Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 19 Issue 2 Version 1.0 Year 2019
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Probability Distribution of Sum of Sides of a Geometric Figure Indexed in Arithmetic Sequence

By Okoli Odilichukwu Christian & Nwosu Dozie Felix
Federal Polythenic Nekede Owerri

Abstract- In this paper we studied the probability distribution associated with sums of sides of any geometric figure indexed in a finite set of Arithmetic Sequence, motivated by the work of researchers in this direction, we derived a probability distribution of an arbitrary sides of geometric figure indexed in a finite set of Arithmetic Sequence with its equivalent recursion form and then give some of its Properties. The results obtained in this paper trivialized and compliment known results in the literatures.

Keywords: *probability distribution, geometric random variable, moments, arithmetic sequence.*

GJSFR-F Classification: *MSC 2010: 11G45*

PROBABILITYDISTRIBUTIONOFSUMOFSIDESOFAGEOMETRICFIGUREINDEXEDINARITHMETICSEQUENCE

Strictly as per the compliance and regulations of:

Notes

Probability Distribution of Sum of Sides of a Geometric Figure Indexed in Arithmetic Sequence

Okoli Odilichukwu Christian [α] & Nwosu Dozie Felix [σ]

Abstract- In this paper we studied the probability distribution associated with sums of sides of any geometric figure indexed in a finite set of Arithmetic Sequence, motivated by the work of researchers in this direction, we derived a probability distribution of an arbitrary sides of geometric figure indexed in a finite set of Arithmetic Sequence with its equivalent recursion form and then give some of its Properties. The results obtained in this paper trivialized and compliment known results in the literatures.

Keywords: probability distribution, geometric random variable, moments, arithmetic sequence.

I. Introduction

We begin with the Bernoulli's distribution. If X is a random variable associated with a random experiment on a set E with two possible outcomes, then X has a Bernoulli distribution $B(p)$ given by

$$b(x;p) = p^x(1-p)^{1-x} \; ; x = 0,1 \qquad (1.1)$$

where the parameter p is the probability of success.

If we extend the range (domain) of the independent variable x to $\{0,1,2,\ldots,n\}$ we have the Binomial distribution $B(n,p)$ given by

$$b(x;n,p) = \binom{n}{x} p^x(1-p)^{n-x} \; ; x = 0,1,2,\ldots,n \qquad (1.2)$$

where the parameter n is the number of independent trials.

Suppose we impose a further condition on the domain of X, that is selection (sampling) is made without replacement, then the number of trials (is no longer independent) gives rise to Hypergeometric distribution $H(k,n,N)$ given by

$$h(x;k,n,N) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}} \quad L \le x \le U \qquad (1.3)$$

where k,n,N are fixed constants, $L = max\{0, k-N+n\}$ and $U = min\{n,k\}$.

Now, suppose we decide to fix the number of successes we require in (1.2) and then observe the random number of trials needed to obtain this number of successes,

Author α: Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria.
e-mails: odicomatics@yahoo.com, odico2003@yahoo.com
Author σ: Department of Mathematics and Statistics, Federal Polythenic Nekede Owerri, Imo State, Nigeria.
e-mail: fedocon2003@gmail.com

then the random number X of trials required to obtain the first success has a geometric distribution given by

$$g(x;p) = pq^{x-1}\;; x = 1,2,3,\dots \tag{1.4a}$$

and if the random variable X is the number of failures before the occurrence of the first success, then we have

$$g(x;p) = pq^{x}\;; x = 0,1,2,3,\dots \tag{1.4b}$$

Observe that the geometric distributions in $(1.4a)$ and $(1.4b)$ are distributions of the number of independent Bernoulli trials required to obtain a single success. Hence, a further generalisation is to seek for the distribution of the random variable X on which the rth success $(r > 1)$ occurs, such a distribution is called the negative binomial distribution $NB(r,p)$ and is given by

$$nb(x;r,p) = \binom{x-1}{r-1}p^r q^{x-r}; x = r, r+1, r+2, \dots. \tag{1.5a}$$

and if the random variable X is the number of failures before the occurrence of the first rth success, then we have

$$nb(x;r,p) = \binom{x+r-1}{r-1}p^r q^{x}; x = 0,1,2,\dots. \tag{1.5b}$$

One of the most important generalizations of (1.2) above is the discrete multivariate distribution function that belong to the (one dimensional) multinomial distribution $M(n\ p_1 \dots,\ p_k)$ is given by

$$m(x_1, x_2, \dots, x_k) = \binom{n}{x_1, x_2, \dots, x_k} p_1^{x_1}\, p_2^{x_2} \dots\, p_k^{x_k}\;; \tag{1.6}$$

where $\sum_{i=1}^{k} p_i = 1$ and $n, p_1, p_2, \dots, p_k$ are the parameters.

To mention but a few, the probability mass functions considered in (1.1) to (1.6) are often referred as the classical or standard discrete probability mass functions. However most of these standard pmf is inadequacy in modeling different types of scenario. Consequent, in recent times, researchers have focused more on generalizing-improving with the aim of making the functions to be more adequate, that is seeking for a probability distribution functions that will accommodate and at the same time applicable in modeling different types of scenario which the former probability distribution functions could not handle.

In order to improve on the discrete models (1.1) to (1.6) we consider some of the important contributors and their results in the sequel.

Philippou and Muwafi (1982) introduced the distribution of order $\boldsymbol{k}$ which gives rise to several studies of distribution of order k as contained in the reference (which reduce to the respective classical probability distribution for $k = 1$) some of these distributions are given by

$$b(x;k,n,p) = \sum_{j=1}^{k-1} \sum_{x_1, x_2, \dots, x_k} \binom{x_1 + x_2 + \cdots + x_k, x}{x_1, x_2, \dots, x_k, x} p^x \left(\frac{q}{p}\right)^{\sum_{i=1}^{k} x_i}; x = 0,1, \dots, \left[\left(\frac{n}{k}\right)\right] \tag{1.7}$$

Notes

Where $x_1 + 2x_2 + \cdots + kx_k = n - kx - j$, $[a]$ is the greatest integer function less than or equal to a

$$g(x;k,p) = \sum_{x_1,x_2,\ldots,x_k} \binom{x_1+x_2+\cdots+x_k}{x_1,x_2,\ldots,x_k} p^x \left(\frac{q}{p}\right)^{\sum_{i=1}^{k} x_i} ; x \geq k \qquad (1.8)$$

Where $x_1 + 2x_2 + \cdots + kx_k = x - k$.

Notes

$$nb(x;k,p) = \sum_{x_1,x_2,\ldots,x_k} \binom{x_1+x_2+\cdots+x_k, r-1}{x_1,x_2,\ldots,x_k, r-1} p^x \left(\frac{q}{p}\right)^{\sum_{i=1}^{k} x_i} ; x \geq rk \qquad (1.9)$$

Where $x_1 + 2x_2 + \cdots + kx_k = x - rk$.

Are the binomial, geometric, negative binomial distribution of order k respectively. The asymptotic properties of some of these distributions give rise to other important distributions as studied by Aki et al (1984), Feller (1956).

In 1986, Panaretos and Evdokia improved on some of the above distributions, in particular (1.2) and (1.3) via sampling from an urn containing a white balls and b black balls. The following *Hypergeometric distribution of order* k was introduced. Assuming that n balls are drawn one at a time;

Without replacement gives rise to;

$$h_1(x;k,n,p) = \sum_{j=1}^{k-1} \sum_{x_1,x_2,\ldots,x_k} \binom{x_1 + x_2 + \cdots + x_k, x}{x_1, x_2, \ldots, x_k, x} \frac{b^{\left(\sum_{i=1}^{k} x_i\right)} a^{\left(n - \sum_{i=1}^{k} x_i\right)}}{(a+b)^{(n)}} ;$$

$$x = 0,1,\ldots,\left[\left(\frac{n}{k}\right)\right] (1.10a)$$

With replacement gives rise to

$$h_2(x;k,n,p) = \sum_{j=1}^{k-1} \sum_{x_1,x_2,\ldots,x_k} \binom{x_1 + x_2 + \cdots + x_k, x}{x_1, x_2, \ldots, x_k, x} \left(\frac{a}{a+b}\right)^{n-\sum_{i=1}^{k} x_i} \left(\frac{b}{a+b}\right)^{\sum_{i=1}^{k} x_i} ;$$

$$x = 0,1,\ldots,\left[\left(\frac{n}{k}\right)\right] (1.10b)$$

With replacement and addition of one ball of the same colour that was selected, before the next draw gives rise to

$$h_3(x;k,n,p) = \sum_{j=1}^{k-1} \sum_{x_1,x_2,\ldots,x_k} \binom{x_1 + x_2 + \cdots + x_k, x}{x_1, x_2, \ldots, x_k, x} \frac{b_{\left(\sum_{i=1}^{k} x_i\right)} a_{\left(n - \sum_{i=1}^{k} x_i\right)}}{(a+b)_{(n)}} ;$$

$$x = 0,1,\ldots,\left[\left(\frac{n}{k}\right)\right] (1.10c)$$

With replacement and addition of c balls of the same color that was selected, before the next draw gives rise to

$$h_4(x;k,n,p) = \sum_{j=1}^{k-1} \sum_{x_1,x_2,\ldots,x_k} \binom{x_1 + x_2 + \cdots + x_k, x}{x_1, x_2, \ldots, x_k, x} \frac{\left(\frac{b}{c}\right)_{\left(\sum_{i=1}^{k} x_i\right)} \left(\frac{a}{c}\right)_{\left(n - \sum_{i=1}^{k} x_i\right)}}{\left(\frac{a+b}{c}\right)_{(n)}} ;$$

$$x = 0,1,\ldots,\left[\left(\frac{n}{k}\right)\right] \quad (1.10d)$$

Where $x_1 + x_2 + \cdots + x_k = n - kx - j,\ a^{(m)} = a(a-1)\cdots(a-m+1)$
$a_{(m)} = a(a+1)\cdots(a+m-1)$

In 1986, Panaretos and Evdokia introduced the *Cluster Binomial Distribution* as an improvement on the classical binomial distribution via sampling from an urn containing i labeled balls $(i = 1,2,\cdots k)$ with p_i denoting the probability that a ball bearing the number i will be drawn, such that $\sum_{i=1}^{k} p_i = p$. Then, $q = 1 - p$ is the probability that a ball bearing a zero will be drawn. Let X be a random variable that count the sum of the numbers on the balls drawn. If the random variable X take the value r for the n balls drawn, r_1 bear the number 1, r_2 bear the number 2 and so on, r_k bear the number k so that $\sum_{i=1}^{k} i r_i = r$ and each of the remaining $n - \sum_{i=1}^{k} r_i$ balls bear the zero. Then the pmf is given by;

$$cb(r;n,k,p_1,\cdots,p_k) = \sum_{r_1,r_2,\ldots,r_k} \binom{n}{r_1,r_2,\ldots,r_k,n-\sum_{i=1}^{k} r_i} \left(\prod_{i=1}^{k} p_i^{r_i}\right) q^{n-\sum_{i=1}^{k} r_i} \quad (1.11)$$

In an attempt to improve on the (one dimensional) multinomial distribution $M(n, p_1,\ldots,p_k)$ given in (1.6), Okoli et al introduce the following parameter; let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_N)$ be a multi-index (or multi-integer), $E=\{x_1,x_2,\ldots,x_N\}$ a finite set and $M(\alpha,E) = \{x_1^{\alpha_1},x_2^{\alpha_2},\ldots,x_N^{\alpha_N}\}$ a finite multi-set induced by $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_N)$. However, he observed that for arbitrary but fixed $d \in \mathbb{N}$, the multinomial distributions in (1.6) do not give adequate description to many important practical problems defined on the more general set given by

$$M\left(\alpha, E^{(d)}\right) = \left\{x_{\underline{i}_d}^{\alpha_{\underline{i}_d}} : i_r \in [k_r], k_r \in \mathbb{N}, r \in [d]\right\} \quad (1.12)$$

where $E^{(d)} = \{x_{\underline{i}_d} : i_r \in [k_r], k_r \in \mathbb{N}, r \in [d]\}, [k_r] = \{1,2,3,\ldots,k_r\}$.

For such case, more adequate and elaborate discrete distribution models are needed which they proved in the theorem that follows

Theorem 1.0

Let $M_{(d)}(p;\ \alpha^{(d)}; X^{(d)})$ denote a pairwise collections of the multiplicity $\alpha_{i_1,i_2,\ldots,i_d}$ and probability $p_{i_1,i_2,\ldots,i_d}$ for each $x_{i_1,i_2,\ldots,i_d} \in X^{(d)}$ on a finite multiset M, then the probability that $x_{i_1,i_2,\ldots,i_d} \in X^{(d)}$ is selected exactly $\alpha_{i_1,i_2,\ldots,i_d}$ times $(i_r \in [k_r],\ k_r \in \mathbb{N},\ r \in [d])$ in n-trials is

$$m_d\left(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_d}\right) = \prod_{r=1}^{d} \cdot \left(\prod_{i_r=1}^{k_r} \binom{\left|\alpha_{\underline{i}_{r-1}}\right| - \sum_{j_r=1}^{i_r-1}\left|\alpha_{\underline{i}_{r-1}j_r}\right|}{\left|\alpha_{\underline{i}_{r-1}i_r}\right|}\right) \prod_{\underline{i}_d} p_{\underline{i}_d}^{\alpha_{\underline{i}_d}}$$

Where $\left|\alpha_{\underline{i}_{r-1}}\right| - \sum_{j_r=1}^{k_r}\left|\alpha_{\underline{i}_{r-1}j_r}\right| = 0\ \ \forall\, r \in [d],\ m_d\left(\alpha_{i_1},\alpha_{i_2},\ldots,\alpha_{i_d}\right) = P\left(X_{i_1,i_2,\ldots,i_d} = \alpha_{i_1,i_2,\ldots,i_d}\right)$
And the associated integer $\left|\alpha^{(d)}\right|$ is given by

Notes

$$|\alpha^{(d)}| = \sum_{r=1}^{d} . \sum_{i_r=1}^{k_r} \alpha_{\underline{i}_d}$$

the associated monomial $x^{\alpha^{(d)}}$ by

$$x^{\alpha^{(d)}} = \prod_{r=1}^{d} . \prod_{i_r=1}^{k_r} x_{\underline{i}_d}^{\alpha_{\underline{i}_d}}$$

Notes

the associated factorial $\alpha^{(d)}!$ by

$$\alpha^{(d)}! = \prod_{r=1}^{d} . \prod_{i_r=1}^{k_r} \alpha_{\underline{i}_d} = |\alpha_{\underline{i}_d}|!$$

Where $|\alpha_{\underline{i}_d}|! = |\alpha_{\underline{i}_{d-1}1}|!\,|\alpha_{\underline{i}_{d-1}2}|!\,|\alpha_{\underline{i}_{d-1}3}|! \cdots |\alpha_{\underline{i}_{d-1}k_d}|!$ $\;and\; \alpha^{(d)} = \alpha\ (for\ short)$

This discrete probability functions has been applied to certain parameter estimation problems in time series and contingency table analysis of arbitrary d-dimensional tables. However, if $d = 1$, we obtain multinomial distribution $M(n, p_1, \dots, p_k)$ given in (1.6).

In 1756 (republished in 1967), Abraham De Moivre studied the probability distribution for a fair (balanced) m-sided die tossed n number of times. Let $X_n^{(m)}$ be a random variable that count the total score in n rolls of an m-sided die, the following probability mass function was obtained

$$P\left(X_n^{(m)} = x\right) = \frac{1}{m^n} \sum_{s=0}^{\beta_1} (-1)^s \binom{n}{s} \binom{n-1+x-ms}{n-1};\ 0 \le x \le (m-1)n \qquad (1.13)$$

Where $\beta_1 = min\left\{n, [\frac{x}{m}]\right\}$ and $[\frac{x}{m}]$ is the greatest integer function less than or equal to $\frac{x}{m}$.

The coefficient of $\frac{1}{m^n}$ often denoted by $C_m(n, x)$ have been studied in detail by Dafnis et al (2007), Freund (1956), who discussed their role in occupancy theory. In particular, $C_m(n, x)$ can be interpreted as "the number of ways of putting n indistinguishable objects into x numbered boxes with each box containing at most $m - 1$ objects. So that if $m = 2$ we have the standard binomial coefficient given by $C_2(n, x) = \binom{n}{x}$; $0 \le x \le n$. A recurrence formula for computing $C_m(n, x)$ is given by

$$C_m(n, x) = \sum_{j=0}^{m-1} C_m(n-1, x-j) \qquad (1.14)$$

One can easily see that for $m = 2$, this recursion reduces to the well-known classical binomial identity.

The number $C_m(n, x)$ has been used extensively in probability studies Ashok, et al (2011); Balasubramanian, et al (1995); De Moivre (1756); Feller (1968); Makri and Philippou (2005); Makri et al (2007a; 2007b) and related areas like reliability and inferential statistics, Ailing (1993); Bollinger and Burchard (1990); Gabai (1970). For more properties on $C_m(n, x)$; generalized Pascal triangles or Pascal triangles of order m, we refer to Bondarenko (1993); Dafnis, et al (2007); Freund (1956); Gabai (1970); Ollerton and Shannon (1998, 2004, 2005) and the references therein.

In 1995, Balasubramanian et al introduced the *extended binomial distribution of order* m *with index* n *and parameter* p as an improved version of the standard binomial distribution and the distribution studied by Abraham De Moivre in 1756 via considering n roll of an m sided die which is not necessarily fair (balanced) with face marked i $(i = 0,1,2,\cdots,m-1)$ and a turn-up side probability p_i $(\sum_{i=0}^{m-1} p_i = 1)$ satisfying the condition $q^m - p^m = q - p$. It was proved that if $X_n^{(m)}$ is a random variable that count the total score in n rolls of an m-sided die then the probability mass function (pmf) is given by

$$P\left(X_n^{(m)} = x; p\right) = \sum_{s=0}^{\beta_1} (-1)^s \binom{n}{s} \binom{n-1+x-ms}{n-1} p^x q^{(m-1)n-x}; 0 \le x \le (m-1)n \quad (1.15)$$

Where $\beta_1 = min\left\{n, [\frac{x}{m}]\right\}$ and $[\frac{x}{m}]$ is the greatest integer function less than or equal to $\frac{x}{m}$.

Observe that if the die is a fair one, then it implies that $p = \left(\frac{1}{m}\right)^{\frac{1}{m-1}} = q$ so that on substitution into equation (1.15) yield the result of Abraham De Moivrein in equation (1.13).

Ashok et al (2011) studied and derived a recursion formula for the probability distribution of the sum of rolling a fair dice (6-sided die) n times (which is equivalently to rolling n several dice once) which is given by;

$$f_j(m) = \frac{1}{6}\left(f_{j-1}(m-1) + f_{j-1}(m-2) + \cdots + f_{j-1}(m-6)\right); j = 1,2,\cdots,n; m \in [j, 6j] \quad (1.16)$$

In 2017, Okoli studied a $(v-u+1)$-sided die with turn-up side probability denoted by $T(x,y) = p^x q^y$: $x,y = 1,2,3,\ldots,m$; $x+y=k; 0 \le p,q \le 1$. The following theorem was proved.

Theorem 1.1

Let $X_n^{(m,m)}$ be a random variable that count the total score in n rolls of an m-sided die with range $x = 1,2,\ldots,m$ and turn-up side probabilities $T(x, m-x)$ $(x \in \{1,2,3,\ldots,m\})$ satisfying the condition $p(q^m - p^m) = (q-p)$ then the probability mass function (pmf) is given by

$$P\left(X_n^{(m,m)} = x; p\right) = \sum_{s=0}^{\beta} (-1)^s \binom{n}{s} \binom{x-ms-1}{n-1} p^x q^{mn-x}; n \le x \le mn$$

Where $\gamma = \min\left\{n, \left[\frac{x-n}{m}\right]\right\}$.

It is important to note that a careful examination of these papers and related works in the literature that dealt with improvement of probability distribution, shows that the improvements, extensions, generalisations so achieved by these authors are mostly, at least in one of the following directions:

(i) Addition of one or more parameters to the original probability function,

(ii) Extension of the domain or space of the parameter(s),

Notes

(iii) Extension of the domain or dimension of the independent variable of the original probability function.

Motivated by the results of the research in this direction via the work of Abraham De Moivre (1756), Balasubramanian et al(1995), Ashok et al (2011) and Okoli (2017), we seek to derived a probability distribution of an arbitrary sides of a geometric figure indexed in a finite set of Arithmetic Sequence. This will take care of some of the computational inadequacies due to the works of Abraham De Moivre (1756), Balasubramanian et al (1995), Ashok et al (2011) and Okoli (2017), in modeling the distribution of sides of geometric figure indexed in an arbitrary finite set of Arithmetic Sequence, which we shall illustrate in the sequel.

Notes

II. Methodology

We shall use the telling example that follows to compare the distribution studied by Balasubramanian et al (1995) and Okoli (2017) in modeling the distribution of a fair die.

For illustrative purpose, Let $X_2^{(6,6)}$ be the sum of scores obtained in the toss of a fair die twice, we wish to construct a probability table for the distribution of $X_2^{(6,6)}$.

First, we consider the sample spaces given below from which we then give the probability table for the distribution of $X_2^{(6,6)}$.

Table I: (Sample space of twice tossed die)

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Table II: (Sample space of sum of scores)

2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

Table III: (Probability distribution table)

x	2	3	4	5	6	7	8	9	10	11	12
$P(X_2^{(6,6)} = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Now let $f_B(x)$ and $f_O(x)$ denotes the probability mass functions due to Balasubramanian et al (1995) and Okoli (2017); that is

$$f_B(x;p) = \frac{1}{m^n}\sum_{s=0}^{\beta}(-1)^s\binom{n}{s}\binom{n-1+x-ms}{n-1}\ ;\ 0 \le x \le (m-1)n, \beta = min\left\{n, \left[\frac{x}{m}\right]\right\}$$

$$f_O(x;p) = \frac{1}{m^n}\sum_{s=0}^{\gamma}(-1)^s\binom{n}{s}\binom{x-ms-1}{n-1}\ ;\ n \le x \le mn, \gamma = min\left\{n, \left[\frac{x-n}{m}\right]\right\}$$

We begin with Balasubramanian *pmf* in Aki et al, (1984) denoted by $f_B(x;p)$. Observe that

(1) In $f_B(x;p)$, $0 \le x \le (m-1)n$ implies that $0 \le x \le 10$ (since $n = 2$ and $m = 6$). However this does not agree with the range of x given in Table II for a fair die tossed twice. Infact, it implies that using $f_B(x;p)$, $f_B(11;p)$ and $f_B(12;p)$ cannot be evaluated since the range of specification of x ($0 \le x \le 10$) does not includes $\{11, 12\}$.

(2) It is important to note also that, even at those values of x ($0 \le x \le 10$) specified in $f_B(x;p)$, the pmf fail to give accurate probability value(s) for such value(s) of x. To see this, in particular, observe from the probability distribution table (Table III).

Notes

$$P\left(X_2^{(6,6)} = 9\right) = \frac{4}{36}$$

Using

$$f_B(x;p) = \frac{1}{m^n}\sum_{s=0}^{\beta}(-1)^s\binom{n}{s}\binom{n-1+x-ms}{n-1} \; ; \; 0 \le x \le (m-1)n$$

$$\Rightarrow f_B(9;p) = \frac{1}{6^2}\sum_{s=0}^{\beta}(-1)^s\binom{2}{s}\binom{10-6s}{1}$$

$$= \frac{1}{6^2}\left[(-1)^0\binom{2}{0}\binom{10-6\times 0}{1} + (-1)^1\binom{2}{1}\binom{10-6}{1}\right] = \frac{1}{6^2}[10-2\times 4] = \frac{2}{36};$$

$$\Rightarrow f_B(9;p) \ne P\left(X_2^{(6,6)} = 9\right)$$

Now with the pmf defined by Okoli (2017), $f_O(x;p)$. Observe that

(1). In $f_O(x;p)$, $n \le x \le mn$ implies that $2 \le x \le 12$ (since $n = 2$ and $m = 6$). This does agree with the range of x given in Table II for a fair die tossed twice. Which is not the case for $f_B(x;p)$.

(2). $f_O(x;p)$, give accurate probability value(s) for each value(s) of x. To see this, observe from the probability distribution table (Table III).

$$P\left(X_2^{(6,6)} = 9\right) = \frac{4}{36}$$

Using

$$f_O(x;p) = \frac{1}{m^n}\sum_{s=0}^{\gamma}(-1)^s\binom{n}{s}\binom{x-ms-1}{n-1} \; ; \; n \le x \le mn$$

$$\Rightarrow f_B(9;p) = \frac{1}{6^2}\sum_{s=0}^{\beta}(-1)^s\binom{2}{s}\binom{8-6s}{1}$$

$$= \frac{1}{6^2}\left[(-1)^0\binom{2}{0}\binom{8-6\times 0}{1} + (-1)^1\binom{2}{1}\binom{8-6}{1}\right] = \frac{1}{6^2}[8-2\times 2] = \frac{4}{36};$$

$$\Rightarrow f_O(9;p) = P\left(X_2^{(6,6)} = 9\right)$$

Notes

Hence we conclude that the pmf we defined $f_O(9;p)$ is more practicable to work with than the one defined by Balasubramanian et al (1995) in modeling the distribution of sums of sides of a standard die. Since the standard die is indexed in the finite arithmetic sequence $\{1,2,3,\cdots,m\}$. It is important to note that if we choose the finite arithmetic sequence $\{0,1,2,3,\cdots,m-1\}$ for the indexing, then $f_O(x;p)$ will no longer be adequate, rather $f_B(x;p)$ will be suitable in modeling the distribution of sums of sides data. Thus, we have seen that the distribution studied by Balasubramanian et al (1995) and Okoli (2017) is rather restrictive and particularized, in the sense that ordinarily it cannot be use in modeling the distribution of sides of geometric figure indexed in an arbitrary finite set of Arithmetic Sequence. As a matter of fact, this constitute a major weakness which we shall address in the sequel.

Let $d,u,v \in \mathbb{N}$, we now proceed to define a probability distribution that will be suitable in modeling the distribution of sides of geometric figure indexed in an arbitrary finite set of Arithmetic Sequence given by $\{u,u+d,u+2d,\ldots,v\}$ where d, u and v denote the common difference, first and last term, this implies that our geometric figure is $\left(\frac{v-u}{d}+1\right)$-sided. Thus, a typical sample space and sample space of sums of scores of such geometric figure tossed twice is given as

Table IV: (Sample space of twice tossed arbitrary geometric figure indexed in arithmetic sequence)

u,u	$u,(u+d)$	$u,(u+2d)$	$\cdots$	u,v
$(u+d),u$	$(u+d),(u+d)$	$(u+d),(u+2d)$	$\cdots$	$(u+d),v$
$(u+2d),u$	$(u+2d),(u+d)$	$(u+2d),(u+2d)$	$\cdots$	$(u+2d),v$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v,u	$v,(u+d)$	$v,(u+2d)$	$\cdots$	v,v

Table V: (Sample space of sums of scores for the geometric figure indexed in arithmetic sequence)

$2u$	$2u+d$	$2u+2d$	$\cdots$	$u+v$
$2u+d$	$2u+2d$	$2u+3d$	$\cdots$	$d+u+v$
$2u+2d$	$2u+3d$	$2u+4d$	$\cdots$	$2d+u+v$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$v+u$	$d+u+v$	$2d+u+v$	$\cdots$	$2v$

Now to introduce a little more perturbation (unfairness) on this geometric figure we let $k \in \mathbb{N}$ (where k is not necessarily equal to v) and then defined the turn-up side probabilities as

$$T(x,y) = p^x q^y:\ x,y = u,u+d,u+2d,\ldots,v;\ x+y=k; 0 \le p,q \le 1. \qquad (2.1)$$

Where $u < k \le v$

Clearly the discrete probability distribution function associated with die models mentioned above is not adequate for modelling the distribution of sums of the turn-up side probability for the geometric figure described in equation (2.1) and table V. Hence there is need to study the model in equation (2.1) going by the very fact that all the other models mentioned above can be easily be deduced from the model.

Global Journal of Science Frontier Research (F) Volume XIX Issue II Version I Year 2019

Now, observe that the generating function $G(t)$ for the $\left(\frac{v-u}{d}+1\right)$-sided figure is given by

$$G(t) = q^{k-v}\, p^u t^u \frac{(q^{v-u+d} - p^{v-u+d} t^{v-u+d})}{q^d - p^d t^d} \tag{2.2}$$

With the normalization condition

$$p^u (q^{v-u+d} - p^{v-u+d}) = q^{v-k} (q^d - p^d) \tag{2.3}$$

If for a fix q, we define the auxiliary function $g(p) = 0$ by

$$g(p) = p^u (q^{v-u+d} - p^{v-u+d}) - q^{v-k} (q^d - p^d) \tag{2.4}$$

It then follows that the first and second derivatives of the function $g(.)$ are given by

$$g^{'}(p) = uq^{v-u+d} p^{u-1} - (v+d) p^{v+d-1} + dq^{v-k}\, p^{d-1} \tag{2.5}$$

$$g^{''}(p\,) = u(u-1) q^{v-u+d} p^{u-2} - (v+d-1)(v+d) p^{v+d-2} + d(d-1) q^{v-k}\, p^{d-2} \tag{2.6}$$

Equation (2.5) is nonlinear function of p whose root can be determined by applying any of the iterative approximation formulas for finding the roots (zeros) of nonlinear equations. Since $p \in (0,1)$ by definition, observe that $g^{''}(p) < 0\ \forall\, p \in (0,1)$. Hence this implies that the function $g(.)$ is strictly increasing for $0 \le p \le q_{k,u,v,d}$ and strictly decreasing for $q_{k,u,v,d} \le p \le 1$. Where $q_{k,u,v,d}$ is the zero of the function $g^{'}(.)$, which in turn correspond to the turning (maximum) point of the function $g(.)$. Consequently it follows that $g(.)$ is monotone (sectionally) and unimodal with the mode occurring at the turning point $p = q_{k,u,v,d}$ [see Balasubramanian, et al (1995); Dharmadhikari and Joag-dev (1988); Hogg and Craig (1978); Okoli et al (2016); Okoli (2017); Okoli (2017)].

However, if in particular we take $k = v$, then there exists $q_{u,v,d}$ for a balanced figure such that $q = q_{u,v,d} = p$. Then the normalization condition also reduces to

$$p^u (q^{v-u+d} - p^{v-u+d}) = (q^d - p^d) \tag{2.7}$$

Equation (2.1) to equation(2.7) implies the results of the authors mentioned in $(a), (b),$ *and* (c) above.

We state the following theorems which unify the results of the authors: Balasubramanian, et al (1995); Okoli (2017); Okoli (2017) in the next section of this work as follows.

III. Main Results

In this section, we now proceed to state some important theorem associated with turn-up side probability for the geometric figure described in equation (2.1) (table V) and their consequences.

Notes

Global Journal of Science Frontier Research (F) Volume XIX Issue II Version I Year 2019

Notes

Theorem 3.1

Let $X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ be a random variable that count the total score in n rolls of a $(\frac{v-u}{d}+d)$-sided geometric figure with turn-up side probabilities $T(x,k-x)$ satisfying the condition $p^u(q^{v-u+d}-p^{v-u+d})=q^{v-k}\,(q^d-p^d)$, with range $x=u,u+d,u+2d,\dots,v$. Then the probability generating function (pgf) is given by

$$G_n(t)=E\left(t^{X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}}\right)=\left[q^{k-v}\ p^u t^u\ \frac{(q^{v-u+d}-p^{v-u+d}t^{v-u+d})}{q^d-p^d t^d}\right]^n \quad (2.8)$$

Proof

Now, observe that the probability generating function (pgf) of $X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ constitute a convolution of $X_j^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ $(j=1,2,3,\dots n)$. Where each $X_j^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ is an independent identically distributed (iid) random variables corresponding to the scores of $\left(\frac{v-u}{d}+1\right)$-sided die and turn-up side probabilities $(x,k-x)$. Thus

$$G_n(t)=E\left(t^{X_j^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}}\right)$$

$$E\left(t^{X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}}\right)=\prod_{j=1}^{n}E\left(t^{X_j^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}}\right)=\left[q^{k-v}\ p^u t^u\ \frac{(q^{v-u+d}-p^{v-u+d}t^{v-u+d})}{q^d-p^d t^d}\right]^n$$

This completes the proof.

Theorem 3.2

Let $X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ be a random variable that count the total score in n rolls of a $(\frac{v-u}{d}+1)$-sided geometric figure with turn-up side probabilities $T(x,k-x)$ satisfying the condition $p^u(q^{v-u+d}-p^{v-u+d})=q^{v-k}\,(q^d-p^d)$, with range $x=u,u+d,u+2d,\dots,v$. Then the probability mass function (pmf) is given by

$$P\left(X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}=x;p\right)=\sum_{s=0}^{\beta_3}(-1)^s\binom{n}{s}\binom{n-1+x-\left(\frac{v-u+d}{d}\right)s-\left(\frac{un}{d}\right)}{n-1}p^x q^{kn-x};$$

$$un\le x\le kn$$

Where $\beta_3=\min\left\{n,\left[\frac{dx-un}{v-u+d}\right]\right\},k\le v$.

Proof

We expand (2.2) in n independent rolls of $(\frac{v-u}{d}+1)$-sides as follows

$$G_n(t)=\left[q^k\ \left(\frac{pt}{q}\right)^u\frac{\left(1-\frac{p^{v-u+d}t^{v-u+d}}{q^{v-u+d}}\right)}{1-\left(\frac{pt}{q}\right)^d}\right]^n=q^{kn}\,T^{un}\left(\frac{1-T^{v-u+d}}{1-T^d}\right)^n;\ T=\frac{pt}{q}$$

$$= q^{kn}\, T^{un} (1 - T^{v-u+d})^n (1 - T^d)^{-n}$$

$$= q^{kn}\, T^{un} \left(\sum_{s=0}^{n} (-1)^s \binom{n}{s} T^{(v-u+d)s} \right) \sum_{v=0}^{\infty} (-1)^v \binom{-n}{v} T^{dv}$$

$$= q^{kn}\, T^{un} \left(\sum_{s=0}^{n} (-1)^s \binom{n}{s} T^{(v-u+d)s} \right) \left(\sum_{v=0}^{\infty} \binom{n-1+v}{v} T^{dv} \right)$$

$$= \sum_{v=0}^{\infty} \sum_{s=0}^{n} (-1)^s\, q^{kn} \binom{n}{s} \binom{n-1+v}{v} T^{(v-u+d)s+dv+un}$$

$$= \sum_{x=un}^{\infty} \sum_{s=0}^{n} (-1)^s\, q^{kn} \binom{n}{s} \binom{n-1+\left(\frac{x}{d}\right)-\left(\frac{v-u+d}{d}\right)s-\left(\frac{un}{d}\right)}{\left(\frac{x}{d}\right)-\left(\frac{v-u+d}{d}\right)s-\left(\frac{un}{d}\right)} T^x$$

$$= \sum_{x=un}^{\infty} \sum_{s=0}^{n} (-1)^s \binom{n}{s} \binom{n-1+\left(\frac{x}{d}\right)-\left(\frac{v-u+d}{d}\right)s-\left(\frac{un}{d}\right)}{n-1} p^x q^{kn-x}\, t^x$$

Where $(v-u+d)s + dv + un = x$. Thus, it follows from the last equation above that the probability mass function pmf is given by.

$$P\left(X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)} = x; p\right) =$$

$$\sum_{s=0}^{\beta_3} (-1)^s \binom{n}{s} \binom{n-1+\left(\frac{x}{d}\right)-\left(\frac{v-u+d}{d}\right)s-\left(\frac{un}{d}\right)}{n-1} p^x q^{kn-x}\,; un \le x \le kn$$

If we are dealing with a fair (balanced) die (i.e. $k = v,\ q = \left(\left(\frac{d}{v-u+d}\right)^{\frac{1}{v}}\right)$ then the corollary that follows is a consequence of theorem 2.2 above.

Corollary 3.3

Let $X_n^{\left(\frac{v-u+d}{d},\frac{v-u+d}{d}\right)}$ be a random variable that count the total score in n rolls of a $\left(\frac{v-u}{d}+1\right)$-sided geometric figure with turn-up side probabilities $T(x, v-x)$ satisfying the condition $p^u(q^{v-u+d} - p^{v-u+d}) = (q^d - p^d)$, with range $x = u, u+d, u+2d, \ldots, v$. Then the probability mass function (pmf) is given by

$$P\left(X_n^{\left(\frac{v-u+d}{d},\ \frac{v-u+d}{d}\right)} = x; p\right) =$$

$$\sum_{s=0}^{\beta_3} (-1)^s \binom{n}{s} \binom{n-1+\left(\frac{x}{d}\right)-\left(\frac{v-u+d}{d}\right)s-\left(\frac{un}{d}\right)}{n-1} p^x q^{vn-x}\,; un \le x \le vn$$

Notes

Notes

Corollary 3.4

Let $X_n^{\left(\frac{v-u+d}{d},\ \frac{v-u+d}{d}\right)}$ be a random variable that count the total score in n rolls of a $(\frac{v-u}{d}+1)$-sided geometric figure with turn-up side probabilities $T(x, v-x)$ satisfying the condition $p^u(q^{v-u+d} - p^{v-u+d}) = (q^d - p^d)$, with range $x = u, u+d, u+2d, \dots, v$. Then the probability mass function (pmf) is given by

$$P\left(X_n^{\left(\frac{v-u+d}{d},\ \frac{v-u+d}{d}\right)} = x; p\right) =$$

$$\left(\frac{d}{v-u+d}\right)^n \sum_{s=0}^{\beta_3} (-1)^s \binom{n}{s} \binom{n-1+\left(\frac{x}{d}\right)-\left(\frac{v-u+d}{d}\right)s-\left(\frac{un}{d}\right)}{n-1};\ un \le x \le vn$$

Theorem 3.5

Let $X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ be a random variable that count the total score in n rolls of a $(\frac{v-u}{d}+1)$-sided geometric figure with turn-up side probabilities $T(x, k-x)$ satisfying the condition $p^u(q^{v-u+d} - p^{v-u+d}) = q^{v-k}\,(q^d - p^d)$, with range $x = u, u+d, u+2d, \dots, v$. Then the mean and variance are determined by

$$(i)\quad G_n{}'(1) = E\left(X_n^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}\right) = un + np^d\left[\frac{d-(v-u+d)p^v q^{(k-v)}}{q^d-p^d}\right]$$

$$(i)\quad G_n{}''(1) = un(un-1) + 2un^2p^d\left[\frac{d-(v-u+d)p^v q^{(k-v)}}{q^d-p^d}\right]^2$$

$$+ n(n-1)\left[\frac{dq^d-(v-u+d)p^{v+d}q^{(k-v)}}{q^d-p^d}\right]$$

$$+ nq^{(k-v)}\left[\frac{-(v-u+d)(v-u+d-1)p^{v+d} + d(d-1)p^d q^{(v-k)}}{q^d-p^d}\right.$$

$$\left.+ \frac{2d^2q^{2d}q^{(v-k)} - 2d(v-u+d)p^{v+2d}}{(q-p)^2}\right]$$

Proof.

Since $G_n(t) = q^{(k-v)n}\, p^{un}\, t^{un}\left(\frac{q^{v-u+d} - p^{v-u+d}t^{v-u+d}}{q^d - p^d t^d}\right)^n$, it follows that the derivative $G_n{}'(t)$ of $G_n(t)$ is

(i)

$$G_n{}'(t) = unq^{(k-v)n}\, p^{un}\, t^{un-1}\left(\frac{q^{v-u+d} - p^{v-u+d}t^{v-u+d}}{q^d - p^d t^d}\right)^n$$

$$+ nq^{(k-v)n}\, p^{un}\, t^{un}\left(\frac{q^{v-u+d} - p^{v-u+d}t^{v-u+d}}{q^d - p^d t^d}\right)^{n-1}\left[\frac{-(v-u+d)p^{v-u+d}t^{v-u+d-1}}{q^d - p^d t^d}\right.$$

$$\left.+ \frac{dp^d t^{d-1}\left(q^{v-u+d} - p^{v-u+d}t^{v-u+d}\right)}{(q^d - p^d t^d)^2}\right]$$

Thus,

$$G_n'(1) = unq^{(k-v)n}\, p^{un} \left(\frac{q^{v-u+d} - p^{v-u+d}}{q^d - p^d}\right)^n$$

$$+ nq^{(k-v)n}\, p^{un} \left(\frac{q^{v-u+d} - p^{v-u+d}}{q^d - p^d}\right)^{n-1} \left[\frac{-(v-u+d)p^{v-u+d}}{q^d - p^d}\right.$$

$$\left. + \frac{dp^d\left(q^{v-u+d} - p^{v-u+d}\right)}{(q^d - p^d)^2}\right]$$

The result follows by applying the normalization condition.

(ii)

$$G_n''(t) = un(un-1)q^{(k-v)n}\, p^{un}\, t^{un-2} \left(\frac{q^{v-u+d} - p^{v-u+d}t^{v-u+d}}{q^d - p^d t^d}\right)^n$$

$$+2un^2 q^{(k-v)n}\, p^{un}\, t^{un-1} \left(\frac{q^{v-u+d} - p^{v-u+d}t^{v-u+d}}{q^d - p^d t^d}\right)^{n-1} \left[\frac{-(v-u+d)p^{v-u+d}t^{v-u+d-1}}{q^d - p^d t^d}\right.$$

$$\left. + \frac{dp^d t^{d-1}(q^{v-u+d} - p^{v-u+d}t^{v-u+d})}{(q^d - p^d t^d)^2}\right]$$

$$+n(n$$

$$-1)q^{(k-v)n}\, p^{un}\, t^{un} \left(\frac{q^{v-u+d} - p^{v-u+d}t^{v-u+d}}{q^d - p^d t^d}\right)^{n-2} \left[\frac{-(v-u+d)p^{v-u+d}t^{v-u+d-1}}{q^d - p^d t^d}\right.$$

$$\left. + \frac{dp^d t^{d-1}(q^{v-u+d} - p^{v-u+d}t^{v-u+d})}{(q^d - p^d t^d)^2}\right]^2$$

$$+ nq^{(k-v)n}\, p^{un}\, t^{un} \left(\frac{q^{v-u+d} - p^{v-u+d}t^{v-u+d}}{q^d - p^d t^d}\right)^{n-1} \left[\frac{-(v-u+d)(v-u+d-1)p^{v-u+d}t^{v-u+d-2}}{q^d - p^d t^d}\right.$$

$$+ \frac{d(d-1)p^d t^{d-2}(q^{v-u+d} - p^{v-u+d}t^{v-u+d}) - 2d(v-u+d)p^{v-u+2d}t^{v-u+2d-2}}{(q^d - p^d t^d)^2}$$

$$\left. + \frac{2(dp^d t^{d-1})^2(q^{v-u+d} - p^{v-u+d}t^{v-u+d})}{(q^d - p^d t^d)^3}\right]$$

Thus,

$$G_n''(1) = un(un-1)q^{(k-v)n}\, p^{un} \left(\frac{q^{v-u+d} - p^{v-u+d}}{q^d - p^d}\right)^n$$

$$+2un^2 q^{(k-v)n}\, p^{un} \left(\frac{q^{v-u+d} - p^{v-u+d}}{q^d - p^d}\right)^{n-1} \left[\frac{-(v-u+d)p^{v-u+d}}{q^d - p^d} + \frac{dp^d(q^{v-u+d} - p^{v-u+d})}{(q^d - p^d)^2}\right]$$

$$+n(n-1)q^{(k-v)n}\, p^{un} \left(\frac{q^{v-u+d} - p^{v-u+d}}{q^d - p^d}\right)^{n-2} \left[\frac{-(v-u+d)p^{v-u+d}}{q^d - p^d}\right.$$

$$\left. + \frac{dp^d(q^{v-u+d} - p^{v-u+d})}{(q^d - p^d)^2}\right]^2$$

Notes

Notes

$$+nq^{(k-v)n}\, p^{un}\left(\frac{q^{v-u+d}-p^{v-u+d}}{q^d-p^d}\right)^{n-1}\left[\frac{-(v-u+d)(v-u+d-1)p^{v-u+d}}{q^d-p^d}\right.$$

$$+\frac{d(d-1)p^d(q^{v-u+d}-p^{v-u+d})-2d(v-u+d)p^{v-u+2d}}{(q^d-p^d)^2}$$

$$\left.+\frac{2(dp^d)^2(q^{v-u+d}-p^{v-u+d})}{(q^d-p^d\ )^3}\right]$$

The result follows by applying the normalization condition and the variance can be computed using the standard definition $Var\left(X_n^{(m,k)}\right)=G_n{''}(1)+G_n{'}(1)-\left(G_n{'}(1)\right)^2$ this completes the proof.

Corollary 3.6

Let $X_n^{\left(\frac{v-u+d}{d},\ \frac{v-u+d}{d}\right)}$ be a random variable that count the total score in n rolls of an $(\frac{v-u}{d}+1)$-sided geometric figure with turn-up side probabilities $T(x,v-x)$ satisfying the condition $p^u(q^{v-u+d}-p^{v-u+d})=(q^d-p^d)$, with range $x=u,u+d,u+2d,\dots,v$. Then the mean and variance are determined by

$$(i)\ \ G_n{'}(1)=E\left(X_n^{\left(\frac{v-u+d}{d},\ \frac{v-u+d}{d}\right)}\right)=un+np^d\left[\frac{d-(v-u+d)p^v}{q^d-p^d}\right]$$

$$(i)\ \ G_n{''}(1)=un(un-1)+2un^2p^d\left[\frac{d-(v-u+d)p^v}{q^d-p^d}\right]$$

$$+n(n-1)\left[\frac{dq^d-(v-u+d)p^{v+d}}{q^d-p^d}\right]^2$$

$$+\,n\left[\frac{-(v-u+d)(v-u+d-1)p^{v+d}+d(d-1)p^d}{q^d-p^d}\right.$$

$$\left.+\frac{2\left(dq^d\right)^2-2d(v-u+d)p^{v+2d}}{(q-p)^2}\right]$$

Corollary 3.7

Let $X_n^{(v-u+1,v-u+1)}$ be a random variable that count the total score in n rolls of a $(v-u+1)$-sided geometric figure with turn-up side probabilities $T(x,v-x)$ satisfying the condition $p^u(q^{v-u+1}-p^{v-u+1})=(q-p)$, with range $x=u,u+1,u+2,u+3,\dots,v$. Then the mean and variance are determined by

$$(i)\ \ G_n{'}(1)=E\left(X_n^{(v-u+1,v-u+1)}\right)=un+np\left[\frac{1-(v-u+1)p^v}{(q-p)}\right]$$

$$(i)\ G_n''(1) = un(un-1) + 2un^2\left[\frac{p}{(q-p)} + \frac{-(v-u+1)p^{v+1}}{q-p}\right]$$

$$+ n(n-1)\left[\frac{p}{(q-p)} + \frac{-(v-u+1)p^{v+1}}{q-p}\right]^2$$

$$+ n\left[\frac{-(v-u+1)(v-u)p^{v+1}}{q-p} + \frac{2p^2(1-(v-u+1)p^v)}{(q-p)^2}\right]$$

Global Journal of Science Frontier Research (F) Volume XIX Issue II Version I Year 2019

Theorem 3.8

Let $S_j^{\left(\frac{v-u+d}{d}\right)}$ be a random variable that count the total score in j rolled of a $(\frac{v-u+d}{d})$-sided geometric figure with turn-up side probabilities $T(x, v-x)$ satisfying the condition $p^u(q^{v-u+d} - p^{v-u+d}) = (q^d - p^d)$, with range $x = u, u+d, u+2d, \ldots, v$. Then the recursion formula for the probability mass function (pmf) is given by

$$P\left(\left\{S_j^{\left(\frac{v-u+d}{d}\right)} = r; d\right\}\right) = \sum_{x=1}^{\frac{v-u+d}{d}} f_{j-1}(r-u-(x-1)d)\, p^x q^{v-x};\ ju \le r \le jv, j = 1,2,\cdots,n$$

Proof.

Now, let $x_j \in \{u, u+d, u+2d, \cdots, v\}$ be the number that turns up when the jth die is rolled for each $\left(\frac{v-u+d}{d}\right)$-sided die for $j = 1,2,\cdots,n$. It then follows that the probability distribution for each x_j is given by

$$f(x; d) = p^x q^{v-x}\ ; x = u, u+d, u+2d, \cdots, v$$

It follows that $P(\{x_j = x\}) = f_1(x; d) = f(x; d)$ for each $\left(\frac{v-u+d}{d}\right)$-sided geometric figure, so that if we define the random variable $S_j^{\left(\frac{v-u+d}{d}\right)} = x_1 + x_2 + \cdots + x_j$ to be the sum of the j rolled of each $\left(\frac{v-u+d}{d}\right)$-sided die such that $P\left(\left\{S_j^{\left(\frac{v-u+d}{d}\right)} = x\right\}\right) = f_j(x; d)$ and $\sum_{x=u}^{v} f_0(x; d) = 1$.

If $r \in [ju, jv]$, then for any event $\left\{S_j^{\left(\frac{v-u+d}{d}\right)} = r\right\}$ we have that

$$\left\{S_j^{\left(\frac{v-u+d}{d}\right)} = r\right\} = \bigcup_{x=u}^{v}\left\{S_{j-1}^{\left(\frac{v-u+d}{d}\right)} = r-x, x_j = x\right\}$$

Which implies that

$$P\left(\left\{S_j^{\left(\frac{v-u+d}{d}\right)} = r\right\}\right) = P\left(\bigcup_{x=u}^{v}\left\{S_{j-1}^{\left(\frac{v-u+d}{d}\right)} = r-x, x_j = x\right\}\right)$$

$$= \sum_{x=u}^{v} P\left(\left\{S_{j-1}^{\left(\frac{v-u+d}{d}\right)} = r - x, x_j = x\right\}\right) = \sum_{x=u}^{v} P\left(\left\{S_{j-1}^{\left(\frac{v-u+d}{d}\right)} = r - x\right\}\right) P(\{x_j = x\})$$

$$= \sum_{x=u}^{v} f_{j-1}(r-x) f(x) = \sum_{x=u}^{v} f_{j-1}(r-x)\, p^x q^{v-x} = \sum_{x=1}^{\frac{v-u+d}{d}} f_{j-1}(r-u-(x-1)d)\, p^x q^{v-x}$$

Notes

If we are dealing with a fair (balanced) die $\left(i.e\ q = \left(\frac{d}{v-u+d}\right)^{\frac{1}{v}} = p\right)$ then the corollary that follows is a consequence of theorem 3.1 above.

Corollary 3.9

Let $S_j^{\left(\frac{v-u+d}{d}\right)}$ be a random variable that count the total score in j rolled of a $(\frac{v-u+d}{d})$-sided fair die and turn-up side probabilities $T(x, v-x)$ satisfying the condition $p^u(q^{v-u+d} - p^{v-u+d}) = (q^d - p^d)$, with range $x = u, u+d, u+2d, \ldots, v$. Then the recursion formula for the probability mass function (pmf) is given by

$$P\left(\left\{S_j^{\left(\frac{v-u+d}{d}\right)} = r; d\right\}\right) = \left(\frac{d}{v-u+d}\right) \sum_{x=1}^{\frac{v-u+d}{d}} f_{j-1}(r-u-(x-1)d);$$

$$ju \le r \le jv, \qquad j = 1,2,\cdots,n.$$

Also, to obtain recurrence formula result that conform to the probability distribution due to Balasubramanian et al (1994) and Okoli (2017) for the case of a fair die, we simply put $d = 1$, $u = 0$, $v = m-1$ and $d = 1$, $u = 1$, $v = m$ to obtain several corollaries which are results of some authors in the literature (see Ashok et al (2011); Balasubramanian, (1995); Okoli (2017); Okoli (2017)).

Theorem 3.10

Let $S_j^{\left(\frac{k-u+d}{d}\right)}$ be a random variable that count the total score in j rolled of a $(\frac{v-u+d}{d})$-sided die and turn-up side probabilities $T(x, k-x)$ satisfying the condition $p^u(q^{v-u+d} - p^{v-u+d}) = q^{v-k}(q^d - p^d)$, with range $x = u, u+d, u+2d, \ldots, v$. Then the moment generating function (mgf) is given by

$$M(t) = E\left(e^{tS_j^{\left(\frac{k-u+d}{d}\right)}}\right) = \left[q^{k-v}\, p^u e^{tu}\, \frac{\left(q^{v-u+d} - p^{v-u+d} e^{t(v-u+d)}\right)}{q^d - p^d e^{td}}\right]^n$$

Proof

Now, observe that the moment generating function (mgf) of $S_j^{\left(\frac{k-u+d}{d}\right)}$ constitute a convolution of $X_j^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ $(j = 1,2,3, \ldots n)$. Where each $X_j^{\left(\frac{v-u+d}{d},\ \frac{k-u+d}{d}\right)}$ is an independent identically distributed (iid) random variables corresponding to the scores of $\left(\frac{k-u+d}{d}\right)$-sided die and turn-up side probabilities $(x, k-x)$. Thus

$$M(t) = E\left(e^{tS_j^{\left(\frac{k-u+d}{d}\right)}}\right) = E\left(e^{\sum_{j=1}^{n} tX_j^{\left(\frac{v-u+d}{d}, \frac{k-u+d}{d}\right)}}\right)$$

$$= \prod_{j=1}^{n} E\left(e^{tX_j^{\left(\frac{v-u+d}{d}, \frac{k-u+d}{d}\right)}}\right) = \left[q^{k-v}\, p^u e^{tu} \frac{\left(q^{v-u+d} - p^{v-u+d} e^{t(v-u+d)}\right)}{q^d - p^d e^{td}}\right]^n$$

This completes the proof.

Observe that we can easily deduce the result of this moment generating function (mgf) for the probability distribution function from theorem 3.1. to give

$$M(t) = E\left(e^{tX_n^{\left(\frac{v-u+d}{d}, \frac{k-u+d}{d}\right)}}\right) = \prod_{j=1}^{n} E\left(e^{tX_j^{\left(\frac{v-u+d}{d}, \frac{k-u+d}{d}\right)}}\right)$$

$$= \left[q^{k-v}\, p^u e^{tu} \frac{\left(q^{v-u+d} - p^{v-u+d} e^{t(v-u+d)}\right)}{q^d - p^d e^{td}}\right]^n$$

IV. Discussion and Conclusion

Observe that several other corollaries can be deduce from the theorems above which reduces to the results obtained in Ashok et al (2011); Balasubramanian, (1995); Okoli (2017); Okoli (2017) as special cases. Succinctly, it follows that; if $u = 0, d = 1$ and $k = v = m - 1$, we obtain the results of Balasubramanian et al (1995), if $u = 0$, $d = 1$ and $k \leq v = m - 1$, we obtain the results of Okoli (2017), if $u = 1, d = 1$ and $k \leq v = m$, we obtain the results of Okoli (2017) and if $u = 1, d = 1$ and $k = v = 6$, we obtain the results of Ashok et al (2011). Hence, the results of this research work unifies and improves the works of several researchers in this direction, haven shown that the existing results in the literature can be deduce easily from the results in this paper.

References Références Referencias

1. Ailing, D. W., (1993). Estimation of Hit Number; *Biometrics*: 27, 605-613.
2. Aki, S., (1985). Discrete distribution of order k on a binary sequence; *Ann. Inst. Statist. Math.*: 36A, 205-224.
3. Aki et al, (1984). On discrete distribution of order k on a binary sequence; *Ann. Inst. Statist. Math.*: 37A, 431-440.
4. Ashok, K., Dalpatadu, J. R., and Lucas, A. F., (2011). The Probability Distribution of the Sum of Several Dice: Slot Applications; *UNLV Gaming Research and Review Journal*: Vol. 15 No. 2, 109-118.
5. Balasubramanian, K., Viveros, R., and Balakrishnan, N., (1995). Some Discrete Distribution Related to Extened Pascal Triangles; *Fibonacci Quarterly*: 33, No.5, 415–425.
6. Bondarenko, B. A., (1993). Generalized Pascal Triangles and Pyramids, Their Fractals, Graphs, and Applications; *The Fibonacci Association*.
7. Bollinger, R. C., (1986). A Note on Pascal-T Triangles, Multinomial Coefficients and Pascal Pyramids; *The Fibonacci Quarterly*: 24, 140–144.
8. Bollinger, R. C., (1984). Fibonacci ˆ-Sequences, Pascal-7" Triangles, and Θ-in-a Row Problems; *The Fibonacci Quarterly*: 17.1, 23-28.

Notes

Notes

9. Bollinger, R. C., (1993). Extended Pascal Triangles; *Math. Magazine:* 66, 87-94.
10. Bollinger, R. C., and Burchard, Ch. L., (1990). Theorem and Some Related Results for Ex- tended Pascal Triangles; *Amer. Math. Monthly*: 97, 198-204.
11. Dafnis, S.D., Makri, F. S., and Philippou, A. N., (2007). Restricted Occupancy of *s* kind of Cells and Generalised Pascal Triangle; *The Fibonacci Quarterly*: 347-356.
12. De Moivre, A., (1756). *The Doctrine of Chances: or, A Method of Calculating the Probabilities of Events in Play*. 3rd ed. London: Millar; rpt. New York: Chelsea, 1967.
13. Derman, C., Lieberman, G., and Ross, S., (1982). On the Consecutive-k-of n:F System; *IEEE Trans. Reliability*: 31, 57–63.
14. Dharmadhikari, S,. and Joag-dev, K., (1988). *Unimodality, Convexity, and Applications*. San Diego: Academic Press.
15. Feller, W., (1968). *An Introduction to Probability Theory and Its Applications.* Vol.1. 3rd ed. New York: Wiley.
16. Fisher, R. A., (1973). *Statistical Methods and Scientific Inference*. New York: Hafner Press.
17. Freund, J. E., (1956). Restricted Occupancy Theory—A Generalization of Pascal's Triangles; *Amer. Math. Monthly:* 63, 20-27.
18. Gabai, H., (1970). Generalized Fibonacci ˆ-Sequences; *The Fibonacci Quarterly*: 8.1,31-38.
19. Hogg, R. V., and Craig, A. T., (1978). *Introduction to Mathematical Statistics.* 4th ed. New York: Macmillan.
20. Kalbfleisch, J. G., and Sprott, D. A., (1974). Inference about Hit Number in a Virological Model; *Biometrics:* 30, 199-208.
21. Makri, F. S., and Philippou, A. N., (2005). On Binomial and Circular Binomial Distributions of Order k for l-overlapping Success Runs of Length k; *Statistical Papers*: 46.3, 411–432.
22. Makri, F. S., Philippou, A. N., and Psillakis, Z. M., (2007a). Polya, Inverse Polya and Circular Polya Distributions of Order k for l-overlapping Success Runs; *Communications in Statistics- Theory and Methods:* 36, 657–668.
23. Makri, F. S., Philippou, A. N., and Psillakis, Z. M., (2007b). Shortest and Longest Length of Success Runs in Binary Sequences; *Journal of Statistical Planning and Inference*: 137, 2226–2239.
24. Okoli, O. C. et 'al., (2015). Construction of Arithmetic Distribution function in a defined interval; *Journal of the Nigerian Association of Mathematical Physics*: Vol.31, No.1, 399-402.
25. Okoli, O. C., Osuji, G. A., Nwosu, D. F. and Njoku, K. N. C., (2016). On the Modified Extended Generalised Exponential Distribution; *European Journal of Statistics and Probability*: Vol.4 No.4, 1-11.
26. Okoli, O.C., (2017). Modified Extended Binomial-Type Probability Distribution Function; *JCOOU*: Vol. 2, (accepted).
27. Okoli, O.C., (2017). On the Extended Binomial-Type Probability Distribution Function; *JCOOU*: Vol. 2, (accepted).
28. Ollerton, R. L. and Shannon, A. G., (1998). Some Properties of Generalized Pascal Squares and Triangles; *The Fibonacci Quarterly*: 36, 140–144.
29. Ollerton, R. L. and Shannon, A. G., (2004). Extensions of Generalized Binomial Coefficients; *In Applications of Fibonacci Numbers*: Volume 9. Edited by F. T. Howard, Dordrecht: Kluwer Academic Publishers, 187–199.

30. Ollerton, R. L. and Shannon, A. G., (2005). Further Properties of Generalized Binomial Coefficient k-extensions; *The Fibonacci Quarterly*: 43, 124–129.
31. Panaretos, J. and Xekalaki, E. (1986b). On some distributions arising certain generalized sampling schemes; *Commun. Statist. -Theor. Meth.:* 15(3), 873-891.
32. Panaretos, J. and Xekalaki, E. (1986c). On generalized binomial and multinomial distributions and their relation to generalized Poisson distributions; *Commun. Statist. -Theor. Meth.*: 15(3), 873-891.
33. Philippou, A. N. and Muwafi, A. A. (1982). Waiting for the kth consecutive success and the Fibonacci sequence of order k; *The Fibonacci Quarterly*: 20, 28-32
34. Philippou, A. N. (1984). The negative binomial distribution of order k and some of its properties; *Biometrical Journal*: 26, 789-794.
35. Philippou, A. N., Georghiou, C. and Philippou, G. N. (1983). A generalized geometric distribution and some of its properties; *Statistics Probability Letters*: 1, 171-175.

Notes