Mathematical Model of Malaria Transmission with Optimal Control in Democratic Republic of the Congo

By Mojeeb AL-Rahman EL-Nor Osman, Cuihong Yang & Isaac Kwasi Adu

Central China Normal University

Abstract- In this paper, we studied the effect of the specific incidence function for the appearance of backward bifurcation in malaria transmission model with standard incidence rate. The stability analysis of disease-free equilibrium (DFE) was investigated, the basic reproduction number R_0, was obtained using the next generation matrix technique, the existence of the endemic equilibrium was also investigated and the existence of feasible region where the model is well-known shows that the model exhibits the backward bifurcation phenomenon when $R_0 < 1$ and the global stability of the endemic equilibrium has been proofed. Further-more, we applied the model to exiting data of the Democratic Republic of the Congo (DRC) to fit some parameters.

Keywords: Stability analysis, standard incidence, simulation; backward bifurcation, optimal control.

GJSFR-F Classification: MSC 2010: 00A71

MATHEMATICALMODELOFMALARIATRANSMISSIONWITHOPTIMALCONTROLINDEMOCRATICREPUBLICOFTHECONGO

Strictly as per the compliance and regulations of:

Mathematical Model of Malaria Transmission with Optimal Control in Democratic Republic of the Congo

Mojeeb AL-Rahman EL-Nor Osman $^{\alpha}$, Cuihong Yang $^{\sigma}$ & Isaac Kwasi Adu $^{\rho}$

Abstract: In this paper, we studied the effect of the specific incidence function for the appearance of backward bifurcation in malaria transmission model with standard incidence rate. The stability analysis of disease-free equilibrium (DFE) was investigated, the basic reproduction number R_0, was obtained using the next generation matrix technique, the existence of the endemic equilibrium was also investigated and the existence of feasible region where the model is well-known shows that the model exhibits the backward bifurcation phenomenon when $R_0 < 1$ and the global stability of the endemic equilibrium has been proved. Furthermore, we applied the model to exiting data of the Democratic Republic of the Congo (DRC) to fit some parameters. In addition to that, we formulated an optimal control problem with an objective function, with three controls, the preventive using Long-Lasting Insecticide Treated Net (LLITN) $u_1(t)$, the treatment with drug of infected individuals $u_2(t)$ and the insecticide spray on the breeds grounds for the mosquitoes $u_3(t)$, has been used as control measures for infected individuals. Numerical simulations that were carried out to support our analytic results also suggest that, two control strategies $u_1(t)$ and $u_2(t)$ together are more effective than other controls in controlling the number of infected individuals in the DRC. Reducing the number of infected individuals and increasing the number of recovered humans with reduce the disease transmission.

Keywords: Stability analysis, standard incidence, simulation; backward bifurcation, optimal control.

I. Introduction

Malaria is a dangerous parasitic disease in less developed countries, especially, in Sub-Saharan Africa, causing high morbidity and mortality. It is estimated that nearly 300 to 400 million malaria cases occur worldwide, out of which 1.52 million die every year [1,2]. In 2016, an estimated 216 million cases of malaria occurred world wide compared with 237 million cases in 2010 and 214 million new cases of malaria and deaths in 2015. Approximately 80% of malaria death are concentrated in 15 countries most of them in Africa [3,4]. Five species of *Plasmodium* can infect humans:*P.falciparum, P.vivax, P.ovale, P.malariae* and *P.knowlesi*. Among these species of human malaria parasites, *P.falciparum* is the most dangerous. The female *Anopheles* mosquito is the primary vector of malaria parasite [1,2]. Malaria is a major health problem in the Democratic Republic of the Congo

Author α: School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PRC. Department of Mathematics and Computer Science, Faculty of Pure and Applied Sciences, International University of Africa, P.O.Box 2469, Khartoum, Sudan.
e-mail: mojeeb osman@yahoo.com
Author σ: School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PRC.
Author ρ: School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, PRC, Department of Mathematics, Valley View University, Techiman Campus, P.O.Box 183 B/A,Ghana.

(DRC) which is one of the two highest leading contributors to the global burden of illness [5]. Daily in the DRC, more than 400 children die, and nearly half the deaths is caused by malaria [6].

Mathematical models of malaria transmission are useful to providing better insights into the behavior of the disease. These models has played a great role in influencing

Figure 1: Areas of malaria transmission- Democratic Republic of the Congo provided by CDC [19]

the decision making processes regarding inter vention strategies for preventing and controlling the resurgence of malaria. The study of malaria using mathematical modeling began in 1911 with Ronald Ross [7, 8]. Many researches have studied the transmission of malaria through mathematical models specifically using SEIR model for humans and SEI for the mosquitoes [9–16]. A lot of works have also been done on modeling the malaria transmission and control using SEIR-SEI model [17, 18].

In this paper, we study SEIR-SEI malaria transmission model with standard incidence rate that was presented by [15] and applied it to estimate parameters with real data of Democratic Republic of the Congo. Furthermore, we modified the model with three different control strategies, $u_1(t)$, $u_2(t)$ and $u_3(t)$. Our goal is to minimize the number of malaria Infected individuals in Democratic Republic of the Congo and advice the government to set a program to reduce and control the disease from the country. The rest of the paper is organized as follows: Section 2 presents the model description, positivity of the solutions, existence of equilibria and bifurcation. Numerical simulations analysis for the model is given in Section 3. Analysis of optimal control is presented in section 4. In Section 5, we perform the numerical simulations of optimal control and the conclusion is given in Section 6.

II. Mathematical Model

a) Model description

We study a $SEIR - SEI$ seven dimensional malaria transmission model consisting of two groups of populations, host (human) and vector (mosquito) populations. The host population is divided into four compartment: Susceptible $S_h(t)$, Expose $E_h(t)$, Infectious $I_h(t)$ and Recovered $R_h(t)$ humans respectively. The vector population is divided into three compartment: Susceptible $S_m(t)$, Exposed $E_m(t)$ and Infectious $I_m(t)$ mosquitoes respectively. We assume that the mosquitoes never recover from malaria infection. Thus,

Ref

6. Humanitarian Health Action, Malaria. Available at: https://www.who.int/hac/donorinfo/campaigns/cod/malaria/en/

the total number of host and vector populations are represented by $N_h = S_h(t) + E_h(t) + I_h(t) + R_h(t)$ and $N_m = S_m(t) + E_m(t) + I_m(t)$ respectively. σ_m is the biting rate of female mosquitoes (assume to be constant), which depends on many number of environmental and climatic factors. The number of bites per humans is $\sigma_m \frac{N_m}{N_h}$, the force of infection from vector to host, denoted by λ_h and it is defined as:

$$\lambda_h = \beta_h \sigma_m \frac{I_m}{N_h} \tag{2.1}$$

By the same way, the force of infection from host to vector is denoted by λ_m and it is defined as:

$$\lambda_m = \sigma_m (\beta_m \frac{I_h}{N_h} + \beta_{mh} \frac{R_h}{N_h}) \tag{2.2}$$

Also, we assume that recovered human can transfer the infection to mosquitoes, but it is transmission rate is less than infected human transmission rate that is $(\beta_{mh} < \beta_h)$. human enter the susceptible compartment through birth or immigration rate (recruitment rate) Λ_h. The parasite will be passed onto human at infection rate λ_h and that human will shift to the exposed compartment. Human enter the infection compartment at the rate α_h, after infection humans recover, they will mo ve to the recovered compartment at rate γ_h and die from malaria at rate δ. The recovered individuals can again joined the susceptible compartment after losing it's temporary immunity at rate ρ. Humans leave the population through the natural death rate μ and the infectious human leaves the population also at malaria death rate δ. The rest of the model parameters are listed in Table 1. Applying all the above assumptions the model is described by the following seven nonlinear system of differential equations.

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda_h - \dfrac{\sigma_m \beta_h I_m}{N_h} S_h - \mu S_h + \rho R_h, \\[2mm] \dfrac{dE_h}{dt} = \dfrac{\sigma_m \beta_h I_m}{N_h} S_h - (\alpha_h + \mu) E_h, \\[2mm] \dfrac{dI_h}{dt} = \alpha_h E_h - (\gamma_h + \mu + \delta) I_h, \\[2mm] \dfrac{dR_h}{dt} = \gamma_h I_h - (\rho + \mu) R_h, \\[2mm] \dfrac{dS_m}{dt} = \Lambda_m - \sigma_m \dfrac{(\beta_m I_h + \beta_{mh} R_h)}{N_h} S_m - \zeta S_m, \\[2mm] \dfrac{dE_m}{dt} = \sigma_m \dfrac{(\beta_m I_h + \beta_{mh} R_h)}{N_h} S_m - (\alpha_m + \zeta) E_m, \\[2mm] \dfrac{dI_m}{dt} = \alpha E_m - \zeta I_m, \end{cases} \tag{2.3}$$

With the initial conditions: $S_h(0) > 0, E_h(0) \geq 0, I_h(0) \geq 0,\ R_h(0) \geq 0,\ S_m(0) > 0,\ E_m(0) \geq 0,\ I_m(0) \geq 0$.

b) Positivity and boundedness of the solutions

Adding all the human and mosquito equations in (2.3) we respectively obtain

$$\frac{N_h}{dt} = \Lambda_h - \mu N_h - \delta I_h, \tag{2.4}$$

and

$$\frac{N_m}{dt} = \Lambda_m - \zeta N_m,\tag{2.5}$$

It is clear that the solutions N_m of the equation (2.5) approach the equilibrium point $\frac{\Lambda_m}{\zeta}$ when $t \to \infty$. Also, it follows from equation (2.4) that

$$\frac{N_h}{dt} \leq \Lambda_h - \mu N_h \quad and \quad \frac{N_h}{dt} \leq 0 \quad if \quad N_h \geq \frac{\Lambda_h}{\mu},\tag{2.6}$$

By using the standard comparison theorem [20, 21], it can be verified that the basic dynamical features of model (2.3) will be discussed in the following lemma.

Lemma 2.1. Let $(S_h, E_h, I_h, R_h, S_m, E_m, I_m)$ be the solution of the model (2.3) with initial conditions $S_h > 0, E_h \geq 0, I_h \geq 0, R_h \geq 0, S_m > 0, E_m \geq 0$ and $I_m \geq 0$. The closed region $\Gamma = \left\{ (S_h, E_h, I_h, R_h, S_m, E_m, I_m) \in \mathbb{R}^7_+ | S_h + E_h + I_h + R_h \leq \frac{\Lambda_h}{\mu}, S_m + E_m + I_m \leq \frac{\Lambda_m}{\zeta} \right\}.$ is positively invariant and attracting under the flow that is explained by the system (2.3).

c) *Existence of equilibria*
i. *Disease-Free Equilibrium and Reproduction Number*

When there is no disease, that is $E_h = I_h = R_h = E_m = I_m = 0$, the system (2.3) has a disease-free equilibrium (DFE) P_0 , which is obtained by setting the right hand side of the model (2.3) to zero and defined by

$$P_0 = (\frac{\Lambda_h}{\mu}, 0, 0, 0, \frac{\Lambda_m}{\zeta}, 0, 0),\tag{2.7}$$

Using the next generation techniques [22, 23], the stability of P_0 can be established as follows. Initially, we define $x = (E_h, I_h, R_h, E_m, I_m, S_h, S_m)^T$. The model (2.3) is rewritten in the following form

$$\frac{d}{dt}x(t) = \mathcal{F}(t, x(t)) - \mathcal{V}(t, x(t)),\tag{2.8}$$

where

$$\mathcal{F}(t, x(t)) = \begin{bmatrix} \frac{\sigma_m \beta_h I_m S_h}{N_h} \\ 0 \\ 0 \\ \frac{\sigma_m(\beta_m I_h + \beta_{mh} R_h)}{N_h} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad and \quad \mathcal{V}(t, x(t)) = \begin{bmatrix} (\alpha_h + \mu)E_h \\ -\alpha_h E_h + (\gamma_h + \mu + \delta)I_h \\ -\gamma_h I_h + (\rho + \mu)R_h \\ (\alpha_m + \zeta)E_m \\ -\alpha_m E_m + \zeta I_m \\ -\Lambda_h + \frac{\sigma_m \beta_h I_m S_h}{N_h} - \rho R_h + \mu S_h \\ -\Lambda_m + \frac{\sigma_m(\beta_m I_h + \beta_{mh} R_h)S_h}{N_h} + \zeta S_m \end{bmatrix}.$$

20. S.M. Garba et al, Backward bifurcarions in dengure transmission dynamics, Math. Biosci.215(2008) 11-25.

Ref

Consequently, it was found that the system (2.8) has DFE, $x_0 = (0, 0, 0, 0, 0, \frac{\Lambda_h}{\mu}, \frac{\Lambda_m}{\zeta})$, which is identical to P_0, of the model (2.3). So, the derivatives $D\mathcal{F}(P_0)$ and $D\mathcal{V}(P_0)$ are written as

$$D\mathcal{F}(P_0) = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}, \quad and \quad D\mathcal{V}(P_0) = \begin{bmatrix} V & 0 \\ J_3 & J_4 \end{bmatrix},$$

where F (non-negative) and V (a non-singular M-matrix) are 5×5 matrices defined as the following respectively, and J_3, J_4 are matrices associated with the transmission terms of the system (2.3) and all eigenvalues of J_4 have positive real parts.

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & \sigma_m\beta_h \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sigma_m\beta_m\mu\Lambda_m}{\zeta\Lambda_h} & \frac{\sigma_m\beta_{mh}\mu\Lambda_m}{\zeta\Lambda_h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad and \quad V = \begin{bmatrix} K_1 & 0 & 0 & 0 & 0 \\ -\alpha_h & K_2 & 0 & 0 & 0 \\ 0 & -\gamma_h & K_3 & 0 & 0 \\ 0 & 0 & 0 & K_4 & 0 \\ 0 & 0 & 0 & -\alpha_m & \zeta \end{bmatrix},$$

Letting

$$\mathcal{R}_0 = \rho(FV^{-1}) = \begin{bmatrix} 0 & 0 & 0 & 0 & \sigma_m\beta_h \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sigma_m\beta_m\mu\Lambda_m}{\zeta\Lambda_h} & \frac{\sigma_m\beta_{mh}\mu\Lambda_m}{\zeta\Lambda_h} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{K_1} & 0 & 0 & 0 & 0 \\ \frac{\alpha_h}{K_1K_2} & \frac{1}{K_2} & 0 & 0 & 0 \\ \frac{\alpha_h\gamma_h}{K_1K_2K_3} & \frac{\gamma_h}{K_2K_3} & \frac{1}{K_3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{K_4} & 0 \\ 0 & 0 & 0 & \frac{\alpha_m}{\zeta K_4} & \frac{1}{\zeta} \end{bmatrix},$$

Thus, $\mathcal{R}_0$ can be defined as

$$\mathcal{R}_0 = \frac{\sigma_m\sqrt{\beta_h\mu\Lambda_h\Lambda_m\alpha_h\alpha_m K_1 K_2 K_3 K_4 K_5}}{\zeta\Lambda_h K_1 K_2 K_3 K_4}, \tag{2.9}$$

Where $K_1 = (\alpha_h + \mu), K_2 = (\gamma_h + \mu + \delta), K_3 = (\rho + \mu), K_4 = (\alpha_m + \zeta)$ and $K_5 = (\beta_m K_3 + \beta_{mh}\gamma_h)$. Hence, according to the above, the following stability result follows.

Lemma 2.2. *The DFE, P_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

ii. *Endemic equilibria and backward bifurcation*

Let $S_h^*, E_h^*, I_h^*, R_h^*, S_m^*, E_m^*, I_m^*$ represent the arbitrary equilibrium points of S_h, E_h, I_h, R_h S_m, E_m, I_m respectively. Furthermore, using the technique employed in [20] and let

$$\lambda_h^* = \frac{\sigma_m\beta_h I_m^*}{N_h^*} \quad and \quad \lambda_m^* = \sigma_m\frac{\beta_m I_h^* + \beta_{mh}R_h^*}{N_h^*}, \tag{2.10}$$

λ_h^*, λ_m^* are the force infection of human and mosquito at the equilibria points, respectively. To express the equilibria point in terms of λ_h^*, λ_m^* setting all the derivatives in model (2.3)

to be zero and solving, we obtained the following

$$S_h^* = \frac{\Lambda_h K_1 K_2 K_3}{K_1 K_2 K_3(\lambda_h^* + \mu) - \rho\gamma_h\lambda_h^*},$$

$$E_h^* = \frac{\Lambda_h K_2 K_3 \lambda_h^*}{K_1 K_2 K_3(\lambda_h^* + \mu) - \rho\gamma_h\lambda_h^*},$$

$$I_h^* = \frac{\Lambda_h \alpha_h K_3 \lambda_h^*}{K_1 K_2 K_3(\lambda_h^* + \mu) - \rho\gamma_h\lambda_h^*},$$

$$R_h^* = \frac{\Lambda_h \gamma_h \alpha_h \lambda_h^*}{K_1 K_2 K_3(\lambda_h^* + \mu) - \rho\gamma_h\lambda_h^*}, \tag{2.11}$$

$$S_m^* = \frac{\Lambda_m}{(\lambda_m^* + \zeta)},$$

$$E_m^* = \frac{\Lambda_m \lambda_m^*}{K_4(\lambda_m^* + \zeta)},$$

$$I_m^* = \frac{\alpha_m \Lambda_m \lambda_m^*}{\zeta K_4(\lambda_m^* + \zeta)},$$

Substituting (2.11) into (2.10) and after simple calculation we obtained

$$\lambda_h^* = \frac{\sigma_m \beta_h \alpha_h \Lambda_m \lambda_m^*(K_1 K_2 K_3(\lambda_h^* + \mu) - \rho\gamma_h\alpha_h\lambda_h^*)}{\zeta K_4 \Lambda_h(\lambda_m^* + \zeta)(K_1 K_2 K_3 + K_6\lambda_h^*)}, \tag{2.12}$$

$$\lambda_m^* = \frac{\sigma_m \alpha_m K_5 \lambda_m^*}{(K_1 K_2 K_3 + K_6\lambda_h^*)}, \tag{2.13}$$

where $K_6 = \alpha_h K_3 + K_2 K_3 + \alpha_h\gamma_h$. Also, substituting (2.13) into (2.12) gives the following equation in terms of λ_h^*:

$$f(\lambda_h^*) = b_0\lambda_h^{*2} + b_1\lambda_h^* + b_2 = 0, \tag{2.14}$$

where $b_0 = \zeta\Lambda_h K_4 K_6(\sigma_m\alpha_h K_5 + \zeta K_6)$, $b_1 = S(\mathcal{R}_1^* - 1)$ and $b_2 = \Lambda_h\zeta^2 K_1^2 K_2^2 K_3^2 K_4(1 - \mathcal{R}_0^2)$, with $K = \sigma_m\alpha_h K_5 + 2\mu_m K_6$, $\mathcal{R}_1^* = \frac{K\mu_m\Lambda_h K_1 K_2 K_3 K_4}{S}$. and $S = \sigma^2\alpha_h\alpha_m\beta_h\Lambda_m K_5(\mu_h(\mu_h K_3 + \delta K_3 + \alpha_h\gamma_h) + \alpha_h K_2 K_5)$. Thus, the positive endemic equilibria of the model (2.3) are given by (2.14), substituting the positive value of λ_h^* into the equations (2.11). Denoting $\mathcal{R}_1^* = \frac{K\mu_m\Lambda_h K_1 K_2 K_3 K_4}{S}$.

Thus, $b_1 = S(\mathcal{R}_1^* - 1)$. Obviously, the factor b_0 is always positive and b_2 is positive if $\mathcal{R}_0$ is less than one and negative if $\mathcal{R}_0$ is greater than one. Since $a_0 > 0$, the existence of the positive solutions of equation (2.14), will depend on the signs of b_1 and b_2. If $\mathcal{R}_0 > 1$ then $b_2 < 0$ and (2.14) has only positive solution. So there is unique endemic equilibrium whenever $\mathcal{R}_0 > 1$. If $\mathcal{R}_0 = 1$. then $b_2 = 0$ and (2.14) has a unique nonzero solution of $\lambda_h^* = \frac{-b_1}{b_0}$, which is positive if and only if $b_1 < 0$ and negative solution if $b_1 > 0$ when $\mathcal{R}_1^* > 1$. Subsequently, no endemic equilibrium exist if $\mathcal{R}_0 = 1$ and $b1 > 0$, the case $\mathcal{R}_0 < 1$ makes $b_2 > 0$. if b_1 which corresponds to $\mathcal{R}_1^*$, (2.14), has two positive solutions.

$$\lambda_{h,Large}^* = \frac{-b_1 - \sqrt{b_1^2 - 4b_0 b_2}}{2b_0},$$

$$\lambda_{h,Small}^* = \frac{-b_1 + \sqrt{b_1^2 - 4b_0 b_2}}{2b_0},$$

Thus, let $b_1^2 - 4b_0 b_2 = 0$. Solving for the critical value of $\mathcal{R}_0$, denoted by $\mathcal{R}_2$, gives

$$\mathcal{R}_2 = \sqrt{1 - \frac{b_1^2}{4b_0\Lambda_h\mu^2 K_1^2 K_2^2 K_3^2 K_4}}, \tag{2.15}$$

from all the above analysis $b_1^2 - 4b_0b_2 < 0 \Leftrightarrow \mathcal{R}_0 < \mathcal{R}_2$, $b_1^2 - 4b_0b_2 = 0 \Leftrightarrow \mathcal{R}_0 = \mathcal{R}_2$, $b^2_1 - 4b_0b_2 > 0 \Leftrightarrow \mathcal{R}_0 > \mathcal{R}_2$. consequently, we have following results on existence of the endemic equilibrium.

Theorem 2.3 The model (2.3) *has*
(i) a unique endemic equilibrium in Γ *if* $\mathcal{R}_0 > 1$,
(ii) a unique endemic equilibrium in Γ *when* $\mathcal{R}_0 = 1$,
(iii) a unique endemic equilibrium of multiplicity in Γ *where* $\mathcal{R}_0 = \mathcal{R}_2 < 1$ *and* $\mathcal{R}_1^* < 1$,
(iv) two endemic equilibria P_{Large}^* *and* P_{Small}^* *in* Γ *when* $\mathcal{R}_2 < \mathcal{R}_0 < 1$ *and* $\mathcal{R}_1^* < 1$,
(v) no endemic equilibrium otherwise.

Theorem 2.3 establishes that $\mathcal{R}_0 = 1$ is the bifurcation value. In fact, a cross $\mathcal{R}_0 = 1$ the disease-free equilibrium, P_0, changes its stability properties, see Lemma 2.2. This implies that when $\mathcal{R}_0$ is less that one, so that an infective replaces itself with less one new infective, then the disease die out in time.

Consider the system

$$\frac{dy}{dt} = f(y, \phi), \tag{2.16}$$

where ϕ is the bifurcation parameter, f is continuously differentiable at least in both y and ϕ. The DFE is the line $(y_0; \phi)$ and the local stability of the DFE changes at the point $(y_0; \phi)$ see [23] our main focus on DFE, P_0 to investigate the appearance of the transcritical bifurcation at $\mathcal{R}_0 = 1$. Clearly, $\mathcal{R}_0 = 1$ is equivalent to

$$\beta_h = \beta_h^* = \frac{\Lambda_h \zeta^2 K_1 K_2 K_3 K_4}{\mu \alpha_h \sigma_m^2 \alpha_m \Lambda_m K_5}, \tag{2.17}$$

By Lemma 2.2, the DFE P_0 is locally stable when $\beta_h < \beta_h^*$ and unstable when $\beta_h > \beta_h^*$. Thus, $\beta_h = \beta_h^*$ is a bifurcation value.
Let $S_h = y_1, E_h = y_2, I_h = y_3, R_h = y_4, S_m = y_5, E_m = y_6$ and $I_m = y_7$, such that $N_h = y_1 + y_2 + y_3 + y_4$ and $N_m = y_5 + y_6 + y_7$. Furthermore, using vector notations we can write $y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)^T$ and $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T$, the system (2.3) can be written as

$$\frac{dy}{dt} = f(y, \beta_h), \tag{2.18}$$

such that:

$$f_1 = \Lambda_h + \rho y_4 - \frac{\sigma_m \beta_h y_7 y_1}{(y_1 + y_2 + y_3 + y_4)} - \mu y_1,$$

$$f_2 = \frac{\sigma_m \beta_h y_7 y_1}{(y_1 + y_2 + y_3 + y_4)} - (\alpha_h + \mu)y_2,$$

$$f_3 = \alpha_h y_2 - (\gamma_h + \mu + \delta)y_3,$$

$$f_4 = \gamma_h y_3 - (\rho + \mu)y_4,$$

$$f_5 = \Lambda_m - \sigma_m \frac{\beta_m y_3 + \beta_{mh} y_4}{(y_1 + y_2 + y_3 + y_4)} y_5 - \zeta y_5,$$

$$f_6 = \sigma_m \frac{\beta_m y_3 + \beta_{mh} y_4}{(y_1 + y_2 + y_3 + y_4)} y_5 - (\alpha_m + \zeta)y_6, \qquad (2.19)$$

$$f_7 = \alpha_m y_6 - \zeta y_7,$$

The Jacobian matrix of the model (2.19) evaluated at, P_0, when $\beta_h = \beta_h^*$ is given by

$$J(P_0, \beta_h^*) = \begin{bmatrix} -\mu & 0 & 0 & \rho & 0 & 0 & -\sigma_m \beta_h^* \\ 0 & -K_1 & 0 & 0 & 0 & 0 & \sigma_m \beta_h^* \\ 0 & \alpha_h & -K_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_h & -K_3 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\sigma_m \beta_m \mu \Lambda_m}{\zeta \Lambda_h} & -\frac{\sigma_m \beta_{mh} \mu \Lambda_h}{\zeta \Lambda_h} & -\zeta & 0 & 0 \\ 0 & 0 & \frac{\sigma_m \beta_m \mu \Lambda_m}{\zeta \Lambda_h} & \frac{\sigma_m \beta_{mh} \mu \Lambda_h}{\zeta \Lambda_h} & 0 & -K_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_v & -\mu \end{bmatrix} \qquad (2.20)$$

It's clear that the eigenvalues of (2.20) admits a simple zero eigenvalue and the other eigenvalues are real and negative. Thus, the P_0, is a nonhyperbolic equilbrium when $\beta_h = \beta_h^*$, that is according to center manifold theorem see [24], there are two important quantities a and b of the normal form representing the dynamics of the system on the center manifold. If $a < 0$ and $b > 0$, then the bifurcation is forward and if $a > 0$ and $b > 0$, then the bifurcation is backward. Using the approach in [24],we establish the following calculations.

Eigenvector of the Jacobian matrix at P_0 when $\beta_h = \beta_h^*$

Let $w = (w_1, w_2, w_3, w_4, w_5, w_6, w_7)^T$ and $v = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)^T$ be a right and left eigenvectors of $J(P_0, \beta_h^*)$, respectively. After simple calculation it follows that the components of the right eigenvectors w are given by

$$w_1 = -\frac{(\mu(K_1 K_3 + \gamma_h K_3 + \alpha_h \gamma_h) + \delta K_1 K_2)}{\mu K_2 K_3} w_2, \quad w_3 = \frac{\alpha_h w_2}{K_2}, \quad w_4 = \frac{\alpha_h \gamma_h w_2}{K_2 K_3},$$

$$w_5 = -\frac{\mu \alpha_h \sigma_h \Lambda_m K_5 w_2}{\zeta^2 \Lambda_h K_2 K_3}, \quad w_6 = \frac{\mu \alpha_h \sigma_m \Lambda_m K_5 w_2}{\zeta \Lambda_h K_2 K_3 K_4}, \quad w_7 = \frac{\mu \alpha_h \sigma_m \alpha_m \Lambda_m K_5 w_2}{\zeta_2 \Lambda_h K_2 K_3 K_4}, \quad w_2 > 0,$$

By the same way, the components of the left eigenvector v are given by

$$v_1 = 0, \quad v_2 = \frac{\mu \alpha_h \sigma_m \alpha_m \Lambda_m K_5 v_7}{\zeta \Lambda_h K_1 K_2 K_3 K_4}, v_3 = \frac{\mu \sigma_m \alpha_m \Lambda_m K_5 v_7}{\zeta \Lambda_h K_2 K_3 K_4},$$

$$v_4 = \frac{\mu \sigma_m \alpha_m \Lambda_m \beta_{mh} v_7}{\zeta \Lambda_h K_3 K_4}, \quad v_5 = 0, \quad v_6 = \frac{\alpha_h v_7}{K_4}, \quad v_7 > 0,$$

The non-zero partial derivatives of f evaluated at P_0 associated with the system (2.19) are given by

$$\frac{\partial^2 f_1}{\partial y_2 \partial y_7} = \frac{\partial^2 f_1}{\partial y_3 \partial y_7} = \frac{\partial^2 f_1}{\partial y_4 \partial y_7} = \frac{\sigma_m \beta_h^* \mu}{\Lambda_h},$$

$$\frac{\partial^2 f_2}{\partial y_2 \partial y_7} = \frac{\partial^2 f_2}{\partial y_3 \partial y_7} = \frac{\partial^2 f_2}{\partial y_4 \partial y_7} = -\frac{\sigma_m \beta_h^* \mu}{\Lambda_h},$$

$$\frac{\partial^2 f_5}{\partial y_1 \partial y_3} = \frac{\partial^2 f_5}{\partial y_2 \partial y_3} = \frac{\sigma_m \beta_m \mu^2 \Lambda_m}{\zeta \Lambda_h^2}, \frac{\partial^2 f_5}{\partial y_3^2} = \frac{2\sigma_m \beta_m \mu^2 \Lambda_m}{\zeta \Lambda_h^2},$$

$$\frac{\partial^2 f_5}{\partial y_1 \partial y_4} = \frac{\partial^2 f_5}{\partial y_2 \partial y_4} = \frac{\sigma_m \beta_{mh} \mu^2 \Lambda_m}{\zeta \Lambda_h^2}, \frac{\partial^2 f_5}{\partial y_3 \partial y_4} = \frac{\sigma_m \mu^2 \Lambda_m (\beta_m + \beta_{mh})}{\zeta \Lambda_h^2},$$

$$\frac{\partial^2 f_5}{\partial y_3 \partial y_5} = -\frac{\sigma_m \beta_m \mu}{\Lambda_h}, \frac{\partial^2 f_5}{\partial y_4 \partial y_5} = -\frac{\sigma_m \beta_{mh} \mu}{\Lambda_h}, \frac{\partial^2 f_5}{\partial y_4^2} = \frac{2\sigma_m \beta_{mh} \mu^2 \Lambda_m}{\zeta \Lambda_h^2},$$

$$\frac{\partial^2 f_6}{\partial y_1 \partial y_3} = \frac{\partial^2 f_6}{\partial y_2 \partial y_3} = -\frac{\sigma_m \beta_m \mu^2 \Lambda_m}{\zeta \Lambda_h^2}, \frac{\partial^2 f_6}{\partial y_1 \partial y_4} = \frac{\partial^2 f_6}{\partial y_2 \partial y_4} = -\frac{\sigma_m \beta_{mh} \mu^2 \Lambda_m}{\zeta \Lambda_h^2},$$

$$\frac{\partial^2 f_6}{\partial y_3^2} = -\frac{2\sigma_m \beta_{mh} \mu^2 \Lambda_m}{\zeta \Lambda_h^2}, \frac{\partial^2 f_6}{\partial y_3 \partial y_4} = -\frac{\sigma_m \mu^2 \Lambda_m (\beta_m + \beta_{mh})}{\zeta \Lambda_h^2},$$

$$\frac{\partial^2 f_6}{\partial y_4^2} = -\frac{2\sigma_m \beta_{mh} \mu^2 \Lambda_m}{\zeta \Lambda_h^2}, \frac{\partial^2 f_6}{\partial y_3 \partial y_5} = \frac{\sigma_m \mu \beta_m}{\Lambda_h}, \frac{\partial^2 f_6}{\partial y_4 \partial y_5} = \frac{\sigma_m \mu \beta_{mh}}{\Lambda_h},$$

$$\frac{\partial^2 f_1}{\partial y_7 \partial \beta_h} = -\sigma_m, \frac{\partial^2 f_2}{\partial y_7 \partial \beta_h} = \sigma_m,$$

Thus, from all of the above expression, the factors a and b are given by:

$$\begin{aligned} a &= \sum_{k,i,j=1}^{7} v_k w_i w_j \frac{\partial^2 f_k}{\partial y_i \partial y_j}(P_0, \beta_h^*) \\ &= \sum_{i,j=1}^{7} v_2 w_i w_j \frac{\partial^2 f_2}{\partial y_i \partial y_j}(P_0, \beta_h^*) + \sum_{i,j=1}^{7} v_3 w_i w_j \frac{\partial^2 f_3}{\partial y_i \partial y_j}(P_0, \beta_h^*) + \\ &\quad \sum_{i,j=1}^{7} v_4 w_i w_j \frac{\partial^2 f_4}{\partial y_i \partial y_j}(P_0, \beta_h^*) + \sum_{i,j=1}^{7} v_6 w_i w_j \frac{\partial^2 f_6}{\partial y_i \partial y_j}(P_0, \beta_h^*) + \\ &\quad \sum_{i,j=1}^{7} v_7 w_i w_j \frac{\partial^2 f_7}{\partial y_i \partial y_j}(P_0, \beta_h^*), \end{aligned} \qquad (2.21)$$

$$= -\frac{2 v_7 w_2^2 \alpha_h \alpha_m \mu \sigma_m \Lambda_m K_5 B}{\zeta^2 \Lambda_h^2 K_2^2 K_3^2 K_4},$$

$$b = \sum_{k,i=1}^{7} v_k w_i \frac{\partial^2 f_k}{\partial y_i \partial \beta_h}(P_0, \beta_h^*)$$

$$= \sum_{i=1}^{7} v_1 w_i \frac{\partial^2 f_1}{\partial y_i \partial \beta_h}(P_0, \beta_h^*) + \sum_{i=1}^{7} v_2 w_i \frac{\partial^2 f_2}{\partial y_i \partial \beta_h}(P_0, \beta_h^*) \qquad (2.22)$$

$$= v_7 w_2 \frac{\sigma_m^3 \mu^2 \alpha_h^2 \Lambda_m^2 \alpha_m^2 K_5^2}{\zeta^3 \Lambda_h^2 K_1 K_2^2 K_3^2 K_4^2},$$

where $B = \mu(2\zeta K_6 + \sigma_m \alpha_h K_5) - \zeta \psi$ and $\quad = \mu(\mu K_3 + \delta K_3 + \alpha_h \gamma_h) + \alpha_h K_2 K_3$. Obviously, b is positive since all the parameters are non-negative. Thus, the local dynamics around the P_0, for $\beta_h = \beta_h^*$ depends on the sign of the factor a and also the sign of B. Rewriting B in term of $\mathcal{R}_1^*$ as $B = \zeta \psi(\mathcal{R}_1^* - 1)$. It follows that, from (2.21), if $\mathcal{R}_1^* < 1$ then $a > 0$ and $a < 0$ if $\mathcal{R}_1^* > 1$.

Theorem 2.4. *The model (2.3) when $\mathcal{R}_0 = 1$, exhibits a backward bifurcation if $\mathcal{R}_1^* < 1$ and a forward bifurcation if $\mathcal{R}_1^* > 1$.*

By Theorem 2.3 (iv) and Theorem 2.4, the following lemma is established.

Lemma 2.5. *The system (2.3) shows the backward bifurcation when $\mathcal{R}_1^* < 1$ and $\mathcal{R}_2 < \mathcal{R}_0 < 1$.*

iii. *Global stability of the endemic equilibrium*

In this subsection, we proof the global stability of the endemic equilibrium as in [15]

Theorem 2.6. *The endemic equilibrium point, P^*, is globally asymptotically stable in Γ when $\mathcal{R}_0 > 0$ provided that*

$$2 - \frac{E_h}{E_h^*} - \frac{E_h^* R_h}{E_h R_h^*} - \frac{S_h^*}{S_h} + \frac{S_h^* R_h}{S_h R_h^*} \geq 0. \tag{2.23}$$

Proof. Let $2 - \frac{E_h}{E_h^*} - \frac{E_h^* R_h}{E_h R_h^*} - \frac{S_h^*}{S_h} + \frac{S_h^* R_h}{S_h R_h^*} \geq 0$. Consider the Lyapunov function

$$L(t) = (S_h - S_h^* - S_h^* \ln \tfrac{S_h}{S_{h_*}^*}) + (E_h - E_h^* - E_h^* \ln \tfrac{E_h}{E_h^*})$$

$$+ \left(\frac{\sigma_m \beta_h I_m^* S_h^*}{(\gamma_h + \mu + \delta) I_h^* N_h^*} + \frac{\rho R_h}{(\rho + \mu) I_h^*} \right)(I_h - I_h^* - I_h^* \ln \tfrac{I_h}{I_h^*})$$

$$+ \frac{\rho}{\rho + \mu}(R_h - R_h^* - R_h^* \ln \tfrac{R_h}{R_h^*}) + (S_m - S_m^* - S_m^* \ln \tfrac{S_m}{S_m^*}) \tag{2.24}$$

$$+ (E_m - E_m^* - E_m^* \ln \tfrac{E_m}{E_m^*}) + \frac{\alpha_m + \zeta}{\alpha_m}(I_m - I_m^* - I_m^* \ln \tfrac{I_m}{I_m^*}),$$

Then, the derivative of $L(t)$ calculated along solutions of the model (2.3) is given by

$$L'(t) = (1 - \tfrac{S_h^*}{S_h})\tfrac{dS_h}{dt} + (1 - \tfrac{E_h^*}{E_h})\tfrac{dE_h}{dt} + \left(\frac{\sigma_m \beta_h I_m^* S_h^*}{(\gamma_h + \mu + \delta) I_h^* N_h^*} + \frac{\rho R_h^*}{(\rho + \mu) I_h^*} \right)(1 - \tfrac{I_h^*}{I_h})\tfrac{dI_h}{dt}$$

$$+ \frac{\rho}{\rho + \mu}(1 - \tfrac{R_h^*}{R_h})\tfrac{dR_h}{dt} + (1 - \tfrac{S_m^*}{S_m})\tfrac{dS_m}{dt} + (1 - \tfrac{E_m^*}{E_m})\tfrac{dE_m}{dt} + \frac{(\alpha_m + \zeta)}{\alpha_m}(1 - \tfrac{I_m^*}{I_m})\tfrac{dI_m}{dt}$$

$$= -\mu \frac{(S_h - S_h^*)^2}{S_h} + \sigma_m \beta_m \frac{I_m^* S_h^*}{N_h^*} \left[4 - \frac{S_h^*}{S_h} - \frac{E_h I_h^*}{E_h^* I_h} - \frac{I_h}{I_h^*} - \frac{S_h E_h^*}{S_h^* E_h} \right]$$

$$+ \rho R_h^* \left[3 - \frac{E_h^* R_h}{E_h R^*} - \frac{I_h R_h^*}{I_h^* R_h} - \frac{E_h I_h^*}{E_h^* I_h} \right] \tag{2.25}$$

$$- \rho R_h^* \left[2 - \frac{E_h}{E_h^*} - \frac{E_h^* R_h}{E_h R_h^*} - \frac{S*_h}{S_h} + \frac{S_h^* R_h}{S_h R_h^*} \right] - \zeta \frac{(S_m - S_m^*)^2}{S_m}$$

$$+ \sigma_m S_m^* f(I_h^*, R_h^*) \left[4 - \frac{S_m^*}{S_m} - \frac{E_m I_m^*}{E_m^* I_m} - \frac{I_m}{I_m^*} - \frac{S_m E_m^*}{S_m^* E_m} \right]$$

$$+ \sigma_m S_m^* f(I_h, R_h) \left[1 - \frac{f(I^*, R_h^*)}{f(I_h, R_h)} [1 - \frac{S_m E_m^*}{S_m^* E_m}] \right],$$

where $f(I_h, R_h) = \frac{\beta_m I_h + \beta_{mh} R_h}{N_h}$ and $f(I_h^*, R_h^*) = \frac{\beta_m I_h^* + \beta_{mh} R_h^*}{N_h^*}$.

Since, the arithmetic mean is greater than or equal to the geometric, clearly

$$\left[4 - \frac{S_h^*}{S_h} - \frac{E_h I_h^*}{E_h^* I_h} - \frac{I_h}{I_h^*} - \frac{S_h E_h^*}{S_h^* E_h} \right] \leq 0,$$

$$\left[3 - \frac{E_h^* R_h}{E_h R^*} - \frac{I_h R_h^*}{I_h^* R_h} - \frac{E_h I_h^*}{E_h^* I_h} \right] \leq 0, \tag{2.26}$$

$$\left[4 - \frac{S_m^*}{S_m} - \frac{E_m I_m^*}{E_m^* I_m} - \frac{I_m}{I_m^*} - \frac{S_m E_m^*}{S_m^* E_m} \right] \leq 0,$$

Furthermore, $f(I_h, R_h)$ is an increasing function which implies that

$$(1 - \frac{f(I_h^*, R_h^*)}{f(I_h, R_h)}) < 0, \tag{2.27}$$

Thus, finally it was found that,

$$\left(1 - \frac{S_m E_m^*}{S_m^* E_m}\right) < 0, \tag{2.28}$$

when $S_m < S_m^*$ and $E_m^* < E_m$. as a consequence, the conditions $(2.23),(2.25),(2.26),(2.27)$ and (2.28) ensure that $\frac{dL(t)}{dt} \leq 0$. Noting that the additional $2 - \frac{E_h}{E_h^*} - \frac{E_h^* R_h}{E_h R_h^*} - \frac{S_h^*}{S_h} + \frac{S_h^* R_h}{S_h R_h^*} \geq 0$. is not necessary if the malaria confers permanent immunity against re-infection [25]. In addition to that the equality $\frac{dV(t)}{dt} = 0$ holds only for $S_h = S_h^*, E_h = E_h^*, I_h = I_h^*, R_h = R_h^*, S_m = S_m^*, E_m = E_m^*$ and $I_m = I_m^*$. Then, the equilibrium point P^* is the only positively invariant set to system (2.3) contained entirely in

$\left\{(S_h, E_h, I_h, R_h, S_m, E_m, I_m) \in \Gamma : S_h = S_h^*, E_h = E_h^*, I_h = I_h^*, R_h = R_h^*, S_m = S_m^*, E_m = E^*, I_m = I_h^*\right\}.$

Subsequentaly, from the LaSalle's invariance principle [26]. that each solutions of the equations (2.25) with the initial conditions in Γ converge to P^*, as t approaches to ∞. Hence, the positive endemic equilibrium is globally asymptotically stable.

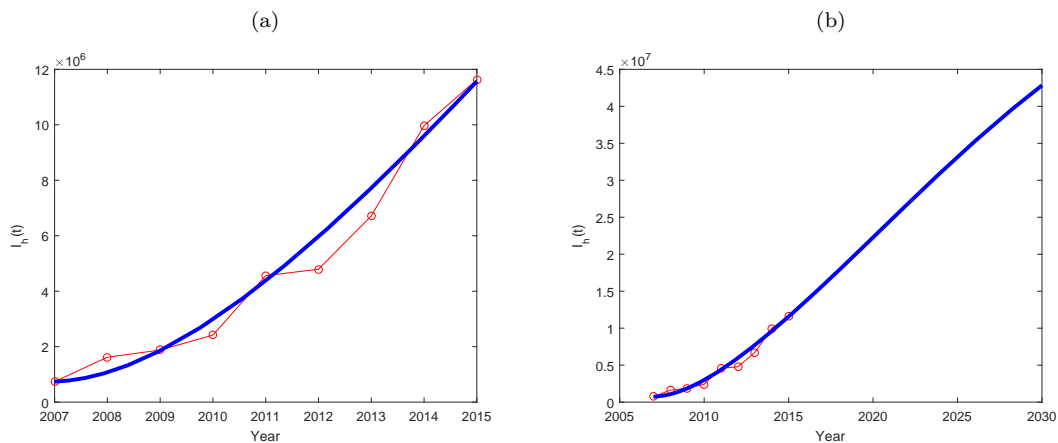## III. Numerical Simulations

Figure 2: Comparisons of the reported malaria cases of Democratic Republic of the Congo in WHO (red curve) and the solution of infectious human $I_h(t)$ for model (2.3). (a): Simulation of malaria reported cases in Democratic Republic of the Congo from 2007 to 2015. (b): Prediction of malaria cases for Democratic Republic of the Congo 2007 to 2030.

In this section, we use our model to simulate the reported malaria cases provided by WHO of DRC. In our simulation we use the data from 2007-2015 because there was many local conflicts from 2000-2006 in the country that may affects the data collection. For this reason, we ignored that period of time. All the parameter values are listed in Table 1. Based on those parameters, we carry out the numerical simulations of our model and obtain a reasonable result between the infected human of model (2.3) and the real data of DRC see Fig2(a),(b), indicates that the transmission of disease in DRC has not reached at a stable period yet and the malaria will become more dangerous in the future. Fig3(a),(b) and Fig4(a),(b), shows the solution of model (2.3) with parameter values from Table 1, for E_m, E_m and E_h, I_h, R_h respectively. Fig5(a),(b) and Fig6(a), shows the effects of changing β_h, γ_h and β_{mh} on the number of infected humans. Fig6(b) displays the influence of the initial conditions of S_m on I_h.
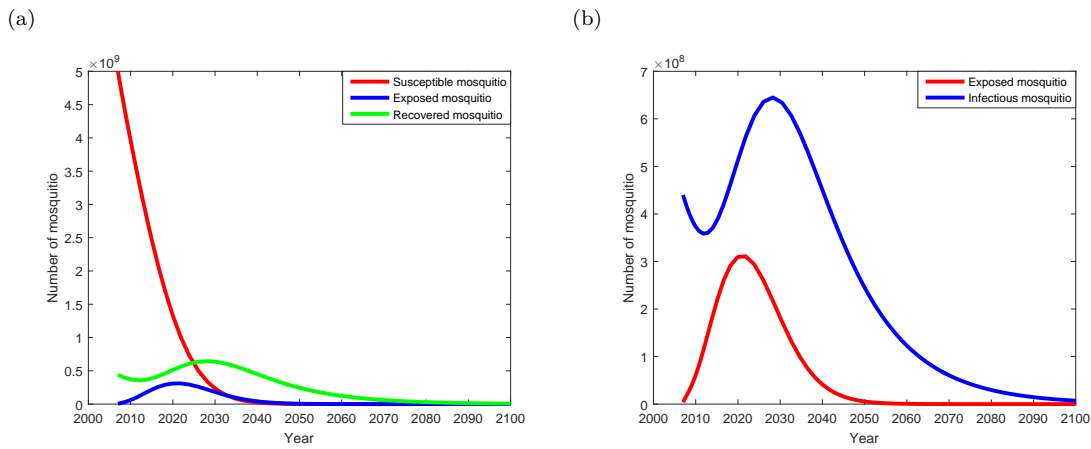
Figure 3: Solution of model (2.3) with parameter values from Table 1 of Democratic Republic of the Congo.
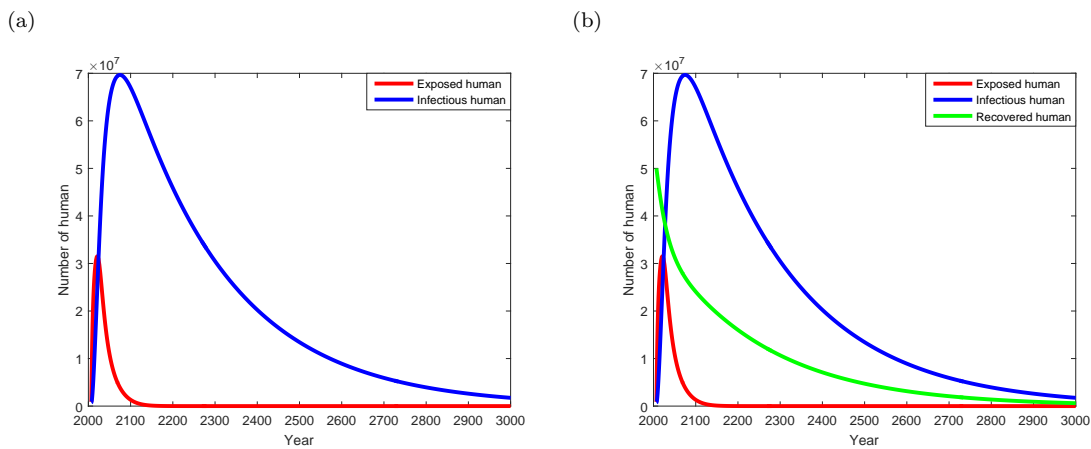

Figure 4: Solution of model (2.3) with parameter values from Table 1 of Democratic Republic of the Congo.
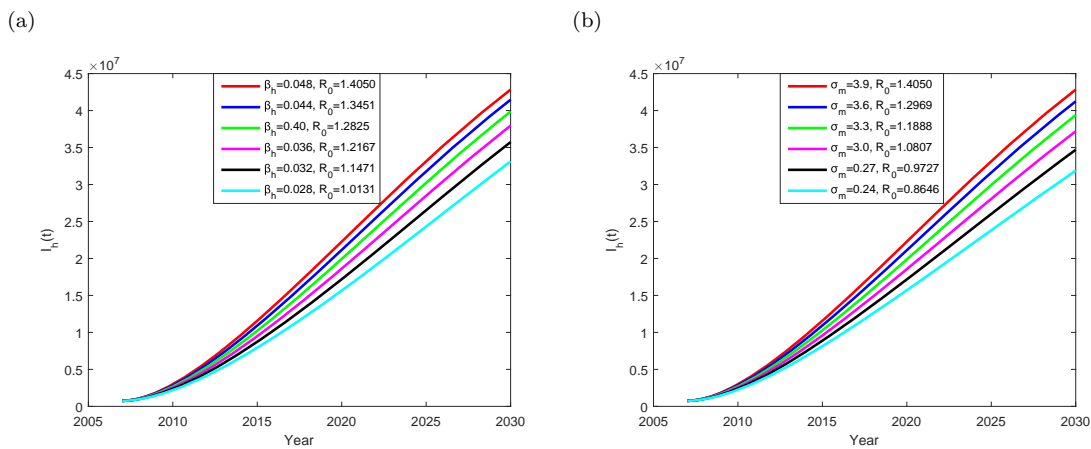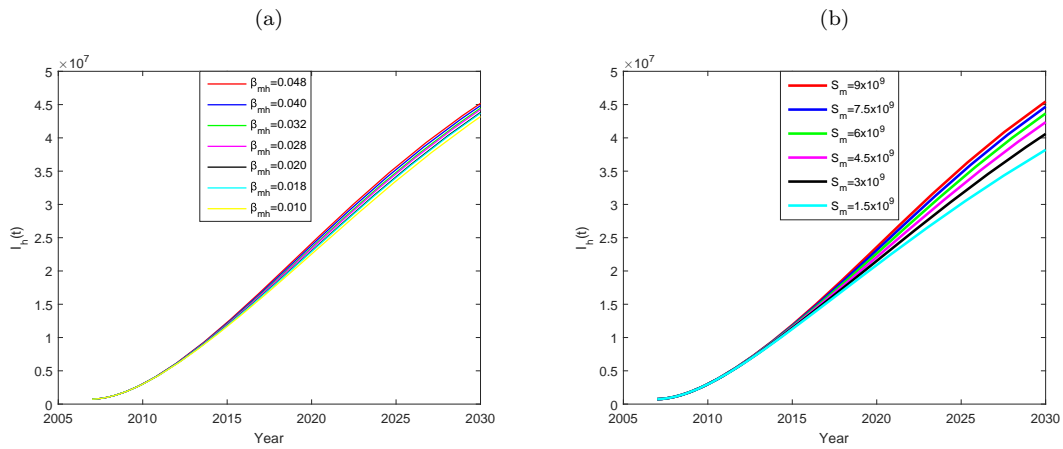

Figure 5: Solution of model (2.3) with parameter values from Table 1 of Democratic Republic of the Congo, (a) The influence of β_h on the number of infectious (b)The influence of σ_m on the number of infectious.

Notes

Figure 6: Solution of model (2.3) with parameter values from Table 1 of Democratic Republic of the Congo, (a) The influence of β_{mh} on the number of infectious (b)The influence of initial size of susceptible mosquit.

IV.   ANALYSIS OF OPTIMAL CONTROL

In this section, the system (4.3) is formulated to estimate the effects of the three control strategies: $u_1(t)$ represents the preventive measure using Long-lasting Insecticide Treated Net(LLITN), $u_2(t)$ is the treatment with drug of infected individuals and the insecticide spray on the breeding grounds of the mosquitoes represented by $u_3(t)$. The control $u_2(t)$ measures the rate at which infected humans are treated with the efficacy of treatments $c \in [0,1]$. Also $d_1, d_2 \in [0,1]$ are constants rate. Our objective is to minimize the number malaria infected individuals, through the optimal control strategies $u_1(t)$, $u_2(t)$ and $u_3(t)$. Malaria is prevalence in the DRC, specially in the rural areas which has many forest and heavy rainfall and that has increased the mosquitoes population. We used three control variables, $u_1(t)$, $u_2(t)$ and $u_3(t)$ which represent the efforts on preventing malaria infections through the use of (LLITN), the treatment with drug and the insecticide spray on the breeds grounds for the mosquitoes respectively, to see the effects of them on malaria transmission on the (DRC). Our objective function defined as:

$$J(u_1, u_2, u_3) = \int_0^{t_f} (A_1 E_h + A_2 I_h + A_3 I_m + \frac{c_1}{2}u_1^2 + \frac{c_2}{2}u_2^2 + \frac{c_3}{2}u_3^2)dt, \qquad (4.1)$$

where A_1, A_2, A_3 are the balancing cost factors due to scale and c_1, c_2 and c_3 denote the weighting constants for making uses of prevention strategies using $u_1(t)$, $u_2(t)$ and $u_3(t)$ controls. Consequently, we attempt to expect an optimal control u_1^*, u_2^* and u_3^* such that,

$$J(u_1^*, u_2^*, u_3^*) = minJ(u_1, u_2, u_3), \Delta = \{(u_1, u_2, u_3)|0 \leq u_i \leq 1, i = 1, 2, 3\}. \qquad (4.2)$$

$$\begin{cases} \dfrac{dS_h}{dt} = \Lambda_h - (1-u_1)\dfrac{\sigma_m\beta_h I_m}{N_h}S_h - \mu S_h + \rho R_h, \\[2mm] \dfrac{dE_h}{dt} = (1-u_1)\dfrac{\sigma_m\beta_h I_m}{N_h}S_h - (\alpha_h+\mu)E_h, \\[2mm] \dfrac{dI_h}{dt} = \alpha_h E_h - (\gamma_h+\mu+\delta+cu_2)I_h, \\[2mm] \dfrac{dR_h}{dt} = (\gamma_h+cu_2)I_h - (\rho+\mu)R_h, \\[2mm] \dfrac{dS_m}{dt} = \Lambda_m - (1-u_3)\sigma_m\dfrac{(\beta_m I_h+\beta_{mh}R_h)}{N_h}S_m - (\zeta+d_1u_1)S_m, \\[2mm] \dfrac{dE_m}{dt} = (1-u_3)\sigma_m\dfrac{(\beta_m I_h+\beta_{mh}R_h)}{N_h}S_m - (\alpha_m+\zeta+d_1u_1)E_m, \\[2mm] \dfrac{dI_m}{dt} = \alpha E_m - (\zeta+d_1u_1)I_m, \end{cases} \tag{4.3}$$

The optimal control must conform the necessary conditions that is emanated from the Pontryagin Maximum Principle [27]. This concept transpose the equations (4.1) and (4.3) into a type of problem characterised with minimizing pointwise a Hamiltonian H, with respect to u_1, u_2 and u_3

$$H = A_1 E_h + A_2 I_h + A_3 I_m + \tfrac{c_1}{2}u_1^2 + \tfrac{c_2}{2}u_2^2 + \tfrac{c_3}{2}u_3^2$$

$$+\lambda_1\left\{\Lambda_h - (1-u_1)\tfrac{\sigma_m\beta_h I_m}{N_h}S_h - \mu S_h + \rho R_h\right\},$$

$$+\lambda_2\left\{(1-u_1)\tfrac{\sigma_m\beta_h I_m}{N_h}S_h - (\alpha_h+\mu)E_h\right\},$$

$$+\lambda_3\left\{\alpha_h E_h - (\gamma_h+\mu+\delta+cu_2)I_h\right\},$$

$$+\lambda_4\left\{(\gamma_h+cu_2)I_h - (\rho+\mu)R_h\right\}, \tag{4.4}$$

$$+\lambda_5\left\{\Lambda_m - (1-u_3)\sigma_m\tfrac{(\beta_v I_m+\beta_{mh}R_h)}{N_h}S_m - (\zeta+d_1u_1)S_m\right\},$$

$$+\lambda_6\left\{(1-u_3)\sigma_m\tfrac{(\beta_m I_h+\beta_{mh}R_h)}{N_h}S_m - (\alpha_m+\zeta+d_1u_1)E_m\right\},$$

$$+\lambda_7\left\{\alpha E_m - (\zeta+d_1u_1)I_m\right\},$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ and λ_7, represents the adjoint variables.
The system solution is attained by suitably taking partial derivatives of the Hamiltonian (4.4) with respect to the associated state variables.

Theorem 4.1. *Given an optimal control u_1^*, u_2^*, u_3^* and the solutions $S_h, E_h, I_h, R_h, S_m, E_m, I_m$ of the corresponding state System (2.3) and (4.3) that minimize $J(u_1, u_2, u_3)$ over Γ. Then there exists adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$, satisfying*

$$\tfrac{-d\lambda_i}{dt} = \tfrac{\partial H}{\partial i} \tag{4.5}$$

Where $i = 1, 2, 3, 4, 5, 6, 7$ and with transversality conditions

$$\lambda_1(t_f) = \lambda_2(t_f) = \lambda_3(t_f) = \lambda_4(t_f) = \lambda_5(t_f) = \lambda_6(t_f) = \lambda_7(t_f) = 0 \tag{4.6}$$

and

$$u_1^* = min\left\{1, max(0, \tfrac{1}{c_1}(\tfrac{\sigma_m \beta_h I_m S_h}{N_h}\left(\lambda_2 - \lambda_1\right) + d_1\left(S_m \lambda_5 + E_m \lambda_6 + I_m \lambda_7\right)))\right\}, \qquad (4.7)$$

$$u_2^* = min\left\{1, max(0, \tfrac{1}{c_2}(cI_h\left(\lambda_3 - \lambda_4\right)))\right\}, \qquad (4.8)$$

$$u_3^* = min\left\{1, max(0, \tfrac{1}{c_3}(\tfrac{\sigma_m(\beta_m I_h + \beta_{mh})}{N_h} S_m)\left(\lambda_6 - \lambda_5\right))\right\}, \qquad (4.9)$$

Proof. Theorem 4.1 and Corollary 4.1 of [27] gives the conditions of possible existence of an optimal control based on the convexity of the integrand of $J(u_1, u_2, u_3)$ with respect to u_1, u_2 and u_3 a priori boundedness of the state solutions, and the resulting Lipschitz characteristics of the state system of the ODE's with the state variables. The Hamiltonian function determines at the optimal control level leads to the adjoint variables. Thus, the adjoint equations can be rearranged as

$$\frac{d\lambda_1}{dt} = ((1-u_1)(\frac{\sigma_m \beta_h I_m}{N_h}))(\lambda_1 - \lambda_2) + ((1-u_1)\frac{\sigma_m \beta_h I_m S_h}{N_h^2})(\lambda_2 - \lambda_1) + \mu \lambda_1$$

$$+ ((1-u_3)\frac{\sigma_m(\beta_m I_h + \beta_{mh} R_h)}{N_h^2} S_m)(\lambda_6 - \lambda5),$$

$$\frac{d\lambda_2}{dt} = -A_1 + ((1-u_1)(\frac{\sigma_m \beta_h I_m}{N_h^2}))(\lambda_2 - \lambda_1) + (\alpha_h + \mu)\lambda_2 - \alpha_h \lambda_3$$

$$+ (\frac{(1-u_3)\sigma_m(\beta_m I_h + \beta_{mh} R_h)}{N_h^2} S_m)(\lambda_6 - \lambda_5),$$

$$\frac{d\lambda_3}{dt} = -A_2 + ((1-u_1)(\frac{\sigma_m \beta_h I_m S_h}{N_h^2}))(\lambda_2 - \lambda_1) + (\gamma_h + \mu + \delta + cu_2)\lambda_3 - (\gamma_h + cu_2)\lambda_4$$

$$+ ((1-u_3)(\frac{\sigma_m \beta_m S_m}{N_h}))(\lambda_5 - \lambda_6) + ((1-u_3)\frac{\sigma_m(\beta_m I_h + \beta_{mh} R_h)}{N_h^2} S_m)(\lambda_6 - \lambda5),$$

$$\frac{d\lambda_4}{dt} = ((1-u_1)(\frac{\sigma_m \beta_h I_m S_h}{N_h^2}))(\lambda_2 - \lambda_1) - \rho \lambda_1 + (\rho + \mu)\lambda_4$$

$$+ (\frac{(1-u_3)\sigma_m \beta_{mh}}{N_h} S_m)(\lambda_5 - \lambda_6) + (\frac{(1-u_3)\sigma_m(\beta_m I_h + \beta_{mh} R_h)}{N_h^2} S_m)(\lambda_6 - \lambda_5),$$

$$\frac{d\lambda_5}{dt} = (\frac{(1-u_3)\sigma_m(\beta_m I_h + \beta_{mh} R_h)}{N_h} S_m)(\lambda_6 - \lambda_5) + (\zeta + d_1 u_1)\lambda_5,$$

$$\frac{d\lambda_6}{dt} = (\alpha_m + \zeta + d_1 u_1)\lambda_6 + \alpha_m \lambda_7,$$

$$\frac{d\lambda_7}{dt} = -A_3 + ((1-u_1)\frac{\sigma_m \beta_h S_h}{N^h})(\lambda_1 - \lambda_2) + (\zeta + d_1 u_1)\lambda_7,$$

V. Numerical Simulations of Optimal Control

In this section, we discuss the numerical outcomes of our various optimal control strategies on the spread of malaria in Democratic Republic of the Congo. The Table 1 presents the parameter values that was used in the simulations.

27. Fleming WH, Wiley RW. Deterministic and stochastic optimal control. Spring Verlag, New York; 1962.

a) Prevention of disease using $u_1(t)$ only

In this strategy, malaria prevention control $u_1(t)$ was used to optimize the objective function $J(u_1(t), u_2(t), u_3(t))$, while we set the other controls u_1, u_2 to zero. In Fig7(a), there is a significant different between the states with controls $(u_1(t) \neq u_2(t) = u_3 = 0)$ and without controls $(u_1(t) = u_2(t) = u_3(t) = 0)$ the number of infected mosquitoes decreases. The strategy is not effective in reducing the number of infected human and increasing the number of recovered human in Fig7 (b),(c).

b) Prevention of disease using $u_2(t)$ only

In this strategy, $u_2(t)$ was employed to optimize, the objective function $J(u_1(t), u_2(t), u_3(t))$, while $u_1 = u_2 = 0$. There is no significant different between the cases with controls and that without control in Fig8(a). In Fig8(b),(c), there is a significant different between the states with control $(u_2(t) \neq 0, u_2(t) = u_3(t) = 0)$, and that without control $(u_1(t) = u_2(t) = u_3(t) = 0)$ after $t = 30$ in infected recovered humans (decreasing ,increasing) respectively.

c) Prevention of disease using $u_3(t)$ only

In this strategy, $u_3(t)$ was used to minimize the objective function $J(u_1(t), u_2(t), u_3(t))$, when $u_1 = u_2 = 0$. In Fig9(a),(b),(c), there is no significant difference between the cases with control and that without the control. Thus, the strategy is not effective to reduce (increase) the number of infected individuals and recovered respectively.

d) Prevention of disease using $u_1(t)$ and $u_2(t)$ only

In this strategy, malaria prevent controls $u_1(t)$ and $u_2(t)$ were used to minimize the objective function $J(u_1(t), u_2(t), u_3(t))$, when $u_3 = 0$. it is clear in Fig10 (a),(b) there is a significant different between the situations with control and without control. Using the controls together $u_1(t)$ and $u_2(t)$ is effective in increasing the number of infected mosquitoes and that of disease infected humans. Also the strategy increases the number of recovered human in Fig10(c).

e) Prevention of disease using $u_1(t)$ and $u_3(t)$ only

In this strategy, malaria prevention controls $u_1(t)$ and $u_3(t)$ were used to optimize the objective function $J(u_1(t), u_2(t), u_3(t))$, when $u_2(t)$. In Fig11(a),(b),(c) are similar to Fig7 (a),(b),(c) only there is a significant different between the cases with control and that without controls in Fig11(a) and there is no difference in the other.

f) Prevention of disease using $u_2(t)$ and $u_3(t)$ only

In this strategy, malaria prevention control $u_2(t)$ and $u_3(t)$ were used to optimize the objective function $J(u_1(t), u_2(t), u_3(t))$, at the same time u_1 set to zero. In Fig12(a), it can be seen that there is no significant different between cases with control and that without control. In Fig12(b),(c), there is a significant different between cases with control and that without control. The number of infected humans is reduced as the result of the intervention. Also increasing the number of recovered humans in Fig12(c). Therefore the strategy is effective in reducing the number of infected individuals and increasing the number of recovered humans.

g) Prevention of disease using $u_2(t)$; $u_2(t)$ and $u_3(t)$ only

In this strategy, malaria prevention controls $u_1(t), u_2(t)$ and $u_3(t)$ were used to optimize the objective function $J(u_1(t), u_2(t), u_3(t))$. It is obvious in Fig13, there is a significant

different between the cases with control and that without controls, the number of infected mosquitoes and infected humans are controlled which is clearly in Fig13(a),(b), and an increasing number of recovered individuals in Fig13(c). Therefore the strategy is effective in controlling I_m and I_h.

Table 1: Parameter values for model (2.3)

| Parameter	Description	Value	Source
ζ	Mosquitoes natural death rate	0.0714	[12]
β_h	Transmission probability from I_m to S_h	0.048	[12]
β_m	Transmission probability from I_h to S_m	0.48	[12]
Λ_m	Mosquitoes recruitment rate	500	[15]
μ	Humans natural death rate	0.0000472	[28]
σ_m	Mosquitoes biting rate	0.39	[29]
Λ_h	Humans recruitment rate	1.5	Fitting
β_{mh}	Transmission probability from R_h to S_m	0.048	Fitting
ρ	Loss of immunity rate for humans	0.0146	Fitting
γ_h	Infectious humans recovery rate	0.003704	Fitting
δ	Humans disease induced death rate	0.0003454	Fitting
α_h	Progression rate from E_h to I_h compartment	0.07333	Fitting
α_m	Progression rate from E_m to I_m compartment	0.21	Fitting
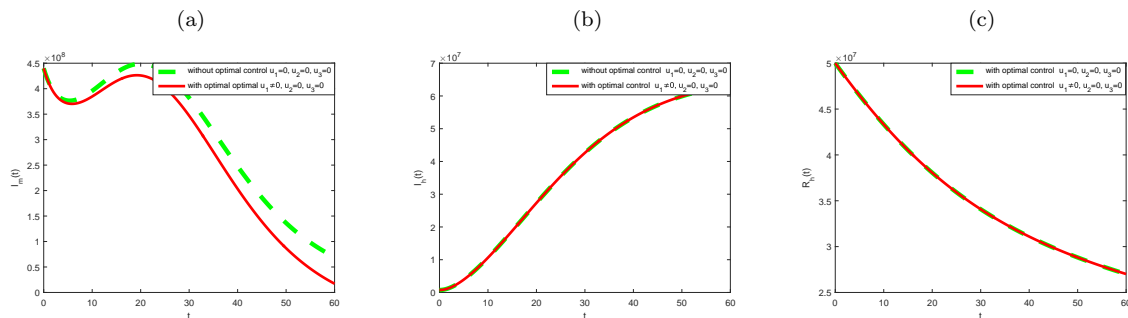
Figure 7: Simulations of the model showing the effect of Long-lasting Insecticide Treated Net(LLITN) only in malaria transmission . Fig 1 (a),(b) and (c) represents the behavior infected mosquitoes, infected humans and recovered humans respectively. Dashed line represents system without optimal control $(u_1 = 0, u_2 = 0, u_3 = 0)$ and solid line shows the system with optimal control $(u_1 \neq 0, u_2 = 0, u_3 = 0.)$.
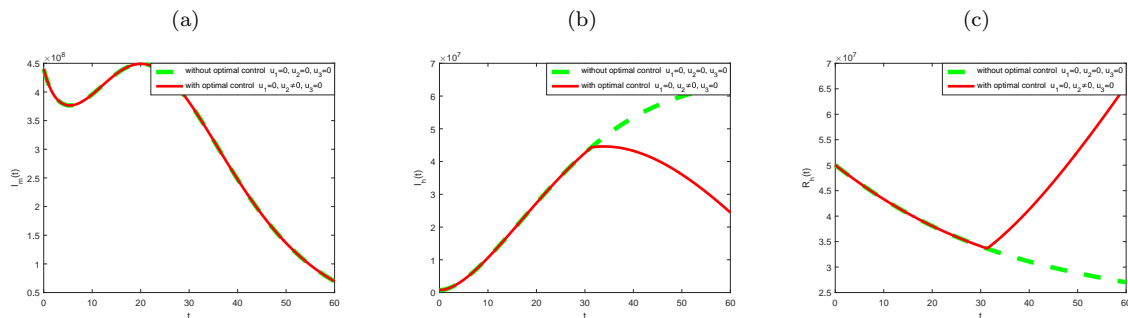

Figure 8: Simulations of the model showing the effect of treatment with drug only in malaria transmission . Fig 2 (a),(b) and (c) represents the behavior infected mosquitoes, infected humans and recovered humans respectively. Dashed line represents system without optimal control $(u_1 = 0, u_2 = 0, u_3 = 0)$ and solid line shows the system with optimal control $(u_1 = 0, u_2 \neq 0, u_3 = 0.)$.
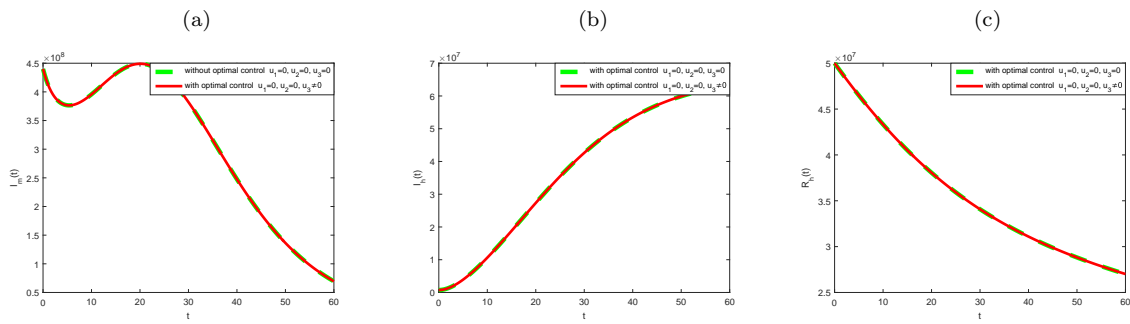
Figure 9: Simulations of the model showing the effect of the insecticide spray on the breeds grounds for the mosquitoes only. Fig 3 (a),(b) and (c) represents the behavior infected mosquitoes, infected humans and recovered humans respectively. Dashed line represents system without optimal control $(u_1 = 0, u_2 = 0, u_3 = 0)$ and solid line shows the system with optimal control $(u_1 = 0, u_2 = 0, u_3 \neq 0.)$.
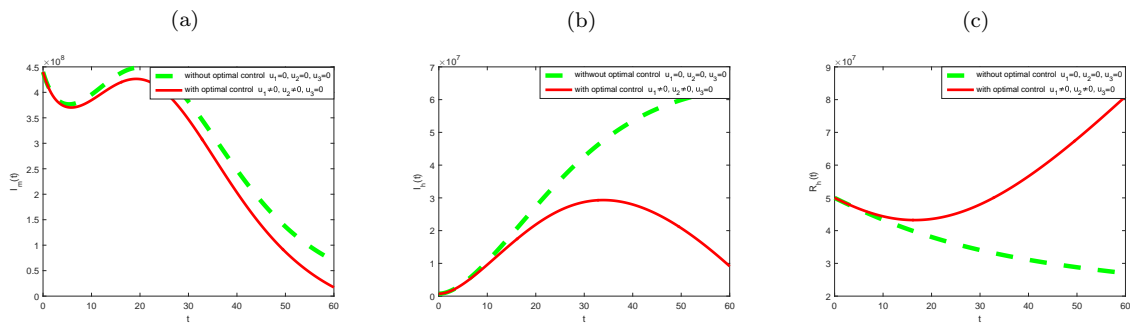

Figure 10: Simulations of the model showing the effect of Long-lasting Insecticide Treated Net(LLITN) and treatment with drug only in malaria transmissio. Fig 4 (a),(b) and (c) represents the behavior infected mosquitoes, infected humans and recovered humans respectively. Dashed line represents system without optimal control $(u_1 = 0, u_2 = 0, u_3 = 0)$ and solid line shows the system with optimal control $(u_1 \neq 0, u_2 \neq 0, u_3 = 0.)$.
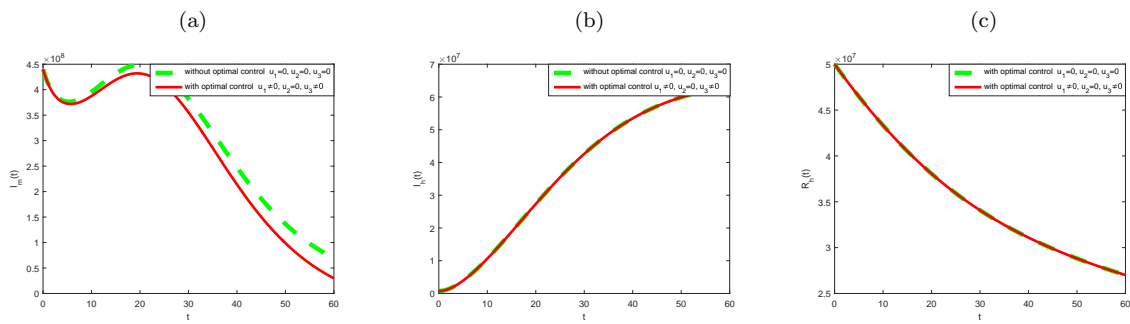

Figure 11: Simulations of the model showing the effect of Long-lasting Insecticide Treated Net(LLITN) and treatment with drug only in malaria transmission . Fig 5 (a),(b) and (c) represents the behavior infected mosquitoes, infected humans and recovered humans respectively. Dashed line represents system without optimal control $(u_1 = 0, u_2 = 0, u_3 = 0)$ and solid line shows the system with optimal control $(u_1 \neq 0, u_2 = 0, u_3 \neq 0.)$.
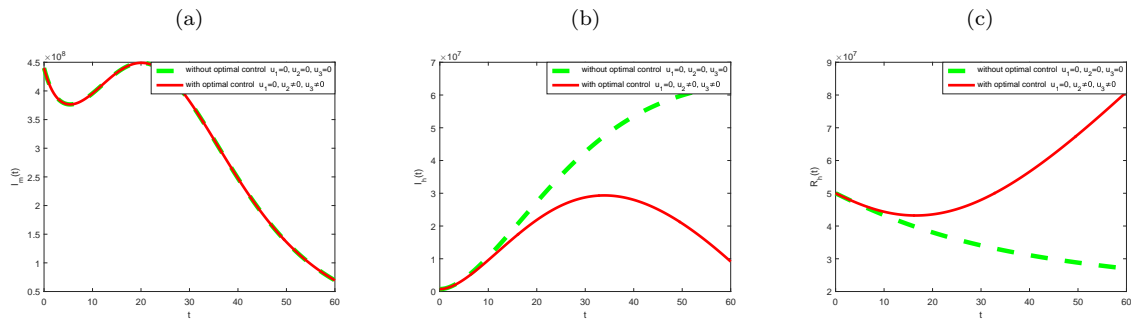
(a) (b) (c)

Figure 12: Simulations of the model showing the effect of treatment with drug and treatment with drug only in malaria transmission . Fig 6 (a),(b) and (c) represents the behavior infected mosquitoes, infected humans and recovered humans respectively. Dashed line represents system without optimal control ($u_1 = 0, u_2 = 0, u_3 = 0$) and solid line shows the system with optimal control ($u_1 = 0, u_2 \neq 0, u_3 \neq 0.$).
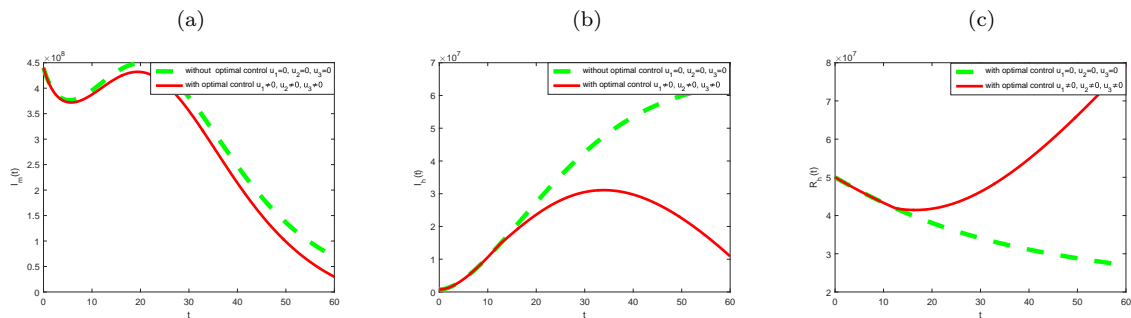
(a) (b) (c)

Figure 13: Simulations of the model showing the effect of Long-lasting Insecticide Treated Net(LLITN), treatment with drug and the insecticide spray on the breeds grounds for the mosquitoes. Fig 4 (a),(b) and (c) represents the behavior infected mosquitoes, infected humans and recovered humans respectively. Dashed line represents system without optimal control ($u_1 = 0, u_2 = 0, u_3 = 0$) and solid line shows the system with optimal control ($u_1 \neq 0, u_2 \neq 0, u_3 \neq 0.$).

VI. CONCLUSION

In this paper, we propose a differential equation model of malaria transmission with standard incidence function with three optimal control strategies $u_1(t), u_2(t)$ and $u_3(t)$. The primary properties of the model was investigated without the controls. The stability analysis for DFE was investigated. $\mathcal{R}_0$ is obtained using next generation matrix technique. Also the stability analysis for EE is verified. The model exhibits the backward bifurcation phenomenon when $\mathcal{R}_0 < 1$. Next, we use the model to fit the confirmed reported malaria cases of DRC. Optimal control strategy was is applied to the model with three controls $u_1(t), u_2(t)$ and $u_3(t)$. Our simulation results predicated that malaria will be reduce in coming years which is illustrated in Fig3 (a),(b) and Fig4(a),(b). The number of infected individuals I_m and I_h always lead to decreases under the use of $u_1(t)$ and $u_2(t)$ controls together and increasing the number of recovered humans in the country.

Appendix A. Castillo-Chavez and Song(2004)

Theorem A 1. Consider the following general system of ordinary differential equations with a parameter ϕ, where 0 is an equilibrium point of the system

$$\frac{dx}{dt} = f(y,t), \quad f : \mathbb{R}^n \times \mathbb{R}, \quad and \quad f \in \mathbb{C}(\mathbb{R}^n \times \mathbb{R}), \tag{6.1}$$

That is ($f(0, \phi) \equiv 0$ for all ϕ) such that:

(1) $A = D_y f(0,0) = \frac{\partial f_i}{\partial y_j}(0,0)$ is the linearization matrix of the system around the equilibrium 0 with ϕ evaluated at 0.

(2) Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real parts.

(3) Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue. Let f_k be the kth- component of f and:

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial y_i \partial y_j}(0,0),$$

$$b = \sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial y_i \partial \phi}(0,0),$$

Then the local dynamics of the system around the equilibrium point 0 are totally determined by signs of a and b.

(i) if $a > 0, b > 0$. when $\phi < 0$ with $|\phi| << 1$, 0 is locally asymptotically stable and there exists a positive unstable equilibrium; when $0 < \phi << 1$, 0 is unstable, and there exists a negative and locally asymptotically stable equilibrium.

(ii) $a < 0, b < 0$. When $\phi < 0$ with $|\phi| << 1, 0$ is unstable, when $0 < \phi << 1, 0$ is locally asymptotically stable, and there exists a positive unstable equilibrium.

(iii) $a > 0, b < 0$. When $\phi < 0$ with $\phi << 1$, 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when $0 < \phi << 1$, 0 is stable, anda positive unstable equilibrium appears.

(iv) $a < 0, b > 0$. When ϕ changes from negative to positive, 0 changes its stability form stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable. Particularly, if $a > 0$ and $b > 0$, then a backward bifurcation occurs at $\phi = 0$.

REFERENCES RÉFÉRENCES REFERENCIAS

1. Marsh K, et al. Indicators of life-threatening malaria Africa children. N.Eng.J.Med. 1995;332(21):1399-1404.
2. Babikera HA, et al. The role of asymptomatic P.falciparum parasitaemia in the evoluation of antimalarial drug resistance in areas of seasonal transmission. Drug Resist Updates. 2013;16(2):1-9.
3. World malaria report 2017. Available: http://www.who.int/malaria/publications /world-malaria-report-2017/report/en/. Accessed date: 15.8.2018.
4. World Malaria Report 2015. Available: http://www.who.int/malaria/publications /world-malaria-report-2015/report/en/. Accessed date: 20.7.2018.
5. Maketa et al. The relationship between Plasmodium infection, anaemia and nutritional status in asymptomatic children aged under ˙ve years living in stable transmission zones in Kinshasa, Democratic Republic of Congo. Malaria Journal (2015)14:83. DOI 10.1186/s12936-015-0595-5. Accessed date: 1.7.2018.
6. Humanitarian Health Action, Malaria. Available at: https://www.who.int/hac/ donorinfo/campaigns/cod/malaria/en/
7. Olaniyi S, Obabiyi OS. Mathematical model for malaria transmission dynamics in human and mosquito populations with nonlinear forces of infection. International Journal of Pure and Applied Mathematics. 2013;88(1):125-156
8. Ross R. The prevention of malaria, John Murray, London; 1911.

9. Anderson RM, May RM. Infectious diseases of humans: Dynamics and control. Oxford University Press, London; 1991.
10. Altaf Khan M, et al. Stability analysis of an SEIR model with non-linear saturated incidence and temporary immunity. Int. J. Adv. Appl. Math and Mech. 2015;2(3):234-7-2529.
11. Ngwa GA, Shu WS. Amathematical model for endemic with variable human and mosquito populations. Available: www.ictp.trieste.it/pub.off.1999
12. Chitnis N, Hyman JM, Cushing JM. Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. Bulletin of Mathematical Biology. 2008;70:1272-1296.
13. Jia li. Malaria model with stage- structured mosquitoes. Mathematical Biosciences and Engineering. 2011;8(3):753-768. DOI:10.39394/8.753
14. Cai L, et al. Optimal control of a malaria model with asymptomatic class and superinfection. Mathematical Biosciences; 2017. DOI:10.1016/j.mbs.2017.03.003
15. P.Roop-o, et al. The effect of incidence function in backward bifurcation for malaria model with temporary immunity. Mathematical Biosciences. 2015, doc:10.1016/j.mbs.2015.04.008.
16. Feng X, et al. Stability and backward bifurcation in a malaria transmission model with applications to the control of malaria in China. Mathematical Biosciences. 2015;266:5264.16
17. Blayneh K, et al. Optimal control of vector-borne diseases: Treatment and prevention. Discrete Contin. Dyn. Syst. Ser. 2009;B11:587-611.
18. Rafikov M, et al. Optimal control strategy of malaria vector using genetically modified mosquitoes. J. Theor. Biol. 2009;258:418-425.
19. Areas of malaria transmission-Democratic Republic of the Congo provided (CDC). Available: www.cdc.gov/malaria/images/map/RepoftheCongo 3.14.16.jpg
20. S.M. Garba et al, Backward bifurcarions in dengue transmission dynamics, Math. Biosci.215(2008) 11-25.
21. V.Lakshmikantham et al, Stability Analysis of Nonlinear systems, Marcel Dekker Inc.New York, Basel, 1989.
22. Hirsch W, Hanisch H, Gabril J. Di˙erential equation models of some parasitic infec-tions: methods for the study of asymptotic behavior. Commun. Pure. Appl. Math. 1985;38:733-753.
23. Van den Driessche P, Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Mathematical Biosciences. 2002;180:29-48.
24. C.Castillo-Chavez, B.Song, Dynamical method of tuberclosis and their applications, Math.Biosci.Eng.1(2)(2004)361-404.
25. S.Sun,C.Guo,C.Li, Global analysis of an SEIRS model with saturating contact rate, Appl.Math.Sci. 6(2012) 3991-4003.
26. J.P. LaSalle, The Stability of Dynamical Systems, Regional Conference Series in Applied Mathematics., SIAM, Philadelphia,1979.
27. Fleming WH, Wiley RW. Deterministic and stochastic optimal control. Spring Verlag, New York; 1962.
28. Democratic Republic of the Congo-Life expectancy. Available: https://knoema.com/atlas/Democratic-Republic-of-the-Congo/topics/Demographics/Population-forecast /Life- expectancy. Accessed date 12.8.2018.
29. Anopheles Mosquitoes. Available: https://www.cdc.gov/malaria/about/biology/mosquitoes/index.html. Accessed date 12.8.2018.

21

Notes

This page is intentionally left blank