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Conjugate Fourier Series of (N, p, q) Summability of Approximation Theory F

By Sanjay Mukherjee & A J Khan
MATS University

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The monotonic non- increasing sequence of real constant that the conjugate Fourier series is almost (N, p, q) summable to

$$\|\tilde{t}_{n,m} - \tilde{f}\| = o(1)$$

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3. G. G. Lorentz, A contribution to the theory of divergent series, Acta Math. 80 (1948), 167-190. MR10, 367e.

Conjugate Fourier Series of (N, p, q) Summability of Approximation Theory F

Sanjay Mukherjee [α] & A J Khan [σ]

Abstract- In this paper the author has obtained a theorem that is:

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I. Introduction

Lorentz [3], for the first time in 1048, defined almost convergence of a bounded sequence. It is easy to see that a convergent sequence is almost convergent [4]. . Here, almost generalized Nörlund summability of method is considered. In 1913, Hardy [1] established $(c, \alpha), \alpha > 0$ summability of the series.Later on in 1948, harmonic summability which is weaker then summability $(c, \alpha), \alpha > 0$ of the series was discussed by Siddiqi[8]. The generalization of Siddiqi has been obtained by several workers, for example, Singh [9, 10], Iyengar[2], Pati[5]' Tripathi[11], Rajagopal[7] for Norlund mean. In an attempt to make an advance study in this direction, in the present paper, a theorem on almost generalized Nörlund summability of conjugate series of Fourier series has been obtained.

II. Definitions and Notations

Let $\sum a_n$ be an infinite series with $\{S_n\}$ as the sequence of its nth partial sums.
Lorentz [3] has given the following definition.
A bounded sequence$\{S_n\}$ is said to be almost convergent to a limit S, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{v=m}^{n+m} S_v = S, \text{ uniformly with respect to m.} \tag{2.1}$$

Let $\{p_n\}$ and $\{q_n\}$ be the two sequences of non-zero real constants such that

$$P_n = p_0 + p_1 + \cdots p_n, \qquad P_{-1} = p_{-1} = 0 \tag{2.2a}$$

$$Q_n = q_0 + q_1 + \cdots q_n, \qquad Q_{-1} = q_{-1} = 0 \tag{2.2b}$$

Author α σ: Research Scholar, MATS University, Raipur (C.G.), India. e-mails: sanjaymukherjeeruma@gmail.com, khanaj@matsuniversity.ac.in

Given two sequences $\{p_n\}$ and $\{q_n\}$, convolution p*q is defined by

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_k q_{n-k} \tag{2.3}$$

It is familiar and can be easily verified that the operation of convolution is commutative and associative, and

$$(p * 1)_n = \sum_{k=0}^{n} p_k \tag{2.4}$$

Notes

The series $\sum a_n$ or the sequence $\{S_n\}$ is said to be almost generalized Nörlund (N,p,q) (Qureshi[6]) summable to S, if

$$t_{n,m} = \frac{1}{R_n} \sum_{v=0}^{n} p_{n-v}\, q_v S_{v,m} \tag{2.5}$$

Tends to S, as $n \to \infty$, uniformly with respect to m, where

$$S_{v,m} = \frac{1}{v+1} \sum_{k=m}^{v+m} S_k \tag{2.6}$$

Let f(t) be a periodic function with period 2π and integrable in the sense over an interval $(-\pi, \pi)$.
Let its Fourier series be given by

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) \tag{2.7}$$

And then the conjugate series of (2.7) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nt - b_n \cos nt) = \sum_{n=1}^{\infty} B_n(t) \tag{2.8}$$

Let $\{p_n\}$ be a nonnegative non-increasing generating sequence for (N, p_n) method such that

$$P_n = P(n) = p_0 + p_1 + p_2 + \cdots p_n \to \infty, as\ n \to \infty \tag{2.9}$$

Particular Cases:

a) Almost (N,p,q) method reduces to almost Nörlund method (N, p_n) if $q_n = 1$ for all n.

b) Almost (N,p,q) method reduces to almost Riesz method $(\overline{N}, q_n)$ if $p_n = 1$ for all n.

c) In the special case when $p_n = \binom{n+\alpha+1}{\alpha-1}, \alpha > 0$, the method (N, p_n) reduces to the well known method of summability (C, α).

d) $p_n = \frac{1}{n+1}$ of the Nörlund mean is known as harmonic mean and is written as (N,1/(n+1)).

Following notations will be used:

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

$$\psi(t) = f(x+t) - f(x-t)$$

$$\Phi(t) = \int_0^t |\phi(u)|\, du$$

$$\Psi(t) = \int_0^t |\psi(u)|\, du$$

$$\tau = \left[\frac{1}{t}\right] = The\ integral\ part\ of\ \frac{1}{t}$$

Notes

III. Known Theorem

If f(x) is periodic and belongs to the class $Lip(\alpha, p) for\ 0 < \alpha \leq 1$, if the sequence $\{p_n\}$ is as defined in (2.9) with other requirements there in and if

$$\int_1^n \left(\frac{(p(y)^q)}{y^{q\alpha+2-\delta q-q}}\right) = o\left(\frac{p(n)}{n^{\frac{\alpha-1}{q-\delta-1}}}\right) \quad (3.1)$$

Then

$$\left\|\tilde{t}_n - \tilde{f}\right\|_p = o\left(\frac{1}{n^{\frac{\alpha-1}{p}}}\right) \quad (3.2)$$

where $\widetilde{t_n}$ are the (N, p_n) means of the series (2.8) and 1/p +1/q =1 such that $1 \leq p$

IV. Main Theorem

Our object of this paper is to prove the following theorem.

Theorem: The monotonic non- increasing sequence of real constant of the conjugate Fourier series is (N,p,q) summable to

$$\left\|\tilde{t}_{n,m} - \tilde{f}\right\| = o(1)$$

V. Lemmas

For the proof of theorem 4, the following lemmas are required

Lemma 5.1: For $0 < t < \frac{1}{(n+m)}$, we have

$$\left|N_{n,m}(t)\right| = o(n+m)$$

Proof: For $0 < t < \frac{1}{(n+m)}$, we have

$$\left|N_{n,m}(t)\right| = \frac{1}{2\pi R_n}\left|\sum_{v=0}^{n} p_{n-v} q_v \frac{sin(v+1)(t/2)\{cos(v+2m+1)(t/2) - cos(t/2)\}}{(v+1)sin^2(t/2)}\right|$$

$$= \frac{1}{2\pi R_n}\left|\sum_{v=0}^{n} p_{n-v} q_v \frac{sin(v+1)\left(\frac{t}{2}\right)\left\{2sin\left(\frac{(v+2m+2)}{2}\right)\left(\frac{t}{2}\right) sin\left(\frac{(v+2m)}{2}\right)\left(\frac{t}{2}\right)\right\}}{(v+1)sin^2\left(\frac{t}{2}\right)}\right|$$

$$\leq \frac{1}{2\pi R_n}\left|\sum_{v=0}^{n} p_{n-v} q_v \frac{(v+1)sin\left(\frac{t}{2}\right)\left\{2sin\left(\frac{(v+2m+2)}{2}\right)\left(\frac{t}{2}\right) sin\left(\frac{(v+2m)}{2}\right)\left(\frac{t}{2}\right)\right\}}{(v+1)sin^2\left(\frac{t}{2}\right)}\right|$$

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$$\leq \frac{1}{2\pi R_n}\left|\sum_{v=0}^{n} p_{n-v}q_v \frac{2((v+2m+2)/2)\left\{sin\left(\frac{t}{2}\right) sin\left(\frac{(v+2m)}{2}\right)\left(\frac{t}{2}\right)\right\}}{(v+1)sin\left(\frac{t}{2}\right)}\right|$$

$$= \frac{1}{2\pi R_n}\left\{\sum_{v=0}^{n} p_{n-v}q_v\right\}(n+2m+2)$$

$$= O(n+m)\frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v$$

$$|N_{n,m}(t)| = o(n+m)$$

Lemma 5.2: For $\frac{1}{(n+m)} < t < \pi$, we have

$$|\bar{N}_{n,m}(t)| = o\left(\frac{1}{t^2 n}\right)$$

Proof: For $\frac{1}{(n+m)} < t < \pi$, we have

$$\bar{N}_{n,m}(t) = \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{cos\,(m+(v+1)/2)t\,sin\,((v+1)/2)t}{(v+1)sin^2(t/2)}$$

$$|\bar{N}_{n,m}(t)| \leq \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{cos(m+(v+1)/2)t\,sin((v+1)/2)t}{(v+1)sin^2(t/2)}$$

$$\leq \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{1}{(v+1)sin^2(t/2)}$$

$$= o\left(\frac{1}{t^2}\right)\frac{1}{R_n}\sum_{v=0}^{n}\left(\frac{p_{n-v}q_v}{(v+1)}\right)$$

$$|\bar{N}_{n,m}(t)| = o\left(\frac{1}{t^2 n}\right)$$

VI. Proof of the Theorem (4)

Let $S_k(x)$ denote the nth partial sum of the series (2.8).
Then we have

$$S_k(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{cos\,(k+(1/2))t - cos\,(t/2)}{sin\,(t/2)}\psi(t)dt \tag{6.1}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \frac{cos\,(k+(1/2))t}{sin\,(t/2)}\psi(t)dt - \frac{1}{2\pi}\int_0^{\pi} cot\left(\frac{t}{2}\right)\psi(t)dt \tag{6.2}$$

Now, by using (2.6) we get

$$S_{v,m} = \frac{1}{v+1}\sum_{k=m}^{v+m}\left\{\frac{1}{2\pi}\int_0^{\pi}\frac{cos\,(k+(1/2))t}{sin\,(t/2)}\psi(t)dt - \frac{1}{2\pi}\int_0^{\pi} cot(t/2)\,\psi(t)dt\right\} \tag{6.3}$$

So that by using (2.5), we have

Notes

$$t_{n,m} = \frac{1}{R_n}\sum_{v=0}^{n} p_{n-v}q_v \left\{\frac{1}{2\pi}\int_0^{\pi}\sum_{k=m}^{v+m}\frac{cos\big(k+(1/2)\big)t}{sin(t/2)}\psi(t)dt - \frac{1}{2\pi}\int_0^{\pi} cot(t/2)\,\psi(t)dt\right\}$$

$$\|\tilde{t}_{n,m} - \tilde{f}\| = \frac{1}{R_n}\sum_{0}^{\pi} p_{n-v}q_v \frac{1}{2\pi(v+1)}\int_0^{\pi}\sum_{k=m}^{v+m}\frac{cos\big(k+(1/2)\big)t}{sin(t/2)}\psi(t)dt$$

$$= \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \int_0^{\pi}\frac{sin(v+m+1)t - sinmt}{2(v+1)sin^2(t/2)}\psi(t)dt$$

$$= \int_0^{\pi}\left\{\frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{cos(v+2m+1)(t/2)sin(v+1)(t/2)}{(v+1)sin^2(t/2)}\right\}\psi(t)dt$$

$$= \int_0^{\pi}\bar{N}_{n,m}(t)\psi(t)dt$$

$$= \left\{\int_0^{1/(n+m)} + \int_{1/(n+m)}^{1/(n+m)^{\delta}} + \int_{1/(n+m)^{\delta}}^{\pi}\right\}\bar{N}_{n,m}(t)\psi(t)dt = I_1 + I_2 + I_3 \qquad (6.4)$$

First we consider,

$$I_1 = \int_0^{1/(n+m)} \bar{N}_{n,m}(t)\psi(t)dt$$

$$= \int_0^{1/(n+m)} \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{cos(v+2m+1)(t/2)sin(v+1)(t/2)}{(v+1)sin^2(t/2)}\psi(t)dt$$

$$= \int_0^{1/(n+m)} \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{sin(v+1)(t/2)\{cos(v+2m+1)(t/2) - cos(t/2)\}}{(v+1)sin^2(t/2)}\psi(t)dt$$

$$+ \int_0^{1/(n+m)} \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{sin(v+1)(t/2)cot(t/2)}{(v+1)sin(t/2)}\psi(t)dt$$

$$= I_{1.1} + I_{1.2} \qquad (6.5)$$

Now

$$|I_{1.1}| \le \int_0^{1/(n+m)} \frac{1}{2\pi R_n}\left|\sum_{v=0}^{n} p_{n-v}q_v \frac{sin(v+1)(t/2)\{cos(v+2m+1)(t/2) - cos(t/2)\}}{(v+1)sin^2(t/2)}\right| |\psi(t)|dt$$

$$= \int_0^{1/(n+m)} |\bar{N}_{n,m}(t)||\psi(t)|dt$$

Notes

$= o(n+m)\int_0^{1/(n+m)}|\psi(t)|\,dt$ by Lemma 5.1

$$= o(n+m)o\left[\frac{\alpha(n+m)}{(n+m)R_{n+m}}\right]$$

$$= o\left[\frac{1}{log(n+m)}\right]$$

$= o(1)$, as $n \to \infty$, uniformly with respect to m (6.6)

Next, for $0 \le t \le \frac{1}{(n+m)}$

$$|I_{1.2}| \le \int_0^{1/(n+m)} \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \frac{sin(v+1)(t/2)cot(t/2)}{(v+1)sin(t/2)}\psi(t)dt$$

$$\le \frac{1}{2\pi R_n}\sum_{v=0}^{n} p_{n-v}q_v \int_0^{1/(n+m)} \frac{(v+1)sin(t/2)cot(t/2)}{(v+1)sin(t/2)}\psi(t)dt$$

$$= \frac{1}{2\pi}\int_0^{1/(n+m)} cot(t/2)\psi(t)\,dt$$

Since the conjugate function exists, therefore

$= o(1), as\ n \to \infty$, uniformly with respect to m (6.7)

Thus from (6.6) and (6.7), we get

$I_1 = o(1), as\ n \to \infty$, uniformly with respect to m (6.8)

Now, we get

$$|I_2| \le \int_{1/(n+m)}^{1/(n+m)^\delta} \left|\bar{N}_{n,m}(t)\right||\psi(t)|\,dt$$

$= o\int_{1/(n+m)}^{1/(n+m)^\delta} \frac{|\psi(t)|}{t^2 n}dt$ by Lemma 5.2

$$= o\left(\frac{1}{n}\right)\int_{1/(n+m)}^{1/(n+m)^\delta} \frac{|\psi(t)|}{t^2}dt$$

$$= o\left(\frac{1}{n}\right)o(n)$$

$I_2 = o(1), as\ n \to \infty$, uniformly with respect to m (6.9)

Finally, we have

Notes

Notes

$$|I_3| \le \int_{1/(n+m)^\delta}^{\pi} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \left| \frac{cos(v+2m+1)(t/2) sin(v+1)(t/2)}{(v+1) sin^2(t/2)} \right| |\psi(t)| dt$$

$$= \int_{1/(n+m)^\delta}^{\pi} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \left| \frac{sin(v+m+1)(t) - sinmt}{(v+1) sin^2(t/2)} \right| |\psi(t)| dt$$

$$= \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \left[\int_{1/(n+m)^\delta}^{\pi} \left| \frac{sin(v+m+1)(t)}{(v+1) sin^2(t/2)} \right| |\psi(t)| dt + \int_{1/(n+m)^\delta}^{\pi} \left| \frac{sinmt}{(v+1) sin^2(t/2)} \right| |\psi(t)| dt \right]$$

$$= I_{3.1} + I_{3.2} \qquad (6.10)$$

Now, by using second mean value theorem, we have

$$|I_{3.1}| \le \frac{1}{2\pi R_n} \sum_{v=0}^{n} \frac{p_{n-v} q_v}{2(v+1) 2 sin^2(1/(n+m)^\delta)} \int_{1/(n+m)^\delta}^{\epsilon} |sin(v+m+1)|\, |\psi(t)| dt$$

Where $\frac{\mathbf{1}}{(\boldsymbol{n}+\boldsymbol{m})^{\boldsymbol{\delta}}} \le \epsilon \le \boldsymbol{\pi}. \mathbf{0} \le \boldsymbol{\delta} \le \frac{\mathbf{1}}{\mathbf{2}}$

$$= o\left(\frac{1}{n}\right) (n+m)^{2\delta} \left(\frac{1/2(n+m)^\delta}{sin(1/2(n+m)^\delta)} \right)^2 \int_{1/(n+m)^\delta}^{\epsilon} |\psi(t)| dt$$

$I_{3.1} = o(1), as\ n \to \infty$, uniformly with respect to m (6.11)

Now

$$|I_{3.2}| \le \int_{1/(n+m)^\delta}^{\pi} \frac{1}{2\pi R_n} \sum_{v=0}^{n} p_{n-v} q_v \left| \frac{sinmt}{(v+1) sin^2(t/2)} \right| |\psi(t)| dt$$

$$\le \frac{1}{2 sin^2(1/2(n+m)^\delta)} \int_{1/(n+m)^\delta}^{\epsilon} |\psi(t)| dt$$

Where $\frac{\mathbf{1}}{(\boldsymbol{n}+\boldsymbol{m})^{\boldsymbol{\delta}}} \le \epsilon \le \boldsymbol{\pi}. \mathbf{0} \le \boldsymbol{\delta} \le \frac{\mathbf{1}}{\mathbf{2}}$

$I_{3.2} = o(1), as\ n \to \infty$, uniformly with respect to m (6.12)

Now combining (6.8), (6.9) and (6.12), we get

$$\left\| \tilde{t}_{n,m} - \tilde{f} \right\| = o(1)$$

Thus completes the theorem

VII. Conclusion

If $\{p_n\}$ and $\{q_n\}$ be the monotonic non- increasing sequence of real constant such that the conjugate Fourier series is almost (N,p,q) summable then

$$\|\tilde{t}_{n,m} - \tilde{f}\| = o(1)$$

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