-1} = \{p_1, p_2.....p_{k-1}\}$. Then $\exists q \in R$ such that $T_{k-1} \subseteq s(q)$. Thus $T_k = \{p_1, p_2.....p_k\} \subseteq s(q) \cup \{p_k\} \subseteq s(\{q, p_k\})$. Put $S = \{q, p_k\}$ and let $m \in \mu(R)$. Then by Proposition 2.1. $R = s(\sigma(m))$. Note that $q \in R \Rightarrow q \in s(\sigma(m)) \Rightarrow \delta(m', x, q) > 0$, for some $(m', x) \in \sigma(m) \times X$ now as $(L, \bullet, e)$ is a monoid without zero divisors. Then $m' \in \sigma(m) \Rightarrow \sigma(m') \subseteq \sigma(\sigma(m)) = \sigma(m) \subseteq R$ (as $m \in \mu(R)$) $\Rightarrow m' \in \mu(R) \Rightarrow s(\sigma(m')) = R$ (by Proposition 2.1). Consequently, $p_k \in R \Rightarrow p_k \in s(\sigma(m')) \Rightarrow \delta(r, y, p_k) > 0$, for some $(r, y) \in \sigma(m') \times X$. From $r \in \sigma(m')$, we get $m' \in s(r)$, whereby $s(m') \subseteq s(s(r)) = s(r)$. This together with the facts that $q \in s(m')$ and $p_k \in s(r)$, gives $q, p_k \in s(r)$. Hence $S = \{q, p_k\} \subseteq s(r)$. Hence $T_k \subseteq s(r)$.

Remark 2.5 *Let (R, X, λ) be a primary subautomaton of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ be a monoid with zero divisors. Then for every finite subset T of R, $T \nsubseteq s(r)$, for some $r \in R$, as the following counter-example shows.*

counter-example 2.3 *Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{m_o, n_o, k_o\}$. Then $\sigma(\{n_o, k_o\}) = \{m_o, n_o\}$, $s(\sigma(\{n_o, k_o\})) = \{m_o, n_o, k_o\}$. Thus $s(\sigma(\{n_o, k_o\})) = A, \forall\{n_o, k_o\}) \in \mu(A)$, which shows that A be a primary of an L-fuzzy automaton Q. But for every finite subset $T = \{n_o\} \subseteq A$, $T = \{n_o\} \nsubseteq s(\{k_0\}) = \{l_0\}$, for some $\{k_0\} \in A$.*

Proposition 2.6 *If $(L, \bullet, e)$ be a monoid without zero divisors. Then a primary of a compact L-fuzzy automaton is a maximal singly generated subautomaton.*

Proof: Let N be a primary of a compact L-fuzzy automaton $M = (Q, X, \delta)$ and let $p \in R$. Then $p \in s(\mu(R))$. So $\exists q \in \mu(R)$ with $p \in s(q)$. As $q \in \mu(R), \sigma(q) \subseteq R$. Now $\sigma(q)$ is finite owing to the compactness of (Q, X, δ), so $\exists q' \in R$ such that $\sigma(q) \subseteq s(q')$ (by Lemma 3.1) and $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(\sigma(q)) \subseteq s(s(q')) = s(q')$. Also, $q \in s(q') \Rightarrow q' \in \sigma(q) \Rightarrow s(q') \subseteq s(\sigma(q'))$. Thus $s(\sigma(q')) = s(q')$, whereby $s(q')$ is a genetically closed subset of R (as $q' \in R$). So by the minimality of R, say $s(q') = R$. Hence the primary (R, X, λ) is singly generated. Let $S = s(t)$ be the state-set of another singly generated subautomaton of M such that $R = s(q') \subseteq S$. To prove that $R = S$. It is enough to show that $t \in s(q')$. Now $s(q') \subseteq S = s(t) \Rightarrow q' \in s(t) \Rightarrow t \in \sigma(q')$, so that $t \in s(q')$ (as $s(q')$ is a primary: cf. Proposition 2.5).

Remark 2.6 *If $(L, \bullet, e)$ be a monoid with zero divisors. Then a primary of a compact L-fuzzy automaton is not a maximal singly generated subautomaton.*

Notes

Proposition 2.7 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $R \subseteq Q$. Then $s(\sigma(R))$ can be written as the union of those primaries of Q, which have nonempty intersection with R.*

Proof: The proof is similar, as given in [19].

Proposition 2.8 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $R \subseteq Q$ be genetic with $(L, \bullet, e)$ be a monoid without zero divisors. Then $s(R)$ is the union of primaries of Q having nonempty intersection with R.*

Proof: As R be genetic, $\sigma(R) \subseteq s(R)$, implying that $s(\sigma(R)) \subseteq s(s(R)) = s(R)$, as $(L, \bullet, e)$ is a monoid without zero divisors. On the other hand , as $R \subseteq \sigma(R) \subseteq s(\sigma(R))$, we have $s(R) \subseteq s(\sigma(R))$. Thus $s(R) = s(\sigma(R))$. Hence by the Proposition 3.2. $s(R)$ is the union of primaries of Q having nonempty intersection with R.

Remark 2.7 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $R \subseteq Q$ be genetic with $(L, \bullet, e)$ be a monoid with zero divisors. Then $s(R)$ is not the union of primaries of Q having nonempty intersection with R, as the following counter-examples shows.*

counter-example 2.2 *Similar to counter-example 2.2.*

III. Primary Decomposition of an L-Fuzzy Automata

Finite state automata admit a primary decomposition (cf. e.g., [1]). Even an infinite state automaton can admit a primary decomposition, for example, when that automaton is compact (cf. [18]). In [19] we extended this for L-fuzzy automaton provided that $(L, \bullet, e)$ is a monoid without zero divisors. Also, we introduce the concept of a source-splitting sub-automaton, including their characterization and a topological description with $(L, \bullet, e)$ is a monoid without zero divisors. lastly, we introduce the concept of f-primaries of an L-fuzzy automaton $M = (Q, X, \delta)$ and provided $(L, \bullet, e)$ is a monoid without zero divisors.

Proposition 3.1 *An L-fuzzy auotmata M is strongly connected if and only if M has no proper subautomaton but if $(L, \bullet, e)$ be a monoid without zero divisors then the convere is true*

Proof: The proof is similar, as given in [23].

Lemma 3.1 *Let (Q, X, δ) be an L-fuzzy automaton and $q \in Q$ such that $R \subseteq s(\sigma(q))$ is a non-empty regular closed (genetically closed) subset of Q. Then $q \in R$ and $(L, \bullet, e)$ be a monoid without zero divisors.*

Proof: Let $p \in \mu(R) \subseteq R \subseteq s(\sigma(q))$. Then $\sigma(p) \subseteq R$ and $p \in s(\sigma(q))$. Now $\sigma(p) \subseteq R \Rightarrow s(\sigma(p)) \subseteq s(R) = R$. Also, $p \in s(\sigma(q)) \Rightarrow p \in s(t)$, for some $t \in \sigma(q) \Rightarrow t \in \sigma(p)$, for some t such that $q \in s(t)$. Again, $t \in \sigma(p) \Rightarrow \sigma(t) \subseteq \sigma(\sigma(p)) = \sigma(p)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Now $q \in s(t) \Rightarrow q \in s(\sigma(t)) \subseteq s(\sigma(p)) \subseteq R$. Thus, $q \in R$.

Remark 3.1 *Let (Q, X, δ) be an L-fuzzy automaton and $q \in Q$ such that $R \subseteq s(\sigma(q))$ is a non-empty regular closed (genetically closed) subset of Q and $(L, \bullet, e)$ be a monoid with zero divisors. Then $q \notin R$, as the following counter-example shows.*

counter-example 3.1 *Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{n_0\}$ and $\{k_0\} \in Q$. Then $\sigma\{k_0\} = \{n_0\}, s(\sigma\{k_0\}) = \{n_0, k_0\}$. Then $A \subseteq s(\sigma\{k_0\})$ is not a non-empty regular closed subset of Q, with $\{k_0\} \notin A$.*

Proposition 3.2 *[20] (Primary Decomposition Theorem) Let the L-fuzzy automaton $M = (Q, X, \delta)$ be compact (having possibly an infinite state-set Q). Then*

(i) $M = \cup_{i=1}^{n} P_i$, and

(ii) for any $j, 1 \leq j \leq n, M = \cup_{i=1, i \neq j}^{n} P_i$,

where $P_1, P_2, P_3, \ldots\ldots, P_n$ are all the distinct primaries of M.

Ref

19. A.K. Srivastava and S.P. Tiwari, A topology for fuzzy automata, in: Proc. 2002 *AFSS Internat. Conf. on Fuzzy Systems, Lecture Notes in Artificial Intelligance, Springer-verlag*, 2275 (2002) 485-490.

Proposition 3.3 *A L-fuzzy automaton $M = (Q, X, \delta)$ is retrievable if and only if M is decomposable and the primaries of M are strongly connected.*

Proof: The proof is similar, as given in [20].

Remark 3.2 *The converse of the above proposition is not true, if $(L, \bullet, e)$ be a monoid with zero divisors.*

Proposition 3.4 *Let $(L, \bullet, e)$ be a monoid without zero divisors M is decomposable and the primaries of M be strongly connected. Then M is retrievable.*

Proof: Let M be decomposable and the primaries of M be strongly connected. Also, let $p, q \in Q$ be such that $p \in \sigma(q)$ and as $(L, \bullet, e)$ be a monoid without zero divisors. Then $\sigma(p) \subseteq \sigma(\sigma(q)) = \sigma(q)$ and so $s(\sigma(p)) \subseteq s(\sigma(q))$. Since M is decomposable and $s(\sigma(p))$ is a regular closed subset of Q, $s(\sigma(p))$ should contain a primary subset of Q, say R. Now, $R \subseteq s(\sigma(p)) \subseteq s(\sigma(q)) \Rightarrow p, q \in R$(cf. Lemma 4.1). But as R is strongly connected, $p \in s(q)$, whereby $q \in \sigma(p)$. Thus, M is retrievable.

Definition 3.1 [20] *A subautomaton $N = (R, X, \lambda)$ of an L-fuzzy automaton $M = (Q, X, \delta)$ is called source-splitting in M if*

(i) R is genetically closed subset of Q, and

(ii) $\forall r \in R, \exists r_1, r_2 \in \sigma(r)$ such that $\sigma(r_1) \cap \sigma(r_2) = \phi$.

Lemma 3.2 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and let $\{P_i : i \in I\}$ be a family of all distinct primaries of M, with respective state-sets R_i. Then $s(Q - \cup_{i \in I} R_i)$ is a source-splitting in M.*

Proof: The proof is similar, as given in [20].

Lemma 3.3 *Let $N = (R, X, \lambda)$ be a source-splitting subautomaton of an L-fuzzy automaton $M = (Q, X, \delta)$ and $(L, \bullet, e)$ be a monoid without zero divisors . Then $\mu(R) \subseteq Q - \cup_{i \in I} R_i$, where $R_i^{,}s$ are in Lemma 4.2.*

Proof: To show that $\mu(R) \subseteq Q - \cup_{i \in I} R_i$, we show that $\mu(R) \cap R_i = \phi, \forall i \in I$. If possible, let $\mu(R) \cap R_i \neq \phi$, for some $i \in I$. Then $\mu(R) \cap s(\mu(R_i)) \neq \phi$ (since $s(\mu(R_i)) = R_i, \forall i \in I$). Now $q \in \mu(R) \cap s(\mu(R_i)) \Rightarrow \sigma(q) \subseteq R$ and $q \in s(t)$, for some $t \in \mu(R_i) \Rightarrow s(\sigma(q)) \subseteq s(R) = R$ and $q \in s(t)$, for some t with $\sigma(t) \subseteq R_i$. Also $\sigma(t) \subseteq R_i \Rightarrow s(\sigma(t)) \subseteq s(R_i) = R_i$. So, as $s(\sigma(t))$ is regular closed and R_i, being a primary subset, is minimal regular closed, we have $s(\sigma(t)) = R_i$. As $(L, \bullet, e)$ is a monoid without zero divisors. Then $q \in s(t) \Rightarrow t \in \sigma(q) \Rightarrow \sigma(t) \subseteq \sigma(\sigma(q)) = \sigma(q) \Rightarrow s(\sigma(t)) \subseteq s(\sigma(q)) \Rightarrow R_i \subseteq R$. This shows that R contains a primary subset, a contradiction, as N, being source-splitting in M, cannot contain any primary. Hence, $\mu(R) \subseteq Q - \cup_{i \in I} R_i$.

Remark 3.3 *Let $N = (R, X, \lambda)$ be a source-splitting subautomaton of an L-fuzzy automaton $M = (Q, X, \delta)$ and $(L, \bullet, e)$ be a monoid with zero divisors . Then $\mu(R) \not\subseteq Q - \cup_{i \in I} R_i$, where $R_i^{,}s$ are in Lemma 4.2, as the following counter-example shows.*

counter-example 3.2 *Consider the L-fuzzy automaton given in counter-example 2.1. Let $A = \{m_0, n_0, k_0\}, A_1 = \{n_0, k_0\}, A_2 = \{k_0, l_0\}$. Then $\mu(A) = \{n_0, k_0\}$, now $\mu(A) \cap A_i = \{n_0, k_0\} \cap \{n_0, k_0\} \cap \{k_0, l_0\} = \{k_0\} \neq \phi, \forall i = 1, 2$. Thus $\mu(R) \not\subseteq Q - \cup_{i \in I} R_i$.*

Proposition 3.5 *An L-fuzzy automaton (Q, X, δ) is compact if and only if their exists a finite subset $Q^{,}$ of Q such that $s(Q^{,}) = Q$, or equivalently, if and only if $\sigma(Q)$ is finite.*

Proof: The proof is similar, as given in [21].

Lemma 3.4 *Let (R, X, λ) be a primary of an L-fuzzy automaton (Q, X, δ). Then $s(\sigma(p)) = R, \forall p \in \mu(R)$.*

Proof: The proof is similar, as given in [21].

Ref

20. S.P. Tiwari and A.K. Srivastava, On a decomposition of fuzzy automata, *Fuzzy Sets and Systems* 151 (2005) 503-511.

Ref

21. S.P. Tiwari and A.K. Srivastava, On another decomposition of fuzzy automata, *Journal of Uncertain Systems*, 5 (2011) 33-37.

Lemma 3.5 *Let (R, X, λ) be a primary of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ be a monoid without zero divisors. Then for every finite subset T of R, $T \subseteq s(r)$ for some $r \in R$.*

Proof: We prove this Lemma by induction on the number of elements in T. The results is obvious if $|T| = 1$. Now assume the result to be true for all T having $k-1$ elements. Consider some $T \subseteq R$ having k elements. Pick any $p \in T$. By induction hypothesis, $\exists q \in R$ such that $T - p \subseteq s(q)$. Thus $T \subseteq s(q) \cup p \subseteq s(\{q, p\})$. Put $S = \{q, p\}$ and let $m \in \mu(R)$. Then by Lemma 4.4, $R = s(\sigma(m))$. Note that $q \in R \Rightarrow q \in s(\sigma(m)) \Rightarrow \delta(m', x, q) > 0$ for some $(m', x) \in \sigma(m) \times X$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $m' \in \sigma(m) \Rightarrow \sigma(\sigma(m')) = \sigma(m') \subseteq \sigma(m) \subseteq R$ (as $m \in \mu(R)) \Rightarrow m' \in \mu(R) \Rightarrow s(\sigma(m')) = R$ (by Lemma 4.4). Consequently, $p \in R \Rightarrow p \in s(\sigma(m')) \Rightarrow \delta(r, y, p) > 0$ for some $(r, y) \in \sigma(m') \times X$. From $r \in \sigma(m')$, we get $m' \in s(r)$, whereby $s(m') \subseteq s(s(r)) = s(r)$. This, together with the facts that $q \in s(m')$ and $p \in s(r)$. Hence $S = \{q, p\} \subseteq s(r)$. Thus $s(\{q, p\}) \subseteq s(s(r))$, i.e., $s(\{q, p\}) \subseteq s(r)$. Hence $T \subseteq s(r)$.

Remark 3.4 *Let (R, X, λ) be a primary of an L-fuzzy automaton (Q, X, δ) and $(L, \bullet, e)$ is a monoid with zero divisors. Then for every finite subset T of R, $T \not\subseteq s(r)$ for some $r \in R$, as the following counter-example shows.*

counter-example 3.3 *Similar to counter-example 3.1.*

Proposition 3.6 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $(L, \bullet, e)$ be a monoid without zero divisors. Then each maximal finitely closed subset of Q is $T(Q)$-open.*

Proof: Let R be a maximal finitely closed subset of Q. We have to show that $s(R) = R$. Let T be a finite subset of $s(R)$. For each $q \in T$, we can find some $p \in R$ with $q \in s(p)$. The set S of all such p's must be finite. Also, $S \subseteq R$. Obviously, $T \subseteq s(S)$. Now as R is finitely closed, $\exists t \in Q$ such that $S \subseteq s(t)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(S) \subseteq s(t)$ (since $s(s(t)) = s(t)$), whereby $T \subseteq s(t)$. Thus for each finite subset T of $s(R), \exists t \in Q$ such that $T \subseteq s(t)$. Hence $s(R)$ is a finitely closed subset of Q. But as R is a maximal finitely closed set and $R \subseteq s(R)$, we find that $s(R) = R$.

Remark 3.5 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $(L, \bullet, e)$ be a monoid with zero divisors. Then each maximal finitely closed subset of Q is not $T(Q)$-open.*

Proposition 3.7 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton. Then every primary of M is an f-primary of M.*

Proof: The proof is similar, as given in [21].

Remark 3.6 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $p \in Q$, if $s(p)$ is a primary of M. Then $s(p)$ is an f-primary of M.*

Proof: In view of Proposition 4.7, $s(p)$ is an f-primary of M.

Remark 3.7 *$s(p)$ is a primary of M if $(L, \bullet, e)$ be a monoid without zero divisors. So, converse is true only if $(L, \bullet, e)$ be a monoid without zero divisors.*

Proposition 3.8 *Let $M = (Q, X, \delta)$ be an L-fuzzy automaton and $p \in Q$, if $s(p)$ is an f-primary of M. Then it is a primary of M and $(L, \bullet, e)$ be a monoid without zero divisors.*

Proof: We only need to prove that if $s(p)$ is an f-primary then it is a primary. So let $s(p)$ be an f-primary of M. In view of Proposition 2.2, it suffices to show that $p \in \mu(s(p))$, or that $\sigma(p) \subseteq s(p)$. Let $q \in \sigma(p)$. Then $p \in s(q)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(p) \subseteq s(s(q)) = s(q)$. As $s(q)$ is finitely closed and $s(p)$, being an f-primary, is maximal finitely closed, $s(q) = s(p)$. Thus $q \in s(p)$, showing that $\sigma(p) \subseteq s(p)$.

Proposition 3.9 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $N = (R, X, \lambda)$ be its subautomaton, if N is a primary of M. Then it is an f-primary of M.*

Proof: Once again, in view of Proposition 4.7, N is an f-primary of M.

Remark 3.8 *If N is a primary of M and $(L, \bullet, e)$ be a monoid with zero divisors. Then it is not an f-primary of M.*

Proposition 3.10 *Let $M = (Q, X, \delta)$ be a compact L-fuzzy automaton and $N = (R, X, \lambda)$ be its subautomaton, if N is an f-primary of M. Then it is a primary of M and $(L, \bullet, e)$ be a monoid without zero divisors.*

Proof: Once again, in view of Proposition 4.7, we only need to prove that if N is an f-primary of M then it is a primary. Now, as M is compact, there exists a finite subset R' of R such that $s(R') = R$ (cf. Proposition 4.5). Also, as R is finitely closed, $\exists q \in Q$ such that $R' \subseteq s(q)$, as $(L, \bullet, e)$ is a monoid without zero divisors. Then $s(R') \subseteq s(s(q)) = s(q)$. Thus $R \subseteq s(q)$. But as $s(q)$ is finitely closed and R is maximal finitely closed (as N is an f-primary), $s(q) = R$. Hence N is a primary of M (cf. Proposition 3.1).

Remark 3.9 *If N is an f-primary of M and $(L, \bullet, e)$ be a monoid with zero divisors. Then it is not a primary of M.*

IV. Conclusion

In this paper, we have introduced and studied here the decomposition of fuzzy automata via their primary and f-primary based on lattice ordered monoids. Interestingly, we found that decomposition concepts for fuzzy automata based on lattice-ordered monoids depends on the associated monoid structure. The obtained results generalize the observations made in [19, 20, 21].

References Références Referencias

1. Z. Bavel, Structure and transition preserving functions of finite automata, *J. Assoc. Comput. Mach.*, 15 (1968) 135-158.
2. X. Guo, Grammar theory based on lattice ordered monoid, *Fuzzy Sets and Systems*, 160 (2009) 1152-1161.
3. J. Ignjatovij*c*, M. J *C*ric, S. Bogdanovij*c*, Determinization of fuzzy automata with membership values in complete residuated lattices, *In-formation Sciences*, 178 (2008) 164-180.
4. Y.B. Jun, Intuitionistic fuzzy finite state machines, *Journal of Ap-plied Mathematics and Computing*, 17 (2005) 109-120.
5. Y.B. Jun, Intuitionistic fuzzy finite switchboard state machines, *Journal of Applied Mathematics and Computing*, 20 (2006) 315-325.
6. Y.B. Jun, Quotient structures of intuitionistic fuzzy finite state machines, *Information Sciences*, 177 (2007) 4977-4986.
7. Y.H. Kim, J.G. Kim and S.J. Cho, Products of *T*-generalized state machines and *T*-generalized transformation semigroups, *Fuzzy Sets and Systems*, 93 (1998) 87-97.
8. H.V. Kumbhojkar, S.R. Chaudhri, On proper fuzzification of fuzzy finite state machines, *International Journal of Fuzzy Mathematics*, 4 (2008) 1019-1027.
9. Y. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice-orderd monoids, *Fuzzy Sets and Systems*, 156 (2005) 68-92.
10. D.S. Malik, J.N. Mordeson, M.K. Sen, Submachines of fuzzy finite state machine, *Journal of Fuzzy Mathematics*, 2 (1994) 781-792.
11. J.N. Mordeson, D.S. Malik, Fuzzy Automata and Languages: Theory and Applications. London/Boca Raton: Chapman and Hall/CRC (2002).

Ref

19. A.K. Srivastava and S.P. Tiwari, A topology for fuzzy automata, in: Proc. 2002 *AFSS Internat. Conf. on Fuzzy Systems, Lecture Notes in Artificial Intelligance, Springer-verlag*, 2275 (2002) 485-490.

Notes

12. J.N. Mordeson and P.S. Nair, Retrievability and Connectedness in Fuzzy Finite State Machines, in: Proc. 1996 *Internat. Conf. on IEEE*, 3 1996 1586-1590.
13. D. Qiu, Characterizations of fuzzy finite automata, *Fuzzy Sets and Systems*, 141 (2004) 391-414.
14. D. Qiu, Automata theory based on complete residuated lattice-valued logic(I), *Sci. China Ser. (F)*, 44 (2001) 419-429.
15. D. Qiu, Automata theory based on complete residuated lattice-valued logic(II), *Sci. China Ser. (F)*, 45 (2002) 442-452.
16. E. S. Santos, Maximin automata, *Information and Control*, 12 (1968) 367-377.
17. A.K. Srivastava, W. Shukla, A topology for automata , *Information and control* A note, 32 (1976) 163-168.
18. A.K. Srivastava, W. Shukla, A topology for automata II, *Internat. J. Math. Math. Sci.*, 9 (1986) 425-428.
19. A.K. Srivastava and S.P. Tiwari, A topology for fuzzy automata, in: Proc. 2002 *AFSS Internat. Conf. on Fuzzy Systems, Lecture Notes in Artificial Intelligance, Springer-verlag*, 2275 (2002) 485-490.
20. S.P. Tiwari and A.K. Srivastava, On a decomposition of fuzzy automata, *Fuzzy Sets and Systems*, 151 (2005) 503-511.
21. S.P. Tiwari and A.K. Srivastava, On another decomposition of fuzzy automata, *Journal of Uncertain Systems*, 5 (2011) 33-37.
22. S.P. Tiwari, Anupam K. Singh, Shambhu Sharan, Fuzzy automata based on lattice-ordered monoid and associated topology, *Journal of Uncertain Systems*, 6 (2012) 51-55.
23. S.P. Tiwari and Shambhu Sharan, Fuzzy automata based on latticeordered monoid with algebraic and topological aspects, *Internation Journal of Fuzzy and Information Engineering*, *Communicated*.
24. S.P. Tiwari, Anupam K. Singh, Fuzzy preorder, fuzzy topology and fuzzy transition system, in: Proc. ICLA 2013, *Lecture Notes in Com-puter Science, Springer, Berlin* 7750 (2013), 210-219.
25. Anupam K. Singh, Fuzzy preordered set, fuzzy topology and fuzzy automaton based on generalized residuated lattice, *Iranian Journal of Fuzzy Systems*,14 (2017) 55-66.
26. W.G. Wee, On generalizations of adaptive algorithm and application of the fuzzy sets concept to pattern classification, Ph. D. Thesis, Purdue University, Lafayette, IN 1967.
27. L.A. Zadeh, Fuzzy Sets, *Information and control*, 8 (1965) 338-353.

This page is intentionally left blank

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

A Numerical Approach to the Solution of the System of Second-Order Boundary-Value Problems

By Muhaiminul Islam Adnan & Ashek Ahmed
United International University

Abstract- In this paper, Galerkin method is presented to obtain the approximate solutions of the system of second order boundary value problems using piecewise continuous and differentiable Bernstein polynomials. Derivation of rigorous matrix formulations is exploited to solve the system of second order boundary value problems where, given boundary conditions are satisfied by Bernstein polynomials. The derived formulation is applied to solve the system of second order boundary value problems numerically. Results of numerical approximate solutions converge to the exact solutions monotonically with desired large significant accuracy.

Keywords: galerkin method, bernstein polynomials, numerical solution of system of second order BVPs.

GJSFR-F Classification: MSC 2010: 35A24

ANUMERICALAPPROACHTOTHESOLUTIONOFTHESYSTEMOFSECONDORDERBOUNDARYVALUEPROBLEMS

Strictly as per the compliance and regulations of:

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version 1 Year 2018

A Numerical Approach to the Solution of the System of Second-Order Boundary-Value Problems

Muhaiminul Islam Adnan α & Ashek Ahmed σ

Abstract- In this paper, Galerkin method is presented to obtain the approximate solutions of the system of second order boundary value problems using piecewise continuous and differentiable Bernstein polynomials. Derivation of rigorous matrix formulations is exploited to solve the system of second order boundary value problems where, given boundary conditions are satisfied by Bernstein polynomials. The derived formulation is applied to solve the system of second order boundary value problems numerically. Results of numerical approximate solutions converge to the exact solutions monotonically with desired large significant accuracy.

Keywords: galerkin method, bernstein polynomials, numerical solution of system of second order BVPs.

I. Introduction

Ordinary differential systems appear recurrently in various applications in engineering, physics, biology and other fields. Due to importance of their frequent appearance in various applied field focus on study of ordinary differential systems have been increased. Ordinary differential systems are essential gear in solving real-world problems. A wide variety of natural phenomena are modeled by second Order differential systems. However, various conventional numerical methods used to solve second order initial value problems which methods cannot be used to solve linear second order boundary value problems. There are few valid methods to obtain numerical solutions for a system of second order boundary value problems. The authors discussed the existence of solutions to second order systems, together with the approximation of solutions through finite difference equations in [1, 5]. T. Valanarasu and N. Ramanujan recommended a method to solve a system of singularly linear second order ordinary differential equations [6]. Geng et al. are represented a new method to obtain the analytical and approximate solutions of linear and non-linear system of second order boundary value problems [7].

There are numerous numerical methods such as least square method, finite difference method, Sinc-Galerkin method, and also others methods using polynomial and non-polynomial spline functions to solve second order boundary value problems (BVPs), recently Bhatti and Bracken [8] used Bernstein polynomials for solving two point BVPs by the Galerkin method, but few of them are used to solve system of second order

Author α: Muhaiminul Islam Adnan Lecturer, Institute of Natural Sciences (INS), United International University, Bangladesh.
e-mail: adnan@ins.uiu.ac.bd
Author σ: Ashek Ahmed Lecturer, Institute of Natural Sciences (INS), United International University, Bangladesh.
e-mail: ahmed@ins.uiu.ac.bd

Ref

7. F.Z. Geng, M.G. Cui, Solving a nonlinear system of second order boundary value problems, J. Math. Anal. Appl. 327 (2007) 1167-1181.

boundary value problems. This paper is concentrated on Galerkin method which is used to solve system of second order BVPs with dirichlet boundary condition of the type

$$\left.\begin{array}{l} a_1(x)u'' + a_2(x)v'' + a_3(x)u' + a_4(x)v' + a_5(x)u + a_6(x)v = f_1(x) \\ b_1(x)u'' + b_2(x)v'' + b_3(x)u' + b_4(x)v' + b_5(x)u + b_6(x)v = f_2(x) \end{array}\right\}$$

Subject to the boundary conditions

$$u(a) = u(b) = 0, \quad v(a) = v(b) = 0$$

where $a < x < b$, $a_i(x)$, $b_i(x)$, $f_1(x)$ and $f_2(x)$ are given functions, and $a_i(x)$, $b_i(x)$ are continuous.

Notes

II. Formulation

Let us consider the one-dimensional system of second order differential equations

$$\left.\begin{array}{l} -u''(x) + q(x)\,u(x) + r(x)\,v(x) = f(x) \\ -v''(x) + s(x)\,v(x) + t(x)\,u(x) = g(x) \end{array}\right\} \tag{1}$$

for the pair of functions $u(x)$ and $v(x)$ in $0 < x < 1$. Since each equation is of second order, two boundary conditions are required to specify each of the solution components $u(x)$ and $v(x)$ uniquely. For convenience, we assume homogeneous Dirichlet data at the ends as boundary conditions

$$u(0) = u(1) = v(0) = v(1) = 0 \tag{2}$$

The data include the prescribed functions f, g, q, r, s and t, which are assumed to be bounded and sufficiently smooth to ensure subsequent variational integrals are well defined and the problem is "well posed". Let consider two trial approximate solutions for the pair of functions $u(x)$ and $v(x)$ of system (1) given by

$$\left.\begin{array}{l} \tilde{u}(x) = \sum_{i=1}^{n} a_i\, p_i(x), n \geq 1 \\ \tilde{v}(x) = \sum_{i=1}^{n} b_i\, p_i(x), n \geq 1 \end{array}\right\} \tag{3}$$

where a_i and b_i are parameter, $p_i(x)$ are co-ordinate function which satisfies boundary condition (2). Now apply Galerkin Method in system (1) we get weighted residual system of equations

$$\left.\begin{array}{l} \int_0^1 (-\tilde{u}''(x) + q\tilde{u}(x) + r\,\tilde{v}(x))\; p_i(x)\,dx = \int_0^1 f\; p_i(x)\,dx \\ \int_0^1 (-\tilde{v}''(x) + s\,\tilde{v}(x) + t\,\tilde{v}(x))\, p_i(x)\,dx = \int_0^1 g\; p_i(x)\,dx \end{array}\right\} \tag{4}$$

Integrating by parts and setting $p_i(x) = 0$ at the boundary $x = 0$ and $x = 1$, then we obtain system of weighted residual equations

$$\left.\begin{array}{l} \int_0^1 \left(\tilde{u}'(x)p_i'(x) + q\,\tilde{u}(x)\,p_i(x) + r\,\tilde{v}(x)\,p_i(x)\right) dx = \int_0^1 f\; p_i(x)\,dx \\ \int_0^1 (\tilde{v}'(x)p_i'(x) + s\,\tilde{v}(x)\,p_i(x) + t\;\tilde{u}(x)\,p_i(x))\; dx = \int_0^1 g\; p_i(x)\,dx \end{array}\right\} \tag{5}$$

Now putting the representation (3) into (5) we get

$$\int_0^1 (\sum_{j=1}^{n} a_j\, p_j'(x)\ p_i'(x) + \mathrm{q} \sum_{j=1}^{n} a_j\, p_j(x)\ p_i(x) + \mathrm{r} \sum_{j=1}^{n} b_j\, p_j(x)\ p_i(x))\ dx = \int_0^1 f\ p_i(x)\, dx$$

$$\int_0^1 (\sum_{j=1}^{n} b_j\, p_j'(x)\ p_i'(x) + \mathrm{s} \sum_{j=1}^{n} b_j\, p_j(x)\ p_i(x) + \mathrm{t} \sum_{j=1}^{n} a_j\, p_j(x)\ p_i(x)\,)\, dx = \int_0^1 g\ p_i(x)\, dx$$

Notes

We can write above equations as

$$\sum_{j=1}^{n} \left\{ \int_0^1 \left[\left(p_j'(x)\, p_i'(x) \right) a_j + \left(q\, p_j(x)\, p_i(x) \right) a_j + \left(r\, p_j(x)\, p_i(x) \right) b_j \right] dx \right\} = \int_0^1 f\ p_i(x) dx$$

$$\sum_{j=1}^{n} \left\{ \int_0^1 \left[\left(p_j'(x)\, p_i'(x) \right) b_j + \left(s\, p_j(x)\, p_i(x) \right) b_j + \left(t\, p_j(x)\, p_i(x) \right) a_j \right] dx \right\} = \int_0^1 g\ p_i(x) dx$$

where, $i = 1,2,3, \dots, n$

Equivalently,

$$\left.\begin{aligned} \sum_{j=1}^{n} \{A_{j,i}\, a_j + B_{j,i}\, b_j\} &= F_i \\ \sum_{j=1}^{n} \{C_{j,i}\, b_j + D_{j,i}\, a_j\} &= G_i \end{aligned}\right\} \tag{6}$$

where, $i = 1,2,3, \dots, n$

Where

$$A_{j,i} = \int_0^1 [\left(p_j'(x)\, p_i'(x) \right) + \left(q\, p_j(x)\, p_i(x) \right)] dx$$

$$B_{j,i} = \int_0^1 \left(r\, p_j(x)\, p_i(x) \right) dx\,, F_i = \int_0^1 f\ p_i(x) dx$$

$$C_{j,i} = \int_0^1 [\left(p_j'(x)\, p_i'(x) \right) + \left(s\, p_j(x)\, p_i(x) \right)] dx$$

$$D_{j,i} = \int_0^1 \left(t\, p_j(x)\, p_i(x) \right) dx\,, G_i = \int_0^1 g\ p_i(x) dx$$

where, $i = 1,2,3, \dots, n$

for $i = 1,2, \dots, n$ we get n system of equations, which involve parameters a_i and b_i and which can be obtained by solving system (6). System (6) can be assembled by element matrix contribution.

III. Bernstein Polynomials

The general form of the Bernstein polynomials of nth degree over the interval [a,b] is defined by [8-10]

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i\, (b-x)^{n-i}}{(b-a)^n}\,, \qquad \mathrm{a} \le \mathrm{x} \le \mathrm{b}$$

where, $i = 1,2,3, \dots, n$

Note that each of these $n+1$ polynomials having degree n satisfies the following properties:

i. $B_{i,n}(x) = 0$ if $i < 0$ or $i > n$
ii. $\sum_{i=0}^{n} B_{i,n}(x) = 1$
iii. $B_{i,n}(a) = B_{i,n}(b) = 0$, $1 \le i \le n$

The first 11 Bernstein polynomials of degree ten over the interval [0,1], are given below:

i. $B_{0,10}(x) = (1-x)^{10}$
ii. $B_{1,10}(x) = 10(1-x)^9 x$
iii. $B_{2,10}(x) = 45(1-x)^8 x^2$
iv. $B_{3,10}(x) = 120(1-x)^7 x^3$
v. $B_{4,10}(x) = 210(1-x)^6 x^4$
vi. $B_{5,10}(x) = 252(1-x)^5 x^5$
vii. $B_{6,10}(x) = 210(1-x)^4 x^6$
viii. $B_{7,10}(x) = 120(1-x)^3 x^7$
ix. $B_{8,10}(x) = 45(1-x)^2 x^8$
x. $B_{9,10}(x) = 10(1-x) x^9$
xi. $B_{10,10}(x) = x^{10}$

All these polynomials satisfy dirichlet boundary conditions. These polynomials and combination of polynomials can be used as trial approximate solutions.

IV. Numerical Example

In this section, we apply the formulation discussed above to solve the system of linear second order BVPs [11]. Consider the following system of equations

$$\left.\begin{aligned} u''(x) + x\,u(x) + x\,v(x) &= f(x) \\ v''(x) + 2x\,v(x) + 2x\,u(x) &= g(x) \end{aligned}\right\} \tag{7}$$

Subject to the boundary conditions

$$u(0) = u(1) = 0, \;\; v(0) = v(1) = 0 \tag{8}$$

where, $0 < x < 1$, $f(x) = 2$ and $g(x) = -2$. The solutions of system (7) are $u(x) = x^2 - x$ and $v(x) = x - x^2$, respectively.

We use combinations of nine Bernstein polynomials as trial approximate solution to solve the system (7). Consider trial approximation solutions of the system (7) are

$$\left.\begin{aligned} \tilde{u}(x) &= \sum_{i=1}^{n} a_i B_{i,10}(x) \\ \tilde{v}(x) &= \sum_{i=1}^{n} b_i\, B_{i,10}(x) \end{aligned}\right\} \tag{9}$$

where, $i = 1,2,3,\ldots,n$

Which satisfy boundary condition (8). Where a_i and b_i are unknown parameter. Solving the system (7) by using derived formula in section (2), we get the values of parameters a_i and b_i By putting these parameters in (9) we obtain our desire approximate solutions.

Ref

11. Jungfeng Lu - Variational Iteration Method for Solving a nonlinear System of Second-order Boundary value problems, Computer and Mathematics with Applications, 54(2007), 1133-1138.

Notes

Table 1: Comparison of approximation solution $\tilde{u}\,(x)$ and exact solution $u\,(x)$ of system (7)

Values of x	Exact solution	Approximate solution $\tilde{u}(x)$	Absolute error $\lvert\tilde{u}(x) - u(x)\rvert$
0	0.00000	0.00000	0.00000
0.1	−0.09000	−0.09000	1.44875×10^{-12}
0.2	−0.16000	−0.16000	5.68325×10^{-12}
0.3	−0.21000	−0.21000	9.00676×10^{-12}
0.4	−0.24000	−0.24000	1.64499×10^{-11}
0.5	−0.25000	−0.25000	3.71680×10^{-11}
0.6	−0.24000	−0.24000	6.82071×10^{-11}
0.7	−0.21000	−0.21000	8.63545×10^{-11}
0.8	−0.16000	−0.16000	6.54453×10^{-11}
0.9	−0.09000	−0.09000	1.86649×10^{-11}
1.0	0.00000	0.00000	0.00000

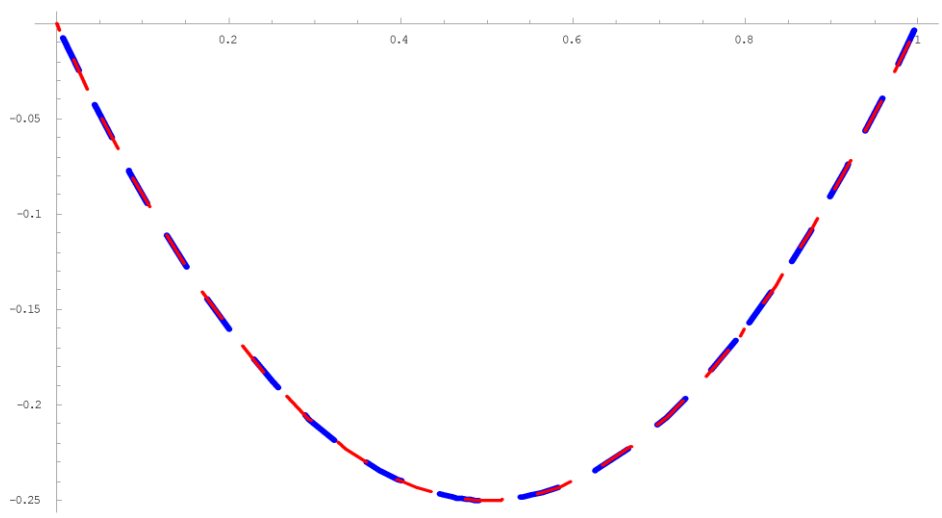

Figure 1: Graphical comparison of Exact solution $\boldsymbol{u}(\boldsymbol{x})$ (- - -) and approximate solution $\tilde{\boldsymbol{u}}\,(\boldsymbol{x})$ (- - -)

Table 2: Comparison of approximation solution $\tilde{\boldsymbol{v}}\,(\boldsymbol{x})$and exact solution $\boldsymbol{v}(\boldsymbol{x})$ of system (7)

Values of x	Exact solution	Approximate solution $\tilde{v}(x)$	Absolute error $\lvert\tilde{v}(x) - v(x)\rvert$
0	0.00000	0.00000	0.00000
0.1	0.09000	0.09000	1.44875×10^{-12}
0.2	0.16000	0.16000	5.68325×10^{-12}
0.3	0.21000	0.21000	9.00676×10^{-12}
0.4	0.24000	0.24000	1.64499×10^{-11}
0.5	0.25000	0.25000	3.71680×10^{-11}
0.6	0.24000	0.24000	6.82071×10^{-11}
0.7	0.21000	0.21000	8.63545×10^{-11}
0.8	0.16000	0.16000	6.54453×10^{-11}
0.9	0.09000	0.09000	1.86649×10^{-11}
1.0	0.00000	0.00000	0.00000

Table 1 and 2 shows the numerical solution and comparison between exact solutions [11] of system of second order BVPs (7). Figure 1 and 2 display comparison with exact and approximate numerical solution. Our numerical approximate solution gives better accuracy compare with other method [11].

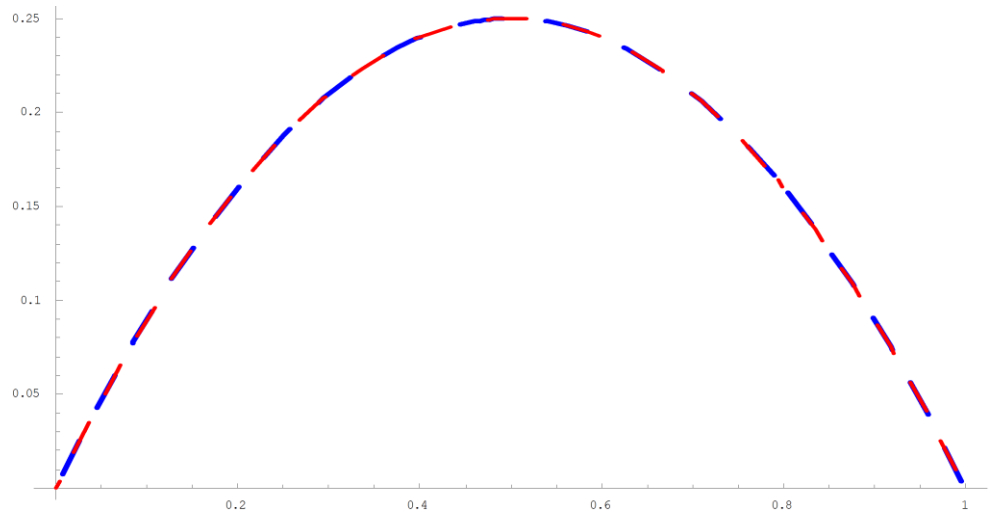

Figure 2: Graphical comparison of Exact solution $\boldsymbol{v}(\boldsymbol{x})$ (- - -) and approximate solution $\widetilde{\boldsymbol{v}}(\boldsymbol{x})$ (- - -).

V. Conclusion

In this paper, a numerical method is developed to solve the system of the system of second order boundary value problems by using the Galerkin method. Developed matrix formulation is a general method which used to solve problems. We introduce a problem in matrix formulation which gives us a better result. The results obtained are very encouraging and this method performs better than other methods.

References Références Referencias

1. H.B. Thompson, C. Tisdell, Systems of difference equations associated with boundary value problems for second order system of ordinary differential equations, J. Math. Appl. 248 (2000) 333-347.
2. Xiyon Cheng, Chengkui Zhong, Existence of positive solutions for a second order ordinary differential system, J. Math. Anal. Appl. 312 (2005) 14-23.
3. H.B. Thompson, C. Tisdell, Boundary value problems for systems of difference equations associated with systems of second-order ordinary differential equations, Appl. Math. Lett. 15(6) (2002) 761-766.
4. H.B. Thompson, C. Tisdell, The nonexistence of spurious solutions to discrete, two-point boundary value problems, Appl. Math. Lett. 16(1) (2003) 79-84.
5. J. Mawhin, C. Tisdell, A note on the uniqueness of solutions to nonlinear, discrete, vector boundary value problems, Nonlinear Anal. Appl. 1-2 (2003) 789-798 (to V. Lakshmikantham on his 80th birthday).
6. T. Valanarasu, N. Ramanujam, An asymptotic initial value method for boundary value problems for a system of singularity perturbed second-order ordinary differential equations, Appl. Math. Comput. 147 (2004) 227-240.
7. F.Z. Geng, M.G. Cui, Solving a nonlinear system of second order boundary value problems, J. Math. Anal. Appl. 327 (2007) 1167-1181.

Ref

11. Jungfeng Lu - Variational Iteration Method for Solving a nonlinear System of Second-order Boundary value problems, Computer and Mathematics with Applications, 54(2007), 1133-1138.

8. Bhatti, M. I., Braken, P., Solutions of differential equations in a Bernstein Polynomial basis, J. Comput. Appl. Math. 205 (2007), 272-280.
9. Reinkenhof, J., Differentiation and integration using Bernstein's polynomials, Int. J. Number. Methods Engrg, 11(1977), 1627-1630.
10. Kreyszig, E., Bernstein polynomials and numerical integration, Int. J. Number. Methods Engrg, 14(1979), 292-295.
11. Jungfeng Lu - Variational Iteration Method for Solving a nonlinear System of Second-order Boundary value problems, Computer and Mathematics with Applications, 54(2007), 1133-1138.

Notes

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version 1 Year 2018

This page is intentionally left blank

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Laplace Transform of Fourier Series of Periodic Functions of a Period P = 2 π

By Shiferaw Geremew Kebede, Awel Seid Geletie, Dereje Legesse Abaire & Mekonnen Gudeta Gizaw
Madda Walabu University

Abstract- The authors establish a set of presumably new results, Which transform a periodic functions of period p = 2$\boldsymbol{\pi}$ to new functions. So in this paper, the authors tries to evaluate Laplace transform of the discontinuous and periodic function that involves in some applications of non-homogeneous differential equations in Physics, electrical engineering, and other many disciplines. Hence, such type of functions expands to a Fourier series, which represent complicated and discontinuous regarding of simpler continuous and periodic functions of cosines and sines.

Keywords: laplace transforms, fourier series.

GJSFR-F Classification: MSC 2010: 44A10

LAPLACETRANSFORMOFFOURIERSERIESOFPERIODICFUNCTIONSOFAPERIODP2

Strictly as per the compliance and regulations of:

Ref

7. Differential Equations for Engineers, Wei-Chau Xie, 2010

Laplace Transform of Fourier Series of Periodic Functions of a Period P = 2 π

Shiferaw Geremew Kebede [α], Awel Seid Geletie [σ], Dereje Legesse Abaire [ρ] & Mekonnen Gudeta Gizaw [ω]

Abstract- The authors establish a set of presumably new results, Which transform a periodic functions of period p = 2π to new functions. So in this paper, the authors tries to evaluate Laplace transform of the discontinuous and periodic function that involves in some applications of non-homogeneous differential equations in Physics, electrical engineering, and other many disciplines. Hence, such type of functions expands to a Fourier series, which represent complicated and discontinuous regarding of simpler continuous and periodic functions of cosines and sines.

Keywords: *laplace transforms, fourier series.*

I. Introduction

The Laplace Transform is a transformation, that it changes a function into a new function. Because of some its properties, it is very important in studying linear differential equations. Laplace transform is named after mathematician and astronomer Pierre-Simon Laplace, who used a similar transform (now called the z-transform) in his work on probability theory.[2] The current widespread use of the transform (mainly in engineering) came about during and soon after World War II [3] although it had been used in the 19th century by Abel, Lerch, Heaviside,and Bromwich. The early history of methods having some similarity to Laplace transform is as follows. From 1744, Leonhard Euler investigated integrals of the form as solutions of differential equations but did not pursue the matter very far.[4] Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form which some modern historians have interpreted within modern Laplace transform theory.[5][6][Clarication needed] These types of integrals attracted Laplace's attention in 1782 where he was following in the spirit of Euler in using the integrals themselves as solutions of equations. [7] However, in 1785, Laplace took the critical step forward, rather than just looking for a solution in the form of an integral. He started to apply the transforms in the sense that was later to become accepted and transform the whole of a difference equation, to look for solutions to the transformed equation. He then went on to apply the Laplace transform in the same way and started to derive some of its properties, beginning to appreciate its potential power. [8] Laplace also recognized that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply

Author α σ ρ ω: *Department of Mathematics, Madda Walabu University, Bale Robe Ethiopia. e-mail: yerosenshiferaw@gmail.com*

to a limited region of space because those solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space.[9] When we solve some applications of non-homogeneous differential equations that involves Physics, electrical engineering, and other many disciplines by using Laplace transformations like Forced oscillations of a body of mass m on a spring of modulus k in Physics, mathematically defined by using ordinary differential equation as:

Where $r(t)$ is a discontinuous and periodic function defined as:

$$r(t) = t + \tfrac{\pi}{2}, \text{ if } -\pi < t < 0 \text{ and } r(t) = -t + \tfrac{\pi}{2}, \text{ if } 0 < t < \pi$$

where, $r(t + 2\pi) = r(t)$

Hence, such type of functions expand to a Fourier series, which represent a complicated and discontinuous functions in terms of simpler continuous functions and periodic functions of cosines and sines.

a) Definition

The Laplace Transform of a function $f(t)$ defined for all $t \geq 0$, is the integral

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

The function F(s) is called the Laplace transform of the function $f(t)$. Denoted by $L(f(t)$.

where $s \in \Re$ is any real number such that the integral above converges.

The operation which yields $F(s)$ from a given $f(t)$ is called Laplace transformation. $f(t)$ is called the inverse transform or inverse of $F(s)$.

b) Definition

A Fourier series of a periodic functions of a period, $p = 2\pi$ is given by:

$$f(t) = a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt]$$

where a_0, a_k and b_k are called the Fourier coefficients and defined as:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt$$

Where, $k = 1, 2, 3, ...$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt$$

Where, $k = 1, 2, 3, ...$

II. Laplace Transform Of Fourier Series With A Period $P = 2\pi$

Theorem 1: (Laplace Transform of Fourier series with a period $p = 2\pi$

Suppose $f(t)$ a periodic function with a period $p = 2\pi$ is given,then its Fourier series expansion is:

$$f(t) = a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt]$$

Ref

9. Historically, how and why was the Laplace Transform invented? Written 18 Oct 2015 From Wikipedia:

Notes

Such that, the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$
$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \frac{s}{s^2+k^2} + b_k \frac{k}{s+k^2}]$$

Where, a_0, a_k and b_k are the Fourier coefficients defined in definition1.2.

proof

Suppose $f(t)$ is a periodic with $p = 2\pi$ and has a Fourier series expansion; Then,

$$f(t) = a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt]$$

$$\Rightarrow L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$

$$= L(a_0) + L(\Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$

$$= \int_0^{\infty} e^{st} a_0 dt + \int_0^{\infty} e^{st}(\Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])dt$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[\int_0^{\infty} e^{st}(a_k \cos kt + b_k \sin kt)dt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[\int_0^{\infty} e^{st} a_k \cos kt dt + \int_0^{\infty} e^{st} b_k \sin kt dt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \int_0^{\infty} e^{st} \cos kt dt + b_k \int_0^{\infty} e^{st} \sin kt dt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \frac{s}{s^2+k^2} + b_k \frac{k}{s+k^2}]$$

Theorem 2: (Laplace Transform of Fourier Cosine and Sine series with a period $p = 2\pi$

A) Suppose $f(t)$ is even periodic function with a period $p = 2\pi$,then its Fourier Cosine series expansion is:

$$f(t) = a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt$$

Where,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(kt)dt$$

Such that, the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt)$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty} a_k \frac{s}{s^2 + k^2}$$

Proof

Assume $f(t)$ is an even periodic function with a period of $p = 2\pi$ then its Fourier coefficients are defined as:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

$$a_k = \frac{2}{\pi} \int_{0}^{\pi} f(t) \cos ktdt$$

Where, $k = 1, 2, 3, ...$

$$b_k = 0$$

Hence,$f(t)$ has a Fourier cosine series expansion;
Then,

$$f(t) = a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt$$

$$\Rightarrow L(f(t)) = L(a_0 + \Sigma_{k=1}^{\infty} a_k \cos kt)$$

$$= L(a_0) + L(\Sigma_{k=1}^{\infty}[a_k \cos kt + b_k \sin kt])$$

$$= \int_0^{\infty} e^{st} a_0 dt + \int_0^{\infty} e^{st} (\Sigma_{k=1}^{\infty} a_k \cos kt) dt$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty} [\int_0^{\infty} e^{st} a_k \cos ktdt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty} [a_k \int_0^{\infty} e^{st} \cos ktdt]$$

$$= \frac{a_0}{s} + \Sigma_{k=1}^{\infty} a_k \frac{s}{s^2 + k^2}$$

B) Suppose $f(t)$ is an odd periodic function with a period $p = 2\pi$,then its Fourier sine series expansion is:

$$f(t) = \Sigma_{k=1}^{\infty} b_k \sin kt$$

Such that, the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = L(\Sigma_{k=1}^{\infty} b_k \sin kt)$$

$$= \Sigma_{k=1}^{\infty} b_k \frac{k}{s^2 + k^2}$$

Proof

Assume $f(t)$ is odd periodic function with a period of $p = 2\pi$, then its Fourier coefficients are defined as:
$a_0 = 0$ and $a_k = 0$

Notes

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(t)\sin ktdt$$

Where, $k = 1, 2, 3, ...$

Hence, $f(t)$ has a Fourier sine series expansion;

Then,

Notes

$$f(t) = \Sigma_{k=1}^{\infty} b_k \sin kt$$

$$\Rightarrow L(f(t)) = L(\Sigma_{k=1}^{\infty} a_k \sin kt)$$

$$= \int_0^{\infty} e^{st}(\Sigma_{k=1}^{\infty} b_k \sin kt)dt$$

$$= \Sigma_{k=1}^{\infty}[\int_0^{\infty} e^{st} b_k \sin ktdt]$$

$$= \Sigma_{k=1}^{\infty}[b_k \int_0^{\infty} e^{st} \sin ktdt]$$

$$= \Sigma_{k=1}^{\infty} b_k \frac{k}{s^2 + k^2}$$

III. Conclusion

When we solve some applications of non-homogeneous differential equations that involves in Physics, electrical engineering and other many disciplines by using Laplace transformations like Forced oscillations, electrical circuit, periodic rectangular wave and half-wave rectifier and so on, are expands to a Fourier series, which represent a complicated and discontinuous functions regarding simpler continuous and periodic functions of cosines and sines. Therefore it is possible to solve a differential equations that involve such type of functions by using Laplace transformation by expanding them in Fourier series.

Therefore the results on Laplace transform of Fourier series are summarized as follows;

1. If $f(t)$ a periodic function with a period $p = 2\pi$ is given,then the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = \frac{a_0}{s} + \Sigma_{k=1}^{\infty}[a_k \frac{s}{s^2 + k^2} + b_k \frac{k}{s + k^2}]$$

2. I $f(t)$ is an even periodic function with a period $p = 2\pi$,then the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) == \frac{a_0}{s} + \Sigma_{k=1}^{\infty} a_k \frac{s}{s^2 + k^2}$$

3.I $f(t)$ is odd periodic function with a period $p = 2\pi$,then the Laplace transform of $f(t)$ is given in the form of series as:

$$L(f(t)) = \Sigma_{k=1}^{\infty} b_k \frac{k}{s^2 + k^2}$$

References Références Referencias

1. Ordinary Differential equations, GABRIEL NAGY, Mathematics Department, SEPTEMBER 14, 2015
2. Introduction to Ordinary Differential Equations and Some Applications, Edward Burkard
3. Advanced Engineering Mathematics, Erwin Kreyszing, Herbert Kreyszing, Edward J. Norminton 10th Edition
4. A first Course in Differential equations, Rudolph E. Longer, 1954
5. Differential Equation and Integral Equations, Peter J. Collins, 2006
6. Differential Equations, James R. Brannan, William E. Boyce, 2nd edition
7. Differential Equations for Engineers, Wei-Chau Xie, 2010
8. Advanced Engineering Mathematics 7th Edition, PETER V. ONEIL
9. Historically, how and why was the Laplace Transform invented? Written 18 Oct 2015 From Wikipedia:

Notes

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Bifurcation for a Class of Fourth-Order Stationary Kuramoto-Sivashinsky Equations under Navier Boundary Condition

By Imed Abid, Soumaya Sâanouni & Nihed Trabelsi
University of Tunis El Manar

Abstract- In this paper, we study the bifurcation of semilinear elliptic problem of fourth-order with Navier boundary conditions. We discuss the existence and the uniqueness of a positive solution and we also prove the existence of critical value and the uniqueness of extremal solutions. We take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point.

Keywords: *bifurcation, regularity, stability, quasilinear.*

GJSFR-F Classification: *MSC 2010: 35B32, 35B65, 35B35, 35J62*

BIFURCATIONFORACLASSOFFOURTHORDERSTATIONARYKURAMOTOSIVASHINSKYEQUATIONSUNDERNAVIERBOUNDARYCONDITION

Strictly as per the compliance and regulations of:

Notes

Bifurcation for a Class of Fourth-Order Stationary Kuramoto-Sivashinsky Equations under Navier Boundary Condition

Imed Abid [α], Soumaya Sâanouni [σ] & Nihed Trabelsi [ρ]

Abstract- In this paper, we study the bifurcation of semilinear elliptic problem of fourth-order with Navier boundary conditions. We discuss the existence and the uniqueness of a positive solution and we also prove the existence of critical value and the uniqueness of extremal solutions. We take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point.

Keywords: bifurcation, regularity, stability, quasilinear.

I. Introduction

Let Ω be a smooth bounded domain in $\mathbb{R}^n$, the Kuramoto-Sivashinsky (KS) equation

$$\begin{array}{rcccl} \partial_t u + \Delta^2 u - \gamma\Delta u + \delta u - \beta|\nabla u|^2 & = & \lambda f(x,u) & \text{in} & \Omega, \\ \Delta u = u & = & 0 & \text{on} & \partial\Omega, \end{array}$$

arises in many applications from mathematical physics, which are usually used to describe some phenomena appearing in physics, engineering, and other sciences.

Moreover, the first bifurcation may be subcritical and bistability then occurs. The features qualitatively agree with the experiments. Finally, wave shapes are compared. The addition of the lighter order terms leads to better agreement with the experiments. The (KS) equation was the first non-linear wave equation to be proposed for describing long interfacial waves of two-layer couette and poiseuille flows. In agreement with previous studies, it is shown that each higher order term has a "laminarizing effect": the solutions of the (KS) equation simplify the benefit of stationary traveling waves.

In the stationary case and for $\gamma = \delta = \beta = 0$, various authors have studied the existence of weak solutions for the bifurcation problem

$$(E_\lambda) \quad \left\{ \begin{array}{rcccl} \Delta^2 u & = & \lambda f(u) & \text{in} & \Omega, \\ \Delta u = u & = & 0 & \text{on} & \partial\Omega, \end{array} \right.$$

Author α: University of Tunis El Manar, higher Institut of medicals technologies of Tunis, 9 street Dr.Zouhair Essa_ 1006 Tunis, Tunisia. e-mail: abidimed7@gmail.com
Author σ: University of Tunis El Manar, Faculty of Sciences of Tunis, Department of Mathematics, University Campus 2092 Tunis,Tunisia. e-mail: saanouni.soumaya@yahoo.com
Author ρ: University of Tunis El Manar, higher Institut of medicals technologies of Tunis, 9 street Dr.Zouhair Essafi 1006 Tunis, Tunisia. e-mail: nihed.trabelsi78@gmail.com

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Abid and al. have proved in [1] that there exists $0 < \lambda^* < \infty$, a critical value of the parameter λ, such as (E_λ) has a minimal, positive, classical solution u_λ for $0 < \lambda < \lambda^*$ and does not have a weak solution for $\lambda > \lambda^*$. When $\lim\limits_{t\to\infty} \dfrac{f(t)}{t} = a < \infty$, it is proved also there exists a unique classical solution u^* of (E_{λ^*}) if and only if $\lim\limits_{t\to+\infty} (f(t) - at) < 0$.

When $\delta = \beta = 0$, several researchers are interested in this type of phenomenon:

$$\begin{array}{rcll} \Delta^2 u - \gamma\Delta u & = & \lambda f(x,u) & \text{in} \quad \Omega, \\ \Delta u = u & = & 0 & \text{on} \quad \partial\Omega. \end{array}$$

In [17, 18], Lazer and Mckenna firstly proposed and studied the problem of periodic oscillations and traveling waves in a suspension bridge. It was pointed out in [23, 28] that the problem provides a good model for the study of the static deflection of an elastic plate in a fluid. Moreover, Ahmed and Harbi in [2] showed that the problem can also be applied to engineering, such as communication satellites, space shuttles and space stations equipped with large antennas mounted on long flexible beams. Such problems appear for example in the micro-electromechanical systems giving the modelization of electrostatic actuation for membranes deflecting on thin plates in the field of nanotechnology detection systems in [19, 24] where the parameter γ represents the constant tension rising in the stretching energy sector in the presence of elastic deformation. It arises in mechanics too, see [11] and in electricity, see [16].

Ben Omrane and Khenissy in [6] show the nonexistence of solutions with Dirichlet boundary conditions. They prove a dichotomy result giving the positivity preserving property for a biharmonic equation arising in MEMS models.

When γ is not constant, other results prove the existence of solutions and study the bifurcation problem with Navier boundary conditions,

$$\Delta^2 u - \operatorname{div}(\gamma(x)\nabla u) = \lambda f(u) \quad \text{in } \Omega.$$

In [26], Sâanouni and Trabelsi show how the critical problem behaves when it is considered with the Navier boundary condition. The function $\gamma(x)$ is smooth positive on $\overline{\Omega}$ and the L^∞-norm of its gradient is small enough in order to assure ensure the application of maximum principle [12].

Our main interest here will be in the study of a bifurcation problem in stationary case for $\beta = 0$ and γ, δ, $\lambda > 0$, we consider the following problem

$$(P_\lambda) \quad \left\{ \begin{array}{rcll} \Delta^2 u - \gamma\Delta u + \delta u & = & \lambda f(u) & \text{in} \quad \Omega, \\ u & > & 0 & \text{in} \quad \Omega, \\ \Delta u = u & = & 0 & \text{on} \quad \partial\Omega, \end{array} \right.$$

where Ω is a smooth bounded domain in $\mathbb{R}^n; (n \geq 2)$ and f is a positive, increasing and convex smooth function on $(0, +\infty)$, which verifies

$$\lim_{t\to\infty} \frac{f(t)}{t} = a \in (0, \infty).$$

This paper is organized as follows: In next section we state our main results contained in this work (see Theorems 1, 2 and 3). The content of section 3 is devoted to the proof of Theorem 1, concerns existence of minimal solutions. It is shown that there exists a limiting parameter λ^* such that one has existence of stable regular minimal solutions to

Ref

2. NU. Ahmed, H. Harbi, *Mathematical analysis of dynamic models of suspension bridges.* SIAM J. Appl. Math. 58, 853-874 (1998).

(P_λ) for $\lambda \in (0, \lambda^*)$, while for $\lambda > \lambda^*$, not even singular solutions exist. In Sections 4 and 5, we devoted to the proofs of Theorems 1 and 2, we take into account the types of problems of bifurcation for a class of elliptic problems we also establish the asymptotic behavior of the solution around the bifurcation point. Finally, in Section 6 we also give some hints on how to proceed for semilinear problems under Dirichlet boundary conditions.

Notes

II. Main Results

Throughout our paper, we denote by $\|\cdot\|_2$, the $L^2(\Omega)$-norm, whereas we denote by $\|\cdot\|$, the $H^2(\Omega) \cap H_0^1(\Omega)$-norm given by

$$\|u\|^2 = \int_\Omega |\Delta u|^2.$$

Definition 1. *We say that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution of the problem (P_λ), if $f(u) \in L^1(\Omega)$ and*

$$\int_\Omega \Delta u.\Delta\varphi + \gamma \int_\Omega \nabla u.\nabla\varphi + \delta \int_\Omega u\varphi = \lambda \int_\Omega f(u)\varphi, \quad \forall \varphi \in C^2(\overline{\Omega}) \cap H^2(\Omega) \cap H_0^1(\Omega).$$

Such solutions are usually known as weak energy solutions. For short, we will refer to them simply as solutions which is assured by the next lemma.

Remark 1. *Since $f(t) \leq at + f(0)$, if $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a weak solution of (P_λ) and $u \in L^1(\Omega)$, we say that u is regular solution. It is easily seen by a standard bootstrap argument that u is always a classical solution.*

For more detail, see [12, Proposition 7.15]. In the rest of this paper, we denote by a solution of problem (P_λ) any weak or classical solution.

Definition 2. *We say that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ is a supersolution of (P_λ) if $f(u) \in L^1(\Omega)$ and*

$$\Delta^2 u - \gamma\Delta u + \delta u \geq \lambda f(u) \text{ in } \mathcal{D}'(\Omega).$$

Reversing the inequality one defines the notion of subsolution.

Next, we recall a version of the Maximum Principle for the biharmonic operator.

Proposition 1. *Let $u \in H^4(\Omega) \cap H_0^1(\Omega)$ be a function such that*

$$\Delta^2 u \geq 0 \text{ in } \Omega, \ \Delta u = u = 0 \text{ on } \partial\Omega.$$

Then

$$u(x) \geq 0, \ \Delta u(x) \leq 0 \text{ in } \Omega.$$

We say that a solution u_λ of problem (P_λ) is *minimal* if $u_\lambda \leq u$ in Ω for any solution u of (P_λ).

Recall that $\lambda_1 \in \mathbb{R}$, be the first eigenvalue and φ_1 the first normalized eigenfunction of $(-\Delta)^2 - \gamma\Delta + \delta$ in Ω with homogeneous Dirichlet boundary data

$$\begin{cases} \Delta^2 \varphi_1 - \gamma \Delta \varphi_1 + \delta \varphi_1 & = \lambda_1 \varphi_1 \quad \text{in} \quad \Omega, \\ \Delta \varphi_1 = \varphi_1 & = 0 \quad \text{in} \quad \partial\Omega \\ \|\varphi_1\|_2 & = 1. \end{cases} \tag{1}$$

an eigenvalue λ_1 which is positive and that can be characterized as follows

$$\lambda_1 = \min_{\varphi \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_\Omega \Big(|\Delta \varphi|^2 + \gamma |\nabla \varphi|^2 + \delta \varphi^2 \Big)}{\displaystyle\int_\Omega |\varphi|^2}; \tag{2}$$

there exists a non-negative function $\varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, which is an eigenfunction corresponding to λ_1, attaining the minimum in (2), that is $\|\varphi_1\|_2 = 1$ and

$$\lambda_1 = \int_\Omega \Big(|\Delta \varphi_1|^2 + \gamma |\nabla \varphi_1|^2 + \delta \varphi_1^2 \Big).$$

A solution u of problem (P_λ) is stable if and only if the first eigenvalue of the linearized operator

$$v \mapsto L_{\lambda,u}(v) := \Delta^2 v - \gamma \Delta v + \delta v - \lambda f'(u) v,$$

given by

$$\eta_1(\lambda, u) := \inf_{v \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \eta_1(\lambda, u)(v),$$

where for any $v \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}$

$$\eta_1(\lambda, u)(v) = \frac{\displaystyle\int_\Omega \Big(|\Delta v|^2 + \gamma |\nabla v|^2 + \delta v^2 \Big) - \lambda \int_\Omega f'(u) v^2 dx}{\|v\|_2^2}$$

is non negative. In other words,

$$\lambda \int_\Omega f'(u) v^2 dx \leq \int_\Omega \Big(|\Delta v|^2 + \gamma |\nabla v|^2 + \delta v^2 \Big), \quad \text{for any } v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{3}$$

If $\eta(\lambda, u) < 0$, the solution u is said to be *unstable*.

Next, we let

$$\Lambda := \{\lambda > 0 \,|\, (P_\lambda) \text{ admits a solution}\} \text{ and } \lambda^* := \sup \Lambda \leq +\infty.$$

and

$$r_0 := \inf_{t>0} \frac{f(t)}{t}.$$

The two values a and r_0 that we have already defined will be important in the bifurcation phenomena. More precisely, in the frame of the critical value λ^*.

We propose to show the following results, first main statement asserts the existence of the critical value λ^*.

Theorem 1. *There exists a critical value $\lambda^* \in (0, \infty)$ such that the following properties hold:*

(i) *For any $\lambda \in (0, \lambda^*)$, problem (P_λ) has a minimal solution u_λ, which is the unique stable solution of (P_λ).*
(ii) *For any $\lambda \in (0, \lambda_1/a)$, u_λ is the unique solution of problem (P_λ).*
(iii) *The function $\lambda \longmapsto u_\lambda$ is a $\mathcal{C}^1$, convex, increasing function.*
(iv) *If (P_{λ^*}) has a solution, then $u^* := \lim_{\lambda \to \lambda^*} u_\lambda$ and $\eta_1(\lambda^*, u^*) = 0$.*

Notes

The next natural obvious object of study gives us more precise information for λ^*. An important role in our arguments will be played by

$$l := \lim_{t\to\infty}\Big(f(t) - at\Big).$$

We distinguish two different situations strongly depending on the sign of l.

Notes

Theorem 2. *Assume that $l \geq 0$. We have three equivalent assertions:*

(i) $\lambda^* = \lambda_1/a$.

(ii) *problem (P_{λ^*}) has no solution.*

(iii) $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of Ω.*

Again the question arises as to what happens when $l < 0$. The following was proved.

Theorem 3. *Assume that $l < 0$. Then we have.*

(i) *the critical value λ^* belongs to $(\lambda_1/a, \lambda_1/r_0)$*

(ii) *(P_{λ^*}) has a unique solution u^*.*

(iii) *The problem (P_λ) has an unstable solution v_λ for any $\lambda \in (\lambda_1/a, \lambda^*)$ and the sequence $(v_\lambda)_\lambda$ satisfies:*

(a) $\lim_{\lambda\to\lambda_1/a} v_\lambda = \infty$ *uniformly on compact subsets of Ω,*

(b) $\lim_{\lambda\to\lambda^*} v_\lambda = u^*$ *uniformly in Ω.*

III. Proof of Theorem 1

We say that a solution u_λ of problem (P_λ) is *minimal* if $u_\lambda \leq u$ in Ω for any solution u of (P_λ).

Lemma 1. *Problem (P_λ) has no solution for any $\lambda > \lambda_1/r_0$, but has at least one solution provided λ is positive and small enough.*

Proof: First, to show that (P_λ) has a solution, we use the barrier method.

Since $f(0) > 0$, $\underline{u} \equiv 0$ is a strict subsolution of (P_λ) for every $\lambda > 0$. To this aim, let $\overline{w} \in H^4(\Omega)$ which satisfies

$$\begin{cases} \Delta^2\overline{w} - \gamma\Delta\overline{w} + \delta\overline{w} &= 1 \quad \text{in} \quad \Omega, \\ \Delta\overline{w} = \overline{w} &= 0 \quad \text{in} \quad \partial\Omega. \end{cases}$$

The choice of $\overline{w}$ implies that $\overline{w}$ is a bounded supersolution of (P_λ) for small λ, more precisely whenever $\lambda < 1/f(\|\overline{u}\|_\infty)$.

Notice that for any $\lambda > 0$, the function $\underline{w} \equiv 0$ is a sub-solution of (P_λ) since $f(0) > 0$. Next, we define a sequence $w_n \in H^4(\Omega)$ by

$$\begin{cases} \Delta^2 w_{n+1} - \gamma\Delta w_{n+1} + \delta w_{n+1} &= \lambda f(w_n) \quad \text{in} \quad \Omega \\ \Delta w_{n+1} = w_{n+1} &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4}$$

The maximum principle (see [7]) implies that

$$\underline{w} \leq w_n \leq w_{n+1} \leq \overline{w} \text{ for all } n \in \mathbb{N},$$

so that the sequence $(w_n)_{n\geq 0}$ is increasing and bounded, then it converges. It follows that problem (P_λ) has a solution.

Assume now that u is a solution of (P_λ) for some $\lambda > 0$. Using φ_1 given in (1) as a test function and integrating by parts, we get

$$\begin{aligned}\lambda_1 \int_\Omega \varphi_1\, u &= \int_\Omega (\Delta^2\varphi_1 - \gamma\Delta\varphi_1 + \delta\varphi_1)u = \int_\Omega \Delta^2 u\ \varphi_1 - \gamma \int_\Omega \Delta u\ \varphi_1 + \delta \int_\Omega u\ \varphi_1 \\ &= \lambda \int_\Omega f(u)\ \varphi_1 \geq \lambda\ r_0 \int_\Omega u\ \varphi_1.\end{aligned}$$

This yields

$$(\lambda_1 - \lambda r_0) \int_\Omega \varphi_1 u \geq 0.$$

Since $\varphi_1 > 0$ and $u > 0$, we conclude that the parameter λ should belong to $(0, \lambda_1/r_0)$.

As a consequence we have that λ^* is a real. Another useful result is stated in what follows.

Lemma 2. *Assume that (P_λ) is resolvable, then a minimal solution u_λ exists. Moreover, $(P_{\lambda'})$ is resolvable for any $\lambda' \in (0, \lambda)$.*

Proof: Fix $\lambda \in (0, \lambda^*)$ and let u be a solution of (P_λ). As above, we use the barrier method to obtain a minimal solution of (P_λ). The basic idea is to prove by induction that the sequence $(w_n)_{n\geq 0}$ defined in (4) is increasing and bounded by u, so it converges to some solution u_λ. Since u_λ is independent of the choice of u, then it is a minimal solution.

Now, if u is a solution of (P_λ), then u is a super-solution for the problem $(P_{\lambda'})$ for any λ' in $(0, \lambda)$ and 0 can be used always as a sub-solution.

Remark 2. *Thanks to lemmas 1 and 2, the set Λ is an interval not empty and bounded.*

a) Proof of Theorem 1.

i. *Proof of (i).* First, we claim that u_λ is stable. Indeed, arguing by contradiction, i.e. the first eigenvalue $\eta_1(\lambda, u_\lambda)$ is negative. Then, there exists an eigenfunction $\in H^4(\Omega)$ such that

$$\begin{aligned}\Delta^2\boldsymbol{\psi} - \gamma\Delta\boldsymbol{\psi} + \delta\boldsymbol{\psi} - \lambda f'(u_\lambda)\,\boldsymbol{\psi} &= \eta_1\boldsymbol{\psi} && \text{in} \quad \Omega, \\ \boldsymbol{\psi} &> 0 && \text{in} \quad \Omega \\ \Delta\boldsymbol{\psi} = \ \boldsymbol{\psi} &= 0 && \text{on} \quad \partial\Omega.\end{aligned}$$

Consider $u^\varepsilon := u_\lambda - \varepsilon\ \psi$. Hence, by linearity, we have

$$\begin{aligned}\Delta^2 u^\varepsilon - \gamma\Delta u^\varepsilon + \delta u^\varepsilon - \lambda f(u^\varepsilon) &= \lambda f(u_\lambda) - \varepsilon(\Delta^2\boldsymbol{\psi} - \gamma\Delta\boldsymbol{\psi} + \delta\psi) - \lambda f(u_\lambda - \varepsilon\psi) \\ &= \lambda f(u_\lambda) - \varepsilon(\lambda f'(u_\lambda)\boldsymbol{\psi} + \eta_1\boldsymbol{\psi}) - \lambda f(u_\lambda - \varepsilon\psi) \\ &= \lambda \left[f(u_\lambda) - f(u_\lambda - \varepsilon\psi) - \varepsilon f'(u_\lambda)\boldsymbol{\psi}\right] - \varepsilon\ \eta_1\boldsymbol{\psi} \\ &= \lambda\ o_\varepsilon(\varepsilon\psi) - \varepsilon\ \eta_1\boldsymbol{\psi} \\ &= \varepsilon\ \psi\ (\lambda\ o_\varepsilon(1)\ - \eta_1).\end{aligned}$$

Since $\eta_1(\lambda, u_\lambda) < 0$, for $\varepsilon > 0$ small enough, we have

$$\Delta^2 u^\varepsilon - \gamma\Delta u^\varepsilon + \delta u^\varepsilon - \lambda f(u^\varepsilon) \geq 0 \quad \text{in } \Omega.$$

Then, for $\varepsilon > 0$ small enough, we use the strong maximum principle (Hopf's lemma, see [14]) to deduce that $u^\varepsilon \geq 0$ is a super-solution of (P_λ). As before, we obtain a solution u such that $u \leq u^\varepsilon$ and since $u^\varepsilon < u_\lambda$, then we contradict the minimality of u_λ.

Now, we show that (P_λ) has at most one stable solution. Assume the existence of another stable solution $v \neq u_\lambda$ of problem (P_λ). Let $w := v - u_\lambda$, then by maximum principle $w > 0$ and from (3) taking w as a test function, we have

Notes

Notes

$$\begin{aligned}
\lambda \int_\Omega f'(v)\, w^2 &\leq \int_\Omega |\Delta w|^2 + \gamma \int_\Omega |\nabla w|^2 + \delta \int_\Omega w^2 \\
&\leq \int_\Omega \Delta^2 w\ w - \gamma \int_\Omega \Delta w\ w + \delta \int_\Omega w\ w \\
&\leq \int_\Omega \Big[\Delta^2 v - \gamma \Delta v + \delta v - \Delta^2 u_\lambda + \gamma \Delta u_\lambda - \delta u_\lambda\Big]\ w \\
&\leq \lambda \int_\Omega \Big[f(v) - f(u_\lambda)\Big]\ w.
\end{aligned}$$

Therefore

$$\int_\Omega \Big[f(v) - f(u_\lambda) - f'(v)(v - u_\lambda)\Big] w \geq 0.$$

Thanks to the convexity of f, the term in the brackets is nonpositive, hence

$$f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) = 0 \ \text{ in } \Omega,$$

which implies that f is affine over $[u_\lambda, v]$ in Ω. So, there exists two real numbers α and β such that

$$f(x) = \alpha x + \beta \quad \text{ in } [0, \max_\Omega v].$$

Finally, since u_λ and v are two solutions to $\Delta^2 w - \gamma \Delta w + \delta w = \lambda \alpha w + \lambda \beta$, we obtain that

$$0 = \int_\Omega \Big(u_\lambda\ \Delta^2 v - v\ \Delta^2 u_\lambda\Big) - \gamma \int_\Omega \Big(u_\lambda\ \Delta v - v\ \Delta u_\lambda\Big) + \delta \int_\Omega \Big(u_\lambda\ v - v\ u_\lambda\Big) = \lambda \beta \int_\Omega (u_\lambda - v).$$

This is impossible since $\beta = f(0) > 0$ and $w = v - u_\lambda$ is positive in Ω.

3.1.2. *Proof of (ii).* Recall that λ_1 is defined in (1). By the convexity of f, we deduce that $a = \sup_{\mathbb{R}_+} f'(t)$. Let u be a solution to (P_λ) for $\lambda \in (0, \lambda_1/a)$, we suppose that u is unstable. Then, we can take $\varphi = \varphi_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ which satisfy

$$\lambda\ a \int_\Omega \varphi^2 \geq \lambda \int_\Omega f'(u)\varphi^2 > \int_\Omega |\Delta \varphi|^2 + \gamma \int_\Omega |\nabla \varphi|^2 + \delta \int_\Omega \varphi^2 = \lambda_1 \int_\Omega \varphi^2,$$

which shows that

$$(\lambda\ a - \lambda_1) \int_\Omega \varphi^2 > 0.$$

Impossible for $\lambda \in (0, \lambda_1/a)$. Then, $\eta_1(\lambda, u) \geq 0$, so we obtain the uniqueness of u.

For the existence, we consider the minimization problem

$$\min_{u \in H^2(\Omega) \cap H_0^1(\Omega)} \mathcal{J}_\lambda(u),$$

where

$$\mathcal{J}_\lambda(u) := \frac{1}{2} \int_\Omega \Big(|\Delta u|^2 + \gamma |\nabla u|^2 + \delta u^2\Big) - \lambda \int_\Omega F(u), \quad \text{for all} \quad u \in H^2(\Omega) \cap H_0^1(\Omega)$$

with

$$u^+ := \max(u, 0) \quad \text{and} \quad F(u) := \int_0^{u^+} f(s) ds.$$

If $\lambda \in (0, \lambda_1/a)$, there exist $\varepsilon > 0$ and $A > 0$ depending on λ such that

$$2\lambda F(t) \leq (\lambda_1 - \varepsilon) t^2 + A, \qquad \forall\, t \in \mathbb{R}.$$

Standard arguments imply that $\mathcal{J}_\lambda(u)$ is coercive, bounded from below and weakly lower semi-continuous in $H^2(\Omega) \cap H_0^1(\Omega)$ (see propositions 2.2 and 2.3 in [29]). It is easy to see that the minimum of $\mathcal{J}_\lambda$ is attained by some function $u \in H^2(\Omega) \cap H_0^1(\Omega)$. So, the critical point u of $\mathcal{J}_\lambda$ gives a solution of (P_λ).

3.1.3. *Proof of (iii).* By sub- and super-solution method, see Lemma 2 we obtain that the mapping $\lambda \mapsto u_\lambda$ is increasing and this proves (iii).

3.1.4. *Proof of (iv).* Now Consider the nonlinear operator

$$\begin{array}{rcl} G : (0,+\infty) \times C^{4,\alpha}(\overline{\Omega}) \cap E & \longrightarrow & C^{0,\alpha}(\overline{\Omega}) \\ (\lambda, u) & \longmapsto & \Delta^2 u - \gamma \Delta u + \delta u - \lambda f(u), \end{array}$$

where $\alpha \in (0,1)$ and E is the function space defined by

$$E := \{u \in W^{4,2}(\Omega) \mid \Delta u \ = \ u \ = \ 0 \text{ on } \partial\Omega\}. \tag{5}$$

Assume that (P_{λ^*}) has a solution u. Then for any $\lambda \in (0, \lambda^*)$, $u_\lambda \leq u$ in Ω.

Then for every $\lambda \in (0, \lambda^*)$ we have $u_\lambda \leq u^*$ in Ω. Using the monotonicity of u_λ, we deduce that the function

$$u^* = \lim_{\lambda \to \lambda^*} u_\lambda$$

is well defined in Ω and is a semi-stable solution of problem (P_{λ^*}). Assuming that the first eigenvalue $\eta_1(\lambda^*, u^*)$ is positive, we can apply the implicit function theorem to the operator G. For any λ in a neighborhood of λ^* and u in a neighborhood of u^*, we have $G(\lambda, u) = 0$, which proves that the problem (P_λ) has a solution for λ in a neighborhood of λ^*. But this contradicts the definition of λ^*. So, $\eta_1(\lambda^*, u^*) = 0$ and this completes the proof of Theorem 1.

IV. Proof of Theorem 2

Remark 3. *Thanks to Lemma 1 and (ii) of Theorem 1, the critical value λ^* satisfies:*

$$\lambda_1/a \leq \lambda^* \leq \lambda_1/r_0.$$

To prove this theorem, we show that the three assertions are equivalent. And finally, we prove that one hoolds. We shall use the following auxiliary result which is a reformulation of Theorem due to Hörmander [15].

Lemma 3. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$ with smooth boundary. Let (u_n) be a sequence of super-harmonic nonnegative functions defined on Ω. Then the following alternative holds:*

- **:** *(i) either $\lim_{n\to\infty} u_n = \infty$ uniformly on compact subsets of Ω,*
- **:** *(ii) or (u_n) contains a subsequence which converges in $L^1_{loc}(\Omega)$ to some function u.*

Remark 4. *The result by Hörmander is also true if (u_n) is a sequence of a super-harmonic nonnegative functions.*

4.1. *Proof.* **(i)⇒(ii).** By contradiction. We assume that $\lambda^* = \frac{\lambda_1}{a}$. If (P_{λ^*}) has a solution u^* then, as we have already observed in (iv) of Theorem 1, $\eta_1(\lambda^*, u^*) = 0$. Thus, there exists $\in H^4(\Omega)$ satisfying:

$$\begin{aligned} \Delta^2 \psi - \gamma \Delta \psi + \delta \psi - \lambda^* f'(u^*)\, \psi \ &= 0 && \text{in } \Omega \\ \psi \ &> 0 && \text{in } \Omega \\ \Delta \psi = \ \psi \ &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Notes

Using φ_1, given in (1), as a test function and integrating by parts, we obtain

$$\int_\Omega \Big(\Delta^2\varphi_1 - \gamma\Delta\varphi_1 + \delta\varphi_1\Big) \quad - \lambda^* \int_\Omega f'(u^*)\ \psi\ \varphi_1 = 0$$

therefore

$$\int_\Omega \Big(\lambda_1 - \lambda^* f'(u^*)\Big)\ \psi\ \varphi_1 = 0.$$

Notes

$\varphi_1 > 0,\ \psi > 0,\ \lambda^* = \frac{\lambda_1}{a}$ and $a = \sup_{t>0} f'(t)$, we have $\lambda_1 - \lambda^* f'(u^*) \geq 0$, the above equation forces $\lambda_1 - \lambda^* f'(u^*) = 0$. Hence

$$f'(u^*) \equiv a \quad \text{in} \quad \Omega.$$

This implies that $f(t) = at + b$ in $[0, \max_\Omega u^*]$ for some scalar $b > 0$. But there is no positive function in Ω such that $u = \Delta u = 0$ on $\partial\Omega$ and

$$\Delta^2 u - \gamma\Delta u + \delta u = \lambda^* a u + \lambda^* b \quad \text{in } \Omega.$$

If not, using φ_1 and integrating by parts, we have

$$\int_\Omega \Delta^2 u\ \varphi_1 - \gamma \int_\Omega \Delta u\ \varphi_1 + \delta \int_\Omega u\ \varphi_1 = \lambda^* a \int_\Omega u\ \varphi_1 + \lambda^* b \int_\Omega \varphi_1$$

then

$$\int_\Omega \Big(\Delta^2\varphi_1 - \gamma\Delta\varphi_1 + \delta\varphi_1\Big)\ u = \lambda_1 \int_\Omega u\ \varphi_1 + \lambda^* b \int_\Omega \varphi_1$$

i.e.

$$0 = \lambda^* b \int_\Omega \varphi_1 \quad \text{which is impossible.}$$

Hence, problem (P_{λ^*}) has no solution and (i) implies (ii).

4.2. *Proof* **(ii) ⇒(iii).** We assume that (ii) occurs and we claim that $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ uniformly on compact subsets of Ω. By contradiction, suppose that (iii) doesn't hold. By Lemma 4 and up to a subsequence, (u_λ) converges locally in $L^1(\Omega)$ to u^* as $\lambda \to \lambda^*$.

Lemma 4. The minimal solution u_λ of the problem (P_λ) is bounded in $L^2(\Omega)$.

Proof. If not, we define

$$u_\lambda := k_\lambda w_\lambda,$$

with

$$\|w_\lambda\|_2 = 1 \quad \text{and} \quad k_\lambda \to +\infty \quad \text{as} \quad \lambda \to \lambda^*.$$

Since $f(t) \leq at + f(0)$, We have

$$\begin{aligned}
\int_\Omega |\Delta w_\lambda|^2 &\leq \int_\Omega |\Delta w_\lambda|^2 + \gamma \int_\Omega |\nabla w_\lambda|^2 + \delta \int_\Omega w_\lambda^2 \\
&= \int_\Omega \Delta^2 w_\lambda\ w_\lambda - \gamma \int_\Omega \Delta w_\lambda\ w_\lambda + \delta \int_\Omega w_\lambda\ w_\lambda = \int_\Omega \frac{\lambda f(u_\lambda)}{k_\lambda} w_\lambda \\
&\leq \lambda^* \int_\Omega \Big(a\, w_\lambda^2 + \frac{f(0)}{k_\lambda} w_\lambda\Big) \ \leq\ \lambda^* a - c \int_\Omega w_\lambda \\
&\leq \lambda^* a - c\sqrt{|\Omega|},
\end{aligned}$$

where c is a positive constant independent on λ.

Recall that w_λ satisfies $\Delta^2 w_\lambda - \gamma\Delta w_\lambda + \delta w_\lambda = \dfrac{\lambda f(k_\lambda w_\lambda)}{k_\lambda}$ and f is quasilinear. These facts imply that (w_λ) is bounded in $H^4(\Omega)$. Hence, up to a subsequence, we have

$$w_\lambda \rightharpoonup w \text{ weakly in } H^4(\Omega) \text{ and } w_\lambda \to w \text{ strongly in } H^3(\Omega) \text{ as } \lambda \to \lambda^*.$$

Moreover, by the trace theorem, $w = \Delta w = 0$ on $\partial\Omega$. We deduce that

$$\Delta^2 w_\lambda - \gamma\Delta w_\lambda + \delta w_\lambda = \frac{\lambda f(u_\lambda)}{k_\lambda} \longrightarrow 0 \ \text{ in } \ L^1_{loc}(\Omega) \ \text{ as } \ \lambda \to \lambda^*.$$

This implies $\Delta^2 w - \gamma\Delta w_\lambda + \delta w_\lambda = 0$ in $\mathcal{D}'(\Omega)$. So, by Navier boundary condition, we deduce that $w \equiv 0$ in Ω. This contradicts the fact that $\|w\|_2 = \lim\limits_{\lambda\to\lambda^*} \|w_\lambda\|_2 = 1$. This complete the proof of lemma.

Hence, (u_λ) is bounded in $L^2(\Omega)$ and by the same arguments as above, it is bounded in $H^4(\Omega)$ and up to a subsequence, we have

$$u_\lambda \rightharpoonup u \ \text{ weakly in } \ H^4(\Omega) \ \text{ and } \ u_\lambda \to u \ \text{ in } \ L^2(\Omega) \ \text{ as } \ \lambda \to \lambda^*.$$

$$\begin{cases} \Delta^2 u - \gamma\Delta u + \delta u &= \lambda^* f(u) \quad \text{in} \quad \Omega \\ \Delta u = u &= 0 \qquad\quad \text{on} \quad \partial\Omega \end{cases}$$

and this impossible by the hypothesis (ii). This shows that (ii) implies (iii). Moreover, this simply shows that (ii) and (iii) are equivalent.

4.3. *Proof* **(iii)⇒(i).** if (P_{λ^*}) has a solution u^* then the sequence (u_λ) converges to u^* as λ tends to λ^*, which cannot happen in the case where $\lim\limits_{\lambda\to\lambda^*} u_\lambda = \infty$. Hence, (iii) implies (i).

Indeed, clearly if (ii) and (iii) occur, we have $\lim\limits_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Set

$$u_\lambda = k_\lambda \, w_\lambda \ \text{ with } \ \|w_\lambda\|_2 = 1.$$

Then, up to a subsequence, we obtain

$$w_\lambda \rightharpoonup w \ \text{ weakly in } \ H^4(\Omega) \ \text{ and } \ w_\lambda \to w \ \text{ strongly in } \ H^3(\Omega) \ \text{ as } \ \lambda \to \lambda^*.$$

Moreover,

$$\Delta^2 w_\lambda - \gamma\Delta w_\lambda + \delta w_\lambda \longrightarrow \Delta^2 w - \gamma\Delta w + \delta w \quad \text{ in } \mathcal{D}'(\Omega) \ \text{ as } \ \lambda \to \lambda^*$$

and

$$\frac{\lambda}{k_\lambda} f(k_\lambda w_\lambda) \longrightarrow \lambda^* a w \quad \text{ in } \ L^2(\Omega) \ \text{ as } \ \lambda \to \lambda^*.$$

Then,

$$\begin{cases} \Delta^2 w - \gamma\Delta w + \delta w &= \lambda^* a w \quad \text{in } \Omega, \\ \Delta w = w &= 0 \qquad\quad \text{on } \partial\Omega. \end{cases}$$

Multiplying by φ_1, which is defined in (1), we obtain

$$\begin{aligned} \int_\Omega \lambda^* a \, w \, \varphi_1 &= \int_\Omega \Delta^2 w \, \varphi_1 - \gamma \int_\Omega \Delta w \, \varphi_1 + \delta \int_\Omega w \, \varphi_1 \\ &= \int_\Omega \Delta^2 \varphi_1 \, w - \gamma \int_\Omega \Delta v_1 \, w + \delta \int_\Omega \varphi_1 \, w = \int_\Omega \lambda_1 \, \varphi_1 \, w. \end{aligned}$$

Notes

This proves (i). To finish the proof of Theorem 2, we need only to show that $(P_{\lambda_1/a})$ has no solution. Indeed, assume that u is a solution of $(P_{\lambda_1/a})$. Since $f(t) - at \geq 0$, we have

$$\Delta^2 u - \gamma\Delta u + \delta u = \frac{\lambda_1}{a} f(u) \geq \lambda_1 u \ \text{ in } \ \Omega.$$

Multiplying the previous equation by φ_1 and integrating by parts, we get $f(u) = a\,u$ in Ω, which contradicts $f(0) > 0$. This concludes the proof of Theorem 2.

Notes

Remark 5. Observe that the equivalence of the assertions of Theorem 2 does not depend on the sign of l.

V. Proof of Theorem 3

5.1. *Proof* **(i).** For the first part of Theorem 3, we have already seen in Remark 3 that $\lambda_1/a \leq \lambda^* \leq \lambda_1/r_0$. Hence it suffices to prove that $\lambda^* \neq \lambda_1/a$ and $\lambda^* \neq \lambda_1/r_0$. First, assume that $\lambda^* = \lambda_1/a$. By Remark 5, we have

$$\lim_{\lambda\to\lambda^*} u_\lambda = \infty \text{ uniformly on compact subsets of } \Omega.$$

Let u_λ be the minimal solution to (P_λ). Then, multiplying (P_λ) by φ_1 and integrating, we obtain

$$0 = \int_\Omega \Big(\lambda_1\, u_\lambda - \lambda\, f(u_\lambda)\Big)\varphi_1 = \int_\Omega \Big((\lambda_1 - a\lambda)u_\lambda - \lambda(f(u_\lambda) - au_\lambda)\Big)\varphi_1$$

and then

$$\lambda \int_\Omega \varphi_1\Big(f(u_\lambda) - au_\lambda\Big) \geq 0.$$

Passing to the limit in the last inequality as λ tends to λ^*, we find

$$0 \leq l\lambda^* \int_\Omega \varphi_1 < 0,$$

which is impossible and then $\lambda^* \neq \frac{\lambda_1}{a} a$.

Now, assume that $\lambda^* = \lambda_1/r_0$ and let u be a solution of problem (P_{λ^*}). Multiplying (P_{λ^*}) by φ_1 and integrating by parts, we have

$$\lambda_1 \int_\Omega u\varphi_1 = \frac{\lambda_1}{r_0} \int_\Omega f(u)\varphi_1$$

that is

$$\int_\Omega (f(u) - r_0 u)\varphi_1 = 0$$

which forces $f(u) = r_0\, u$ in Ω, so that $f(t) = r_0 t$ in $[0, \max_\Omega u]$. As above, this contradicts the fact that $f(0) > 0$.

5.2. *Proof* **(ii).** Since $\lambda^* > \lambda_1/a$, the existence of a solution to (P_{λ^*}) is assured by Remark 5. Then, it remains to prove the uniqueness. Assume that u is another solution to (P_{λ^*}) and let $w := u - u^*$. Since $u_\lambda < u$ and $\lim_{\lambda\to\lambda^*} u_\lambda = u^*$, we have $w \geq 0$. Then by convexity of f we have

$$\Delta^2 w - \gamma\Delta w + \delta w = \lambda^*\Big(f(u) - f(u^*)\Big) \geq \lambda^* f'(u^*)w \ \text{ in } \Omega.$$

Recall that $\eta_1(\lambda^*, u^*) = 0$, so let be the corresponding eigenfunction. Multiplying the last inequality by and integrating by parts, we find

$$0 = \int_\Omega \lambda^* \Big(f(u) - f(u^*) - f'(u^*)w \Big) \quad \geq 0.$$

Therefore, we must have equality $f(u) - f(u^*) = f'(u^*)w$ in Ω, which implies that f is linear in $[0, \max_\Omega u]$ and this leads a contradiction as in the proof of Theorem 1.

5.3. *Proof* **(iii).** concerning the existence of a non stable solution v_λ of (P_λ) will be proved by using the mountain pass theorem of Ambrosetti and Rabinowitz [3] in the following form:

Theorem 4. *Let E be a real Banach space and $J \in \mathcal{C}^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions:*
$(*)$ *there exist positive constants R and ρ such that*

$$J(u) \geq J(u_0) + \rho, \text{ for all } u \in E \text{ with } \|u - u_0\| = R.$$

$(**)$ *there exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$.*

Then the functional J possesses at least a critical point. The critical value is characterized by

$$c := \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where

$$\Gamma := \Big\{ g \in C([0,1], E) \,|\, g(0) = u_0,\, g(1) = v_0 \Big\}$$

and satisfies

$$c \geq J(u_0) + \rho.$$

In our case,

$$\begin{aligned} \mathcal{J}_\lambda : \quad E &\longrightarrow \mathbb{R} \\ u &\longmapsto \frac{1}{2}\Big(\int_\Omega |\Delta u|^2 + \gamma |\nabla u|^2 + \delta u^2 \Big) - \int_\Omega F(u), \end{aligned}$$

where

$$F(t) = \lambda \int_0^t f(s) ds, \quad \text{for all } t \geq 0.$$

We take u_0 as the stable solution u_λ for each $\lambda \in (\lambda_1/a, \lambda^*)$.

Remark 6. *The energy functional $\mathcal{J}_\lambda$ belongs to $\mathcal{C}^1(E, \mathbb{R})$ and*

$$\langle \mathcal{J}'_\lambda(u), v \rangle = \int_\Omega \Delta u \Delta v + \gamma \int_\Omega \nabla u \nabla v + \delta \int_\Omega uv - \lambda \int_\Omega f(u) v dx, \text{ for all } u,\, v \in E.$$

Since $\eta_1(\lambda, u_\lambda) > 0$, the function u_λ is a strict local minimum for $\mathcal{J}_\lambda$, we apply the mountain pass theorem for $\mathcal{J}_\lambda$. We show in the next lemma that $\mathcal{J}_\lambda$ satisfies the Palais-Smale compactness condition.

Notes

Notes

Lemma 5. *Let $(u_n) \subset E$ be a Palais-Smale sequence; that is,*

$$\sup_{n\in\mathbb{N}} |\mathcal{J}_\lambda(u_n)| < +\infty, \tag{6}$$

$$\|\mathcal{J}'_\lambda(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{7}$$

Then (u_n) is relatively compact in E.

Proof: Since any subsequence of (u_n) verifies (6) and (7) it is enough to prove that (u_n) contains a convergent subsequence. It suffices to prove that (u_n) contains a bounded subsequence in E. Indeed, suppose we have proved this. Then, up to a subsequence, $u_n \to u$ weakly in E, strongly in $L^2(\Omega)$. Now (7) gives

$$\Delta^2 u_n - \gamma\Delta u_n + \delta u_n - \lambda f(u_n) \to 0 \quad \text{in } \mathcal{D}'(\Omega)$$

Note that $f(u_n) \to f(u)$ in $L^2(\Omega)$ because $|f(u_n) - f(u)| \le a|u_n - u|$. This shows that

$$\Delta^2 u_n - \gamma\Delta u_n + \delta u_n \to \lambda f(u) \quad \text{in } \mathcal{D}'(\Omega).$$

That is

$$\Delta^2 u - \gamma\Delta u + \delta u - \lambda f(u) = 0.$$

The above equality multiplied by u gives

$$\int_\Omega |\Delta u|^2 + \gamma \int_\Omega |\nabla u|^2 + \delta \int_\Omega u^2 - \lambda \int_\Omega f(u)u = 0. \tag{8}$$

Now (7) multiplied by (u_n) gives

$$\int_\Omega |\Delta u_n|^2 + \gamma \int_\Omega |\nabla u_n|^2 + \delta \int_\Omega u_n^2 - \lambda \int_\Omega f(u_n)u_n \to 0 \tag{9}$$

in view of the boundedness of (u_n) and the $L^2(\Omega)$-convergence of u_n and $f(u_n)$, we have

$$\lambda \int_\Omega f(u_n)u_n \to \lambda \int_\Omega f(u)u$$

Hence, (8) and (9) give

$$\int_\Omega |\Delta u_n|^2 \to \int_\Omega |\Delta u|^2 \quad \text{and} \quad \gamma \int_\Omega |\nabla u_n|^2 \to \gamma \int_\Omega |\nabla u|^2$$

which insures us that $u_n \to u$ in E.

Actually, it is enough to prove that (u_n) is (up to a subsequence) bounded in $L^2(\Omega)$. Indeed, the $L^2(\Omega)$-boundedness of (u_n) implies that E-boundedness of (u_n) as it can be seen by examining (6).

We shall conclude the proof obtaining a contradiction from the supposition that $\|u_n\|_2 \to \infty$. Let $u_n = k_n w_n$ with $k_n > 0$, $k_n \to \infty$ and $\|w_n\|_2 = 1$. Then

$$0 = \lim_{n\to\infty} \frac{\mathcal{J}_\lambda(u_n)}{k_n^2} = \lim_{n\to\infty} \left[\frac{1}{2}\int_\Omega |\Delta w_n|^2 + \frac{\gamma}{2}\int_\Omega |\nabla w_n|^2 + \frac{\delta}{2}\int_\Omega w_n^2 - \frac{1}{k_n^2}\int_\Omega F(u_n)\right]$$

However, since $|f(t)| \le a|t| + b$, we have

$$|F(u_n)| = |F(k_n w_n)| \le \frac{a\lambda}{2} k_n^2 w_n^2 + b\lambda |k_n w_n|.$$

This shows that

$$\frac{1}{k_n^2}\int_\Omega F(u_n) \le \frac{a\lambda}{2}\int_\Omega w_n^2 + \frac{b\lambda}{k_n}\int_\Omega w_n < \infty.$$

We claim that

$$\Delta^2 w - \gamma\Delta w + \delta w = a\lambda w^+ \quad \text{where } w^+ := \max\{0, w\}. \tag{10}$$

Indeed, (7) divided by k_n gives

$$\int_\Omega \Delta w_n \cdot \Delta v + \gamma\int_\Omega \nabla w_n \cdot \nabla v + \delta\int_\Omega w_n\, v - \lambda\int_\Omega \frac{f(u_n)}{k_n} v \to 0 \tag{11}$$

for each $v \in E$. Now

$$\int_\Omega \Delta w_n \cdot \Delta v + \gamma\int_\Omega \nabla w_n \cdot \nabla v + \delta\int_\Omega w_n\, v \to \int_\Omega \Delta w \cdot \Delta v + \gamma\int_\Omega \nabla w \cdot \nabla v + \delta\int_\Omega w\, v$$

Hence (10) can be concluded from (11) if we show that $1/k_n f(u_n)$ converges (up to a subsequence) to aw^+ in $L^2(\Omega)$. Now $1/k_n f(u_n) = 1/k_n f(k_n w_n)$ and it is easy to see that the required limit is equal to aw in the set $\{x \in \Omega : w_n(x) \to w(x) \neq 0\}$.

If $w(x) = 0$ and $w_n(x) \to w(x)$, let $\varepsilon > 0$ and n_0 be such that $|w_n(x)| < \varepsilon$ for $n \ge n_0$. Then

$$\frac{f(k_n w_n)}{k_n} \le a\varepsilon + \frac{b}{k_n} \quad \text{for such } n,$$

that is the required limit is 0. Thus, $f(u_n)/k_n \to aw^+$ a.e. Here $b = f(0)$. Now $w_n \to w$ in $L^2(\Omega)$ and, thus, up to a subsequence, w_n is dominated in $L^2(\Omega)$ (see [7, Theorem IV.9]).

Since $1/k_n f(u_n) \le a|w_n| + 1/k_n b$, it follows that $1/k_n f(u_n)$ is also dominated. Hence (10) is now obtained. Now (10) and the maximum principle imply that $w \ge 0$ and (10) becomes

$$\begin{aligned} \Delta^2 w - \gamma\Delta w + \delta w &= \lambda a w \quad \text{in } \Omega, \\ w &\ge 0 \quad \text{in } \Omega, \\ \|w\|_2 &= 1 \quad \text{in } \Omega. \end{aligned} \tag{12}$$

Thus from (1), we have $\lambda a = \lambda_1$ and $w = \varphi_1$, which contradicts the fact that $\lambda \neq \lambda_1/a$. This contradiction finishes the proof of the lemma 5.

Now, we need only to check that the two geometric assumptions of theorem 4 are fulfilled.

First, since u_λ is a local minimum of $\mathcal{J}_\lambda$, there exists $R > 0$ such that for all $u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $\mathcal{J}_\lambda(u) \ge \mathcal{J}_\lambda(u_\lambda)$. Then

$$\mathcal{J}_\lambda(u) - \mathcal{J}_\lambda(u_\lambda) = \mathcal{J}_\lambda''(u_\lambda)(u - u_\lambda, u - u_\lambda) + \rho \quad \text{where } \rho > 0.$$

This makes u_λ becomes a strict local minimal for J, which proves $(*)$.

Recall that $\lim_{t\to+\infty}(f(t) - a\,t)$ is finite, then there exists $\beta \in \mathbb{R}$ such that

$$f(t) \ge a\,t + \beta, \quad \forall t > 0.$$

Hence

$$F(t) \ge \frac{a\,\lambda}{2}t^2 + \beta\lambda t, \quad \forall t > 0.$$

Notes

This yields, using the definition of φ_1 mentioned in (1),

$$\mathcal{J}_\lambda(t\varphi_1) = \frac{\lambda_1 - a\lambda}{2} t^2 \int_\Omega \varphi_1^2 - \beta\lambda t \int_\Omega \varphi_1,$$

since $\|\varphi_1\|_2 = 1$, then we have

$$\frac{\mathcal{J}_\lambda(t\varphi_1)}{t^2} = \frac{\lambda_1 - a\lambda}{2} - \frac{\beta\lambda}{t} \int_\Omega \varphi_1 \tag{13}$$

Notes

which implies

$$\limsup_{t\to+\infty} \frac{1}{t^2}\, \mathcal{J}_\lambda(t\varphi_1) \le \frac{\lambda_1 - a\lambda}{2} < 0, \quad \forall \lambda > \lambda_1/a.$$

Therefore

$$\lim_{t\to+\infty} \mathcal{J}_\lambda(t\varphi_1) = -\infty.$$

So, there exists $v_0 \in E$ such that $\mathcal{J}_\lambda(v_0) \le \mathcal{J}_\lambda(u_\lambda)$ and $(**)$ is proved.

Finally, let $\tilde{v}$ (respectively $\tilde{c}$) be the critical point (respectively critical value) of $\mathcal{J}_\lambda$, we recall that the function $\tilde{v}$ belongs to E and satisfies

$$\Delta^2\tilde{v} - \gamma\Delta\tilde{v} + \delta\tilde{v} = \lambda f(\tilde{v}) \quad \text{in } \Omega \quad \text{and} \quad \mathcal{J}_\lambda(\tilde{v}) = \tilde{c}.$$

The next lemma states that the limit of a sequence of unstable solutions is also unstable.

Lemma 6. *Let $u_n \rightharpoonup u$ in $H^2(\Omega)\cap H_0^1(\Omega)$ and $\mu_n \to \mu$ be such that $\eta_1(\mu_n, u_n) < 0$. Then, $\eta_1(\mu, u) < 0$.*

Proof: The fact that $\eta_1(\mu_n, u_n) < 0$ is equivalent to the existence of a $\varphi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\int_\Omega |\Delta\varphi_n|^2 + \gamma \int_\Omega |\nabla\varphi_n|^2 + \delta \int_\Omega \varphi_n^2 \le \mu_n \int_\Omega f'(u_n)\varphi_n^2 \quad \text{with} \quad \int_\Omega \varphi_n^2 = 1 \tag{14}$$

Since $f' \le a$, (14) shows that (φ_n) is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$. Let $\varphi \in E$ be such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $H^2(\Omega) \cap H_0^1(\Omega)$. Then

$$\mu_n \int_\Omega f'(u_n)\varphi_n^2 \to \mu \int_\Omega f'(u)\varphi^2$$

This can be seen by extracting from (φ_n) a subsequence dominated in $L^2(\Omega))$ as in [7, Theorem IV.9]. Now we have

$$\int_\Omega |\Delta\varphi|^2 \le \liminf \int_\Omega |\Delta\varphi_n|^2 \text{ and } \int_\Omega |\nabla\varphi|^2 \le \liminf \int_\Omega |\nabla\varphi_n|^2$$

finally, since $\|\varphi\|_2 = 1$, we obtain

$$\int_\Omega |\Delta\varphi|^2 + \int_\Omega |\nabla\varphi|^2 + \delta \int_\Omega \varphi^2 \le \mu \int_\Omega f'(u)\varphi^2.$$

Obviously, the fact that the function v belongs to $C^4(\bar{\Omega}) \cap E$ follows from a bootstrap argument.

Actually, the next paragraph said a good deal more, giving additional information on precisely the comportment of the instable solution v_λ.

5.4. *Proof* **(iii) (a).** By contradiction, thanks to Lemma 3, there is a sequence of positives scalars (λ_n) and a sequence (v_n) of unstable solutions to (P_{λ_n}) such that $v_n \to v$ in $L^1_{\text{loc}}(\Omega)$ as $\lambda_n \to \lambda_1/a$ for some function v.

We first claim that (v_n) cannot be bounded in E. Otherwise, let $w \in E$ be such that, up to a subsequence,

$$v_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad v_n \to w \text{ strongly in } L^2(\Omega).$$

Therefore,

$$\Delta^2 v_n - \gamma\Delta v_n + \delta v_n \to \Delta^2 w - \gamma\Delta w + \delta w \text{ in } \mathcal{D}'(\Omega),$$

$$f(v_n) \to f(w) \text{ in } L^2(\Omega),$$

which implies that $\Delta^2 w - \gamma\Delta w + \delta w = \frac{\lambda_1}{a} f(w)$ in Ω. It follows that $w \in E$ and solves (P_λ) with λ_1/a in stead of λ. From Lemma 6, we deduce that

$$\eta_1\Big(\frac{\lambda_1}{a}, w\Big) \leq 0. \tag{15}$$

Relation (15) shows that $w \neq u_{\lambda_1/a}$ which contradicts the fact that (P_λ) with λ_1/a in stead of λ has a unique solution. Now, since $\Delta^2 v_n - \gamma\Delta v_n + \delta v_n = \lambda_n f(v_n)$, the unboundedness of (v_n) in E implies that this sequence is unbounded in $L^2(\Omega)$, too. To see this, let

$$v_n = k_n w_n, \quad \text{where } k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \to \infty.$$

Then

$$\Delta^2 w_n - \gamma\Delta w_n + \delta w_n = \frac{\lambda_n}{k_n} f(v_n) \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega).$$

So, we have convergence also in the sense of distributions and (w_n) is seen to be bounded in E with standard arguments. We obtain

$$\Delta^2 w - \gamma\Delta w + \delta w = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

The desired contradiction is obtained since $w \in E$.

5.5. *Proof* **(iii) b.** We end the proof by showing that v_λ tends to u^* uniformly in Ω when λ tends to λ^*.

As before, it is sufficient to prove the $L^2(\Omega)$ boundedness of v_λ near λ^* and to use the uniqueness property of u^*. Assume that $\|v_n\|_2 \to \infty$ as $\lambda_n \to \lambda^*$, where v_n is a solution to (P_{λ_n}). We write again $v_n = k_n w_n$. Then,

$$\Delta^2 w_n - \gamma\Delta w_n + \delta w_n = \frac{\mu_n}{k_n} f(v_n). \tag{16}$$

The fact that the right-hand side of (16) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in E. Let (w_n) be such that (up to a subsequence)

$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \to w \text{ strongly in } L^2(\Omega).$$

A computation already done shows that

$$\Delta^2 w - \gamma\Delta w + \delta w = \lambda^* a w, \quad w \geq 0 \text{ and } \|w\|_2 = 1,$$

which forces λ^* to be λ_1/a. This contradiction concludes the proof.

Notes

Notes

In conclusion, all these results give us a rather clear schema of solutions for the quasilinear case $a \in (0, +\infty)$. An important role in our arguments has played by $l := \lim_{t\to\infty}\Big(f(t) - at\Big)$. We distinguish two different situations strongly depending on the sign of l.

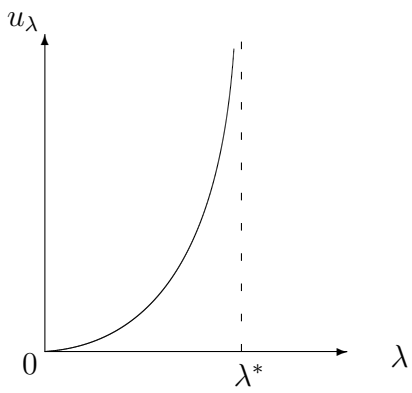

Fig. 1: Behavior of the minimal solution, l > 0

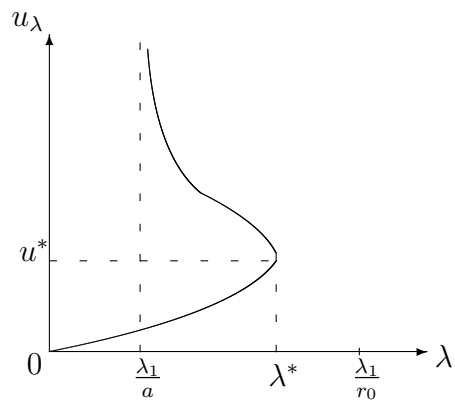

Fig. 2: Bifurcation branches, l < 0

VI. The Dirichlet Problem

References Références Referencias

1. Abid, M. Jleli and N. Trabelsi, *Weak solutions of quasilinear biharmonic problems with positive, increasing and convex nonlinearities.* Analysis and applications. vol.06, No. 03 pp-213-227 (2008).
2. NU. Ahmed, H. Harbi, *Mathematical analysis of dynamic models of suspension bridges.* SIAM J. Appl. Math. 58, 853-874 (1998).
3. A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Funct. Anal. 14 (1973), 349-381.
4. S. Baraket, M. Khtaifi, T. Ouni, *Singular limit solutions for 4-dimensional stationary Kuramoto-Sivashinsky equations with exponential nonlinearity,* EJDE, Vol. 2015 (2015), No. 187, pp. 1–26.
5. S. Baraket, M. Khtaifi, T. Ouni, *Singular limit for 4-dimensional general stationary q-Kuramoto-Sivashinsky (q-KS) equations with exponential nonlinearity,* accepted in Anal Stiinti¨ce ale Universitatii Ovidius Constanta.
6. H. Ben Omrane and S. Khenissy, *Positivity preserving results for a biharmonic equation under Dirichlet boundary conditions,* Opuscula Math. 34, no. 3 (2014), 601-608.
7. H. Brezis, *Analyse Fonctionnelle. Théeorie et Applications,* (Masson, Paris, 1992).
8. H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow up for* $u_t - \Delta u = g(u)$ *revisited,* Adv. Diff. Eq. 1 (1996), 73-90.
9. F.J.S. Correa, J.V.A. Goncalves, Angelo Roncalli, *On a class of fourth order nonlinear elliptic equations under Navier boundary conditions,* Analysis and applications. 8 (2010) 185-197
10. M. Filippakis and N. Papageorgiou, *Multiple solutions for nonlinear elliptic problems with a discontinuous nonlinearity,* Analysis and applications. (Singap.) 4 (2006), 1-18.
11. S. Fucik, A.Kufner, *Nonlinear Differential Equations.* Studies in Applied Mechanics 2. (Amsterdam-New York, Elsevier Scienti¨c Publ. Co. 1980. 360)

12. F. Gazzola, Hans-Christoph Grunau and G.Sweers, *Polyharmonic Boundary Value Problems, Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains,* Springer(1991).
13. M. Ghergu and V. R˘adulescu, *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis,* Oxford Lecture Series in Mathematics and its Applications, vol. 37, (Oxford University Press, 2008).
14. D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* (Springer-Verlag, Heidelberg, 2001).
15. L. Hörmander, *The Analysis of Linear Differential Operators I,* (Springer-Verlag, Berlin, 1983).
16. H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to Partial Differential Equations,* (Springer-Verlag, Berlin, 2003).
17. AC. Lazer, PJ. Mckenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis,* (SIAM Rev. 32(4), 537-578 (1990).
18. AC. Lazer, PJ. Mckenna, *Global bifurcation and a theorem of Tarantello,* Analysis and applications. 181(3), 648-655 (1994).
19. F.H. Lin, Y.S. Yang, *Nonlinear non-local elliptic equation modelling electrostatic actuation,* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. A 463 (2007), 1323-1337.
20. Y. Martel, *Uniqueness of weak solution for nonlinear elliptic problems,* Houston Journal Math. 23 (1997), 161-168.
21. P. Mironescu and V. R˘adulescu, *A bifurcation problem associated to a convex, asymtotically linear function,* C. R. Acad. Sci. Paris, Ser. I 316 (1993), 667-672.
22. P. Mironescu and V. R˘adulescu, *The study of a bifurcation problem associated to an asymptotically linear function,* Nonlinear Analysis 26 (1996), 857-875.
23. CV. Pao, *On fourth-order elliptic boundary value problems,* Proc. Am. Math. Soc. 128, 1023-1030 (2000).
24. J.A. Pelesko, D.H. Bernstein, *Modelling MEMS and NEMS,* New York, NY: Chapman and Hall/CRC, 2003.
25. V. Rădulescu, *Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations,* Contemporary Mathematics and Applications, Vol. 6 (Hindawi Publ. Corp., 2008).
26. S. Sâanouni, N. Trabelsi, *Bifurcation for elliptic forth-order problems with quasilinear source term* Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 92, pp. 1–16.
27. M. Sanchón, *Boundedness of the extremal solution of some p-Laplacian problems,* Nonlinear Anal. 67(1) (2007) 281-294. Universidade de Coimbra, Preprint No. 06-04.
28. G. Tarantella, *A note on a semilinear elliptic problem,* Differ. Integral Equ. 5(3), 561-565 (1992).
29. J. Velin, *A criterion for existence of a positive solution of a nonlinear elliptic system,* Analysis and applications. vol.6, No.3 (2008) 299-321.

Notes

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 18 Issue 8 Version 1.0 Year 2018
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Solutions on Generalized Non-Linear Cauchy-Euler ODE

By Wasihun Assefa Woldeyes & Beletu Worku Beyene
Madda Walabu University

Abstract- In this note, the authors generalize the linear Cauchy-Euler ordinary differential equations (ODEs) into nonlinear ODEs and provide their analytic general solutions. Some examples are presented in order to clarify the applications of interesting results.

Keywords: *Cauchy-Euler equation, nonlinear ODE.*

GJSFR-F Classification: *MSC 2010: 45E05*

SOLUTIONSONGENERALIZEDNONLINEARCAUCHYEULERODE

Strictly as per the compliance and regulations of:

Global Journal of Science Frontier Research (F) Volume XVIII Issue VIII Version 1 Year 2018

Solutions on Generalized Non-Linear Cauchy-Euler ODE

Wasihun Assefa Woldeyes [α] & Beletu Worku Beyene [σ]

Abstract- In this note, the authors generalize the linear Cauchy-Euler ordinary differential equations (ODEs) into nonlinear ODEs and provide their analytic general solutions. Some examples are presented in order to clarify the applications of interesting results.

Keywords: *Cauchy-Euler equation, nonlinear ODE.*

I. Introduction and Preliminary Results

The Cauchy-Euler equation is often one of the first higher order differential equations with variable coefficients (see [1], p. 281, see also [2, 3, 4, 5]). If the independent variable is changed from x to t (via the transformation $x = e^t$), then the resulting equation becomes a linear constant coefficient ODE. The standard technique for solving a linear constant coefficient ODE is to look for exponential solutions. A special class of homogeneous second order Cauchy-Euler ODE has the form

$$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0, \tag{1.1}$$

for constants a and b. This equation actually has what it called a singular point at $x = 0$ which yields trivial solution but we are focus to find non-trial solutions. To solve the equation, alternatively, a solution of the form $y = x^m$ can be tried directly into (1.1) and deduce the characteristic (or auxiliary or indicial) equation

$$m^2 + (a-1)m + b = 0. \qquad (1.2).$$

The general solution of (1.1) depends on the nature of the roots of (1.2) (see, e.g., [6, 7, 8, 9]). That is, if (1.2) has two distinct real roots say m_1 and m_2, then the general solution of (1.1) is $y = c_1x^{m_1} + c_2x^{m_2}$. If (1.2) has double real roots (or the unique real root) m, then the general solution of (1.1) is $y = c_1x^m + c_2x^m \ln x$. And also, if (1.2) has complex conjugates roots $m_{1,2} = \alpha \pm i\beta$, then the general solution of (1.1) is $y = x^\alpha(c_1 \sin(\beta \ln x) + c_2 \cos(\beta \ln x))$, where c_1 and c_2 are some constants.

Author α σ: Department of Mathematics, College of Natural and Computational Science, Madda Walabu University, Bale-Robe, Ethiopia.
e-mails: wasihune2011@gmail.com, bityworku21@gmail.com

Ref

1. Daniel Zwillinger, *Handbook of Differential Equations 3rd edition*, Academic Press, 1997.

The linear none-homogeneous Cauchy-Euler equation has the form

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=r(x). \tag{1.3}$$

Lemma: Let $r(x)$ be a piecewise continuous function on I and let y_1 and y_2 be two linearly independent solutions of (1.1) on I. Then a particular solution y_p of (1.3), [7, 10, 11, 12] is given by

$$y_p=-y_1\int\frac{y_2r(x)}{x^2W(x;y_1,y_2)}dx+y_2\int\frac{y_1r(x)}{x^2W(x;y_1,y_2)}dx, \tag{1.4}$$

where $W(x;y_1,y_2)$ is the Wronskian of y_1 and y_2.

Proof. Let $u(x)$ and $v(x)$ be continuously differentiable functions (to be determined) for some x in I such that

$$y_p=uy_1+vy_2, \tag{1.5}$$

is a particular solution of (1.3). Differentiation of (1.5) leads to

$$\frac{dy_p}{dx}=u\frac{dy_1}{dx}+y_1\frac{du}{dx}+v\frac{dy_2}{dx}+y_2\frac{dv}{dx}. \tag{1.6}$$

We choose u and v so that

$$y_1\frac{du}{dx}+y_2\frac{dv}{dx}=0. \tag{1.7}$$

Using (1.7) in (1.6), we have

$$\frac{dy_p}{dx}=u\frac{dy_1}{dx}+v\frac{dy_2}{dx}\ \ and\ \ \frac{d^2y_p}{dx^2}=u\frac{d^2y_1}{dx^2}+v\frac{d^2y_2}{dx^2}+\frac{du}{dx}\frac{dy_1}{dx}+\frac{dv}{dx}\frac{dy_2}{dx}. \tag{1.8}$$

Since y_p is a particular solution of (1.3), y_1 and y_2 are solutions of (1.1), using (1.5) and (1.8) in (1.3), we obtain the condition

$$\frac{du}{dx}\frac{dy_1}{dx}+\frac{dv}{dx}\frac{dy_2}{dx}=\frac{r(x)}{x^2}. \tag{1.9}$$

Thus from (1.7) and (1.9), we have system of equations as

$$\begin{cases} y_1\frac{du}{dx}+y_2\frac{dv}{dx}=0 \\ \frac{du}{dx}\frac{dy_1}{dx}+\frac{dv}{dx}\frac{dy_2}{dx}=\frac{r(x)}{x^2} \end{cases} \tag{1.10}$$

Applying Cramer's rule for (1.10) and after simplification, we get

$$u=-\int\frac{y_2r(x)}{x^2W(x;y_1,y_2)}dx\ \ and\ \ v=\int\frac{y_1r(x)}{x^2W(x;y_1,y_2)}dx. \tag{1.11}$$

So (1.5) and (1.11) yield the desired results. We complete proof of lemma.

In the next section, we have showed that a general solution of the generalized second order nonlinear Cauchy-Euler equation and examples are presented to clarify the results.

Notes

Notes

II. The Main Results

Theorem. Let $f(y(x))$ be continuously differentiable function. The second order nonlinear homogeneous ordinary differential equation of the form

$$x^2\left(p(y)\frac{d^2y}{dx^2}+q(y)\left(\frac{dy}{dx}\right)^2\right)+axp(y)\frac{dy}{dx}+bf(y)=0, \tag{2.1}$$

has the general solution $f(y)=c_1x^{m_1}+c_2x^{m_2}$ if (1.2) has two distinct real roots. And also (2.1) has the general solution $f(y)=c_1x^m+c_2x^m\ln x$ if (1.2) has double real roots; and (2.1) has the general solution $f(y)=x^\alpha(c_1\sin(\beta\ln x)+c_2\cos(\beta\ln x))$ whenever (1.2) has complex conjugates roots, where a, b, c_1 and c_2 are real constants, $p(y)=\frac{d}{dy}(f(y))$ and $q(y)=\frac{d}{dy}(p(y))$.

If $r(x)$ be a piecewise continuous function and $f_1(y(x))$ and $f_2(y(x))$ be two linearly independent solutions of (2.1), then the second order nonlinear non-homogeneous ODE of the form

$$x^2\left(p(y)\frac{d^2y}{dx^2}+q(y)\left(\frac{dy}{dx}\right)^2\right)+axp(y)\frac{dy}{dx}+bf(y)=r(x), \tag{2.2}$$

has a particular solution $f_p(y(x))$ which is given by

$$f_p(y(x))=-f_1(y)\int\frac{f_2(y)r(x)}{x^2W(x;f_1,f_2)}dx+f_2(y)\int\frac{f_1(y)r(x)}{x^2W(x;f_1,f_2)}dx. \tag{2.3}$$

Where a, b are real constants, $p(y)=\frac{d}{dy}(f(y))$, $q(y)=\frac{d}{dy}(p(y))$, and $W(x;f_1(y),f_2(y))\neq 0$ is the Wronskian of $f_1(y)$ and $f_2(y)$.

Proof. To prove the first identity (2.1), let $\xi=f(y)$, so that $\frac{d\xi}{dx}=p(y)\frac{dy}{dx}$ and $\frac{d^2\xi}{dx^2}=q(y)(\frac{dy}{dx})^2+p(y)\frac{d^2y}{dx^2}$. Then, we have

$$x^2\frac{d^2\xi}{dx^2}+ax\frac{d\xi}{dx}+b\xi=0. \tag{2.4}$$

To solve the equation, plug $\xi=x^m$ into (2.4) and we get the characteristic equation (1.2). Applying (1.1), (1.2) and after simplification, we obtain the general corresponding solutions of (2.1). Hence we complete proof of (2.1).

Next, to prove the second identity (2.2), let $\xi=f(y)$, then we obtain

$$x^2\frac{d^2\xi}{dx^2}+ax\frac{d\xi}{dx}+b\xi=r(x). \tag{2.5}$$

Let $\xi_1=f_1(y)$ and $\xi_2=f_2(y)$ be two linearly independent solutions of (2.4) so that they are also solutions for (2.1). Let $u(x)$ and $v(x)$ be continuously differentiable functions (to be determined) such that

$$\xi_p=f_p(y)=u\xi_1+v\xi_2, \tag{2.6}$$

is a particular solution of (2.5) which is also a particular solution of (2.2). Then applying (1.6)-(1.10) and using (2.6) with its first and second derivatives in (2.5), we have system of equations as

$$\begin{cases} \xi_1 \frac{du}{dx} + \xi_2 \frac{dv}{dx} = 0 \\ \frac{du}{dx}\frac{d\xi_1}{dx} + \frac{dv}{dx}\frac{d\xi_2}{dx} = \frac{r(x)}{x^2} \end{cases} \tag{2.7}$$

Using (1.11) and (2.7) after simplification with little algebra, we get

$$u = -\int \frac{\xi_2 r(x)}{x^2 W(x;\xi_1,\xi_2)} dx \quad and \quad v = \int \frac{\xi_1 r(x)}{x^2 W(x;\xi_1,\xi_2)} dx;$$

$$u = -\int \frac{x^{-2} f_2(y) r(x)}{W(x; f_1(y), f_2(y))} dx \quad and \quad v = \int \frac{x^{-2} f_1(y) r(x)}{W(x; f_1(y), f_2(y))} dx. \tag{2.8}$$

Using (2.6) and (2.8), we arrive at (2.2).

III. Examples

Now let us show the usefulness of the theorem through some examples.

Example 3.1. Solve the following nonlinear ODE for $x \in (0, \frac{\pi}{2})$.

$$2x^2\left(\frac{1}{1+4y^2}\frac{d^2y}{dx^2} - \frac{8y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2\right) - \frac{4x}{1+4y^2}\frac{dy}{dx} + 2\arctan(2y) = x^3,$$

Solution. Let $\xi = \arctan(2y)$ so that

$$\frac{d\xi}{dx} = \frac{2}{1+4y^2}\frac{dy}{dx} \quad and \quad \frac{d^2\xi}{dx^2} = \frac{2}{1+4y^2}\frac{d^2y}{dx^2} - \frac{16y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2.$$

Then the given equation reduce to the form

$$x^2\frac{d^2\xi}{dx^2} - 2x\frac{d\xi}{dx} + 2\xi = x^3.$$

From this, the two linearly independent solutions of the corresponding homogeneous part are $\xi_1 = x = f_1(y)$ and $\xi_2 = x^2 = f_2(y)$. Here the Wronskian $W(x, x^2) = x^2 \neq 0$. Clearly $y_1(x) = \frac{1}{2}\tan(x)$ and $y_2(x) = \frac{1}{2}\tan(x^2)$ are the two linearly independent solutions of the corresponding homogeneous part.

From the above theorem using (2.3), a particular solution ξ_p is given by

$$\xi_p = f_p(y) = -x\int \frac{x^2.x^3}{x^2.x^2}dx + x^2\int \frac{x.x^3}{x^2.x^2}dx = \frac{x^3}{2}.$$

Thus by applying the above theorem (2.2), a particular solution of the given equation is $f_p(y) = \arctan(2y) = \frac{x^3}{2}$ implies $y_p = \frac{1}{2}\tan(\frac{x^3}{2})$.

Notes

Notes

Thus obviously the general solution of the given nonlinear non-homogeneous second order ODE is simply the sum of the general solutions of its corresponding homogeneous part and the particular solution of the non-homogeneous part. Hence $y(x) = c_1 \tan(x) + c_2 \tan(x^2) + \frac{1}{2}\tan(\frac{x^3}{2})$ is the required general solution, for some integral constants c_1 and c_2.

Example 3.2. Find a particular solution of the nonlinear ODE

$$2x^2\left(\frac{1}{1+4y^2}\frac{d^2y}{dx^2} - \frac{8y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2\right) - \frac{4x}{1+4y^2}\frac{dy}{dx} + 2\arctan(2y) = x^4.$$

Solution. Let $\xi = \arctan(2y)$ and using example (*3.1*), we obtain

$$x^2\frac{d^2\xi}{dx^2} - 2x\frac{d\xi}{dx} + 2\xi = x^4.$$

By the above theorem and example (*3.1*), a particular solution is $\xi_p = f_p(y) = \frac{x^4}{6}$, implies that $y_p = \frac{1}{2}\tan(\frac{x^4}{6})$ is the desired solution.

Example 3.3. Find a particular solution of the nonlinear ODE

$$2x^2\left(\frac{1}{1+4y^2}\frac{d^2y}{dx^2} - \frac{8y}{(1+4y^2)^2}\left(\frac{dy}{dx}\right)^2\right) - \frac{4x}{1+4y^2}\frac{dy}{dx} + 2\arctan(2y) = x^4 + x^3.$$

Solution. By the principle of superposition, a particular solution should be

$$y_p = y_{p1} + y_{p2} = \frac{1}{2}\tan\left(\frac{x^3}{2}\right) + \frac{1}{2}\tan\left(\frac{x^4}{6}\right)$$

is the desired solution.

Example 3.4. Find the general solution of the nonlinear ODE

$$x^2\left(\frac{2}{y^3}\left(\frac{dy}{dx}\right)^2 - \frac{1}{y^2}\frac{d^2y}{dx^2}\right) - \frac{7x}{y^2}\frac{dy}{dx} + \frac{13}{y} = \frac{4}{x^3}, \quad for \quad x \in (1,2].$$

Solution. Let $\xi = \frac{1}{y}$. Using ξ and its first and second derivatives in the given equation, we obtain

$$x^2\frac{d^2\xi}{dx^2} + 7x\frac{d\xi}{dx} + 13\xi = \frac{4}{x^3},$$

which has two linearly independent solutions $\xi_1 = x^{-3}\cos(2\ln x)$ and $\xi_2 = x^{-3}\sin(2\ln x)$. Here the Wronskian $W(x;\xi_1,\xi_2) = 2x^{-7} \neq 0 \ for \ x \in (1,2]$. Thus by the above theorem (2.2) and (2.3), we get a particular solution

$$\xi_p = x^{-3}\sin^2(2\ln x) - x^{-3}\cos^2(2\ln x).$$

$$y_p = \frac{x^3}{\sin^2(2\ln x) - \cos^2(2\ln x)}.$$

Hence the required general solution for some constants c_1 and c_2 is

$$y(x) = c_1 \frac{x^3}{\cos(2\ln x)} + c_2 \frac{x^3}{\sin(2\ln x)} + y_p.$$

IV. Concluding Remarks and Observations

In this paper besides having an important history background, it also has interesting applications. Using this paper, we can find the general solutions of a nonlinear Cauchy-Euler equation that can be reduced to the general form of a linear Cauchy-Euler equation [1], which is given as

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = f(x),$$

by using appropriate methods. In particular, the ideas of this paper may be a base to obtain a generalized version of other first order ODEs. Moreover, the approach adopted in this paper was meant to reach both researchers and undergraduate students.

References Références Referencias

1. Daniel Zwillinger, *Handbook of Differential Equations 3rd edition,* Academic Press, 1997.
2. N. Finizio and G. Ladas, *Ordinary Differential Equations with Modern Applications,* Wadsworth Publishing Company, Belmont, CA, 1982.
3. L. Jodar, *Boundary value problems for second order operator differential equations,* Linear Algebra and Its Appls, 91, 1987, 1-12.
4. G. F. Simmons, *Differential Equations with Applications and Historical Notes*, McGraw-Hill Book Company, New York, 1972.
5. G. Valiron, *The Geometric Theory of Ordinary Differential Equations and Algebraic Functions,* Math Sci Press, Brookline, MA, 1950.
6. Peeyush Chandra, A. K. Lal, V. Raghavendra, G. Santhanam, *Notes on Mathematics-102*[1], [1]Supported by a grant from MHRD.
7. Edward Burkard, *Introduction to Ordinary Differential Equations and Some Applications,* 2010.
8. Jeffrey R. Chasnov, *Introduction to Differential Equations: Lecture notes for MATH 2351/2352,* Hong Kong University, 2014.
9. Thomas Kecker, *On the singularity structure of differential equations in the complex plane,* University College London, 2014.
10. N. Euler, T. Wolf[1], PGL Leach[2] and M. Euler, *Linearisable Third Order Ordinary Differential Equations and Generalised Sundman Transformations*, Sweden, 2002.
11. P. K. Subramanian and Melisa *Hendrata, Lecture Notes on Ordinary Differential Equations,* California State University, Los Angeles, 2011.
12. Paul Dawkins, *Differential Equations*, 2007.

Ref

1. Daniel Zwillinger, *Handbook of Differential Equations 3rd edition,* Academic Press, 1997.

Global Journals Guidelines Handbook 2018

www.GlobalJournals.org

FELLOWS

FELLOW OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (FARSS)

Global Journals Incorporate (USA) is accredited by Open Association of Research Society (OARS), U.S.A and in turn, awards "FARSS" title to individuals. The 'FARSS' title is accorded to a selected professional after the approval of the Editor-in-Chief/Editorial Board Members/Dean.

- The "FARSS" is a dignified title which is accorded to a person's name viz. Dr. John E. Hall, Ph.D., FARSS or William Walldroff, M.S., FARSS.

FARSS accrediting is an honor. It authenticates your research activities. After recognition as FARSB, you can add 'FARSS' title with your name as you use this recognition as additional suffix to your status. This will definitely enhance and add more value and repute to your name. You may use it on your professional Counseling Materials such as CV, Resume, and Visiting Card etc.

The following benefits can be availed by you only for next three years from the date of certification:

FARSS designated members are entitled to avail a 40% discount while publishing their research papers (of a single author) with Global Journals Incorporation (USA), if the same is accepted by Editorial Board/Peer Reviewers. If you are a main author or co-author in case of multiple authors, you will be entitled to avail discount of 10%.

Once FARSB title is accorded, the Fellow is authorized to organize a symposium/seminar/conference on behalf of Global Journal Incorporation (USA). The Fellow can also participate in conference/seminar/symposium organized by another institution as representative of Global Journal. In both the cases, it is mandatory for him to discuss with us and obtain our consent.

You may join as member of the Editorial Board of Global Journals Incorporation (USA) after successful completion of three years as Fellow and as Peer Reviewer. In addition, it is also desirable that you should organize seminar/symposium/conference at least once.

We shall provide you intimation regarding launching of e-version of journal of your stream time to time.This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.

The FARSS can go through standards of OARS. You can also play vital role if you have any suggestions so that proper amendment can take place to improve the same for the benefit of entire research community.

As FARSS, you will be given a renowned, secure and free professional email address with 100 GB of space e.g. johnhall@globaljournals.org. This will include Webmail, Spam Assassin, Email Forwarders,Auto-Responders, Email Delivery Route tracing, etc.

The FARSS will be eligible for a free application of standardization of their researches. Standardization of research will be subject to acceptability within stipulated norms as the next step after publishing in a journal. We shall depute a team of specialized research professionals who will render their services for elevating your researches to next higher level, which is worldwide open standardization.

The FARSS member can apply for grading and certification of standards of their educational and Institutional Degrees to Open Association of Research, Society U.S.A. Once you are designated as FARSS, you may send us a scanned copy of all of your credentials. OARS will verify, grade and certify them. This will be based on your academic records, quality of research papers published by you, and some more criteria. After certification of all your credentials by OARS, they will be published on your Fellow Profile link on website https://associationofresearch.org which will be helpful to upgrade the dignity.

The FARSS members can avail the benefits of free research podcasting in Global Research Radio with their research documents. After publishing the work, (including published elsewhere worldwide with proper authorization) you can upload your research paper with your recorded voice or you can utilize chargeable services of our professional RJs to record your paper in their voice on request.

The FARSS member also entitled to get the benefits of free research podcasting of their research documents through video clips. We can also streamline your conference videos and display your slides/ online slides and online research video clips at reasonable charges, on request.

The FARSS is eligible to earn from sales proceeds of his/her researches/reference/review Books or literature, while publishing with Global Journals. The FARSS can decide whether he/she would like to publish his/her research in a closed manner. In this case, whenever readers purchase that individual research paper for reading, maximum 60% of its profit earned as royalty by Global Journals, will be credited to his/her bank account. The entire entitled amount will be credited to his/her bank account exceeding limit of minimum fixed balance. There is no minimum time limit for collection. The FARSS member can decide its price and we can help in making the right decision.

The FARSS member is eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get remuneration of 15% of author fees, taken from the author of a respective paper. After reviewing 5 or more papers you can request to transfer the amount to your bank account.

MEMBER OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (MARSS)

The ' MARSS ' title is accorded to a selected professional after the approval of the Editor-in-Chief / Editorial Board Members/Dean.

The “MARSS” is a dignified ornament which is accorded to a person’s name viz. Dr. John E. Hall, Ph.D., MARSS or William Walldroff, M.S., MARSS.

MARSS accrediting is an honor. It authenticates your research activities. After becoming MARSS, you can add 'MARSS' title with your name as you use this recognition as additional suffix to your status. This will definitely enhance and add more value and repute to your name. You may use it on your professional Counseling Materials such as CV, Resume, Visiting Card and Name Plate etc.

The following benefitscan be availed by you only for next three years from the date of certification.

MARSS designated members are entitled to avail a 25% discount while publishing their research papers (of a single author) in Global Journals Inc., if the same is accepted by our Editorial Board and Peer Reviewers. If you are a main author or co-author of a group of authors, you will get discount of 10%.

As MARSS, you will be given a renowned, secure and free professional email address with 30 GB of space e.g. johnhall@globaljournals.org. This will include Webmail, Spam Assassin, Email Forwarders,Auto-Responders, Email Delivery Route tracing, etc.

We shall provide you intimation regarding launching of e-version of journal of your stream time to time.This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.

The MARSS member can apply for approval, grading and certification of standards of their educational and Institutional Degrees to Open Association of Research, Society U.S.A.

Once you are designated as MARSS, you may send us a scanned copy of all of your credentials. OARS will verify, grade and certify them. This will be based on your academic records, quality of research papers published by you, and some more criteria.

It is mandatory to read all terms and conditions carefully.

Auxiliary Memberships

Institutional Fellow of Global Journals Incorporation (USA)-OARS (USA)

Global Journals Incorporation (USA) is accredited by Open Association of Research Society, U.S.A (OARS) and in turn, affiliates research institutions as "Institutional Fellow of Open Association of Research Society" (IFOARS).

The "FARSC" is a dignified title which is accorded to a person's name viz. Dr. John E. Hall, Ph.D., FARSC or William Walldroff, M.S., FARSC.

The IFOARS institution is entitled to form a Board comprised of one Chairperson and three to five board members preferably from different streams. The Board will be recognized as "Institutional Board of Open Association of Research Society"-(IBOARS).

The Institute will be entitled to following benefits:

The IBOARS can initially review research papers of their institute and recommend them to publish with respective journal of Global Journals. It can also review the papers of other institutions after obtaining our consent. The second review will be done by peer reviewer of Global Journals Incorporation (USA) The Board is at liberty to appoint a peer reviewer with the approval of chairperson after consulting us.

The author fees of such paper may be waived off up to 40%.

The Global Journals Incorporation (USA) at its discretion can also refer double blind peer reviewed paper at their end to the board for the verification and to get recommendation for final stage of acceptance of publication.

The IBOARS can organize symposium/seminar/conference in their country on behalf of Global Journals Incorporation (USA)-OARS (USA). The terms and conditions can be discussed separately.

The Board can also play vital role by exploring and giving valuable suggestions regarding the Standards of "Open Association of Research Society, U.S.A (OARS)" so that proper amendment can take place for the benefit of entire research community. We shall provide details of particular standard only on receipt of request from the Board.

The board members can also join us as Individual Fellow with 40% discount on total fees applicable to Individual Fellow. They will be entitled to avail all the benefits as declared. Please visit Individual Fellow-sub menu of GlobalJournals.org to have more relevant details.

We shall provide you intimation regarding launching of e-version of journal of your stream time to time. This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.

After nomination of your institution as "Institutional Fellow" and constantly functioning successfully for one year, we can consider giving recognition to your institute to function as Regional/Zonal office on our behalf.
The board can also take up the additional allied activities for betterment after our consultation.

The following entitlements are applicable to individual Fellows:

Open Association of Research Society, U.S.A (OARS) By-laws states that an individual Fellow may use the designations as applicable, or the corresponding initials. The Credentials of individual Fellow and Associate designations signify that the individual has gained knowledge of the fundamental concepts. One is magnanimous and proficient in an expertise course covering the professional code of conduct, and follows recognized standards of practice.

Open Association of Research Society (US)/ Global Journals Incorporation (USA), as described in Corporate Statements, are educational, research publishing and professional membership organizations. Achieving our individual Fellow or Associate status is based mainly on meeting stated educational research requirements.

Disbursement of 40% Royalty earned through Global Journals : Researcher = 50%, Peer Reviewer = 37.50%, Institution = 12.50% E.g. Out of 40%, the 20% benefit should be passed on to researcher, 15 % benefit towards remuneration should be given to a reviewer and remaining 5% is to be retained by the institution.

We shall provide print version of 12 issues of any three journals [as per your requirement] out of our 38 journals worth $ 2376 USD.

Other:

The individual Fellow and Associate designations accredited by Open Association of Research Society (US) credentials signify guarantees following achievements:

- The professional accredited with Fellow honor, is entitled to various benefits viz. name, fame, honor, regular flow of income, secured bright future, social status etc.

- In addition to above, if one is single author, then entitled to 40% discount on publishing research paper and can get 10%discount if one is co-author or main author among group of authors.
- The Fellow can organize symposium/seminar/conference on behalf of Global Journals Incorporation (USA) and he/she can also attend the same organized by other institutes on behalf of Global Journals.
- The Fellow can become member of Editorial Board Member after completing 3yrs.
- The Fellow can earn 60% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.
- Fellow can also join as paid peer reviewer and earn 15% remuneration of author charges and can also get an opportunity to join as member of the Editorial Board of Global Journals Incorporation (USA)
- • This individual has learned the basic methods of applying those concepts and techniques to common challenging situations. This individual has further demonstrated an in–depth understanding of the application of suitable techniques to a particular area of research practice.

Note :

"

- In future, if the board feels the necessity to change any board member, the same can be done with the consent of the chairperson along with anyone board member without our approval.
- In case, the chairperson needs to be replaced then consent of 2/3rd board members are required and they are also required to jointly pass the resolution copy of which should be sent to us. In such case, it will be compulsory to obtain our approval before replacement.
- In case of "Difference of Opinion [if any]" among the Board members, our decision will be final and binding to everyone. "

Preferred Author Guidelines

We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

Before and during Submission

Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

1. Authors must go through the complete author guideline and understand and *agree to Global Journals' ethics and code of conduct,* along with author responsibilities.
2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
5. Authors should submit paper in a ZIP archive if any supplementary files are required along with the paper.
6. Proper permissions must be acquired for the use of any copyrighted material.
7. Manuscript submitted *must not have been submitted or published elsewhere* and all authors must be aware of the submission.

Declaration of Conflicts of Interest

It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

Policy on Plagiarism

Plagiarism is not acceptable in Global Journals submissions at all.

Plagiarized content will not be considered for publication. We reserve the right to inform authors' institutions about plagiarism detected either before or after publication. If plagiarism is identified, we will follow COPE guidelines:

Authors are solely responsible for all the plagiarism that is found. The author must not fabricate, falsify or plagiarize existing research data. The following, if copied, will be considered plagiarism:

- Words (language)
- Ideas
- Findings
- Writings
- Diagrams
- Graphs
- Illustrations
- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

Copyright

During submission of the manuscript, the author is confirming an exclusive license agreement with Global Journals which gives Global Journals the authority to reproduce, reuse, and republish authors' research. We also believe in flexible copyright terms where copyright may remain with authors/employers/institutions as well. Contact your editor after acceptance to choose your copyright policy. You may follow this form for copyright transfers.

Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- Single section and succinct.
- An outline of the job done is always written in past tense.
- Concentrate on shortening results—limit background information to a verdict or two.
- Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

Index

save our planet

Global Journal of Science Frontier Research

Visit us on the Web at www.GlobalJournals.org | www.JournalofScience.org
or email us at helpdesk@globaljournals.org

ISSN 9755896