Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 19 Issue 3 Version 1.0 Year 2019
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On the Solution of Wave-Schrodinger Equation

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Abstract- In this paper, we are finding solution of fraction Wave-Schrodinger equation by Laplace transform in sense of Caputo fractional derivative. It was found that the fundamental solution of the equation related to Wright function.

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GJSFR-F Classification: *MSC 2010: 35L05*

Global Journal of Science Frontier Research (F) Volume XIX Issue III Version I Year 2019

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I. Introduction

The Laplacian operator $\triangle^k$ iterated $k-$ times is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k, \tag{1.1}$$

where n is the dimension of space $\mathbb{R}^n$, k is a nonnegative integer. A. Kananthai[1] has proved that the generalized function $(-1)^k S_{2k}(x)$ is an elementary solution of the operator $\triangle^k$, that is

$$\triangle^k(-1)^k S_{2k}(x) = \delta,$$

where δ is the Dirac-delta distribution and $S_{2k}(x)$ is defined by

$$S_{2k}(x) = \frac{\pi^{-\frac{n}{2}} 2^{-2k} \Gamma\left(\frac{n-2k}{2}\right)\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{\frac{2k-n}{2}}}{\Gamma(k)}, \tag{1.2}$$

In 2002, A.Kananthai, S. Suantai, V. Longani[2] have first introduced the operator $\triangle_i^k$ and is defined by

$$\triangle_i^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right), i = \sqrt{-1} \tag{1.3}$$

They have proved the function $(-1)^k(-i)^{\frac{q}{2}} S_{2k}(x)$ is an elementary solution of the operator $\triangle_i^k$ and $S_{2k}(x)$ is defined by (1.2).It is well known the linear Schrodinger equation can be written as the following form

Ref

1. Kananthai, *On the Solution of the n-Dimensional Diamond Operator*, Applied Mathematics and Computational, Elsevier Science Inc. New York, 1997, pp. 27-37.

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$$\frac{\partial}{\partial t}u(x,t) = i\frac{\partial^2}{\partial x^2}u(x,t), i = \sqrt{-1} \tag{1.4}$$

with the initial condition

$$u(x,0) = f(x).$$

The Schrodinger equation has been widely in application in science and engineering, there are several integral transform such as Laplace transform, Fourier transform, Wavelet transform etc. for solving the equation.

The purpose of this work is to introduce a new function where related the Wright function [3] and studied Laplace transform of a new function.After that,we are solving the fundamental solution of the wave-schrodinger equation as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0 \ , \ i = \sqrt{-1} \ , \ 1 < \alpha \leq 2 \tag{1.5}$$

with the initial condition

$$\phi(x,0) = 0 \ , \ \phi_t(x,0) = \delta(x),$$

where δ is the Dirac-delta distribution and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative. Before going that point, the following definitions and some important concepts are needed.

II. Preliminaries

Definition 2.1 Let $f(t)$ be a function an exponential order and piecewise continuous. The Laplace transform of the function f is given by

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)dt \tag{2.1}$$

Definition 2.2 Let $f(t)$ be a function of the Schwart space the Fourier transform of $f(t)$ is given by

$$\widehat{f}(w) = \int_{\mathbb{R}} f(t)e^{iwt}dt \tag{2.2}$$

Definition 2.3 For m to be the smallest integer that exceeds α , the Caputo fractional derivatives of order is defined by

$$D^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-n-1}\frac{\partial^m}{\partial t^m}u(x,t) \\ \frac{\partial^m}{\partial t^m}u(x,t) \ , \ n = m \end{cases} \tag{2.3}$$

Definition 2.4 The Laplace transform of the Caputo fractional derivative is defined by

$$\mathcal{L}[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0) \ , \ n-1 < \alpha < n \tag{2.4}$$

Ref

3. I. Podlubny. *Fractioanl Differential Equation*, An introduction to fractional derivatives. Academic Press 1999.

Notes

Definition 2.5 The Wright function $W_{\alpha,\beta}$ is defined by

$$W_{\alpha,\beta} = \sum_{n=0}^{\infty} \frac{Z^n}{n!\Gamma(n\alpha + \beta)} \ , \ \alpha > -1 \ , \ \beta \in \mathbb{C} \tag{2.5}$$

where $\Gamma(x)$ is the Euler Gamma function is given by the integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-6} dt \tag{2.6}$$

Lemma 2.1 The function $\gamma(a,t)$ is defined by the following expressions

$$\gamma(a,t) = t^{-\alpha+1} W_{-\alpha \ , \ 2-\alpha}(at^{-\alpha}) \tag{2.7}$$

and Laplace transform of $\gamma(a,t)$ is given by

$$\mathcal{L}[\gamma(a,t)] = s^{\alpha-2} e^{as^{\alpha}}.$$

Proof: By (2.1) *, we have*

$$\begin{aligned}
\mathcal{L}[\gamma(a,t)] &= \int_0^{\infty} e^{-st} t^{-\alpha+1} W_{-\alpha \ , \ 2-\alpha}(at^{-\alpha}) dt \\
&= \int_0^{\infty} e^{-st} t^{-\alpha+1} \sum_{k=0}^{\infty} \frac{(at^{-\alpha})^k}{k!\Gamma(-\alpha k + 2 - \alpha)} dt \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\alpha k + 2 - \alpha)} \int_0^{\infty} e^{-st} t^{-\alpha-\alpha k+1} dt \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\alpha k + 2 - \alpha)} \mathcal{L}[t^{-\alpha-\alpha k+1}] \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\alpha k + 2 - \alpha)} \frac{\Gamma(-\alpha - \alpha k + 2)}{s^{-\alpha-\alpha k+2}} \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k! s^{-\alpha-\alpha k+2}} \\
&= s^{\alpha-2} \sum_{k=0}^{\infty} \frac{(as^{\alpha})^k}{k!} \\
&= s^{\alpha-2} e^{as^{\alpha}}
\end{aligned} \tag{2.8}$$

That completes the proof.

III. Main Results

Theorem 3.1 Consider the Fractional Wave-Schrodinger equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} \phi(x,t) + i \frac{\partial^2}{\partial x^2} \phi(x,t) = 0 \ , \ i = \sqrt{-1} \ , \ 1 < \alpha \leq 2 \tag{3.1}$$

with the initial condition

$$\phi(x,0)=0\ ,\ \ \phi_t(x,0)=\delta(x)$$

where $\delta(x)$ is the dirac delta distribution.By the Laplace and Fourier transform,we obtain the fundamental solution of the equation (3.1) is given by

$$\phi(x,t)=\frac{1}{2}\sqrt{i}t^{-\frac{\alpha}{2}+1}W_{-\frac{\alpha}{2}\ ,\ 2-\frac{\alpha}{2}}(-\sqrt{i}|x|t^{-\frac{\alpha}{2}}) \tag{3.2}$$

where $W_{\alpha,\beta}$ is the Wright function is defined by (2.5). If we put $\alpha=2$ in (3.1) the fractional Wave-Schrodinger equation reduced to

$$\frac{\partial^2}{\partial t^2}\phi(x,t)+i\frac{\partial^2}{\partial x^2}\phi(x,t)=0 \tag{3.3}$$

and the solution of (3.3) is given by

$$\phi(x,t)=\frac{1}{2}\sqrt{i}W_{-1,1}(-\sqrt{i}|x|t^{-1}) \tag{3.4}$$

Proof: By (3.1) , we have

$$\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t)+i\frac{\partial^2}{\partial x^2}\phi(x,t)=0 \tag{3.5}$$

Taking Laplace transform both sides of (3.5) and we get by definition 2.1

$$\mathcal{L}\Big[\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t)\Big]+i\mathcal{L}\Big[\frac{\partial^2}{\partial x^2}\phi(x,t)\Big]=0$$

$$s^\alpha\phi(x,s)-s^{\alpha-2}\delta(x)=-i\frac{\partial^2}{\partial x^2}\phi(x,s). \tag{3.6}$$

Applying Fourier transform respect to variable x both sides of (3.6),we obtained

$$\begin{aligned} s^\alpha\mathcal{F}\phi(x,s)-s^{\alpha-2}\mathcal{F}[\delta(x)] &= -i\mathcal{F}\frac{\partial^2}{\partial x^2}\phi(x,s) \\ s^\alpha\phi(\omega,s)-s^{\alpha-2} &= i\omega^2\phi(\omega,s) \\ \phi(\omega,s) &= \frac{s^{\alpha-2}}{s^\alpha+(-i)\omega^2} \\ &= \frac{is^{\alpha-2}}{is^\alpha+\omega^2}. \end{aligned} \tag{3.7}$$

Applying inverse Fourier transform both sides of (3.7) , we obtain

$$\begin{aligned} \phi(x,s) &= \mathcal{F}^{-1}\Big[\frac{is^{\alpha-2}}{is^\alpha+\omega^2}\Big] \\ &= \frac{\sqrt{i}s^{\alpha-2}e^{-|x|\sqrt{i}s^{\frac{\alpha}{2}}}}{2s^{\frac{\alpha}{2}}} \\ &= \frac{1}{2}\sqrt{i}s^{\frac{\alpha}{2}-2}e^{-|x|\sqrt{i}s^{\frac{\alpha}{2}}}. \end{aligned}$$

Notes

By Lemma 2.1,we obtain the solution of (3.1) as follows

$$\begin{aligned}\phi(x,t) &= \frac{1}{2}\sqrt{i}r(-\sqrt{i}|x|,t) \\ &= \frac{1}{2}\sqrt{i}t^{-\frac{\alpha}{2}+1}W_{-\frac{\alpha}{2},2-\frac{\alpha}{2}}(-\sqrt{i}|x|t^{-\frac{\alpha}{2}}) \end{aligned} \tag{3.8}$$

Notes

If we put $\alpha = 2$ in (3.1) and (3.8) respectively, the equation reduced to the Wave-Schrodinger equation

$$\frac{\partial^2}{\partial t^2}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0, \tag{3.9}$$

and the solution of (3.9) is given by

$$\phi(x,t) = \frac{1}{2}\sqrt{i}W_{-1,1}(-\sqrt{i}|x|t^{-1}). \tag{3.10}$$

That completes the proof.

Acknowledgement

The authors would like to thank the Thailand Research Fund , Graduate School, Maejo University, Chiang Mai, Thailand for financial support and Prof. Luis Guillermo Romero for the helpful of discussion.

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