Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 19 Issue 5 Version 1.0 Year 2019
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Functional Calculus for the Series of Semigroup Generators via Transference

By Shawgy Hussein, Simon Joseph, Ahmed Sufyan, Murtada Amin, Ranya Tahire & Hala Taha
Upper Nile University

Abstract- In this paper, apply an established transference principle to obtain the boundedness of certain functional calculi for the sequence of semigroup generators. It is proved thatif $-A_j$ be the sequence generates C_0- semigroups on a Hilbert space, then for each $\varepsilon > -1$ the sequence of operators A_j has bounded calculus for the closed ideal of bounded holomorphic functions on right half–plane. The bounded of this calculus grows at most logarithmically as$(1+\varepsilon) \searrow 0$. As a consequence decay at ∞. Then showed that each sequence of semigroup generator has a so-called (strong) m-bounded calculus for all m $\in \mathbb{N}$, and that this property characterizes the sequence of semigroup generators. Similar results are obtained if the underlying Banach space is a UMD space. Upon restriction to so-called $\gamma_j - \boldsymbol{bounded}$ semigroups, the Hilbert space results actually hold in general Banach spaces.

Keywords: functional calculus, transference, operator semigroup, fourier multiplier, γ-boundedness.

GJSFR-F Classification: MSC 2010: 47A60

FUNCTIONALCALCULUSFORTHESERIESOFSEMIGROUPGENERATORSVIATRANSFERENCE

Strictly as per the compliance and regulations of:

© 2019. Shawgy Hussein, Simon Joseph, Ahmed Sufyan, Murtada Amin, Ranya Tahire & Hala Taha. This is a research/review paper, distributed under the terms of the Creative Commons Attribution-Noncommercial 3.0 Unported License http://creativecommons.org /licenses/by-nc/3.0/), permitting all non commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

eJournal Print Journal epaper open standard 3d virtual journal

Ref

2. C.Batty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.

Functional Calculus for the Series of Semigroup Generators via Transference

Shawgy Hussein α, Simon Joseph σ, Ahmed Sufyan ρ, Murtada Amin Ѡ, Ranya Tahire ¥ & Hala Taha §

Abstract- In this paper, apply an established transference principle to obtain the boundedness of certain functional calculi for the sequence of semigroup generators. It is proved thatif $-\boldsymbol{A_j}$ be the sequence generates $\boldsymbol{C_0}$- semigroups on a Hilbert space, then for each $\boldsymbol{\varepsilon} > -1$ the sequence of operators $\boldsymbol{A_j}$ has bounded calculus for the closed ideal of bounded holomorphic functions on right half–plane. The bounded of this calculus grows at most logarithmically as$(\mathbf{1}+\boldsymbol{\varepsilon}) \searrow \mathbf{0}$. As a consequence decay at ∞. Then showed that each sequence of semigroup generator has a so-called (strong) m-bounded calculus for all m ∈ ℕ, and that this property characterizes the sequence of semigroup generators. Similar results are obtained if the underlying Banach space is a UMD space. Upon restriction to so-called$\boldsymbol{\gamma_j} - \boldsymbol{bounded}$ semigroups, the Hilbert space results actually hold in general Banach spaces.

Keywords: *functional calculus, transference, operator semigroup, fourier multiplier, γ-boundedness.*

I. Introduction

Functional calculus for thesequence of operators A_j on a Banach space X is a "method" of associating a closed sequence of operators $f_j(A_j)$ to all $f_j = f_j(z_j)$ taken from a Set of functions in such a way that formulae valid for the functions turn into valid formulae for the operators upon replacing the independent variables Z_j by A_j. A common way to establish such a calculus is to start with an algebra of "good" functions f_j where definitions of $f_j(A_j)$ as bounded sequence of operators are more or less straightforward, and then extend this "primary" or "elementary calculus" by means of multiplicative in[1,Chapter 1] and [2]. It is then natural to ask which of the so constructed closed sequence of operators $f_j(A_j)$ are actually bounded, a question particularly relevant in applications, e.g., to evolution equations,see, [3,4].

The latter question links functional calculus theory to the theory of vector-valued singular integrals, best seen in the theory of sectorial operators with a bounded H^∞-calculus, see, [5]. It appears there that in order to obtain nontrivial results the

Author α: Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan. e-mail: shawgy2020@gmail.com
Author σ: Upper Nile University, Faculty of Education, Department of Mathematics, South Sudan. e-mail: s.j.u.khafiir@gmail.com
Author ρ: Ministry of Education, Department of Mathematics, Sultanate Oman. e-mail: shibphdoman@gmail.com
Author Ѡ: Ministry of Education and Higher Education, Department of Mathematics, Qatar. e-mail: mlgandi2@gmail.com
Author ¥: Jazan University, Faculty of Science, Department of Mathematics, Kingdom of Saudi Arabia. e-mail: rayya2014@hotmail.com
Author §: Princess Nourah bint Abdulrahman University, College of Science, Department of Mathematics, Kingdom of Saudi Arabia. e-mail: hala_taha2011@hotmail.com

© 2019 Global Journals

underlying Banach space must allow for singular integrals to converge, i.e., be a UMD space. Furthermore, even if the Banach space is a Hilbert space, it turns out that simple resolvent estimates are not enough for the boundedness of an H^{∞} -calculus.

However, some of the central positive results in that theory — show that the presence of a C_0-group of operators does warrant the boundedness of certain H^{∞}-calculi. In [6], the underlying structure of these results was brought to light, namely a transference principle, a factorization of the sequence of operators $f_j(A_j)$ in terms of vector-valued Fourier multiplier operators. Finally, in [7], it was shown that C_0 -semigroups also allow for such transference principles.

Markus Haase and Jan Rozendaal [8] developed this approach further. They apply the general form of the transference principle for semigroups given in [9] to obtain bounded functional calculi for the sequence of generators of C_0 -semigroups. These results, in theorems 3.3, 3.7, and 4.3,are proved for general Banach spaces. However, they make use of the analytic $L^{1+\varepsilon}(\mathbb{R};\ X)$ Fourier multiplier algebra, and hence are useful only if the underlying Banach space has a geometry that allows for nontrivial Fourier multiplier operators. In case X = H is a Hilbert space one, obtains particularly nice results, which want to summarize here.

Theorem 1.1: Let $-A_j$ be the sequence of generators of bounded C_0–semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ on a Hilbert space H with M:= $sup_{t\in\mathbb{R}_+}$. Then the following assertions hold.

(a) For $\omega_j<0$ and $f_j \in H^{\infty}(R_{\omega_j})$ one has $f_j\ (A_j)T^j(1+\varepsilon) \in \mathcal{L}\,(\mathrm{H}\,)$ with

$$\left\|\sum_j f_j\left(A_j\right)T^j\ (1+\varepsilon)\right\| \leq c(1+\varepsilon)\mathrm{M}^2 \sum_{\mathrm{j}} \|f_j\|_{\mathrm{H}^{\infty}\left(\mathrm{R}_{\omega_j}\right)} \tag{1}$$

where c(1+ε) = O(|log(1 + ε) |) as(1 + ε) ↘ 0, and c(1+ε) = O(1) as $(1+\varepsilon) \to \infty$.

(b) For $\omega_j < 0 < \beta + \varepsilon$ and $\lambda_j \in \mathbb{C}$ with Re $\lambda_j < 0$ there is $\varepsilon \geq -1$ such that

$$\left\|\sum_j f_j(A_j)\,(A_j-\lambda_j)^{-(\beta+\varepsilon)}\right\| \leq (1+\varepsilon)M^2 \sum_j \|f_j\|_{H^{\infty}\left(R_{\omega_j}\right)} \tag{2}$$

For all $f_j \in H^{\infty}(R_{\omega_j})$. In particular, dom($A_j^{\beta+\varepsilon}$) ⊆dom($f_j(A_j)$).

(c) A_j has strong m-bounded H^{∞} -calculus of type 0 for each m ∈ ℕ.

When X is a UMD space, one can derive similar results, we extend the Hilbert space results to general Banach spaces by replacing the assumption of boundedness of the semigroup by its γ_j -boundedness, a concept strongly put forward by Kalton and Weis [9]. In particular, Theorem 1.1 holds true for γ_j -bounded semigroups on arbitrary Banach spaces with M being the γ_j -bound of the semigroups.

Stress the fact that in contrast to [1], where sectorial operators and, accordingly, functional calculi on sectors, were considered, deals with general sequence of semigroup generators and with functional calculi on half-planes. The abstract theory of (holomorphic) functional calculi on half-planes can be found in [2 corollaries 6.5and 7.1]

Ref

6. M.Haase: A transference principle for general groups and functional calculus on UMD spaces. Math.Ann.345(2)(2009) 245-265.

© 2019 Global Journals

The starting point of the present work was the article [10] by Hans Zwart. There is shown that one has an estimate (1) with $c(1+\varepsilon) = O((1+\varepsilon)^{-1/2})$ as $(1+\varepsilon) \searrow 0$. (The case $\beta + \varepsilon > 1/2$) in (2) is an immediate consequence, however, that case is essentially trivial)

In [7] and its sequal paper [11], the functional calculus for a semigroup generator is constructed in a rather unconventional way using ideas from systems theory. However, a closer inspection reveals that transference is present there as well, hidden in the very construction of the functional calculus.

Ref

11. F.L.Schwenninger, H.Zwart: Weakly admissible $\mathcal{H}_\infty^-$-calculus on reflexive Banach spaces, Indag. Math. (N.S) 23 (4) (2012) 796-815.

a) Notation and terminology

Write $\mathbb{N} := \{1, 2, \ldots\}$ for the natural numbers and $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals.The letters X and Y are used to denote Banach spaces over the complex number field. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. For a closed sequence of operatorsA_j on X their domainsare denoted by dom(A_j) and their ranges by ran(A_j). The spectrums of A_j are $\sigma(A_j)$ and the resolvent sets $\rho(A_j) := \mathbb{C}$ " $\sigma(A_j)$. For all $z_j \in \rho(A_j)$ the operators$R(Z_j, A_j) := (z_j - A_j)^{-1} \in \mathcal{L}(X)$ is the resolvents of A_j at z_j.

For $\varepsilon > 1$, $L^{1+\varepsilon}$ ($\mathbb{R}$; X) is the Bochner space of equivalence classes of X - valued $(1+\varepsilon)$ –Lebesgue integrable functions on $\mathbb{R}$.The Hölder conjugate of $(1+\varepsilon)$ is $\left(\frac{1+\varepsilon}{\varepsilon}\right)$. The norm on $L^{1+\varepsilon}(\mathbb{R}, X)$is usually denoted by$\|\cdot\|_{1+\varepsilon}$.

For $\omega_j \in \mathbb{R}$ and $z_j \in \mathbb{C}$, let $e_{\omega_j}(z_j) := e^{\omega_j z_j}$. By M($\mathbb{R}$) (resp. M($\mathbb{R}_+$)), denote the space of complex-valued Borel measures on $\mathbb{R}$ (resp. $\mathbb{R}_+$) with the total variation norm, and write$M_{\omega_j}(\mathbb{R}_+)$ for the distributions μ^j on $\mathbb{R}_+$ of the form $\mu^j(\mathrm{ds}) = e^{\omega_j s}\nu^j(\mathrm{ds})$ for some $\nu^j \in$ M($\mathbb{R}_+$). Then $M_{\omega_j}(\mathbb{R}_+)$ is a Banach algebra under convolution with the series of norms

$$\sum_j \|\mu^j\|_{M_{\omega_j}(\mathbb{R}_+)} = \sum_j \left\|e_{-\omega_j}\mu^j\right\|_{M(\mathbb{R}_+)}$$

For $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$, let supp$(\mu^j)$ be the topological support of $e_{-\omega_j}\mu^j$, functions g^j such that $e_{-\omega_j} g^j \in L^1(\mathbb{R}_+)$ are usually identified with its associated measures $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$ given by $\mu^j(\mathrm{ds}) = g^j(\mathrm{s})\,\mathrm{ds}$. Functions and measures defined on $\mathbb{R}_+$ are identified with their extensions to $\mathbb{R}$ by setting them equal to zero outside $\mathbb{R}_+$.

For an open subset $\Omega \neq \emptyset$ of $\mathbb{C}$, let $H^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω, until Banach algebra concerning to the series of norms

$$\sum_j \|f_j\|_\infty = \sum_j \|f_j\|_{H^\infty(\Omega)} = sup_{z_j \in \Omega} \sum_j |f_j(z_j)| \quad (f_j \in H^\infty(\Omega))$$

Consider the case where Ω is equal to a right half-planes

$$R_{\omega_j} = \{z_j \in \mathbb{C} | Re(z_j) > \omega_j\}$$

for some $\omega_j \in \mathbb{R}$ (we write $\mathbb{C}_+$ for R_0).

© 2019 Global Journals

For convenience abbreviate the coordinate functions $Z_j \longmapsto z_j$ simply by the letters z_j. Under this convention, $f_j = f_j\ (z_j)$ for functions f_j defined on some domain $\Omega \subseteq C$.

The Fourier transform of an X-valued tempered distribution Φ on $\mathbb{R}$ is denoted by $\mathcal{F}\Phi$. If $\mu^j \in \mathrm{M}(\mathbb{R})$ then $\mathcal{F}_{\mu^j} \in L^\infty(\mathbb{R})$ are given by

$$\sum_j \mathcal{F}_{\mu^j}(\xi) = \int_{\mathbb{R}} \sum_j e^{-i\xi s}\, \mu^j(ds) \qquad (\xi \in \mathbb{R})$$

For $\omega_j \in \mathbb{R}$ and $\mu^j \in M_{\omega_j}\ (\mathbb{R}_+)$, let $\widehat{\mu^j} \in H^\infty(R_{\omega_j}) \cap \mathrm{C}(\overline{\mathrm{R}_{\omega\ j}})$,

$$\sum_j \widehat{\mu^j}(z_j) = \int_0^\infty \sum_j e^{-z_j s}\, \mu^j(ds) \qquad (z_j \in R_{\omega_j})$$

Be the Laplace–Stieltjes transforms of μ^j.

II. Fourier Multipliers and Functional Calculus

Discuss some of the concepts that will be used in what follows (see,e.g., [8]).

a) Fourier multipliers

Fix a Banach space X and let m $\in L^\infty(\mathbb{R};\ \mathcal{L}(\mathrm{X}))$ and $\varepsilon \geq 0$. Then m is a bounded $L^{1+\varepsilon}(\mathbb{R};\ \mathrm{X})$-Fourier multiplier if there exists $\varepsilon \geq$ -1 such that

$$T_m^j(\varphi_j) = \mathcal{F}^{-1}(m.\mathcal{F}\varphi_j) \in L^{1+\varepsilon}(\mathbb{R};\ \mathrm{X})\ \ and\ \left\|\sum_j T_m^j(\varphi_j)\right\|_{1+\varepsilon} \leq (1+\varepsilon)\sum_j \|\varphi_j\|_{1+\varepsilon}$$

for each X-valued Schwartz functions φ_j. In this case, the mappings T_m^j extends uniquely to bounded sequence of operators on $L^{1+\varepsilon}(\mathbb{R};\ \mathrm{X})$ if $\varepsilon < \infty$ and on $C_0(\mathbb{R};\ \mathrm{X})$ if $\varepsilon = \infty$. Let $\|m\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$ be the norms of the operators T_m^j and let $\mathcal{M}_{1+\varepsilon}(\mathrm{X})$ be the unital Banach algebra of all bounded $L^{1+\varepsilon}(\mathbb{R};\ \mathrm{X})$-Fourier multipliers, endowed with the norm $\|\cdot\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$.

For $\omega_j \in \mathbb{R}$ and $\varepsilon \geq 0$, we let

$$A_j M_{1+\varepsilon}^X\left(R_{\omega_j}\right) = \left\{f_j \in H^\infty\left(R_{\omega_j}\right) \middle| f_j(\omega_j + i\,\cdot) \in \mathcal{M}_{1+\varepsilon}(X)\right\} \tag{3}$$

be the analytic L^(1+ε)(R; X)-Fourier multiplier algebras on R˙(ω˙j), endowed the series of norms

$$\sum_j \|f_j\|_{A_j M_{1+\varepsilon}^X} = \sum_j \|f_j\|_{A_j M_{(1+\varepsilon)}^X(R_{\omega_j})} = \sum_j \|f_j(\omega_j + i\,\cdot)\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

Here $f_j\ (\omega_j + i\,\cdot) \in L^\infty(\mathbb{R})$ denotes the trace of the holomorphic functions f_j on the boundary $\partial R_{\omega_j} = \omega_j + i\mathbb{R}$. By classical Hardy space theories,

Ref

8. M.Haase, J.Rozendaal: Functional calculus for the semigroup generators via transference.

© 2019 Global Journals

$$f_j\left(\omega_j + is\right) = lim_{\acute{\omega}_j \searrow \omega_j} f_j\left(\acute{\omega}_j + is\right) \tag{4}$$

Exists for almost all s ∈ ℝ, with $\sum_j \|f_j(\omega_j + i\cdot)\|_{L^\infty(\mathbb{R})} = \sum_j \|f_j\|_{H^\infty(R_{\omega_j})}$.

Remark 2.1: (Important!). To simplify notation sometimes omit the reference to the Banach space X and write $A_j M_1 (R_{\omega_j})$ instead of $A_j M_1^X(R_{\omega_j})$, whenever it is convenient.

The spaces $A_j M_{1+\varepsilon}^X(R_{\omega_j})$ are until Banach algebra, constructively embedded in $H^\infty\left(R_{\omega_j}\right)$, and $A_j M_1^X (R_{\omega_j}) = A_j M_\infty^X \left(R_{\omega_j}\right)$ are contractively embedded in $A_j M_{1+\varepsilon}^X \left(R_{\omega_j}\right)$ for all $\varepsilon > 0$,

Need two lemmas about the analytic multiplier algebra.

Lemma 2.2: For every Banach space X, all $(0 \le \varepsilon \le \infty)$,

$$\sum_j A_j M_{1+\varepsilon}^X \left(R_{\omega_j}\right) = \left\{ f_j \in H^\infty\left(R_{\omega_j}\right) \mid sup_{\acute{\omega}_j > \omega_j} \sum_j \|f_j(\acute{\omega}_j + i\cdot)\|_{\mathcal{M}_{1+\varepsilon}(X)} < \infty \right\}$$

With

$$\sum_j \|f_j\|_{A_j M_{1+\varepsilon}^X\left(R_{\omega_j}\right)} = sup_{\acute{\omega}_j > \omega_j} \sum_j \|f_j(\acute{\omega}_j + i\cdot)\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

for all $f_j \in A_j M_{1+\varepsilon}^X(R_{\omega_j})$

Proof. Let $\omega_j \in \mathbb{R}$, $f_j \in A_j M_{1+\varepsilon}(R_{\omega_j})$. For all $\acute{\omega}_j > \omega_j$ and s ∈ ℝ,

$$\sum_j f_j\left(\acute{\omega}_j + is\right) = \sum_j \frac{\acute{\omega}_j - \omega_j}{\pi} \int_{\mathbb{R}} \frac{f_j\left(\omega_j - ir\right)}{(s-r)^2 + (\acute{\omega}_j - \omega_j)^2}\, dr$$

The right-hand side is the series of the convolutions of $f_j\left(\omega_j - i\cdot\right)$ and the Poisson kernel

$$P_{\acute{\omega}_j - \omega_j}(r) = \frac{\acute{\omega}_j - \omega_j}{\pi(r^2 + (\acute{\omega}_j - \omega_j)^2)}$$

Since $\sum_j \left\|P_{(\acute{\omega}_j - \omega_j)}\right\|_{L^1(\mathbb{R})} = 1$,

$$\left\|\sum_j f_j\left(\acute{\omega}_j + i\cdot\right)\right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} \le \sum_j \|f_j(\omega_j - i\cdot)\|_{\mathcal{M}_{1+\varepsilon}(X)} = \sum_j \|f_j\|_{A_j M_{(1+\varepsilon)}^X(R_{\omega_j})}$$

The converse follows from (4) ■

For $\mu^j \in$ M(ℝ) and $\varepsilon \ge 0$, let $L_{\mu^j} \in \mathcal{L}(L^{1+\varepsilon}(\mathbb{R}; X))$,

$$L_{\mu^j}(f_j) := \mu^j * f_j, \quad \left(f_j \in L^{1+\varepsilon}(\mathbb{R};\, X) \right), \tag{5}$$

be the convolution sequence of operators associated with μ^j.

© 2019 Global Journals

Lemma 2.3: For each $\omega_j \in \mathbb{R}$ the Laplace transform induces an isometric algebra isomorphism from $M_{\omega_j}(\mathbb{R}_+)$ onto $A_j M_1^{\mathbb{C}}(R_{\omega_j}) = A_j M_1^X\ (R_{\omega_j})$. Moreover,

$$\sum_j \left\|\widehat{\mu^j}\right\|_{A_j M_{1+\varepsilon}^X\ (R_{\omega_j})} = \sum_j \left\|L_{e_{-\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{(1+\varepsilon)}(X))}$$

for all $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$, $\varepsilon \geq 0$

Proof: The mappings $\mu^j \mapsto e_{-\omega_j}\mu^j$ and $f_j \mapsto f_j(\cdot\ + \omega_j)$ are isometric algebra isomorphisms $M_{\omega_j}(\mathbb{R}_+) \to M(\mathbb{R}_+)$ and $A_j M_{1+\varepsilon}(R_{\omega_j}) \to A_j M_{1+\varepsilon}(\mathbb{C}_+)$, respectively. Hence it suffices to let$\omega_j = 0$.The Fourier transform induces an isometric isomorphism from $\mathrm{M}(\mathbb{R})$ onto $\mathcal{M}_1(\mathrm{X})$. If $\mu^j \in \mathrm{M}(\mathbb{R}_+)$ and $f_j = \widehat{\mu^j} \in H^\infty(\mathbb{C}_+)$ then $f_j(i\ \cdot) = \mathcal{F}\mu^j \in \mathcal{M}_1(\mathrm{X})$ with $\sum_j \|f_j(i\ \cdot)\|_{\mathcal{M}_1(X)} = \sum_j \|\mu^j\|_{M(\mathbb{R}_+)}$ Moreover, for $\varepsilon \geq 0$,

$$\sum_j \|f_j(i\ \cdot)\|_{\mathcal{M}_{1+\varepsilon}(X)} = \sum_j sup_{\|g^j\|_{1+\varepsilon} \leq 1} \|\mathcal{F}^{-1}(f_j(i\ \cdot)\mathcal{F}\, g^j)\|_{1+\varepsilon} = sup_{\|g^j\|_{1+\varepsilon} \leq 1} \sum_j \|\mu^j * g^j\|_{1+\varepsilon} = \sum_j \|L_{\mu^j}\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

If $f_j \in A_j M_1(\mathbb{C}_+)$ then $f_j(i\ \cdot) = \mathcal{F}\mu^j$ for some $\mu^j \in \mathrm{M}(\mathbb{R})$. An application of Liouville's theorem shows that $\mathrm{supp}(\mu^j) \subseteq \mathbb{R}_+$, hence $f_j = \widehat{\mu^j}$. ■

b) *Functional Calculus*

Assume that we are familiar with the basic notions and results of the theory of C_0-semigroups as developed, e.g., in [5]

All C_0–semigroups $T^j = (T^j(t))_{t\in\mathbb{R}_+}$ on a Banach space X has the type (M, ω_j) for some $\mathrm{M} \geq 1$ and $\omega_j \in \mathbb{R}$, which means that $\|\sum_j T^j\ (\mathrm{t})\| \leq \mathrm{M}\sum_j e^{\omega_j t}$ for all $\mathrm{t} \geq 0$. The generators of T^j are the uniqueclosed sequence of operators $-A_j$ such that

$$\sum_j (\lambda_j + A_j)^{-1}x = \int_0^\infty \sum_j e^{-\lambda_j t}\, T^j(t)x dt \quad (x \in X)$$

for $\mathrm{Re}(\lambda_j)$ large. The Hille–Phillips (functional) calculus for A_j are defined as follows. Fix $\mathrm{M} \geq 0$ and $(\omega_j)_0 \in \mathbb{R}$ such that T^j has types$\left(\mathrm{M}, -\left(\omega_j\right)_0\right)$. For $\mu^j \in M_{(\omega_j)_0}(\mathbb{R}_+)$ defines $T^j_{\mu^j} \in \mathcal{L}(\mathrm{X})$ by

$$\sum_j T^j_{\mu^j} x = \int_0^\infty \sum_j T^j(t)x\mu^j(dt), \quad (x \in X) \tag{6}$$

For $f_j = \widehat{\mu^j} \in A_j M_j\left(R_{(\omega_j)_0}\right)$ sets $f_j(A_j) := T^j_{\mu^j}$. The mappings $f_j \mapsto f_j(A_j)$ is an algebra homomorphism. In a second stepthe definitions of $f_j(A_j)$ is extended to a larger class of functions via regularization, i.e.,

$$f_j\ (A_j)\ := \mathrm{e}(A_j)^{-1}\, (\mathrm{e}f_j)(A_j)$$

Ref

5. P. Kunstmann, L. Weis: Maximal L_p–regularity for parabolic equations, Fourier multiplier theorems and H^∞ – functional calculus, vol.1855, Springer BBerlin, 2004, pp.65-312.

© 2019 Global Journals

Ref
2. C.Batty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.

If there exists $e \in A_j M_1\left(R_{(\omega_j)_0}\right)$ such that $e(A_j)$ is injective and $ef_j \in A_j M_1\left(R_{(\omega_j)_0}\right)$. Then $f_j(A_j)$ is closed and unbounded operator on X and the definition of $f_j(A_j)$ are independents of the choice of regularize. The following lemma shows in particular that for $\omega_j < (\omega_j)_0$ the sequence of operators $f_j(A_j)$ are defined for all $f_j \in H^\infty(R_{\omega_j})$ by virtue of the regularizes $e(z_j) = (Z_j - \lambda_j)^{-1}$,
where $\mathrm{Re}(\lambda_j) < \omega_j$.

Lemma 2.4: Let $\beta + \varepsilon > \frac{1}{2}$, $\lambda_j \in \mathbb{C}$ *and* ω_j , $(\omega_j)_0 \in \mathbb{R}$, $\varepsilon \geq 0$. Then

$$f_j\,(z_j)(z_j - \lambda_j)^{-(\beta+\varepsilon)} \in A_j M_1(R_{\omega_j})_0 \text{ . for all } f_j \in H^\infty R_{(\omega_j)}$$

Proof: After shifting suppose that $\omega_j = 0$. Sets $h_j(z_j) := f_j\,(z_j)(z_j - \lambda_j)^{-(\beta+\varepsilon)}$ for $z_j \in \mathbb{C}_+$. Then $h_j(i\,\cdot) \in L^2(\mathbb{R})$ with

$$\left\|\sum_j h_j\,(i\,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} \sum_j \frac{|f_j(is)|^2}{|is - \lambda_j|^{2(\beta+\varepsilon)}} ds \leq \int_{\mathbb{R}} \sum_j \frac{\|f_j\|^2_{M^\infty(\mathbb{C}_+)} ds}{|is - \lambda_j|^{2(\beta+\varepsilon)}}$$

Hence $h_j = \widehat{g^j}$ for some $g^j \in L^2(\mathbb{R}_+)$. Then $e_{-(\omega_j)_0} g^j \in L^1(\mathbb{R}_+)$ and $\widehat{e_{-\omega_0} g^j}\,(z_j) = h_j\,(z_j + (\omega_j)_0\,)$ for $z_j \in \mathbb{C}_+$. Lemma 2.3 yields $h_j \in A_j M_1 R_{(\omega_j)_0}$ with

$$\sum_j \|h_j\|_{A_j M_1 R_{(\omega_j)_0}} = \sum_j \|h_j(\cdot + (\omega_j)_0)\|_{A_j M_1(\mathbb{C}_+)} = \sum_j \left\|e_{-(\omega_j)_0} g^j\right\|_{L^1(\mathbb{R}_+)} \blacksquare$$

The Hille–Phillips calculus is an extension of the holomorphic functional calculus for the sequence of operators of half-plane type discussed in [2]. The sequence operators of A_j are of the half-plane types $(\omega_j)_0 \in \mathbb{R}$ if $\sigma(A_j) \subseteq \overline{R_{(\omega_j)_0}}$ with

$$sup_{\lambda_j \in \mathbb{C} \setminus R_{(\omega_j)}} \sum_j \|R(\lambda_j, A_j)\| < \infty,$$

for all $\varepsilon > 0$

One can associate the sequence of operators $f_j(A_j) \in \mathcal{L}(X)$ to certain elementary functions via Cauchy integrals and regularize as above to extend the definitions to all $f_j \in H^\infty\,(R_{\omega j})$. If $-A_j$ generates C_0 -semigroups of types (M, $-(\omega_j)_0$) then A_j are of half-plane types$(\omega_j)_0$, for $\omega_j < (\omega_j)_0$, $\varepsilon > 0$ and$f_j \in H^\infty(R_{\omega_j})$ the definitions of $f_j(A_j)$ via the Hille–Phillips calculus and the half-plane calculus coincide.

Lemma 2.5: (Convergence Lemma). Let A_j be densely defined sequence of operators of half-plane types$(\omega_j)_0 \in \mathbb{R}$ on a Banach space X. Let $\omega_j < (\omega_j)_0$ and $(f_j)_{j \in J} \subseteq H^\infty(R_{\omega_j})$ be satisfying the following conditions:
(1) $\sup\{|(f_j)_j(z_j)|\mid z_j \in R_{\omega_j}, j \in J\} < \infty$;

© 2019 Global Journals

(2) $(f_j)_j(A_j) \in \mathcal{L}(\mathrm{X})$ for all j ∈ J and $sup_{j\in J}\|(f_j)_j(A_j)\| < \infty$;

(3) $f_j(z_j) := Lim_{j\in J} f_j(z_j)$ exists for all $z_j \in R_{\omega_j}$.

Then $f_j \in H^\infty(R_{\omega_j})$, $f_j(A_j) \in \mathcal{L}(\mathrm{X})$, $(f_j)_j(A_j) \to f_j\ (A_j)$ strongly and

$$\left\|\sum_j f_j\ (A_j)\right\| \leq \lim sup_{j\in J} \sum_j \|(f_j)_j(A_j)\|$$

Let A_j be the sequence of operators of half-plane types $(\omega_j)_0$ and $\omega_j < (\omega_j)_0$. For a Banach algebra F of functions continuously embedded in $H^\infty(R_{\omega_j})$, say that A_j has bounded F -calculus if there exists a constant $\varepsilon \geq$-1 such that $f_j(A_j) \in \mathcal{L}(\mathrm{X})$ with

$$\left\|\sum_j f_j\ (A_j)\right\|_{\mathcal{L}(X)} \leq (1+\varepsilon)\sum_j \|f_j\|_F \text{ for all } f_j \in F \quad (7)$$

The sequence of operators $-A_j$ generates a C_0-semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of types (M, ω_j) if and only if - $(A_j + \omega_j)$ generates the semigroups sequence of $(e^{-\omega_j t}T^j(t))_{t\in\mathbb{R}_+}$ of types (M, 0). The functional calculi for A_j and $A_j + \omega_j$ are linked by the simple composition rules "$f_j(A_j + \omega_j) = f_j(\omega_j + z_j)(A_j)$". Henceforth we shall mainly consider bounded semigroups; all results carry over to general semigroups by shifting.

III. Functinal Calculus for Semigroup Generators

Define the function η : (0, ∞) × (0, ∞) × [1, ∞] → $\mathbb{R}_+$ by

$$\eta(\beta+\varepsilon, t, 1+\varepsilon) = inf\left\{\|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} | \psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)} \; on\; [t, \infty)\right\} \quad (8)$$

The set on the right-hand side is not empty: choose for instance $\psi_j := \mathbf{1}_{[0,t]} e_{-(\beta+\varepsilon)}$ and $\varphi_j = \frac{1}{t} e_{-(\beta+\varepsilon)}$. By Lemma A.1,

$$\eta(\beta+\varepsilon, t, 1+\varepsilon) = O(|\log((\beta+\varepsilon)t)|) \text{ as } (\beta+\varepsilon)t \to 0, \text{ for } \varepsilon > 0.$$

For the following result recall the definitions of the operators L_{μ^j} from (5) and $T^j_{\mu^j}$ from (6).

Proposition 3.1: Let $(T^j(t))_{t\in\mathbb{R}_+}$ be C_0-semigroup of type (M, 0) on a Banach space X. Let $\varepsilon \geq 0$, 1+ε , $\omega_j > 0$ and $\mu^j \in M_{-\omega_j}(\mathbb{R}_+)$ with supp(μ^j) ⊆ [1+ε , ∞). Then

$$\left\|\sum_j T^j_{\mu^j}\right\|_{\mathcal{L}(X)} \leq M^2\eta\sum_j(\omega_j, 1+\varepsilon, 1+\varepsilon)\left\|L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))} \quad (9)$$

Proof: Factorizes $T^j_{\mu^j}$ as $T^j_{\mu^j} = \mathrm{P} \circ L_{e_{\omega_j}\mu^j} \circ \iota$, where

a) $\iota : \mathrm{X} \to L^{1+\varepsilon}(\mathbb{R}; \mathrm{X})$ is given by

Notes

Global Journal of Science Frontier Research (F) Volume XIX Issue V Version I Year 2019

© 2019 Global Journals

$$\iota(x)(s) = \begin{cases} \psi_j(-s)T^j(-s)x & if \ s \le 0, \\ 0 & if \ s > 0, \end{cases} \qquad (x \in X)$$

b) $P : L^{1+\varepsilon}(\mathbb{R}; \mathrm{X}) \rightarrow \mathrm{X}$ is given by

$$\sum_j P(f_j) = \int \sum_j \varphi_j(t) T^j(t) f_j(t)\, dt \quad (f_j \in L^{1+\varepsilon}(\mathbb{R}, X))$$

c) $\psi_j \in L^{1+\varepsilon}(\mathbb{R}_+)$ and $\varphi_j \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}_+)$ are such that $\psi_j * \varphi_j \equiv e_{-\omega_j}$ on $[1+\varepsilon, \infty)$. This is deduced that $\mu^j = (\psi_j * \varphi_j)e_{\omega_j}\, \mu^j$. Hölder's inequality then implies

$$\left\| \sum_j T^j_{\mu^j} \right\| \le M^2 \sum_j \|\psi_j\|_{1+\varepsilon} \left\| L_{e_{\omega_j}\mu^j} \right\|_{\mathcal{L}(L^{1+\varepsilon}(X))} \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

and taking the infimum over all such ψ_j and φ_j yields (9). ■

Define, for a Banach space X, $\omega_j \in \mathbb{R}$, and $\varepsilon >$ -1, the spaces

$$A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}) = \left\{ f_j \in A_j M^X_{(1+\varepsilon)}\left(R_{\omega_j}\right) \middle| f_j(z_j) = O\left(e^{-(1+\varepsilon)Re(z_j)}\right) \text{ as } |z_j| \rightarrow \infty \right\}$$

end owed with the norms of $A_j M^X_{1+\varepsilon}(R_{\omega_j})$.

Lemma 3.2: For every Banach space X, $\omega_j \in \mathbb{R}$, $1 \le \varepsilon \le \infty$, and , ε ¿ -1

$$A_j M^X_{(1+\varepsilon),(1+\varepsilon)}\left(R_{\omega_j}\right) = A_j M^X_{(1+\varepsilon)}\left(R_{\omega_j}\right) \cap e_{-(1+\varepsilon)} H^\infty\left(R_{\omega_j}\right) = e_{-(1+\varepsilon)} A_j M^X_{(1+\varepsilon)}\left(R_{\omega_j}\right) \quad (10)$$

In particular, $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ are closed ideal in $A_j M^X_{(1+\varepsilon)}(R_{\omega_j})$.

Proof: The first equality in (10) is clear, and so are the inclusions $e_{-(1+\varepsilon)} A_j M_{(1+\varepsilon)}(R_{\omega_j}) \subseteq A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$. Conversely, if $f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j}) \cap e_{-(1+\varepsilon)} H^\infty(R_{\omega_j})$ then $e_{(1+\varepsilon)} f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j})$, since

$$\sum_j \left\| e^{(1+\varepsilon)(\omega_j + i\cdot)} f_j(\omega_j + i\cdot) \right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} = \sum_j e^{(1+\varepsilon)\omega_j} \|f_j(\omega_j + \mathrm{i}\cdot)\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

Suppose that $((f_j)_n)_{n\in\mathbb{N}} \subseteq A_j M_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ converges to $f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j})$. The Maximum Principle implies

$$\sum_j \|e_{(1+\varepsilon)}(f_j)_n\|_{H^\infty(R_{\omega_j})} = \sum_j e^{(1+\varepsilon)\omega_j} \|(f_j)_n\|_{H^\infty(R_{\omega_j})},$$

hence $(e_{(1+\varepsilon)}(f_j)_n)_{n\in\mathbb{N}}$ is Cauchy in $H^\infty(R_{\omega_j})$. Since it converges pointwise to $e_{(1+\varepsilon)} f_j$, (10) implies $f_j \in A_j M_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$. ■

To prove the main result [8] of this section. Note that the union of the ideals $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ for $\varepsilon >$ -1 is densest in $A_j M^X_{(1+\varepsilon)}(R_{\omega_j})$ with respect to pointwise and bounded convergence of sequences. If there was a single constant independent of $\varepsilon > -1$

Ref

8. M.Haase, J.Rozendaal: Functional calculus for the of semigroup generators via transference.

© 2019 Global Journals

bounding the $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ - calculus for all, the Convergence Lemma would imply that A_j has bounded $A_j M^X_{(1+\varepsilon)}\ (R_{\omega_j})$ -calculus, but this is known to be false in general [1, Corollary 9.1.8].

Theorem 3.3: For each $0 < \varepsilon < \infty$, there exists a constant $c_{1+\varepsilon} \geq 0$ such that the following holds. Let $-A_j$ the sequence of generates C_0 -semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of type (M, 0) on a Banach space X and let$(1+\ \varepsilon)$, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}(X)$ and

$$\left\|\sum_j f_j(A_j)\right\| \leq \begin{cases} c_{(1+\varepsilon)}M^2\sum_j \left|\log(\omega_j(1+\ \varepsilon))\right| \|f_j\|_{A_j M^X_{(1+\varepsilon)}} & \text{if } \omega_j\ (1+\ \varepsilon)\ \leq\ \min\left(\frac{1}{1+\ \varepsilon},\ \frac{\varepsilon}{1+\varepsilon}\right) \\ 2M^2\sum_j e^{-\omega_j(1+\varepsilon)}\|f_j\|_{A_j M^X_{(1+\varepsilon)}} & ,if\ \omega_j(1+\ \varepsilon) > min\left(\frac{1}{1+\ \varepsilon},\ \frac{\varepsilon}{1+\varepsilon}\right) \end{cases}$$

for all $f_j \in A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{-\omega_j})$. In particular, A_j has bounded $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{-\omega_j})$ - calculus.

Proof: First consider $f_j \in A_j M_{1,(1+\varepsilon)}(R_{-\omega_j})$. Let $\delta_{(1+\varepsilon)} \in M_{-\omega_j}(\mathbb{R}_+)$ be the unit point mass at $\varepsilon > -1$. By Lemmas 3.2 and 2.3 there exists $\mu^j \in M_{-\omega_j}\ (\mathbb{R}_+)$ such that $f_j\ =\ e_{-(1+\varepsilon)}\widehat{\mu^j}\ = \widehat{\delta_{(1+\varepsilon)} * \mu^j}$. Since $\delta_{(1+\varepsilon)} * \mu^j \in M_{-\omega_j}\ (\mathbb{R}_+)$ with supp $(\delta_{(1+\varepsilon)} * \mu^j) \subseteq$ [1+ε, ∞), Proposition 3.1 and Lemma 2.3 yield

$$\left\|\sum_j f_j\ (A_j)\right\| \leq M^2\eta\sum_j\Big(\omega_j, (1+\ \varepsilon), (1+\ \varepsilon)\Big)\|f_j\|_{A_j M^X_{1+\varepsilon}} \tag{11}$$

Suppose $f_j \in A_j M_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ are arbitrary. For ε ¿ 0, k ∈ ℕ and $z_j \in R_{-\omega_j}$

Set $s\ g^j_{k,}(z_j) := \frac{k}{z_j - \omega_j + k}$ and $(f_j)_{k,\epsilon}(z_j) = f_j(z_j + \epsilon)g^j_k(z_j + \epsilon)$. Lemma 2.4 yields $(f_j)_{k,\epsilon} \in A_j M_{1,(1+\varepsilon)}(R_{-\omega_j})$, hence, by what have shown,

$$\left\|\sum_j (f_j)_{k,\epsilon}\ (A_j)\right\| \leq M^2\eta\sum_j(\omega_j, 1+\varepsilon, 1+\varepsilon)\ \|(f_j)_{k,\epsilon}\|_{A_j M^X_{1+\varepsilon}}$$

The inclusions $A_j M_1(R_{-\omega_j}) \subseteq A_j M_{1+\varepsilon}(R_{-\omega_j})$ are contractive, so Lemma 2.3 implies that $g^j_k \in A_j M_{1+\varepsilon}\ (R_{-\omega_j})$ with

$$\left\|\sum_j g^j_k\right\|_{A_j M^X_{(1+\varepsilon)}} \leq \sum_j \|g^j_k\|_{A_j M_1} = k\|e_{-k}\|_{L^1(\mathbb{R}_+)} = 1$$

Combining this with Lemma 2.2 yields

$$\left\|\sum_j (f_j)_{k,\epsilon}\right\|_{A_j M^X_{(1+\varepsilon)}} \leq \sum_j \|f_j(\cdot+\epsilon)\|_{A_j M^X_{(1+\varepsilon)}}\|g^j_k(\cdot+\epsilon)\|_{A_j M^X_{(1+\varepsilon)}} \leq \sum_j \|f_j\|_{A_j M^X_{(1+\varepsilon)}}$$

Notes

Global Journal of Science Frontier Research (F) Volume XIX Issue V Version I Year 2019

© 2019 Global Journals

In particular, $sup_{k,\epsilon}\left\|\sum_j (f_j)_{k,\epsilon}\right\|_{\infty} < \infty$ and $sup_{k,\in}\left\|\sum_j (f_j)_{k,\epsilon}(A_j)\right\| < \infty$. The Convergence Lemma 2.5 implies that $f_j(A_j) \in \mathcal{L}(\mathrm{X})$ satisfies (11). Lemma A.1 concludes the proof. ∎

Remark 3.4: Because $A_j \mathrm{M}_1(R_{-\omega_j}) = A_j M_\infty(R_{-\omega_j})$ are contructively embedded in $A_j M_{(1+\varepsilon)}(R_{-\omega_j})$ Theorem 3.3 also holds for $\varepsilon \geq 0$. However, A_j trivially has abounded $A_j M_1$-calculus bylimma 2.3 and the Hille-Phillips calculus.

Notes

Note that the exponential decays of $\sum_j |f_j(z_j)|$ are only required as the real parts of z_j tends to infinity. If $\sum_j |f_j(z_j)|$ decays exponentially as $|z_j| \to \infty$ the result is not interesting by lemma 2.4.

Equivalently formulate Theorem 3.3 as a statement about composition with sequence semigroupoperators.

Corollary 3.5: Under the assumptions of Theorem 3.3, $f_j\ (A_j)T^j(1+\varepsilon) \in \mathcal{L}(\mathrm{X})$ and

$$\left\|\sum_j f_j\ (A_j)T^j\ (1+\varepsilon)\right\|$$

$$\leq \begin{cases} c_{1+\varepsilon}M^2 \sum_j |\log(\omega_j(1+\varepsilon))| e^{\omega_j(1+\varepsilon)} \|f_j\|_{A_j M^X_{1+\varepsilon}}, & if\, \omega_j(1+\varepsilon) \leq \min\left(\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right) \\ 2M^2 \sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}}, & if\ \ \omega_j(1+\varepsilon) > \min\left(\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right) \end{cases}$$

For all $f_j \in A_j M^X_{1+\varepsilon}(R_{-\omega_j})$.

Proof. Note that $\sum_j f_j(A_j)T^j(1+\varepsilon) = \sum_j (e_{-(1+\varepsilon)}f_j)(A_j)$ and $\sum_j \|e_{-(1+\varepsilon)}f_j\|_{A_j M^X_{\varepsilon+1}}$

$$= \sum_j e^{\omega_j(1+\varepsilon)} \|f_j\|_{A_j M^X_{1+\varepsilon}} \ ∎$$

a) Additional results

As the first corollary of Theorem 3.3 we obtain a sufficient condition for a semigroupgenerator to have a bounded $A_j M_{1+\varepsilon}$- calculus (see,e.g.,[8]).

Corollary 3.6: Let $-A_j$ be the sequence of generates bounded C_0 -semigroups $(T^j(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(\mathrm{X})$ with

$$\bigcup_{\varepsilon > -1} \sum_j ran(T^j(1+\varepsilon)) = X$$

Then A_j has bounded $A_j M^X_{1+\varepsilon}(R_{\omega_j})$-calculus for all ω_j ¡ 0, $\varepsilon \geq 0$.

Proof: Using Corollary 3.5 note that $f_j(A_j)T^j(1+\varepsilon) \in \mathcal{L}(\mathrm{X})$ implies ran $(T^j(1+\varepsilon)) \subseteq$ dom$(f_j\ (A_j))$. An application of the Closed Graph Theorem (using the Convergence Lemma) yields (7). ∎

Global Journal of Science Frontier Research (F) Volume XIX Issue V Version I 67 Year 2019

© 2019 Global Journals

Theorem 3.7: Let $0 < \varepsilon < \infty$, , $\omega_j > 0$ and $\beta + \varepsilon$, $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}(\lambda_j) < 0 < \mathrm{Re}(\beta + \varepsilon)$. There exists a constant $C = C(1{+}\varepsilon,\ \beta + \varepsilon,\ \lambda_j\ ,\ \omega_j) \geq 0$ such that the following holds. Let $-A_j$ be the sequence of generates C_0- semigroups$(T^j(t))_{t\in\mathbb{R}_+}$of type (M,0) on a Banach space X. Then dom $\left((A_j - \lambda_j)^{(\beta+\varepsilon)}\right) \subseteq$dom $(f_j(A_j))$ and

$$\left\|\sum_j f_j(A_j)\,(A_j - \lambda_j)^{-(\beta+\varepsilon)}\right\| \leq (1+\varepsilon)M^2 \sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}}$$

for all $f_j \in A_j M^X_{1+\varepsilon}(R_{-\omega_j})$.

Proof: First note that $-(A_j - \lambda_j)$ generates the exponentially stable semigroups $(e^{\lambda_j t}(T^j(t))_{t\in\mathbb{R}_+}$. Hence to write

$$\sum_j (A_j - \lambda_j)^{-(\beta+\varepsilon)} x = \frac{1}{\Gamma(\beta+\varepsilon)} \int_0^\infty t^{(\beta+\varepsilon)-1} \sum_j e^{\lambda_j t} T^j(t)\, x dt \qquad (x \in X)$$

Fix $f_j \in A_j M_{1+\varepsilon}\ (R_{-\omega_j})$ and set a:= $\frac{1}{\omega_j}$ min $\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\}$. By Corollary 3.5,

$$\int_0^\infty t^{Re(\beta+\varepsilon)-1}\, e^{Re(\lambda_j)t} \left\|\sum_j f_j\ (A_j) T^j(t)(x)\right\| dt \leq (1+\varepsilon)M^2 \sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}} \|x\| < \infty$$

for all x $\in$ X, where

$$C = c_{1+\varepsilon} \int_0^a t^{Re(\beta+\varepsilon)-1} \sum_j \left|\log(\omega_j t)\right| e^{(Re(\lambda_j)+\omega_j)t} dt + 2\int_a^\infty t^{Re(\beta+\varepsilon)-1} \sum_j e^{\left(Re(\lambda_j)\right)t} dt$$

are independents of f_j, M, and x. Since $f_j(A_j)$ are closed operators, this implies that$(A_j - \lambda_j)^{-(\beta+\varepsilon)}$maps into dom $f_j(A_j)$ with

$$\sum_j f_j(A_j)\,(A_j - \lambda_j)^{-(\beta+\varepsilon)} = \frac{1}{\Gamma(\beta+\varepsilon)} \int_0^\infty t^{(\beta+\varepsilon)-1} \sum_j e^{\lambda_j t} f_j(A_j) T^j(t) dt$$

as a strong integral. ■

Remark 3.8: Theorem 3.7 shows that for all analytic multiplier functions f_j the domainsdom $(f_j(A_j))$ are relatively large, it contains the real interpolation spaces $(\mathrm{X}, \mathrm{dom}(A_j))_{(\theta,1+\varepsilon)}$ and the complex interpolation spaces $[\mathrm{X}, \mathrm{dom}(A_j)]_\theta$ for all $\theta \in (0,1)$ and $\varepsilon \geq 0$.

Remark 3.9: Describe the ranges of $f_j(A_j)(A_j - \lambda_j)^{-(\beta+\varepsilon)}$ in Theorem 3.7. More explicitly. In fact

$$\mathrm{ran}\left(f_j(A_j)(A_j - \lambda_j)^{-(\beta+\varepsilon)}\right) \subsetneq dom(A_j - \lambda_j)^\beta$$

for all Re$(\beta) < Re(\beta+\varepsilon)$. Indeed, this follows if show that

Notes

© 2019 Global Journals

Notes

$\operatorname{ran}\left((A_j-\lambda_j)^{-(\beta+\varepsilon)}\right) \subsetneq dom\left((A_j-\lambda_j)^{\beta} f_j(A_j)\right)$implies

$$dom\left((A_j-\lambda_j)^{\beta} f_j(A_j)\right) = dom(f_j(A_j)) \cap dom\left([(z_j-\lambda_j)^{\beta} f_j(z_j)](A_j)\right)$$

The inclusion ran $\left((A_j-\lambda_j)^{-(\beta+\varepsilon)}\right) \subsetneq dom(f_j(A_j))$ follows from Theorem 3.7. Since

$$\left[(z_j-\lambda_j)^{\beta} f_j(z_j)\right](A_j)(A_j-\lambda_j)^{-(\beta+\varepsilon)} = [(z_j-\lambda_j)^{-\varepsilon} f_j(z_j)](A_j) = f_j(A_j)(A_j-\lambda_j)^{-\varepsilon}$$

The same holds for the inclusion $\operatorname{ran}\left((A_j-\lambda_j)^{-(\beta+\varepsilon)}\right) \subseteq dom\left([(z_j-\lambda_j)^{\beta} f_j(z_j)](A_j)\right)$

b) Semigroups on Hilbert spaces

If X = H is a Hilbert space, Plancherel's Theorem implies $A_j M_2^H = H^\infty$ with equality of norms. Hence the theory above specializes to the following result, implying (a) and (b) of Theorem (1.1),

Corollary 3.10: Let $-A_j$ be the sequence ofgenerates bounded C_0-semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of type (M, 0) on aHilbert space H. Then the following assertions hold.

(a) There exists a universal constant c ≥ 0 such that the following holds. Let $1+\varepsilon$, $\omega_j > 0$. Then$f_j(A_j) \in \mathcal{L}(\mathrm{H})$ and

$$\left\|\sum_j f_j(A_j)\right\| \le \begin{cases} cM^2 \sum_j \left|\log(\omega_j(1+\varepsilon))\right| \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) \le \frac{1}{2} \\ 2M^2 \sum_j e^{-\omega_j(1+\varepsilon)} \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) > \frac{1}{2} \end{cases}$$

for all $f_j \in e_{-(1+\varepsilon)} H^\infty\ (R_{-\omega_j})$. Moreover, $f_j(A_j)T^j\ (1+\varepsilon) \in \mathcal{L}(\mathrm{H}\ )$ with

$$\left\|\sum_j f_j(A_j)\, T^j\ (1+\varepsilon)\right\| \le \begin{cases} cM^2 \sum_j \left|\log(\omega_j(1+\varepsilon))\right| e^{\omega_j(1+\varepsilon)} \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) \le \frac{1}{2} \\ 2M^2 \sum_j \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) > \frac{1}{2} \end{cases}$$

for all $f_j \in H^\infty(R_{-\omega_j})$.

(b) If

$$\bigcup_{\varepsilon>-1} \sum_j ran\left(T^j(1+\varepsilon)\right) = H$$

then A_j has bounded $H^\infty(R_{\omega_j})$-calculus for all $\omega_j < 0$.

(c) For $\omega_j < 0$ and$\beta+\varepsilon$, $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}(\lambda_j) < 0 < \mathrm{Re}(\beta+\varepsilon)$ there is $\mathrm{C} = \mathrm{C}(\beta+\varepsilon, \lambda_j, \omega_j)$ such that

$$\left\|\sum_j f_j(A_j)(A_j - \lambda_j)^{-(\beta+\varepsilon)}\right\| \le CM^2 \sum_j \|f_j\|_\infty$$

© 2019 Global Journals

for all $f_j \in H^{\infty}(R_{\omega_j})$. In particular, $\mathrm{dom}(A_j^{\beta+\varepsilon}) \subseteq \mathrm{dom}(f_j(A_j))$.

Note: We can deduce that:

$$C\sum_j \|f_j\|_\infty \leq \frac{(1+\varepsilon)}{C}\sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}} \ ,$$

From Theorem 3.7 and Corollary 3.10 Part (c).

Part (c) shows that, even though the sequence of semigroup generators on Hilbert spaces do not have abounded H^∞-calculus in general, each functions f_j that decays with polynomial rate $\varepsilon > 0$ at infinity yields bounded sequence of operators $f_j(A_j)$. For $\beta+\varepsilon > \frac{1}{2}$ this is already covered by Lemma 2.4, but for $\beta+\varepsilon \in (0,\frac{1}{2}]$ it appears to be new.

Remark 3.11: Part (c) of Corollary 3.10 yields a statement about stability of numerical methods. Let $-A_j$ be the sequence generates an exponentially stable semigroups $(T^j(t))_{t\geq 0}$ on a Hilbert space,

Let r $\in H^\infty(\mathbb{C}_+)$ be such that $\|r\|_{H^\infty(\mathbb{C}_+)} \leq 1$, and let $\beta+\varepsilon,\ h_j > 0$. Then

$$sup\left\{\left\|r(h_j A_j)^n x\right\| \middle| n \in \mathbb{N}, x \in dom(A_j^{\beta+\varepsilon})\right\} < \infty \tag{12}$$

Follows from (c) in Corollary 3.10 after shifting the generator. Elements of the form $r(h_j A_j)^n\, x$ are often used in numerical methods to approximate the solution of the abstract Cauchy problem associated to $-A_j$ with initial value x, and (12) shows that such approximations are stable whenever x $\in \mathrm{dom}(A_j^{\beta+\varepsilon})$.

IV. M-Bounded Functional Calculus

Describe another transference principle for semigroups, one that provides estimates for the norms of the sequence ofoperators of the form $f_j^{(m)}(A_j)$ for f_j analytic multiplier functions and $f_j^{(m)}$its m-th derivatives, m $\in \mathbb{N}$. Moreover, recall our notational simplifications $A_j M_{1+\varepsilon}\ (R_{\omega_j}) := A_j M^X_{1+\varepsilon}\ (R_{\omega_j})$ (Remark 2.1).

Let $\omega_j < (\omega_j)_0$ be real numbers. The sequence operators of A_j of half-plane types $(\omega_j)_0$ a Banach space X, has an m-bounded $A_j M^X_{1+\varepsilon}\ (R_{\omega_j})$-calculus if there exists $\varepsilon \geq$ -1, such that $f_j^{(m)}(A_j) \in \mathcal{L}$(X) with

$$\left\|\sum_j f_j^{(m)}(A_j)\right\| \leq (1+\varepsilon)\sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}} \qquad for\ all\ f_j \ \in A_j M^X_{1+\varepsilon}(R_{\omega_j})$$

This is well defined since the Cauchy integral formula implies that $f_j^{(m)}$is bounded on every half-planes $R_{\acute{\omega}_j}$ with $\acute{\omega}_j > \omega_j$.

© 2019 Global Journals

Say that A_j has a strongm-bounded $A_j M_{1+\varepsilon}^X$ -calculus of types $(\omega_j)_0$ if A_j has an m-bounded $A_j M_{1+\varepsilon}^X\ (R_{\omega_j})$ -calculus for every $\omega_j < (\omega_j)_0$ such that for some $\varepsilon \geq 0$ one has

$$\left\|\sum_j f_j^{(m)}(A_j)\right\| \leq (1+\varepsilon)\sum_j \frac{1}{\left((\omega_j)_0-\omega_j\right)^m}\|f_j\|_{A_j M_{1+\varepsilon}^X(\mathrm{R}_{\omega_j})} \tag{13}$$

Notes

for all $f_j \in A_j M_{1+\varepsilon}^X\left(\mathrm{R}_{\omega_j}\right)$and ω_j¡$(\omega_j)_0$.

Lemma 4.1: Let A_j be the sequence of operators of half-plane types $(\omega_j)_0 \in \mathbb{R}$ on a Banach space X, and let $0 \leq \varepsilon \leq \infty$, and m$\in \mathbb{N}$. If A_j has a strong m-bounded $A_j M_{1+\varepsilon}^X$ -calculus of types$(\omega_j)_0$, then A_j has a strong n-bounded $A_j M_{1+\varepsilon}^X$ -calculus of types$(\omega_j)_0$ for all n, $\varepsilon > 0$,

Proof: Let $\omega_j < \beta+\varepsilon < (\omega_j)_0$,$f_j \in A_j M_{1+\varepsilon}(R_{\omega_j})$ and $n \in \mathbb{N}$. Then

$$\sum_j f_j^{(n)}(\beta+\mathrm{i}s) = \frac{(n)!}{2\pi!}\int_{\mathbb{R}}\sum_j \frac{f_j\left((\beta+\varepsilon)+ir\right)}{\left((\beta+\varepsilon)+ir\right)-(\beta+is)^{n+1}}dr$$

$$= \frac{(n)!}{2\pi!}\sum_j \left(f_j((\beta+\varepsilon)+i\,\cdot) * ((\varepsilon-i\,\cdot)^{-n-1})\right)(s)$$

For some s $\in \mathbb{R}$, by the Cauchy Integral formula. Hence, using lemma 2.2,

$$\left\|\sum_j f_j^{(n)}(\beta+i\,\cdot)\right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} \leq \frac{(n)!}{2\pi!}\|(\varepsilon-i\,\cdot)^{-n-1}\|_{L^1(\mathbb{R})}\sum_j \left\|f_j\left((\beta+\varepsilon)+i\,\cdot\right)\right\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

$$\leq \frac{C}{(-\varepsilon)^n}\sum_j \|f_j\|_{A_j M_{1+\varepsilon}(R_{\omega_j})}$$

for some C $=$ C$(n) \geq 0$ independents of f_j , β, $\beta+\varepsilon$ and ω_j. Letting $\beta+\varepsilon$ tend to ω_j yields

$$\left\|\sum_j f_j^{(n)}\right\|_{A_j M_{(1+\varepsilon)}(R_\beta)} = \left\|\sum_j f_j^{(n)}(\beta+i\,\cdot)\right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} \leq C\sum_j \frac{1}{\left(\beta-\omega_j\right)^n}\|f_j\|_{A_j M_{(1+\varepsilon)}\left(R_{\omega_j}\right)} \tag{14}$$

Let $\varepsilon \geq 0$. Applying (14) with $n-m$ in place of n shows that $f_j^{(n-m)} \in A_j M_{1+\varepsilon}\,(R_\beta)$ with

$$\left\|\sum_j f_j^{(n)}(A_j)\right\| \leq C'\sum_j \frac{1}{((\omega_j)_0-\beta)^m}\left\|f_j^{(n-m)}\right\|_{A_j M_{1+\varepsilon}(R_\beta)}$$

$$\leq CC'\sum_j \frac{1}{((\omega_j)_0-\beta)^m(\beta-\omega_j)^{n-m}}\|f_j\|_{A_j M_{(1+\varepsilon)}(R_{\omega_j})}$$

Finally, letting $\beta+\varepsilon = \frac{1}{2}((\omega_j + (\omega_j)_0)$,

© 2019 Global Journals

$$\left\|\sum_j f_j^{(n)}(A_j)\right\| \le C'' \sum_j \frac{1}{((\omega_j)_0 - \omega_j)^{(n)}} \|f_j\|_{A_j M_{(1+\varepsilon)}(R_{\omega_j})}$$

for some $C'' \ge 0$ independents of f_j and ω_j.■

For the transference principle in Proposition 3.1 it is essential that the support of $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$ are contained in some interval $[1+\varepsilon, \infty)$ with $\varepsilon > -1$. One cannot expect to find such a transference principle for arbitraries μ^j, as this would allow one to prove that the sequence of semigroup generators has a bounded analytic multiplier calculus. However, if we let $t\mu^j$ be given by $(t\mu^j)(dt) := t\,\mu^j(dt)$ then we can deduce the following transference principle. Use the conventions $1/\infty := 0$, $\infty^0 := 1$.

Proposition 4.2: Let $-A_j$ be the sequence of generators of a C_0 –semigroups $(T^j(t))_{t\in\mathbb{R}+}$ of type (M, 0) On a Banach space X. Let $0 \le \varepsilon \le \infty$, ω_j ¡ 0 and $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$. Then

$$\left\|\sum_j T^j_{t\mu^j}\right\| \le M^2 \sum_j \frac{1}{|\omega_j|}(1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)}\left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} \left\|L_{e_{-\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

Proof: Factorizes $T^j_{t\mu^j}$ as $T^j_{t\mu^j} = P \circ L_{e_{-\omega_j}\mu^j} \circ \iota$, where ι and P are as in the proof of Proposition 3.1 with $\psi_j, \varphi_j := \mathbf{1}_{\mathbb{R}_+} e_{\omega_j}$. Since $(\psi_j * \varphi_j)e_{-\omega_j}\mu^j = t\,\mu^j$. Then

$$\left\|\sum_j T^j_{t\mu^j}\right\| \le M^2 \sum_j \|e_{\omega_j}\|_{\frac{1+\varepsilon}{\varepsilon}} \left\|L_{e_{-\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))} \|e_{\omega_j}\|_{1+\varepsilon}$$

$$= M^2 \sum_j \frac{1}{|\omega_j|}(1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)}\left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} \left\|L_{e_{-\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

by Holder's inequality. ■

To prove our main result m - bounded functional calculus, a generalization of theorem 7.1 in [2] to arbitrary Banach spaces.

Theorem 4.3: Let A_j be densely defined sequence of operators of half-plane type 0 on a Banach space X.Then the following assertions are equivalent:

(i) $-A_j$ is the sequence of generators of bounded C_0 -semigroup on X.

(ii) A_j has a strong m-bounded $A_j M^X_{1+\varepsilon}$ -calculus of type 0 for some/all $\varepsilon \ge 0$ and some/allm $\in \mathbb{N}$.

If $-A_j$ be the sequence of generates bounded C_0 -semigroup then A_j has an m-bounded $A_j M^X_{1+\varepsilon}(R_{\omega_j})$-calculus for all $\omega_j < 0$, $\varepsilon \ge 0$ and m$\in \mathbb{N}$.

Proof. (i) ⇒ (ii) By Lemma 4.1 it suffices to let m = 1. Proceed along the same lines as the proof of Theorem 3.3. Let $(T^j(t))_{t\epsilon\mathbb{R}_+} \subseteq \mathcal{L}(X)$ be the sequence semigroups generated by $-A_j$ and fix $\omega_j < 0, \varepsilon \ge 0$ and $f_j \in A_j M_{1+\varepsilon}(R_{\omega_j})$. Define the functions $(f_j)_{k,\epsilon} := f_j(\cdot + \epsilon) g^j_k(\cdot + \epsilon)$ for k $\in \mathbb{N}$ and $\varepsilon > 0$, where $g^j_k(z_j) := \frac{k}{z_j - \omega_j + k}$ for $z_j \in R_{\omega_j}$. Then

Ref

2. C.Baty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.

© 2019 Global Journals

$(f_j)_{k,\epsilon} \in A_j M_1(R_{\omega_j})$ by Lemma 2.4, and Lemma 2.3 yields $(\mu^j)_{k,\epsilon} \in M_{\omega_j}(\mathbb{R}_+)$ with $(f_j)_{k,\epsilon} = \widehat{\mu^j_{k,\epsilon}}$. Then

$$\sum_j (f_j')_{k,\epsilon}(z_j) = lim_{h_j \to 0} \sum_j \frac{(f_j)_{k,\epsilon}(z_j + h_j) - (f_j)_{k,\epsilon}(z_j)}{h_j}$$

$$= lim_{h_j \to 0} \int_0^\infty \sum_j \frac{e^{-(z_j+h_j)t} - e^{-z_j t}}{h_j} (\mu^j)_{k,\epsilon}(dt) = -\int_0^\infty \sum_j t e^{-z_j t} \mu^j_{k,\epsilon}(dt)$$

$$= -\sum_j \widehat{t\mu^j_{k,\epsilon}}(z_j)$$

for $z_j \in R_{\omega_j}$, by the Dominated Convergence Theorem. Hence $(f_j')_{k,\epsilon}(A_j) = -T^j_{t\mu^j_{k,\epsilon}}$, and Proposition 4.2 and Lemma 2.3 imply

$$\left\| \sum_j (f_j')_{k,\epsilon}(A_j) \right\| \leq (1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)} \left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} M^2 \sum_j \frac{\|(f_j)_{k,\epsilon}\|_{A_j M^X_{1+\varepsilon}}}{|\omega_j|}$$

Furthermore, $sup_{k,\epsilon} \|\sum_j (f_j)_{k,\epsilon}\|_{A_j M^X_{1+\varepsilon}}$. the $(f_j)_{k,\epsilon}$ are uniformly bounded. By the Cauchy Cauchy integral formula, so are the derivatives $(f_j')_{k,\epsilon}$ on every smaller half-plane. Since $(f_j')_{k,\epsilon}(z_j) \to (f_j')(z_j)$ for all $z_j \in R_{\omega_j}$ as k $\to \infty$, $\epsilon \to 0$, the Convergence Lemma yields $f_j'(A_j) \in \mathcal{L}(X)$ with

$$\left\| \sum_j f_j'(A_j) \right\| \leq (1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)} \left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} M^2 \sum_j \frac{\|f_j\|_{A_j M^X_{1+\varepsilon}}}{|\omega_j|}$$

which is (4.1) for m = 1.

For (ii) $\Rightarrow$ (i) assume that A_j has a strong m-bounded $A_j M_{1+\varepsilon}$-calculus of type 0 for some $\varepsilon \geq 0$ and some m $\in \mathbb{N}$. Then

$$e_{-t} \in A_j M_1(R_{\omega_j}) \subseteq A_j M_{1+\varepsilon}(R_{\omega_j})$$

for all t > 0 and $\omega_j < 0$, with

$$\left\| \sum_j e_{-t} \right\|_{A_j M_{1+\varepsilon}(R_{\omega_j})} \leq \sum_j \|e_{-t}\|_{A_j M_1(R_{\omega_j})} = \sum_j e^{-t\omega_j}$$

Then, $(e_{-t})^{(m)} = (-t)^m e_{-t}$ implies

$$t^m \left\| \sum_j e^{-tA_j} \right\| \leq C \sum_j \frac{1}{|\omega_j|^m} e^{-t\omega_j}$$

Letting $\omega_j := -\frac{1}{t}$] yields the required statement ■

© 2019 Global Journals

If X = H is a Hilbert space then Plancherel's theorem yields the following result.

Corollary 4.4: Let A_j be densely defined sequence of operators of half-plane type 0 on a Hilbert space H. Then the following assertions are equivalent:
(i) $-A_j$ is the sequence of generators of a bounded C_0 -semigroup on H.
(ii) A_j has strong m-bounded H^∞ -calculus of type 0 for some/all m ∈ ℕ.

In particular, if $-A_j$ be the sequence of generates bounded C_0-semigroup then A_j has m-bounded $H^\infty(R_{\omega_j})$-calculus for all $\omega_j < 0$ and m ∈ ℕ.

V. Semigroups on UMD Spaces

A Banach space X is a UMD space if the function t ⟼ sgn(t) is a bounded L^2(X)-Fourier multiplier. For $\omega_j \in \mathbb{R}$, let

$$H_1^\infty\left(R_{\omega_j}\right) = \left\{f_j \in H^\infty(R_{\omega_j}) \middle| (Z_j - \omega_j)\acute{f}_j(z_j) \in H^\infty(R_{\omega_j})\right\}$$

be the analytic Mikhlin algebras on R_{ω_j}, a Banach algebra endowed with the series of norms

$$\sum_j \|f_j\|_{H_1^\infty} = \sum_j \|f_j\|_{H_1^\infty\left(R_{\omega_j}\right)} = sup_{z_j \in R_{\omega_j}} \sum_j |f_j(z_j)| + \sum_j |(Z_j - \omega_j)\acute{f}_j(z_j)| \left(f_j \in H_1^\infty\left(R_{\omega_j}\right)\right)$$

Lemma 2.2 yield the continuous inclusion

$$H_1^\infty\left(R_{\omega_j}\right) \hookrightarrow A_j M_{1+\varepsilon}^X\left(R_{\omega_j}\right)$$

For each $\varepsilon > 0$, if X is a UMD space. Combining this with Theorems 3.3 and 4.3 and Corollaries 3.5 and 3.6 proves the following theorem (see ,e.g., [8]).

Theorem 5.1: Let $-A_j$ be the sequence of generates C_0-semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of type (M, 0) on a UMD space X. Then the following assertions hold.

(a) For each $\varepsilon > 0$, there exists a constant $c_{\varepsilon+1}$ = c(1 + ε, X) ≥ 0 such that the following holds.

Let 1 + ε, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}(X)$ with

$$\left\|\sum_j f_j(A_j)\right\| \le \begin{cases} c_{\varepsilon+1} M^2 \sum_j |\log(\omega_j(1+\varepsilon))| \, \|f_j\|_{H_1^\infty} & if\ \omega_j(1+\varepsilon) \le \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \\ 2c_{\varepsilon+1} M^2 \sum_j e^{-\omega_j(1+\varepsilon)} \|f_j\|_{H_1^\infty} & if\ \omega_j(1+\varepsilon) > \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \end{cases}$$

for all $f_j \in H_1^\infty(R_{-\omega_j}) \cap e_{-(1+\varepsilon)} H^\infty(R_{-\omega_j})$, and $f_j(A_j) T^j(1+\varepsilon) \in \mathcal{L}(X)$ with

8. M.Haase, J.Rozendaal: Functional calculus for the semigroup generators via transference.

© 2019 Global Journals

Notes

$$\left\|\sum_j f_j(A_j)T^j(1+\varepsilon)\right\|$$

$$\leq \begin{cases} c_{\varepsilon+1}M^2 \sum_j \left|\log(\omega_j(1+\varepsilon))\right| e^{\omega_j(1+\varepsilon)} \|f_j\|_{H_1^\infty} & if\ \omega_j(1+\varepsilon) \leq \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \\ 2c_{\varepsilon+1}M^2 \sum_j \|f_j\|_{H_1^\infty} & if\ \omega_j(1+\varepsilon) > \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \end{cases}$$

for all $f_j \in H_1^\infty(R_{-\omega_j})$.

(b) If

$$\bigcup_{\varepsilon>-1} \sum_j ran\left(T^j(1+\varepsilon)\right) = X$$

then A_j has bounded $H_1^\infty(R_{\omega_j})$ -calculus for all $\omega_j < 0$.

(c) A_j has a strong m-bounded H_1^∞ -calculus of type 0 for all m $\in \mathbb{N}$.

Remark 5.2: Theorem 3.7 yields the domain inclusions dom $(A_j^{\beta+\varepsilon}) \subseteq \mathrm{dom}(f_j(A_j))$ for all $\beta+\varepsilon \in \mathbb{C}_+, \omega_j < 0$ and $f_j \in H_1^\infty(R_{\omega_j})$, on a UMD space X. However, this inclusion in fact, holds true on a general Banach space X. Indeed, for $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}(\lambda_j) < 0$, Bernstein's Lemma [12, Proposition 8.2.3] implies $\frac{f_j(z_j)}{(\lambda_j - z_j)^{\beta+\varepsilon}} \in A_j \mathrm{M}_1(\mathbb{C}_+)$, hence $f_j(A_j)(\lambda_j - A_j)^{-(\beta+\varepsilon)} \in \mathcal{L}(\mathrm{X})$ and $\mathrm{dom}(A_j^{\beta+\varepsilon}) \subseteq \mathrm{dom}(f_j(A_j))$. Series estimates

$$\left\|\sum_j f_j(A_j)(\lambda_j - A_j)^{-(\beta+\varepsilon)}\right\| \leq (1+\varepsilon)\sum_j \|f_j\|_{H_1^\infty(R_{\omega_j})}$$

then follows from an application of the Closed Graph Theorem and the Convergence Lemma.

Remark 5.3: To apply Theorem 5.1 one can use the continuous inclusion

$$H^\infty\left(R_{\omega_j} \cup (S_{\varphi_j}+a)\right) \subseteq H_1^\infty\left(R_{\acute{\omega}_j}\right) \tag{15}$$

For $\acute{\omega}_j > \omega_j$, a $\in \mathbb{R}$ and $\varphi_j \in (\frac{\pi}{2}, \pi]$. Here $R_{\omega_j} \cup \left(S_{\varphi_j} + \mathrm{a}\right)$ are the union of R_{ω_j} and the translated sectors S_{φ_j} + a, where

$$S_{\varphi_j} = \left\{z_j \in \mathbb{C} \,\middle|\, |\arg(z_j)| < \varphi_j\right\}$$

Indeed, to derive (15) it suffices to let a = 0, yields the desired result.

VI. γ_j – Bounded Semigroups

The geometry of the underlying Banach space X played an essential role in the results of properties of the analytic multiplier algebras $A_j M_{1+\varepsilon}^X$. To wit, in to identify

© 2019 Global Journals

nontrivial functions in $A_j M_{1+\varepsilon}^X$ one needs a geometric assumption on X, for instance that it is a Hilbert or a UMD space. Take a different approach and make additional assumptions on the semigroup instead of the underlying space. Show that if the semigroups in questions are γ_j -bounded then one can recover the Hilbert space results on an arbitrary Banach space X.

Assume to be familiar with the basics of the theory of γ_j -radonifying sequence of operators and γ_j -boundedness as collected in the survey article by van Neerven[13].

Let H be a Hilbert space and X a Banach space. The linear sequence of operators T^j : H → X is γ_j –summing if

$$\sum_j \|T^j\|_{\gamma_j} = sup_F \sum_j \left(\mathbb{E} \left\| \sum_{h_j \in F} (\gamma_j)_{h_j} T^j h_j \right\|_X^2 \right)^{1/2} < \infty \ ,$$

Where the supremum is taken over all finite orthonormal systems F ⊆ H and $((\gamma_j)_{h_j})_{h_j \in F}$ is an independent collection of complex-valued standard Gaussian random variables on some probability space. Endow

$$(\gamma_j)_\infty \ (\mathrm{H};\mathrm{X}) := -T^j : \mathrm{H} \to \mathrm{X} \mid T^j \text{ are } \gamma_j \text{ -summing}\}$$

with the norms $\|\cdot\|_{\gamma_j}$ and let the spaces γ_j (H ; X) of all γ_j -radonifying sequence of operators be the closure in $(\gamma_j)_\infty$(H ; X) of the finite-rank sequence of operators H ⊗ X.

For a measure spaces (Ω, μ^j) let γ_j(Ω; X) (resp. $(\gamma_j)_\infty$(Ω; X)) be the space of all weakly L^2–functions f_j : Ω → X for which the integration sequence of operators of $(J)_{f_j}$: L^2(Ω) → X,

$$\sum_j (J)_{f_j}(\mathrm{g}^j) = \int_\Omega \sum_j \mathrm{g}^j . f_j \, d\mu^j \qquad (\mathrm{g}^j \in L^2(\Omega))$$

Is γ_j -radonifying (γ_j–summing), and endow it with the norms $\|f_j\|_{\gamma_j} = \left\|(J)_{f_j}\right\|_{\gamma_j}$.

Collections $\mathcal{T}^j \subseteq \mathcal{L}(\mathrm{X})$ are γ_j -bounded if there exists a constant C ≥ 0 such that

$$\left(\mathbb{E} \left\| \sum_j \sum_{T^j \in \acute{\mathcal{T}}^j} (\gamma_j)_{T^j} T^j x_{T^j} \right\|^2 \right)^{1/2} \leq C \sum_j \left(\mathbb{E} \left\| \sum_{T^j \in \acute{\mathcal{T}}^j} (\gamma_j)_{T^j} x_{T^j} \right\|^2 \right)^{1/2}$$

for all finite subsets $\acute{\mathcal{T}}^j \subseteq \mathcal{T}^j$, sequences $((x)_{T^j})_{T^j \in \acute{\mathcal{T}}^j} \subseteq$ X and independent complex-valued standard Gaussian random variables $((\gamma_j)_{T^j})_{T^j \in \acute{\mathcal{T}}^j}$. The smallest such C is the γ_j -bound of $\mathcal{T}^j$ and is denoted by.$[\![T^j]\!]^{\gamma_j}$ Every γ_j -bounded collections are uniformly bounded with supremum boundless than orequal to the γ_j -bound, and the converse holds if X is a Hilbert space.

An important result involving γ_j -boundedness is the multiplier theorem. State a version that is tailored to the purposes. Given a Banach space Y, a function g^j: ℝ → Y

Notes

© 2019 Global Journals

is piecewise $W^{1,\infty}$ if $g^j \in W^{1,\infty}(\mathbb{R}$ " $\{a_1 , . . . , a_n\}$; Y) for some finite set $\{a_1 , . . . , a_n \} \subseteq \mathbb{R}$.

Theorem 6.1 (Multiplier Theorem): Let X and Y be Banach spaces and $T^j : \mathbb{R} \to \mathcal{L}(X, Y)$ a strongly measurable mappings such that

$$\mathcal{T}^j = -T^j \ (s) \mid s \in \mathbb{R}\}$$

Notes

are γ_j -bounded. Suppose furthermore that there exists a dense subset D ⊆ X such that s $\mapsto T^j(s)x$ is piecewise $W^{1,\infty}$for all x ∈ D. Then the multiplication sequence of operators

$$\mathcal{M}_{T^j} : L^2(\mathbb{R}) \otimes X \to L^2(\mathbb{R}; Y), \mathcal{M}_{T^j} \ (f_j \otimes x) = f_j \ (\cdot)T^j \ (\cdot)x$$

Extends uniquely to boundedsequence of operators

$$\mathcal{M}_{T^j} : \gamma_j(L^2(\mathbb{R}); X) \to \gamma_j(L^2(\mathbb{R}); Y)$$

with

$$\left\| \sum_j \mathcal{M}_{T^j} \right\| \leq \sum_j [\![\mathcal{T}^j]\!]^{\gamma_j}$$

Proof: That $\mathcal{M}_{T^j}$ extends uniquely to bounded sequence of operators into $(\gamma_j)_\infty(L^2(\mathbb{R});Y)$ with $\|\sum_j \mathcal{M}_{T^j}\| \leq \sum_j [\![\mathcal{T}^j]\!]^{\gamma_j}$. To see that in facts ran$(\mathcal{M}_{T^j}) \subseteq \gamma_j(\mathbb{R}; Y)$, employ a density argument. For x ∈ D let $a_1 , . . . , a_n \in \mathbb{R}$ be such that s $\mapsto T^j$ (s)x $\in W^{1,\infty}(\mathbb{R}$ "$\{a_1 , . . . , a_n \}$; Y), and set $a_0 := -\infty$, $a_{n+1} := \infty$. Let $f_j \in C_c(\mathbb{R})$ be given and note that

$$\sum_j \int_{a_j}^{a_{j+1}} \|f_j\|_{L^2(s,a_{j+1})} \|T^j(s)' x\| ds < \infty$$

for all j ∈ {1, . . . , n}. Furthermore,

$$\int_{-\infty}^{a_1} \sum_j \|f_j\|_{L^2(-\infty,s)} \|T^j(s)' x\| ds < \infty$$

yields $(\mathbf{1}_{(a_j,a_{j+1})} f_j)(\cdot)T^j(\cdot)x \in \gamma_j(\mathbb{R}; Y)$ for all 0 ≤ j ≤ n, hence $f_j(\cdot)T^j(\cdot)x \in \gamma_j(\mathbb{R}; Y)$. Since $C_c(\mathbb{R}) \otimes D$ is dense in $L^2(\mathbb{R}) \otimes X$, which in turn is dense in$\gamma_j(L^2(\mathbb{R}); X)$, theresult follows. ■

To prove a generalization of part (a) of Corollary 3.10, recall that

$$e_{-(1+\varepsilon)} H^\infty(R_{\omega_j}) = \{ f_j \in H^\infty(R_{\omega_j}) \mid f_j(z_j) = O(e^{-(1+\varepsilon) R(z_j)} \text{as } |z_j| \to \infty\}$$

for $\varepsilon > -1, \omega_j \in \mathbb{R}$.

Global Journal of Science Frontier Research (F) Volume XIX Issue V Version I Year 2019

© 2019 Global Journals

Theorem 6.2: There exists a universal constant c≥ 0 such that the following holds. Let$-A_j$ be sequence of generatesγ_j- bounded C_0-semigroups $(T^j(t))_{t\epsilon\mathbb{R}_+}$with M:=$[\![T^j]\!]^{\gamma_j}$ on Banach space X, and let 1+ ε, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}$(X) with

$$\left\|\sum_j f_j(A_j)\right\| \leq \begin{cases} cM^2\sum_j |\log(\omega_j(1+\varepsilon))| \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) \leq \frac{1}{2} \\ 2M^2\sum_j e^{-\omega_j(1+\varepsilon)}\|f_j\|_\infty & if\ \omega_j(1+\varepsilon) > \frac{1}{2} \end{cases}$$

for all $f_j \in e_{-(1+\varepsilon)}H^\infty(R_{-\omega_j})$.

In particular, A_j has a bounded $e_{-(1+\varepsilon)}H^\infty(R_{-\omega_j})$-calculus.

Proof: Only need to show that the estimate (9) in Proposition 3.1 can be refined to

$$\left\|\sum_j T^j_{\mu^j}\right\| \leq M^2\eta\sum_j(\omega_j, 1+\varepsilon, 2)\left\|L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}(\gamma_j(\mathbb{R},X))} \tag{16}$$

for $\mu^j \in M_{-\omega_j}(\mathbb{R}_+)$ with supp$\mu^j \subseteq [1+\varepsilon, \infty)$. Then one uses that

$$\left\|\sum_j L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}(\gamma_j(\mathbb{R},X))} \leq \sum_j \left\|\widehat{e_{\omega_j}\mu^j}\right\|_{H^\infty(\mathbb{C}_+)} = \sum_j \|\widehat{\mu^j}\|_{H^\infty(R_{-\omega_j})}$$

by the ideal properties of $\gamma_j(L^2(\mathbb{R}); X)$ [13, Theorem 6.2], and proceeds as in the proof of Theorem 3.3 to deduce the desired result.

To obtain (16) we factorizes $T^j_{\mu^j}$ as $T^j_{\mu^j} = P\circ L_{e_{-\omega_j}\mu^j}\circ\iota$, where ι: X→ $\gamma_j(\mathbb{R}; X)$ and P: $\gamma_j(\mathbb{R}; X) \to X$ are given by

$$\iota x(s) := \psi_j(-s)T^j(-s)x \quad (x \in X,\ s \in \mathbb{R}),$$

$$\sum_j Pg^j = \int_0^\infty \sum_j \varphi_j(t)T^j(t)\, g^j(t)dt \qquad (g^j \in \gamma_j(\mathbb{R}, X))$$

for $\psi_j, \varphi_j \in L^2(\mathbb{R}_+)$ such that $\psi_j * \varphi_j \equiv e_{-\omega_j}$ on $[1+\varepsilon, \infty)$. Show that the maps ι and P are well-defined and bounded. To this end, first note that s ⟼ $T^j(-s)x$ is piecewise $W^{1,\infty}$ for all x in the dense subset dom$(A_j) \subseteq X$ and that

$$\psi_j(-\cdot) \otimes x \in L^2(-\infty, 0) \otimes X \subseteq \gamma_j(L^2(\mathbb{R}); X).$$

Therefore Theorem 6.1 yields ıx ∈ $\gamma_j(\mathbb{R}, X)$ with

$$\left\|\sum_j l\,x\right\|_{\gamma_j} = \left\|\sum_j J_{l\,x}\right\|_{\gamma_j} \leq M\sum_j \|\psi_j(-\cdot)\otimes x\|_{\gamma_j} = M\sum_j \|\psi_j\|_{L^2(\mathbb{R}_+)}\|x\|_X$$

As for P, write

Notes

© 2019 Global Journals

$$\sum_j Pg^j = \int_0^\infty \sum_j \varphi_j(t)T^j(\mathrm{t})\,g^j(t)dt = \sum_j J_{T^j g^j}(\varphi_j)$$

And use Theorem 6.1 once again to see that $T^j g^j \in \gamma_j(\mathbb{R}; X)$. Hence

$$\left\|\sum_j P\,g^j\right\|_X \le \sum_j \left\|J_{T^j g^j}\right\|_{\gamma_j} \|\varphi_j\|_{L^2(\mathbb{R}_+)} \le M \sum_j \|\varphi_j\|_{L^2(\mathbb{R}_+)} \|g^j\|_{\gamma_j}$$

Finally, estimating the norms of $T^j_{\mu^j}$ through this factorization and taking the infimum over all ψ_j and φ_j yields (16).■

Note: In putting μ^j by $t\,\mu^j$ in the proof of Theorem 6.2 we have,

$$\sum_j (\omega_j, 1+\varepsilon, 2)\left\|L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}(\gamma_j(\mathbb{R},X))} \le \frac{1}{n}\sum_j \frac{1}{|\omega_j|}(1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)}\left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\left\|L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

Corollary 6.3: Corollary 3.10 generalizes to γ_j –bounded semigroups on arbitrary Banach spaces upon replacing the uniform bound M of T^j by $[\![T^j]\!]^{\gamma_j}$.

Theorem 4.3 can be extended in an almost identical manner to γ_j–versions (see,e.g., [8]).

Theorem 6.4: Let $-A_j$ be the sequence generates γ_j -bounded C_0-semigroup on a Banach X. Then A_j has a strongm-bounded H^∞-calculus of type 0 for all m ∈ ℕ.

Appendix A. Growth estimates

In this appendix we examine the function η: $(0,\infty)\times(0,\infty)\times[1,\infty]\to\mathbb{R}_+$ from (3.1):

$$\eta(\beta+\varepsilon, t, 1+\varepsilon) = inf\left\{\|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} \middle| \psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)}\ on\ [t,\infty)\right\}$$

Use the notation $f_j \lesssim g^j$ for real-valued functions $f_j, g^j : Z \to \mathbb{R}$ on some set Z to indicate that there exists a constant $c \ge 0$ such that $f_j(z_j) \le cg^j(z_j)$ for all $z_j \in Z$.

Lemma A.1: For each $\varepsilon > 0$ there exist constants $c_{1+\varepsilon}, d_{1+\varepsilon} \ge 0$ such that

$$d_{1+\varepsilon}|\log((\beta+\varepsilon)t)| \le \eta(\beta+\varepsilon,t,1+\varepsilon) \le c_{1+\varepsilon}|\log((\beta+\varepsilon)t)| \qquad \text{(A.1)}$$

If $(\beta+\varepsilon)t \le min\left\{\frac{1}{1+\varepsilon},\frac{\varepsilon}{1+\varepsilon}\right\}$ $If\ (\beta+\varepsilon)t > min\left\{\frac{1}{1+\varepsilon},\frac{\varepsilon}{1+\varepsilon}\right\} then$

$$e^{-(\beta+\varepsilon)t} \le \eta(\beta+\varepsilon,t,1+\varepsilon) \le 2e^{-(\beta+\varepsilon)t} \qquad \text{(A.2)}$$

Proof: First note that $\eta(\beta+\varepsilon, \mathrm{t}, 1+\varepsilon) = \eta((\beta+\varepsilon)\mathrm{t}, 1, 1+\varepsilon) = \eta(1, (\beta+\varepsilon)\mathrm{t}, 1+\varepsilon)$, for all $\beta+\varepsilon$, t and $1+\varepsilon$. Indeed, for $\psi_j \in L^{1+\varepsilon}(\mathbb{R}_+)$, $\varphi_j \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}_+)$ with $\psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)}$ on $[1,\infty)$ defines $(\psi_j)_t(s) := t^{-\left(\frac{1}{\varepsilon+1}\right)}\psi_j\left(\frac{s}{t}\right)$ and $(\varphi_j)_t(s) := t^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\psi_j(s/t)$ for some $s \ge 0$. Then

Ref

8. M.Haase, J.Rozendaal: Functional calculus for the of semigroup generators via transference.

© 2019 Global Journals

$$\sum_j (\psi_j)_t * (\varphi_j)_t(r) = \int_0^\infty \sum_j \psi_j\left(\frac{r-s}{t}\right)\varphi_j\left(\frac{s}{t}\right)\frac{ds}{t} = \sum_j \psi_j * \varphi_j\left(\frac{r}{t}\right)$$

for all r ≥ 0, so $(\psi_j)_t * (\varphi_j)_t \equiv e_{-(\beta+\varepsilon)}$ on [t, ∞). Moreover,

$$\sum_j \|(\psi_j)_t\|_{1+\varepsilon}^{1+\varepsilon} = \int_0^\infty \sum_j \left|\psi_j\left(\frac{s}{t}\right)\right|^{1+\varepsilon}\frac{ds}{t} = \int_0^\infty \sum_j |\psi_j(s)|^{1+\varepsilon}\, ds = \sum_j \|\psi_j\|_{1+\varepsilon}^{1+\varepsilon}$$

and similarly $\sum_j \|(\varphi_j)_t\|_{\frac{1+\varepsilon}{\varepsilon}} = \sum_j \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$. Hence $\eta(\beta+\varepsilon,\ \mathrm{t}, 1+\varepsilon) \leq \eta((\beta+\varepsilon)\mathrm{t}, 1, 1+\varepsilon))$. Considering $(\psi_j)_{(1/t)}$ and $(\varphi_j)_{(1/t)}$ yields $\eta(\beta+\varepsilon,\ \mathrm{t},\ 1+\varepsilon) = \eta((\beta+\varepsilon)\mathrm{t},\ 1,\ 1+\varepsilon)$. The other equality follows immediately. Hence, to prove all of the inequalities in (A.1) or (A.2), assume either that $\beta+\varepsilon = 1$ or that t = 1 (but not both).

For the left-hand inequalities, assume that $\beta+\varepsilon = 1$ and first consider the left-hand inequality of (A.1). Let t < 1 and $\psi_j \in L^{1+\varepsilon}$ $(\mathbb{R}_+)$, $\varphi_j \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}_+)$ such that $\psi_j * \varphi_j \equiv e_{-1}$ on [t, ∞). Then

$$|\log(t)| = -\log(t) = \int_t^1 \frac{ds}{s} \leq e\int_t^1 e^{-s}\frac{ds}{s} = e\int_t^1 \sum_j |\psi_j * \varphi_j(s)|\frac{ds}{s}$$

$$\leq e\int_t^1\int_0^s \sum_j |\psi_j(s-r)|\cdot|\varphi_j(r)|\, dr\frac{ds}{s}$$

$$\leq e\int_0^\infty\int_r^\infty \sum_j \frac{|\varphi_j(s-r)|}{s}\, ds|\psi_j(r)|dr$$

$$= e\int_0^\infty\int_0^\infty \sum_j \frac{|\psi_j(r)||\varphi_j(r)|}{s+r}\, dsdr \leq \frac{e\pi}{\sin(\pi/1+\varepsilon)}\sum_j \|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

where used Hilbert's absolute inequality [14, Theorem 5.10.1]. It follows that

$$\eta(1,t,1+\varepsilon) \geq \frac{\sin(\pi/1+\varepsilon)}{e\pi}|\log(t)|$$

For the left-hand inequality of (A.2), assume that $\beta+\varepsilon = 1$ and let t > 0 be arbitrary. Then

$$e^{-t} = \sum_j (\psi_j * \varphi_j)(t) \leq \int_0^t \sum_j |\psi_j(t-s)||\varphi_j(s)|\, ds \leq \sum_j \|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

By Hölder's inequality, hence $e^{-t} \leq \eta(1,t,1+\varepsilon)$.

For the right-hand inequalities in (A.1) and (A.2), assume that t = 1 and first consider the right-hand inequality in (A.1) for $\beta+\varepsilon \leq min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\}$. In the proof of Lemma A.1, it is shown that

$$((\psi_j)_0 * (\varphi_j)_0)(s) = \begin{cases} s & , s \in [0,1) \\ 1, & s \geq 1 \end{cases}$$

Notes

Notes

for

$$\sum_j (\psi_j)_0 = \sum_{j=0}^{\infty} \beta_j \, \mathbf{1}_{(j,j+1)} \; and \; (\varphi_j)_0 = \sum_{j=0}^{\infty} \acute{\beta}_j \, \mathbf{1}_{(j,j+1)}$$

where $(\beta_j)_j$ and $(\acute{\beta}_j)_j$ are sequences of positive scalars such that $\beta_j = O\left((1+j)^{-\left(\frac{1}{1+\varepsilon}\right)}\right)$ and $\beta_j' = O\left((1+j)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right)$ as $j \to \infty$. Let $\psi_j := e_{-(\beta+\varepsilon)}(\psi_j)_0$ *and* $\varphi_j := e_{-(\beta+\varepsilon)}(\varphi_j)_0$. Then $\psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)}$ on $[1,\infty)$ and

$$\left\|\sum_j \psi_j\right\|_{1+\varepsilon}^{1+\varepsilon} = \left\|\sum_j e_{-(\beta+\varepsilon)} (\psi_j)_0\right\|_{1+\varepsilon}^{1+\varepsilon} = \sum_{j=0}^{\infty} \beta_j^{1+\varepsilon} \int_j^{j+1} e^{-(\beta+\varepsilon)(1+\varepsilon)s} \, ds \lesssim \sum_{j=0}^{\infty} \frac{e^{-(\beta^2+\beta(1+\varepsilon)+\varepsilon)j}}{1+j}$$

$$\leq 1 + \int_0^{\infty} \frac{e^{-(\beta^2+\beta(1+\varepsilon)+\varepsilon)s}}{1+s} \, ds = 1 + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)} \int_{\alpha q}^{\infty} \frac{e^{-s}}{s} \, ds$$

The constant in the first inequality depends only on $1+\varepsilon$. Since $(\beta^2 + \beta(\varepsilon+1) + \varepsilon) \leq 1$,

$$\left\|\sum_j \psi_j\right\|_{1+\varepsilon}^{1+\varepsilon} \lesssim 1 + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)} \left(\int_{(\beta+\varepsilon)(1+\varepsilon)}^{1} \frac{e^{-s}}{s} \, ds + \int_1^{\infty} \frac{e^{-s}}{s} \, ds \right)$$

$$\leq 1 + \int_{(\beta+\varepsilon)(1+\varepsilon)}^{1} \frac{1}{s} \, ds + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)} \int_1^{\infty} e^{-s} \, ds$$

$$= 1 - \log(\beta^2 + \beta(1+\varepsilon) + \varepsilon) + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)-1} \leq log\left(\frac{1}{(\beta+\varepsilon)}\right) + 2$$

Moreover, $\frac{1}{(\beta+\varepsilon)} \geq 1 + \varepsilon > 1$ *hence* $\log\left(\frac{1}{\beta+\varepsilon}\right) \geq \log(1+\varepsilon) > 0$

and

$$\log\left(\frac{1}{\beta+\varepsilon}\right) + 2 \leq \left(1 + \frac{2}{\log(1+\varepsilon)}\right) \log\left(\frac{1}{\beta+\varepsilon}\right)$$

Therefore

$$\left\|\sum_j \psi_j\right\|_{1+\varepsilon} \lesssim \log\left(\frac{1}{\beta+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} = |\log(\beta+\varepsilon)|^{\frac{1}{1+\varepsilon}}$$

For a constant depending only on $1+\varepsilon$. Similarly deduce

$$\left\|\sum_j \varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}} \lesssim |\log(\beta+\varepsilon)|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}$$

for a constant depending only on $\frac{1+\varepsilon}{\varepsilon}$ (and thus on $1+\varepsilon$). This yields (A.1).

Global Journal of Science Frontier Research (F) Volume XIX Issue V Version I Year 2019

© 2019 Global Journals

For the right-hand side of (A.2) we assume that t = 1 and, without loss of generality (Since$(\beta+\varepsilon,t,1+\varepsilon)=\eta(\beta+\varepsilon,t,\frac{1+\varepsilon}{\varepsilon})$), that $\beta+\varepsilon>\frac{1}{1+\varepsilon}$ let $\varphi_j=\mathbf{1}_{[0,1]}e_{(\beta+\varepsilon)(\varepsilon)}$ and $\psi_j=\frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\mathbf{1}_{\mathbb{R}_+}e_{-(\beta+\varepsilon)}$. Then

$$\sum_j \psi_j * \varphi_j(r)=\frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\int_0^1 e^{(\beta+\varepsilon)(\varepsilon)s}e^{-(\beta+\varepsilon)(r-s)}\,ds=e^{-(\beta+\varepsilon)r}$$

For $r\geq 1$. Hence

$$\eta(\beta+\varepsilon,1,1+\varepsilon)\leq\sum_j\|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

$$=\frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\left(\int_0^\infty e^{-(\beta^2+\beta(1+\varepsilon)+\varepsilon)s}\,ds\right)^{\left(\frac{1}{1+\varepsilon}\right)}\left(\int_0^1 e^{\left(\beta+\varepsilon)((\varepsilon)\left(\frac{1+\varepsilon}{\varepsilon}\right)s\right)}ds\right)^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}$$

$$=\frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\left(\int_0^1 e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)s}\,ds\right)^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}=\left(e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1\right)^{-\left(\frac{1}{1+\varepsilon}\right)}$$

$$\leq 2^{\left(\frac{1}{1+\varepsilon}\right)}e^{-(\beta+\varepsilon)}\leq 2e^{-(\beta+\varepsilon)}$$

Where haveused the assumption $(\beta^2+\beta(1+\varepsilon)+\varepsilon)>1$ in the penultimate inequality. ■

Note: Deduce that:

(1) $\left\|\sum_j\psi_j\right\|_{1+\varepsilon}\leq M_{\beta,\varepsilon}\sum_j\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$

(2) $e^{-t}\leq\|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}\leq 2e^{-(\beta+\varepsilon)}$

When $\beta+\varepsilon=1$, $t>0$

Proof. (1) Since

$$\left\|\sum_j\psi_j\right\|_{1+\varepsilon}\leq|\log(\beta+\varepsilon)|^{\frac{1}{1+\varepsilon}}\qquad(a)$$

And

$$\left\|\sum_j\varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}}\leq|\log(\beta+\varepsilon)|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}\qquad(b)$$

Divide we have

$$\left\|\sum_j\psi_j\right\|_{1+\varepsilon}\leq M_{\beta,\varepsilon}\sum_j\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

Notes

© 2019 Global Journals

Where we have $M_{\beta,\varepsilon} = |\log(\beta+\varepsilon)|^{\left(\frac{1-\varepsilon}{1+\varepsilon}\right)}$

(2)From (A.2), we can get

$$e^{-t} \le \sum_j \|\psi_j\|_{1+\varepsilon} \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} \quad (c)$$

$$\le \left(\sum_j \|\psi_j\|_{1+\varepsilon}^2\right)^{\frac{1}{2}} \left(\sum_j \|\varphi_j\|^2\right)^{\frac{1}{2}} = \|\psi_j\|_{1+\varepsilon} \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} \le 2e^{-(\beta+\varepsilon)} \quad \blacksquare.$$

Notes

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgments

The authors wish to think the knowledgeable referee for their corrections and remarks.

References Références Referencias

1. Markus Haase: The Functional Calculus for Sectorial Operators. Oper. Theory Adv.Appl., vol.169, Birkhauser Verlag, 2006.
2. C.Batty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.
3. W.Arendt: Semigroups and evolutions: Functional calculus, regularity and kernel estimates, Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2004.
4. N.J.Kalton, L.Weis: The H^∞-calculus and sums of closed operators, Math. Ann.321 (2) (2001) 391-345.
5. P. Kunstmann, L. Weis: Maximal L_p–regularity for parabolic equations, Fourier multiplier theorems and $H^\infty -$ functional calculus, vol.1855, Springer BBerlin, 2004, pp.65-312.
6. M.Haase: A transference principle for general groups and functional calculus on UMD spaces. Math.Ann.Ann.345(2)(2009) 245-265.
7. Simon Joseph, Ahmed Syfyan, Hala Taha: and Ranya Tahir: TransferencePrincipales for the series of Semigroups with a Theorem of Peller. Scientific Research Publishing, Advance of Pure Mathematics, 9(2) (2019). http://doi.org/10.4236/apm.2019.92009
8. M.Haase, J.Rozendaal: Functional calculus for the of semigroup generators via transference. J. Funct. Anal. 265 (12) (2013) 3345-3368. http://dx.doi.org/10.1016/j.jfa.2013.08.019
9. Nigel Kalton: Lutz Weis, The H^∞- functional calculus and square function estimate. un published manuscript, 2004.
10. H.Zwart: Toeplitz operators and $\mathcal{H}^\infty$-calculus, J.Funct.Anal.263 (1) (2012) 167-182 .
11. F.L.Schwenninger, H.Zwart: Weakly admissible $\mathcal{H}_\infty^-$-calculus on reflexive Banach spaces, Indag. Math. (N.S) 23 (4) (2012) 796-815.
12. W. Arendt, CharlesJ.K.Batty, Matthias Hieber, and Frank Neubrander: Vector Valued - LaplaceTransforms and Cauchy Problems. Springer Monogr. Math., vol. 96, Birkhauser/Springer Basel AG, Basel, 2011.

© 2019 Global Journals

13. Jan van Neervenγ-Radoniifying operators – A Survey in: The AMSI – ANU Workshop on spectral Theory and Harmonic Analysis, in: Proc. Centre Math. Appl. Austral. Nat.Univ., vol.44, Austral. Nat. Univ, Canberra, 2010, pp.1-61.
14. D.I.H.Garling: Inequilities: A journey in to linear Analysis. Combridge University Press, Combridge, 2007.

Notes

© 2019 Global Journals

Global Journals Guidelines Handbook 2019

www.GlobalJournals.org