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Laser Acupuncture Therapy

Nonlinear Mathematical Model

Highlights

Wave-Schrodinger Equation

Riemann-Liouville Fractional

Discovering Thoughts, Inventing Future

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By Prof. Maria Kuman

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Keywords: nonlinear math-model of acupuncture; math-model explains how acupuncture works; acu-math-model explains clinical observations; acu-math-model predicts waves; acu-waves experimentally confirmed, gender differences in acupuncture response.

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Keywords: nonlinear math-model of acupuncture; math-model explains how acupuncture works; acu-math-model explains clinical observations; acu-math-model predicts waves; acu-waves experimentally confirmed, gender differences in acupuncture response.

I. Introduction

The nonlinear mathematical model offered here aims to explain what exactly happens when an acupuncture point on the surface of the skin is punctured with a needle. *Measurements of the skin conductivity revealed that the acupuncture points are ellipses about one inch long and a half-inch wide*. They are oriented with their long axes along lines called *acupuncture meridians* and separated [3] by distances of about 3/4 of an inch.

1/ Each acupuncture point has conductivity, which is 2-3 mA higher than the surrounding tissue [3,4]. This is because under each acupuncture point there is a dense set of nerve fibers without myelin cover, which are like wires without insulation (which explains why the *acupuncture points are more tender to the touch*). The higher conductivity of the acupuncture points [3,4,5] allows us to consider them in this mathematical model as neuronal pools.

The "pool" approximation means that the discrete systems of neurons are considered a continuum. Mathematically, this means replacement of the summation over the system of neurons with integration, which considerably simplifies the mathematical description of the acupuncture points.

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Ref

3. Mann F, *Acupuncture - the Ancient Chinese Art of Healing*, Cambridge, 1971.

2/ Also, since there is a dense set of thin blood vessels under each acupuncture point [3], the acupuncture points have a few tenths of a degree higher temperature [3] and higher oxygen consumption [2][3] than the surrounding tissue.

Thus, *neurologically* the acupuncture points are conducting ellipses imbedded in a semiconducting tissue. *Hematologically*, the acupuncture points are ellipses with higher temperature and oxygen consumption imbedded in a medium with lower temperature and oxygen consumption. In both cases the medium is inhomogeneous.

The propagation of waves in inhomogeneous media depends more on the topography of the inhomogeneous medium than on the kind of waves propagating in it - electric or temperature waves. Since the meridian topography is the same, should we consider the acupuncture points as neuronal or blood pools, mathematically they should be described by the same type nonlinear integral equations.

Shuiski [6] was the first to measure volt-ampere characteristics at the acupuncture points. He found that the acupuncture points have *hysteresis* type of behavior, while the surrounding tissue does not. Hysteresis type of volt-ampere characteristics means that at gradually increased voltage one electric current is measured, but when the voltage is gradually decreased another electric current is measured.

Thus, the electric response to the same strength of voltage is different depending on whether the voltage was increased or decreased. This is so because the voltage increase had induced long-range long-lasting correlations, which still exist when the voltage is decreased. *This makes history or initial conditions important, i.e. it matters what has happened to the system before the measurement.*

'*Hysteresis*' means '*delayed effect*' (hysteros=later), i.e. after the system has been influenced the first time, residual changes remained in it. Each inhomogeneous medium after a certain degree of inhomogeneity starts exhibiting hysteresis type of behavior [6] and the pulses and waves propagating in it become nonlinear [8]. Thus, each inhomogeneous media exhibits nonlinear properties and only nonlinear equations could describe it.

II. Nonlinear Mathematical Model

The conductivity of the acupuncture points is usually measured with different equipment specially designed for this purpose and with them their shape and size was found. *With these equipment, it was discovered that the acupuncture points that signal pathology (which are more painful to the touch) have electrical potential either higher or lower than that of adjacent acupuncture points. It was also found that with each acupuncture treatment the electric potential of these acupuncture points changes toward normal and their painfulness decreases.*

Let us consider each acupuncture meridian, which is a chain of conducting acupuncture points, as a chain of neuronal "pools" with higher conductivity. To describe mathematically a single acupuncture meridian, we will use the system theory [13] used for mathematical modeling of neuronal activity, which was developed for neuronal pools [9].

At the beginning for simplicity, let us assume that each acupuncture treatment stimulates only one neuron A making it to fire an impulse X_A. For simplicity, let the impulse X_A is a Dirac δ -function,

$$X_A(t) = \delta(t); \qquad -\infty < t < \infty \tag{1}$$

Ref

6. Shuiski V, Biophysical Aspects of Acupuncture, *VIII World Congress of Acupuncture*, Sofia, May 15-20, 1983.

and let this stimulus influences only one neuron B. The response of this neuron B to the stimulus X_A will be an electric potential $V_B(t)$ called generator potential (see [9]).

$$V_B(t) = V_{B0} + \int X_A(t')h_{BA}(t - t')\,dt' = V_{B0} + X_A @ h_{BA}(t), \quad (2)$$

Each acupuncture treatment can either excite and then $h_{BA} > 0$, or inhibit and then $h_{BA} < 0$, depending on the type of treatment and type of needles. According to the system theory [13], the response (generator) potential $V_B(t)$ to the stimulus X_A can be obtained from the convolution of the impulse X_A with response $h_{BA}(t)$

$$V_B(t) = V_{B0} + X_A @ h_{BA}(t) \quad (3)$$

The dependence of the generated potential $V_B(t)$ on the stimulus X_A should be nonlinear because: i) there are thresholds for activation of the synapses; ii) at large X_A the secretion of neurotransmitters would be exhausted; and iii) there is a postsynaptic saturation effect and the superposition of synaptic influences is not fully additive [9]. Hence, $V_B(t)$ should be represented by

$$V_B(t) = V_B(0) + S_{BA}(X_A) @ h_{BA}(t) \quad (4)$$

where $S_{BA}(X_A)$ is a nonlinear function of X_A that reflects the average generator potential S_{BA}, produced in cell B by the stimulus X_A.

Since each acupuncture treatment influences more than one neuron, we should replace X_A with X_{Aj} and consider each acupuncture point as a discrete set of n neurons, numbered j= 1,2,...,n. Let us assume that the answers to each of these neurons add up. Considering the interaction only between couples of neurons ij we can write the potential V_B as

$$V_B(t) = V_B(0) + \Sigma\, S_{BAj}(X_{Aj}) @ h_{BAj}(t) \quad (5)$$

Let us now consider each acupuncture point as a continuum of neurons or as a neuronal pool - the high concentration of nerve fibers without myelin cover in each acupuncture point allows us to do this. Then we can replace the summation in eqn. (5) with integration. As a result, eqn. (5) transforms into:

$$V_B(t) = V_{B0} + \int S_{BA}(X_A) @ h_{BA}(t)d\tau \quad (6)$$

Then each meridian becomes a one-dimensional chain R_i of neuronal pools s_i (the index i numbers the neuronal pools or the acupuncture points in the chain). The system theory [13] allows the effect of each acupuncture treatment to be presented either in potential representation or in frequency representation.

In potential representation (eqn. 6), the potential $v(s_i,t)$ of the treated acupuncture point is considered a basic variable. In frequency representation, the frequency $x(s_i, t)$ of the treated acupuncture point is considered a basic variable. The theory allows a transfer from potential to frequency representation and vice versa. Through the following transformation

$$x(s_i, t) = T(s_i, v(s_i, t))$$

our equation (6) could be written in frequency representation

$$x(s_i, t) = T(s_i, v(s_i)) + \Sigma \int\int S_{BA}(s_i, s_j', v(s_j', t'))h(s_i, s_j', t')\,dt'\,ds_j'. \quad (7)$$

The needles, used for treatment of acupuncture points, generate electric potential (experimental fact [1]). Massage of the acupuncture points (acupressure) also generates

Ref

13. Varju D, *Systemtheorie*, Springer-Verlag, Heidelberg - NY, 1977.

electrical potential, because of the piezoelectric properties of the skin (electric potential resulting from rubbing of the skin).

In the cases of electro stimulation when the acupuncture point is stimulated with electric current or external potential V = E, eqn. (7) transform to eqn. (7')

$$x(s_i, t) = T(s_i, v(s_i)) + \Sigma \int S_{BA}(s_i, s_j', X(s_j', t')) @ h_{BA}(s_i, s_j', t')ds_j' + S(s_i, E(s_i, t)) @ h(s_i, t)) \quad (7')$$

The equivalent potential representation of eqn. (7') is:

$$v(s_i, t) = E + v_0 + \Sigma \iint U(s_i, s_j', v(s_j', t)) @ h(s_i, s_j, t)ds_j' \, dt' \quad (8)$$

Equation (8) is a reasonable approximation of the more general and more difficult model

$$V(t) = V_0 + \int K(X(t'), t, t') \, h(t - t')dt' \quad (9)$$

Our attention will be restricted to a special class of Kernel functions $K(s_i, s_j', t')$ allowing transformation of the integral equation into a partial differential equation of parabolic type. This type of equation has been studied extensively in the last two - three decades in various areas, such as ecology, nerve impulse theory, etc.

Let an acupuncture treatment at point s=0, at time t=0 creates an impulse shaped as Dirac function δ with strength a, which influences another acupuncture point, located at a distance s Є R at time t > 0. The direct influence is described by

$$h(s, t) = U(a)(1/2)(Dt)^{-1/2} \exp(-s^2/4Dt)\exp(-d't), \quad (10)$$

where U(a) > 0 means stimulation, U(a) < 0 - inhibition. D is a diffusion constant, d' - a constant of decrease of the signal with time.

If the mutual coupling of two points depends only on the distance between them |s - s'|, then

$$h(s, s', t) = h(|s - s'|, t); \quad U(s, s', v) = U(v); \quad (11)$$

By substituting (11) in (8) we get

$$w(s, t) = \int U(v(s, t)) @ h(|s-s'|, t-t')ds', \quad (12)$$

where w = v - E obeys *the parabolic equation*

$$w_t - d\,w - D\,w_{ss} = U(w + E). \quad (13)$$

This transfer integral -> differential equation is possible only at the special choice of h(s, t) described by (10). Is this choice suitable for an acupuncture theory?

According to formula (10), at a distance s from the origin of the signal (s=0), the response is maximal after time t = t max

$$0 = h/t = U(a)(1/2)(\pi D)^{-1/2} h(s, t)(-1/2t + s^2/4Dt^2 - d')$$

i.e.

$$t_{\text{max resp.}} = -1/4d' + ((1/4d')^2 + (1/4Dd')s^2)^{1/2}; \ (d' > 0);$$

or

$$t_{\text{max resp.}} = s^2/2D; \ (d' = /0); \quad (14)$$

Notes

In formula (14), the time response depends reciprocally on the diffusion constant D, i.e. the time response is shorter when the diffusion constant is larger. Since the processes in a child's body run faster, their constant of active diffusion D will be larger. Then formula (14) can explain an event observed in the acupuncture clinical practice: the cure of children is faster because their constant of active diffusion is larger.
At E = const., eqn. (13) transforms to:

$$v_t - D\, v_{ss} = U(v) - d'.\, v + I, \tag{15}$$

where I= d'.E. Therefore, d' describes the dissipation observed in a body tissue during the propagation of external input E=const.
By introducing the function

$$f(v) = U(v) - d'.\, v + I$$

we can write (15) in the form

$$v_t - Dv_{ss} = f(v) \tag{16}$$

known as *Fisher equation*. The equation has two types of solutions: stable (stationary) and unstable [10].

The stationary solutions of eqn. (16) v(s, t) = v(s) are determined by the ordinary differential equations

$$v''(s) + f(v(s)) = 0; \qquad -\infty \le s \le \infty \tag{17}$$

The unstable solutions of eqn. (16) correspond to a traveling front and can be written in the form

$$v(s, t) = u(s - ct) = u(z) \tag{17'}$$

$$u(\infty) = \lim\ u(z)$$

$$u(-\infty) = \lim\ u(z)$$

The two asymptotes of the function coincide with two of the stable (stationary) solutions (c - speed of the traveling front). If c > 0, the front travels to the right, if c ¡ 0 the front travels to the left, c=0 corresponds to a stationary wave.
The traveling front u is a solution of the ordinary differential equation

$$u'' + c\, u' + f(u) = 0, \tag{18}$$

which becomes eqn.(17) with substitution (17') and at

$$u(-\infty) = y_i; \qquad u(\infty) = y_j;$$

where y_i , y_j are different zeros of f.

Aronson and Weinberger in 1975 [11] have proved a theorem for a system of neurons: A sufficiently large local excitation can lead to a global excitation; a sufficiently large local depression can lead to a global depression.

This can explain how local excitation from local treatment (or inhibition when the treatment is done with a thick needle) can lead to a global excitation (or inhibition) of the processes in the whole body.

If we include self-inhibition [9] in our model, which has been proven to exist in nerve propagation and is included in all nerve-propension models, then instead of traveling fronts, we will have as solutions traveling pulses.
Equation (16) then becomes

Ref

10. Fisher B, A Neuron Field Theory: Mathematical Approaches to the Problem of Large Numbers of Interacting Cells, *Bull. Math. Biol.*, 1973; 38: 253-267.

$$v(s, t) = \int [U(v(s, t) - w(s, t)) @ h(| s-s'|, t-t')]ds' + E; \quad (19)$$

where

$$w (s, t) = v(s, t) @ h(t) = \int v(s, t) h(t - t') dt';$$

$$h (t - t') = b \exp (-d_2 (t - t')); \quad b > 0; \; d_2 > 0.$$

Differentiating (19) we get

$$v_t - D v_{ss} = f(v) - w; \quad w_t (t) = b v - d_2 w. \quad (20)$$

Equation (20) is identical with the *Fitzhugh - Nagumo equation, which is simplification of the Hodgkin - Huxley model for nerve pulse propagation* [9]. The unstable solution of (20) is a traveling pulse, which can be represented in two-coordinate vector form

$$(v(s,t), w(s,t)) = (u (s - ct), g(s - ct)) = Z (z) \quad (21)$$

It is time now to take into consideration the natural DC potentials at rest V_N, which exist between the upper and lower body and determine the direction of the acupuncture meridians. When the arms are up, the acupuncture meridians run downward on the outer side of arms, legs, and on the back and they run upward on the inner side of arms, legs, and the chest and abdomen.

Without these natural DC potentials at rest that determine the direction of the acupuncture meridians, the pulses created during acupuncture treatment of point j would run with the same speed in both directions of the meridian. The DC potentials at rest make the electric pulses run in the direction of the meridian.

When acupuncture point is treated with a needle, some patients feel the electric current running in the direction of the meridian (especially during rotation of the needle). Based on their sensation, they can tell in which direction the meridians run. Prof. Zhu experimentally measured these traveling electric pulses [1].

Whenever the equation (20) has solutions traveling electrical pulses, it has solutions traveling waves. The nonlinear equations have more than one solution and all solutions need to be considered on an equal basis. This gives to the traveling waves equal right of existence with the traveling electric pulse.

The *traveling-wave solutions* are described by a function

$$Z(z) = Z (k s - t), \quad (22)$$

where k is the wave number ($k = 2 \pi / \lambda$), λ - the wavelength, and the speed of wave propagation is $c = \omega /k$.

The traveling waves, also called *wave trains*, can be represented as two component vectors

$$Z(z) = (v (s, t), w (s, t)) = (v (z), w(z)). \quad (23)$$

In this representation eqn. (20) can be written as

$$k^2 D u'' + u' + f(u) = g(z) \quad (24)$$

$$u - g(z) d = -b u$$

In the case of nerve propagation, the existence of wave trains has been proved by Conley, Hasting and Carpenter [12] independently. Numerical calculations showed that

Ref

1. Zhu Z X, *Biophysical Approaches in Acupuncture*, Conference in Budapest, 1985.

there are two types of waves with frequencies ν and ν' (with wave numbers k and k') propagating with different speeds c (slow) and c' (fast).

The train-waves, predicted by our mathematical model, must be running from the treated acupuncture point in the direction of the meridian. This was first reported at the 8th World Congress of Acupuncture. The Hungarians showed interest and a year later they proved experimentally that waves do propagate along the meridian [2]. However, more detailed experiments are necessary to prove that both slow and fast waves are present.

III. Auto-Waves

The offered here nonlinear mathematical model also allows us to explain the following fact known to every acupuncturist: treatment of distant points of the acupuncture meridian has maximal effect on the organ. It has never been explained before, but it can find its natural explanation in the frames of this model, if we think of the waves propagating along the acupuncture meridian as auto-waves.

Auto-wave means that the wave generated at each acupuncture treatment moves in the direction of the meridian and when it meets the next acupuncture point, it makes it a new center of wave propagation. This could happen only if the distances between the acupuncture points are such that the echo - reflected waves from the second point annihilate the forwarding signal from the first point, but enhance the signal from the second point.

In this way, the wave center moves in the direction of the meridian and the further it goes from the initial point, the larger the amplitude of the signal becomes. This explains why the more distant is the treated acupuncture point of each acupuncture meridian, the stronger is the effect on the organ represented by this meridian.

If waves are involved in each acupuncture treatment, this means that the acupuncture works through the waves of our nonlinear electromagnetic field (NEMF), which is located in the Subconscious and from the Subconscious rules and regulates the functioning of all the organs and everything else in the body.

Our fast response is done through the waves of this NEMF. If we don't have conscious awareness of the existence of this NEMF, it is because when our life is threatened and we need to respond fast to survive, we don't want to be bothered with information about the functioning of our organs.

The waves of this NEMF are the basis of our Quantum Computer, which operates in the Subconscious, and from the Subconscious rules and regulates everything in the body. If acupuncture works through waves, it influences the regulatory work of our Quantum Computer.

IV. Gender Differences in Acupuncture Response

Ancient acupuncture books say that the acupuncture meridians are like rivers, but along them runs energy instead of water. In the way the rivers flow into sea, the acupuncture meridians flow into 6 spinning seas. They are called chakras in Hindu sources because "chakra" means "spinning wheel" in Sanskrit [15].

Each of these chakras is related to a gland of internal secretion secreting hormones into the blood stream. If so, these spinning energy centers, called chakras, rule our endocrine system, the balance of which determines our physical and mental

Ref

2. Eory A, *International Acupuncture Congress*, Bratislava, February 1985; see also: *Acupuncture and Electro-Therapeutic Research (Int. J.)* 1984; 9: 217-223.

health. Since the chakras spin in opposite directions in males and females, the acupuncture meridians are expected to run in opposite direction in males and females.

If so, there will be a difference in the response of males and females to acupuncture. Indeed, ancient texts recommend: when treating females with acupuncture to use points on the left hand because the effect on the organ will be stronger; when treating males with acupuncture to use points on the right hand because the effect on the organ will be stronger [15].

Also, ancient texts when speaking about pulse diagnosis claim that the best way to diagnose a woman is to measure the pulse on her left hand, while the best way to diagnose a man is to measure the pulse on his right hand. This is because the pulse on the left woman's hand is stronger and the pulse on the right man's hand is stronger [15].

There is also a difference left – right in the strength of our eyes. For women the left eye is stronger, for men the right eye is stronger. The difference is about ¼ of diopter, but it is always present [15]. The cited book of the author [15] contains a lot more interesting facts including the offered here mathematical model.

V. Conclusion

Let us summarize. The nonlinear mathematical model offered here was able to explain many of the bizarre features observed in the clinical acupuncture practice: i) why the treatment of distant acupuncture points has maximal effect on the organ, ii) why for children the cure is faster, and iii) why some patients feel electric current is running from the treated acupuncture point in the direction of the acupuncture meridian.

The nonlinear mathematical model describing one acupuncture meridian, being nonlinear, has two types of solutions - electric pulse and electromagnetic wave. Based on this, *each acupuncture treatment is expected to generate electric impulse and electromagnetic wave. They will both run from the treated acupuncture point in the direction of the acupuncture meridian.*

The Chinese Prof. Zhu [1] already experimentally measured *electric pulses running from the treated acupuncture point in the direction of the acupuncture meridian*. This is also consistent with clinical observations - some patients do sense the electric pulse generated at the treated acupuncture point and propagating in the direction of the acupuncture meridian.

The Hungarian scientist Dr. A. Eory [2] confirmed the theoretically predicted by us waves. This was the first attempt to measure *traveling waves running from the treated acupuncture point in direction of the acupuncture meridian*. He found that waves do run in the direction of the acupuncture meridians all the time, but when an acupuncture point of the acupuncture meridian is treated with a needle, a wave is generated which modifies the constantly running wave.

However, more detailed measurements are necessary to determine the speeds of propagation of these waves. *The model predicts two types of traveling waves*, called slow and fast wave trains. Obviously, two types of waves (slow and fast) are generated at each acupuncture treatment. They propagate along the acupuncture meridian, where the conductivity is higher, and they propagate in the direction of the meridian determined by the DC potential gradient of the body at rest.

We shouldn't finish this article without emphasizing again the fact that if acupuncture works through waves, this means that the acupuncture works through the waves of our nonlinear electromagnetic field (NEMF), which is located in the

Ref

15. Kuman M, Modern Aspects of Ancient Acupuncture, Health and Happiness Books, 1997.

Subconscious. From the Subconscious, it rules and regulates the functioning of all the organs and everything else in the body and determines our physical and mental health.

References Références Referencias

1. Zhu Z X, *Biophysical Approaches in Acupuncture*, Conference in Budapest, 1985.
2. Eory A, *International Acupuncture Congress,* Bratislava, February 1985; see also: *Acupuncture and Electro-Therapeutic Research (Int. J.)* 1984; 9: 217-223.
3. Mann F, *Acupuncture - the Ancient Chinese Art of Healing*, Cambridge, 1971.
4. Podshibjakin A K in: *Medical Problems of Physical Exercises*, 1971; 1: 45.
5. Bossy J, Bases morphologiques et fonctionelles de l'analgesie acupunctural, *Giorn. Accad. Med. Torino*, 1973; 136, 3-23.
6. Shuiski V, Biophysical Aspects of Acupuncture, *VIII World Congress of Acupuncture,* Sofia, May 15-20, 1983.
7. Antonez V A, Antonez M A, Kudrjashov A V, Possibility for Autowaves in Thin Blood Vessel Nets, in: *Autowave Processes*, Gorki, 1981 (Russ.)
8. Hirota R, and Suzuki K, Theoretical and Experimental Studies of Solitons in Nonlinear Lumped Networks, *Proc. IEEE*, 1973; 61:1483.
9. Uwe an der Heiden, *Analysis of Neural Network*, Springer-Verlag, Heidelberg - N.Y., 1980.
10. Fisher B, A Neuron Field Theory: Mathematical Approaches to the Problem of Large Numbers of Interacting Cells, *Bull. Math. Biol.*, 1973; 38: 253-267.
11. Aronson D G, Weinberger H F, Nonlinear Diffusion in. Nerve Propagation, *Lecture Notes in Math.*, v.446, Springer-Verlag, Heidelberg - NY, 1975.
12. All three papers on traveling wave solutions are cited in [9].
13. Varju D, *Systemtheorie*, Springer-Verlag, Heidelberg - NY, 1977.
14. Ivanitski G R, Krinski V I, Selkov E E, *Math. Biophysics of Cells,* Moscow, 1978.
15. Kuman M, Modern Aspects of Ancient Acupuncture, Health and Happiness Books, 1997.

Notes

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Economic Determinants Affecting Military Expenditures: Panel Data Analysis

By Mahmoud Mourad & Bilal Nehme

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Abstract- Our research revolves around the topic of considering the military expenditures per capita as a dependent variable and the GDP per capita and CO2 Emissions per capita as two explanatory variables. The study is made up of ten sections addressing several points, each of which clarifies the research method in order to reach a conclusion revealing the importance of the findings. Beginning with the basic statistical characteristics, such as averages, standard deviations, minimums, maximums and the Compound Annual Growth Rate (CAGR), a benefit use of the graph of each variable for each country has been highlighted for a better understanding of the rising and falling during its temporal evolution. The various aspects of the panel analysis have been completed as the questions of individual specific heterogeneity in panel data, the panel unit root tests using the most famous from the first and second generations, and the co-integration analysis according to the Pedroni's approach, which has led to the rejection of the null hypothesis of no co-integration for each country and for the group as a whole.

Keywords: principal component analysis, panel co-integration, long-run equilibrium, forecasts.

GJSFR-F Classification: MSC 2010: 62N86

ECONOMICDETERMINANTSAFFECTINGMILITARYEXPENDITURESPANELDATAANALYSIS

Strictly as per the compliance and regulations of:

Economic Determinants Affecting Military Expenditures: Panel Data Analysis

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Abstract- Our research revolves around the topic of considering the military expenditures per capita as a dependent variable and the GDP per capita and CO2 Emissions per capita as two explanatory variables. The study is made up of ten sections addressing several points, each of which clarifies the research method in order to reach a conclusion revealing the importance of the findings. Beginning with the basic statistical characteristics, such as averages, standard deviations, minimums, maximums and the Compound Annual Growth Rate (CAGR), a benefit use of the graph of each variable for each country has been highlighted for a better understanding of the rising and falling during its temporal evolution. The various aspects of the panel analysis have been completed as the questions of individual specific heterogeneity in panel data, the panel unit root tests using the most famous from the first and second generations, and the co-integration analysis according to the Pedroni's approach, which has led to the rejection of the null hypothesis of no co-integration for each country and for the group as a whole. The long-run equilibrium relationships are carried out using both the FMOLS and the DOLS estimators. The performance of these relationships has been measured over the period 2014−2017 by considering a linear adjustment that links the forecast and the observed values associated with the ten countries of linear correlation coefficients as positive and near to one. This research attempts to deal with a point considered to be innovative that consists of using Principal Component Analysis (PCA) applied to the residuals of the ADF equations used in the panel unit root tests. In this respect, an algorithm is being proposed showing the importance of a certain ordering of countries which could be informative on the degree of heterogeneity of the panel vis-à-vis the masked factors of military expenditures.

This link between PCA is considered as econometric without model and the panel's analysis with adequate model. This link can be better exploited by considering a panel with a large number of individuals.

Keywords: principal component analysis, panel co-integration, long-run equilibrium, forecasts.

I. Introduction

After World War II (1939-1945) and the eruption of the Cold War (1947-1991), the strategies of the political leaders were devoted to defend their own territories and those of their allies. This opened the door to ensure that the military expenditure to be used for national self-defence to face any eventual danger from outside the country. However, where did the arms race amongst all developed and developing countries come from ? Unfortunately, instead of spending the necessities for the welfare of the whole planet, the states regrouped

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themselves and built alliances (ANZUS [1], NATO [2], EDC [3], WEU [4],...). Moreover, the conflicts on the planet did not come to an end, but they are clearly seen in the Middle East conflict between the Arab states and the Israel, in southern Asia between Pakistan and India, and in eastern Asia between South-Korea and Vietnam. All these conflicts have created an impulse that boosted the military expenditure in all countries. In this regard, several factors mask the military expenditure among different states, for instance, the factors that masked the US military expenditure were not the same in Singapore, because for the former it is the whole planet that was being targeted, while the latter wants to protect the society's welfare. For this reason, we cannot generalize or suggest the same hidden factors for different countries, but certainly there will be some common hidden factors such as the self-defence against a possible external danger. Thus studying a panel of countries should take into account a certain degree of heterogeneity in their individual behaviors with respect to any economic variable, military expenditure or other factor. For example, the economic growth of a country depends on many factors such as the power of the industry sector, the degree of corruption, and others. A panel is an importer of heterogeneity and the experts must reduce this heterogeneity by a suitable choice of this panel.

The purpose of this article is to carry out an in-depth analysis of the panel of three macroeconomic variables, where Military Expenditure per Capita (MEXPC) is considered as a dependent variable, and both the Gross Domestic Product per Capita (GDPPC) and the CO2 Emissions per Capita (EMCO2PC) are independent variables. The research aims at performing a unit root panel analysis using first generation tests, without taking into account any dependence between the panel's sections (Harris and Tzavalis (1999)-HT, Breitung (2000)-λ, Im et al. (2003)-IPS) and the second generation, while considering the dependence using the Cross-sectionally Augmented IPS Panel Unit Root Test (CIPS). It is true that in any study of the panel, researchers start by testing the homogeneity, i.e. can we stay in the context of Pooled Regression Model, fixed model or random effect model. However econometric methodologies have progressed especially with Pesaran (2004, 2007); Im et al. (2003) and Pedroni (1995, 1996, 1999, 2001, 2004, 2007) in the analysis of co-integration at the level of a country taken alone and with the whole group. That is there may be two types of co-integration relationship : one relationship associated with each country in the panel and another for the entire panel. This leads us to be careful in making a quick decision about the heterogeneity of the panel.

The use of Principal Component Analysis (PCA) is now becoming essential to clarify the aspect of heterogeneity. So we will take the residues associated with the panel - Unit Root test (see Table (3)) to perform two types of analysis : The first is to use the cross section dependence proposed by (brush and Pagan 1980) CD_{BP} and the famous test CD_P proposed by Pesaran (2004). The second exploits the importance of masked factors that are fixed at two only, due to the two explanatory variables GDPPC and EMCO2PC that we have imposed. With this use of the PCA, we will propose an algorithm that could enrich

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1. The Australia, New Zealand, United States Security Treaty (ANZUS or ANZUS Treaty) is the 1951 collective security non-binding agreement between Australia and New Zealand and, separately, Australia and the United States, to co-operate on military matters in the Pacific Ocean region, although today the treaty is taken to relate to conflicts worldwide.
2. The North Atlantic Treaty Organization also called the North Atlantic Alliance, is an intergovernmental military alliance between 29 North American and European countries. The organization implements the North Atlantic Treaty that was signed on April 4, 1949.
3. The Treaty establishing the European defence Community, also known as the Treaty of Paris, is an unratified treaty signed on 27 May 1952 by the six inner countries of European integration : the Benelux countries, France, Italy and West Germany.
4. The Western European Union was the international organization and military alliance that succeeded the Western Union (WU) after the 1954 amendment of the 1948 Treaty of Brussels.

the panel econometric methods with model by better exposing the aspect of heterogeneity with a suitable choice of the panel ordering, see Appendix B. The seven co-integration tests proposed by Pedroni will be used in this paper and the long-term equilibrium relationships will be estimated by FMOLS and DOLS methods taking into account this heterogeneity.

This paper is organized as follows : Section 1 comprises the introduction and Section 2 presents a descriptive statistical study of the three random variables to better appreciate their temporal evolution and to highlight the interesting events that have affected the evolution. In Section 3, we test the homogeneity of the panel according to the Fisher test by taking the raw data, the log-transformed data and the first difference data. Section 4 is dedicated for the panel unit root tests. In Section 5, the Panel co-integration tests of Pedroni will be performed. In Section 6, we address the estimation of the long-run equilibrium relationship according to the FMOLS and DOLS estimators. While the section 7, the predictive model performance is performed using the findings of the FMOLS Estimator. The conclusion and discussions are presented in 8.

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II. Statistical Description of Variables

For the purpose of the analysis, it is very useful to perform a basic statistical description when we have a panel of several countries in order to see the temporal evolution of each variable of the panel on one hand, and to check whether an individual effect is found among the countries on another hand. These are the basic statistical characteristics such as averages, standard deviations, minimums, maximums and the Compound Annual Growth Rate (CAGR) which is a specific term for the geometric progression ratio that provides a constant rate of evolution over the time period $(1968, 2014)$. For a time series X_t, the CAGR between 1968 and 2014 is calculated by :

$$\text{CAGR } (1968{,}2014) \ = \left(\frac{X_{2014}}{X_{1968}}\right)^{\frac{1}{46}} - 1.$$

It is clear that this description would have a meaning especially if the individual time series are realized with the stationary Gaussian random processes. Any way, we hope that this section will better explain the temporal evolution of each of the variables, especially if we can repair rupture of time due to a political or economic intervention that had an impact on the variables in question. As we stated in the introduction, we have a panel of (N=10) countries and three variables of which two are explanatory that are studied over the period $1968 - 2014$ (47 years).These countries are :

Arab world, Israel, USA, Canada, Japan, South Korea (Korea), France, United Kingdom, India and Pakistan. Figures of the individual in primary time series and in log-transformed data are presented in Appendix (A).

Let us have a close look at the elementary statistics in Appendix (A), Tables (8, 9, 10, 11, 12, 13). The lowest values are observed in India ($14), Pakistan ($25) and Arab world ($168). Israel and USA showed the highest military expenditure per capita of averages $1361 and $1117 respectively. The important difference between the Min and Max supports the idea that the two countries follow a very similar defence policy because each of them has enormous concerns about national security and domination by force. Both countries apply a policy based on the importance of military power to impose control over other states or enemies, ensuring the superiority of their military strength. The high standard deviations reflect a large variation over time. The annual growth over the period $1968 - 2014$ measured by CAGR is of the order of 4.58% in Israel and 3.45% in the USA which is the lowest rate.

These growth rates are relatively low when compared to rates in other countries. The highest CAGR are found in South Korea (10.05%), Japan (7.82%) and Arab world (7.57%). These results tell us about the situation of the conflict in the Korean Peninsula where South Korea, Japan and their allies line up on one side, and North Korea and its allies line up on the other side, not to mention the reality of the Arab-Israeli conflict, which has increased military expenditure in both directions.

Regarding GDP Per Capita, the three countries having the highest averages are USA ($26285), Japan ($24122) and Canada ($21945) while the lowest are located in India ($486), Pakistan ($501) and the Arab world ($2525). Considering the CAGR values, the highest value is recorded in South Korea (11.34%) followed by Arab world (7.93%). Indeed, South Korea has experienced an exceptional growth and integration in the world economy over the past fifty years Carroué (1997). For the Arab world, it is an informal way of admitting that the oil boom of the 1970s is responsible for this high value of CAGR. The lowest CAGR values are recorded in Pakistan (4.88%) and USA (5.48%).

For the third variable EMIPCA, since high values of this variable will have negative impacts on human life on the terrestrial globe, let us try to read carefully the CAGR values. Negative values indicate a decrease in the period 1968 − 2014. They were observed in USA (0.32%), France (1.07%) and United Kingdom (1.14%). This is a positive sign for these three countries, but also insufficient given the high averages measured by metric tons per capita of the variable EMIPCA, which are (19.51), (6.92) and (9.61) respectively. The three lowest positive values of CAGR are (0.07%) in Canada, (1.02%) in Israel, and (1.18%) in Japan. We choose this variable in the belief that it has an effect (positive or negative) on military expenditures per capita in a given country.

For raw data and logarithm transformed data, the elementary statistics such as Average, Std.Dev, Min and Max provide almost similar information for all variables. There will be only a difference in the CAGR because the trend evolution is generally weak in log-transformed data. For the MEXPCA variable, in raw data, the Max and Min were in South Korea and Pakistan, while the log-transformed data are observed in India and the USA. For GDPPCA, the highest CAGR values remain in South Korea and Arab world for raw data, while in log-transformed data, the lowest CAGR values appear in USA and Canada. For EMIPCA variable, in both raw data and log-transformed data, the negative CAGR are observed in the same countries : USA, France and United Kingdom, while the two highest positive values of CAGR are (5.73%) in South Korea and (2.52%) in Arab world.

Let us review the graphs of the time series associated with each country and start with the Military expenditures per capita (MEXPCA). The first graph shows the gap between the Arab world and Israel. A year after the Arab-Israeli War that took place between 5 and 10 of June 1967 between the Israel and the neighbouring states (Egypt, Jordan and Syria), the Israeli military expenditure per capita rose from ($287.19) in 1968 to ($823.62) in 1973, when the war known as the 1973 Arab-Israeli war erupted in October (6th - 25th), then it reached ($1145.35) in 1974. This graph shows two interesting peaks, the first ($1709.33) took place in 1988 marking the ending of the Iran-Iraq War (20 August 1988) and the second ($2114.21) in 1991 marking the Gulf War (17 January-28 February 1991). The MEXPCA of Arab world grew very slowly from ($16.44) in 1968 to ($224.92) in 1982 then he reached ($472.04) in 2014 while for Israel, the graph shows the value ($2249.5) at the end of this year.

For USA, we observe a graph with a growing cycle of length (23 or 24 years) : from 1968 to 1990 where the variable MEXPCA increased from ($402.24) in 1968 to ($1226.53) in 1990, then from 1991 to 2014 where it arose from ($1107.96) to ($1914.22) with a very large

Notes

growth peak between 2001 (September 11 attacks) and 2011 from ($1097.46) to ($2282.53). For Canada, there is a similarity with the USA but a much lower variability in the time series going from ($86.64) in 1968 to ($502.42) in 2014. The military expenditures per capita, in Japan and Korea, were very close to each other until the year 2004, and a net difference in growth was observed between the two countries to reach ($740) and ($368.52) in 2014 respectively. This growth is due to the buildup of Chinese military expenditure, which, in turn, encourages defence expenditure of some possible adversaries (e.g., Japan and Republic of Korea Todd and Justin (2016). France and United Kingdom followed a very similar policy of military expenditure per capita over the period 1968 − 2014, going from ($119.55) to ($959.25) in France and from ($100.76) to ($915.31) in the United Kingdom, with (4.63%) and (4.93%) CAGR values respectively. The interstate war that took place between India and Pakistan (2001−2003) has encouraged defence expenditure for each of them. The graphs show that in Pakistan, the variable MEXPCA was stronger than in India and the two graphs intersect in 2011 reaching the value around ($40).

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For Israel, an investigation of the GDP per capita graph shows that important growth took place after the 2006 Lebanon-Israel war, also called the 2006 Israel-Hezbollah war, a 34-day military conflict. The GDPPCA moved from ($21827.82) in 2006 to ($37539.95) in 2014 while, in the Arab world, the overview shows a CAGR (7.93%) going from ($222.62) in 1968 to ($7452.81) in 2014. For the USA, an almost linear trend is noticed indicating an increase from ($4695.92) in 1968 to (54696.73) in 2014. However, in Canada, we witness a convex evolution over 1991−2008 from ($21664.6) to ($46596.34) with a very increasing trend between 2002 and 2008 due to the largest increases for agricultural products such as wheat, corn and canola. With an overview of the Japanese economy, we can distinguish between three phases : 1968−1985 ($1450.62−$11584.65), 1986−1995 ($17111.85−$43440.37), 1996−2014 ($38436.93−$38109.41) with a maximum of ($48603.48) in 2012. While in Korea, there were small fluctuations over the entire period from ($198.37) in 1968 to ($27811.37) in 2014. In fact, the GDPPCA in Japan began a period of expansion in 1986 that continued until 1995, marking the start of the end of the Cold War with the arrival of Mikhail Gorbachev as leader of the Soviet Union (1985 − 1991). For France and United Kingdom, there were two small troughs in 1984 and 2001 (September 11 attacks) where GDPPCA went from ($9397.495 to $22433.56) and ($8179.194 to $27427.59) respectively. In France, economic instability marked the Giscard d'Estaing government and the early years of the presidency of François Mitterrand. Moreover, in United Kingdom, a substantial increase in unemployment from 5.3% in 1979 to a peak at nearly 11.9% in 1984 Bell and Blanchflower (2010). Finally, for India and Pakistan, the GDPPCA are very comparable in their trends from 1968 to 2014, going from ($98.83) to ($1576.00) and from ($146.98) to ($1316.98) respectively.

We still need to quickly describe the third variable CO2 emissions measured by metric tons per capita. The fluctuations are small in all countries, with a net decrease in both France and United Kingdom passing between 1968 and 2014 from 7.50 to 4.57 and from 10.99 to 6.5 respectively. The highest values are observed in both USA and Canada with a slight decline in the period 1968 − 2014, for the USA (from 19.09 to 16.49) and for Canada (from 14.63 to 15.12). A simple comparison between the Israel and the Arab world shows a wide gap between the two parties : at the beginning of the period, we observe the values 4.93, 1.54, and at the end of the period we read the values 7.86 and 4.86. Among the ten countries, Korea and India recorded a net increase over the period from 1.21 to 11.57 in Korea, and from 0.35 to 1.73 in India. Japan reveals a growth of 5.57 to 9.54 with CAGR of 1.18%. Finally, in Pakistan, the variable EMIPCA had the lowest values compared with the other countries.

The evolution moved from 0.45 in 1968 to 0.9 in 2014. For the logarithm data graphs, the readers are left to appreciate the temporal evolution of each variable in each country.

III. Homogeneous or Heterogeneous Panel ?

Considering the following model :

$$Y_{it} = \beta_{0,i} + \sum_{j=1}^{k} \beta_{1,i} X_{j(it)} + \epsilon_{it}, \quad k = 2, i \in [1, N], \ t \in [1968, 2014] = [1, T]. \tag{1}$$

Notes

In matrix form $Y_{it} = \beta_{0,i} + \mathcal{X}'_{it}\mathcal{B}^i + \epsilon_{it}$, such that $\mathcal{X}_{it} = \left(X_{1i,t}, X_{2i,t}\right)', \mathcal{B}_i = \left(\beta_{0,1}, \mathcal{B}^i\right)'$ and $\mathcal{B}^i = \left(\beta_{1,i}, \beta_{2,i}\right)$. In the literature Hsiao (1986); Hurlin (2010) and (Mourad (2019), p. 150-154), we have three tests which represent the first steps in a panel data study. Indeed, the researcher in this field is invited to ask questions about the nature of the panel : Is it a homogeneous or heterogeneous panel ? Indeed, the three tests will lead together to a decision around the existence of a panel structure or take each country separately without taking care of the panel itself.

First test :

$$\begin{cases} H_0^1 : \beta_{0,i}, \mathcal{B}^i = \mathcal{B}, \ \forall i = 1, N \\ H_a^1 : \exists (i, j) \in [1, N]; \ \beta_{0,i} \neq \beta_{0,j} \text{ or } \mathcal{B}^i \neq \mathcal{B}^j \end{cases}$$

Under the null hypothesis H_0^1, we consider a pooled Regression Model (PRM) with respect to the number of the imposed restrictions $[\nu_1 = (k+1)(N-1)]$. Using the OLS method, we estimate the PRM and we save the residual sum of squares $\text{RSS}_{(\text{pooled},r_1)}$, where r_1 designates the restrictions under H_0^1. Then we consider N models of multiple linear regressions, with a model for each country, and we keep the residual sum of squares associated with each country $\text{RSS}_i^1, \ i = 1, \ldots, N$ and finally we calculate $\text{RSS}_{1,\nu_2} = \sum_{i=1}^{N} \text{RSS}_1^i$, where $[\nu_2 = N(T-K-1)]$ degrees of freedom. If the null hypothesis is true, then we calculate the F-statistic given by :

$$F_1 = \frac{\frac{\left[\text{RSS}_{\text{pooled},r_1} - \text{RSS}_{1,\nu_2}\right]}{\nu_1}}{\frac{\text{RSS}_{1,\nu_2}}{\nu_2}}$$

And we compare it to the tabulated value $F_{0.05;\nu_1;\nu_2}$. If $F_1 < F_{0.05;\nu_1;\nu_2} \approx 1.51$, then we accept H_0^1 and by consequence we obtain a homogeneous panel data model. If we reject H_0^1, we move on to the second step which consists of determining whether the heterogeneity comes from the coefficients $\mathcal{B}^i$ or not.

Second test :

$$\begin{cases} H_0^2 : \mathcal{B}^i = \mathcal{B}, \ \forall i = 1, N \\ H_a^2 : \exists (i, j) \in [1, N]; \ \mathcal{B}^i \neq \mathcal{B}^j \end{cases}$$

In this test, no restriction is imposed on the parameters $(\beta_{0,i}, \ i = 1, N)$. Using the so-called method (Within estimation) we obtain the residual sum of squares $\text{RSS}_{(\text{pooled},r_2)}$, where r_2 designates the restrictions under H_0^2 with restrictions $[\nu_1 = K(N-1)]$. Like our path in

the first test, by the same method we estimate a model for each country and we retain the $(\mathrm{RSS}_2 = \sum_{i=1}^{N} \mathrm{RSS}^i_{\mathrm{within}})$, with $[\nu_2 = (NT - (K+1)N)]$ as degrees of freedom. Under the null hypothesis H_0^2, we calculate the following F-statistic :

$$F_2 = \frac{\frac{\left[\mathrm{RSS}_{\mathrm{within},r_2} - \mathrm{RSS}_{2,\nu_2}\right]}{\nu_1}}{\frac{\mathrm{RSS}_{2,\nu_2}}{\nu_2}}$$

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If $F_2 > F_{0.05;\nu_1;\nu_2} \approx 1.63$, then we reject the panel structure and, by consequence, the estimated vector $\mathcal{B}^i$ will be made for the countries one by one. If we accept H_0^2, then we retain the panel structure and we then seek to determine in a third step if the coefficients $(\beta_{0,i})$ have an individual dimension.

Third test :

$$\begin{cases} H_0^3 : \beta_{0,i} = \beta_0, \ \forall i = 1, N \\ H_a^3 : \exists (i,j) \in [1, N]; \ H_a^3 : \beta_{0,i} \neq \beta_{0,j} \end{cases}$$

Under H_0^3, we impose $(\mathcal{B}^i = \mathcal{B}, \ \forall i = 1, N)$. There will be available $(\nu_1 = (N-1))$ of restrictions. Under H_a^3, the $(\mathcal{B}^i, i = 1, N)$ are the same, but the $\beta_{0,i}$ differs according to the countries. Using the Pooled estimation method, we guarantee $\mathrm{RSS}_{\mathrm{pooled},r_3}$ and using Within Estimation Method, we retain $\mathrm{RSS}_3 = \mathrm{RSS}_{2,r_2}$, with $(\nu_2 = NT - N - K = N(T-1) - K)$ as degrees of freedom. Under the null hypothesis H_0^3, we calculate the following F-statistic :

$$F_2 = \frac{\frac{\left[\mathrm{RSS}_{3,r_3} - \mathrm{RSS}_{2,r_2}\right]}{\nu_1}}{\frac{\mathrm{RSS}_{2,r_2}}{\nu_2}}$$

If $F_3 > F_{0.05;\nu_1;\nu_2} \approx 1.90$, then we reject H_0^3 and we get a panel model with individual effects. Contrariwise, if we accept H_0^3, we retrieve an homogeneous panel data model. The findings of the three tests above are given in the following Table (1) :

Table 1: Homogeneous or Heterogeneous panel : Fisher's test

Null hypothesis	Primary data$^{(1)}$	Data in logarithm$^{(2)}$	Data in first difference$^{(1)}$	Data in first difference$^{(2)}$
H_0^1	253.76^r	346.00^r	2.209^r	0.705^a
H_0^2	45.25^r	23.14^r	2.532^r	0.249^a
H_0^3	45.25^r	23.14^r	2.532^a	0.249^a

^{r} rejection the null hypothesis at a 5% significant level.
^{a} acceptance the null hypothesis at a 5% significant level.

This change in the responses of the $H_0^i, \ i = 1, 2, 3$ tests led us directly to examine the Panel Unit Root Tests.

IV. Panel Unit Root Tests (PURT)

Over the past two decades, research was carried out on the PURT. The first generation of testing was demonstrated by Levin and Lin (1992) as working papers at the University of

California, and then they were published by Proceeding with the application of LL (or LLC) technique. It is important to draw the reader's attention to the importance of individual and temporal dimensions in the unit root study of a panel data. The co-integration tests for short-time series are known to be inefficient in distinguishing between stationary and non-stationary time series. The issue of co-integration is complicated especially if the time series experienced a rupture in the trend. This is true of the time series associated with the exchange rates if we examine them over a period of time before and after the cancellation of the Bretton-Woods system. Therefore, the experts of econometrics propose to study a number of countries benefiting from the information related to each country, which contributes to the establishment of a broad analysis in the long and short run. Hence, the adoption of the panel data will provide a more objective analysis of the acceptance or rejection of the null hypothesis of co-integration, while we cannot do it at the level of each country separately. Another advantage of the use of the panel data, in both time and individual dimensions, is that the unit root test follows a normal distribution, while the latter is not available in the time dimension study alone. The researcher must move from the unit root test in a single time series to several multi-time series ; and therefore DF, ADF, PP, KPSS and the modified version of the DF test proposed by Elliott and Stock (1996) will need improvement to deal with the time and individual dimensions of the time series. Thus, even if the size of a time series is small for each country, the increase in the number of countries will increase the total number of observations and thus avoid falling into the rupture of trend. See a recent study Jaunky and Lundmark (2017).

Comparing the two approaches with the unit root, the traditional approach that takes only the time dimension and the approach that takes the time and individual dimensions Hurlin and Mignon (2006) reveals two fundamental differences :

- The first is related to the non-standard asymptotic distribution in the time dimension and how it varies with constant and/or trend in the deterministic component. In the case of panel data, the unit root tests will follow the normal distribution except for Fischer tests. This is a fundamental difference between the two approaches, and these normal distributions will remain conditionally related to the deterministic components of the model used.
- The second difference would be the possibility of heterogeneity among individuals in the case of panel data, whereas there is no such possibility in the case of time dimension.

Through this observation, we come to the following question : Can we use the same model in the case of time series for one individual, i.e. with only a time dimension, and in the case of a panel data ? If yes, this means that we have assumed a homogeneity among individuals in terms of dynamic characteristics and their consequences for the stationarity or non-stationarity of the time series.

The random use of the same model to test the unit root on all individuals will often lead to spurious results. Therefore, we must first resolve the issue of heterogeneity among individuals before embarking on testing the panel unit root. This is secured through what we have studied in the previous section. There are many tests in literature review that talk about testing the panel unit root, where the most famous of the first generation are Levin and Lin (1992); Harris and Tzavalis (1999); Maddala (1999); Maddala and Wu (1999), (λ test Breitung (2000)), (Lagrange multiplier (LM) test Hadri (2000)), Levin et al. (2002)

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and Im et al. (2003). The heterogeneity also affects the alternative hypothesis in terms of the panel unit tests. If we study the GDP per capita in several countries over a certain period of time, we can accept the existence of the unit root in a group and reject it in other countries. Given the importance of studying the heterogeneity among individuals, it was necessary to ask this question : Is it logical to consider the null hypothesis of cross-sectional independence among individuals ? This hypothesis is considered as a nuisance parameter for the researchers. Thus rejecting it suggests the use of appropriate new panel unit root test as a second generation. Many tests have been proposed, for example, Choi (2001); Breuer et al. (2002); Phillips and Hansen (1990); Chang (2002); Moon and Perron (2004); J. and S (2004); Breitung and Das (2005); Hurlin and Mignon (2006) and the most famous test is the Cross-Sectionally Augmented IPS (CIPS) Panel Unit Root proposed by Pesaran (2007). For more information about these tests see (Mourad (2019), p. 255-288). In our approach to the PURT, we will limit ourselves to the four tests : IPS, HT, λ and CIPS.

The critical values of the HT technique change with the change of the deterministic i.e. without intercept, with intercept only, and with intercept and trend. If we investigate Tables 1a, 1b and 1c of Harris and Tzavalis (1999) (p. 211, 212 and 213 respectively), we find by interpolation the critical values of the Z statistic as shown in the following table :

For λ and IPS, the output statistics are compared to the $1\%, 5\%$, and 10% significance levels with the one-tailed (negative) of a standard Normal with the critical values of (-2.326),

	Significance levels $T = 47,\ N = 10$		
Critical values	1%	5%	10%
Without intercept	−3.96	−2.41	−1.79
Intercept only	−3.15	−2.09	−1.65
Intercept + trend	−2.82	−1.97	−1.54

(-1.645), and (-1.282) correspondingly. For the CIPS test, the critical values of average of individual cross-sectionally Augmented Dickey-Fuller are around $-2.55, -2.33$ and -2.21 at $1\%, 5\%$ and 10% significant levels (constant only) respectively (Source : Table 3b-Pesaran (2007)).

Cross section dependence (CD) test :

From the findings in Table (1), for each variable, we have decided the measurement of the correlation coefficients among the **Cross-sectional analysis** using the ADF at order $(p = 3)$ in the level :

$$\Delta X_{it} = \alpha_i + \gamma_i X_{it-1} + \sum_{k=1}^{p-1} \varphi_{ik} \Delta X_{i,t-k} + \epsilon_{it} \tag{2}$$

Retaining the estimate residues $(e_{it},\ t \geq 4)$ for each section, for each country, we calculate the Pearson's correlation coefficient $\hat{\rho}_{ij}$ (correlations between panel units) :

$$corr(e_{it}, e_{jt}) = \hat{\rho}_{ij} = \frac{\sum_{t=1}^{T} e_{it}e_{jt}}{\sqrt{\left(\sum_{t=1}^{T} e_{it}^2\right)^2 \times \left(\sum_{t=1}^{T} e_{jt}^2\right)^2}}, \quad i, j = 1, \ldots, N \text{ and } i \neq j. \tag{3}$$

If $\hat{\rho}_{ij} = 0, \forall t, i \neq j$, then there is no correlation between e_i and e_j. For that, we test the null hypothesis H_0 versus the alternative H_a :

$$\begin{cases} H_0 : \text{there is no correlation between } e_i \text{ and } e_j \\ H_a : \text{there is correlation between } e_i \text{ and } e_j \end{cases}$$

By consulting the literature review Rafael E. De Hoyos and Sarafidis (2006) and (Mourad (2019), p. 355-357), if N is relatively small and T is large enough, it is possible to estimate the model above using the OLS method and saving the associated residues as proposed Breusch and Pagan (1980), and then we calculate the statistic :

$$\mathrm{CD_{BP}} = T \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\rho}_{ij}^2$$

$$\mathrm{CD_{BP}} \xrightarrow[T\to\infty]{} \chi^2_{\frac{N(N-1)}{2};N=10}$$

df	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.01$
45	57.505	61.656	69.957

One disadvantage of this test is that it is inappropriate when N is large ($N \to \infty$). To treat better the cross-sectional analysis, Pesaran (2004) proposed the following statistic :

$$\mathrm{CD_P} = \sqrt{\frac{2T}{N(N-1)}} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \hat{\rho}_{ij} \Rightarrow \mathrm{CD_P} \xrightarrow[T\to\infty, N\to\infty]{} N(0,1),$$

with the two-tailed of a standard normal, the critical values are (1.96), (2.58), and (3.29) for $10\%, 5\%$ and 1% respectively.

In Table (2), for each variable both $\mathrm{CD_{BP}}$ and $\mathrm{CD_P}$ statistics are highly significant and suggest to take into account highly correlated countries.

Table 2: Testing for cross-section dependence in panel

	$\mathrm{CD_{BP}}$		$\mathrm{CD_P}$	
Variables	X	ΔX	X	ΔX
Y	109.19^a	115.38^a	6.12^a	07.05^a
X_1	170.93^a	187.23^a	10.17^a	10.26^a
X_2	122.60^a	144.43^a	04.53^a	05.21^a

^{a}, ^{b} and ^{c} indicate that the test is significant at $1\%, 5\%$ and 10% significant levels respectively.

Notes

CIPS test :

Pesaran suggests the following equation :

$$\text{CADF}_{p_i} : \Delta Y_{it} = a_i + b_i Y_{it-1} + c_i \bar{Y}_{t-1} + \sum_{j=0}^{p_i} d_{ij} \Delta \bar{Y}_{t-j} + \sum_{j=1}^{p_i} \delta_{ij} \Delta Y_{it-j} + e_{it}, \tag{4}$$

Notes

where $\bar{Y}_t = N^{-1} \sum_{j=1}^{N} Y_{jt}; \bar{Y}_{t-k} = N^{-1} \sum_{j=1}^{N} Y_{jt-k}, \bar{Y}_0 = N^{-1} \sum_{j=1}^{N} Y_{j0}$ and Y_{j0} is fixed or random, considering that the data generating process (DGP) is a simple dynamic linear heterogeneous panel data model. This test is entitled also *Cross-sectionally Augmented version of the IPS Panel Unit Root Test* entitled (CIPS), which is a simple average of the individual CADF-tests. In fact, practically, we maintain the t-statistics $\widehat{\text{CADF}}_{p_i}$ of the estimate parameters $(\hat{b}_i, i = 1, \ldots, N)$ then we calculate CIPS $= \overline{\text{CADF}} = \frac{1}{N} \sum_{i=1}^{N} \widehat{\text{CADF}}_{p_i}$.

The findings in Table (3) reveal that all variables are stationary in first difference.

V. Panel Co-Integration Tests of Pedroni

In this section, the methodology carried out by Pedroni (1995, 1996, 1999, 2001, 2004, 2007) will be used. The use of co-integration techniques to test the presence of long run

Table 3: Panel unit root test-Data in natural logarithm Individua l Specific Components : Constant Average p chosen from 3 by AIC

	HT		λ		IPS		CIPS	
Variables	X	ΔX	X	ΔX	X	ΔX	X	ΔX
Y	0.40	-42.3^a	2.48	-9.05^a	-1.87^b	-7.73^a	-2.45^b	-4.31^a
X_1	1.18	-43.46^a	4.66	-10.90^a	-1.45^c	-8.51^a	-2.01	-3.88^a
X_2	1.53	-45.95^a	5.71	-07.14^a	1.12	-11.0^a	-1.72	-4.29^a

^{a}, ^{b} and ^{c} indicate that the test is significant at $1\%, 5\%$ and 10% significant levels respectively.

relationships among integrated variables has enjoyed growing popularity in the empirical literature (see Mourad (2019) and Mourad (2018a,b)). In this section, we focus on the long-run relationship which could exist between the military expenditures per capita $(Y_{i,t})$ as a dependent variable in ten counties from $1968 - 2014$ and two independent variables GDP per capita $(X_{1i,t})$ and CO2 emissions per capita $(X_{2i,t})$. The Pedroni procedure will be used respecting the following steps (see Mourad (2019), p. 296-301).

Briefly, we consider the hypothesized long-run regression between the dependent variable Y_{it} and two independent variables $(M = 2)$ as the following :

$$Y_{it} = \alpha_i + \beta_{1i} X_{1i,t} + \beta_{2i} X_{2i,t} + e_{it}, \ i = 1, \ldots, 10, \ t = 1, \ldots, T. \tag{5}$$

$$Y_{it} = \alpha_i + \mathcal{X}'_{it} \mathcal{B} + e_{it}, \ \mathcal{X}_{it} = (X_{1i,t}, X_{2i,t}), \ \mathcal{B} = (\beta_{1i}, \beta_{2i})'$$

Assuming that there is a homogeneity of the parameters of the long run relationship, i.e. $\mathcal{B}_i = \mathcal{B}, \ \forall i = 1, \ldots, N$. In the equation (5), T is the number of observations over time and N denotes the number of individual members in the panel. It is quite clear to assume that the slope coefficients $(\beta_{ij}, \ j = 1, 2)$ and the member specific intercept α_i can vary across each cross-section. In the equation (5), we have adopted a regression equation with

a heterogeneous intercept. Note that it could also be estimated without a heterogeneous intercept, or with time trend and/or common time dummies. By OLS method, we estimate the model in the equation (5) and we save the residues $\hat{e}_{it}$. Using the estimate residuals $\hat{e}_{it}$ in (5) to estimate the model :

$$\hat{e}_{it} = \gamma_i \hat{e}_{it-1} + u_{it} \tag{6}$$

Pedroni suggests the nearest integer $k_i = 4\left(\frac{T}{100}\right)^{2/9}$ as truncation lag parameter for the Newey-West kernel estimator recommended in Newey and West (1994). His tests take into account the heterogeneity through the parameters that may differ between individuals. Such heterogeneity can be located at both the long-run regression i.e. the co-integration relations, and the short-run dynamics. Pedroni accepts the null hypothesis of no intra-individual co-integration for both homogeneous and heterogeneous panels. Thus, under the alternative hypothesis, exists a co-integration relation which is specific for each individual. He proposes seven statistics, four of which are based on the within-dimension and three on the between-dimension. Statistically speaking, for all tests, the null hypothesis of no co-integration is :

$$\mathrm{H}_0 : \gamma_i = 1, \ \forall i = 1, \ldots, N,$$

where the parameter γ_i is estimated in (6). Whereas the alternative hypothesis changes according to the within (intra) or between (inter) dimension vision.

In the within-dimension :

$$\mathrm{H}_a : \gamma_i = \gamma < 1, \ \forall i = 1, \ldots, N,$$

where γ is a common value. The alternate to no co-integration must be that if the individuals are co-integrated, then they will exhibit the same long run co-integrating relationships.

In the between-dimension :

$$\mathrm{H}_a : \gamma_i < 1, \ \forall i = 1, \ldots, N,$$

where a common value γ is not required. Under this alternative hypothesis, the individual cross sections contain co-integrating relationships that are free to take on different values for different members of the panel. In other words, we allow the presence of heterogeneity between individuals. Since it is rare in practice to find an identical co-integration vectors for all individuals, because a considered heterogeneity through parameters may differ among individuals.

VI. FMOLS and DOLS Estimators of the Long-Run Equilibrium

When the residues of the co-integration relationship are correlated with the innovations of regressors, then the ordinary least squares estimators (OLS) of the co-integration vector parameters are biased. This bias entitled as long-term endogeneity or a bias of the second order implies non-standard distributions of the main usual tests statistics. Given the evidence of panel co-integration, the long-run relationships between the different variables can be further estimated by several methods proposed in the literature, e.g. the Fully-Modified Ordinary Least Squares (FMOLS) which is a semi-parametric procedure suggested by Phillips and Hansen (1990); Phillips (1995); Pedroni (1995) and the dynamic OLS (DOLS) estimator proposed by Stock and Watson (1993); Kao and Chiang (2000); Mark and Sul (2003). In both cases, the FMOLS and DOLS procedures estimate both individual-specific cointegrating vectors and aggregated estimator.

Notes

Notes

FMOLS procedure :

The Fully Modified Ordinary Least Square (FMOLS) method is one of the methods that permits a correction of the long term endogenous bias particularly for the finite sample size. The idea is to bring a new representation of the co-integration relationship in which the residues verify well the orthogonality properties. In other words, the FMOLS regression estimates a linear regression, then it adjusts the estimates and covariance matrix for endogeneity Mourad (2019). When the individual dimension is sufficiently large and even for the short time series, the FMOLS estimator is consistent and it has a relatively well performance controlling the likely endogeneity of the regressors and serial correlation.

Table 4: Pedroni Panel co-integration Tests Results
Natural logarithm of data in deviations from time period means

Alternative hypothesis : Common AR coefficients (within-dimension)		
	Statistics	
Tests	Det = constant $p = 6$	Det = trend $p = 4$
Panel-ν statistic (non-parametric)	0.31	0.85
Panel ρ-statistic (non-parametric)	-1.04	-0.73
Panel pp-statistic (non-parametric)	-1.58^c	-1.99^b
Panel ADF-statistic (parametric)	-1.58^c	-1.90^b
Alternative hypothesis : Common AR coefficients (within-dimension)		
Group ρ-statistic (non-parametric)	-0.14	0.15
Group pp-statistic (non-parametric)	-1.46^c	-1.65^b
Group ADF-statistic (parametric)	-2.10^b	-1.39^c

The variance ratio test is right-sided, while the other Pedroni tests are left-sided. All reported values are distributed $N(0,1)$ under the null unit root or no co-integration. For the left-sided tests, the rejection of the null will take place in the left tail. The critical values are $-1.28, -1.64$ and 2.33 at 10%, 5% and 1% significance levels respectively.
Conclusion : The estimation proceeds on the basis that the demeaned series are co-integrated.
^{b} and ^{c} indicate the rejection of the null hypothesis of no co-integration on the 5% and 10% significance levels respectively.
Note 1 : The data have been demeaned with respect to common time effects to accommodate some forms of cross-sectional dependency, so that in place of y_{it}, x_{1it} and x_{2it}, we use : $\tilde{y}_{it} = Y_{it} - \bar{Y}_t$; $\bar{Y}_t = \frac{1}{N}\sum_{i=1}^{N} y_{it}$ and $\tilde{x}_{jit} = x_{jit} - \bar{x}_{jt}$; $\bar{x}_{jt} = \frac{1}{N}\sum_{i=1}^{N} x_{jit}$; $j = 1, 2$.
Note 2 : A variable on the right hand side (RHS) of your model may be endogenous. This endogeneity means that the explanatory variable is correlated with the model's error term. The correlation of a RHS variable with the error term means that OLS is neither unbiased nor consistent.
Note 3 : Kernel width = 4.

DOLS procedure :

The DOLS procedure consists in including lags and leads of the regressors in the long-run equilibrium relationship to eliminate feedback effects and endogeneity. This has the consequence of eliminating the correlations between the explanatory variables and residues. Thus in our case, we obtain :

$$\tilde{Y}_{it} = \alpha_i + \beta_{1i}\tilde{X}_{1i,t} + \beta_{2i}\tilde{X}_{2i,t} + e_{it}.$$

Let's consider

$$\tilde{X}_{i,t} = \left(\tilde{X}_{1i,t}, \tilde{X}_{2i,t}\right)' \text{ and } \mathcal{B}_i = \left(\beta_{1i}, \beta_{2i}\right)'.$$

If we choose truncation at $lagp$, we obtain :

$$\tilde{Y}_{it} = \alpha_i + \tilde{X}'_{it}\mathcal{B}_i + \sum_{s=-p}^{p} c_{is}^{1}\Delta\tilde{X}_{1i,t+s} + \sum_{s=-p}^{p} c_{is}^{6}\Delta\tilde{X}_{2i,t+s} + e_{it},\ \ i \in [1,10],\ \ t \in [1975, 2008].$$

The DOLS can very quickly exhaust the degrees of freedom in a data set. If we choose truncation at lag p, there are $2p+1$ added regressors in the differences for each right side endogenous variable, plus we lose $2p+1$ data points allowing for lags, leads and first differences. So with 47 observations per individual, the order $(p = 6)$ leaves us with 34 usable observations, and 29 regressors.

For all of the group mean FMOLS estimates and standard errors in Table (5), we have considered the case in which the data have been demeaned over the cross-sectional dimension in order to account for some of the likely cross-sectional dependence through common time effects. The FMOLS and DOLS group mean estimators for the panel as a whole provide credible estimates for the parameters using the RATS option (AVERAGE=sqrt) [5].

In Tables (5), we present the estimation results associated to the long-run equilibrium individually and aggregately according to the two methods FMOLS and DOLS.

VII. Evaluating Forecast Performance

The long-term relationships were estimated over the period $1968 - 2014$. We will make forecasts for the military expenditures per capita $(Y_{i,t})$ for the years $2015-2017$ and compare with the observed values that are available for these three years. To make these forecasts, we need the observed values for the variables $(X_{1i,t})$ and $(X_{2i,t})$. In fact the observations are available for $(X_{1i,t})$ and not available for $(X_{2i,t})$.

For this, we will predict the CO2 emissions per capita $(X_{2i,t})$ over the period $2015-2017$ and for each country taken separately using the ARIMA technique. The Augmented Dickey-Fuller test (ADF) will be used to test the null hypothesis which a unit root presents in a univariate time series in logarithm. The choice of the ADF order p was made by AIC ensuring that the residues behaved like a white noise for lags $6, 12$ and 18. In other words, the Ljung-Box statistic is significant for a level of 5% . The null hypothesis of unit root is tested against alternative of absence of unit root and the results are presented in Tables (6) and (7). The inspection of these tables shows the acceptance of the null hypothesis (there is a root unit in the level variables) but this hypothesis is rejected if we consider the data in first difference for both primary data and natural logarithm of data. For a sample size $(T = 50)$ and at 5% and 10% significance levels, the critical values for the ADF tests are respectively -2.93 and -2.60 for τ_μ test statistic (intercept only), -3.50 and -3.18 for τ_τ test statistic (intercept and trend). The findings of ADF tests support the idea of taking EMIPCA variables in first difference.

If we closely investigate the long-term equilibrium relationship for each country, we find different results for both FMOLS and DOLS techniques. For all 10 countries, Arab world, Israel, USA, Canada, Japan, South Korea, France, United Kingdom, India and Pakistan, we found when using FMOLS positive effect of GDPPCA on MEXPCA. Thus an increase of 1%

Notes

5. Considering the Manual RATS, AVERAGE=SQRT weights each individual by the diagonal matrix formed by taking the square roots of the precision matrix (inverse covariance matrix) of the estimates for that individual. This matches up with the averaging done in computing the t-statistics, that is, the coefficients and covariance matrix from AVERAGE=SQRT will reproduce the average t-statistics.

in the variable GDPPCA leads to an increase in MEXPCA of 1.22%, 2.77%, 0.99%, 1.14% and 1.02%, 0.27%, 1.13%, 0.65%, 0.87% and 0.66% respectively. However, using DOLS esti-

Table 5: Estimation of the long-run equilibrium

Country (i)	FMOLS			DOLS		
	Intercept	$\tilde{X}_{1it}$	$\tilde{X}_{2it}$	Intercept	$\tilde{X}_{1it}$	$\tilde{X}_{2it}$
Arab world	0.896	1.222	-0.192	0.172	0.967	-1.184
	(8.19)	(12.92)	(-0.97)	(3.55)	(29.80)	(-21.98)
Israel	0.803	2.768	-2.359	-0.400	4.697	-2.757
	(3.15)	(7.03)	(-4.59)	(-3.07)	(26.66)	(-31.35)
USA	0.536	0.991	-0.198	1.946	-0.574	0.291
	(1.59)	(2.46)	(-0.70)	(7.78)	(-1.80)	(2.07)
Canada	-0.849	1.139	-0.122	-1.111	1.397	-0.138
	(-6.32)	(5.79)	(-0.70)	(-18.66)	(12.72)	(-1.74)
Japan	-0.095	1.017	-2.367	-0.208	1.055	-2.293
	(-0.33)	(8.61)	(-6.13)	(-2.18)	(56.99)	(-12.58)
South Korea	-0.203	0.267	0.760	-0.950	-2.120	4.183
	(-2.41)	(1.02)	(2.29)	(-3.99)	(-3.02)	(3.58)
France	-0.263	1.129	-0.071	-0.703	1.558	-0.183
	(-3.23)	(14.56)	(-2.80)	(-16.87)	(39.40)	(-19.87)
United Kingdom	0.089	0.653	0.252	0.951	-0.037	-0.009
	(0.87)	(7.83)	(5.60)	(2.76)	(-0.15)	(-0.06)
India	0.254	0.867	0.406	1.494	1.268	0.435
	(0.90)	(10.91)	(6.07)	(12.94)	(55.90)	(12.37)
Pakistan	-0.925	0.660	-0.228	-2.723	-0.025	-0.301
	(-1.38)	(4.65)	(-1.22)	(-12.87)	(-0.72)	(-5.41)
Group	0.015	0.978	-0.030	-0.294	1.046	-0.345
	(0.32)	(23.96)	(-0.99)	(-9.68)	(68.23)	(-23.70)

mator, the effects will be positive and they are evaluated at 0.97%, 4.7%, 1.4%, 1.06%, 1.56% and 1.27% for each of Arab World, Israel, Canada, Japan, France, and India successively. Moreover, we observe a negative impact (significant at level 10%) evaluated at 0.57% for USA and for the other countries, but we did not notice a significant effect of GDPPCA on MEXPCA for United Kingdom and Pakistan.

Using both methods FMOLS and DOLS, the investigation of the impact of the EMIPCA variable on the MEXPCA variable revealed a negative effect for Arab World, Israel, Japan, France, and Pakistan. An increase of 1% in the variable EMIPCA leads to a decrease in MEXPCA of 0.19%(1.18%), 2.36%(2.76%), 2.37%(2.29%)0.07%(0.18%), and 0.3% (with DOLS only) respectively.

For South Korea, United Kingdom, and India, we found a positive impact of EMIPCA on MEXPCA using FMOLS estimator, so an increase of 1% in the variable EMIPCA leads

Notes

Table 6: The Augmented Dickey-Fuller test (ADF)-Primary data

Country (i)		Intercept		Intercept & trend	
	p	$X_{2i,t}$	$\Delta X_{2i,t}$	$X_{2i,t}$	$\Delta X_{2i,t}$
Arab world	5	-0.28	-4.28^a	-1.86	-4.22^a
Israel	3	-1.22	-3.89^a	-0.42	-4.02^a
USA	5	-0.63	-4.41^a	-1.68	-4.30^a
Canada	4	-2.08	-4.16^a	-2.44	-4.25^a
Japan	5	-1.08	-3.71^a	-2.43	-3.55^a
South Korea	5	-0.42	-3.46^a	-1.96	$-3.35b$
France	5	-0.75	-3.71^a	-2.03	-3.62^a
United Kingdom	5	0.38	-3.72^a	-1.62	-3.78^a
India	3	3.53	-3.11^a	1.05	-4.79^a
Pakistan	3	-0.65	-3.20^a	-3.33	-3.08

^{a,b} : the null hypothesis of a unit root is rejected at 5% and 10% significance levels respectively.

Table 7: The Augmented Dickey-Fuller test (ADF)-log-transformed data

Country (i)		Intercept		Intercept & trend	
	p	$X_{2i,t}$	$\Delta X_{2i,t}$	$X_{2i,t}$	$\Delta X_{2i,t}$
Arab world	5	-1.52	-3.99^a	-3.00	-4.01^a
Israel	3	-1.21	-4.26^a	-0.30	-4.43^a
USA	5	-0.56	-4.14^a	-1.63	-4.07^a
Canada	4	-2.08	-4.13^a	-2.43	-4.22^a
Japan	5	-1.08	-3.89^a	-2.53	-3.70^a
South Korea	4	-2.80	-3.13^a	-0.76	-4.30^a
France	5	-0.33	-3.47^a	-2.09	-3.43^a
United Kingdom	5	1.08	-3.26^a	-0.57	-3.52^a
India	3	1.00	-4.38^a	-1.80	-4.56^a
Pakistan	3	-0.69	-3.45^a	-3.06	-3.37^b

^{a,b} : the null hypothesis of a unit root is rejected at 5% and 10% significance levels respectively.

to an increase in MEXPCA of $0.76\%, 0.25\%$ and 0.41% respectively. However, we found a positive impact of EMIPCA on MEXPCA while using DOLS estimator, so an increase of 1% in the variable EMIPCA leads to an increase in MEXPCA of $0.29\%, 4.18\%$ and 0.44% for USA, South Korea and India respectively. For the other countries not mentioned, no significant effect of EMIPCA on MEXPCA was found.

Finally, for the group of 10 countries, both FMOLS and DOLs methods revealed a positive impact of GDPPCA on MEXPCA. In this way, a growth of 1% in GDPPCA leads to a growth in MEXPCA of 0.99% and 1.05% respectively. The variable EMIPCA revealed a negative

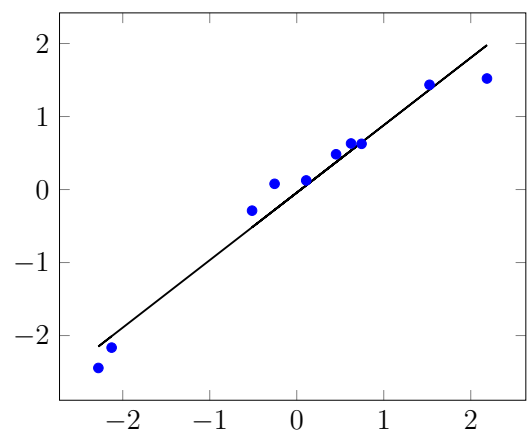
Notes

impact (0.35%) on MEXPCA if the DOLS estimator is used. Based on these results, it is necessary to choose between the two methods for each country calculating the forecasts for the period $2015-2017$, noting that the period data $1968-2014$ have been used to estimate the predictive models.

In the following, the forecasts will be calculated using the models estimated by FMOLS only. In fact with the DOLS method, we lose a lot of observations. Let's designate by P_t the forecasted value and A_t the real value of a time series. If $P_t = A_t$ then the forecasts are perfectly exact and the linear correlation coefficient between P_t and A_t is equal to 1. In the following, for each year on the period $2015-2017$, the forecasted and observed values associated with the ten countries are plotted and a simple regression model is estimated. If the coefficient of determination is near to 1, then the accuracy of the forecasts is considered as very good.

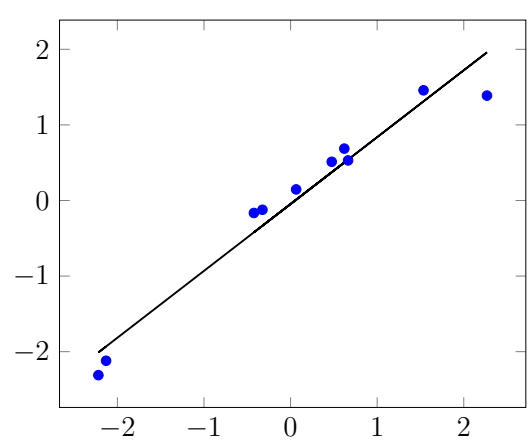
Notes

$$\left| \begin{array}{c} \hat{A}_t = 0.9242P_t - 0.0431 \\ \text{t-stat} \quad (15.37) \quad (-0.53) \\ R^2 = 0.9673 \quad \hat{\rho} = 0.9835 \end{array} \right|$$



Observed (A_t) vs. predicted (P_t) values : 2015.

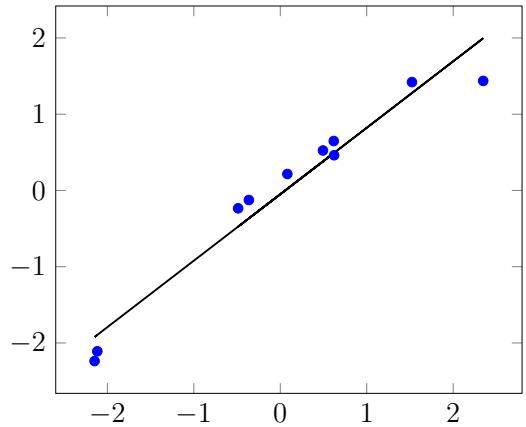
$$\left| \begin{array}{c} \hat{A}_t = 0.8847P_t - 0.0469 \\ \text{t-stat} \quad (13.20) \quad (-0.52) \\ R^2 = 0.9561 \quad \hat{\rho} = 0.9778 \end{array} \right|$$



Observed (A_t) vs. predicted (P_t) values : 2016.

Figure 1: Forecast accuracy for Military Expenditure per Capita.

$$\left| \begin{array}{c} \hat{A}_t = 0.8716 P_t - 0.0498 \\ \text{t-stat} \quad (12.78) \quad (-0.54) \\ R^2 = 0.9533 \quad \hat{\rho} = 0.9764 \end{array} \right|$$



Observed (A_t) vs. predicted (P_t) values : 2017.

VIII. Conclusion

The model estimated by FMOLS estimator reveals such interesting results. In fact, the use of this nonparametric estimator is justified comparing with the DOLS estimator leading to a significant reduce in the degree of freedom in the size of panel data. A positive effect of GDP per capita appeared for all countries except Korea. Indeed, if GDP per capita increases 1%, then the military expenditure increases 1.22%, 2.77%, 0.99%, 1.14%, 1.02%, 1.13%, 0.65%, 0.87% and 0.66% respectively for the Arab word, Israel, USA, Canada, Japan, France, UK, India and Pakistan. It should be noted that the Israel comes in the first place among this group of 10 countries with regard to an increase military expenditure resulting in an increase in the GDP per capita. Furthermore, we have observed an almost similar behaviour regarding the impact of GDP per capita on the military expenditure, a different behaviour notices CO2 emissions per capita and its impact on military expenditure has been observed. Indeed, we found a positive effect for Korea, UK, India i.e. if the CO2 emissions per capita increases 1%, then the military expenditure increases 0.76%, 0.25% and 0.41% respectively. However, a negative effect is revealed for Israel, Japan, France only, an increase of 1% reduces the military expenditure per capita of 2.36%, 2.37% and 0.071% respectively, signalling the DOLS estimator yields the same findings for this countries. For other countries, a nonsignificant effect was observed. For the entire group, there is a very positive effect only for GDP per capita, and a 1% increase in GDP increases the military expenditure per capita of 0.98%. The importance of our proposal and especially the suggested *ordering algorithm* (see Appendix B) is the optimal choice of countries to have a high degree of homogeneity to be together in an econometric study of the panel. The problem is not limited to military expenditure, but for each economic variable we can use this algorithm leading to a better choice of such a tuple. Econometrics without model is seen for us as an approach of great interest to help the econometrics with model to better choose the panels by gaining homogeneity and consequently in the planning especially the predictions.

Appendix A

Table 8: Military Expenditure per Capita-Primary data

	Arab world	Israel	USA	Canada	Japan	Korea	France	UK	India	Pakistan
Average	168	1361	1117	307	225	256	580	559	14	25
Std.	100	494	578	152	148	205	283	288	10	10
Min	16	287	361	84	12	9	113	100	3	7
Max	472	2249	2283	623	475	740	1034	1076	39	46
CAGR	0.0757	0.0458	0.0345	0.0389	0.0782	0.1005	0.0463	0.0491	0.0576	0.0361

Notes

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Table 9: Military Expenditure per Capita-Data in logarithm

	Arab world	Israel	USA	Canada	Japan	Korea	France	UK	India	Pakistan
Average	4.90	7.13	6.87	5.59	5.01	5.03	6.19	6.14	2.38	3.11
Std.	0.79	0.48	0.58	0.57	1.09	1.24	0.66	0.70	0.70	0.46
Min	2.80	5.66	5.89	4.43	2.45	2.20	4.73	4.61	1.10	1.97
Max	6.16	7.72	7.73	6.43	6.16	6.61	6.94	6.98	3.67	3.82
CAGR	0.0173	0.0068	0.0050	0.0072	0.0193	0.0242	0.0079	0.0085	0.0266	0.0122

Table 10: GDP per Capita-Primary Data

	Arab world	Israel	USA	Canada	Japan	Korea	France	UK	India	Pakistan
Average	2525	14556	26285	21945	24122	9064	20819	20769	486	501
Std.	2018	10126	15791	14704	15418	8467	13350	15178	410	336
Min	223	1648	4696	3411	1451	198	2532	1896	99	100
Max	7509	37540	54697	52497	48603	27811	45334	50134	1576	1317
CAGR	0.0793	0.0703	0.0548	0.0604	0.0736	0.1134	0.0635	0.0722	0.0620	0.0488

Table 11: GDP per Capita-Data in logarithm

	Arab world	Israel	USA	Canada	Japan	Korea	France	UK	India	Pakistan
Average	7.50	9.28	9.94	9.75	9.72	8.37	9.67	9.56	5.89	6.01
Std.	0.91	0.86	0.75	0.76	1.03	1.50	0.84	0.99	0.76	0.66
Min	5.41	7.41	8.45	8.13	7.28	5.29	7.84	7.55	4.59	4.61
Max	8.92	10.53	10.91	10.87	10.79	10.23	10.72	10.82	7.36	7.18
CAGR	0.0109	0.0077	0.0056	0.0062	0.0081	0.0144	0.0067	0.0077	0.0103	0.0079

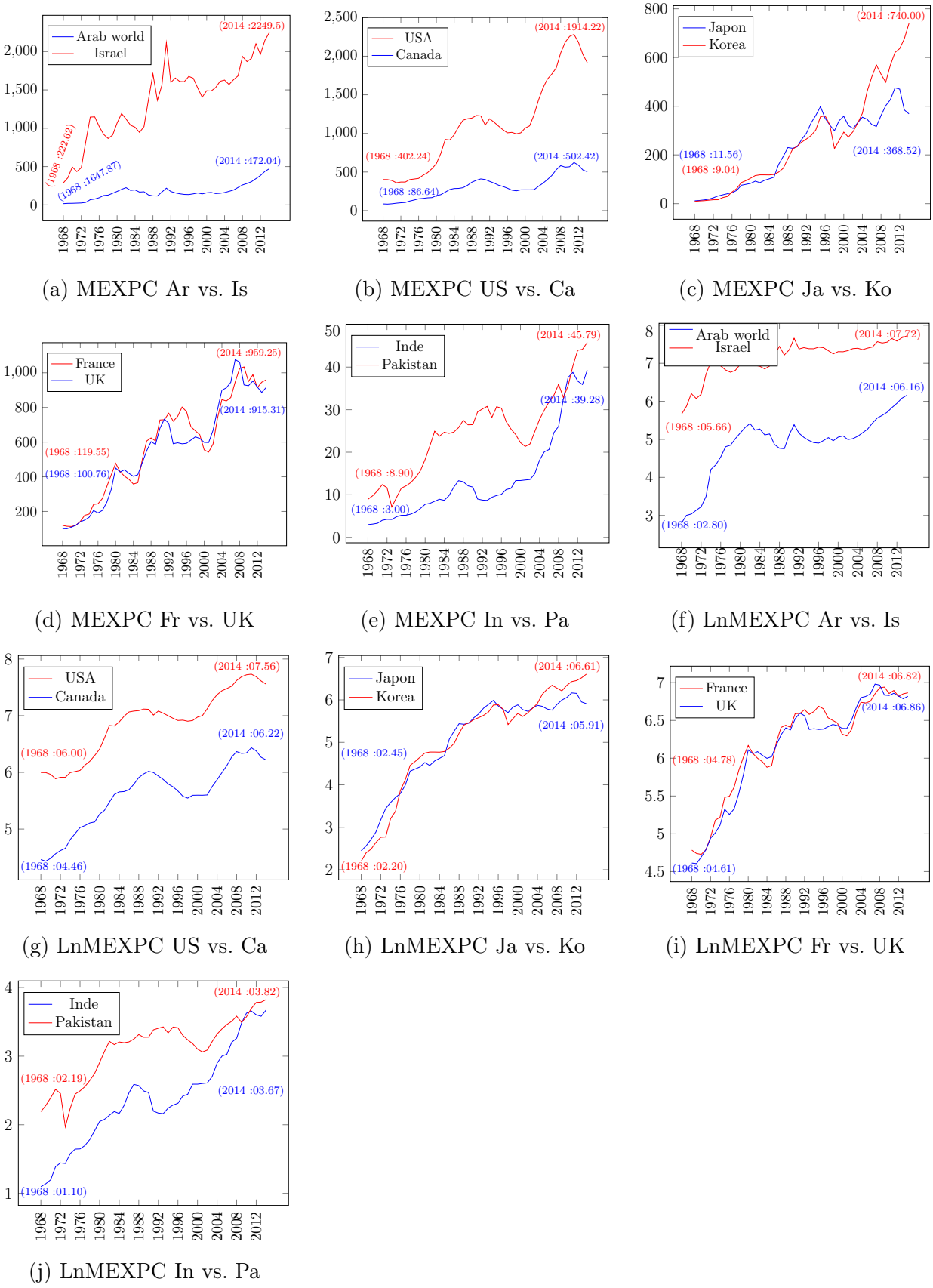
Table 12: Emission CO2 per Capita-Primary Data

	Arab world	Israel	USA	Canada	Japan	Korea	France	UK	India	Pakistan
Average	3.28	7.41	19.51	16.41	8.61	6.44	6.92	9.61	0.79	0.63
Std.	0.84	1.68	1.44	0.89	1.00	3.47	1.41	1.28	0.39	0.22
Min	1.54	4.93	16.30	14.62	5.57	1.21	4.57	6.50	0.35	0.31
Max	4.86	9.88	22.51	18.21	9.91	11.80	9.67	11.82	1.73	0.99
CAGR	0.0252	0.0102	-0.0032	0.0007	0.0118	0.0504	-0.0107	-0.0114	0.0351	0.0151

Table 13: Emission CO2 per Capita- Data in logarithm

	Arab world	Israel	USA	Canada	Japan	Korea	France	UK	India	Pakistan
Average	1.15	1.98	2.97	2.80	2.15	1.68	1.91	2.25	-0.35	-0.53
Std.	0.28	0.23	0.08	0.05	0.12	0.67	0.20	0.14	0.49	0.37
Min	0.43	1.59	2.79	2.68	1.72	0.19	1.52	1.87	-1.04	-1.18
Max	1.58	2.29	3.11	2.90	2.29	2.47	2.27	2.47	0.55	-0.01
CAGR	0.0285	0.0056	-0.0011	0.0003	0.0059	0.0573	-0.0061	-0.0054	(a)	(a)

(a) : The data in natural logarithm contains negative values.



(a) MEXPC Ar vs. Is (b) MEXPC US vs. Ca (c) MEXPC Ja vs. Ko

(d) MEXPC Fr vs. UK (e) MEXPC In vs. Pa (f) LnMEXPC Ar vs. Is

(g) LnMEXPC US vs. Ca (h) LnMEXPC Ja vs. Ko (i) LnMEXPC Fr vs. UK

(j) LnMEXPC In vs. Pa

Figure 2: Military Expenditures per Capita-Primary Data & Data in logarithm.

Notes

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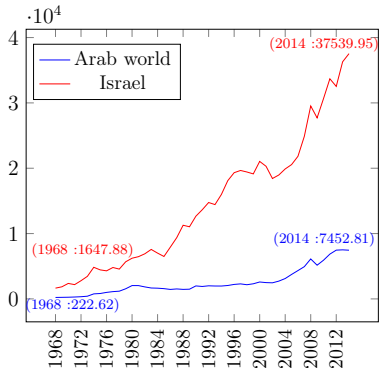
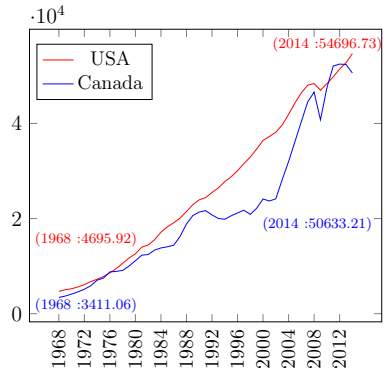
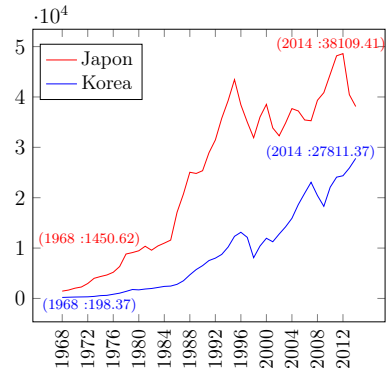


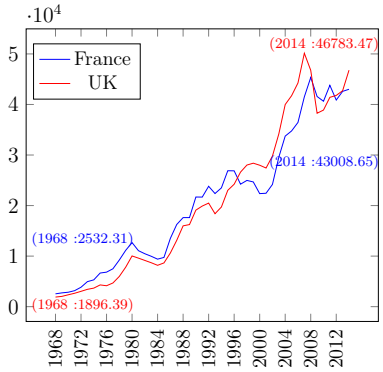
(a) GDPPC Ar vs. Is



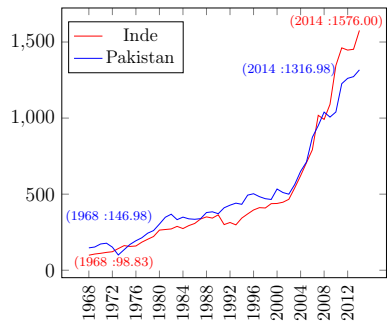
(b) GDPPC US vs. Ca



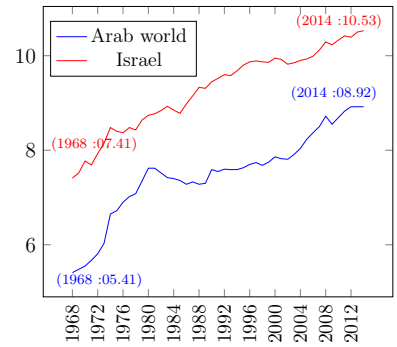
(c) GDPPC Ja vs. Ko



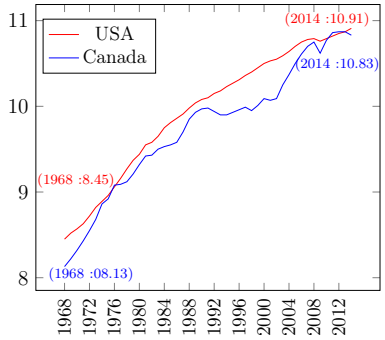
(d) GDPPC Fr vs. UK



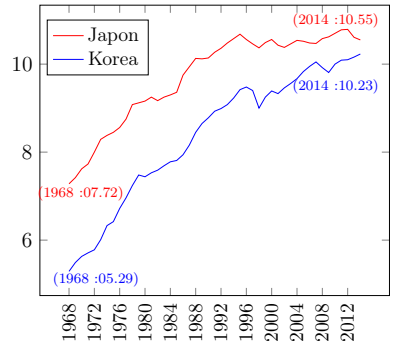
(e) GDPPC In vs. Pa



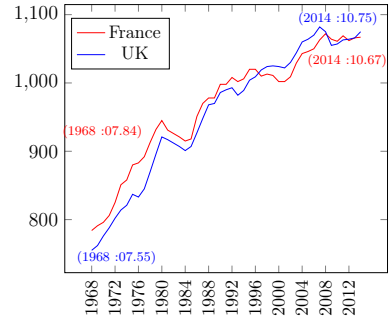
(f) LnGDPPC Ar vs. Is



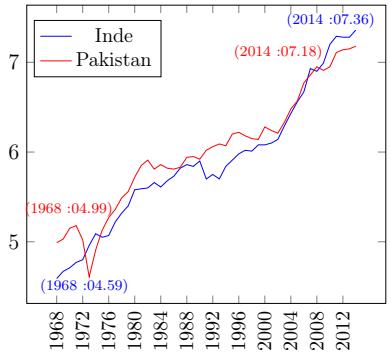
(g) LnGDPPC US vs. Ca



(h) LnGDPPC Ja vs. Ko

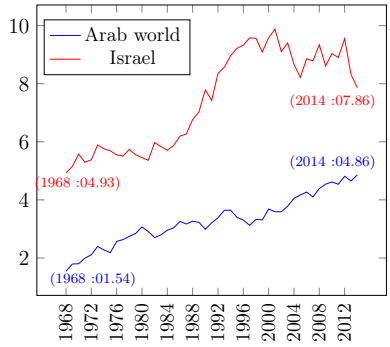


(i) LnGDPPC Fr vs. UK

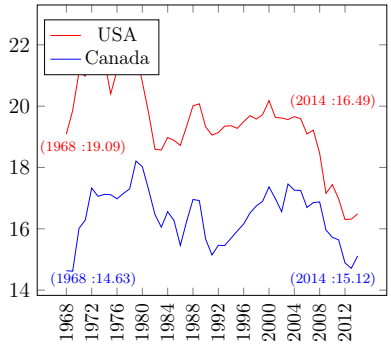


(j) LnGDPPC In vs. Pa

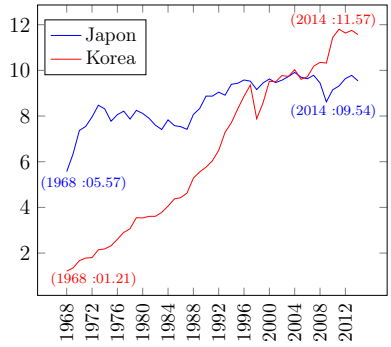
Figure 3: GDP per Capita- Primary Data & Data in logarithm.



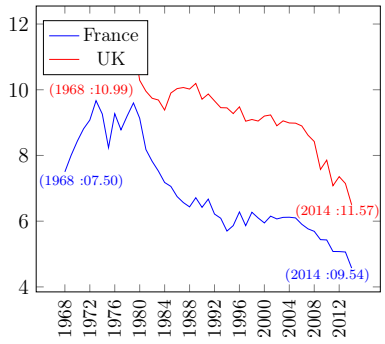
(a) EMIPC Ar vs. Is



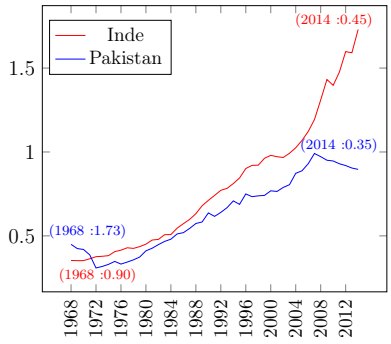
(b) EMIPC US vs. Ca



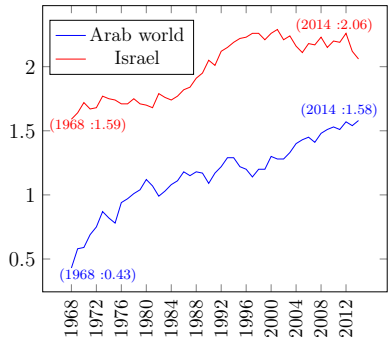
(c) EMIPC Ja vs. Ko



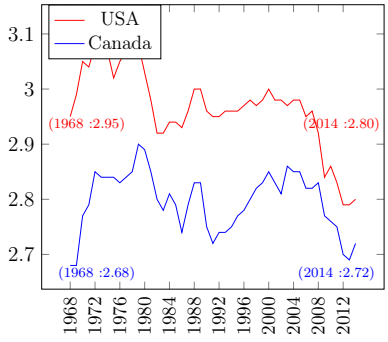
(d) EMIPC Fr vs. UK



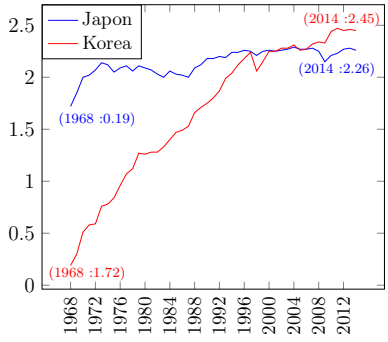
(e) EMIPC In vs. Pa



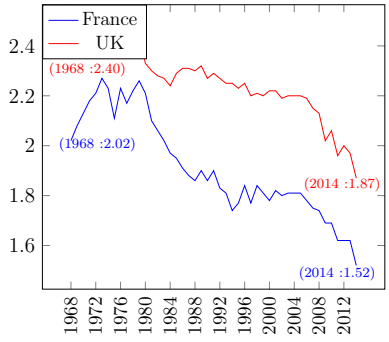
(f) LnEMIPC Ar vs. Is



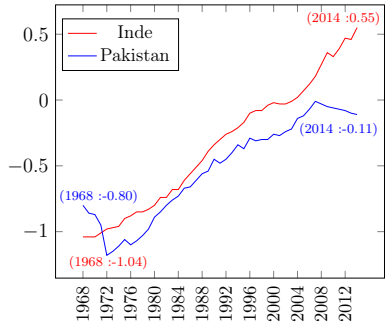
(g) LnEMIPC US vs. Ca



(h) LnEMIPC Ja vs. Ko



(i) LnEMIPC Fr vs. UK



(j) LnEMIPC In vs. Pa

Figure 4: CO2 Emissions per Capita - Primary Data & Data in logarithm

Notes

Appendix B

Notes

Large data tables usually contain a large amount of information, which is partly hidden because the data are too complex to be easily interpreted. Principal Component Analysis (PCA) is a projection method that helps to extract the important information from the statistical data to represent it as a set of new orthogonal variables called *principal components*. So, in this way, the first principal component retains maximum variation that was present in the original components. The principal components are the *eigenvectors* of a covariance matrix, and hence they are orthogonal. The eigenvectors determine the directions of the new feature space, and the *eigenvalues* determine their magnitude. In other words, the eigenvalues explain the variance of the data along the new feature axes.

How do we choose the order of the countries using a PCA analysis? When we talk about a panel of several individuals, we do not give an order of individuals but we discuss the different topics of the analysis, such as the unit root tests, the individual and global co-integration, i.e. for the set of panel. After the estimation of residuals due to the Augmented Dickey-Fuller (ADF) equation and after using the Cross-sectional Dependence (CD) test, a strong dependence appeared between cross residuals and validated by CD_P and CD_{BP}. But this work did not take into account the order of the countries in the panel. Often, the PCA users look for hidden factors in a time series without respecting the order. In order to explain and understand the military expenditure per capita for a country, certainly we can consider many hidden factors to discriminate countries because the hidden factors are not the same. Cavatorta (2010) considers four factors that influence the military expenditure.

For example, in 2017, the United States, alone, has an annual military expenditure about 35.85% as a share of the total military expenditure of the world and its occupations go beyond its national borders to reach the whole planet. So there is a significant number of hidden factors, including GDP! It is not only the protection of its territory but the domination of the entire continent. A country like Singapore, for example, has military expenditure, but it is to protect its national achievements, especially economic development and the social welfare system. Briefly, the number of hidden factors is relative to each country. In other words, there is a *country effect* in the military expenditure. This effect would be fixed or not, and it needs a specific analysis considering the place of the country in the proposed ordering.

Since a factor reflects a criterion of homogeneity between individuals, and since we have fixed two explanatory variables (GDP per capita and CO2 emission per capita) to explain military expenditure, we dedicate this section to the PCA analysis applied on the cross residual data of the military expenditure, for 10 countries form 1968 to 2014, due to the ADF equation used in Section 5 by proposing an algorithm of the order for the panel components that we call *Ordering Algorithm*. In this section, the PCA method was used to extract a fixed number of components (two) and the computation of each ordering presented. The power to detect the heterogeneity i.e. how the two factors measure the two common factors among the ten countries. Without doubt, the two factors do not explain military expenditures in the same way, because it can have four factors for one country but only one factor for another.

The proposed Ordering Algorithm is a sequential procedure based on five steps. In this Algorithm, ${}^{i}_{j}\lambda_{max}$ (rsp. ${}^{i}_{j}\lambda_{min}$) represents the max (rsp. min) value of the Initial Eigenvalues-Cumulative Percentage Variance (CPV) for j-tuple of ordering i. Each eigenvalue represents the amount of variance in the original variables accounted for by each component. The Percentage of Variance is the ratio, expressed in percentage, of the variance accounted for by each component to the total variance in all of the variables. Finally, the Cumulative Percentage Variance (CPV) gives the percentage of variance represented by the first 2 components. The

Initial Eigenvalues-CPV presented in Tables 1 to 10 are computed by using the SPSS (Statistical Package for Social Sciences) program. The Ordering Algorithm involves the following five steps :

- **Step 1 :** We take the countries two by two with order, then we calculate ${}^{1}_{2}\lambda_{max}$ and ${}^{1}_{2}\lambda_{min}$ associated with all the 2-tuple ; knowing that, in this step will be only one factor. Then we calculate ${}^{1}_{2}\lambda_{max}$ and ${}^{1}_{2}\lambda_{min}$ for the nine 2-tuple. We choose the 2-tuple corresponds to ${}^{1}_{2}\lambda_{max}$.

- **Step 2 :** Now, we set the number of factors to two. We take the couple chosen in step 1 then we introduce the remaining eight countries one by one. We calculate ${}^{1}_{3}\lambda_{max}$ and ${}^{1}_{3}\lambda_{min}$ for the eight 3-tuple. We choose the 3-tuple corresponds to ${}^{1}_{3}\lambda_{max}$.

- **Step 3 :** For each j-tuple, $j = 3, \ldots, 10$, we obtain the couples $({}^{1}_{j}\lambda_{min}, {}^{1}_{j}\lambda_{max})$.

- **Step 4 :** We repeat the same previous steps to choose the couple $({}^{i}_{j}\lambda_{min},\ {}^{i}_{j}\lambda_{max})$, for $i = 1, \ldots, 10;\ j = 3, \ldots, 10$, associated to the j-tuple from each i-ordering.

- **Step 5 :** We calculate the associated average of ${}_{j}\lambda_{min}$ and ${}_{j}\lambda_{max}$ by

$$ {}_{j}\bar{\lambda}_{min} = \frac{1}{10}\sum_{i=1}^{10} {}^{i}_{j}\lambda_{min} \text{ and } {}_{j}\bar{\lambda}_{max} = \frac{1}{10}\sum_{i=1}^{10} {}^{i}_{j}\lambda_{max}. $$

Algorithm 1: Computation of $\left\{ {}_{j}\bar{\lambda}_{min},\ {}_{j}\bar{\lambda}_{max} \right\}_{j\in\{3,\ldots,n\}}$.

Data: The considered n-countries, the Initial Eigenvalues-CPV ${}^{i}_{j}\lambda$ and the couple $({}^{i}_{j}\lambda_{min},\ {}^{i}_{j}\lambda_{max})$, for $i \in \{1, \ldots, n\}$ and $j \in \{3, \ldots, n\}$.

Result: $\left\{ {}_{j}\bar{\lambda}_{min},\ {}_{j}\bar{\lambda}_{max} \right\}_{j\in\{3,\ldots,n\}}$.

```
begin
    for i = 1 to n do
        ^i_j λ_min   0, ^i_j λ_max   0, _j λ̄_min   0, _j λ̄_max   0
        for j = 3 to n do
            ^i_j λ_min = min(^i_j λ_1, ..., ^i_j λ_{n-j+1})
            ^i_j λ_max = max(^i_j λ_1, ..., ^i_j λ_{n-j+1})
            _j λ̄_min = _j λ̄_min + ^i_j λ_min
            _j λ̄_max = _j λ̄_max + ^i_j λ_max
        end
        _j λ̄_min = (1/n) ^i_j λ_min
        _j λ̄_max = (1/n) ^i_j λ_max
    end
end
```

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The Tables 1 to 10 represent, for each country, the Ordering graph and the computation of ${}^{i}_{j}\lambda_{max}$, ${}^{i}_{j}\lambda_{min}$ and the range for each j-tuple. We can based on the Tables 1 to 10 to compute, for example, the ${}_{3}\bar{\lambda}_{min}$. We have the following Table[6] :

3-tuple	${}^{i}_{3}\lambda_{min}$
(Ar, Is, Ja)	81.182
(Is, Ar, Ja)	81.182
(US, In, Fr)	74.639
(Ca, UK, Fr	80.392
(Ja, Fr, UK)	82.887
(Ko, Pa, Ja)	81.092
(Fr, UK, Ja)	89.213
(UK, Fr, Ja)	89.213
(In, Is, Ar)	76.959
(Pa, Ko, Ja)	81.092
	$\sum = 817.851$

Then ${}^{i}_{3}\bar{\lambda}_{min} = \frac{\sum_{i=1}^{10} {}^{i}_{3}\lambda_{min}}{10} = \frac{817.851}{10} \approx 81.785$. This means that, for a 3-tuple, for all applied ordering on the 10 countries, the power of discrimination of two imposed factors is of the order 81.78%.

Arab World Ordering	Initial Eigenvalues-CPV (%)		
Ar	${}^{1}_{j}\lambda_{max}$	${}^{1}_{j}\lambda_{min}$	Range
(Ar, Is)	71.773	50.698	21.075
(Ar, Is, Ja)	86.974	81.182	05.792
(Ar, Is, Ja, Fr)	73.607	67.124	06.483
(Ar, Is, Ja, Fr, UK)	66.781	59.071	07.710
(Ar, Is, Ja, Fr, UK, Ca)	60.541	55.841	04.700
(Ar, Is, Ja, Fr, UK, Ca, Pa)	54.595	52.174	02.421
(Ar, Is, Ja, Fr, UK, Ca, Pa, Ko)	50.431	48.370	02.061
(Ar, Is, Ja, Fr, UK, Ca, Pa, Ko, In)	46.435	45.287	01.148
(Ar, Is, Ja, Fr, UK, Ca, Pa, Ko, In, US)	42.391	42.391	00.000

Table 14: ${}^{1}_{j}\lambda_{max}$, ${}^{1}_{j}\lambda_{min}$ and the range for each j-tuple for Arab World Ordering.

6. In all Tables, we use the abbreviations : Ar for Arab Word ; Is for Israel ; US for United States ; Ca for Canada; Fr for France; UK for United Kingdom; Ja for Japan ; Ko for South-Korea ; In for India ; Pa for Pakistan.

Israel Ordering Is	Initial Eigenvalues-CPV (%) ${}^{2}_{j}\lambda_{max}$	${}^{2}_{j}\lambda_{min}$	Range
(Is, Ar)	71.773	52.564	19.209
(Is, Ar, Ja)	86.974	81.182	05.792
(Is, Ar, Ja, Fr)	73.607	67.124	06.483
(Is, Ar, Ja, Fr, UK)	66.781	59.071	07.710
(Is, Ar, Ja, Fr, UK, Ca)	60.541	55.841	04.700
(Is, Ar, Ja, Fr, UK, Ca, Pa)	54.595	52.174	02.421
(Is, Ar, Ja, Fr, UK, Ca, Pa, Ko)	50.431	48.370	02.061
(Is, Ar, Ja, Fr, UK, Ca, Pa, Ko, In)	46.435	45.287	01.148
(Is, Ar, Ja, Fr, UK, Ca, Pa, Ko, In, US)	42.391	42.391	00.000

Table 15: ${}^{2}_{j}\lambda_{max}$, ${}^{2}_{j}\lambda_{min}$ and the range for each j-tuple for Israel Ordering.

United States Ordering US	Initial Eigenvalues-CPV (%) ${}^{3}_{j}\lambda_{max}$	${}^{3}_{j}\lambda_{min}$	Range
(US, In)	61.839	52.910	08.929
(US, In, Fr)	77.672	74.639	03.033
(US, In, Fr, UK)	73.773	62.269	11.504
(US, In, Fr, UK, Is)	65.704	60.950	04.754
(US, In, Fr, UK, Is, Ar)	59.420	54.985	04.435
(US, In, Fr, UK, Is, Ar, Ca)	50.702	49.711	00.991
(US, In, Fr, UK, Is, Ar, Ca, Pa)	49.058	48.201	00.857
(US, In, Fr, UK, Is, Ar, Ca, Pa, Ko)	45.969	44.836	01.133
(US, In, Fr, UK, Is, Ar, Ca, Pa, Ko, Ja)	42.391	42.391	00.000

Table 16: ${}^{3}_{j}\lambda_{max}$, ${}^{3}_{j}\lambda_{min}$ and the range for each j-tuple for United States Ordering.

Canada Ordering Ca	Initial Eigenvalues-CPV (%) ${}^{4}_{j}\lambda_{max}$	${}^{4}_{j}\lambda_{min}$	Range
(Ca, UK)	70.503	55.630	14.873
(Ca, UK, Fr)	90.193	80.392	09.801
(Ca, UK, Fr, Pa)	79.655	72.973	06.682
(Ca, UK, Fr, Pa, Ko)	72.638	65.081	07.557
(Ca, UK, Fr, Pa, Ko, In)	63.781	61.471	02.310
(Ca, UK, Fr, Pa, Ko, In, Ja)	55.915	54,843	01.072
(Ca, UK, Fr, Pa, Ko, In, Ja, Ar)	51.524	49.239	02.285
(Ca, UK, Fr, Pa, Ko, In, Ja, Ar, US)	46.695	46.435	00.260
(Ca, UK, Fr, Pa, Ko, In, Ja, Ar, US, Is)	42.391	42.391	00.000

Table 17: ${}^{4}_{j}\lambda_{max}$, ${}^{4}_{j}\lambda_{min}$ and the range for each j-tuple for Canada Ordering.

Notes

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Japan Ordering Ja	Initial Eigenvalues-CPV (%) ${}^{5}_{j}\lambda_{max}$	${}^{5}_{j}\lambda_{min}$	Range
(Ja, Fr)	68.470	52.510	15.960
(Ja, Fr, UK)	91.501	82.887	08.614
(Ja, Fr, UK, Ar)	76.876	72.241	04.635
(Ja, Fr, UK, Ar, Ca)	67.655	61.606	06.049
(Ja, Fr, UK, Ar, Ca, Pa)	61.668	57.900	03.768
(Ja, Fr, UK, Ar, Ca, Pa, Ko)	56.271	54.094	02.177
(Ja, Fr, UK, Ar, Ca, Pa, Ko, In)	51.524	50.019	01.505
(Ja, Fr, UK, Ar, Ca, Pa, Ko, In, US)	46.695	46.435	00.260
(Ja, Fr, UK, Ar, Ca, Pa, Ko, In, US, Is)	42.391	42.391	00.000

Table 18: ${}^{5}_{j}\lambda_{max}$, ${}^{5}_{j}\lambda_{min}$ and the range for each j-tuple for France Ordering.

Korea Ordering Ko	Initial Eigenvalues-CPV (%) ${}^{6}_{j}\lambda_{max}$	${}^{6}_{j}\lambda_{min}$	Range
(Ko, Pa)	71.628	52.093	19.535
(Ko, Pa, Ja)	83.760	81.092	02.668
(Ko, Pa, Ja, Ar)	72.324	64.052	08.272
(Ko, Pa, Ja, Ar, Ca)	64.995	58.763	06.232
(Ko, Pa, Ja, Ar, Ca, Fr)	58.271	55.278	02.993
(Ko, Pa, Ja, Ar, Ca, Fr, UK)	56.271	50.846	05.425
(Ko, Pa, Ja, Ar, Ca, Fr, UK, In)	51.524	50.019	01.505
(Ko, Pa, Ja, Ar, Ca, Fr, UK, In, US)	46.695	46.435	00.260
(Ko, Pa, Ja, Ar, Ca, Fr, UK, In, US, Is)	42.391	42.391	00.000

Table 19: ${}^{6}_{j}\lambda_{max}$, ${}^{6}_{j}\lambda_{min}$ and the range for each j-tuple for United Kingdom Ordering.

France Ordering Fr	Initial Eigenvalues-CPV (%) ${}^{7}_{j}\lambda_{max}$	${}^{7}_{j}\lambda_{min}$	Range
(Fr, UK)	83.814	52.834	30.980
(Fr, UK, Ja)	91.501	89.213	02.288
(Fr, UK, Ja, Ar)	76.876	72.241	04.635
(Fr, UK, Ja, Ar, Ca)	67.655	61.606	06.049
(Fr, UK, Ja, Ar, Ca, Is)	60.541	57.900	02.641
(Fr, UK, Ja, Ar, Ca, Is, Pa)	54.595	52.174	02.421
(Fr, UK, Ja, Ar, Ca, Is, Pa, Ko)	50.431	48.370	02.061
(Fr, UK, Ja, Ar, Ca, Is, Pa, Ko, In)	46.435	45.287	01.148
(Fr, UK, Ja, Ar, Ca, Is, Pa, Ko, In, US)	42.391	42.391	00.000

Table 20: ${}^{7}_{j}\lambda_{max}$, ${}^{7}_{j}\lambda_{min}$ and the range for each j-tuple for Japan Ordering.

United Kingdom Ordering UK	Initial Eigenvalues-CPV (%) ${}^{8}_{j}\lambda_{max}$	${}^{8}_{j}\lambda_{min}$	Range
(UK, Fr)	83.814	52.093	31.721
(UK, Fr, Ja)	91.501	89.213	02.288
(UK, Fr, Ja, Ar)	76.876	72.241	04.635
(UK, Fr, Ja, Ar, Ca)	67.655	61.606	06.049
(UK, Fr, Ja, Ar, Ca, Is)	60.541	57.900	02.641
(UK, Fr, Ja, Ar, Ca, Is, Pa)	54.595	52.174	02.421
(UK, Fr, Ja, Ar, Ca, Is, Pa, Ko)	50.431	48.370	02.061
(UK, Fr, Ja, Ar, Ca, Is, Pa, Ko, In)	46.435	45.287	01.148
(UK, Fr, Ja, Ar, Ca, Is, Pa, Ko, In, US)	42.391	42.391	00.000

Table 21: ${}^{8}_{j}\lambda_{max}$, ${}^{8}_{j}\lambda_{min}$ and the range for each j-tuple for Korea Ordering.

India Ordering In	Initial Eigenvalues-CPV (%) ${}^{9}_{j}\lambda_{max}$	${}^{9}_{j}\lambda_{min}$	Range
(In, Is)	65.286	50.698	14.588
(In, Is, Ar)	84.623	76.959	07.664
(In, Is, Ar, Ca)	72.771	68.012	04.759
(In, Is, Ar, Ca, UK)	65.135	58.758	06.377
(In, Is, Ar, Ca, UK, Fr)	61.616	54.775	06.841
(In, Is, Ar, Ca, UK, Fr, Pa)	54.731	53.974	00.757
(In, Is, Ar, Ca, UK, Fr, Pa, Ko)	50.745	49.058	01.687
(In, Is, Ar, Ca, UK, Fr, Pa, Ko, Ja)	46.435	45.969	00.466
(In, Is, Ar, Ca, UK, Fr, Pa, Ko, Ja, US)	42.391	42.391	00.000

Table 22: ${}^{9}_{j}\lambda_{max}$, ${}^{9}_{j}\lambda_{min}$ and the range for each j-tuple for India Ordering.

Pakistan Ordering Pa	Initial Eigenvalues-CPV (%) ${}^{10}_{j}\lambda_{max}$	${}^{10}_{j}\lambda_{min}$	Range
(Pa, Ko)	71.628	52.510	19.118
(Pa, Ko, Ja)	83.760	81.092	02.668
(Pa, Ko, Ja, Ar)	72.324	64.052	08.272
(Pa, Ko, Ja, Ar, Ca)	64.995	58.763	06.232
(Pa, Ko, Ja, Ar, Ca, Fr)	58.271	55.278	02.993
(Pa, Ko, Ja, Ar, Ca, Fr, Ca)	56.271	50.846	05.425
(Pa, Ko, Ja, Ar, Ca, Fr, Ca, In)	51.524	50.019	01.505
(Pa, Ko, Ja, Ar, Ca, Fr, Ca, In, US)	46.695	46.435	00.260
(Pa, Ko, Ja, Ar, Ca, Fr, Ca, In, US, Is)	42.391	42.391	00.000

Table 23: ${}^{10}_{j}\lambda_{max}$, ${}^{10}_{j}\lambda_{min}$ and the range for each j-tuple for Pakistan Ordering.

Notes

References Références Referencias

Notes

1. Bell, D. N. F. and Blanchflower, D. G. (2010). Uk unemployment in the great recession. *JEL Classifications: D1; I31; E300; E320; J110; J210*, 1(214):3–25.
2. Breitung, J. (2000). The local power of some unit root tests for panel data. In *Advances in Econometrics, Vol. 15: Nonstationary Panels, Panel Cointegration, and Dynamic Panels, JAI,* pages 161–178. Press.
3. Breitung, J. and Das, S. (2005). Panel unit root tests under cross-sectional dependence. *Statistica Neerlandica*, 4(59): 414–433.
4. Breuer, B., Mc Nown, R., and Wallace, M. (2002). Series specific unit root test with panel data. *Oxford Bulletin of Economics and Statistics*, 5(64): 527–546.
5. Breusch, T. S. and Pagan, A. (1980). The lagrange multiplier test and its applications to model specification in econometrics. Review of Economic Studies, 47(1): 239–253.
6. Carroué, L. (Février 1997). Les travailleurs coréens à lássaut du dragon. *Le Monde diplomatique,* pages 1,18 et 19.
7. Cavatorta, E. (2010). Unobserved common factors in military expenditure interactions across mena countries. *Defence and Peace Economics*, 21(4): 301–331.
8. Chang, Y. (2002). Nonlinear iv unit root tests in panels with cross-sectional dependency. *Journal of Econometrics*, 110: 261–292.
9. Choi, I. (2001). Unit root tests for panel data. *Journal of International Money and Finance*, 20 :249–272.
10. Elliott, G. Rothenberg, T. and Stock, J. (1996). Efficient tests for an autoregressive unit root. *Econometrica*, 4(64): 813–836.
11. Hadri, K. (2000). Testing for unit roots in heterogeneous panel data. *Econometrics Journal*, 3: 148–161.
12. Harris, R. and Tzavalis, E. (1999). Inference for unit roots in dynamic panels where the time dimension is fixed. *Journal of Econometrics,* 91: 201–226.
13. Hsiao, C. (1986). *Analysis of Panel Data*. Cambridge University Press, 2 edition.
14. Hurlin, C. (2010). Panel data econometrics. Technical report, https:// studylib.net/doc/.../panel-data-econometrics-christophe-hurlin–Université d'Orleans.
15. Hurlin, C. and Mignon, V. (2006). Panel data econometrics. Technical report, http ://halshs.archives-ouvertes.fr/docs.
16. Im, K., Pesaran, M., and Shin, Y. (2003). Testing for unit roots in heterogenous panels. *Journal of Econometrics*, 115: 53–74.
17. J., B. and S, N. (2004). A panic attack on unit roots and cointegration. *Econometrica*, 4(72):1127–1178.
18. Jaunky, V. and Lundmark, R. (2017). Dynamics of wood pulp production : Evidence from oecd countries. *Forests,* 8(4): 1–10.
19. Kao, C. and Chiang, M. (2000). Dynamics of wood pulp production: Evidence from oecd countries. *Advances in Econometrics,* 15, edited by B. Baltagi et C. Kao: 179–222.
20. Levin, A. and Lin, C. (1992). Unit root test in panel data: Asymptotic and finite sample properties. In *Discussion Paper,* volume 92-93. University of California at San Diego.
21. Levin, A., Lin, C., and Chu, C. (2002). Unit root test in panel data: Asymptotic and finite sample properties. *Journal of Econometrics,* 108: 1–24.

22. Maddala, G. (1999). On the use of panel data methods with cross-country data. Annales *D'économie et de Statistique,* 55-56: 430–449.
23. Maddala, G. and Wu, S. (1999). A comparative study of unit root tests with panel data and a new simple test. *Oxford Bulletin of Economics and Statistics,* 61(0): 631–52.
24. Mark, N. and Sul, D. (2003). Cointegration vector estimation by panel dynamic ols and long-run money demand. *Oxford Bulletin of Economics and Statistics,* 65(5): 655–680.
25. Moon, H. and Perron, B. (2004). Testing for a unit root in panels with dynamic factors. *Journal of Econometrics,* 122(1): 81–126.
26. Mourad,M. (2018a). Trade and GDP in world's top economies: Panel co-integration analysis. *Arabian Journal of Business and Management Review*, 8(5):1–8.
27. Mourad, M. (2018b). Vital economic determinants and real GDP in GCC countries: Panel co-integration analysis. *Review of Economics and Business Administration*, 2(2):141–168.
28. Mourad, M. (2019). *Econometrics from Theory to Practice-volume II.* Lebanese University Press.
29. Newey, W. and West, K. (1994). Automatic lag selection in covariance matrix estimation. *Review of Economic Sudies,* 3: 631–653.
30. Pedroni, P. (1995). Panel cointegration; asymptotic and finite sample properties of pooled time series tests, with an application to the ppp hypothesis. *Working Papers in Economics,* 95-013.
31. Pedroni, P. (1996). Fully modified ols for heterogeneous cointegrated panels and the case of purchasing power parity. *Working Papers in Economics,* 96-020.
32. Pedroni, P. (1999). Critical values for cointegration tests in heterogeneous panels with multiple regressors. *Oxford Bulletin of Economics and Statistics,* 61: 653–670.
33. Pedroni, P. (2001). Purchasing power parity tests in cointegrated panels. *Re-view of Economics and Statistics,* 83:727–731.
34. Pedroni, P. (2004). Panel cointegration: Asymptotic and finite sample properties of pooled time series tests with an application to the ppp hypothesis. *Econometric Theory,* 20:597–625.
35. Pedroni, P. (2007). Social capital, barriers to production and capital shares: Implications for the importance of parameter heterogeneity from a nonstationary panel approach. *Journal of Applied Econometrics,* 22: 429–451.
36. Pesaran, M. (2007). A simple panel unit root test in the presence of cross-section dependence. *Journal of Applied Econometrics,* 2(22):265–312.
37. Pesaran, M. H. (2004). General diagnostic tests for cross section dependence in panels. *Cambridge Working Papers in Economics,* 0435.
38. Phillips, P. (1995). Fully modified least squares and vector autoregression. *Econometrica,* 63(5):1023–1078.
39. Phillips, P. C. B. and Hansen, B. E. (1990). Statistical inference in instrumental variables regression with i(1) processes. *Review of Economic Studies*, 57(1): 99–125.
40. Rafael E. De Hoyos, R. and Sarafidis, V. (2006). Testing for cross-sectional dependence in panel-data models. *The Stata Journal,* 6(4): 482–496.

Notes

41. Stock, J. and Watson, M. (1993). A simple estimator of cointegrating vectors in higher order integrated systems. *Econometrica,* 61(4): 783–820.
42. Todd, S. and Justin, G. (2016). Military expenditure trends for 1960-2014 and what they reveal. *Defence and Peace Economics,* 7(2):174–184.

Notes

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The Limits of the Laser Acupuncture Therapy

By Maria Kuman

Abstract- The above author was the first to predict through mathematical model that waves must propagate from the treated acupuncture point in direction of the acupuncture meridian. The waves were recorded experimentally one year later. After almost 40 years of measurements with developed and patented by her sensitive equipment allowing her to measure the weak nonlinear electromagnetic field (NEMF) of our body, the author was the first to explain that acupuncture works through the waves of our weak (NEMF), which rules and regulates everything in the body. The article also explains why we have holographic representations of the acupuncture points of the whole body on the palms, feet, ears, and irises – it is because of the laser type coherent waves of our weak NEMF. Based on the adequacy of acupuncture treatment on any of these holographic representations, the article strongly suggests with laser to be treated only the acupuncture points (of the holographic representation) on the ears, which are the shallowest because the depth of penetration of the lasers used for acupuncture treatment is limited.

Keywords: laser acupuncture therapy; limits of laser acupuncture; limits of mild laser; laser for auricular therapy.

GJSFR-F Classification: MSC 2010: 78A60

THELIMITSOFTHELASERACUPUNCTURETHERAPY

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The Limits of the Laser Acupuncture Therapy

Maria Kuman

Abstract- The above author was the first to predict through mathematical model that waves must propagate from the treated acupuncture point in direction of the acupuncture meridian. The waves were recorded experimentally one year later. After almost 40 years of measurements with developed and patented by her sensitive equipment allowing her to measure the weak nonlinear electromagnetic field (NEMF) of our body, the author was the first to explain that acupuncture works through the waves of our weak (NEMF), which rules and regulates everything in the body. The article also explains why we have holographic representations of the acupuncture points of the whole body on the palms, feet, ears, and irises – it is because of the laser type coherent waves of our weak NEMF. Based on the adequacy of acupuncture treatment on any of these holographic representations, the article strongly suggests with laser to be treated only the acupuncture points (of the holographic representation) on the ears, which are the shallowest because the depth of penetration of the lasers used for acupuncture treatment is limited.

Keywords: laser acupuncture therapy; limits of laser acupuncture; limits of mild laser; laser for auricular therapy.

I. Acupuncture Cures Through the Waves of our Weak NEMF

Nonlinear mathematical model of the author [1] predicted waves running from the treated acupuncture point in the direction of the acupuncture meridian and Hungarian scientist experimentally found the waves a year later [2]. The article of the author [3] explains that acupuncture cures through the waves of our weak nonlinear electromagnetic field (NEMF), which rules and regulates everything in the body.

Ancient Chinese sources explain that the acupuncture meridians are like rivers, but along them energy runs, not water. In the way the rivers flow into seas, the acupuncture meridians flow into 6 spinning seas. They are called "chakras" in Indian sources, which in Sanskrit means "spinning wheel" [2]. The chakras of the human body are aligned along the backbone as a chain of alternating vortices (spinning clockwise) and anti-vortices (spinning counterclockwise) of our weak nonlinear EMF.

Thus, the chakras, the acupuncture meridians (along which waves run), and the acupuncture points on them are the anatomy of our NEMF. Our ability to respond fast is done through the waves of this weak NEMF. Should our life depend only on our slow nervous system, we would be dead long time ago [3]. Acupuncture works through influencing our weak NEMF, which rules and regulates everything in the body.

If Wikipedia still announces that acupuncture cannot cure because there is no scientific explanation of how it works, this is because acupuncture works through the waves of our weak NEMF, which is 1,000 times weaker than the biocurrents in our body and our science does not have sensitive enough equipment to measure this weak NEMF.

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Ref

2. Kuman M, Modern Aspects of Ancient Acupuncture, Health and Happiness Books, 1997.

While the biocurrents are measured in milliamperes (one thousand of the ampere), the currents of the weak NEMF are measured in microamperes (one millionth of the ampere). However, if Prof. Harold Burr could measure this weak EMF in 1937 [4], it could be measured now. The field is weak, but its measurement is possible at our level of technology. With my patented sensitive energy meter, I have measured this weak NEMF for almost 40 years.

II. Holographic Representation of all Organs on Many Places of the Body

Holographic images are tri-dimensional images created with laser light, which is coherent light. Light is electromagnetic waves in visible diapason of 800 – 380 nm. If waves are involved, quantum behavior can be expected. Specific feature of the holographic images is that you can cut the image into many pieces and each piece when activated by laser light will reproduce the whole image.

Since holographic images are created with lasers, which emit coherent waves, as we shall see in this section the holographic representation of all acupuncture points of the body on the palms, feet, ears, and irises means that the acupuncture points and acupuncture meridians are also pathways of running waves – the waves of our weak NEMF [2].

Our sleep is a constant alternative switch between light sleep when our Conscious is active and deep sleep when the Conscious has zero activity, but a galvanic response is measured, which is specific for emotional response and our emotional brain called Limbic system is in the Subconscious. If so, the constant alternative switches between light sleep and deep sleep are constant alternative switches between active Conscious Mind and active Subconscious Mind.

Everything we have seen, heard, smelled, and emotionally experienced during the day is recorded in our Conscious Mind, which memorizes on biochemical principle. During light sleep the information recorded during the day is read, copied and during deep sleep recorded in the Subconscious Mind [4], where it is stored for safety and other reasons.

Since we don't have awareness of the functioning of our organs, they must be subordinated to the Subconscious. It is deliberately done so because when our life is threatened and we need to react fast to survive, we don't want to be bothered with information about the functioning of our organs [4].

Since the acupuncture points of the whole body are holographically represented on our palms, feet, ears, and irises, and all the organs are subordinated to the Subconscious, the Subconscious record must be on holographic principle. Since holographic images are created with coherent light, which is electromagnetic waves, the holographic record at subconscious level must be done with the waves of our weak nonlinear electromagnetic field (NEMF) [4].

By all means, it seems that this weak NEMF, which I spent almost 40 years of my life measuring, and which rules and regulates the functioning of our organs, is not the field produced by their functioning. It seems that the NEMF is the one that creates the organs, and later rules, regulates, and synchronizes their function. It also rules (controls) the growing of our skin, nails, and hair and everything else in the body.

Dr. Keith Floyd, psychologist at Virginia Intermonth College said: "If reality is nothing but a holographic illusion, the physical brain does not produce the mind. Rather, it is the mind that creates the appearance of the brain, as well as the body..." [4]. Indeed, the nonlinear electromagnetic field (NEMF) (of which the mind is part)

Ref

4. Kuman M, Research in Medical and Engineering Sciences, 5 (3) 2018.

carries the information under the guidance of which the whole body is created, including the brain.

The ingenious measurements of Prof. Burr showed that unfertilized salamander egg contains EMF, which he measured as high electric potential at one end of the egg and low electric potential at the other end of the egg. After fertilization, from the end with higher potential the brain of the salamander developed, from the end with lower potential the tail developed [5].

This means that unfertilized egg has its NEMF (male or female), but only when the egg is fertilized and the NEMF of the opposite-gender is added to it, can the development of the egg into an embryo start. And since the development is ruled by the waves of the total NEMF, which operates on the holographic principle, the created organs operate on the holographic principle.

III. Whole Life Holographic Memory

Everything we see, hear, smell, or emotionally experience every day is recorded in our Conscious. Every night, during sleep this information is copied and sent for permanent storage in the Subconscious, where it is stored as holographic record through the waves of our NEMF. In this way we have in the Subconscious a compact holographic record of everything we have seen, heard, smelled, and emotionally experienced during our lifetime.

Since nonlinear electromagnetic fields do not dissipate and they can pass through walls or screens, at the end of life when the person dies, this non-dissipating NEMF leaves the body, but it carries a compact three-dimensional holographic record of the whole lifetime of the individual [4].

IV. Holographic Vision

Can we really see through waves? According to Hindu and Yoga texts, we can also see with our mind, not only with our eyes. Russian scientists investigated V. Bronnikov, who could see with his mind [6]. First, when he sees with his eyes, the equipment registers impulse from his eyes to his brain. Second, when he is blindfolded and the equipment does not register impulse from his eyes to his brain, he could still sees.

He could even see the object when it is behind a dense screen [6], which proves that nonlinear waves are involved because only nonlinear waves can pass through a screen. He could even see the object from different angles [6], which indicates that the vision is three-dimensional or holographic. Obviously, the seeing with the mind involves nonlinear coherent (laser type) waves and since our NEMF is rich of such waves, it must be involved in it.

V. The Limits of Laser Acupuncture

But let's go back to the laser acupuncture. Our organs are holographically represented on the surface of the body: on our palms, feet, ears, and the irises of our eyes and the acupuncturists know that they can treat with equal success the acupuncture points of the sick organ on any of these holographic representations.

Every acupuncturist using laser for acupuncture therapy must know that the mild laser used to treat the acupuncture points as a substitute for needles has the power of 3 milliwatts and its depth of penetration is 1 - 2 cm. Knowing this, every acupuncturist who use laser for acupuncture therapy should know the limits of application of the laser acupuncture [2].

Ref

5. Burr HS, The Fields of Life, Ballantine, New York, 1972.

Of all these representations, the holographic representation on the ears has the shallowest acupuncture points – they are mms shallow [1]. Therefore, knowing the limited penetration of the laser light (1 - 2 cm) of the lasers used for acupuncture treatment, the acupuncturists should choose to treat with laser only the shallow points of the ears [2]. Then their laser treatments would enjoy full success.

VI. Conclusion

The article explained:
1/ how acupuncture works,
2/ why the acupuncture points of the organs to be treated are holographically represented at many places in the body,
3/ why based on the adequacy of treatment of acupuncture points of any of these holographic representations, with laser should be treated only the acupuncture points (of the holographic representation) on the ears, which are the shallowest because the depth of penetration of the lasers used in acupuncture is limited.

References Références Referencias

1. Kuman M, 8th World Congress on Acupuncture, Sofia, Bulgaria, 1983.
2. Kuman M, Modern Aspects of Ancient Acupuncture, Health and Happiness Books, 1997.
3. Kuman M, International Journal of Integrated Medicine, 42 (2) 97-112, 2017.
4. Kuman M, Research in Medical and Engineering Sciences, 5 (3) 2018.
5. Burr HS, The Fields of Life, Ballantine, New York, 1972.
6. Tihoplav V, Tihoplav T, Soznanie Vselenoi (The Conscious of the Universe), Krilov Publisher, Russia, 2007.

Ref

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On the Solution of Wave-Schrodinger Equation

By Wanchak Satsanit

Maejo University

Abstract- In this paper, we are finding solution of fraction Wave-Schrodinger equation by Laplace transform in sense of Caputo fractional derivative. It was found that the fundamental solution of the equation related to Wright function.

Keywords: *dirac delta distribution, laplacian operator, wright function.*

GJSFR-F Classification: *MSC 2010: 35L05*

ONTHESOLUTIONOFWAVE-SCHRODINGEREQUATION

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On the Solution of Wave-Schrodinger Equation

Wanchak Satsanit

Abstract- In this paper, we are finding solution of fraction Wave-Schrodinger equation by Laplace transform in sense of Caputo fractional derivative. It was found that the fundamental solution of the equation related to Wright function.

Keywords: *dirac delta distribution, laplacian operator, wright function.*

I. Introduction

The Laplacian operator $\triangle^k$ iterated $k-$ times is defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}\right)^k, \tag{1.1}$$

where n is the dimension of space $\mathbb{R}^n$, k is a nonnegative integer. A. Kananthai[1] has proved that the generalized function $(-1)^k S_{2k}(x)$ is an elementary solution of the operator $\triangle^k$, that is

$$\triangle^k (-1)^k S_{2k}(x) = \delta,$$

where δ is the Dirac-delta distribution and $S_{2k}(x)$ is defined by

$$S_{2k}(x) = \frac{\pi^{-\frac{n}{2}} 2^{-2k} \Gamma\left(\frac{n-2k}{2}\right) \left(x_1^2 + x_2^2 + \ldots + x_n^2\right)^{\frac{2k-n}{2}}}{\Gamma(k)}, \tag{1.2}$$

In 2002, A.Kananthai, S. Suantai, V. Longani[2] have first introduced the operator $\triangle_i^k$ and is defined by

$$\triangle_i^k = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right), i = \sqrt{-1} \tag{1.3}$$

They have proved the function $(-1)^k (-i)^{\frac{q}{2}} S_{2k}(x)$ is an elementary solution of the operator $\triangle_i^k$ and $S_{2k}(x)$ is defined by (1.2).It is well known the linear Schrodinger equation can be written as the following form

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Ref

1. Kananthai, *On the Solution of the n-Dimensional Diamond Operator*, Applied Mathematics and Computational, Elsevier Science Inc. New York, 1997, pp. 27-37.

$$\frac{\partial}{\partial t}u(x,t) = i\frac{\partial^2}{\partial x^2}u(x,t), i = \sqrt{-1} \tag{1.4}$$

with the initial condition

$$u(x,0) = f(x).$$

The Schrodinger equation has been widely in application in science and engineering, there are several integral transform such as Laplace transform, Fourier transform, Wavelet transform etc. for solving the equation.

The purpose of this work is to introduce a new function where related the Wright function [3] and studied Laplace transform of a new function.After that,we are solving the fundamental solution of the wave-schrodinger equation as follows:

$$\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0 \ , \ i = \sqrt{-1} \ , \ 1 < \alpha \leq 2 \tag{1.5}$$

with the initial condition

$$\phi(x,0) = 0 \ , \ \phi_t(x,0) = \delta(x),$$

where δ is the Dirac-delta distribution and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the Caputo derivative. Before going that point, the following definitions and some important concepts are needed.

II. Preliminaries

Definition 2.1 Let $f(t)$ be a function an exponential order and piecewise continuous. The Laplace transform of the function f is given by

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}f(t)dt \tag{2.1}$$

Definition 2.2 Let $f(t)$ be a function of the Schwart space the Fourier transform of $f(t)$ is given by

$$\widehat{f}(w) = \int_{\mathbb{R}} f(t)e^{iwt}dt \tag{2.2}$$

Definition 2.3 For m to be the smallest integer that exceeds α , the Caputo fractional derivatives of order is defined by

$$D^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-n-1}\frac{\partial^m}{\partial t^m}u(x,t) \\ \frac{\partial^m}{\partial t^m}u(x,t) \ , \ n = m \end{cases} \tag{2.3}$$

Definition 2.4 The Laplace transform of the Caputo fractional derivative is defined by

$$\mathcal{L}[D^\alpha f(t)] = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0) \ , \ n-1 < \alpha < n \tag{2.4}$$

Ref

3. I. Podlubny. *Fractioanl Differential Equation*, An introduction to fractional derivatives. Academic Press 1999.

Notes

Definition 2.5 The Wright function $W_{\alpha,\beta}$ is defined by

$$W_{\alpha,\beta} = \sum_{n=0}^{\infty} \frac{Z^n}{n!\Gamma(n\alpha+\beta)} \ , \ \alpha > -1 \ , \ \beta \in \mathbb{C} \tag{2.5}$$

where $\Gamma(x)$ is the Euler Gamma function is given by the integral

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-6} dt \tag{2.6}$$

Lemma 2.1 The function $\gamma(a,t)$ is defined by the following expressions

$$\gamma(a,t) = t^{-\alpha+1} W_{-\alpha \ , \ 2-\alpha}(at^{-\alpha}) \tag{2.7}$$

and Laplace transform of $\gamma(a,t)$ is given by

$$\mathcal{L}[\gamma(a,t)] = s^{\alpha-2} e^{as^{\alpha}}.$$

Proof: By (2.1) *, we have*

$$\begin{aligned}
\mathcal{L}[\gamma(a,t)] &= \int_0^{\infty} e^{-st} t^{-\alpha+1} W_{-\alpha \ , \ 2-\alpha}(at^{-\alpha}) dt \\
&= \int_0^{\infty} e^{-st} t^{-\alpha+1} \sum_{k=0}^{\infty} \frac{(at^{-\alpha})^k}{k!\Gamma(-\alpha k+2-\alpha)} dt \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\alpha k+2-\alpha)} \int_0^{\infty} e^{-st} t^{-\alpha-\alpha k+1} dt \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\alpha k+2-\alpha)} \mathcal{L}[t^{-\alpha-\alpha k+1}] \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k!\Gamma(-\alpha k+2-\alpha)} \frac{\Gamma(-\alpha-\alpha k+2)}{s^{-\alpha-\alpha k+2}} \\
&= \sum_{k=0}^{\infty} \frac{a^k}{k! s^{-\alpha-\alpha k+2}} \\
&= s^{\alpha-2} \sum_{k=0}^{\infty} \frac{(as^{\alpha})^k}{k!} \\
&= s^{\alpha-2} e^{as^{\alpha}}
\end{aligned} \tag{2.8}$$

That completes the proof.

III. Main Results

Theorem 3.1 Consider the Fractional Wave-Schrodinger equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} \phi(x,t) + i \frac{\partial^2}{\partial x^2} \phi(x,t) = 0 \ , \ i = \sqrt{-1} \ , \ 1 < \alpha \leq 2 \tag{3.1}$$

with the initial condition

$$\phi(x,0) = 0 \ , \ \phi_t(x,0) = \delta(x)$$

where $\delta(x)$ is the dirac delta distribution.By the Laplace and Fourier transform,we obtain the fundamental solution of the equation (3.1) *is given by*

$$\phi(x,t) = \frac{1}{2}\sqrt{i}t^{-\frac{\alpha}{2}+1}W_{-\frac{\alpha}{2}\ ,\ 2-\frac{\alpha}{2}}(-\sqrt{i}|x|t^{-\frac{\alpha}{2}}) \tag{3.2}$$

where $W_{\alpha,\beta}$ is the Wright function is defined by (2.5). *If we put $\alpha = 2$ in* (3.1) *the fractional Wave-Schrodinger equation reduced to*

$$\frac{\partial^2}{\partial t^2}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0 \tag{3.3}$$

and the solution of (3.3) *is given by*

$$\phi(x,t) = \frac{1}{2}\sqrt{i}W_{-1,1}(-\sqrt{i}|x|t^{-1}) \tag{3.4}$$

Proof: By (3.1) , we have

$$\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0 \tag{3.5}$$

Taking Laplace transform both sides of (3.5) and we get by definition 2.1

$$\mathcal{L}\Big[\frac{\partial^\alpha}{\partial t^\alpha}\phi(x,t)\Big] + i\mathcal{L}\Big[\frac{\partial^2}{\partial x^2}\phi(x,t)\Big] = 0$$

$$s^\alpha\phi(x,s) - s^{\alpha-2}\delta(x) = -i\frac{\partial^2}{\partial x^2}\phi(x,s). \tag{3.6}$$

Applying Fourier transform respect to variable x both sides of (3.6),we obtained

$$\begin{aligned} s^\alpha\mathcal{F}\phi(x,s) - s^{\alpha-2}\mathcal{F}[\delta(x)] &= -i\mathcal{F}\frac{\partial^2}{\partial x^2}\phi(x,s) \\ s^\alpha\phi(\omega,s) - s^{\alpha-2} &= i\omega^2\phi(\omega,s) \\ \phi(\omega,s) &= \frac{s^{\alpha-2}}{s^\alpha + (-i)\omega^2} \\ &= \frac{is^{\alpha-2}}{is^\alpha + \omega^2}. \end{aligned} \tag{3.7}$$

Applying inverse Fourier transform both sides of (3.7) , we obtain

$$\begin{aligned} \phi(x,s) &= \mathcal{F}^{-1}\Big[\frac{is^{\alpha-2}}{is^\alpha + \omega^2}\Big] \\ &= \frac{\sqrt{i}s^{\alpha-2}e^{-|x|\sqrt{i}s^{\frac{\alpha}{2}}}}{2s^{\frac{\alpha}{2}}} \\ &= \frac{1}{2}\sqrt{i}s^{\frac{\alpha}{2}-2}e^{-|x|\sqrt{i}s^{\frac{\alpha}{2}}}. \end{aligned}$$

Notes

By Lemma 2.1,we obtain the solution of (3.1) as follows

$$\begin{aligned}\phi(x,t) &= \frac{1}{2}\sqrt{i}r(-\sqrt{i}|x|,t)\\ &= \frac{1}{2}\sqrt{i}t^{-\frac{\alpha}{2}+1}W_{-\frac{\alpha}{2},2-\frac{\alpha}{2}}(-\sqrt{i}|x|t^{-\frac{\alpha}{2}})\end{aligned} \tag{3.8}$$

Notes

If we put $\alpha = 2$ in (3.1) and (3.8) respectively, the equation reduced to the Wave-Schrodinger equation

$$\frac{\partial^2}{\partial t^2}\phi(x,t) + i\frac{\partial^2}{\partial x^2}\phi(x,t) = 0, \tag{3.9}$$

and the solution of (3.9) is given by

$$\phi(x,t) = \frac{1}{2}\sqrt{i}W_{-1,1}(-\sqrt{i}|x|t^{-1}). \tag{3.10}$$

That completes the proof.

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References Références Referencias

1. Kananthai, *On the Solution of the n-Dimensional Diamond Operator*, Applied Mathematics and Computational, Elsevier Science Inc. New York, 1997, pp. 27-37.
2. A. Kananthai, S. Suantai, V, Longani. *On the operator Φk related to the wave equation and Laplacian*, Applied Mathematics and Computational, Elsevier Science Inc. New York, 2002, pp 219-229.
3. I. Podlubny. *Fractioanl Differential Equation*, An introduction to fractional derivatives. Academic Press 1999.

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Composition of Mixed Riemann-Liouville Fractional Integral and Mixed Fractional Derivative

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Abstract- We study the question of the composition of a mixed fractional integral and a mixed fractional derivative in a fairly wide class of functions. The treatment formula for mixed fractional derivative is obtained.

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Composition of Mixed Riemann-Liouville Fractional Integral and Mixed Fractional Derivative

T. Mamatov

Abstract- We study the question of the composition of a mixed fractional integral and a mixed fractional derivative in a fairly wide class of functions. The treatment formula for mixed fractional derivative is obtained.

I. Introduction

Various forms of fractional integrals and derivatives are known. Fractional integrals and Riemann-Liouville derivatives are the most common in the scientific literature [1]. Operators of generalized fractional integro-differentiation with Gauss hypergeometric function.

Direct extension of the Riemann-Liouville fractional integro-differentiation operations to the case of many variables, when these operators are applied for each variable or for some of them, gives the so-called partial and mixed fractional integrals and derivatives. They are known [1], as well as [4] - [12]. Thus, in [2], using the two-dimensional Laplace transform, a solution of the two-dimensional Abel integral equation was obtained.

In this paper, we study the question of the composition of a mixed fractional integral and a mixed fractional derivative in a fairly wide class of functions. The treatment formula for mixed fractional derivative is obtained. The results obtained can be applied in the theory of differential equations containing mixed fractional derivatives.

Lemma 3 on the representability of a function $f(x,y)\in AC^{n,m}(\overline{\Omega})$ in the form of (6) and Lemma 4 proved in general are the previously known Lemmas 1 and 2 for the two-dimensional case. Lemmas 3, 4 allow us to prove the theorem (a necessary and sufficient condition for the representability of a function $f(x,y)$ as a mixed fractional integral of a summable function) and Theorems 2, 3 about the composition of a mixed fractional integral and a mixed fractional derivative. Note that Theorems 2, 3 generalize the results of Theorem 2.4 [1, p. 44] for the two-dimensional case.

II. Preliminary Information and Notation

An absolutely important role in the theory of fractional integro differentiation is played by absolutely continuous functions.

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Let be $\Omega = \{(x,y): a < x < b,\ c < y < d\},\ -\infty \le a < b \le +\infty,\ -\infty \le c < d \le +\infty$.

Definition 1: [1, p. 2]. A function $f(x)$ is called absolutely non-discontinuous into segments $[a,b]$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite set of pairwise non-intersecting intervals $[a_k, b_k] \in [a,b]$, $k = \overline{1,m}$, such that $\sum_{k=1}^{m}(b_k - a_k) < \delta$, the inequality $\sum_{k=1}^{m}|f(b_k) - f(a_k)| < \varepsilon$ holds. The space of these functions is denote by $AC([a,b])$.

Definition 2: [1, p. 2]. Let us denote by $AC^n([a,b])$, where $n = 1,2,\ldots$, the spaces of functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ with $f^{(n-1)}(x) \in AC([a,b])$.

Definition 3: A function $f(x,y)$ is called absolutely continuous in Ω, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite set of pairwise non-intersecting intervals $\Delta_k = \{(x,y): x_{1k} \le x \le x_{2k},\ y_{1k} \le y \le y_{2k}\}$, the sum of the areas of which is less δ, the inequality holds

$$\sum_{k=1}^{n}|f(x_{2k}, y_{2k}) - f(x_{2k}, y_{1k}) - f(x_{1k}, y_{2k}) + f(x_{1k}, y_{1k})| < \varepsilon, \tag{1}$$

and if, moreover, $f(a,y) \in AC([c,d])$ and $f(x,c) \in AC([a,b])$. The class of all such functions is indicated $AC(\overline{\Omega})$.

Definition 4: By $AC^{n,m}(\overline{\Omega})$, where $n = 1,2,\ldots$, let us denote the class of functions continuously differentiable on $\overline{\Omega}$ up to order $(n-1, m-1)$, and its mixed partial derivative $\dfrac{\partial^{n+m-2} f}{\partial x^{n-1} \partial y^{m-1}}$ is absolutely continuous in $\overline{\Omega}$.

It is known that the class $AC^n([a,b])$ belongs to those and only those functions $f(x)$ that are representable as antiderivatives of Lebesgue summable functions:

$$f(x) = \int_a^x \psi(x)dx + C, \quad \psi(x) \in L_1(a,b). \tag{2}$$

Lemma 1: [1, p. 39]. The space $AC^n([a,b])$ consists of those and only those functions $f(x)$, which are represented in the form

$$f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\varphi(t)dt + \sum_{k=0}^{n-1} C_k (x-a)^k, \tag{3}$$

Notes

where $\varphi(x) \in L_1([a,b])$, C_k being arbitrary constants.
In the formula (3)

$$\varphi(t) = f^{(n)}(t), \quad C_k = \frac{f^{(k)}(a)}{k!}. \tag{4}$$

Notes

The last equality uses the notation $f^{(n)}(x) = \dfrac{d^n f(x)}{dx^n}$.

A similar property of the functions $f(x,y) \in AC(\overline{\Omega})$ is as follows.

Lemma 2: [3, p. 238]. The class $AC(\overline{\Omega})$ consists of those and only those functions $f(x,y)$ which are represented in the form

$$f(x,y) = \int_a^x \int_c^y \varphi(t,s)dtds + \int_a^x \psi(t)dt + \int_c^y \eta(s)ds + C, \tag{5}$$

Where $\varphi(x,y) \in L_1(\Omega)$, $\psi(x) \in L_1([a,b])$, $\eta(y) \in L_1([c,d])$, and C is an arbitrary constant.

In order to generalize the last lemma to the case of a class $AC^{n,n}(\overline{\Omega})$, we need the following lemma.

Lemma 3: Let $f(x,y) \in AC(\overline{\Omega})$, then

$$f(x,y) = \frac{1}{(n-1)!(m-1)!}\int_a^x \int_c^y \frac{f^{(n,m)}(t,s)dtds}{(x-t)^{1-n}(y-s)^{1-m}} + \sum_{i=0}^{n-1} \frac{f^{(i,0)}(a,y)}{i!}(x-a)^i +$$

$$+ \sum_{k=0}^{m-1} \frac{f^{(0,k)}(x,c)}{k!}(y-c)^k - \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} \frac{f^{(i,k)}(a,c)}{i!k!}(x-a)^i (y-c)^k. \tag{6}$$

In formula (6) the notation used $f^{(i,k)}(x,y) = \dfrac{d^{i+k} f(x,y)}{dx^i dy^k}$.

Proof: Let be $\dfrac{\partial^{n+m-2} f}{\partial x^{n-1} \partial y^{m-1}} \in AC(\overline{\Omega})$. By virtue of Lemma 2, we have

$$\frac{\partial^{n+m-2} f}{\partial x^{n-1} \partial y^{m-1}} = \int_a^x \int_c^y \varphi(t,s)dtds + \int_a^x \psi(t)dt + \int_c^y \eta(s)ds + C_0 \tag{7}$$

Integrating sequentially (7) times $n-1$ by x and times $m-1$ by y, we get

$$f(x,y) = \frac{1}{(n-1)!(m-1)!}\int_a^x \int_c^y (x-t)^{n-1}(y-s)^{m-1}\varphi(t,s)dtds +$$

$$+\frac{(y-c)^{m-1}}{(n-1)!(m-1)!}\int_a^x (x-t)^{n-1}\psi(t)dt+\frac{(x-a)^{n-1}}{(n-1)!(m-1)!}\int_c^y (y-s)^{m-1}\eta(s)ds+$$

$$+\sum_{i=0}^{n-1}\overline{\tau}_i(y)(x-a)^i+\sum_{k=0}^{m-1}\tilde{\tau}_k(x)(y-c)^k, \tag{8}$$

where $\overline{\tau}_i(y)$ $\left(i=\overline{0,n-1}\right)$, $\tilde{\tau}_k(x)$ $\left(k=\overline{0,m-1}\right)$ is arbitrary function. When integrating, the well-known for n - multiple integral formula is used [1]

$$\int_a^x dx\int_a^x dx...\int_a^x F(x)dx=\frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}F(t)dt, \tag{9}$$

proof, which is easy to implement by mathematical induction. It will be clear from the proof that an arbitrary constant in formula (7) is associated with arbitrary functions of formula (8) by the relation

$$(n-1)!\overline{\tau}_{n-1}^{(m-1)}(c)+(m-1)!\tilde{\tau}_{m-1}^{(n-1)}(a)=C_0.$$

Since $f(x,y)\in AC^{n,m}(\overline{\Omega})$, then derivatives $\frac{\partial^{i+k}f}{\partial x^i\partial y^k}$ $(0\le i<n,\ 0\le k<m)$ exist and are continuous in Ω. Calculating the derivatives with x respect to the order $\overline{0,n-1}$ of the function $f(x,y)$ given by formula (8), and assuming in them $x=a$, we obtain the equalities

$$\frac{\partial^i f(a,y)}{\partial x^i}=i!\overline{\tau}_i(y)+\sum_{k=0}^{m-1}\tilde{\tau}_k^{(i)}(a)(y-c)^k,\quad i=\overline{0,n-2}, \tag{10}$$

$$\frac{\partial^{n-1}f(a,y)}{\partial x^{n-1}}=\frac{1}{(m-1)!}\int_c^y\frac{\eta(s)ds}{(y-s)^{1-m}}+(n-1)!\overline{\tau}_{n-1}(y)+\sum_{k=0}^{m-1}\tilde{\tau}_k^{(n-1)}(a)(y-c)^k. \tag{11}$$

Similarly, differentiating (8) by y and assuming $y=c$, we obtain the equality

$$\frac{\partial^k f(x,c)}{\partial y^k}=k!\tilde{\tau}_k(x)+\sum_{i=0}^{n-1}\overline{\tau}_i^{(k)}(c)(x-a)^i,\quad k=\overline{0,m-2}, \tag{12}$$

$$\frac{\partial^{m-1}f(x,c)}{\partial y^{m-1}}=\frac{1}{(n-1)!}\int_a^x\frac{\psi(t)dt}{(x-t)^{1-n}}+(m-1)!\tilde{\tau}_{m-1}(y)+\sum_{i=0}^{n-1}\overline{\tau}_i^{(m-1)}(c)(x-a)^i. \tag{13}$$

Expressing from formulas (10) - (13) $\overline{\tau}_i(y)$ and $\tilde{\tau}_k(x)$ respectively, we get

$$\sum_{i=0}^{n-1}\overline{\tau}_i(y)(x-a)^i+\sum_{k=0}^{m-1}\tilde{\tau}_k(x)(y-c)^k=\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!}\left(\frac{\partial^i f(a,y)}{\partial x^i}-\sum_{k=0}^{m-1}\tilde{\tau}_k^{(i)}(x)(y-c)^k\right)+$$

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$$+\sum_{k=0}^{m-1}\frac{(y-c)^k}{k!}\left(\frac{\partial^k f(x,c)}{\partial y^k}-\sum_{i=0}^{n-1}\overline{\tau}_i^{(k)}(y)(x-a)^i\right)-\frac{(x-a)^{n-1}}{(n-1)!(m-1)!}\int_c^y\frac{\eta(s)ds}{(y-s)^{1-m}}-$$

$$-\frac{(y-c)^{m-1}}{(n-1)!(m-1)!}\int_a^x\frac{\psi(t)dt}{(x-t)^{1-n}}=\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!}\frac{\partial^i f(a,y)}{\partial x^i}+\sum_{k=0}^{m-1}\frac{(y-c)^k}{k!}\frac{\partial^k f(x,c)}{\partial y^k}-$$

$$-\frac{(x-a)^{n-1}}{(n-1)!(m-1)!}\int_c^y\frac{\eta(s)ds}{(y-s)^{1-m}}-\frac{(y-c)^{m-1}}{(n-1)!(m-1)!}\int_a^x\frac{\psi(t)dt}{(x-t)^{1-n}}-$$

$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}(x-a)^i(y-c)^k\left(\frac{\overline{\tau}_i^{(k)}(c)}{k!}+\frac{\widetilde{\tau}_k^{(i)}(a)}{i!}\right). \tag{14}$$

Calculating the mixed derivatives $\frac{\partial^{i+k}f}{\partial x^i\partial y^k}$ of the function (8) at a point (a,c), we get

$$\frac{1}{i!k!}\frac{\partial^{i+k}f(a,c)}{\partial x^i\partial y^k}=\frac{\overline{\tau}_i^{(k)}(c)}{k!}+\frac{\widetilde{\tau}_k^{(i)}(a)}{i!}. \tag{15}$$

Substituting (14), (15) into (8), we get

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y\frac{\varphi(t,s)dtds}{(x-t)^{1-n}(y-s)^{1-m}}+\sum_{i=0}^{n-1}\frac{1}{i!}\frac{\partial^i f(a,y)}{\partial x^i}(x-a)^i+$$

$$+\sum_{k=0}^{m-1}\frac{1}{k!}\frac{\partial^k f(x,c)}{\partial y^k}(y-c)^k-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{1}{i!k!}\frac{\partial^{i+k}f(a,c)}{\partial x^i\partial y^k}(x-a)^i(y-c)^k. \tag{16}$$

Equality (6) follows from (16) and from the fact that $\varphi(x,y)=\frac{\partial^{m+n}f(a,c)}{\partial x^n\partial y^m}$. The lemma is proved.

The following lemma gives a description of the class $AC^{n,m}(\overline{\Omega})$. It generalizes Lemma 1 to the case of two variables and Lemma 2 to the case $n+m>2$.

Lemma 4: The space $AC^{n,m}(\overline{\Omega})$ consists of those and only those functions $f(x,y)$, which are represented in the form

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y(x-t)^{n-1}(y-s)^{m-1}\varphi(t,s)dtds+$$

$$+\sum_{k=0}^{m-1}\frac{(y-c)^k}{(n-1)!k!}\int_a^x(x-t)^{n-1}\psi_k(t)dt+\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!(m-1)!}\int_c^y(y-s)^{m-1}\eta(s)ds+$$

$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}C_{ik}(x-a)^i(y-c)^k, \tag{17}$$

where $\varphi(x,y)\in L_1(\Omega)$, $\psi_k(x)\in L_1([a,b])$ $\left(k=\overline{0,m-1}\right)$, $\eta(y)\in L_1([c,d])$, $\left(i=\overline{0,n-1}\right)$, C_{ik} being arbitrary constants.

Proof: Necessity. Let $f(x,y)\in AC^{n,m}(\overline{\Omega})$. According to the lemma 3

$$f(x,y)=\frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y\frac{f^{(n,m)}(t,s)}{(x-t)^{1-n}(y-s)^{1-m}}dtds+\sum_{i=0}^{n-1}\frac{f^{(i,0)}(a,y)}{i!}(x-a)^i+$$

$$+\sum_{k=0}^{m-1}\frac{f^{(0,k)}(x,c)^i}{k!}(y-c)^k-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{i!k!}(x-a)^i(y-c)^k. \tag{18}$$

Because $f^{(n-1,m-1)}(x,y)\in AC(\overline{\Omega})$, then $f^{(n-1,m-1)}(a,y)\in AC([c,d])$, consequently, $f^{(n-1,0)}(a,y)\in AC^m([c,d])$, from here $f^{(i,0)}(a,y)\in AC^m([c,d])$ $\left(i=\overline{0,n-1}\right)$. Use lemma [1, c.39]

$$f^{(i,0)}(a,y)=\frac{1}{(m-1)!}\int_c^y\frac{\eta_i(s)}{(y-s)^{1-m}}ds+\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{k!}(y-c)^k \tag{19}$$

where $\eta_i(y)\in L_1([c,d])$. Then

$$\sum_{i=0}^{n-1}\frac{f^{(i,0)}(a,y)}{i!}(x-a)^i=\sum_{i=0}^{n-1}\frac{(x-a)^i}{i!\,(m-1)!}\int_c^y\frac{\eta_i(s)}{(y-s)^{1-m}}ds+$$
$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{i!\,k!}(x-a)^i(y-c)^k. \tag{20}$$

Similarly, it is proved that

$$\sum_{k=0}^{m-1}\frac{f^{(0,k)}(x,c)}{k!}(y-c)^k=\sum_{k=0}^{m-1}\frac{(y-c)^k}{ik!\,(n-1)!}\int_a^x\frac{\psi_k(t)}{(x-t)^{1-m}}dt+$$
$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}(a,c)}{i!\,k!}(x-a)^i(y-c)^k. \tag{21}$$

where $\psi_k(x)\in L_1([a,b])$. Substituting (20), (21) into (18), we obtain the formula (17), in which

$$C_{ik}=\frac{1}{i!k!}f^{(i,k)}(a,c). \tag{22}$$

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Sufficiency. When calculating directly $\frac{\partial^{i+k} f}{\partial x^i \partial y^k}$ $(0 \le i < n, 0 \le k < m)$, it is easy to make sure that they are all continuous in $\overline{\Omega}$, and

$$\frac{\partial^{n+m-2} f}{\partial x^{n-1} \partial y^{m-1}} = \int_a^x \int_c^y \varphi(t,s)dtds + \int_a^x \psi(t)dt + \int_c^y \eta(s)ds + \\ + (n-1)!\,(m-1)!\,c_{n-1,m-1}. \tag{23}$$

Obviously $\frac{\partial^{n+m-2} f}{\partial x^{n-1} \partial y^{m-1}} \in AC(\overline{\Omega})$, from where it follows $f(x,y) \in AC^{n,m}(\overline{\Omega})$.

The theorem is proven completely.

Notice, that

$$\varphi(x,y) = f^{(n,m)}(x,y); \tag{24}$$

$$\psi_k(x) = f^{(n,k)}(x,c), \quad k = \overline{0, m-1}; \tag{25}$$

$$\eta_i(y) = f^{(i,m)}(a,y), \quad i = \overline{0, n-1}; \tag{26}$$

$$C_{ik} = \frac{1}{i!k!} f^{(i,k)}(a,c). \tag{27}$$

Definition 5: [1, c. 459]. Let $f(x,y) \in L_1(\Omega)$. The integral

$$\left(I^{\alpha,\beta}_{a+,c+} f\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y \frac{f(t,s)dtds}{(x-t)^{1-\alpha}(y-s)^{1-\beta}}, \tag{28}$$

where $\alpha > 0, \beta > 0$, is called a left-hand sided mixed Riemann-Liouville fractional integral of order (α, β).

The fractional integral (28) is obviously defined on functions $f(x,y) \in L_1(\Omega)$, existing almost everywhere.

Using the Fubini theorem, the semigroup property is proved.

Let $f(x,y) \in L_1(\Omega)$, $\alpha, \beta, \gamma, \delta$ be positive numbers, then equality holds almost everywhere in Ω

$$I^{\alpha,\beta}_{a+,c+} I^{\gamma,\delta}_{a+,c+} f = I^{\alpha+\gamma,\beta+\delta}_{a+,c+} f. \tag{29}$$

It can be shown that if $\alpha > 0$ function $f(x,y)$ is defined in Ω and $f(x,y) \in L_1(\Omega)$, then

$$\left(I^{\alpha}_{a+,x}\right)(x,y) \in L_1([c,d]) \quad \forall x \in (a,b); \quad \left(I^{\alpha}_{a+,y}\right)(x,y) \in L_1([a,b]) \quad \forall y \in (c,d).$$

In the last equations $I^{\alpha}_{a+,x}f$, $I^{\alpha}_{a+,y}f$ are partial Riemann – Liouville fractional integrals with respect to the variables x and y, respectively.
Taking these equalities into account, it is directly verified that

$$\left(I^{\alpha}_{a+,x}I^{\beta}_{c+,y}f\right)(x,y)=\left(I^{\beta}_{c+,y}I^{\alpha}_{a+,x}f\right)(x,y)=\left(I^{\alpha,\beta}_{a+,c+}f\right)(x,y). \tag{30}$$

Definition 6: [1, c. 460]. For function $f(x,y)$, given on Ω, formula

$$\left(\mathcal{D}^{\alpha,\beta}_{a+,c+}f\right)(x,y)=\frac{1}{\Gamma(n-\alpha)\Gamma(m-\beta)}\frac{\partial^{n+m}}{\partial x^n\partial y^m}\int_a^x\int_c^y\frac{f(t,s)dtds}{(x-t)^{\alpha-n+1}(y-s)^{\beta-m+1}} \tag{31}$$

where $\alpha>0,\ \beta>0$, is called a mixed Riemann-Liouville fractional derivative of order (α,β), $n=[\alpha]+1,\ m=[\beta]+1$.

If the function $f(x,y)$ has a property $I^{n-\alpha,m-\beta}_{a+,c+}f\in AC^{n,m}(\overline{\Omega})$, then the order of taking the derivatives in (31) does not matter, and

$$\left(\mathcal{D}^{\alpha,\beta}_{a+,c+}f\right)(x,y)\in L_1(\Omega)\ .$$

Definition 7 is a two-dimensional analogue of Definition 2.3 [1, p. 43].

III. Compositions of Mixed Fractional Integral and Mixed Fractional Derivative of the same Order

Following [1, p. 44], we define the following classes of functions.

Definition 7: Let $I^{\alpha,\beta}_{a+,c+}(L_1)$ denote the space of function $f(x,y)$, represented by the left-sided mixed fractional integral of order (α,β) of a summable function: $f=I^{\alpha,\beta}_{a+,c+}\varphi,\ \ \varphi\in L_1(\Omega)$.

Definition 8: Let $0<\alpha<1,\ 0<\beta<1$. A function $f(x,y)\in L_1(\Omega)$ is said to have a summable fractional derivative $\mathcal{D}^{\alpha,\beta}_{a+,c+}f$, if $I^{n-\alpha,m-\beta}_{a+,c+}f\in AC^{n,m}(\overline{\Omega})$.

The following theorem defines the necessary and sufficient condition for the unique solvability of the two-dimensional Abel integral equation.

Theorem 1: In order that $f(x,y)\in I^{\alpha,\beta}_{a+,c+}(L_1),\ \alpha>0,\ \beta>0$, it is necessary and sufficient that

$$f_{n-\alpha,m-\beta}\in AC^{n,m}(\overline{\Omega}), \tag{32}$$

where $n=[\alpha]+1,\ m=[\beta]+1$, and that

$$f^{(i,0)}_{n-\alpha,m-\beta}(a,y)\equiv 0,\quad i=\overline{0,n-1}; \tag{33}$$

$$f^{(0,k)}_{n-\alpha,m-\beta}(x,c)\equiv 0,\quad k=\overline{0,m-1}\ \ ; \tag{34}$$

$$f^{(i,k)}_{n-\alpha,m-\beta}(a,c)\equiv 0,\quad i=\overline{0,n-1},\ \ k=\overline{0,m-1}. \tag{35}$$

Notes

Notes

Proof: Necessity. Let $f = I_{a+,c+}^{\alpha,\beta}\varphi,\ \ \varphi \in L_1(\Omega)$. In view of the semigroup property

$$f_{n-\alpha,m-\beta}(x,y) = I_{a+,c+}^{n-\alpha,m-\beta} f = I_{a+,c+}^{\alpha,\beta}\varphi\,, \tag{36}$$

where $\varphi \in L_1(\Omega)$. From here follow feasibility conditions (33) – (35). Feasibility condition (32) follow from Lemma 4.

This implies the fulfillment of conditions (33) - (35). The fulfillment of condition (32) follows from Lemma 4.

Sufficiency. Under condition (32), we can present $f_{n-\alpha,m-\beta}$ according to Lemma 3, in the form

$$f_{n-\alpha,m-\beta}(x,y) = \frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y \frac{f_{n-\alpha,\,m-\beta}^{(n,m)}(t,s)}{(x-t)^{1-n}(y-s)^{1-m}}\,dtds +$$

$$+\sum_{i=0}^{n-1}\frac{f_{n-\alpha,m-\beta}^{(i,0)}(a,y)}{i!}(x-a)^i + \sum_{k=0}^{m-1}\frac{f_{n-\alpha,m-\beta}^{(0,k)}(x,c)}{k!}(y-c)^k -$$

$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f_{n-\alpha,m-\beta}^{(i,k)}(a,c)}{k!}(x-a)^i(y-c)^k\,, \tag{37}$$

where $f_{n-\alpha,m-\beta}^{(n,m)} \in L_1(\Omega)$. Taking into account conditions (33) - (35), the last equality is written in the form

$$f_{n-\alpha,m-\beta}(x,y) = \frac{1}{(n-1)!(m-1)!}\int_a^x\int_c^y \frac{f_{n-\alpha,m-\beta}^{(n,m)}(t,s)}{(x-t)^{1-n}(y-s)^{1-m}}\,dtds\,. \tag{38}$$

Using the semigroup property (29), we can write

$$I_{a+,c+}^{n-\alpha,m-\beta} f = I_{a+,c+}^{n,m} f_{n-\alpha,m-\beta}^{(n,m)} = I_{a+,c+}^{n-\alpha,m-\beta} I_{a+,c+}^{\alpha,\beta} f_{n-\alpha,m-\beta}^{(n,m)}\,. \tag{39}$$

From here $I_{a+,c+}^{n-\alpha,m-\beta}\left(f - I_{a+,c+}^{\alpha,\beta} f_{n-\alpha,m-\beta}^{(n,m)}\right) = 0$. Applying the integral to this equality $I_{a+,c+}^{\alpha,\beta}$, we get

$$I_{a+,c+}^{n,m}\left(f - I_{a+,c+}^{\alpha,\beta} f_{n-\alpha,m-\beta}^{(n,m)}\right)dxdy = 0\,. \tag{40}$$

From here $f = I_{a+,c+}^{\alpha,\beta} f_{n-\alpha,m-\beta}^{(n,m)},\quad f_{n-\alpha,m-\beta}^{(n,m)} \in L_1(\Omega)$. The theorem is proved.

Note that Theorem 1 is a generalization of Theorem 2.3 [1, p. 43] in the case of two variables. From it, in particular, it follows that the class of functions having a summable fractional derivative $\mathcal{D}_{a+,c+}^{\alpha,\beta} f$ in the sense of Definition 8 is wider than the class of functions $I_{a+,c+}^{\alpha,\beta}(L_1)$. Namely, the class $I_{a+,c+}^{\alpha,\beta}(L_1)$ owns only those functions that have a summable fractional derivative $\mathcal{D}_{a+,c+}^{\alpha,\beta} f$, for which equalities (33) - (35) hold.

Theorem 2: Let $\alpha > 0,\ \beta > 0$. Then equality

$$\mathcal{D}^{\alpha,\beta}_{a+,c+} I^{\alpha,\beta}_{a+,c+} f = f(x,y) \tag{41}$$

performed for any summable function $f(x,y)$.

Proof: We have

$$\mathcal{D}^{\alpha,\beta}_{a+,c+} I^{\alpha,\beta}_{a+,c+} f = \frac{\partial^{n+m}}{\partial x^n \partial y^m} I^{n-\alpha,m-\beta}_{a+,c+} I^{\alpha,\beta}_{a+,c+} f =$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha)\Gamma(m-\beta)} \frac{\partial^{n+m}}{\partial x^n \partial y^m} \int_a^x \int_c^y \frac{dtds}{(x-t)^\alpha (y-s)^\beta} \int_a^t \int_c^s \frac{f(u,v)dudv}{(t-u)^{n-\alpha}(s-v)^{m-\beta}} \tag{42}$$

Changing the order of integration, we get

$$\mathcal{D}^{\alpha,\beta}_{a+,c+} I^{\alpha,\beta}_{a+,c+} f = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n-\alpha)\Gamma(m-\beta)} \times$$

$$\times \frac{\partial^{n+m}}{\partial x^n \partial y^m} \int_a^x \int_c^y f(u,v)dudv \int_u^x \int_v^y \frac{dtds}{(x-t)^\alpha (y-s)^\beta (t-u)^{n-\alpha}(s-v)^{m-\beta}} =$$

$$= \frac{\partial^{n+m}}{\partial x^n \partial y^m} \int_a^x \int_c^y f(u,v)dudv \frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_u^x \frac{dt}{(x-t)^\alpha (t-u)^{n-\alpha}} \frac{1}{\Gamma(\beta)\Gamma(m-\beta)} \times$$

$$\times \int_v^y \frac{ds}{(y-s)^\beta (s-v)^{m-\beta}} = \frac{1}{\Gamma(n)\Gamma(m)} \frac{\partial^{n+m}}{\partial x^n \partial y^m} \int_a^x \int_c^y \frac{f(u,v)}{(x-u)^{1-n}(y-v)^{1-m}} dudv =$$

$$= f(x,y), \tag{43}$$

Q.E.D.

Theorem 3: For any function $f(x,y) \in I^{\alpha,\beta}_{a+,c+}(L_1)$ the equality

$$I^{\alpha,\beta}_{a+,c+} \mathcal{D}^{\alpha,\beta}_{a+,c+} f = f(x,y), \tag{44}$$

and for any function that has a summable derivative $\mathcal{D}^{\alpha,\beta}_{a+,c+} f$ (in the sense of definition 8), the equality

$$I^{\alpha,\beta}_{a+,c+} \mathcal{D}^{\alpha,\beta}_{a+,c+} f = f(x,y) - \sum_{i=0}^{n-1} \frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)} f^{(n-i-1,0)}_{n-\alpha,0}(a,y) - \sum_{k=0}^{m-1} \frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)} f^{(0,m-k-1)}_{0,m-\beta}(x,c) +$$

$$+ \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} \frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)} f^{(n-i-1,m-k-1)}_{n-\alpha,m-\beta}(a,c), \tag{45}$$

where $f_{\gamma,\delta}(x,y) = I^{\gamma,\delta}_{a+,c+} f$.

Notes

Proof: Let $f(x,y)\in I^{\alpha,\beta}_{a+,c+}(L_1)$, then $f(x,y)=I^{\alpha,\beta}_{a+,c+}\varphi$, $\varphi(x,y)\in L_1(\Omega)$. Based on Theorem 2, we have

$$I^{\alpha,\beta}_{a+,c+}\mathcal{D}^{\alpha,\beta}_{a+,c+}f=I^{\alpha,\beta}_{a+,c+}\mathcal{D}^{\alpha,\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}\varphi=I^{\alpha,\beta}_{a+,c+}\varphi=f(x,y). \tag{46}$$

Notes

Let now $I^{1-\alpha,1-\beta}_{a+,c+}f\in AC(\overline{\Omega})$. According to Lemma 3, the integral $f_{n-\alpha,m-\beta}(x,y)=I^{n-\alpha,m-\beta}_{a+,c+}f$ can be represented as

$$f_{n-\alpha,m-\beta}(x,y)=I^{n,m}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}+\sum_{i=0}^{n-1}\frac{f^{(i,0)}_{n-\alpha,m-\beta}(a,y)}{i!}(x-a)^i+\sum_{k=0}^{m-1}\frac{f^{(0,k)}_{n-\alpha,m-\beta}(x,c)}{k!}(y-c)^k-$$

$$-\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{f^{(i,k)}_{n-\alpha,m-\beta}(a,c)}{k!i!}(x-a)^i(y-c)^k. \tag{47}$$

By the semigroup property, the equality

$$I^{n,m}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}=I^{n-\alpha,m-\beta}_{a+,c+}I^{\alpha,\beta}_{a+,c+}f^{(n,m)}_{n-\alpha,m-\beta}. \tag{48}$$

Further,

$$\frac{(x-a)^i}{i!}f^{(i,0)}_{n-\alpha,m-\beta}(a,y)=I^{n-\alpha,m-\beta}_{a+,c+}\left(\mathcal{D}^{n-\alpha}_{a+,x}\frac{(x-a)^i}{i!}\mathcal{D}^{m-\beta}_{c+,y}f^{(i,0)}_{n-\alpha,m-\beta}(a,y)\right)+$$

$$+\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f^{(i,0)}_{n-\alpha,1}(a,c)=I^{n-\alpha,m-\beta}_{a+,c+}\left(\frac{(x-a)^{i-n+\alpha}}{\Gamma(1+i-n+\alpha)}f^{(i,0)}_{n-\alpha,0}(a,y)\right)+$$

$$+\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f^{(i,0)}_{n-\alpha,1}(a,c). \tag{49}$$

From the last equality it follows that

$$\sum_{i=0}^{n-1}\frac{f^{(i,0)}_{n-\alpha,m-\beta}(a,y)}{i!}(x-a)^i=I^{n-\alpha,m-\beta}_{a+,c+}\left(\sum_{i=0}^{n-1}\frac{(x-a)^{i-n+\alpha}}{\Gamma(1+i-n+\alpha)}f^{(i,0)}_{n-\alpha,0}(a,y)\right)+$$

$$+\sum_{i=0}^{n-1}\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f^{(i,0)}_{n-\alpha,1}(a,c), \tag{50}$$

from where, redesignating the summation index, we get

$$\sum_{i=0}^{n-1}\frac{f^{(i,0)}_{n-\alpha,m-\beta}(a,y)}{i!}(x-a)^i=I^{n-\alpha,m-\beta}_{a+,c+}\left(\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f^{(n-i-1,0)}_{n-\alpha,0}(a,y)\right)+$$

$$+\sum_{i=0}^{n-1}\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f_{n-\alpha,1}^{(i,0)}(a,c). \tag{51}$$

Equality is obtained similarly

$$\sum_{i=0}^{n-1}\frac{f_{n-\alpha,m-\beta}^{(0,k)}(x,c)}{i!}(y-c)^k = I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)\right)+$$

$$+\sum_{k=0}^{m-1}\frac{(x-a)^{n-\alpha-1}(y-c)^k}{k!\Gamma(n-\alpha)}f_{1,m-\beta}^{(0,k)}(a,c). \tag{52}$$

It is not difficult to see that

$$\sum_{i=0}^{n-1}\sum_{i=0}^{n-1}\frac{f_{n-\alpha,m-\beta}^{(i,k)}(a,c)}{i!k!}(x-a)^i(y-c)^k =$$

$$= I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}I_{a+,c+}^{n-\alpha,m-\beta}\left(\frac{(x-a)^i(y-c)^k}{i!\,k!}\right)f_{n-\alpha,m-\beta}^{(i,k)}(a,c)\right)=$$

$$= I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right). \tag{53}$$

Taking into account equalities (48), (51) - (53), equality (47) is written in the form

$$I_{a+,c+}^{n-\alpha,m-\beta}f = I_{a+,c+}^{n-\alpha,m-\beta}I_{a+,c+}^{\alpha,\beta}\mathcal{D}_{a+,c+}^{\alpha,\beta}f + I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)\right)+$$

$$+\sum_{i=0}^{n-1}\frac{(x-a)^i(y-c)^{m-\beta-1}}{i!\Gamma(m-\beta)}f_{n-\alpha,1}^{(i,0)}(a,c)+I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)\right)+$$

$$+\sum_{k=0}^{m-1}\frac{(x-a)^{n-\alpha-1}(y-c)^k}{k!\Gamma(n-\alpha)}f_{1,m-\beta}^{(0,k)}(a,c)-$$

$$-I_{a+,c+}^{n-\alpha,m-\beta}\left(\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right). \tag{54}$$

By grouping the terms, we get

$$I_{a+,c+}^{n-\alpha,m-\beta}\left(f - I_{a+,c+}^{\alpha,\beta}\mathcal{D}_{a+,c+}^{\alpha,\beta}f - \sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)-\right.$$

$$\left.-\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right)=$$

Notes

$$=\sum_{i=0}^{n-1}\frac{(x-a)^{i}(y-c)^{m-\beta-1}}{i!\,\Gamma(m-k)}f_{n-\alpha,1}^{(i,0)}(a,c)+\sum_{k=0}^{m-1}\frac{(y-c)^{k}(x-a)^{n-\alpha-1}}{k!\,\Gamma(n-\alpha)}f_{1,m-\beta}^{(0,k)}(a,c) \qquad (55)$$

In the right-hand side of equality (55), under the integral is a summable function. Applying the operator $I_{a+,c+}^{\alpha,\beta}$ to both parts of equality (55), we obtain

Notes

$$I_{a+,c+}^{n,m}\left(f-I_{a+,c+}^{\alpha,\beta}\mathcal{D}_{a+,c+}^{\alpha,\beta}f-\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)-\right.$$

$$\left.-\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)\right)=$$

$$=\sum_{i=0}^{n-1}\frac{(x-a)^{i+\alpha}(y-c)^{m-1}}{\Gamma(i+\alpha+1)\Gamma(m)}f_{n-\alpha,1}^{(i,0)}(a,c)+\sum_{k=0}^{m-1}\frac{(y-c)^{k+\beta}(x-a)^{n-1}}{\Gamma(k+1+\beta)\Gamma(n)}f_{1,m-\beta}^{(0,k)}(a,c). \qquad (56)$$

Under the integral on the left side of the equality is the summable function, and the right side of the equality is absolutely continuous. Finding the mixed derivative $\frac{\partial^{n+m}}{\partial x^{n}\partial y^{m}}$ of both parts of the equality, we get

$$f-I_{a+,c+}^{\alpha,\beta}\mathcal{D}_{a+,c+}^{\alpha,\beta}f-\sum_{i=0}^{n-1}\frac{(x-a)^{\alpha-i-1}}{\Gamma(\alpha-i)}f_{n-\alpha,0}^{(n-i-1,0)}(a,y)-\sum_{k=0}^{m-1}\frac{(y-c)^{\beta-k-1}}{\Gamma(\beta-k)}f_{0,m-\beta}^{(0,m-k-1)}(x,c)+$$

$$+\sum_{i=0}^{n-1}\sum_{k=0}^{m-1}\frac{(x-a)^{\alpha-i-1}(y-c)^{\beta-k-1}}{\Gamma(\alpha-i)\Gamma(\beta-k)}f_{n-\alpha,m-\beta}^{(n-i-1,m-k-1)}(a,c)=0. \qquad (57)$$

The theorem is proved.

References Références Referencias

1. Samko S.G., A.A. Kilbas and O.I. Marichev. Fractional Integrals and Derivatives. Theory and Applications. Gordon and Breach. Sci. Publ., N. York - London, 1993, 1012 pp. (Russian Ed. - Fractional Interals and Derivatives and Some of Their Applications, Nauka i Tekhnika, Minsk, 1987).
2. Vasilache S. Asupra unei ecuatii integrale de tip Abel cu doua variable.//Comun. Acad.R.P.Romane, 1953. Vol. 3. P. 109-113
3. Smirnov V.I. Higher mathematics course. T. 5. M.: OGIZ, 1947. 584 p.
4. Mamatov T. Mixed fractional integration operators in Hölder spaces (Russian). «Science and World¸. Volgograd, № 1 (1). 2013, P. 30-38
5. Mamatov T, S. Samko. Mixed Fractional Integration Operators in Mixed Weighted Hölder Spaces. FC & AA. Vol.13, Num 3. 2010, p. 245-259
6. Mamatov T. Weighted Zygmund estimates for mixed fractional integration. Case Studies Journal. Volume 7, Issue 5, 2018. p. 82-88
7. Mamatov T. Mixed Fractional Integration In Mixed Weighted Generalized Hölder Spaces. Case Studies Journal. Volume 7, Issue 6. 2018, P. 61-68

8. Mamatov T. Mixed Fractional Integro-Differentiation Operators in Hölder Spaces. The latest research in modren science: experience, traditions and innovations. Proceedings of the VII International Scientific Conference. Section I. North Charleston, SC, USA. 20-21 June, 2018. P. 6-9
9. Mamatov T. Mapping Properties Of Mixed Fractional Integro-Differentiation in Hölder Spaces. Journal of Concrete and Applicable Mathematics (JCAAM). Volume 12, Num. 3-4. 2014. p. 272-290
10. Mamatov T., Rayimov D., Elmurodov M. Mixed Fractioanl Differentiation Operators in Hölder Spaces. Journal of Multidisciplinary Engineering Science and Technology (JMEST), Vol. 6 Issue 4, April – 2019. P. 9855-9857
11. Mamatov T., Homidov F., Rayimov D. On Isomorphism Implemented By Mixed Fractional Integrals In Hölder Spaces. "International Journal of Development Research", Vol. 09, Issue, 05, pp. 27720-27730, May 2019.
12. Mamatov T. Mapping Properties of Mixed Fractional Differentiation Operators in Hölder Spaces Defined by Usual Hölder Condition. Journal of Computer Science & Computational Mathematics, Volume 9, Issue 2, June 2019. DOI: 10.20967/jcscm.2019.02.003

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A Comparative Study between Finite Difference Method and Finite Volume Method for Shallow Water-Dam Break Flow Problem

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Abstract- In this paper we perform numerical simulation of shallow water equations for dam break flow problem. Finite difference method and finite volume method are applied for the numerical solution of the shallow water equations. We estimate water height and water velocity for the test case of shallow water-dam break flow problem at different height ratio. The comparisons between the finite difference method and finite volume method for 1-D dam break problem have been shown. The agreement of numerical solution with analytic solution by finite volume method is better than finite difference method. We compare the computational efficiency for both schemes in subcritical flow.

Keywords: finite difference method, finite volume method, shallow water dam break flow, water height and water velocity.

GJSFR-F Classification: MSC 2010: 65L12

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I. Introduction

Many problems of river management and civil protection consist of the evaluation of the maximum water levels and discharges that may be attained at particular locations during the development of an exceptional meteorological event. There is also prevision of the scenario subsequent to the almost instantaneous release of a great volume of liquid. The situation is that of breaking of a man made of dam. Free surface flows common in hydraulics are usually described by means of the shallow water equations, provided that the representative vertical dimensions are small with respect to horizontal dimensions. Despite their simplicity, this description is valid in many practical applications, rendering worthwhile the efforts in developing good numerical methods to solve the corresponding system of differential equations. Numerical methods are nowadays a common tool to predict flow properties both for steady and unsteady situations of practical interest in hydraulics. The applications of finite differences and finite volumes have been widely reported in particular. The shallow water equations (SWEs) describe the evolution of hydrostatic homogeneous (constant density), incompressible fluid in response to gravitational and rotational accelerations and they are derived from the principles of conservation of mass and conservation of momentum. The SWEs (also called Saint- Venant equations) are one of the simplest form of the equations of motion that can be used to describe the horizontal structure of an atmosphere and ocean that model the propagation of disturbances in fluid. They are

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widely used to model the free surface water flows such as periodic (tidal) flows, transient wave phenomena [10] (tsunamis, flood waves and dam break waves) etc. In fluid dynamics the flow of the fluid is known as the Navier- Stocks equation. The shallow water equations are good approximation to the fluid motion equation when fluid density is homogeneous and depth is small in comparison to characteristic horizontal distance.

Most flows on the surface of the Earths, for examples in rivers, seas and the atmosphere, are Shallow water flows in which the horizontal length and velocity scales of interest are much larger than the vertical one's. The Mathematical formulation of this flows, the so called Shallow water equations, are already known for over a century. At presents, these models are used in all kinds of applications such as flood warning system, impact of changes of water system, climate predictions and reducing water pollution. The analysis of dam break flow is important to capture spatial and temporal evolution of flood event and safety analysis. The dam break basically is catastrophic failure of dam, leading to uncontrolled release of water causing flood in the downstream region. Many Numerical methods are available in the literature to solve the shallow water equations. Bellos et al. [1] reported a two dimensional numerical methods for dam break problem by using the combination of finite element and finite difference method. Fayssal and Mohammed [2] proposed a new simple finite volume method for the numerical solution of shallow water equations. Rahaman et al. [4] have studied the finite difference method to simulate the water height and velocity through shallow water equations. Bagheri and Das [5] developed an implicit high order compact scheme for shallow water equations with dam break problem. Saiduzzaman & Ray [7] examined some numerical methods for shallow water equations. Ahmed et al. [8] developed Godunov type finite volume method for dam break problem.

In section 2, provides shallow water dam break flow problem. We investigate the finite difference Lax Friedrichs method and finite volume method for the numerical solution of shallow water equations in section 3. Numerical results are presented in section 4. Finally the conclusions of the work are given in the last section.

II. Mathematical Model

The Mathematical model that describes the water flow in a river are defined as

$$\begin{aligned} &\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hv)=0 \\ &\frac{\partial v}{\partial t}+\frac{\partial}{\partial x}(\frac{1}{2}v^2+gh)=0 \end{aligned} \tag{1}$$

Where $h(x,t)$ is the water height at the time t and at the space x, g is the acceleration due to gravity, $v(x,t)$ is the flow velocity in the x-direction. With initial and Neumann boundary condition.

$$h(x,0)=\begin{cases} h_l & ,x<=L/2 \\ h_r & ,others \end{cases}$$

$$v(x,0)=v_0(x)\,;0\le x\le L$$

III. Numerical Method

We present here the discretization of the shallow water dam break flow model by finite difference formula, which leads to formulate explicit finite

Ref

1. Bellos V and Hrissanthou V, "Numerical simulation of Dam break flood wave". European water 33: Page: 45-53, 2011.

difference methods for the numerical solution of the governing equation as a nonlinear partial differential equation. The finite volume method is a numerical technique that transforms the partial differential equations representing conservation laws over differential volumes into discrete algebraic equations over finite volumes or cells.

Notes

a) Finite difference method

The numerical discretization of (1) is as follows

$$h_i^{n+1} = \frac{1}{2}(h_{i+1}^n + h_{i-1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}(h_{i+1}^n v_{i+1}^n - h_{i-1}^n v_{i-1}^n) \tag{3}$$

$$v_i^{n+1} = \frac{1}{2}(v_{i+1}^n + v_{i-1}^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}[(\frac{1}{2}v_{i+1}^n v_{i+1}^n + gh_{i+1}^n) - (\frac{1}{2}v_{i-1}^n v_{i-1}^n + gh_{i-1}^n)] \tag{4}$$

b) Finite volume method

We consider conservation law to formulate finite volume method by the following PDE.

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{5}$$

$$\frac{\partial U}{\partial t} + (\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}) \bullet \vec{H} = 0$$

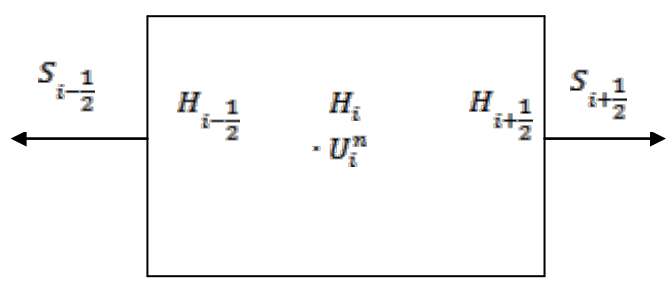
where $\vec{H} = F\vec{i} + 0\vec{j}$ is the flux density vector

$$\frac{\partial U}{\partial t} + \vec{\nabla} \bullet \vec{H} = 0$$

$$\iint_R \frac{\partial U}{\partial t} dR + \iint_R \nabla \bullet \vec{H} dR = 0$$

$$\iint_R \frac{\partial U}{\partial t} dR + \int_C \vec{H} \bullet \vec{n} ds = 0$$

$$\frac{\partial}{\partial t}\iint_R U dR + \int_C \vec{H} \bullet \vec{n} ds = 0$$



When $\vec{n}$ is outward pointing unit normal vector at each point on C. Δx is cell length in x_direction and Δy cell length in y_direction and area of the cell $A = \Delta x \Delta y$. For each cell the two side vectors in the x_direction are $S_{i+\frac{1}{2}} = \Delta y\vec{i} + 0\vec{j}$ and $S_{i-\frac{1}{2}} = -\Delta y\vec{i} + 0\vec{j}$.

$$\frac{\partial}{\partial t} AU_i + \int_C \vec{H} \bullet \vec{n} ds = 0$$

$$\frac{\partial U_i}{\partial t} = -\frac{1}{A}\sum_{sides} \vec{H} \bullet \vec{s}$$

$$U_{next}(k,i) = 0.5(U(k,i-1) + U(k,i+1)) - \frac{\Delta t}{A}[H_{i+\frac{1}{2}} \bullet S_{i+\frac{1}{2}} + H_{i-\frac{1}{2}} \bullet S_{i-\frac{1}{2}}] \tag{6}$$

IV. Numerical Results and Discussion

We start by assuming that in both sides of the dam there are water with corresponding heights h_l and h_r, $h_l > h_r$. We use the three depth ratio (i) $h_r / h_l > 0.5$ (ii) $h_r / h_l < 0.5$ (iii) $h_r / h_l << 0.5$ for simulation. The dam is situated at 1000 m in channel and at time $t = 0$ the dam collapses. The flow consists of a shock wave travelling downstream and a rarefaction travelling upstream. Comparisons are carried out only for wet bed condition with respect to velocity and water height. The results are shown in the following figures.

Notes

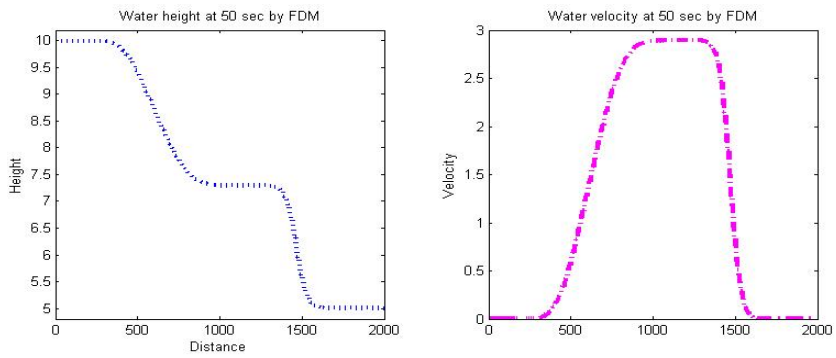


Fig. 1: Water height (left) and water velocity (right) for dam break on wet bed at 50 sec using height ratio 0.5

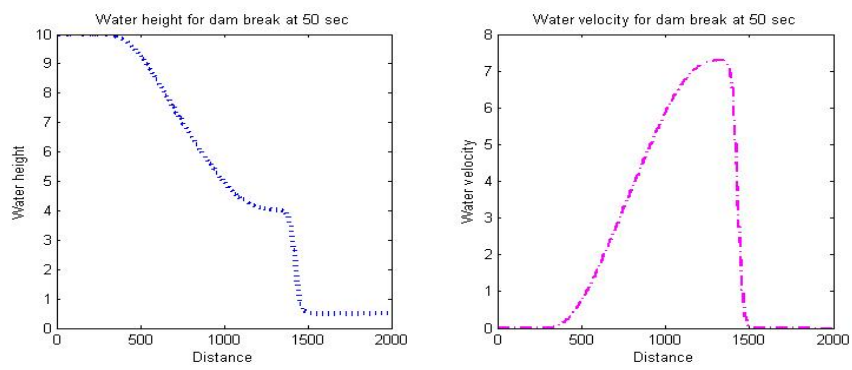


Fig. 2: Water height (left) and water velocity (right) in dam break at 50 sec using height ratio 0.05.

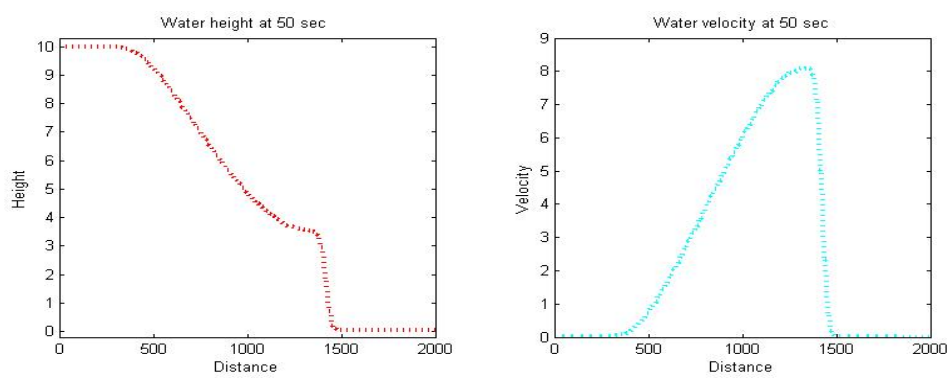


Fig. 3: Water height (left) and water velocity (right) in dam break at 50 sec using height ratio 0.005.

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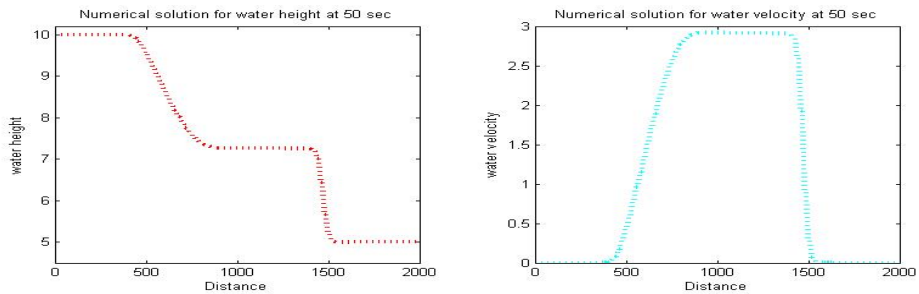

Fig. 4: Water height (left) and water velocity (right) in dam break at 50 sec using height ratio 0.5.

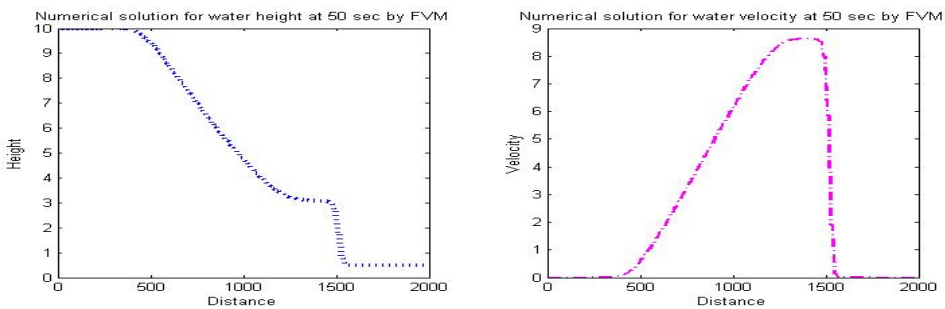

Fig. 5: Water height (left) and water velocity (right) in dam break at 50 sec using height ratio 0.05.

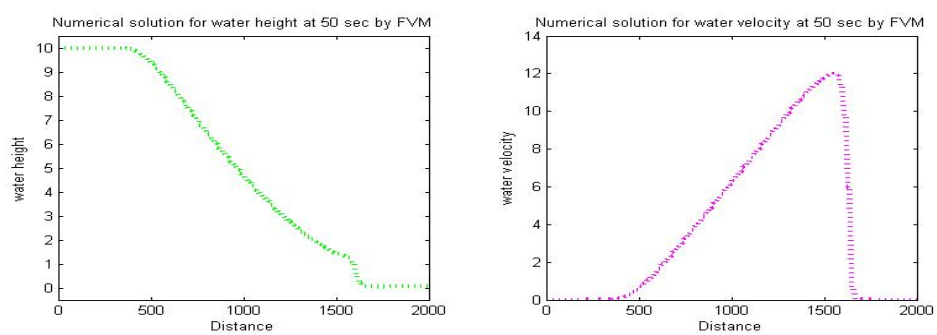

Fig. 6: Water height (left) and water velocity (right) in dam break at 50 sec for height ratio 0.005.

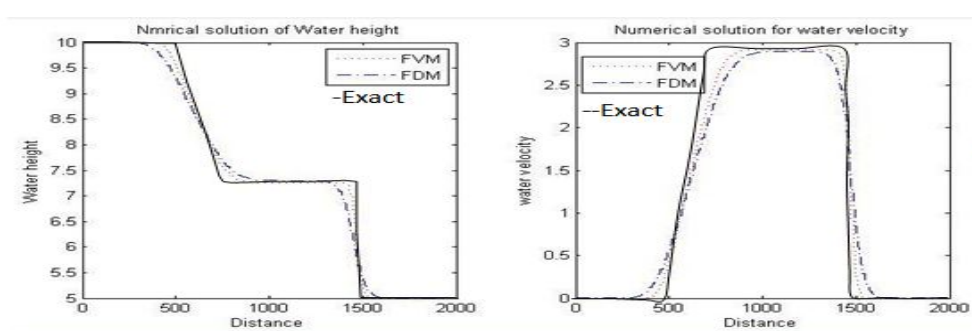

Fig. 7: Finite difference method, finite volume method and analytic solution ([2], [3]) of the dam break test problem for 50 sec using height ratio 0.5.

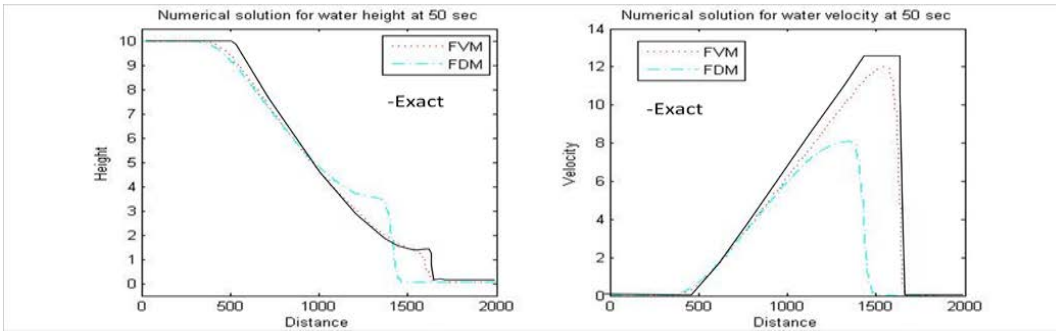

Fig. 8: Finite difference method, finite volume method and analytic solution ([2], [3]) of the dam break test problem for 50 sec using height ratio 0.005.

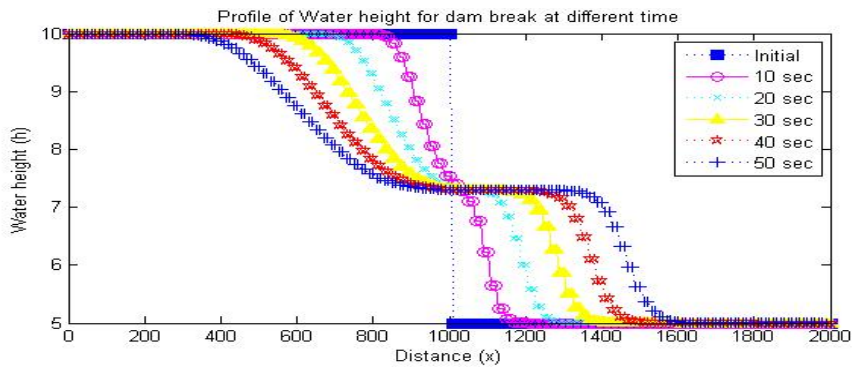

Fig. 9: Water heights in dam break at different time for subcritical flow by FDM.

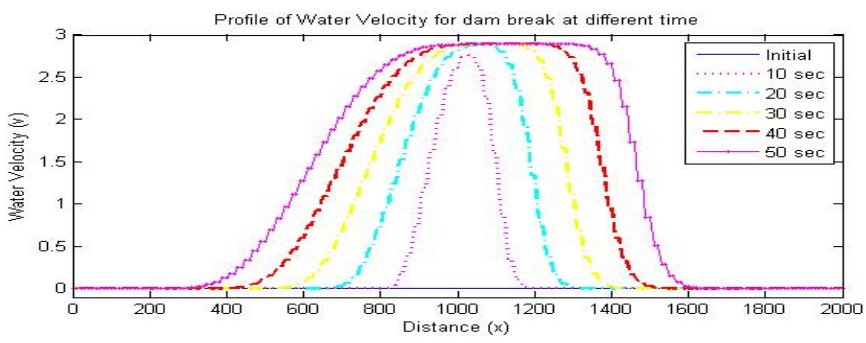

Fig. 10: Water velocity in dam break at different time for subcritical flow by FDM.

a) CPU times for FDM and FVM

We compare the computational efficiency for both numerical schemes.

Table 1: Efficiency test

	Time Step size	Stability condition	CPU time (sec)
FVM	0.05	0.0567	23.72
FDM	0.05	0.0567	2.75
FVM	0.5	0.567	2.586
FDM	0.5	0.567	0.43

Table 1 shows the CPU time of different method for different time step size. We observed that the CPU time of the problem is reduced very well. Table shows for fixed time step size the CPU time of FDM is less than the other method. FDM performs more efficiently than FVM.

Notes

Fig. 1-6 shows the water height and velocity profiles for different height ratio at time 50 sec after the dam-break for both schemes. Fig 7-8 shows comparison of height and velocity profiles for different height ratio of channel at time 50 sec after the dam-break. In fig.7-8, the comparison of numerical and analytical solution corresponding to water height as well as water velocity profile shows good agreement. Finite volume method provides more accurate results than Finite difference method. The analytical reference solutions for these test problems are due to [2], [3]. Therefore the above realistic phenomenons are well described by our implementation.

V. Conclusion

We have demonstrated numerical simulation using FDM and FVM for solving shallow water equations with initial boundary conditions. We have implemented the numerical scheme to simulate water height and water velocity at different condition through shallow water dam break flow after the dam break. We have shown that the numerical result based on the FDM and FVM agrees with some qualitative / realistic behavior of SWE. The solution of finite volume method shows good agreement with exact. Finite difference method is more efficient than finite volume method. The analysis of dam break flow is important to capture spatial and temporal evolution of flood event and safety analysis.

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References Références Referencias

1. Bellos V and Hrissanthou V, "Numerical simulation of Dam break flood wave". European water 33: Page: 45-53, 2011.
2. Fayssal Benkhaldoun and Mohammed Seaid, "A simple finite volume method for shallow water equations". Journal of computational and Applied Mathematics 234(2010), Page: 58-72.
3. J. J. Stoker, Water Waves, Interscience Publishers, Inc., New York, 1986.
4. M. M. Rahaman, L. S. Andallah & M. S. Alam, "Numerical solution of advection diffusion reaction equation coupled with shallow water equations". International Journal of Scientific and Engineering Research , Vol. 8, Issue-11, Nov. 2017, Page-256-262.
5. Jafar Baghari and Samir K Das, "Modeling of shallow water equations by using implicit high order compact scheme with application to dam break problem". Applied and computational Mathematics, 2013, research article.
6. Samir K Das and Jafar Baghari, "Modeling of shallow water equations by using compact Mac-Cormack type schemes with application to dam break problem". International journal of advances in Applied Mathematics and Mechanics, 2015, research article
7. Md. Saiduzzaman & Sobuj Kumar Ray, "Comparison of numerical schemes for shallow water equation". Global Journal of Science Frontier Research Mathematics and Decision Science, Vol-13, Issue-4, Version 1, 2013, Page: 29-46.

Ref

3. J. J. Stoker, Water Waves, Interscience Publishers, Inc., New York, 1986.

8. M.F. Ahmed, M. Mamat, W.B Wan Nik and A. Kartono, "Numerical method for dam break problem by using Godunov approach".App. Math.and Comp.Intel., Vol-2(1), 2013, Page: 95-107.
9. Randall J. Leveque, "Numerical methods for conservation laws", second edition, 1992, Springer.
10. Pilar García-Navarro & Pilar Brufau, "One Dimensional Dam Break Flow Modeling"

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2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

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Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- Single section and succinct.
- An outline of the job done is always written in past tense.
- Concentrate on shortening results—limit background information to a verdict or two.
- Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

Index

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