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Common Best Proximity Point Theorem on the Context of Bimetric Spaces

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COMMONBESTPROXIMITYPOINTTHEOREMONTHECONTEXTOFBIMETRICSPACES

Common Best Proximity Point Theorem on the Context of Bimetric Spaces

M. Iadh. Ayari [α], M. Ali. Ayari [σ] & Sahbi. Ayari [ρ]

Abstract- In this paper, we obtained some best proximity point results for continuous proximal **β**-quasi contractive mappings in the setting of two metrics space. We illustrate our main theorem with an example. As application, we establish several results on existence and uniqueness of fixed points in complete metric spaces.

Keywords and phrases: best proximity points, proximal β-quasi contractive mappings on metric spaces.

I. Introduction

Several fixed point theorem results are obtained in the case of a metric space as shown in figures [**13**]-[**25**]. Khan, Berzig and Chandok have obtained in [**10**] some fixed point results for self-mappings in the case of two metric space endowed with binary relation. Generalization of the contraction principle to the case of non-self- mappings $T : P \longrightarrow Q$ is presented and is natural to look for in an element p_* such that $d(p_*, Tp_*)$ is minimum. Such element a_* is called best proximity point of the non-self-mapping T.

The best approximation t heorem was introduced by Fan [**2**] in 1669. Sadi q Basha [**8**] revisited the the-orem and proposed necessary and sufficient conditions for existence for proximal contractions of first and second kind of such points. Furthermore, several variants of the non-self-contractions for the existence of a best proximity point were studied in [**3**]-[**7**].

The paper discusses the existence and the uniqueness of best proximity points for a class of continuous non-self-mapping in bimetric space. More precisely, let (X, d, δ) a bimetric space. Under additional hy-potheses, we obtain a common best proximity point result of both metrics d and δ by introducing the class of proximal β-quasi-contraction with respect to one metric δ.

The paper is divided into six sections; section two introduces the notation used herein, it presents some definitions and recalls some useful results. The best proximity point theorem is stated in section 3 with the proof. While a conformation of the theorem is described in section 4. Several consequences are derived in Section 5 such as the Maia's Theorem of existence and uniqueness of fixed points in bimetric spaces [**17**] and the theorem of Khan, and Berzik and Chandok [**1**] for continuous generalized quasi-contractive mappings in bimetric spaces.

II. Preliminaries and Definitions

Let (X, d, δ) be a bimetric space such that $d(x, y) \le \delta(x, y)$ for all $x, y \in X$. Let (P, Q) a pair of nonempty subsets of X. We consider the following notations

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Ref

2. Fan. K. *Extentions of two fixed point theorems of F.E. Brower.Math. Z..*, 112 (1669), 234–240.

$d(P,K) := \inf\{d(p,k) : p \in P, q \in Q\}$;

$d(x,Q) := \inf\{d(x,q) : q \in Q\}$;

$P_0^d := \{p \in P :$ there exists $q \in Q$ such that $d(p,q) = d(P,Q)\}$;

$Q_0^d := \{n \in N :$ there exists $p \in P$ such that $d(p,q) = d(P,Q)\}$.

$P_0^\delta := \{p \in P :$ there exists $q \in Q$ such that $\delta(p,q) = \delta(P,Q)\}$;

$Q_0^\delta := \{q \in Q :$ there exists $p \in P$ such that $\delta(p,q) = \delta(P,Q)\}$.

Definition 2.1. [**8**] *Let (X,d) a metric space and (P,Q) a pair of nonempty subset s of X. Let $T : P \to Q$ be a non-self-mapping. An element $p_* \in P$ is said to be a best proximity point of T if*

$$d(p_*, Tp_*) = d(P,Q).$$

Definition 2.2. [**11**] *Let $\beta \in (0,+\infty)$. A $\beta-$comparison function is a map $\varphi : [0,+\infty) \to [0, +\infty)$ satisfying the fol lowing properties:*

(P1) *φ is nondecreasing;*

(P2) $\lim_{n\to\infty} \varphi_\beta^n(t) = 0$ *for all $t > 0$, where φ_β^n denote the n-th iterate of φ_β and $\varphi_\beta(t) = \varphi(\beta t)$;*

(P3) *there exists $s \in (0,+\infty)$ such that $\sum_{n=1}^{\infty} \varphi_\beta^n(s) < \infty$.*

The set of all $\beta-$comparison functions φ satisfying (P1)–(P3) *will be denoted by Φ_β.*

Remark 2.1. Let $\alpha, \beta \in (0,+\infty)$. If $\alpha < \beta$, then $\Phi_\beta \subset \Phi_\alpha$.

We recall some useful lemmas concerning the comparison functions φ.

lemma 2.1. [**11**] *Let $\beta \in (0,+\infty)$ and $\varphi \in \Phi_\beta$. Then*

(i) *φ_β is nondecreasing;*

(ii) *$\varphi_\beta(t) < t$ for all $t > 0$;*

(iii) $\sum_{n=1}^{\infty} \varphi_\beta^n(t) < \infty$ *for all $t > 0$.*

lemma 2.2. [**9**] *Let (X,δ) a metric space and (P,Q) a pair of nonempty subsets of X. Let $T : P \longrightarrow Q$ be a non-self-mapping. Suppose that the following conditions hold:*

(1) $P_0 \neq \emptyset$

(2) $T(P_0 \subseteq Q_0$.

Then, for all $p \in P_0^\delta$, there exists a sequence $\{x_n\} \subset P_0^\delta$ such that

$$\begin{cases} x_0 = p \\ \delta(x_{n+1}, Tx_n) = \delta(P,Q) \qquad \forall\, n \in \mathbb{N}. \end{cases} \tag{1}$$

III. Preliminaries and Definitions

To begin, we have the following lemma:

Lemma 3.1. *Let (X,d,δ) be a bimetric space such that $d(x,y) \le \delta(x,y)$ for all $x, y \in X$. Let (P,Q) a pair of nonempty subsets of X. Suppose that $d(P,Q) = \delta(P,Q)$, then $P_0^\delta \subset P_0^d$ and $Q_0^\delta \subset Q_0^d$.*

Ref

8. Sadiq. Bacha. *Extentions of Banach's contraction principle.* J. Num. Func. Anal. Optim Theory Appl., 31 (2010), 569–576.

Proof. Let $p \in P_0^\delta$. Then $\delta(p,q) = \delta(P,Q)$ for some $q \in Q$. Therefore,

$$d(P,Q) = \delta(P,Q) \leq d(p,q) \leq \delta(p,q) = \delta(P,Q) = d(P,Q).$$

This means $p \in P_0^d$.
The same idea for $Q_0^\delta \subset Q_0^d$.

Let recall the following concept introduced in [**12**] .

Definition 2.1. Let (X,δ) be a metric space . Let $\beta \in (0,+\infty)$. A non-self-mapping $T : P \to Q$ is said to be a proximal β-quasi-contractive iff there exist $\varphi \in \Phi_\beta$ and positive numbers $\alpha_0,\dots,\alpha_4$ such that:

$$\delta(u,v) \leq \varphi(\max\{\alpha_0\delta(x,y), \alpha_1\delta(x,u), \alpha_2\delta(y,v), \alpha_3\delta(x,v), \alpha_4\delta(y,u)\}).$$

For all $x,y,u,v \in P$ satisfying, $\delta(u,Tx) = \delta(P,Q)$ and $\delta(v,Ty) = \delta(P,Q)$.

Our main result is giving by the following best proximity point theorem.

Theorem 3.1. Let (X,d,δ) be a bimetric space and (P,Q) be a pair of non empty subsets of the complete metric space (X,d). Let $T : P \longrightarrow Q$ be a non-self-mapping satisfying the following conditions :

(A1) *The set P_0^δ is nonempty closed in (X,d);*

(A2) *$d(x,y) \leq \delta(x,y)$ for all $x,y \in P$;*

(A3) *$d(P,Q) = \delta(P,Q)$;*

(A4) *T is continuous with respect to d;*

(A5) *there exists $\beta \geq \max_{0\leq k\leq 4}\{\alpha_k, 2\alpha_4\}$ such that T is a proximal β-quasi contractive with respect to δ;*

(A6) *A best proximity for T with respect to d is a best proximity point of T with respect to δ.*

Moreover, assume that $\beta > \max\{\alpha_0,\alpha_3,\alpha_4\}$.
Then T has a unique common best proximity point for both metrics $p_* \in P$ such that

$$d(p_*,Tp_*) = \delta(p_*,Tp_*) = d(P,Q).$$

Proof. Let $x_0 \in P_0{}^\delta$. Using Lemma (2.2), we can find $x_{n+1} \in P_0{}^\delta$ such that

$$\delta(x_{n+1},Tx_n) = \delta(P,Q),$$

for every positive integer k.

If $x_n = x_{n+1}$ are equals for some non negative integer n, then nothing to prove since

$$d(P,Q) \leq d(x_n,Tx_n) \leq \delta(x_n,Tx_n) = \delta(P,Q) = d(P,Q).$$

Then, we assume that $x_n \neq x_{n+1}$ for every negative integer n. Let us prove that the sequence $\{x_n\} \in P_0^\delta \subset P_0^d$ is a Cauchy sequence in (X,δ).

Since $\delta(x_{n+1},Tx_n) = \delta(P,Q)$ and $\delta(x_n,Tx_{n-1}) = \delta(P,Q)$ and T is a proximal β-quasi contractive with respect to the metric δ , we get using triangular inequalities

$$\begin{aligned}\delta(x_{n+1},x_n) &\leq \varphi(\max\{\alpha_0\delta(x_n,x_{n-1}), \alpha_1\delta(x_n,x_{n+1}), \alpha_2\delta(x_{n-1},x_n), \alpha_4\delta(x_{n-1},x_{n+1})\}) \\ &\leq \varphi(\max\{\alpha_0\delta(x_n,x_{n-1}), \alpha_1\delta(x_n,x_{n+1}), \alpha_2\delta(x_{n-1},x_n), \alpha_4\delta(x_{n-1},x_n) + \alpha_4\delta(x_n,x_{n+1})\}) \\ &\leq \varphi(\beta\max\{\delta(x_n,x_{n-1}), \delta(x_n,x_{n+1})\}).\end{aligned}$$

Where $\beta \geq \max\{\alpha_0,\alpha_1,\alpha_2,\alpha_3,2\alpha_4\}$.
If $\delta(x_n,x_{n-1}) \leq \delta(x_n,x_{n+1})$. Then by Lemma (2.1) it follows that

$$\delta(x_{n+1},x_n) \leq \varphi(\beta\delta(x_{n+1},x_n)) = \varphi_\beta(\delta(x_{n+1},x_n)) < \delta(x_{n+1},x_n).$$

Ref

12. M. Iadh. Ayari, M.M.M. Jaradat, ZeadMustfa. Generalization of Best Proximity Points Theorem for Non-Self Proximal Contractions of First Kind Submitted.

Which is a contradiction. So $\forall n \geq 1$,

$$\delta(x_{n-1}, x_n) > \delta(x_{n+1}, x_n),$$

then,

$$\delta(x_{n+1}, x_n) \quad \leq \quad \varphi_\beta(\delta(x_n, x_{n-1})) \quad \forall n.$$

Thus by Mathematical induction we obtain that

$$\delta(x_{n+1}, x_n) \quad \leq \quad \varphi_\beta^n(\delta(x_0, x_1)) \quad \forall n.$$

In addition, for $k < j$ and using triangle inequalities we obtain,

$$\begin{aligned} \delta(x_k, x_j) &\leq \textstyle\sum_{l=k}^{m-1} \delta(x_l, x_{l+1}), \\ &\leq \textstyle\sum_{l=k}^{j-1} \varphi_\beta^l(\delta(x_0, x_1)). \end{aligned}$$

Since $\sum\limits_{l=1}^{\infty} \varphi_\beta^l(t) < \infty$, for every $\epsilon > 0$ there exists $K > 0$ such that

$$\sum_{l=k}^{j-1} \varphi_\beta^l(t) < \epsilon \quad \text{for all } j > k > K.$$

Thus, $d(x_k, x_j) \leq \delta(x_k, x_j) < \epsilon$. This imply that the sequence $\{x_k\}$ is δ-Cauchy, so by (A2), $\{x_k\}$ is d-Cauchy too.

As the space (X, d) is complete, and the sequence $\{x_n\} \in P_0^\delta$ who closed in (X, d) by (A1), then there exists p_* such that $\lim\limits_{n \longrightarrow +\infty} d(x_n, p_*) = 0$.

So $d(P, Q) \leq d(x_{n+1}, Tx_n) \leq \delta(x_{n+1}, Tx_n) = \delta(P, Q) = d(P, Q)$ From (A4), we have T is continuous with respect to d, and, so it follows that

$$\lim_{n \longrightarrow +\infty} d(x_{n+1}, Tx_n) = d(p_*, Tp_*) = d(P, Q).$$

For the uniqueness, suppose that p_* and s_* two distinct best proximity points of T with respect to d on M_0. By hypothesis (A6), we deduce that they are also best proximity points of T with respect to δ. Let $\rho = \delta(p_*, s_*) \geq d(p_*, s_*) > 0$. Since T is a proximal β-quasi contractive with respect to the metric δ, we obtain the following inequality

$$\rho \leq \varphi(\max\{\alpha_0, \alpha_3, \alpha_4\}\rho) \leq \varphi(\beta\rho) = \varphi_\beta(\rho) < \rho,$$

which is a contradiction. Thus $\rho = 0$ and therefore $\delta(p_*, s_*) = 0$.

Remark 3.1. The conditions (A3) and (A6), which occur in Theorem 3.1, are always satisfied in finite dimensional normed spaces.

IV. Example

Consider the complete Euclidian space $X = \mathbb{R}^2$ with the metrics $\delta((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ and the Euclidian distance $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Using the inequality $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$, we get

$$d((x_1, y_1), (x_2, y_2)) \leq \delta((x_1, y_1), (x_2, y_2)).$$

So the assertion $A2$ of Theorem 3.1 is satisfied.

Notes

Let $P = \{(\gamma,0) : \gamma \in [0,1]\}$ and $Q = \{(\delta,1) : \delta \in [0,1]\}$. Also, let $T : P \longrightarrow Q$ be defined by $T(\gamma,0) = (\frac{\gamma}{7},1)$. Then, it is easy to see that $\delta(P,q) = d(P,Q) = 1$ and $P_0^\delta = P_0^d = P$ which is closed in (X,d). Thus, the hypotheses $(A1)$ and $(A2)$ of Theorem 3.1 are satisfied as well. As mentioned in Remark 3.1, the conditions $(A3)$ and $(A6)$ are satisfied since $X = \mathbb{R}^2$ is a finite dimensional normed space. The non-self-mapping T is continuous with respect to the metric d which confirm that the a ssertion $(A4)$ holds.

Now, we shall show that T is β-quasi-contractive mapping with respect to the metric δ with $\varphi(t) = \frac{3}{7}t$, $\beta = \frac{3}{7}$ and $\alpha_i = \frac{1}{3^{i+1}}$ for $i = 0,1,2,3,4$.

Let $x, y, u, v \in P$ where $x = (\gamma_1,0)$, $y = (\gamma_2,0)$, $u = (u_1,0)$ and $v = (v_1,0)$ such that $\delta(u,Tx) = \delta(v,Ty)$. This means that $u_1 = \frac{\gamma_1}{7}$ and $u_2 = \frac{\gamma_2}{7}$. Therefore, we get

$$\begin{aligned}
\delta(u,v) &= \delta((\frac{\gamma_1}{7},0),(\frac{\gamma_2}{7},0))\\
&= \frac{1}{7}|\gamma_1-\gamma_2|\\
&= \frac{1}{7}\delta(x,y)\\
&= \frac{3}{7}(\frac{1}{3}\delta(x,y))\\
&\leq \frac{3}{7}\max\{\frac{1}{3}\delta(x,y),\frac{1}{9}\delta(x,u),\frac{1}{27}\delta(y,v),\frac{1}{81}\delta(x,v),\frac{1}{243}\delta(y,u)\}.
\end{aligned}$$

So, since $\beta = \frac{3}{7} \geq \max_{0\leq k\leq 3}\{\alpha_k, 2\alpha_4\} = \max\{\frac{1}{3},\frac{1}{9},\frac{1}{27},\frac{2}{81}\}$, then T is proximal β-quasi-contractive mapping with respect to the metric δ with $\varphi(t) = \frac{3}{7}t$, $\beta = \frac{3}{7}$ and $\alpha_i = \frac{1}{3^{i+1}}$ for $i = 0,1,2,3,4$.

Hence, all conditions of Theorems 3.1 are satisfied and so T has a unique common proximity point with respect to the metrics δ and d which is $p_* = (0,0) \in P$,

$$\delta((0,0),T(0,0)) = d((0,0),(0,1)) = 1 = d(P,Q).$$

V. Consequences

As a consequence of our main theorem is Maia's fixed point theorem:

Theorem 5.1. [**17**] *Let (X,d,δ) be a bimetric space. Assume that for $T : X \longrightarrow X$, the following conditions are satisfied:*

(1) $d(x,y) \leq \delta(x,y)$ for all $x,y \in X$;
(2) X is complete with respect to d;
(3) T is continuous with respect to d;
(4) there exits $\alpha \in [0,1)$ such that $\delta(Tx,Ty) \leq \alpha\delta(x,y)$.

Then T has a unique fixed point in X

Proof. All hypotheses of our main Theorem 3.1 are satisfied by taking $P = Q = X$. In addition, the function $\varphi(t) = \alpha t$ belongs to Φ_1. Then there exists a unique fixed point in X.

Also we reproduce the results of Khan, Berzik and Chandok [1]. We recall first the definition of generalized contractive mapping.

Ref

1. M.S. Khan, M. Berzig and S. Chandok, *Fixed point theorems in bimetric space endowed with binary relation and applications.* Miskolc Mathematical Notes, 16 (2015), 939–951

Definition 5.1. [1] *Assume* $T: X \longrightarrow X$*, there exists* $\varphi \in \Phi$ *such that*

$$\delta(Tx, Ty) \leq \varphi(M_\delta(x, y)).$$

A mapping T *is called a generalized contractive with respect to* δ*, if*

$$M_\delta(x, y) = \max\{\delta(x, y), \frac{1}{2}[\delta(x, Tx) + \delta(y, Ty)], \frac{1}{2}[\delta(x, Ty) + \delta(y, Tx)]\}.$$

A mapping T *is called a generalized quasi-contractive with respect to* δ*, if*

$$M_\delta(x, y) = \max\{\delta(x, y), \delta(x, Tx), \delta(y, Ty), \delta(x, Ty), \delta(y, Tx)\}.$$

Let recall the following concept introduced in [**11**].

Definition 5.2. [**11**] *Let X a non empty set. A mapping* $T: X \longrightarrow X$ *is called* β*-quasi-contractive, if there exists* $\beta > 0$ *and* $\varphi \in \Phi_\beta$ *such that*

$$\delta(Tx, Ty) \leq \varphi(M_T(\delta(x, y))$$

where

$$M_T(x, y) = \max\{\alpha_0\delta(x, y), \alpha_1\delta(x, Tx), \alpha_2\delta(y, Ty), \alpha_3\delta(x, Ty), \alpha_4\delta(y, Tx)\},$$

with $\alpha_k \geq 0$ *for* $0 \leq k \leq 4$.

Theorem 5.2. [1] *Let* (X, d, δ) *be a bimetric space. Assume that for* $T: X \longrightarrow X$*, the following conditions are satisfied:*

(1) $d(x, y) \leq \delta(x, y)$ *for all* $x, y \in X$*;*
(2) X *is complete with respect to* d*;*
(3) T *is continuous with respect to d;*
(4) T *is a generalized contractive with respect to* δ .

Then T *has a unique fixed point in* X

Proof. In our main Theorem 3.1, taking $P = Q = X$. In addition, since T is a generalized contractive with respect to δ,

$$M_\delta(x, y) \leq \max\{\delta(x, y), \delta(x, Tx), \delta(y, Ty), \delta(x, Ty), \delta(y, Tx)\}.$$

The function $\varphi(t)$ belongs to Φ_2. Then there exists a unique fixed point in X.

Theorem 5.3. [1] *Let* (X, d, δ) *be a bimetric space. Assume that for* $T: X \longrightarrow X$*, the following conditions are satisfied:*

(1) $d(x, y) \leq \delta(x, y)$ *for all* $x, y \in X$*;*
(2) X *is complete with respect to* d*;*
(3) T *is continuous with respect to d;*
(4) T *is a generalized quasi-contractive with respect to* δ .

Then T *has a unique fixed point in* X.

Proof. Also the same idea with $\varphi \in \Phi_2$.

We suggest another corollary, which is an immediate consequence of our Theorem 3.1.

Ref

11. M. I. Ayari, M. Berzig, I. Kedim. *Coincidence and Common Fixed Point Results for β-Quasi Contractive Mappings on Metric Spaces Endowed with Binary Relation.* Math. Sci. (Springer) 10 (2016), no. 3, 105–114; MR3532376

Notes

Corollary 5.1. Let (X, d, δ) be a bimetric space. Assume that for $T : X \longrightarrow X$, the following conditions are satisfied:

(1) $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$;

(2) X is complete with respect to d;

(3) T is continuous with respect to d;

(4) T is a β- quasi-contractive with respect to δ, where $\beta \geq \max_{0 \leq k \leq 3} \{\alpha_k, 2\alpha_4\}$.

Moreover, assume $\beta > \max\{\alpha_0, \alpha_3, \alpha_4\}$.

Then T has a unique fixed point in X.

PROOF. All hypotheses of Theorem 3.1 are satisfied by taking $P = Q = X$.

VI. CONCLUSION

A new class of non-self-mappings in bimetric space is given in this paper. This has been achieved by introducing the proximal β-quasi-contraction mappings involving β- comparison functions. Under some additional conditions, we establish the existence and uniqueness of best proximity points for such mappings. As an application, we derive some fixed point results which are known in the literature. We believe that the approach used in the current contribution, may be extended for non-self-mappings which are not necessarily continuous.

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Notes

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