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Diabetic Retinopathy Fundus Images

Discovering Thoughts, Inventing Future

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Gaussian Kernel Prompted Fuzzy C Means Algorithm with Multi-Object Contouring Method for Segmenting NPDR Features in Diabetic Retinopathy Fundus Images

By Shalini. R & Sasikala. S
University of Madras

Abstract- Diabetic retinopathy is an ophthalmic inflammation caused by diabetes, which ends in visual defacement if not diagnosed earlier, and that has two types, namely Non-Proliferative Diabetic Retinopathy (NPDR) and Proliferative Diabetic Retinopathy (PDR). NPDR features are present in the earliest stage, and systematic detection of these features can improve the diagnosis of the disease severity formerly. Several detection methods exist previously. Still, there is performance lack on large datasets. The objective of this study is detecting NPDR features from diabetic retinopathy fundus images of large datasets with performance level. The study has investigated different fuzzy-based systems and to execute the objective; the GK_FCM approach was proposed, which integrates Gaussian Kernel function in conventional FCM. The execution has four phases. Initially, the input image undergoes preprocessing using green channel extraction, median filter to enhance the image quality and background removal is performed with extended minima transform technique, mathematical arithmetic operation and pixel replacement method to remove the outlier called Fovea (FV).

Keywords: non-proliferative diabetic retinopathy, minima transform technique, gaussian kernel, fuzzy C means, multiclass contour tracking algorithm.

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Notes

Gaussian Kernel Prompted Fuzzy C Means Algorithm with Multi-Object Contouring Method for Segmenting NPDR Features in Diabetic Retinopathy Fundus Images

Shalini. R [α] & Sasikala. S [σ]

Abstract- Diabetic retinopathy is an ophthalmic inflammation caused by diabetes, which ends in visual defacement if not diagnosed earlier, and that has two types, namely Non-Proliferative Diabetic Retinopathy (NPDR) and Proliferative Diabetic Retinopathy (PDR). NPDR features are present in the earliest stage, and systematic detection of these features can improve the diagnosis of the disease severity formerly. Several detection methods exist previously. Still, there is performance lack on large datasets. The objective of this study is detecting NPDR features from diabetic retinopathy fundus images of large datasets with performance level. The study has investigated different fuzzy-based systems and to execute the objective; the GK_FCM approach was proposed, which integrates Gaussian Kernel function in conventional FCM. The execution has four phases. Initially, the input image undergoes preprocessing using green channel extraction, median filter to enhance the image quality and background removal is performed with extended minima transform technique, mathematical arithmetic operation and pixel replacement method to remove the outlier called Fovea (FV).

Further, it is segmented for extracting NPDR features such as Micro-aneurysms (MA), Intra-retinal Hemorrhages (IHM), and Hard Exudates (HEXU) using Gaussian kernel with FCM of multiple parameters. Finally, the extracted features are visually enhanced on the original input image using post-processing operation of multi-class contour tracking (MCT) algorithm comprising different contouring measures. The experiments were done on two available online databases, namely DIARETDB0 and DIARETDB1. The performance of the proposed method is evaluated using the validation measures and compared with kernel induced fuzzy algorithms like MKFCM and LKFCM, comparatively the proposed GK_FCM method outperforms. Hence,the Gaussian kernel-based technique has been used for the analysis of the diabetic retinopathy fundus images to detect NPDR features of Diabetic retinopathy. The proposed work has given better results with an accuracy of 98.21%.

Keywords: non-proliferative diabetic retinopathy, minima transform technique, gaussian kernel, fuzzy C means, multiclass contour tracking algorithm.

I. Introduction

Diabetes mellitus ordinarily referred to as diabetes is a protracted disease that occurs when the pancreas is no longer able to create insulin, so the glucose in the blood are not being transferred into cells, which leads to high blood glucose. The prolonged blood glucose levels in the human body will cause several complications such as blindness, kidney failure, amputations, heart failure, stroke, etc. But among these conditions, blindness due to diabetes is considered an issue as the eyes is the essential organs of our body. The human eye is a significant body organ, but the care taken for

Author α σ: Department of Computer Science, IDE, University of Madras, Chennai, India. e-mail: rshalini1990@gmail.com

this organ is emphasized very less in healthcare. There is no awareness among people about related complications like blindness caused due to diabetes. For instance, if people get blurry vision, they go for a computerized eye tests and wear specs considering it as a usual eye vision problem but not aware of the fact that it had been caused due to any internal disease like diabetes.

According to the Global statistics countersigned by the World Health Organization (WHO) [1], among 7.9 Billion of the current population, about 285.3 million people are visually impaired, out of which 246 million have low vision, and 39.3 million are blind. The reasons for blindness include glaucoma (12.3percent), age-related macular degeneration (8.7percent), diabetic retinopathy (4.8percent), childhood blindness (3.9percent) and trachoma (3.6percent). Among these eye problems, the one which harms the retina part of eyes due to diabetes is referred to as Diabetic Retinopathy (DR) [2]. There are numerous eye retinal disorders, but the most severe causes which doctors see in the retina are hypertension (High blood pressure level) and diabetes (high blood sugar level).

To be more precise, the complication in the retina due to high blood glucose level is more critical since they are symptomless. As per the review given by ophthalmology studies, the clinical and experimental evidence suggests that diabetic retinopathy and associated vision loss have several debilitating effects, including disruption of family functioning, relationships and roles, and deterioration of work prospects resulting in increased financial strain [3].

The tenacious high blood glucose level famishes the small blood vessels with in the retina due to an improper supply of oxygen. Hence this distortion to the retinal part of human eyes due to diabetes is called “Diabetic Retinopathy”, which results in cloudy or blurred vision, and it is caused possibly among people with all types of diabetes such as type 1, type 2 and gestational. This complication results in visual impairment and even leads to blindness if undiagnosed and untreated.

There are two types of Diabetic Retinopathy [4], namely Non-Proliferative Diabetic Retinopathy (NPDR) and Proliferative Diabetic Retinopathy (PDR). The first type of DR disease is Non-Proliferative Diabetic Retinopathy [5], which is the earlier stage that weakens the walls of the blood vessels in retina, consequently the frail retinal blood vessels, begins to dilate and become irregular in diameter that leads to partial retinal mutilation. And this type can progress from mild to severe stage; as more blood vessels become leaky, then the retina begins to deteriorate, which leads to the advanced stage known as Proliferative Diabetic Retinopathy of Diabetic Retinopathy. And this stage is titled as second type of DR disease. It refers to the formation of new, abnormal blood vessels in the retina and these fragile new vessels often bleed, if it bleeds a little, a few dark floaters are seen, and if the bleeding is more, it might block all vision, at a point it can spoil both the central and peripheral (side) vision of the eyes.

Detection of the disease in its earlier stage can reduce the risk of disease severity by 100%. This study detects the first type of DR disease called Non-Proliferative Diabetic Retinopathy that causes different types of illnesses in the eye, such as Micro-aneurysms (MA), Intra-retinal Hemorrhages (IHM) and Hard Exudates (HEXU). The micro-aneurysm is tiny swellings that protrude from the blood vessel, which is the first sign of the NPDR type that appears as small red dots, and it is localized capillary dilatation which is usually s accular (round)[6].

The intra-retinal hemorrhages leaks blood into the retina, which is the second sign of the NPDR type; it is a ‘dot’ or’ blot’ or ‘flame’ shaped depending upon their depth within the retina. There are two layers of the capillary network in the posterior

Ref

1. https://www.geographyandyou.com/population/health/expansion-eye-health-services--essential-combating-diabetic-retinopathy/

retina called nerve fiber layer and inner nuclear layer. Hemorrhage that occurs in the nerve fiber layer tends to be shape of 'flame'. In the inner layer, hemorrhages appear dot or blot shaped, aligned at right angles to the retinal surface, which is consequently viewed using an ophthalmoscope; the clinical differentiation between dot hemorrhages and micro-aneurysms is difficult and of little consequence since both are occurrences of background retinopathy[7].

The hard exudates are the protein fluid that oozed out from the blood vessel, which is the third sign of the NPDR type, and it forms a distinct yellow-white intra-retinal deposit, which varies from specks to larger patches, and that may evolve into rings known as circulates. Ultimately large confluent plaques can form. Hard exudates are extracellular lipid, which leaks from abnormal retinal capillaries, and forms a ring pattern around the leaking vessels. Hard exudates are found in the macular region, and as the lipids coalesce and extend into the central macula, vision can be severely affected[8].

So there is a necessity of an efficient system to discriminate and detect the affected regions with higher accuracy to assist the experts in diagnosing the disease severity earlier. In associate to spot the NPDR features from the fundus image, Non - Diabetic Retinopathy (Non-DR) features in the retinal fundus images have to be spotted and removed for the betterment of lesion identification. The Non-DR features are Blood vessels (BV), Optic disc (OD) and Fovea (FV) to be removed because the blood vessels and fovea features appear dark in color, so it falls in mismatch with the NPDR features like micro aneurysms and hemorrhages and the optic disc is the bright feature which falls in mismatch with the white color feature called exudates.

The retinal blood vessels are the central artery and vein in the retina, which provide and drain blood to and from the eye[9].The main blood vessels are supplied to the retina through the entry point called 'optic disc'. It is a vertical oval, with average dimensions of 1.76mm horizontally by 1.92mm vertically and placed at 3 to 4 mm to the nasal side of the fovea part of the eye[10]. The fovea is a tiny pit located in the macula of the retina that provides the clearest vision of all, and it is a small depression in the retina. The fovea is a black region inside the eye, lies in a fixed orientation and location relative to the optic disc. In the fovea, the layers of the retina spread aside to allow the light to fall directly on the cones that give the sharpest vision. So it is also called as the central fovea or fovea centralis[11].

In general, DR is assessed with single-field non-mydriatic fundus photography and graded according to the International Clinical Diabetic Retinopathy Disease Severity Scale 'HbA1c'[12]. HbA1c is glycated hemoglobin measured by a standardized tests using high-performance liquid chromatography. If higher the HbA1c value, then greater the risk of diabetes-related complications. The optimal HbA1c cutoff for detecting diabetic retinopathy is 49mmol/mol (6.6%) for mild and is 52mmol/mol ((6.9%) for moderate or severe. This grading is done twice in a year to detect the disease severity. But this conventional eye exam becomes a huge and complicated task as the number of patients suffering from the disease is increasing rapidly. Hence considering the importance of the disease severity and the complexity of the manual grading method, an emphasized screening system have to be developed with integrated and hybrid methods for accomplishing accurate diagnosis of the disease.

This proposed work detects the first type of Diabetic Retinopathy (DR) disease called Non-Proliferative Diabetic Retinopathy (NPDR) with its features from retinal fundus images. The task is very challenging because detecting the disease signs in the input includes major issues like noise (illumination or contrast) present in the image

Ref

8. https://www.glycosmedia.com/education/diabetic-retinopathy/diabetic-retinopathy-features-of-diabetes-hard-exudates/

and also the variability in size, color, texture, and shape of the ROIs. Before detecting NPDR features, certain unwanted background features have to be removed to make the detection process more accurate. The study aims to find an appropriate segmentation method with better performance and to overcome the limitations mentioned earlier. The PCI, PEI, DSC measures of the proposed method and the existing works are being compared on two online databases, namely DIARETDB0 and DIARETDB1 [13]. The novelty of this work is comparing the proposed work with the performance of different kernel induced algorithms for segmenting the NPDR features in the Diabetic Retinopathy fund us images.

The GK_FCM algorithm incorporates the Gaussian Kernel function in conventional FCM to achieve the objective of this work. Initially, the input image undergoes preprocessing with green channel extraction and median filtering then background subtraction using extended minima transforms technique, mathematical arithmetic operation, pixel replacement method to eliminate the outlier called Fovea. Further, it is segmented for extracting NPDR features such as Micro aneurysms (MAs), Intraretinal Haemorrhages (IHMs), and Hard Exudates (HEXUs) by integrating Gaussian kernel with FCM on applying multiple parameters. The segmented features are dappled in the original input image using a multi-class contour tracking algorithm with different contouring measures as a post-processing operation.

II. Literature Review

Sasikala et al. [14] have proposed a novel medical image segmentation technique using the optimal threshold Reaction-Diffusion Active Contour model (RD-ACM) to identify Attention Deficit Hyperactive Disorder and cervical cancer-affected areas. In this method, the acquired input images are segmented using Thresholding, the connected components with label matrix algorithm, Heaviside and Dirac delta function; Level set evolution – Two- step splitting method. The proposed method shows better segmentation results. But the proposed RD-ACM gives better results for brain images when compared to cervical cytology images. So it has been found that theRD – ACM method can play a vital role in segmenting the regions of the brain images.

Sasikala et al. [15] has presented a review on various segmentation techniques used on hemorrhage images of both MRI and CT of the brain and analyzed the classification performance of different existing algorithms. Initially Preprocessing approaches are used to denoise the input, and numerous clustering techniques are applied to portray the existence of hemorrhage. Then Machine learning techniques are utilized to focus on issues that manipulate the prediction performance. The methods used for the hemorrhage detection in the input images are Decision Tree classifiers, Support Vector Machine, K-Nearest Neighbours, Thresholding techniques, Fuzzy C Means, Voxel-based outlier detection, Multilayer Perceptron. Among these methods, hemorrhage detection done with Fuzzy C Means results suggested that, to process more training samples, the prospect of this approach have to be modified.

Shyni et al. [16] have surveyed on segmentation algorithms for medical images of spinal cord tumor. The analysis carries various algorithms and techniques used on the medical images such as Fuzzy C-Means, Structural Similarity Index, Hybrid method (Text-Mining, cross-citation based). Data Mining techniques, Genetic Algorithm, support vector machine (SVM), vertebra object boundaries, learning algorithms optimization technique, Propagation segmentation (Prop Seg), level set(Dice similarity coefficient and Hausdorff distance), minimal path search algorithm, subsequent random-walk methods to identify the similarity and variations on the Spinal cord image analysis.

Ref

13. http://www2.it.lut.fi/project/imageret/

Shyni et al. [17] have proposed a work on spinal cord abnormality detection using preprocessing techniques like Median, Arithmetic, Gaussian, and Weiner. The preprocessed image underwent segmented with means, and fuzzy c means clustering algorithm followed by morphological operations and image manipulations have been performed. The performance comparison indices values of two segmentation algorithms witnessed that the proposed FCM method gives improved segmentation results with 84.5% precision.

Aafreen et al. [18] have developed an automatic system that can segment hemorrhage from brain MRI dataset using the Otsu and Watershed segmentation algorithm. For preprocessing the input MRI brain image, median filtering, and morphological operations like dilation and erosion are applied. The ROI have been segmented using Otsu and watershed algorithms. The proposed watershed algorithm have been validated with measures and resulted in an average 0.97overlap metric, average 0.94 precision, and average 0.94 recall, respectively. The results can be improved with variations in the preprocessing methods.

Shalini et al. [19] have presented a survey on the detection of diabetic retinopathy, which gives a review on different algorithms and techniques that have been used for detecting the lesions caused by diabetic retinopathy and also classifying its stages with higher accuracy. From this survey study, it is concluded that the DR lesion detection can be done using preprocessing techniques like green channel extraction, median filtering, and for the segmentation of DR lesions, the FCM algorithm performs better than other segmentation algorithms. Some unwanted features like blood vessels, optic disc needs to be removed to achieve better segmentation results. Finally, grading of lesions can be accomplished using classification algorithms like support vector machine, K nearest neighbor, etc.

Shalini et al. [20] have proposed a comparison work on the detection of hard exudates in diabetic retinopathy fundus images using the principles of Fuzzy-C Means and K-means algorithm. The method involves techniques like green channel extraction, median filter, Binary thresholding, K-means, Fuzzy-C-Means. The proposed comparison work shows that the segmentation of hard exudates using Fuzzy-C-Means is better with an accuracy of 95.05%. The results can be improved by inducing different types of filtering with the fuzzy method.

Alexandre et al. [21] have proposed an approach to segment the fovea a vascular zone of the retina images. The approach involves methods like grey-scale conversion, alternating sequential filtering, H-minima, Regional minima, connected component analysis, distance transform, watershed marker, and the final results have been evaluated in terms of accuracy, specificity, and sensitivity respectively of 0.9947, 0.9972 and 0.8442.

Hosanna et al. [22] have presented a paper to detect hard exudates feature in diabetic retinopathy affected image. Initially, the image is resized, contrast-enhanced with contrast limited adaptive histogram equalization, and intensity of enhanced image have been extracted. Further blood vessels are detected using green channel extraction, adaptive histogram equalization, and morphological operations. In the end, Fuzzy c means clustering (FCM) method segments the exudates in the preprocessed image. The performance measure results about 97.67% of accuracy, 91.108% of sensitivity, 97.95% of specificity.

Pallavi et al. [23] have proposed a segmentation algorithm using fuzzy-based algorithms. The input brain images have been preprocessed with Gaussian noise, salt, and pepper noise. Then the region of interest is segmented using mercer function-based

Ref

17. Shyni Carmel Mary S, Sasikala S, Identification of Abnormality in Spinal cord Using IP-FCM Clustering Algorithm, International Journal of Pure and Applied Mathematics. (2018) 119(16): 439-446.

fuzzy c means (KFCM) and Generalized Spatial kernel-based fuzzy c means (GSKFCM). The proposed methods KFCM and GSKFCM achieved an accuracy of 94.92% and 95.38%.

Ravindraiah et al. [24] have presented a paper for the detection of hard exudates in Diabetic Retinopathy images using Laplacian Kernel Induced Spatial FCM Clustering Algorithm. In this algorithm, laplacian kernel metric is induced into the kernel spatial FCM clustering algorithm for the segmentation of retinal fundus images. In existing methods, FCM and KFCM algorithms are very delicate to noise and other image artifacts because it doesn't have spatial information. To overcome this problem, the author has presented Laplacian kernel spatial FCM, which incorporates spatial information into its objective function and the fuzzy membership function. The performances of this algorithm have been evaluated on different Diabetic Retinopathy images, and the methodology is assessed using statistical measures like Sensitivity and Specificity. Thus LKSFCM method achieved Sensitivity of 99% and Specificity 89%.

Surendiran J et al. [25] have proposed a method to analyze the abnormal retinal images. In this work, the input images are subjected to hard exudates segmentation using the preprocessing techniques like grey-scale conversion; contrast limited adaptive histogram equalization then FCM clustering is applied for segmenting the candidate region. The results obtained have been compared with K-Means clustering, where FCM outperforms with an accuracy of 91.95%.

Rubya et al. [26] have proposed an automatic system that detects and classifies the Diabetic retinopathy lesions using fuzzy logic. Initially, the retinal fundus image is preprocessed with green channel extraction, median filter, contrast limited adaptive histogram equalization, and contrast stretching. Then linear spatial filtering, morphological filtering, transform operations, and binary Thresholding are applied to extract the features like blood vessels, optic disc, hard exudates, micro-aneurysms, and textural features like contrast, homogeneity. The extracted features are classified into respective classes using the fuzzy level set algorithm. The proposed system has higher performance with sensitivity, specificity, and accuracy up to 95.77%, 94.44%, and 95.63%, respectively.

Ganesh et al. [27] have proposed a new efficient system for the detection of microaneurysms in the retinal images. The technique uses Fuzzy-C-Means with the NLM-ADF algorithm. Initially, Fuzzy clustering is done for segmenting the pixels information further NLM in terms of the anisotropic filter is applied to improve identification of micro aneurysms in retinal images. The results show that the method improved the micro-aneurysms detection rate and got a ROC score of 0.427. The proposed methods have been tested on various simulated retina data repositories.

Sergio et al. [28] have proposed an effective method for detecting Non-Proliferative diabetic retinopathy features in color eye fundus images. The algorithm carries preprocessing of image using Green channel extraction, and contrast limited adaptive histogram equalization, the features like optic disc, blood vessels, fovea have been eliminated then the disease signs like micro-aneurysms and hemorrhages are detected by applying the image processing techniques such as alternative sequential filtering, H-minima transform, region minimum, Sobel and Prewitt filters along with morphological operations with the outcome of 87.69% sensitivity and 92.44% specificity.

Ganesh et al. [29] have presented a paper on identifying the microaneurysm feature in retinal images using the grey- scale conversion, Rotational Cross-Section Analysis, and Fuzzy C-Means Clustering Algorithm. The proposed approach has scored 0.435 ROC.

Ref

26. Rubya Afrin, Pintu Chandra Shill, Automatic Lesions Detection and Classification of Diabetic Retinopathy Using Fuzzy Logic, International Conference on Robotics, Electrical and Signal Processing Techniques (ICREST), IEEE. (2019): 527–532.

Venkatraman et al. [30] have proposed a system for the detection of Non-Proliferative Diabetic Retinopathy in Fundus Images by Wavelet Features. The system utilizes histogram equalization, candidate region extraction, wavelet features for detecting the diabetic retinopathy features by applying Mercer kernel, 2nd-degree polynomial kernel, 3rd-degree polynomial kernel and Gaussian kernel with the accuracy of 96.0%, 78.0%, 86.0%, and 84.0%.

Lama et al. [31] have presented work on dark lesion detection for Diabetic retinopathy using preprocessing methods like spatial calibration, illumination equalization, Mean Filter, Adaptive Contrast Equalization, color normalization then entropy-based Thresholding and multi-scale ring-shaped matched filter for optic disc removal. Finally, dynamic shape features like Relative area, Elongation, Eccentricity, Circularity, Rectangularity, Solidity are extracted, which is classified into lesions using a random forest algorithm with AUC of 0.899, 0.916, 0.976, 0.941 by testing four different databases.

Manoj et al. [32] have implemented a computer-aided detection system for the segmentation of Non-Proliferative diabetic retinopathy features and retinal features in color fundus images. The implementation comprises of algorithms like green channel, median filter; contrast limited adaptive histogram equalization, shade correction, Matched Filter-First Order derivative of Gaussian, Mathematical filtering, morphological operations, watershed segmentation, in-painting, h-extended minima algorithm, Selective Binary, and Gaussian Filtering Regularized Level Set and Signed Pressure Force algorithm. The proposed methodology for the segmentation of micro-aneurysms feature attained 90% accuracy, and exudate feature detection has given 93.41% accuracy.

The review done on detecting and segmenting the NPDR and Non-DR features renders various image processing methods. And these existing works have performed the segmentation process with preprocessed inputs, and some executions have been done on non-preprocessed image, and others employed kernels, parameter values to identify the object of interest (OOI) still there is some inability in achieving the accuracy of medical experts' outcome. There are a number of challenges in distinguishing and categorizing DR features; such as the presence of noise and outliers like the blood vessel, optic disc and fovea that are present in input images, the vacillating location of features, the similarity of shape and texture among some deformations (the micro-aneurysms and hemorrhages happen to occur with matching surface), which may direct to extracting redundant or ineffective features and results in low segmentation accuracy. This low performance consequently leads to improper diagnosis of the patient at the time of emergency states by the Physicians, which ultimately causes the severity of the retinal disease. The proposed FCM based segmentation evolves some initiatives to manage the inabilities found in the existing works.

The structures of the proposed work have been organized as follows; section 3 presents the theoretical background of the techniques used. Section 4 describes the experiments conducted to compare the different fuzzy algorithms adopted for NPDR features segmentation. The dataset descriptions have been given in division 4.1. Then the results are presented in division 4.2 and discussed in section 4.3 followed by conclusions and future work in section 5.

III. Theoretical Background

This paper has considered a fuzzy based algorithm for detecting the ROIs. This section presents a brief explanation of the proposed methodologies.

Ref

31. Lama Seoud, Thomas Hurtut, Jihed Chelbi, Farida Cheriet, Pierre Langlois JM "Red Lesion Detection using Dynamic Shape Features for Diabetic Retinopathy Screening, IEEE Transactions on Medical Imaging. (2015): 1 – 11.

a) Fuzzy clustering algorithm

The conventional Fuzzy based algorithms are used for partition the data points, where data point assigns memberships to each center as a result of which, for each iteration data point, belong to more than one middle point. It segments the ROI based on data point, which is chosen precisely among many data points, so the segmentation is more accurate.

For a given data set X={x1, x2, xn}, clustering algorithms partition the n data objects in X into c groups C={C1, C2, , Cc} based on similarity/dissimilarity metric [33]. The standard Fuzzy C Means algorithm [34] uses the Euclidean distance as an objective function to be minimized and expressed as the following equation:

$$J_{FCM}(U,V) = \sum_{i=1}^{c} \sum_{j=1}^{n} \mu_{ij}^{m} \| x_j - v_i \|^2 \quad (1)$$

Where vi is the cluster center of cluster Ci, m is the weighting exponent or the degree of fuzzifier in FCM. The fuzzy partition matrix have been expressed as

$$U = [\mu_{ij}]_{c\times n}, \ \mu_{ij} \in [0,1] \quad (2)$$

It is the membership degree of data object xjto cluster v_i, and

$$\sum_{i=1}^{c} \mu_{ij} = 1, \ \forall j = 1,2,3,\dots,n \quad (3)$$

In the iterations, the membership degree Pij and the cluster centers vihave been updated as

$$\mu_{ij} = \frac{1}{\sum_{k=1}^{c} \left(\frac{d_{ij}}{d_{kj}}\right)^{2/(m-1)}} \quad (4)$$

$$C_i = \frac{\sum_{j=1}^{n} \mu_{ij}^{m} x_j}{\sum_{j=1}^{n} \mu_{ij}^{m}} \quad (5)$$

We iterate (8) and (9) until the changes in the fuzzy partition matrix are very small, or some other stopping criterion has met.

III. Material and Proposed Methodology

In this section, we present the methodology adopted in the work, dataset, and proposed approach.

a) Material

i. *Dataset and Tools used*

The Experimentation of Non-Proliferative Diabetic Retinopathy features detection is conducted on The Processor AMD A8-7410 APU with AMD Radeon R5 Graphics HP Platform, 64-bit operating system, x64-based processor, 2.20 GHz Processor Speed and 4 GB Memory. The Segmentation Algorithm has been developed in the Matlab2014b-32 bit version. The dataset taken for this segmentation process have been obtained from eye care clinics and online repositories, namely DIARETDB0 and DIARETDB1 database, with a resolution of 93 x 71 in 24- bit depth PNG format. The databases contain 200 number of color fundus images for the experiment in which 189 contain signs of diabetic retinopathy.

ii. *Data Preparation*

Resizing: To standardize the image, resizing is carried out by the Bi-Cubic interpolation Area method [35], uses the biased average of four translated pixel values for each output

Ref

33. Kaile Zhou, Shanlin Yang, Exploring the uniform effect of FCM clustering: A data distribution perspective, Knowledge-Based Systems. (2016) 96: pp. 76-83.

pixel value then the input image is zero-padded and transformed into the positive horizontal direction by five-tenths of pixels.

$$p(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j \quad (6)$$

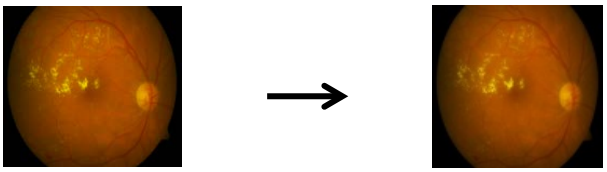


Fig. 1: Fundus retinal image *Fig. 2:* Resized image

b) Methodology

The fundus input contains the unwanted features such as BV, OD, FV, and the NPDR features MA, IRH, HEXU, which are to be segmented, and they have been shown in Fig 2. The preprocessing technique improves the image quality and removes the Non-DR (unwanted) features in the image, then segmentation algorithms segments the NPDR features. The purpose of eliminating unwanted portions is reducing the false detection rates to achieve more accurate results in NPDR features segmentation.

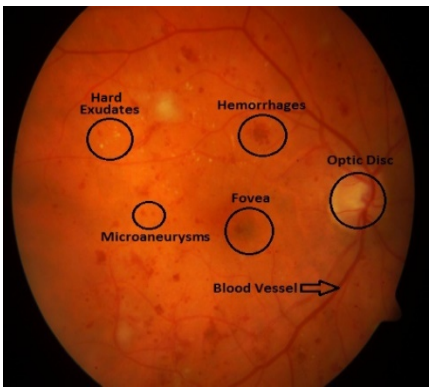


Fig. 3: DR-Fundus retinal image with NPDR and Non-DR Features

The goal of this work is implemented by following the proposed methodology represented in Fig 4, which consists of four-phases, namely standardization, preprocessing, segmentation, and feature recognition. The fundus retinal input image acquired from the DR database is standardized using the bi-cubic interpolation area method [35] for the resizing of the image. In the first phase, the resized image undergoes preprocessing by using the techniques like Green channel extraction, median filter for image enhancement then Binarized contour tracing (BCT), hybrid BINI Thresholding [36], extended minima transform algorithms are applied to detect the Non-DR features like blood vessels, optic disk, and fovea. In the second phase, the detected Non-DR features are removed from the input image using mathematical arithmetic operation (MAO) and pixel replacement method (PRM). The third phase carries out the segmentation on the preprocessed image to isolate the NPDR features like micro-aneurysms, intra-retinal hemorrhages, and hard exudates on applying Gaussian kernel prompted Fuzzy c means method. The fourth phase comprises marking of the segmented features in the input image using a multi-class contour tracking (MCT) algorithm to outline multiple regions of interests. The detailed executions of the algorithm have been explained below.

i. *Design*

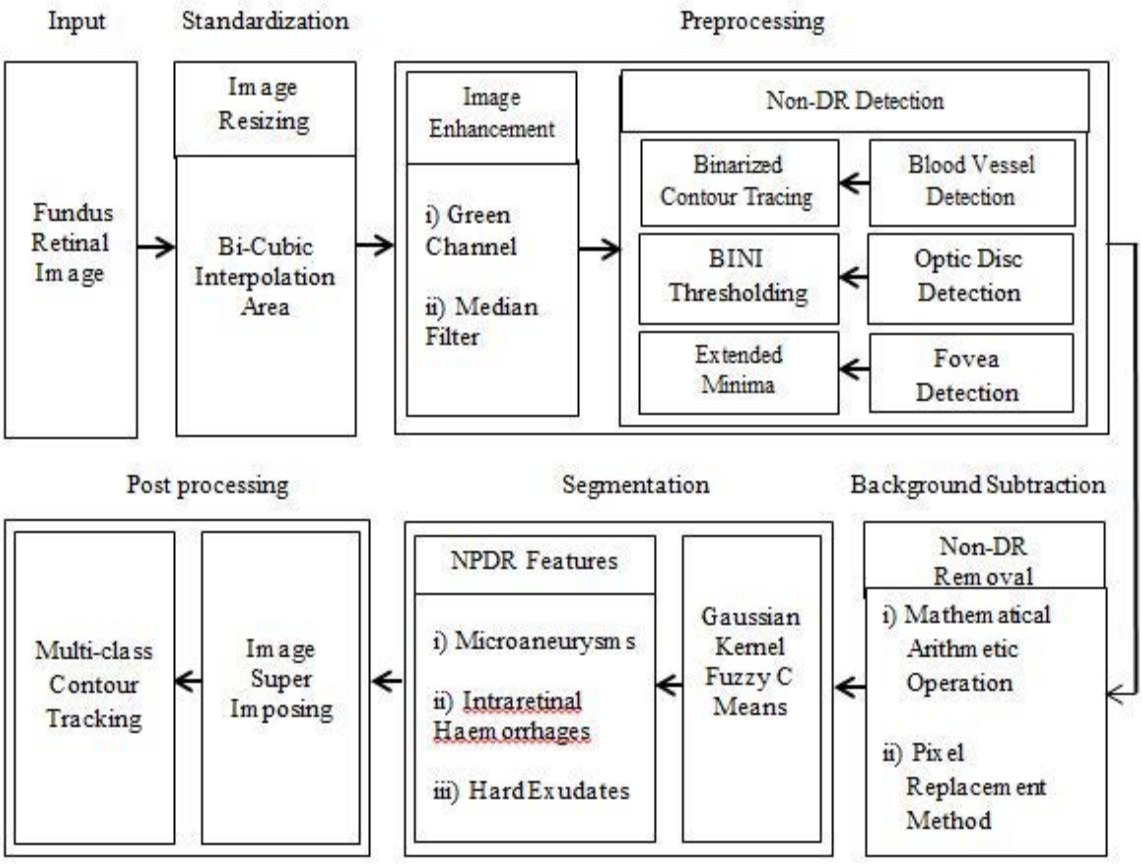


Fig. 4: Segmentation of NPDR features

ii. *Preprocessing*

Conventionally, the image data recorded or obtained through imaging systems like satellites, digital cameras. Though the images captured by a high configured fundus camera, there is a lack in contrast and brightness because of the illumination conditions. These errors are improved using applicable mathematical models, which are either definite or statistical models. This process has been termed as image preprocessing, and it has been performed for enhancing image structures for consequent analysis or image display. Image enhancement is the alteration of the pixel brightness values in an image to improve its visible effect, which is suited for human or machine interpretation. The enhancement process does not upturn the needed information in the data but just emphasizes certain specified image features. Hence the preprocessing is done for making the image more suitable for further processing. The enhancement techniques chosen for DR feature detection is resizing, channel extraction, noise filtering.

To enhance the fundus image segmentation process, the preprocessing operations are carried out using the green channel [37] of the RGB fundus image which project the DR features (ROI) more prominent than the Blue and Red Channels and unlike the other two channels (red, green), the green channel is neither lower illuminated nor over-saturated.

$$g = \frac{G}{R+G+B} \quad (7)$$

The equation represents red channel (R), green channel (G), and blue channel (B), respectively. The resulting image for the normalized green channel has been denoted by g.

The Median Filtering is a non-linear filtering technique [38] which removes noise while preserving the edges to enhance the region of interest.

$$y[m, n] = \text{median}\{x[i, j], (i, j) \in \mathcal{w} \tag{8}$$

Where $\mathcal{w}$ represents a neighborhood value, given by the user, which is centered around the location [m, n] in the image.

The extended-minima transform (SMT) is a Thresholding technique which segments the fovea region. It is the local minima of h-minima transform. The regional transform replaces the pixel values to zero. The h-minima transform subdues all the minima in the intensity image whose depth is less than or equal to a predefined threshold value [39].

$$EM_{(x,y)} = t(I, T) \tag{9}$$

Where, t is minima transform function
I denotes image
T is a threshold value

iii. *Segmentation*

The segmentation process is the significant difficulty in image processing, which is performed to dissect the ROIs. It subdivides the preprocessed image into some parts or objects until the object of interests is isolated, e.g. initially, dissection of the background from the image, then the foreground is segmented. Segmentation of images involves not only the discrimination between regions of interest and the unwanted portions but also the separation of more than one object of interest. One of the methods for such separation is known as FCM segmentation algorithm as follows;

Gaussian Kernel- based fuzzy clustering algorithm:

The kernel-based fuzzy clustering [40] introduced the kernel method into the FCM algorithm, which overcomes FCM's shortcomings in terms of insufficiency caused by data distribution characteristics to clustering results. Define a nonlinear map as

$$\phi : x \rightarrow \phi(x) \in F, \quad \text{where } x \in X, \ X \tag{10}$$

X denotes the data space, and F is the transformed feature space with a higher or even infinite dimension [41]. The objective function of KFCM has been defined as

$$J_{KFCM} = \sum_{i=1}^{c} \sum_{j=1}^{n} \mu_{ij}^{m} \| \phi(x_j) - \phi(v_i) \|^2 \tag{11}$$

Where

$$\| \phi(x_j) - \phi(v_i) \|^2 = K(x_j, x_j) + K(v_i, v_i) - 2K(x_j, v_i) \tag{12}$$

We adopt the Gaussian function [42] as a kernel function,

$$\text{i.e } K(x, v) = \exp\left[\frac{-(x - v)^2}{\sigma^2}\right], K(x, x) = 1 \tag{13}$$

Where σ is Gaussian kernel with multiple parameters, according to Eq.(12), Eq.(11) can be rewritten as:

$$J_{GK\_FCM}(U,V) = 2\sum_{i=1}^{c}\sum_{j=1}^{n}\mu_{ij}^{m}\left[1-K(x_j,v_i)\right] \quad (14)$$

Minimizing Eq. (14) under the constraint of μij.

$$\mu_{ij} = \frac{\left(1-K(x_j,\ v_i)\right)^{-1/(m-1)}}{\sum_{i=1}^{c}\left(1-K(x_j,\ v_i)\right)^{-1/(m-1)}} \quad (15)$$

$$v_i = \frac{\sum_{j=1}^{n}\mu_{ij}^{m}\ K\ (x_j,\ v_i)\ x_j}{\sum_{j=1}^{n}\ \mu_{ij}^{m}\ K(x_j,\ v_i)} \quad (16)$$

iv. *Post-processing*

The post-process has been performed for marking the segmented ROI in the input image. The segmented NPDR features are marked in the original input image using Contour-Base Object Tracking Algorithm [43]. Object tracking is considered to be an essential task in the computer vision field. The state of the contour, which shows the position of the segmented object, is defined using the centroid points. In the proposed work, six different segmented features have been pointed so the multi-class contour tracking algorithm is applied to mark the multiple areas of interest in the fundus input image.

c) Results

i. *Evaluation Metrics*

For internal and external evaluation of the proposed segmentation techniques, validation measures like Partition Coefficient Index (PCI), Partition Entropy Index (PEI), and Disc similarity Coefficient (DSC) have been calculated.

Partition Coefficient is the index value that determines the cluster partitions of two different techniques. The index value ranges between 0.894–0.9160.

$$PCI = \frac{1}{N}\sum_{p=1}^{M}\ \sum_{i=1}^{N}\mu_{iM}^{2} \quad (17)$$

Partition Entropy is the index value that determines the entropy of cluster partitions of two different techniques. The index value ranges between 0.1989–0.2703.

$$PEI = \frac{1}{N}\sum_{p=1}^{M}\ \sum_{i=1}^{N}\mu_{iM}^{2}\log_2(\mu_{iM}) \quad (18)$$

Dice Similarity Coefficient is a performance analysis method based on the spatial overlap between two different segmentation processes of the same image. It is the same as f-score, considered as an accuracy measure that counts all the true positives, false positives and true negatives.

$$DSC = \frac{2.TP}{2.TP+FP+FN} \quad (19)$$

Where TP, FP, TN, FN are True Positive, False Positive, True Negative, and False Negative, which have been defined as the number of pixels classified correctly and incorrectly in abnormal existence and normal image by the proposed method.

Ref

43. Chirag I. Patel, Ripal Patel, Contour Based Object Tracking." International Journal of Computer and Electrical Engineering. (2012) 4(4) : 525-52

Table 1: Performance measures of NPDR Features Segmentation

Methods	PCI	PEI	DSC
MK_FCM	0.90	0.23	0.78%
LK_FCM	0.91	0.48	0.89%
Proposed GK_FCM	0.94	0.79	**0.98%**

Notes

ii. *Implementation Outcome*

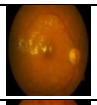
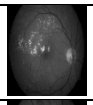
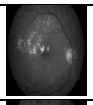
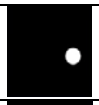
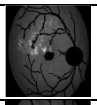
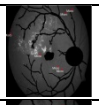
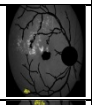
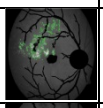
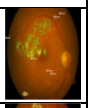
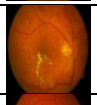
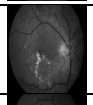
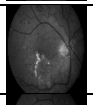
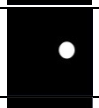
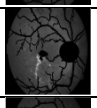
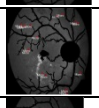
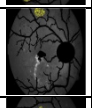
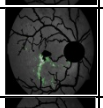
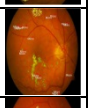
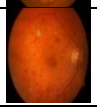
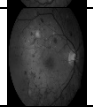
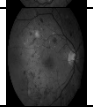
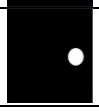
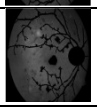
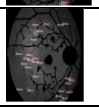
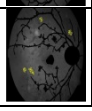
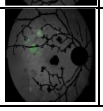
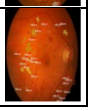
S.no	Input	Preprocess					Backgro-und subtract-ion	Segmentation			Post proce-ss
	Resized Image	Green channel	Median filter	Non-DR Detection			Non-DR removal	NPDR Features Detection			MCT
				BCT: BV	BINI: OD	exMT: FV	MAO & PRM	GK_FCM			
								MA	IRH	HEXU	
1											
2											
3											

Fig. 4: NPDR features Segmentation

iii. *Discussions*

NPDR stage is the sign of leaking blood vessels that drop out blood, fatty deposits, and fluids on the retina. Segmentation of NPDR features is a necessary process to support the expert in the analysis of disease to obstruct its severity as earlier as possible. At first, the acquired inputs have been standardized by resizing it to 512 X 512 dimensions, as portrayed in fig.3, column 2. The purpose of resizing is to make images more receptive to accomplishing further processing and for complete visibility on screens of different devices. Then the resized images undergo the preprocessing operation using green scale conversion as it enhances the fundus image; it is done by extracting the green channel of the color fundus image, which projects the DR features more noticeable than the Blue and Red Channels. Then median filtering is performed on the green scale image to suppress the noise present in the inputs that have been represented in fig.2, column 3.1, and column 3.2.

This green Channel image is applied with the background subtraction process using numerous image processing techniques to detect and remove the unwanted Non-DR features like Blood Vessel, Optic Disc, and Fovea so that the NPDR features are more projective. The first feature Blood Vessel is detected using the binarized contour tracing (BCT) method and the second feature Optic disc is segmented using BINI Thresholding are shown in columns 3.3 and 3.4, which has been already done, and

described in the previous work [36]. The proposed work: detected the third feature called Fovea feature using extended minima transform method, as shown in column 3.5. Then these three detected features are removed from the input image using the mathematical arithmetic operation (MAO) and pixel replacement method (PRM) in column 4. Further, this image is given as input for the segmentation process for segmenting the NPDR features like MA, IRH, and HEXU using Gaussian kernel-based fuzzy c mean algorithm, which have been shown in column 5.1, 5.2, 5.3. Finally, the segmented features are plotted in the fundus input image using a multi-class contour tracking (MCT) algorithm and the result have been shown in column 6.

The first algorithm [30] in table 1 called Mercer-Kernel induced Fuzzy C Means, where clustering is done by FCM integrating with Mercer function to cluster the data points. The mercer function is the kernel method used in the segmentation algorithms to segment the ROI that is unlabeled, and it is suitable for a cluster with spherical ring shape by default. Also the function needs prior information of the cluster shape. If the cluster shape is not specified priory, and ROI outline have not been fixed with default cluster shape, then a grouping of data points in segmentation process flops. The second algorithm [24] in table 1 is Laplacian-kernel based Fuzzy C Means, which uses the kernel with Cauchy distribution to deploy more frequency components which overlook the noises present in the image. But this distribution is not time adaptive in handling the large dataset since it uses a single parameter. The fourth algorithm in table 1 is the proposed Gaussian-kernel based Fuzzy C Means, and this algorithm carries normal distribution of pixels to handle the noise present in the image that makes a grouping of identical pixels more contented. Here the kernel is employed with multi-parameters, which is suitable for handling large datasets with less time and also performs better in multiple ROI segmentation. Hence, the proposed GK˙FCM method has given better results than the existing fuzzy C means algorithms for segmenting multiple ROI and achieved accuracy of 98.21%.

The validation measures of the proposed segmentation algorithms have been evaluated in terms of PCI and PEI. The values are 0.89 and 0.51 for FCM, 0.90 and 0.23 for MKFCM, 0.91 and 0.48 for LKFCM, 0.98 and 0.79 for GK˙FCM. The performance analysis of NPDR feature segmentation using FCM gives 91.95% accuracy, MKFCM gives 78.0% accuracy, LKFCM gives 90.88% accuracy, and the GK˙FCM algorithm gives 98.21% accuracy. The evaluation results have been shown in table 1, and the graph for the resulted values have been given in Fig 4. The results show that the proposed method GK˙FCM gives a better result.

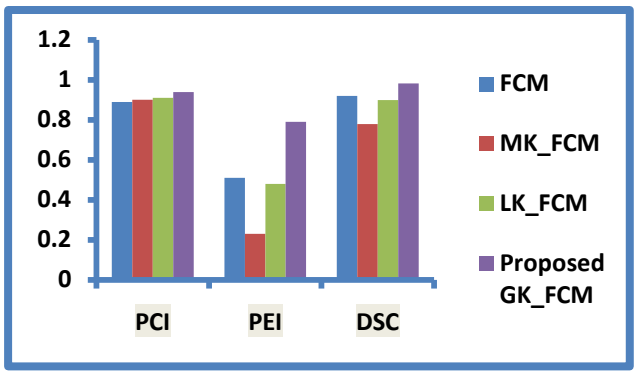


Fig. 3: Comparison of accuracy of NPDR feature segmentation algorithms

Ref

30. Venkatraman K, Programmed Detection of Diabetic Retinopathy in Fundus Images utilizing Wavelet Features, Journal of Chemical and Pharmaceutical Sciences. (2016) 9(2): E59 – E63.

Notes

V. Conclusion and Future Work

The earlier identification of the diabetic retinopathy and its features is more necessary to avoid the precarious condition. So, the segmentation of NPDR features using Fuzzy based algorithm in the fundus images has been implemented by resizing the input image using a bi-cubic interpolation method. Then preprocessing techniques like green channel extraction and median filter have been used for highlighting the image features for subsequent exploration. Further background subtraction have been done, which applies algorithms like binary contour tracing, BINI Thresholding, extended minima transform, mathematical arithmetic operation, and pixel replacement for detecting and removing unwanted features like blood vessels, the optic disc which ignores the false positives and enhances the area of interest to be segmented. For segmenting the ROI so-called NPDR features like micro-aneurysms, intra-retinal hemorrhages, and hard exudates, Gaussian kernel-based Fuzzy C means algorithm have been applied. In this segmentation algorithm, the Gaussian function identifies all the pixels with equal distribution also filter has been set, which improves the features detection process more efficient and accurate. The proposed work has achieved 98.21% accuracy. Future work focuses on feature extraction of Diabetic retinopathy fundus images with improved performance.

Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgement

Dataset used in this work are available in online DIARETDB0 database also gathered from Sankara Nethralaya Hospital, Chennai. "All procedures performed in studies involving human participants were in accordance with the ethical standards of the institutional and/or national research committee and with the 1964 Helsinki declaration and its later amendments or comparable ethical standards."

References Références Referencias

1. https://www.geographyandyou.com/population/health/expansion-eye-health-services-essential-combating-diabetic-retinopathy/
2. Anupama Pattanashetty, Suvarna Nandyal, Diabetic Retinopathy Detection using Image Processing: A Survey, International Journal of Computer Science and Network. (2016): 661–666.
3. Eva Fenwick, Gwyn Rees, Konrad Pesudovs, Mohamed Dirani, Ryo Kawasaki, Tien Y Wong Franzco, Ecosse Lamoureux, Social and emotional impact of diabetic retinopathy: a review, Clinical and Experimental Ophthalmology Φ 2011 Royal Australian and New Zealand College of Ophthalmologists. (2012): 420–427.
4. https://www.aao.org/eye-health/diseases/what-is-diabetic-retinopathy
5. https://healthengine.com.au/info/diabetic-eye-disease-non-proliferative-diabetic-retinopathy
6. https://www.sciencedirect.com/topics/medicine-and-dentistry/microaneurysm
7. https://www.sciencedirect.com/topics/medicine-and-dentistry/intraretinal-microvascular-abnormalities
8. https://www.glycosmedia.com/education/diabetic-retinopathy/diabetic-retinopathy-features-of-diabetes-hard-exudates/
9. https://www.sciencedirect.com/topics/medicine-and-dentistry/retinal-blood-vessel

10. https://www.sciencedirect.com/topics/medicine-and-dentistry/optic-disc
11. https://www.sciencedirect.com/topics/medicine-and-dentistry/fovea-centralis
12. Nam H Cho, Tae Hyuk Kim, Se Joon Woo, Kyu Hyung Park, Soo Lim, Young Min Cho Kyong Soo Park, Hak C Jang, Sung Hee Choi, Optimal HbA1c cut off for detecting diabetic retinopathy, https://www.researchgate.net/publication/235376641 Φ Springer-Verlag Italia.(2013): 661–666.
13. http://www2.it.lut.fi/project/imageret/
14. Sasikala S, Thilagam M, Medical image segmentation using optimum thresholded Reaction diffusion active contour model, Indian Journal of Innovations and Developments. (2015) 4(1): 93-99.
15. Sasikala S, Aafreen Nawresh A, A Review on the segmentation of Brain Haemorrhage images using Brain MRI and CT Scans, International Journal of Innovative Research in Science, Engineering and Technology. (2017) 6(1): 537-540.
16. Shyni Carmel Mary S, Sasikala S, Survey on Segmentation Techniques for Spinal Cord Images, International Journal of Data Mining Techniques and Applications. (2016) 5(2): 121-124.
17. Shyni Carmel Mary S, Sasikala S, Identification of Abnormality in Spinal cord Using IP-FCM Clustering Algorithm, International Journal of Pure and Applied Mathematics. (2018) 119(16): 439-446.
18. Aafreen Nawresh A, Sasikala S, Identification of Haemorrhage in Brain MRI using Segmentation Techniques, International Journal of Engineering Research in computer science and Engineering (IJERCSE). (2018) 5(3): 156-161.
19. Shalini R, Sasikala S, A Survey on Detection of Diabetic Retinopathy, IEEE. (2018): 626-630.
20. Shalini R, Sasikala S, Segmentation of Hard exudates using Fuzzy-C-Means in Diabetic Retinopathy Fundus images, IEEE. (2019): 121–130.
21. Alexandre G Silva, Marina Fouto S, Andre Tda Silva, Rangel Arthur, Angjelica M Arthur, Segmentation of Foveal Avascular Zone of the Retina Based on Morphological Alternating Sequential Filtering, IEEE 28th International Symposium on Computer-Based Medical Systems. (2015) : 38–43.
22. Hosanna Princye P, V.Vijayakumari, Detection of Exudates and feature extraction of retinal images using Fuzzy clustering method, IET publications. 388 – 394.
23. Pallavi Thakur, Chelpa Lingam, Generalized Spatial Kernel based Fuzzy C Means Clustering Algorithm for Image Segmentation, International Journal of Science and Research (IJSR). (2013) 2(5): 165 – 169.
24. Ravindraiah R, Chandra Mohan Reddy S, Rajendra Prasad P, Detection of Exudates in Diabetic Retinopathy Images using Laplacian Kernel Induced Spatial FCM Clustering Algorithm, Indian Journal of Science and Technology. (2016) 9(15): 1–6.
25. Surendiran J, Mohammad Jabirullah, Subramanyamchari K, Detection and Classification of Hard Exudates in Human Retinal Fundus Images Using Clustering and RVM Methods, International Journal of Pure and Applied Mathematics. (2018) 119(12): 14321-14326.
26. Rubya Afrin, Pintu Chandra Shill, Automatic Lesions Detection and Classification of Diabetic Retinopathy Using Fuzzy Logic, International Conference on Robotics, Electrical and Signal Processing Techniques (ICREST), IEEE. (2019): 527–532.
27. Ganesh Naga Sai, Habibulla Khan, Gopinathan E, Reduction of False Microaneurysms in Retinal Fundus Images using Fuzzy C-Means Clustering in terms

Notes

NLM Anisotropic Filter, International Journal of Computer Applications. (2015)113(19): 20 – 25.
28. Sergio Bortolin Junior, Daniel Welfer, "Automatic Detection of Microaneurysms and hemorrhages in color eye fundus images, International Journal of Computer Science & Information Technology (IJCSIT). (2013) 5(5): 21– 37.
29. Ganesh Naga Sai Prasad V, Habibulla Khan, Gopinathan E, Identifying microaneurysms in retinal images using Fuzzy C-Means Clustering, ARPN Journal of Engineering and Applied Sciences. (2015)10(6): 2366 – 2372.
30. Venkatraman K, Programmed Detection of Diabetic Retinopathy in Fundus Images utilizing Wavelet Features, Journal of Chemical and Pharmaceutical Sciences. (2016) 9(2): E59 – E63.
31. Lama Seoud, Thomas Hurtut, Jihed Chelbi, Farida Cheriet, Pierre Langlois JM "Red Lesion Detection using Dynamic Shape Features for Diabetic Retinopathy Screening, IEEE Transactions on Medical Imaging. (2015): 1 – 11.
32. Manoj Kumar, Manikandan, Malaya Kumar Nath, Detection of Microaneurysms and Exudates from Color Fundus Images by using SBGFRLS Algorithm, ICIA. 2016.
33. Kaile Zhou, Shanlin Yang, Exploring the uniform effect of FCM clustering: A data distribution perspective, Knowledge-Based Systems. (2016) 96: pp. 76-83.
34. https://sites.google.com/site/dataclusteringalgorithms/fuzzy-c-means-clustering-algorithm.
35. https://in.mathworks.com/help/vision/ug/interpolation-methods.html.
36. Shalini R, Sasikala S, Segmentation of retinal features using hybrid BINI Thresholding in diabetic retinopathy fundus images, Scopus indexed Springer AISC Series. (2019): 121–130.
37. Noratikah Mazlan, Haniza Yazid, An improved retinal blood vessel segmentation for diabetic retinopathy detection, Computer Methods in Biomechanics and Biomedical Engineering: Imaging & Visualization. (2017): 1 – 10.
38. https://www.sciencedirect.com/topics/engineering/median-filtering.
39. https://edoras.sdsu.edu/doc/matlab/toolbox/images/imextendedmin.html
40. Qiuping Wang, Yiran Zhang, Yanting Xiao, Jidong L, Kernel-based Fuzzy C-means Clustering Based on Fruit Fly Optimization Algorithm, IEEE. (2017): 251 – 256.
41. Yi Ding, Xian Fu, Kernel-based fuzzy c-means clustering algorithm based on genetic algorithm, Neurocomputing. (2016) 188:233-238.
42. Rehna Kalam, Ciza Thomas, Abdul Rahiman M, Gaussian Kernel based fuzzy C-Means clustering algorithm for image segmentation, CS & IT-CSCP.(2016): 47-56.
43. Chirag I. Patel, Ripal Patel, Contour Based Object Tracking." International Journal of Computer and Electrical Engineering. (2012) 4(4): 525-52

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Search for a Mathematical Model of the Kinetics of Saccharomyces Cerevisiae Yeast Cultivation with Oxygen Deficiency

By V. B. Tishin & I. A. Shomrina

Abstract- This article presents research data of the kinetics of Saccharomyces cerevisiae yeasts aerobic cultivation without forced air supply to the cultivator. Oxygen penetrates into the culture medium through its free surface and spreads throughout the liquid volume only due to molecular diffusion. Culture medium mixing occurs with pop-up bubbles of carbon dioxide and thermal energy released by the cells during their development.

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GJSFR-F Classification: *MSC 2010: 92C45*

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Search for a Mathematical Model of the Kinetics of Saccharomyces Cerevisiae Yeast Cultivation with Oxygen Deficiency

Поиск математической модели кинетики развития дрожжей Saccharomyces cerevisiae при недостатке кислорода

V. B. Tishin [α] & I. A. Shomrina [σ]

Abstract- This article presents research data of the kinetics of Saccharomyces cerevisiae yeasts aerobic cultivation without forced air supply to the cultivator. Oxygen penetrates into the culture medium through its free surface and spreads throughout the liquid volume only due to molecular diffusion. Culture medium mixing occurs with pop-up bubbles of carbon dioxide and thermal energy released by the cells during their development.

The result of the research was a generalized mathematical model of the kinetics of the yeast cells development, composed of two special models - the growth of biomass and carbohydrate consumption. The combination of the two models was carried out by introducing the relative specific velocity $\overline{\gamma}_{1o} = \overline{\gamma} / \overline{\gamma}_s$ into the mathematical model. There are specific rates of biomass growth and carbohydrate consumption in this equation.

The obtained generalized mathematical model allows us to take into account the effect on the biological process of the initial values of biomass concentrations in the inoculation and the initial concentration of carbohydrates in the culture fluid, and to predict its progress outside the boundaries of the experiment.

Experimental studies have confirmed the validity of this approach to the search for mathematical models of the kinetics of the development of yeast cells.

Keywords: kinetics, mathematical model, specific rate, biomass.

Аннотация: **В данной статье приводятся данные исследований кинетики аэробного культивирования дрожжей Saccharomyces cerevisiae без принудительной подачи воздуха в культиватор. Кислород воздуха проникает в культуральную среду через её свободную поверхность и распространяется по объёму жидкости только за счёт молекулярной диффузии. Перемешивание среды происходит всплывающими пузырьками диоксида углерода и тепловой энергией, выделяемыми клетками в процессе их развития.**

Результатом исследований стала обобщённая математическая модель кинетики развития популяции дрожжевых клеток, составленная из двух частных моделей - прироста биомассы и потребления углеводов. Объединение двух моделей производилось путём введения в математическую модель относительной удельной скорости $\overline{\gamma}_{1o} = \overline{\gamma} / \overline{\gamma}_s$, где $\overline{\gamma}$ и $\overline{\gamma}_s$ удельные скорости прироста биомассы и потребления углеводов.

Полученная обобщённая математическая модель позволяет учесть влияние на биологический процесс начальных значений концентраций биомассы в засевном материале и углеводов в культуральной жидкости, и прогнозировать его ход за пределами границ эксперимента.

Правомерность такого подхода к поиску математических моделей кинетики развития дрожжевых клеток подтверждена экспериментальными исследованиями.

Ключевые слова: кинетика, математическая модель, удельная скорость, биомасса.

Author: e-mail: irinashomrina@gmail.com

I. Введение

Поиск математических моделей кинетических закономерностей развития клеток микроорганизмов – одна из сложнейших задач микробиологии. Основная сложность заключается во множестве связанных между собой факторов, влияющих на скорости протекания биологических процессов в любой биологической системе. Эту связь и должна установить математическая модель. Многофакторность определяется состоянием среды (температура, pH), в которой клетки развиваются, состоянием самих клеток, видом микроорганизма, гидродинамической обстановкой в культиваторе и т.п.

В данной статье рассматривается несколько упрощённая задача аэробного культивирования определённого штамма дрожжей Saccharomyces cerevisiae, без принудительной подачи воздуха в культиватор и отсутствии какого-либо перемешивающего устройства при постоянной температуре и pH.

Кислород воздуха проникает в культуральную среду через её свободную поверхность и распространяется по объёму жидкости только за счёт молекулярной диффузии. Сказать, что перемешивание вообще отсутствует, будет не верно. Возможными источниками образования конвективных токов в среде могут быть всплывающие пузырьки диоксида углерода и тепловая энергия, выделяемые клетками в процессе их развития.

В практике периодическое культивирование микроорганизмов без принудительной подачи воздуха в культиватор и без перемешивания встречается редко. Видимо это является причиной того, что сведений об исследованиях на эту тему, к тому же ещё и с целью поиска математических моделей биологических процессов, крайне мало. Помощь в раскрытии проблем, затронутых в данной статье можно найти в работах [1, 2, 3, 4, 5].

В определённой мере, этот вариант в производственных условиях имеет место на различных стадиях культивирования чистых культур дрожжей, когда воздух либо вообще не подаётся, либо его расход не велик. Другим примером может служить развитие пивных дрожжей при сбраживании пивного сусла в открытых ёмкостях [6, 7].

В целом приведённые примеры развития дрожжей в условиях дефицита кислорода не являются лимитирующими в общем цикле производства конечного продукта, но представляют общенаучный интерес, и исследования в этой области позволят более глубоко понять влияние различных факторов на скорости протекания биологических процессов. В частности, в дальнейшем они могут помочь в поиске математических моделей кинетики аэробного развития микроорганизмов в условиях принудительной подачи воздуха в культиватор.

В исследованиях использовался штамм хлебопекарных дрожжей Л-12. Опыты проводились при температуре $T = 31-32\,^{o}C$ на мелассных растворах с кратностью разбавления K_{pm} = 4; 8 и 12, что соответствует начальному содержанию углеводов (сахара) в культуральной среде в массовых долях $S_0 = 0.115$ $S_0 = 0.0575$ и $S_0 = 0.0383$ при шести начальных значениях концентраций дрожжей: x_0 = 0.925; 2.5; 4.73; 7.5 и 12,5 кгАСБ/м^3 (АСБ – абсолютно сухая биомасса); pH-среды поддерживалось на уровне 4,2 – 4,6.

Notes

Цель исследований – во-первых, изучение кинетики протекания биологического процесса на различных стадиях культивирования микроорганизмов и при различных начальных значениях концентраций дрожжевых клеток в засевной культуре и углеводов в питательной среде;

во-вторых, поиск уравнений математических моделей, адекватно отражающих развитие биологических процессов, позволяющих прогнозировать их течение за пределами эксперимента и рассчитывать осреднённые по времени культивирования удельные скорости прироста биомассы и потребления субстрата.

Культивирование проводили в течение восьми – девяти часов. Через каждый час отбирались пробы на предмет определения концентрации локальных значений дрожжей x и углеводов S. За начальное время отсчёта брали время $\tau_1 = 0$. Этому времени соответствовали начальные значения концентрации дрожжей x_0 и субстрата S_0.

II. Кинетика прироста биомассы.

В качестве примера на рис.1 представлены результаты экспериментальных исследований кинетики культивирования дрожжей при $S_0 = 0.115$. При других значения S_0 графики выглядят аналогичным образом.

Анализ многочисленных математических моделей [1, 3, 4, 8] показал, что для описания опытных данных можно принять простую, так называемую модель степенного вида (1) [8], достаточно точно отражающую характер протекания биологического процесса во времени. Кроме того она даёт неплохую сходимость опытных и расчётных значений концентраций биомассы в культуральной среде в широком диапазоне изменения времени культивирования:

$$x_b = 1 + (\gamma\tau)^n, \qquad (1)$$

где $x_b = x / x_0$ - безразмерное значение массовой текущей концентрации биомассы x в единице объёма культуральной среды; γ - удельная степенная локальная скорость (в отличие от удельной логарифмической скорости μ [1, 2]) прироста биомассы, 1/время. Из уравнения (1) следует, что отношение $1/\gamma$ имеет вполне определённый физико-биологический смысл - время удвоения биомассы, параметр, величина которого имеет большое значение в технологических расчётах и в процессе культивирования, по сути, остаётся постоянной.

Показатель степени n - величина безразмерная, определяет темп протекания биологического процесса, или иными словами, характеризует изменение скорости его протекания во времени.

Следует различать локальные значения удельных скоростей прироста биомассы и потребления углеводов γ и γ_s и их осреднённые в промежутке времени культивирования величины $\bar{\gamma}$ и $\bar{\gamma}_s$. Локальные значения будут изменяться во времени, т.к. будут изменяться концентрации биомассы, субстрата и продуктов метаболизма в культуральной среде. Осреднённые значения остаются постоянными в пределах времени осреднения, но

меняются в случае изменения x_0 и S_0. В данной статье мы будем иметь дело с осреднёнными величинами.

Дальнейшая задача будет заключаться в том, чтобы на основе полученных данных подобрать уравнения математических моделей, адекватно отражающие изменение прироста биомассы во времени и устанавливающие функциональную связь, $\bar{\gamma}$ $\bar{\gamma}_s$ и n с S_0 и x_0.

Уравнение кинетики прироста биомассы, выраженное через $\bar{\gamma}$, выглядит так же, как и уравнение (1), с заменой в нём γ на $\bar{\gamma}$.

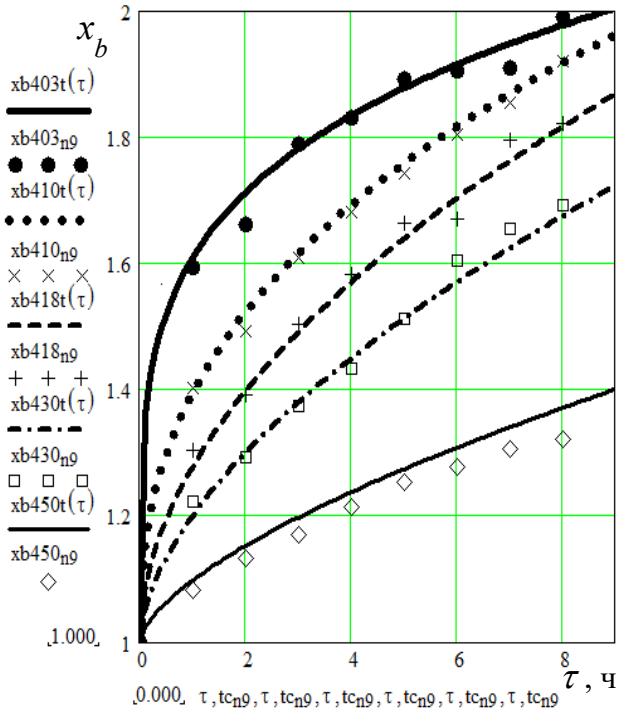

Рис. 1: Изменение концентрации биомассы в процессе культивирования при S_0 = 0,115.

Уравн.: (1)-(7): — x_0 = 0.925; ●●● x_0 =2.5;

— x_0 =4.73; —●— x_0 =7.5; —— x_0 =12.5

В дальнейших наших рассуждениях в поисках математических моделей рассматриваемого варианта культивирования мы, в основном, будим опираться на экспериментальные данные, представленные на рис. 1. Подробный анализ результатов экспериментов при иных значениях S_0 не приводится, т.к. они аналогичны рисунку 1 и их особенности будут отражены в математических моделях.

III. Осреднённая Степенная Удельная Скорость.

В результате компьютерной обработки экспериментальных данных из нескольких предложенных моделей были выбраны следующие функциональные зависимости $\bar{\gamma}(x_0)$ и $n(x_0)$:

$$\bar{\gamma} = \bar{a} - \bar{b}x_0, \tag{2}$$

$$n = \frac{\bar{a}_1 x_0}{\bar{b}_1 + x_0} \tag{3}$$

Notes

В равенствах (2) и (3) эмпирический коэффициент $\bar{a}$ и произведение $\bar{b}x_0$, также как и $\bar{\gamma}$, имеют размерность $1/ч$. Поскольку показатель степени n безразмерен, то и величина $\bar{a}_1$ должна быть безразмерной, а коэффициент $\bar{b}_1$ иметь такую же размерность, как и x_0. Коэффициенты - $\bar{b} = 0.0075$ м^3/(кг ч), $b_1 = 2.2$ оказались постоянными, а коэффициенты $\bar{a}$ и $\bar{a}_1$ -зависимыми от начального содержания углеводов - S_0. Для их расчёта выбраны следующие уравнения:

$$\bar{a} = 0.12(1 - e^{-52 \cdot S_0}), \tag{4}$$

$$\bar{a}_1 = \frac{0{,}3}{S_0^{0{,}43}}. \tag{5}$$

С учётом равенств (4) и (5) и коэффициентов b и b_1 уравнения (2) и (3) примут вид:

$$\bar{\gamma} = 0.12(1 - e^{-52 S_0}) - 0.0075 x_0, \tag{6}$$

$$n = \frac{0.3}{S_0^{0.43}(b_2 + 1)}, \tag{7}$$

где $\bar{b}_2 = 2.2 / x_0$.

Для наглядности, система уравнений (2), (3), (4), (5) представлена в графическом виде на рис. 2а и 2б.

По уравнениям (2) - (7) и рисункам 2 следует сделать несколько комментариев. Первое, что привлекает внимание - близкая к линейной зависимость $\bar{\gamma}$ от x_0. Причиной тому является слабая зависимость $\bar{\gamma}$ от S_0, что отражено в уравнении (6), в котором выражение в скобках в широком диапазоне изменения S_0 близко к единице.

Здесь возникает другой вопрос. Правильно ли уравнение (2) вообще отражает изменение $\bar{\gamma}$ за пределами экспериментальных исследований? Рассмотрим вариант $x_0 \to 0$. В этом случае $\bar{\gamma} \to const = \bar{a}$, чего в реальности быть не может, т.к. без внесения в культуральную жидкость чистой культуры нечему не будет развиваться, поэтому осреднённая удельная скорость $\bar{\gamma}$ должна быть равна нулю.

Рассмотрим другой крайний случай - $x_0 \to \infty$. В этом случае при определённых значениях x_0 удельная скорость становится отрицательной. Придать такому варианту развития биологического процесса какой-то физико-биологический смысл вряд ли удастся. Скорее всего, при $x_0 \to \infty$ $\bar{\gamma}$ будет также стремиться к нулю.

Причины снижения удельной скорости с увеличением концентрации биомассы в культуральной среде различны и объяснения этому имеются в литературе [3, 8]. Уравнение (6) лишь конкретизирует связь удельной скорости $\bar{\gamma}$ с параметрами x_0 и S_0 только для условий эксперимента.

Проанализируем теперь влияние концентрации углеводов на развитие биологических процессов. Согласно уравнению (4) при $S_0 = 0$ коэффициенте $\bar{a} = 0$, и из уравнения (2) следует, что $\bar{\gamma}$ становится величиной отрицательной при любом положительном значении x_0. Однако выше уже показана сомнительность такой ситуации. Развитие клеток в отсутствии углеводов быть не может, значит при $S_0 \to 0$ $\bar{\gamma}$ должно стремится к нулю. Наоборот, засеянные в культуральную жидкость дрожжевые клетки станут гибнуть. Можно, конечно, предположить, что отрицательные значения $\bar{\gamma}$ будут каким-то образом характеризовать скорость гибели клеток, но это требует серьёзного экспериментального подтверждения.

При $S_0 \to 1$ (S_0 не может быть больше единицы), из уравнения (4) следует, что коэффициент $\bar{a}$ стремится к постоянной максимальной величине $\bar{a}_m$, зависящей от S_0 и близкой к 0.12 (см. рис.2а). В этом случае, примерно, при $x_0 = 16$ кг/м^3, согласно уравнению (6) $\bar{\gamma} = 0$ и при дальнейшем увеличении x_0 становиться величиной отрицательной. То, что при $S_0 \to 1$ $\bar{\gamma} \to 0$, ничего удивительного нет, так как при определённой концентрации углеводы начинают проявлять консервирующие свойства, и клетки микроорганизмов перестают развиваться.

Таким образом, $\bar{\gamma} \to 0$ при $x_0 \to 0$ и $S_0 \to 0$, а также и при $x_0 \to \infty$ и $S_0 \to 1$. Если это так, то функция $\bar{\gamma}(x_0)$ при определённых значениях x_0 и S_0 должна иметь максимум. Однако математическая модель, построенная только на основе осреднённых значениях удельной скорости прироста биомассы такого вывода не подтвердила.

Причиной снижения $\bar{\gamma}$ с ростом x_0 в рассматриваемом варианте культивирования может быть падение концентрации кислорода, который находится в растворённом виде в исходной культуральной среде. Его равновесная концентрация в жидкости будет определяться её физическими свойствами, изменяющимися в процессе культивирования.

Notes

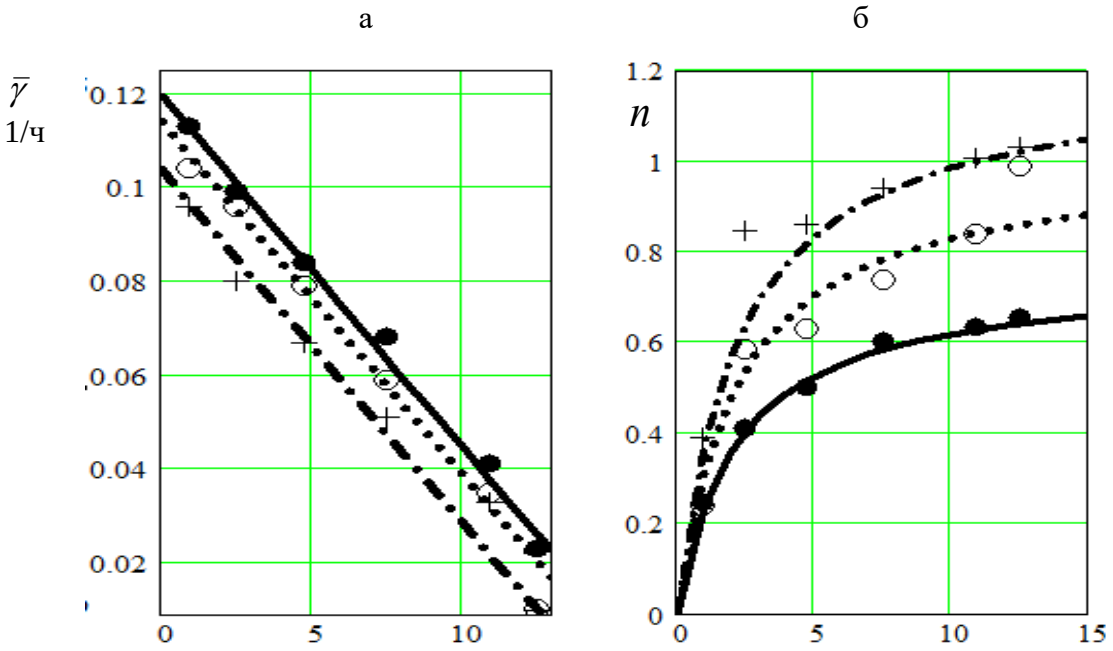

Рис. 2: Зависимость $\bar{\gamma}$ и n от x_0. Линии: урав. (2): — $S_0 = 0{,}115$; ●●● $S_0 = 0{,}0575$; — — $S_0 = 0.0383$;

В момент внесения в культуральную жидкость засевной культуры концентрация растворённого кислорода будет максимальной и условия для развития клеток будут максимально благоприятными. Однако в процессе размножения клеток концентрации кислорода будет падать. Причиной тому могут быть два обстоятельства. Во-первых, потребление кислорода самими клетками и, во-вторых, нарушение равновесного состояния системы из-за выделения клетками в жидкую среду продуктов метаболизма. Но, так как скорость снижения кислорода, как правило, выше, чем скорость его молекулярной диффузии в культуральную жидкость через свободную поверхность, то концентрация кислорода в среде резко подает. Условия жизнедеятельности клеток ухудшаются, что может служить дополнительной причиной снижения удельной скорости прироста биомассы. К сожалению, авторы работы [5] кинетику снижения концентрации кислорода в культуральной среде в процессе культивировании не снимали.

Подобные исследования проводились при культивировании чайного гриба [9], и с пивными дрожжами в процессе брожения сусла [10]. Несмотря на различие видов исследуемых микроорганизмов, кинетические закономерности потребления кислорода у авторов работ [9, 10] оказались схожими. Вполне возможно, что и при культивировании хлебопекарных дрожжей без принудительной аэрации кинетические закономерности изменения концентрации кислорода будут такими же.

Показатель степени n, названный темпом прироста биомассы, как уже сказано ранее, характеризует изменение скорости прироста биомассы во времени, являющейся производной функции (1)

Ref

10. Тишин В. Г.А., Тамазян, В.Г.,Оганнисян В.Г., Т.В. Меледина Т.В. Влияние кислорода на кинетику биологических процессов при сбраживании сусла. Хранение и переработка сельхозсырья, №4, 2010 г., с. 29–32.

$$x'_b = \frac{dx}{d\tau} = \bar{\gamma}^n n \tau^{n-1}. \quad (8)$$

Вид функции (8) будет зависеть от значения n. При $n=1$ удельная скорость будет постоянной - $x'_b = \bar{\gamma}$, функция (1) примет линейный характер, и на рис. 1 линии станут прямыми. При $n >$ линии будут иметь вид восходящих кривых.

Вариант с $n < 1$ представлен на рис. 1. Согласно уравнению (8), скорость прироста биомассы в процессе культивирования падает. Для наглядности производная (8) в графическом виде представлена на рис.3.

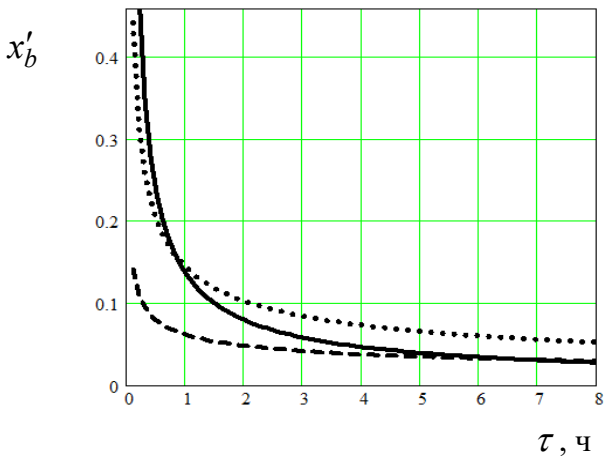

Рис. 3: Изменение скорости прироста биомассы во время культивирования при S_0=0,115; линии: x_0=0.925 ▬; x_0=4.73 ●●●; x_0=12.5 ▬ ▬

Из рис. 3 видно, что скорость прироста биомассы x'_b резко падает уже в первые минуты культивирования. Причём, чем выше начальная концентрация биомассы, тем резче падение скорости во времени. В общем-то, это понятно – происходит быстрое потребление субстрата клетками и при $\tau \to \infty$ $x'_b \to 0$. Но с другой стороны, согласно того же уравнения (1), при $\tau \to 0$ $x'_b \to \infty$, что невозможно в принципе. Это означает, что исходное степенное уравнение (1) не достаточно точно отражает ход биологического процесса в начальный период (начальные минуты, а то и секунды) времени. В принципе, можно подобрать более сложный явный вид уравнения функции $x_b(\tau)$, которое удовлетворяло бы условию - $\tau \to 0$ $x'_b \to const$. Однако проверить это экспериментально довольно сложно.

Сложность проведения экспериментальных исследований прироста биомассы в первый час культивирования заключается в том, что для установления закономерности развития микроорганизмов в указанное время, необходимо взять, по крайней мере, четыре пробы. Сделать их полноценный анализ в течение одного часа, у авторов работы [5] не было просто технической возможности. Выход оставался один – подобрать такие

Ref

5. Тишин И.Б., Мелединa Т.В., Головинская О.В. О выборе математических моделей кинетики культивирования дрожжей Saccharomyces cerevisiae в условиях дефицита кислорода. - Вестник Воронежского государственного университета инженерных технологий - 2015. - № 3(65). - С. 32-37.

уравнения математических моделей, которые позволили бы описать кинетику прироста биомассы, как в пределах времени эксперимента, так и за его пределами.

IV. Кинетика Потребления Углеводов.

Во время культивирования субстрат постоянно потребляется, и концентрация углеводов будет постоянно падать. Изменение концентрации углеводов определённым образом будет сказываться на ходе процесса культивирования в целом. Чтобы ответить на вопрос, как сказываться, необходимо прежде установить закономерность потребления клетками сахара в процессе культивирования на основе экспериментальных данных, представленных на рис. 4 (опытные данные обозначены знаками) при начальной концентрации субстрата S_0 = 0,115. При S_0 = 0,0575 и S_0 = 0,0383 характер изменения содержания сахаров в среде во времени такой же.

На рисунке 4 представлено изменение во времени концентрации углеводов в культуральной жидкости, выраженной в безразмерном виде $S_b = S / S_0$. Из рисунка наглядно видно падение количества субстрата в культуральной среде по мере развития популяции клеток. Причём крутизна наклона кривых зависит как от начального засева дрожжей, так и от начального содержания субстрата в культуральной среде.

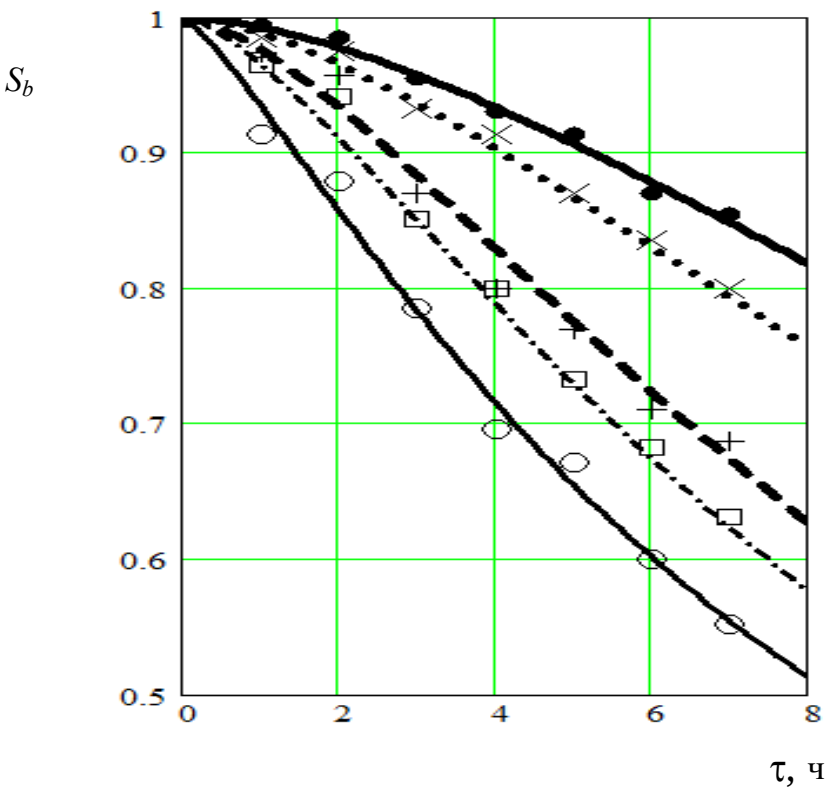

Рис. 4: Изменение концентрации углеводов в процессе культивирования при S_0 = 0,115. Линии соответствуют уравнениям (9)-(1.6):— x_0 = 0.925; ●●● x_0 =2.5; — — x_0 =4.73;—●— x_0 =7.5;;—— x_0 =12.5 кг АСБ/м^3

Можно предположить, что изменение концентрации углеводов в культуральной среде опытные данные будут подчиняться уравнением степенного вида [8]

$$S_b = \frac{1}{1+(\gamma_s \tau)^{n_s}} \tag{9}$$

В уравнении (9) величина $1/\gamma_s$ имеет вполне определённый биологический смысл. Это время снижения концентрации субстрата в два раза.

В результате обработки экспериментальных данных были подобраны эмпирические уравнения, позволившие установить функциональную зависимость $\bar{\gamma}_s$ и n_s от начального засева и начальной концентрации субстрата, следующего вида:

$$\bar{\gamma}_s = a_s x_0^{\,b_s} \tag{10}$$

$$a_s = 0.111 - 0.54 S_0 \tag{11}$$

$$b_s = 0.425 - (3.62 S_0)^{2.41} \tag{12}$$

$$n_s = a_4 b_4^{\,x_0} x_0^{\,c_s} \tag{13}$$

$$a_3 = 3.6 - 3.74 S_0^{0.127} \tag{14}$$

$$b_3 = 0.99 - 1.61 S_0^{1.31} \tag{15}$$

$$c_s = 4.29 \cdot 1.076^{\frac{1}{S_0}} S_0^{1.084} \tag{16}$$

На рис. 5а и 5б дано графическое изображение изменения удельной скорости потребления субстрата $\bar{\gamma}_s$ и показателя степени n_s при различных концентрациях биомассы и углеводов.

На рисунках обращает на себя внимание сравнительно большое отклонение опытных значений $\bar{\gamma}_s$ и n_s от рассчитанных по уравнениям (13) – (16). В зоне малых значений x_0 функции $\gamma_s(x_0)$ и $n_s(x_0)$ имеют экстремумы - с увеличением x_0 наблюдается их падение, затем резкий подъём. Имеет ли этот факт какой-либо физико-биологический смысл, или это ошибка эксперимента, сказать трудно. Уравнения (10) и (13) получены без учёта обнаруженных эффектов.

Ref
8. Tishin V.B., Ismailova Y.N. Mathematical Models of the Kinetics of the Cultivation of Microorganisms. Biophysics, 2018, V 63, №2, pp. 197-200.

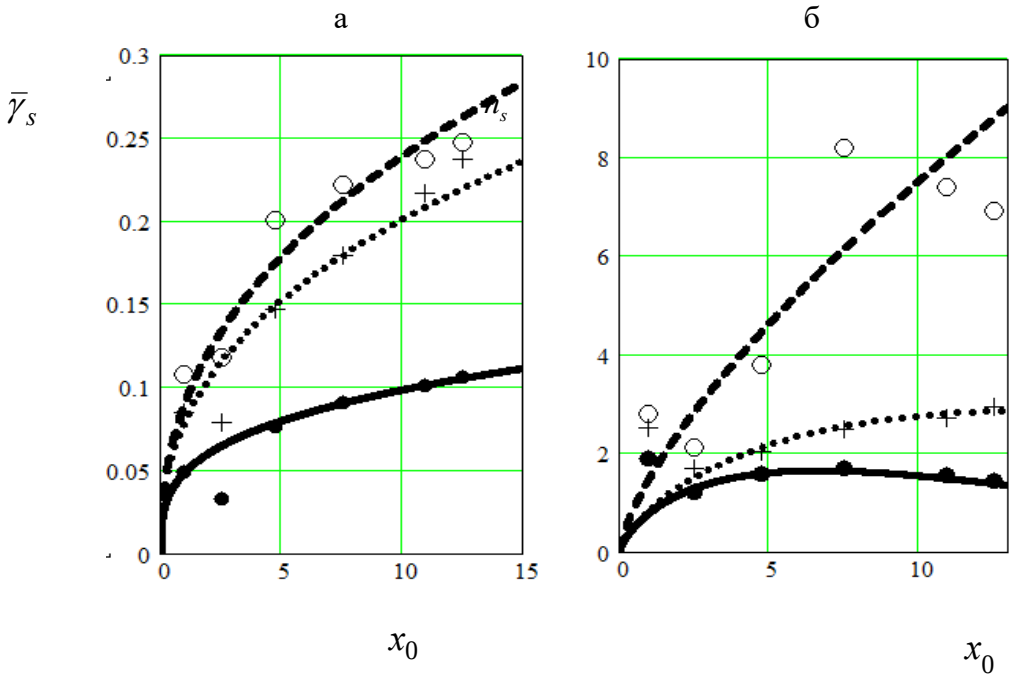

Рис. 5: Зависимость $\bar{\gamma}_s$ и η_s от x_0. Линии: урав. (2): ▬ $S_0 = 0{,}115$; ●●● $S^0 =$ 0,0575; ▬ ▬ $S^0 = 0.0383$;

Несмотря на некоторые неточности и неясности всё же можно принять систему уравнений (9) - (16) в качестве математической модели потребления углеводов дрожжевыми клетками в пределах условий эксперимента. Рисунок 4 наглядно демонстрирует удовлетворительную сходимость опытных значений концентрации углеводов и вычисленных с помощью математической модели.

Указанные модели раздельно не раскрывают особенности развития микроорганизмов в различные моменты времени, в частности от момента засева чистой культуры до времени отбора первой пробы. В это время, согласно рис. 3, в течение полутора - двух часов происходят самые интересные события.

Частично, указанных недостатков, можно избежать, объединив две модели в одну, положив в основу отношение опытных значений локальных параметров $\bar{\gamma}$ и $\bar{\gamma}_s$, введя новый параметр $\bar{\gamma}_o$.

V. Обобщённая Модель Кинетики Культивирования Дрожжей

Таким образом, на основе полученных двух моделей необходимо создать одну единую математическую модель, которая адекватно отражала бы процесс развития дрожжевых клеток с учётом потребления ими углеводов и различных начальных концентраций клеток в засевной культуре. Поскольку главной нашей задачей является получение биомассы дрожжей, то в основу обобщённой математической модели кинетики прироста биомассы следует положить уравнение (1).

Параметр $\bar{\gamma}_o$ можно найти двумя способами. Первый – делением уравнения (2) на уравнение(10):

$$\bar{\gamma}_o = \bar{\gamma} / \bar{\gamma}_s \qquad (17)$$

С учётом уравнения (17) уравнение (1) примет вид

$$x_b = 1 + (\bar{\gamma}_o \cdot \bar{\gamma}_s \cdot \tau)^n . \qquad (18)$$

Второй метод заключается в нахождении уравнения для расчёта удельной скорости непосредственно на основе экспериментальных данных, обработка которых показала, что относительная удельная скорость имеет сложную зависимость от x_0 и S_0. Математически эта зависимость будет выглядеть следующим образом:

$$\bar{\gamma}_{1o} = \frac{a_4 x_0}{1 - b_4 x_0 + (c_3 x_0)^2} . \qquad (19)$$

В уравнении (19) коэффициенты a_4, b_4 и c_3 зависят от S_0 и имеют размерность обратную x_0. Для их расчёта получены эмпирические формулы:

$$a4 = 2.455 \cdot S_0^{\,0.366}, \qquad b4 = 0.85, \qquad c3 = \frac{0.321}{S_0^{\,0.265}} . \qquad (19a)$$

В графическом виде функция $\bar{\gamma}_{1o}(x_0)$ - (19) изображена рис. 6 чёрными линиями. Для сравнения на нём же цветными линиями показана функция $\bar{\gamma}_o(x_0)$ - (17), полученная делением уравнения (2) на уравнение (10).

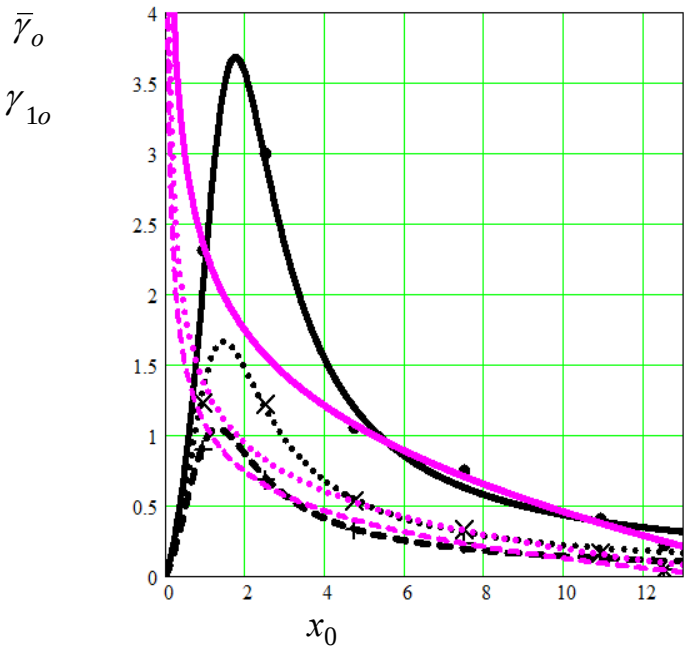

Рис. 6: Зависимость $\bar{\gamma}_o$ и γ_{1o} от x_0 при S_0:

▬ - 0.115; •••• - 0.0575; ▬ ▬ - 0.0383

Как видно по рисунку, согласно уравнению (17) при $x_0 \to 0$ $\bar{\gamma}_o \to \infty$, что не соответствует реальной действительности. Уравнения (19) в большей степени согласуется с опытными данными и согласно им при $x_0 = 0$ $\bar{\gamma}_{1o} = 0$ и этот факт соответствует

действительности. При высоких значениях x_0 (примерно при $x_0 \geq 4$ кг/м^3) уравнения (17) и (19) дают близкие значения $\bar{\gamma}_o$ и $\bar{\gamma}_{1o}$.

Ответить на вопрос – почему при постоянном значении S_0 удельная скорость вначале резко увеличивается с ростом x_0 и достигнув максимума резко начинает падать, довольно сложно. Скорее всего, это связано с теми же причинами, о которых говорится в работах [2, 3, 8] при обсуждении влияния концентрации клеток на их развитие – конкуренция за субстрат, накопление продуктов метаболизма и т.п. Видимо при малых количествах засевного материала конкуренция за субстрат, скорость накопления продуктов метаболизма не велики и увеличение x_0, до определённого предела, может приводить к росту удельной скорости прироста биомассы. Дальнейшее увеличение x_0 приводит к быстрому снижению концентрации углеводов и, соответственно, к росту скорости накопления продуктов метаболизма, и влечёт, конечном итоге, к резкому падению $\bar{\gamma}_{1o}$. В определённой мере эти соображения согласуются с работой Kono T, на которую ссылаются авторы работы [2].

С учётом вновь введённого параметра $\bar{\gamma}_{1o}$ уравнение (1) представим следующим образом:

$$x_b = 1 + (\bar{\gamma}_{1o} \cdot \bar{\gamma}_s \cdot \tau)^n \qquad (20)$$

Правомерность гипотез, высказанных на основе, уравнений (19) подтверждается и рисунком 7, на котором видно, что модели (1) – (7) и (19) – (20) дают не плохую сходимость опытных и теоретических данных по кинетике культивирования.

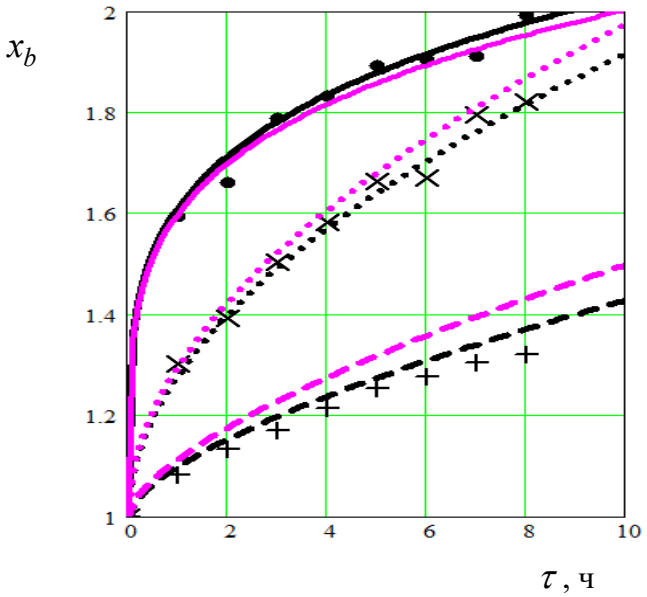

Рис. 7: Кинетика культивирования при $S_0 = 0.115$. Чёрный цвет уравнений (1) - (7), фиолетовый – (19) – (20): — $x_0 = 0.925$; ●●● $x_0 = 4.73$; — — $x_0 = 12.5$.

Ref

2. Басканьян И.А., Мельникова В.А. Периодическое культивирование как основа прогнозирования некоторых аспектов непрерывного культивирования микроорганизмов. – М.: Микробиология, т. 5, ВИНИТИ, 1976, с. 76–91.

Таким образом, на основе двух частных моделей - кинетики культивирования биомассы (1) – (7) и потребления субстрата (9) – (16) построена обобщающая модель (19) – (20).В основу модели положена степенная зависимость (1) с заменой в ней удельной скорости $\bar{\gamma}$ прироста биомассы на произведение - $\bar{\gamma}_{1o} \cdot \bar{\gamma}_s$, позволяющее учесть влияние на кинетику прироста биомассы x_0 и S_0, как в пределах границ эксперимента, так и за их пределами.

Ко всему сказанному можно добавить, что параметр $\bar{\gamma}_{1o}$ является своеобразным критерием подобия, характеризующим взаимное влияние на кинетику развития биологического процесса увеличение концентрации клеток в культуральной среде и снижение концентрации углеводов.

Заслуживающим внимание является то обстоятельство, что функции (19) предполагает резкое изменение критерия $\bar{\gamma}_{1o}$ от нуля до некого максимального значения $\bar{\gamma}_{1om}$ при изменении x_0 от нуля до его критического значения x_{0k}, после чего начинается спад и $\bar{\gamma}_{1o} \to 0$ при $x_0 \to \infty$.

Можно предположить, что подобная зависимость будет наблюдаться и между локальными значениями γ_{1o} и x в процессе культивирования при постоянных значениях x_0 и S_0. Все высказанные гипотезы требуют отдельного экспериментального подтверждения.

VI. Выводы

Математические модели (1) – (7) и (9) – (16) ограничены в возможностях моделирования биологических процессов в широких диапазонах изменения x_0 и S_0 из-за неточностей в расчётах значений $\bar{\gamma}$ и $\bar{\gamma}_s$.

Таких недостатков лишена обобщённая модель (10) и (19) - (20). Модель, во-первых, адекватно отражает влияние на развитие дрожжевых клеток во времени их начальных концентраций в засевной культуре, и начальных значений концентраций углеводов в культуральной среде в диапазоне изменения времени культивирования от $\tau = 0$ до времени окончания процесса культивирования;

во-вторых, раскрывает некоторые кинетические закономерности развития популяции дрожжевых клеток в условиях дефицита кислорода, представленных на рис. 6. Введённый в уравнение (20) параметр γ_{1o} можно принять в качестве критерия подобия, характеризующего взаимное влияние на кинетику развития биологического процесса увеличение концентрации клеток в культуральной среде и снижение концентрации углеводов.

Полученная математическая модель даёт возможность, при заданных значениях температуры культивирования, pH среды, начальной концентрации биомассы и углеводов в культуральной жидкости, определять значения средних за время культивирования удельных скоростей прироста биомассы и потребления углеводов, необходимых в технологических расчётах.

Notes

Notes

Литература

1. Басканьян И.А., Бирюков В.А., Крылов Ю.М. Математическое описание основных кинетических закономерностей процесса культивирования микроорганизмов.– М.: Микробиология, т. 5, ВИНИТИ, 1976, с. 5–75.
2. Басканьян И.А., Мельникова В.А. Периодическое культивирование как основа прогнозирования некоторых аспектов непрерывного культивирования микроорганизмов. – М.: Микробиология, т. 5, ВИНИТИ, 1976, с. 76–91.
3. Васильев Н.Н., Амбросов В.А., Складнев А.А. Моделирование процессов микробиологического синтеза.– М.: Лесн. пром., 1975. – 341
4. Рубин А. Б., Биофизика. Изд. 2 – М.: МГУ, 1997
5. Тишин И.Б., Меледина Т.В., Головинская О.В. О выборе математических моделей кинетики культивирования дрожжей Saccharomyces cerevisiae в условиях дефицита кислорода. - Вестник Воронежского государственного университета инженерных технологий - 2015. - № 3(65). - С. 32-37
6. Плевако У.А. Технология дрожжей. – М.: Пищевая пром-сть, 1970. – 300 с.
7. Семихатова Н.М., Лозенко Н.Ф., Буханова В.М. и др. Производство хлебопекарных дрожжей. – М Пищевая пром-сть, 1978. – 193 с.
8. Tishin V.B., Ismailova Y.N. Mathematical Models of the Kinetics of the Cultivation of Microorganisms. Biophysics, 2018, V 63, №2, pp. 197-200.
9. Исмаилова Ю.Н., Тишин В.Б. Формирование кислотного состава культуральной жидкости чайного гриба Medusomyces gisevi./ В сборнике: Альманах научных работ молодых ученых Университета ИТМО, 2016, Т 5, с. 242-245.
10. Тишин В. Г.А., Тамазян, В.Г.,Оганнисян В.Г., Т.В. Меледина Т.В. Влияние кислорода на кинетику биологических процессов при сбраживании сусла. Хранение и переработка сельхозсырья, №4, 2010 г., с. 29–32.

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Solitary Wave Solutions of Chafee-Infante Equation and (2+1)-Dimensional Breaking Soliton Equation by the Improved Kudryashov Method

By Umme Habiba, Md. Abdus Salam, Md. Babul Hossain & Mousumi Datta
Mawlana Bhashani Science and Technology University

Abstract- In this paper, we apply the improved Kudryashov method for finding exact solution and then solitary wave solutions of the Chafee-Infante equation and (2+1)-dimensional breaking soliton equation, where mathematical software Maple-13 is used as an important mathematical tool for removing calculation complexity, justification of the solutions and its graphical representations.

Keywords: improved kudryashov method; chafee-infante equation; (2+1) dimensional breaking soliton equation; solitary wave solutions.

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Solitary Wave Solutions of Chafee-Infante Equation and (2+1)-Dimensional Breaking Soliton Equation by the Improved Kudryashov Method

Umme Habiba [α], Md. Abdus Salam [σ], Md. Babul Hossain [ρ] & Mousumi Datta [ω]

Abstract- In this paper, we apply the improved Kudryashov method for finding exact solution and then solitary wave solutions of the Chafee-Infante equation and (2+1)-dimensional breaking soliton equation, where mathematical software Maple-13 is used as an important mathematical tool for removing calculation complexity, justification of the solutions and its graphical representations.

Keywords: improved kudryashov method; chafee-infante equation; (2+1) dimensional breaking soliton equation; solitary wave solutions.

I. Introduction

Nonlinear partial differential equations (NPDEs) describe many complex physical phenomena in different fields of science and engineering especially in fluid mechanics, plasma physics, chemical kinematics, chemical physics and geochemistry. It is important to note that many equations contain empirical parameters or empirical functions. Exact solutions allow us to determine these parameters or functions by using various techniques. So many techniques of obtaining exact and then solitary wave solutions have been explored and developed, such as $\exp(\Phi(\xi))$-expansion[1], Exp-function method[2]-[4], F-expansion method[5], modified Kudryashov method[6], modified Simple equation method[7]-[9], the extended tan-method[10], simplest equation method[11] and so on. The objective of this paper is to apply improved Kudryashov method [12] and to explore new exact solutions of nonlinear partial differential equations. This paper is organized as follows: in section 2, we give the description of the improved Kudryashov method. In section 3, we use this method to find the solitary wave solutions of nonlinear partial differential equations pointed out above. In section 4, we try to write the results and future directions. Last of all in section 5 conclusion is given.

II. Description of the Improved Kudryashov Method

The algorithm of the improved Kudryashov method for finding exact solutions of nonlinear partial differential equations is given below

Author α σ ρ ω: Department of Mathematics, Mawlana Bhashani Science and Technology University, Santosh, Tangail-1902, Bangladesh. e-mail: mahinspr@gmail.com

Step-1: Suppose the nonlinear PDE in the following form:

$$p(u, u_t, u_x, u_y, u_{tt}, u_{xt}, u_{xx}, u_{xxy} \dots\dots\dots\dots) = 0 \quad (2.1)$$

Now we use the traveling wave variable

$$u(x,t) = u(\xi),\ \xi = kx - ct \text{ [for (1+1)-dimensional equations]} \quad (2.2)$$

$u(x,y,t) = u(\xi),\ \xi = kx + wy - ct$ [for (2+1)-dimensional equations]

Then eq. (2.1) can be converted to nonlinear ordinary differential equation (ODE) by using eq.(2.2)

$$p(u, -cu', u', c^2u'', -cu'', u'', \dots\dots\dots\dots) = 0 \quad (2.3)$$

Step-2: We seek for the exact solution of eq. (2.3) in the following form:

$$u(\xi) = \frac{\sum_{i=0}^{M} a_i Q^i}{\sum_{j=0}^{N} b_j Q^j}, Q = Q(\xi) \quad (2.4)$$

where $a_i, b_j, i = 1,2,3, \dots\dots M$ and $j = 1,2,3, \dots\dots N$ are unknown constants and $Q(\xi)$ are the following functions: $Q(\xi) = 1/\sqrt{\lambda + c_1 e^{2\xi}}$ or, $Q(\xi) = -1/\sqrt{\lambda + c_1 e^{2\xi}}$ (2.5)
Above functions satisfy to the first order differential equation

$$\frac{dQ}{d\xi} = \lambda Q^3 - Q \quad (2.6)$$

To calculate the necessary number of derivatives of function $u(\xi)$, equation (2.6) is necessary. We can obtain the positive integers M and N by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in eq. (2.3).

Step-3: Substitute $u(\xi)$ and its various derivatives in eq. (2.3) and then we collect all terms with the same powers of function $Q(\xi)$ and equate the resulting expression to zero. Then we obtain a system of algebraic equations. Solving this system, we get values for the unknown parameters.

Step-4: We put these values of unknown parameters and use the solutions of eq. (2.6) to construct the exact solutions of the eq. (2.1). And finally particular choices of arbitrary constants in exact solutions give many solitary wave solutions.

III. Applications

Now we will apply the improved Kudryashov method described in section 2 to find thesolitary wave solutions of nonlinear partial differential equations.

Example-1: Chafee-Infante equation

Here the improved Kudryashov method is used for finding the solitary wave solutions of the Chafee-Infante equation[13]

$$u_t - u_{xx} = \alpha u(1 - u^2) = 0 \quad (3.1)$$

Where α is an arbitrary constant. The parameter α adjust the relative balance of the diffusion term and the nonlinear term.

Applying the travelling wave variable $\xi = kx - ct$ we obtain the following ODE

$$-cu' - k^2u'' + \alpha(u^3 - u) = 0 \quad (3.2)$$

Ref
13. Sakthivel R and Chun C.New Soliton Solutions of Chaffee-Infante Equations Using the Exp-Function Method. Z. Naturforsch. 2010; 65a: 197–202.

where the prime denotes the differentiation with respect to ξ.
We suppose that eq. (3.2) has the travelling wave solution of the form

$$u(\xi) = \frac{\sum_{i=0}^{M} a_i Q^i}{\sum_{j=0}^{N} b_j Q^j}\ ,\ Q = Q(\xi) \tag{3.3}$$

Considering the homogeneous balance between u'' and u^3 in eq. (3.2),we obtain $M = N + 2$. Suppose $N = 1$and then $M = 3$.
Thus the exact travelling wave solution takes the following form:

$$u(\xi) = \frac{a_0 + a_1 Q + a_2 Q^2 + a_3 Q^3}{b_0 + b_1 Q} \tag{3.4}$$

where $a_0, a_1, a_2, a_3$and b_0, b_1 are unknown constants. Substituting eq. (3.4) into eq. (3.2) and taking into account relations eq. (2.6), we get a polynomial of $Q(\xi)$. Collecting all the terms with the same power of $Q(\xi)$ together and equating each coefficient to zero, we can obtain a system of algebraic equations. Solving the resulting system by using Maple, we get the following sets of values of unknown constants.

Case-1: $c = \frac{3}{4}\alpha, k = \pm\frac{1}{2}\sqrt{\frac{\alpha}{2}}, a_0 = 0, a_1 = 0, a_2 = a_2, a_3 = \pm b_1\lambda, b_0 = \pm\frac{a_2}{\lambda}, b_1 = b_1$

The exact solution of eq. (3.1) is: $u(x,t) = \frac{\lambda}{\lambda + c_1 e^{\sqrt{\frac{\alpha}{2}}x - \frac{3}{2}\alpha t}}$ or $\frac{-\lambda}{\lambda + c_1 e^{\sqrt{\frac{\alpha}{2}}x - \frac{3}{2}\alpha t}}$ (3.5)

And for example, two of the solitary wave solutions and their corresponding graphs respectively are:

$u(x,t) = \frac{1}{1+e^{x-3t}}$ when $\lambda = 1, c_1 = 1$and $\alpha = 2$.

$u(x,t) = -\frac{5}{5+3e^{x-3t}}$ when $\lambda = -10, c_1 = -6$ and $\alpha = 2$.

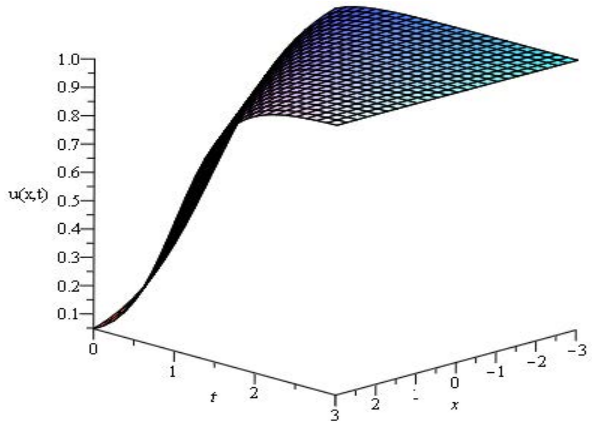


Fig. 1(3d plot): Kink type wave profile
when $\lambda = 1$, $c_1 = 1$, $\alpha = 2$.

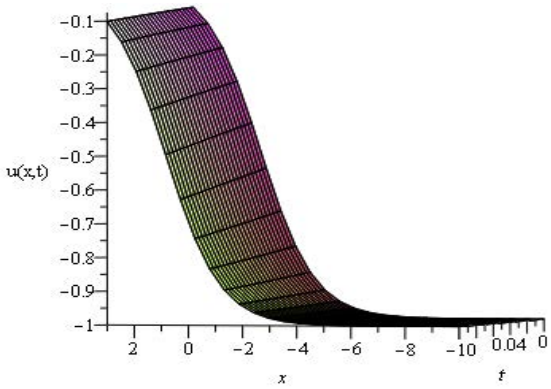


Fig. 2(3d plot): Kink type wave profile
when $\lambda = -10$, $c_1 = -6$, $\alpha = 2$.

Case-2: $c = -\frac{3}{4}\alpha, k = \pm\frac{1}{2}\sqrt{\frac{\alpha}{2}}, a_0 = -\frac{a_2}{\lambda}, a_1 = \pm b_1, a_2 = a_2, a_3 = \pm b_1\lambda, b_0 = \pm\frac{a_2}{\lambda}, b_1 = b_1$

The exact solution of eq. (3.1) is: $u(x,t) = 1 - \frac{\lambda}{\lambda + c_1 e^{\sqrt{\frac{\alpha}{2}}x + \frac{3}{2}\alpha t}}$ or $-1 + \frac{\lambda}{\lambda + c_1 e^{\sqrt{\frac{\alpha}{2}}x + \frac{3}{2}\alpha t}}$ (3.6)

And for example, two of the solitary wave solutions and their corresponding graphs respectively are:

$u(x,t) = 1 - \frac{1}{1+e^{x+3t}}$ when $\lambda = 1, c_1 = 1$ and $\alpha = 2$.

$u(x,t) = -1 + \frac{5}{5+3e^{x+3t}}$ when $\lambda = -10, c_1 = -6$ and $\alpha = 2$.

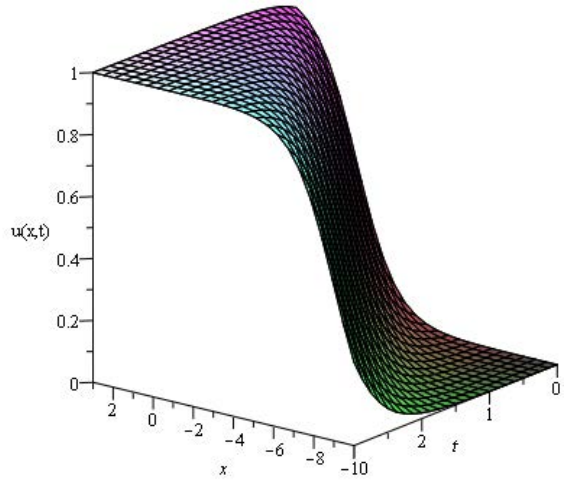


Fig. 3(3d plot): Kink type wave profile when $\lambda = 1$, $c_1 = 1, \alpha = 2$

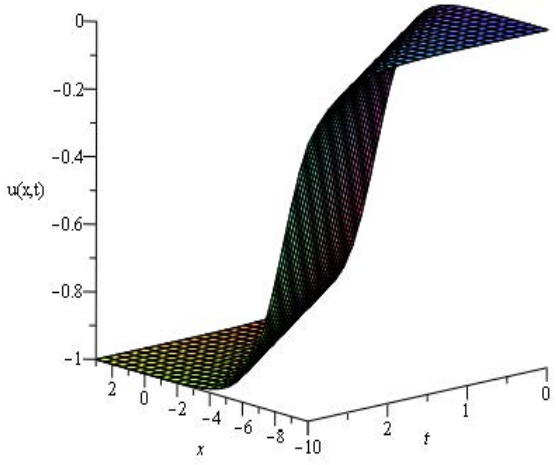
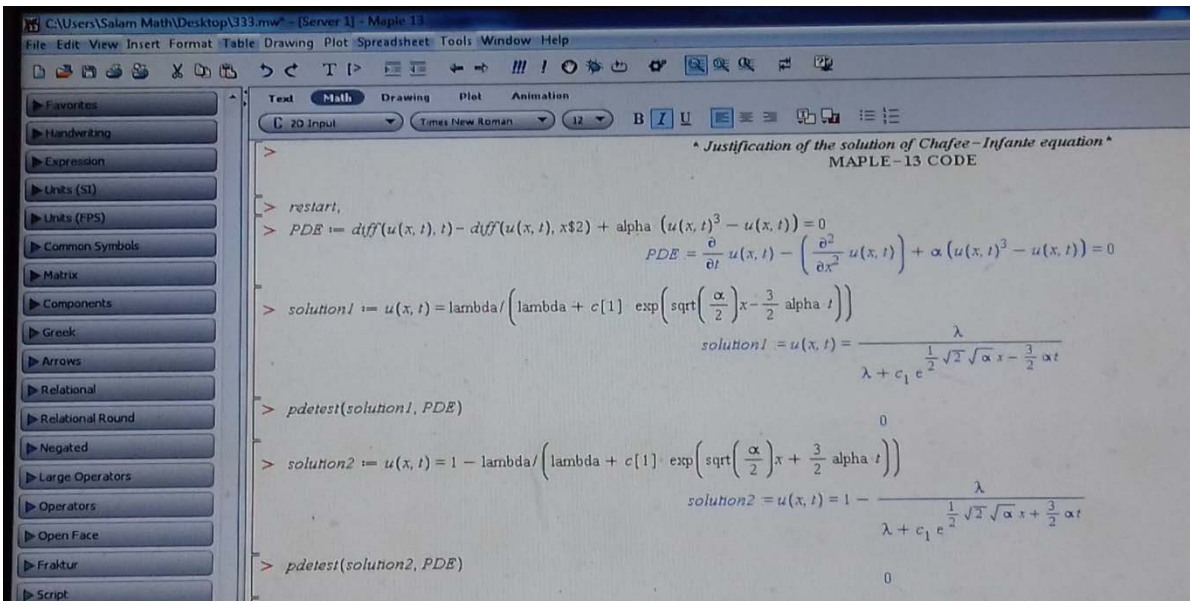


Fig. 4(3d plot): Kink type wave profile when $\lambda = -10$, $c_1 = -6$, $\alpha = 2$.

Justification of the solutions of Chafee-Infante equation by Maple-13



Example 2: The (2+1)-dimensional Breaking Soliton (BS) equation

Now, we will investigate explicit solitary wave solutions of the following (2+1)-dimensional breaking soliton equations

$$u_t + \alpha u_{xxy} + 4\alpha(uv)_x = 0 \tag{3.7}$$

$$u_y = v_x \tag{3.8}$$

Where α is a nonzero constant. Equation (3.7) and eq. (3.8) describe the (2 + 1)-dimensional interaction of a Riemann wave propagation along the *y*-axis with a long wave propagated along the *x*-axis.

If we follow the similar solution procedure of example-1, we get the following sets of constants and corresponding exact solutions.

Case-1: Values of constants

$$c = 4\alpha, a_0 = 0, a_1 = 0, a_2 = 6b_0, a_3 = 6b_1, a_4 = -6b_0, a_5 = -6b_1, b_0 = b_0, b_1 = b_1$$

The exact solution of eq. (3.7) and (3.8) are:

$$u(x,y,t) = v(x,y,t) = \frac{6}{\lambda + c_1 e^{2(x+y-4\alpha t)}} - \frac{6}{[\lambda + c_1 e^{2(x+y-4\alpha t)}]^2} \tag{3.9}$$

And for example, a solitary wave solution and its graphs is:

$$u = v = \frac{6}{1 + e^{2(x+4t)}} - \frac{6}{[1 + e^{2(x+4t)}]^2}$$

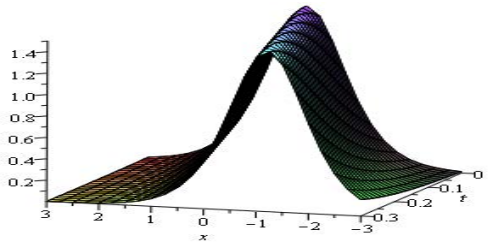


Fig. 5(3d plot): Kink Type wave profile at y=0 when $\lambda = 1, c_1 = 1, \alpha = -1$.

Case-2: Values of constants

$$c = -4\alpha, a_0 = -b_0, a_1 = -b_1, a_2 = 6b_0, a_3 = 6b_1, a_4 = -6b_0, a_5 = -6b_1, b_0 = b_0, b_1 = b_1$$

The exact solution of eq. (3.7) and (3.8) are:

$$u = v = -1 + \frac{6}{\lambda + c_1 e^{2(x+y+4\alpha t)}} - \frac{6}{[\lambda + c_1 e^{2(x+y+4\alpha t)}]^2} \tag{3.10}$$

And for example, a solitary wave solution and its graphs is:

$$u = v = -1 + \frac{6}{1 + 2e^{2(x-8t)}} - \frac{6}{[1 + 2e^{2(x-8t)}]^2}$$

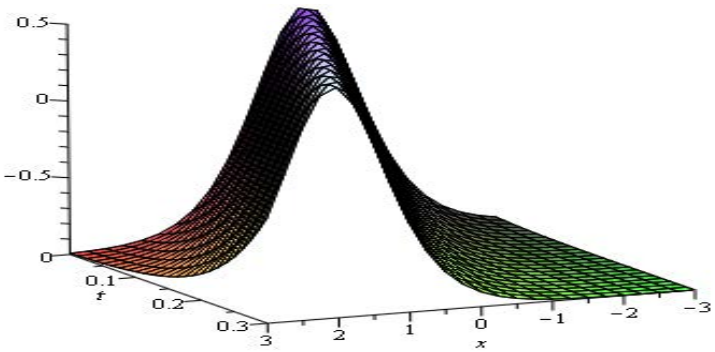


Fig.6(3d plot): Kink type wave profile at y=0 when $\lambda = 1, c_1 = 2, \alpha = -2$

IV. Results and Future Directions

In example-1 of section 3, we find the exact sloutions of Chafee-Infante equation by improved Kudryashov method. From fig. 1-fig.4 we get kink type wave profile for different particular values of parameters choosing in eq.(3.5) and eq.(3.6). These graphs increase or fall down from one asymptotic state to another. The kink solution approaches a constant at infinity. In example-2, using this method wesolve the (2+1)-dimensional breaking soliton equations and get also kink type wave profile. From fig. 5(3d plot) and fig.6(3d plot) give its graphyical representations. In future, various partial differential equations of higher order can be solved by using the improved Kudryashov method. Besides, obtained results can be used for practical applications in later research.

V. Conclusion and Future Research

We have properly applied the improved Kudryashov method to establish exact solutions and then solitary wave solutions of the Chafee-Infante equation and the (2+1)-dimensional breaking soliton equation. The result discover that nonlinear partial differential equations can be easily handled by the improved Kudryashov method and that the performance of this method is authentic and efficient. The method is short and straightforward, and we can also apply this to other nonlinear problems. Also, thephysical interpretation of these solutions and actual applications in reality will be investigated in future papers.

References Références Referencias

1. Khan K, and Akbar MA. Application of exp (Φ(ξ))-expansion method to find the exact solutions of modified Benjamin-Bona-Mahony equation. World Appl. Sci. J. 2013; 24(10): 1373-1377. DOI: 10.5829/idosi.wasj.2013.24.10.1130.
2. Bekir, Boz A. Exact solutions for nonlinear evolution equation using Exp-function method. Physics Letters A2008; 372(10):1619–1625. DOI: 10.1016/j.physleta. 2007.10.018.
3. Soliman A A. Exact solutions of the KdV- Burgers equation by Exp-function method. Chaos, Solitons& Fractals 2009; 41(2):1034-1039. DOI: 10.1016/J.chaos. 2008. 04.038.
4. Chun C.Soliton and periodic solutions for the fifth-order KDV equation with the Exp - function method. Physics Letters A 2008; 372: 2760 -2766.
5. Sheng Zhang et al. A Generalized F-expansion Method and its Application to (2 + 1)-dimensional Breaking Solition Equations. Int. J. of Nonlinear Science 2008; 5(1): 25-32.
6. Kabir MM. Modified Kudryashov method for generalized forms of the nonlinear heat conduction equation. International Journal of the Physical Sciences 2011;6(25): 6061-6064. DOI: 10.5897/IJPS11.871.
7. ZayedME and Yasser A. The modified simple equation method for solving diffusive predator-prey system and Bogoyavlenskii equations. International journal of physical sciences 2015; 10(4): 133-14. DOI: 10.5897/IJPS2014.4244.
8. Jawad AJM, PetkovicMD, and Biswas A. Modified simple equation method for nonlinear evolution equations. App. Math. andCompu. 2010; 217(2): 869-877.
9. AshrafuzzamanMK, Akbar MA. Exact and Solitary Wave Solutions to the Generalized Fifth-order KdV Equation by Using the Modified Simple Equation

Method.Applied and Computational Mathematics 2015; 4(3): 122-129. DOI: 10.11648/j.acm.20150403.14.

10. Abdou MA. The extended tan-method and its applications for solving nonlinear physical model. Appl. Math. Comput. 2007; 190: 988-996. DOI:10.1016/ j.amc.2007.01.070.
11. Kudryashov NA. Simplest equation method to look for exact solutions of nonlinear differential equations. Chaos, Solitons and Fractals 2005; 24: 1217 – 1231. DOI: 10.1016 /j.*chaos*.2004.09.109.
12. SalamMAObayedullah M and Musa Miah M. Application of Improved Kudryashov Method to Solve Nonlinear Partial Differential Equations. Journal of Computer and Mathematical Sciences 2016; 7(4): 175-180.
13. Sakthivel R and Chun C.New Soliton Solutions of Chaffee-Infante Equations Using the Exp-Function Method. Z. Naturforsch. 2010; 65a: 197–202.

Notes

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Performance Assessment of Mean Methods in Estimating Process Capability for Non-Normal Process for Weibull Family Life Distribution

By Braimah, Joseph Odunayo
Ambrose Alli University

Abstract- This paper compares the performances of Gini Mean, Clements and Box- Cox transformation methods for estimating process capability Indices when the distribution of the process data is (skewed) non-normal. The use of Process Performance Index (PPI) is implored for process capability analysis (PCA) using Weibull distribution. Simulation of data was also carried out using R software using a decision interval (target point) of 1.0 and 1.5. Performance assessment was carried out using Boxplots, descriptive statistics and the root mean square deviation. The following were the findings from the results. The Gini mean difference based process capability indices performs best in estimating the process capability indices closest to a set target for varying distribution parameters at different sample sizes, followed by Clements and lastly, the Box-Cox transformation method [10, 19].

Keywords: process control, capability indices, performance index, standard error, skewed.

GJSFR-F Classification: MSC 2010: 11H60

Strictly as per the compliance and regulations of:

Ref

14. Montgomery, D. C and Runger, G. C. (2009). Applied Statistics and Probability for Engineers.3Ed. John Wiley & Sons, New York.

Performance Assessment of Mean Methods in Estimating Process Capability for Non-Normal Process for Weibull Family Life Distribution

Braimah, Joseph Odunayo

Abstract- This paper compares the performances of Gini Mean, Clements and Box- Cox transformation methods for estimating process capability Indices when the distribution of the process data is (skewed) non-normal. The use of Process Performance Index (PPI) is implored for process capability analysis (PCA) using Weibull distribution. Simulation of data was also carried out using R software using a decision interval (target point) of 1.0 and 1.5. Performance assessment was carried out using Boxplots, descriptive statistics and the root mean square deviation. The following were the findings from the results. The Gini mean difference based process capability indices performs best in estimating the process capability indices closest to a set target for varying distribution parameters at different sample sizes, followed by Clements and lastly, the Box-Cox transformation method [10, 19].

Keywords: process control, capability indices, performance index, standard error, skewed.

I. Introduction

Statistical Process Control is the application of statistical tools and techniques in monitoring variation in a continuous process in order to detect variations that are of assignable causes, and therefore make recommendations for corrective check on the process. Control charts are used to monitor processes in order to detect assignable cause(s) that change the process parameters. [6, 7] emphasized the importance of identification of assignable cause. When the distribution of the output quality of the process variable is continuous, the combination of two control charts such as an X -chart and an R-chart are usually required to monitor both the process mean and the process variance [14]. However, recently [17] have shown that the two combined charts are not always reliable in identifying the nature of the change.

Measuring a process performance and acting upon the assessments based on the measurements are critical elements of any continuous quality improvement efforts [15], however, companies make assessments of process performance based on different indicators. Most common of these indicators can be described in terms of process yield, process expected loss and capability indices of a particular process characteristic [4]. Among these indicators, Process Capability Indices (PCIs) have gained substantial attention both in academic community and several types of manufacturing industries

Author: Department of Mathematics and Statistics, Faculty of Physical Sciences, Ambrose Alli University, Ekpoma, Edo State, Nigeria.
e-mail: ojbraimah2014@gmail.com

since 1980s [13]. The first process capability index proposed in the literature more than three decades ago is the C_p index, which is defined as:

$$C_p = \frac{USL - LSL}{6\sigma} \quad (1)$$

where USL and LSL denote the upper and lower specification limits respectively and σ is the standard deviation of the process characteristic of interest [2]. In order to overcome this problem, a second generation PCI, the Cpk index, is introduced. The Cpk is defined as:

$$C_{pk} = \min\left[\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right] \quad (2)$$

where μ and σ are the mean and the standard deviation of the quality characteristic studied, respectively. The mean of the process characteristic has an influence on the C_{pk} index and therefore it is more sensitive to departures from centrality than the Cp index [1, 11].

Pp and Ppk are measures of process performance from a customer perspective [4, 12].

Non-normally distributed processes are not uncommon in practice. Combining this fact with the misleading results of applying basic PCIs to non-normal processes while treating them as normal distributions forced academicians and practitioners to investigate the characteristics of process capability indices with non-normal data [10, 16, 20].

There two approaches adopted in estimating PCI for non-normal process situation include:

(1) Data Transformation Approach: Data transformation approach is aimed at transforming the non-normal process data into normal process data [3, 5, 10].

(2) Distribution Fitting Method for Empirical Data: Distribution fitting methods use the empirical process data, of which the distribution is unknown [10]. These methods later fit the empirical data set with a non-normal distribution based on the parameters of the empirical distribution. Clements' Method is one of the most popular distribution approaches. Therefore, the percentile-based Cp is obtained by:

$$C_p = \frac{USL - LSL}{\xi_{0.99865} - \xi_{0.00135}} \quad (3)$$

where $\xi_{0.99865}$ and $\xi_{0.00135}$ denote the upper and lower 0.135th percentiles of the process distribution, respectively.

Following the same logic, the Cpkindex can be obtained using a percentile approach:

$$C_{pk} = \min\left[\frac{USL - \xi_{0.5}}{\xi_{0.99865} - \xi_{0.5}}, \frac{\xi_{0.5} - LSL}{\xi_{0.5} - \xi_{0.00135}}\right] \quad (4)$$

where $\xi_{0.5}$ is the median of the process distribution, which is used instead of the process mean, because the process mean is not indicative of the centrality of a non-normal distribution specially when skewness of the distribution is taken into account [1].

The mean difference is independent of any central measure of localization, which can be seen from its definition.

$$\Delta_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|x - y|dF(x)dF(y) \quad (5)$$

When the random variable X is discrete (a case more often considered) the formula has the form

Ref

10. Kotz, S and Johnson, N.L (1993). *Process Capability Indices.* 2nd Ed. Chapman & Hall, Suffolk.

$$\Delta_1 = \sum_{i=-\infty}^{i=+\infty} \sum_{j=-\infty}^{j=+\infty} |x - y| p_i p_j \quad (6)$$

The analytic investigation of the discussed characteristic is made difficult because of the absolute value occurring in the formula. However, it facilitates the computations on numerical data, which also concerns, as is well known, the mean deviation.

This paper therefore compares the performances of Gini Mean, Clements and Box- Cox transformation methods for estimating process capability Indices for a non-normal case.

Notes

II. Methodology

a) Process Capability Indices

The process capability index, the Cp index, which is defined as:

$$\text{Cp} = \frac{USL - LSL}{6\sigma} \quad (7)$$

The Cpk can be defined as:

$$min\left[\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right] \quad (8)$$

where USL and LSL denote the upper and lower specification limits, respectively, and σ is the standard deviation of the process characteristic of interest.

Process Capability relative to one sided specification limit

$C_{pu} = \frac{\text{USL- } \mu}{3\sigma}$ Process Capability relative to Upper specification limit

$C_{pl} = \frac{\mu\text{-LSL}}{3\sigma}$ Process Capability relative to lower specification limit

$P_{pu} = \frac{\text{USL- } \mu}{3\sigma}$ Process performance relative to Upper specification limit

$P_{pl} = \frac{\mu\text{-LSL}}{3\sigma}$ Process performance relative to lower specification limit

b) Clements Method (CM)

For non-normal Pearsonian distribution (which includes a wide class of "populations" with non-normal characteristics), [3,18] proposed a method of non-normal percentiles to calculate process capability Cp and process capability for off center process Cpk indices based on the mean, standard deviation, skewness and kurtosis. Clements utilized the table of the family of Pearson curves as a function of skewness and kurtosis [8, 9].

Clements replaced 6σ by (U_P - L_P) in the below equation,

$$\text{Cp} = \frac{USL - LSL}{U_P - L_P} \quad (9)$$

where, U_P is the 99.865 percentile and L_P isthe 0.135 percentile, For Cpk, the process mean u is estimated by median M, and thetwo 305 are estimated by (U_P — M) and (M—L_P) respectively,

$$\text{Cpk} = \min\left[\frac{USL - M}{U_P - M}, \frac{M - LSL}{M - L_P}\right] \quad (10)$$

i. *Algorithm for calculating PCIs using Clements method*

(1) Obtain the specification limits USL and LSL for a given quality characteristic

(2) Estimate sample statistics for the given sample data: sample size, mean, standard deviation, skewness and kurtosis Calculate estimated 0.135percentile L_P

(3) Calculate estimated 99.865 percentile U_P

(4) Calculate estimated median M

(5) Calculate non-normal process capability indices using equations.

$$\mathrm{Cp} = \frac{USL-LSL}{U_P-L_P} \tag{11}$$

$$\frac{\text{USL- M}}{U_P - M}, \frac{\text{M-LSL}}{M - L_P}$$

$$\mathrm{Cpu} = \frac{\text{USL- M}}{U_P-M} \tag{12}$$

$$\mathrm{Cpl} = \frac{\text{M-LSL}}{M-L_P} \tag{13}$$

c) Box-Cox power Transformation (BCT)

The Box-Cox transformation was proposed by Box and Cox in 1964 and used for transforming non-normal data [9]. The Box-Cox transformation uses the parameter λ. In order to transform the data as closely as possible to normality, the best possible transformation should be performed by selecting the most appropriate value of λ. In order to obtain the optimal λ value, Box-Cox transformation method requires maximization of a log-likelihood function. After the transformation, process capability can be evaluated. They proposed a useful family of power transformations on the necessarily positive response variable X.

$$X^{(\lambda)} = \begin{cases} \frac{X^{\lambda}-1}{\lambda}, for\ \lambda \neq 0 \\ \ln X, for\ \lambda = 0 \end{cases} \tag{14}$$

where the variable X takes positive values. If the variable X takes negative values, then a constant value will be added in order to make the values positive. This continuous family depends on a single parameter λ that can be estimated by using maximum likelihood estimation.

Firstly, a value of λ from a pre-assigned range is collected. Then L_{max} is computed as in

$$L_{max} = -\frac{1}{2}\ln\hat{\sigma}^2 + \ln J(\lambda, X) = -\frac{1}{2}\ln\hat{\sigma}^2 + (\lambda-1)\sum_{i=1}^{n}\ln X_i \tag{15}$$

For all $\lambda, J(\lambda, X)$is evaluated as in Equation

$$J(\lambda, X) = \prod_{i-}^{n}\frac{\partial W_i}{\partial X_i} = \prod_{i=1}^{n} X_i^{\lambda-1} \tag{16}$$

$$\ln J(\lambda, X) = (\lambda-1)\sum_{i=1}^{n}\ln X_i \tag{17}$$

For fixed λ, σ^2 is estimated by using $S(\lambda)$, which is the residual sum of squares of $X^{(\lambda)}.\sigma^2$ is estimated by the formula in the equation below [15].

$$\hat{\sigma}^2 = \frac{S(\lambda)}{n} \tag{18}$$

Ref

9. Hoerl, R.W and Snee, R. D. (2010). Flexible Process Capability Indices *The American Statistician*, 64(1), pp.10-14.

d) Gini's Mean Difference (GM)

The Gini's mean difference for a set of n ordered observations, $\{x_1, x_{2,\ldots,}x_n\}$, of a random variable X is defined as:

$$\mathrm{Gn} = \frac{2}{n(n-1)}\sum_{j=1}^{n}\sum_{i=1}^{n}\left|x_i - x_j\right| \tag{18}$$

$$\mathrm{Gn} = \frac{2}{n(n-1)}\sum_{i=1}^{n}[(x_i - x_1) + (x_i - x_2) + \cdots + (x_i - x_{n-1})] \tag{19}$$

$$\mathrm{Gn} = \frac{2}{n(n-1)}\sum_{i=1}^{n}(2i_i - n - 1)x_{(i)} \tag{20}$$

If the random variable X follows normal distribution with mean μ and variance σ^2, then [21] suggests a possible unbiased estimator of standard deviation (σ) as:

$$\sigma * = \mathrm{c}\ \frac{\sum_{i=1}^{n}[(2i_i - n - 1)x_i]}{n(n-1)} \tag{21}$$

where $c = \sqrt{\pi} = 1.77245, \sigma^* = 0.8862$ G_n is an unbiased measure of variability. Gini's mean difference can be rewritten as:

$$\mathrm{Gn} = \frac{2}{n(n-1)}\sum_{i=1}^{n}(2i - n - 1)x_{(i)} \tag{23}$$

If we write this as

$$\mathrm{Gn} = \frac{2}{n(n-1)}\sum_{i=1}^{n}[(i-1) - (n-1)]x_{(i)} \tag{24}$$

$$\mathrm{Gn} = \frac{2}{n(n-1)}\left[\sum_{i=1}^{n}(i-1)x_{(i)} - \sum_{i=1}^{n}(n-1)x_{(i)}\right] \tag{25}$$

$$\mathrm{Gn} = \frac{2}{n(n-1)}[U - V] \tag{26}$$

where U= and V=

The unbiased estimator of Gini Mean difference for Weibull distribution is

$$E(\mathrm{Gn}) = \left(2 - 2^{1-\frac{1}{\beta}}\right)\frac{\Gamma\left(1+\frac{1}{\beta}\right)}{\lambda} = \sigma_{gw} \tag{27}$$

The Weibull probability density function is given as:

$$f(x) = \lambda\beta(\lambda x)^{\beta 00000-1}e^{-(\lambda x)^{\beta}} \tag{28}$$

To compute Cp and Cpk using Gini's mean difference as a measure of variability when the data follow a Weibull distribution

$$C_{npg} = \frac{USL - LSL}{5.3172\sigma_{gw}} \tag{29}$$

$$C_{npkg} = \frac{\min\,(USL - m, m - LSL)}{2.6586\sigma_{gw}}$$

$$C_{npug} = \frac{USL - m}{2.6586\sigma_{gw}} \tag{30}$$

$$C_{nplg} = \frac{m - LSL}{2.6586\sigma_{gw}} \tag{31}$$

Ref

21. Yitzhaki, S (2010). Gini's mean difference: A superior measure of variability for non-normal distributions, *Metron*, 61(2), pp. 285-316.

e) Performance Comparison of the Clements, Box-Cox transformation and the Gini Methods

The performance comparison is carried out by generating Weibull data through simulation and for this reason, process performance indices (PPIs) are executed for computing process capability rather than process capability indices (PCIs).

Weibull distribution is used or modeling most industrial processes especially in reliability field which is concerned with the failure of a product or the time to failure of the product. Only one sided (USL) process performance index P_{Pu} is considered. The USL is computed from the equation below using a targeted *Ppu* of 1.0 and 1.5. The targeted Ppu of 1.0 is indicating the process is marginally capable of meeting the specifications and the Ppu of 1.5 is indicating the process is good and very capable of meeting the specification limits [14].

Box plots, descriptive statistics, the root-mean-square deviation (RMSD), which is used as a measure of error, are utilized for evaluating the performances of the methods. In addition, the bias of the estimated values is important as the efficiency measured by the mean square error.

f) The Root-Mean-Square Deviation (RMSD)

The root-mean-square deviation (RMSD) is used to measure the differences between the targeted Ppu values and the estimates obtained by BCT, Clements and Gini mean difference based methods.

$$\text{RMSD} = \sqrt{\frac{\sum_{i=1}^{r}(Estimated\ \ Ppu_i - Targeted\ \ Ppu_i)^2}{r}} \tag{32}$$

where r is the number of data sets generated randomly for each Weibull distribution with specified parameters. The RMSD serves to aggregate the magnitudes of the errors in the predictions for various times into a single measure of predictive power and a measure of accuracy [8].

III. Result and Data Analysis

a) The Descriptive Statistics

The tables below show the corresponding quantiles, mean, median along with skewness and kurtosis based on the specified parameter values of Weibull distribution. Kurtosis gives information about the relative concentration of values in the center of the distribution as compared to the tails.

Table 1: Summary statistics of Weibull distribution at $\alpha = 1$ and $\beta = 1$ for different sample sizes

	Weibull(α,β)	$X_{0.99865}$	Median = $X_{0.50}$	Mean	Skewness	Kurtosis
n=25	Weibull(1,1)	3.8939	0.7469	1.0296	1.5138	2.7650
n=50	Weibull(1,1)	4.3501	0.6915	0.9989	1.6654	3.2510
n=75	Weibull(1,1)	4.8491	0.6803	0.9733	1.8448	4.7940
n=100	Weibull(1,1)	5.1722	0.7167	1.0130	1.8384	4.6222

Ref
14. Montgomery, D. C and Runger, G. C. (2009). Applied Statistics and Probability for Engineers.3Ed. John Wiley & Sons, New York.

Notes

Table 2: Summary statistics of Weibull distribution at $\alpha = 1$ and $\beta = 2$ for different sample sizes

	Weibull(α,β)	$X_{0.99865}$	Median = $X_{0.50}$	Mean	Skewness	Kurtosis
n=25	Weibull(1,2)	8.4542	1.4737	2.0537	1.7316	3.9214
n=50	Weibull(1,2)	8.9368	1.4228	2.0194	1.6610	3.7344
n=75	Weibull(1,2)	9.5834	1.3896	2.0026	1.7548	4.1068
n=100	Weibull(1,2)	10.2353	1.4067	2.0281	1.7980	4.2492

Table 3: Summary statistics of Weibull distribution at $\alpha = 2$ and $\beta = 1$ for different sample sizes

	Weibull(α,β)	$X_{0.99865}$	Median= $X_{0.50}$	Mean	Skewness	Kurtosis
n=25	Weibull(2,1)	1.9352	0.8284	0.8743	0.5722	0.3388
n=50	Weibull(2,1)	2.0577	0.8495	0.8916	0.5426	0.0124
n=75	Weibull(2,1)	2.0978	0.8295	0.8774	0.5650	0.0292
n=100	Weibull(2,1)	2.2503	0.8158	0.8719	0.6686	0.3788

Table 4: Summary statistics of Weibull distribution at $\alpha = 2$ and $\beta = 2$ for different sample sizes

	Weibull(α,β)	$X_{0.99865}$	Median = $X_{0.50}$	Mean	Skewness	Kurtosis
n=25	Weibull(2,2)	1.6916	1.7779	1.7770	0.5108	0.1892
n=50	Weibull(2,2)	1.7177	1.5458	1.7537	0.5820	0.1896
n=75	Weibull(2,2)	4.2828	1.6516	1.7703	0.5638	-0.0554
n=100	Weibull(2,2)	4.4739	1.6662	1.7733	0.5898	0.1090

The distribution plot of Weibull distribution for various shape and scale parameter is shown below.

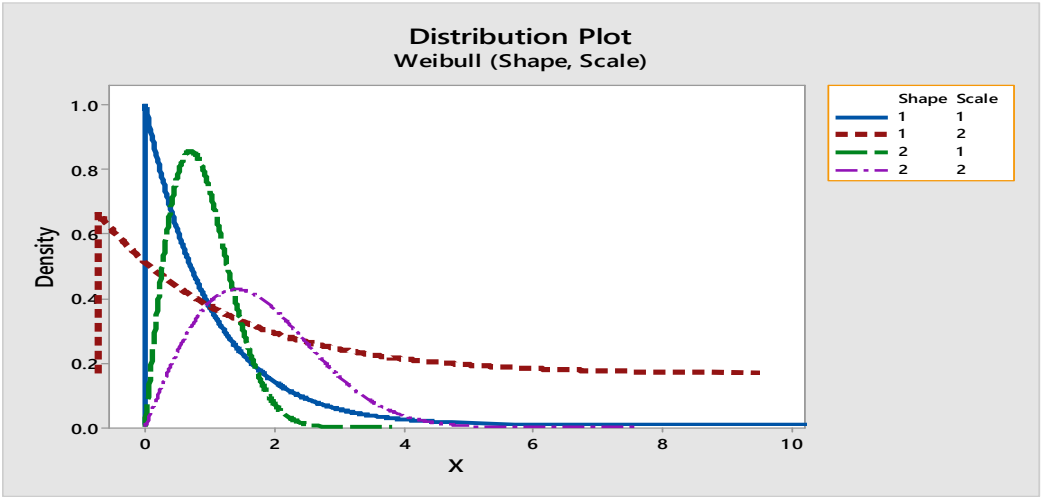


Figure 1: Distribution plot of Weibull distribution using different shape and scale parameters

From the distribution plot in Figure 1, the distribution plots are positively skewed (non-normal) for the combinations of the shape and scale parameters with Weibull (1, 1) the most peaked.

b) Process Capability Analysis

i. *Gini Mean Difference based Process Capability Analysis*

The table of parameter estimation is given below using the generated data from Weibull distribution with varying shape and scale parameters of (1,1), (1,2) (2,1) and (2,2) at different sample sizes of n = 25, 50,75 and 100.

Table 5: Gini's Estimated USL obtained from the data

GMD	C_{npug}	USL FOR GINI			
		n=25	n=50	n=75	n=100
Weibull(1,1)	1.0	3.4055	3.3501	3.3389	3.3753
	1.5	4.7348	4.6794	4.6682	4.7046
Weibull(1,2)	1.0	3.8298	3.7789	3.7457	3.7629
	1.5	5.0079	4.9570	4.9238	4.9409
Weibull (2,1)	1.0	2.2086	2.2297	2.2096	2.1960
	1.5	2.8986	2.9198	2.8997	2.8861
Weibull (2,2)	1.0	3.1118	3.0668	3.0633	3.0778
	1.5	3.8176	3.7726	3.7691	3.7837

Table 6: Clements's Mean Difference based Process Capability Analysis

CA	C_{npug}	USL FOR CLEMENTS ANALYSIS			
		n=25	n=50	n=75	n=100
Weibull 1,1	1.0	3.8939	4.3501	4.8491	5.1722
	1.5	5.4674	6.1794	6.9335	7.3999
Weibull 1,2	1.0	8.4542	8.9368	9.5834	10.2353
	1.5	11.9445	12.6937	13.6802	14.6495
Weibull 2,1	1.0	1.9352	2.0577	2.0978	2.2503
	1.5	2.4886	2.6618	2.7319	2.9675
Weibull 2,2	1.0	1.6916	1.7177	4.2828	4.4739
	1.5	1.6484	1.8036	5.5983	5.8777

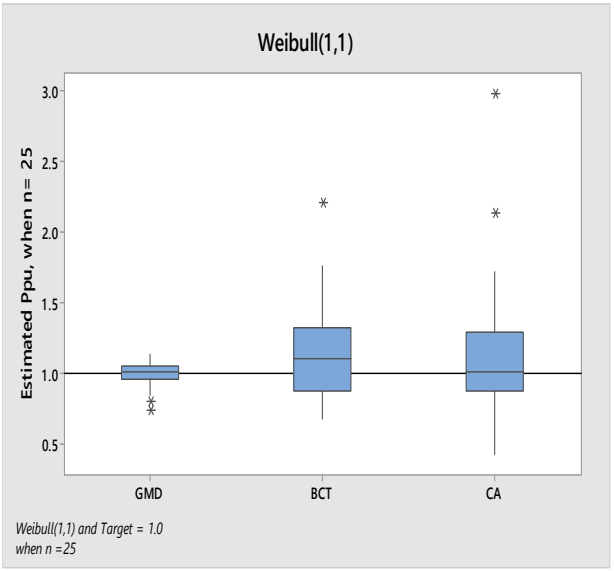
Table 7: Box - Cox's Mean Difference based Process Capability Analysis

BCT	C_{npug}	USL FOR BCT			
		n=25	n=50	n=75	n=100
Weibull 1,1	1	2.6189	2.3984	2.0027	1.8314
	1.5	3.7938	3.4562	2.7452	2.3774
Weibull 1,2	1	3.4239	3.0202	2.5266	2.1994
	1.5	4.7835	4.1252	3.3355	2.8110
Weibull 2,1	1	1.6913	1.6490	1.6384	1.6528
	1.5	2.2016	2.1041	2.0553	2.0790
Weibull 2,2	1	2.6310	2.3786	2.3879	2.3661
	1.5	3.3389	2.9853	2.9895	2.9084

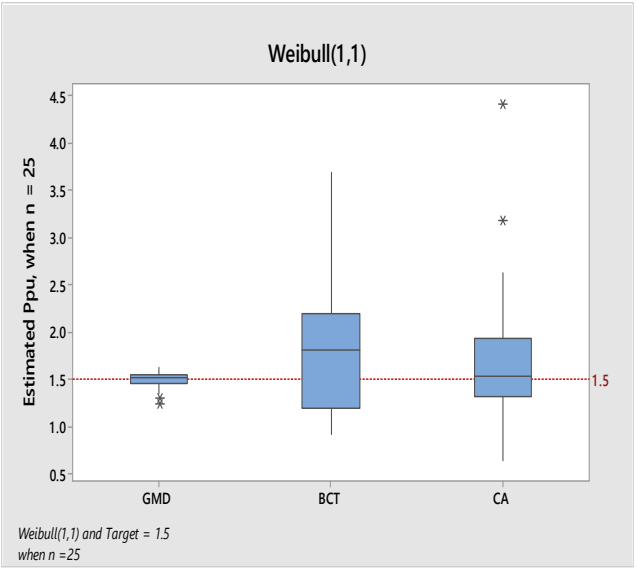
c) Graphical Comparison of the computed Process Capabilities

In order to compare the process capability methods graphically at each targeted Ppu (1.0 and 1.5), box plot or whisker plot is used to show the shape of the distribution, its central value (0.50), variability (0.75 – 0.25) and outliers by star symbol if it exists. The position of the median line in a box plot indicates the location of the values. The figures below shows the comparison

Notes



a.) Weibull(1,1), target Ppu 1.0 and n= 25



b.) Weibull(1,1), target Ppu 1.5 and n= 25

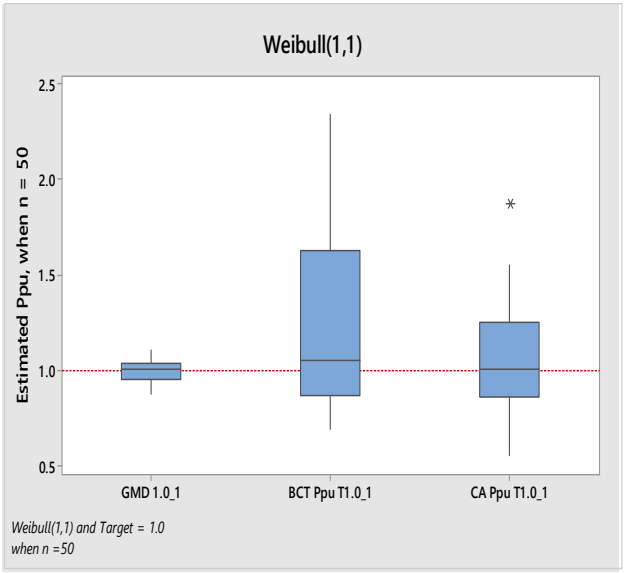


Figure 2: Weibull (1,1), target Ppu 1.0 and n= 50

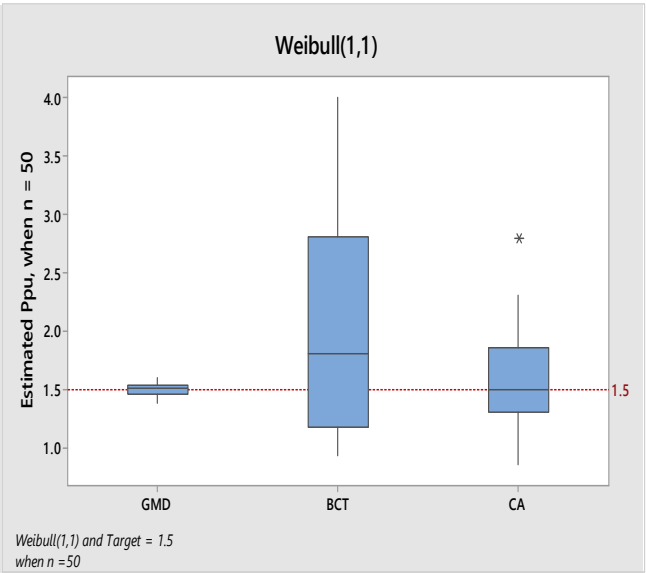


Figure 3: Weibull (1,1), target Ppu 1.5 and n= 50

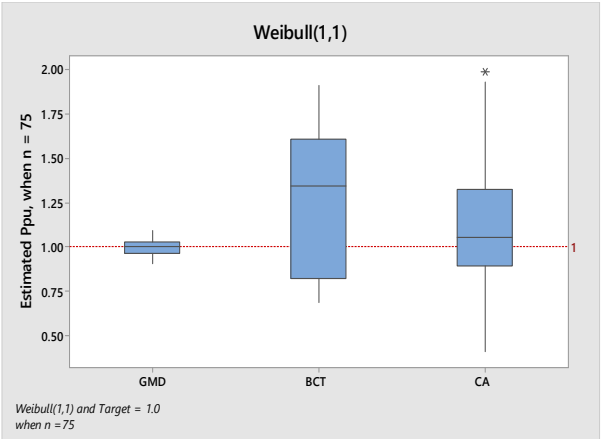


Figure 4: Weibull(1,1), target Ppu 1.0 and n= 75

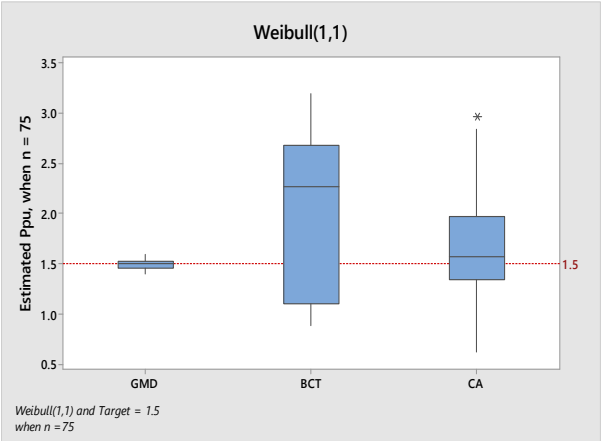


Figure 5: Weibull(1,1), target Ppu 1.5 and n= 75

From the boxplots, the results show that for different distribution parameters at different sample sizes, GMD methods is the best of the three methods for computing process capability for when the process is non-normal.

d) Mean and Standard deviation of Computed Capability Indices

To confirm the result shown from the boxplots above, the mean values and the standard deviation (which shows how concentrated the data are around the mean) of the computed process capabilities are computed in the tables below.

Table 8: Descriptive statistics for CA, BCT, and GMD methods when n = 25

n = 25						
Target Ppu	**Statistics**	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1	Mean	CA	1.1230	1.1930	1.0702	1.1163
		BCT	1.1314	1.0663	1.0125	1.1055
		GMD	1.0000	1.0000	0.9999	1.0000
	Standard Deviation	CA	0.4235	0.5112	0.3542	0.4386
		BCT	0.3097	0.2478	0.2126	0.3483
		GMD	0.0758	0.1706	0.0832	0.1696
1.5	Mean	CA	1.6849	1.7897	1.6191	1.6774
		BCT	1.7820	1.6351	1.5815	1.6583
		GMD	1.5000	1.5000	1.5000	1.5000
	Standard Deviation	CA	0.6275	0.7665	0.5182	0.6465
		BCT	0.6087	0.4404	0.3048	0.4828
		GMD	0.0758	0.1706	0.0832	0.1696

Notes

Notes

Table 9: Descriptive statistics for CA, BCT, and GMD methods when n = 50

n = 50						
Target Ppu	**Statistics**	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1.0	Mean	CA	1.0577	1.1384	1.0262	1.0616
		BCT	1.2337	1.1833	0.9929	1.0356
		GMD	1.0000	1.0000	1.0000	1.0000
	Standard Deviation	CA	0.2644	0.3955	0.1658	0.2691
		BCT	0.4849	0.5377	0.1333	0.1778
		GMD	0.0525	0.1204	0.0603	0.1172
1.5	Mean	CA	1.5856	1.7068	1.5418	1.5930
		BCT	1.9822	1.8171	1.5461	1.5546
		GMD	1.5000	1.5000	1.5000	1.5000
	Standard Deviation	CA	0.3907	1.5000	0.2517	0.4002
		BCT	0.9365	0.8660	0.1953	0.2396
		GMD	0.0525	0.1204	0.0603	0.1172

Table 10: Descriptive statistics for CA, BCT, and GMD methods when n = 75

n = 75						
Target Ppu	**Statistics**	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1	Mean	CA	1.1044	1.0783	1.0396	1.0223
		BCT	1.2484	1.3056	0.9957	1.0203
		GMD	1.0000	1.0762	1.0000	1.0000
	Standard Deviation	CA	0.3418	0.3200	0.2112	0.1540
		BCT	0.3859	0.5790	0.0945	0.1513
		GMD	0.0459	0.3183	0.0523	0.1025
1.5	Mean	CA	1.6550	1.6168	1.5589	1.5345
		BCT	2.0129	2.0084	1.5295	1.5439
		GMD	1.5000	1.5000	1.5000	1.5055
	Standard Deviation	CA	0.5053	0.4764	0.3071	0.2295
		BCT	0.7514	0.9211	0.1434	0.2088
		GMD	0.0459	0.0965	0.0523	0.1025

Table 11: Descriptive statistics for CA, BCT, and GMD methods when n = 100

n = 100						
Target Ppu	**Statistics**	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1	Mean	CA	1.0627	1.0552	1.0311	1.0287
		BCT	1.1825	1.2099	0.9890	1.0056
		GMD	1.0000	1.0000	1.0000	1.0000
	Standard Deviation	CA	0.2580	0.2305	0.1834	0.1856
		BCT	0.3095	0.3763	0.0844	0.0865
		GMD	0.0394	0.0707	0.0436	0.0859
1.5	Mean	CA	1.5936	1.5829	1.5476	1.5428

		BCT	1.8554	1.8531	1.5156	1.5081
		GMD	1.5000	1.5000	1.5000	1.5000
	Standard Deviation	CA	0.3850	0.3449	0.2776	0.2700
		BCT	0.5458	0.5991	0.1000	0.1215
		GMD	0.0394	0.0707	0.0436	0.0859

At Weibull (1, 1) and Weibull (1, 2) at sample size of 25, 50, 75 and 100, the Gini Mean Difference based process capability estimates approximately the the target Ppu of 1.0 and 1.5, the Clements method estimates is also close to the target Ppu while the Box-Cox transformation method is at deviance from the target (overestimated) the Ppu of 1.0 and1.5 as the sample size increases.

At Weibull (2,1) and Weibull (2,2) which indicate low symmetry and at sample size of 25, 50, 75 and 100, the three method estimates are all approximately target Ppu of 1.0 and 1.5 with the Gini Mean Difference based process capability estimates the best (closest).

e) The Root-Mean-Square Deviation (RMSD)

The root-mean-square deviation (RMSD) is used to measure the differences between the targeted Ppu values and the estimates obtained by Box-Cox Transformation, Clements and Gini mean difference based methods.

The tables below summaries the result obtained for each of the distribution parameter at different sample sizes

Table 12: The root-mean-square deviations for CA, BCT, and GMD methods when n = 25

n = 25					
Target Ppu	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1	CA	0.4369	0.5416	0.3576	0.4495
	BCT	0.4794	0.2541	0.2108	0.3606
	GMD	0.0750	0.1688	0.0824	0.1679
1.5	CA	0.6482	0.8122	0.5266	0.6641
	BCT	0.6653	0.4564	0.3125	0.5035
	GMD	0.0750	0.1688	0.0824	0.1679

Table 13: The root-mean-square deviations for CA, BCT, and GMD methods when n = 50

n = 50					
Target Ppu	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1	CA	0.3749	0.4153	0.1662	0.2734
	BCT	0.5339	0.5629	0.1321	0.1795
	GMD	0.0525	0.1192	0.0597	0.1160
1.5	CA	0.3961	0.6173	0.2526	0.4069
	BCT	1.0048	0.9141	0.1988	0.2434
	GMD	0.0525	0.1192	0.0597	0.1160

Table 14: The root-mean-square deviations for CA, BCT, and GMD methods when n = 75

n = 75					
Target Ppu	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1	CA	0.3541	0.3263	0.2128	0.1540
	BCT	0.4557	0.6496	0.0937	0.1512
	GMD	0.0455	0.0956	0.0518	0.1015
1.5	CA	0.5237	0.4859	0.3097	0.2298
	BCT	0.9036	1.0440	0.1450	0.2113
	GMD	0.0455	0.0956	0.0518	0.1016

Table15: The root-mean-square deviations for CA, BCT, and GMD methods when n = 100

n = 100					
Target Ppu	**Method**	**Weibull(1,1)**	**Weibull(1,2)**	**Weibull(2,1)**	**Weibull(2,2)**
1	CA	0.2630	0.2348	0.1841	0.1860
	BCT	0.3566	0.4276	0.0843	0.0858
	GMD	0.0390	0.0700	0.0432	0.0850
1.5	CA	0.3925	0.3514	0.2789	0.2707
	BCT	0.6467	0.6902	0.1002	0.1205
	GMD	0.0390	0.0700	0.0432	0.0850

Results from the root-mean-square deviation (RMSD) in Table 12 to Table 15 shows that the GMD methods have the lowest RMSDs across all the different distribution parameters and sample sizes

IV. Conclusion

In order to examine the impact of non-normal data, the parameter values of Weibull distribution were specified as (1, 1), (1, 2), (2, 1), and (2, 2) corresponding to (shape, scale) at different sample sizes of 25, 50, 75 and 100. These parameters of Weibull distributions are specified such that the effects of the tail behaviour on process capability could be examined. When the Weibull shape parameter is equal to 1, Weibull distribution reduces to Exponential distribution. Hence, this study covers all the Exponential family distributions as well.

Conclusively, from our results and findings, the Gini Mean difference based approach is the best among three methods in estimating process capability in skewed (non-normal) situations. In general, methods involving transformation seem more burdensome in terms of calculation, though it provide estimates of PCIs that truly reflect the capability of the process when there is low symmetry as in Weibull (2, 2).

References Références Referencias

1. Anis, M.Z. (2008). Basic Process Capability Indices: An Expository Review, *International Statistical Review*, 76(3), pp.347-367.
2. Bordignon, S and Scagliarini, M. (2002). Statistical Analysis of Process Capability Indices with Measurement Errors. *Quality and Reliability Engineering International*, 18(4), pp.321–332.

3. Chang, Y.S., Choi, I.S. and Bai, D.S (2002). Process capability indices for skewed populations, *Quality and Reliability Engineering International*, 18(5), pp. 383–393.
4. Chen, J. P. (2000). Re-evaluating the Process Capability Indices for Non-normal Distributions. *International Journal of Production Research*, 38 (6), pp. 1311–1324.
5. Chen, K.S., Huang, M.L and Li, R.K. (2001). Process Capability Analysis for an Entire Product, *International Journal of Production Research*, 39(17), pp. 4077–4087.
6. Ding, J. (2004). A Method of Estimating the Process Capability Index from the First Moments of Non-normal Data, *Quality and Reliability Engineering International*, 20(8) pp. 787–805.
7. Golafshani, N. (2003). Understanding Reliability and Validity in Qualitative Research, *The Qualitative Report*, 8(4) pp.597–607.
8. Hoerl, R.W and Snee, R. D (2010). Statistical Thinking and Methods in Quality Improvement: A Look to the Futu re, *Quality Engineering*, 22 (3), pp.119-129.
9. Hoerl, R.W and Snee, R. D. (2010). Flexible Process Capability Indices *The American Statistician,* 64(1), pp.10-14.
10. Kotz, S and Johnson, N.L (1993). *Process Capability Indices*. 2nd Ed. Chapman & Hall, Suffolk.
11. Kotz, S and Johnson, N. L. (2002). Process Capability Indices: A review, *Journal of Quality Technology*, 34(1), pp.2–53.
12. Lovelace, C. R and Swain, J. J (2009). Process Capability Analysis Methodologies for Zero-Bound and Non-normal Process Data, *Quality Engineering*, 21(2), pp.190–202
13. McCormack Jr, D.W, Harris, I.R, Hurwitz, A.M and Spagon, P.D. (2000). Capability Indices for Non-Normal Data, *Quality Engineering*, 12(4), pp.489–495.
14. Montgomery, D. C and Runger, G. C. (2009). Applied Statistics and Probability for Engineers.3Ed. John Wiley & Sons, New York.
15. Spiring, F., Leung, B., Cheng, S and Yeung, A (2003). A Bibliography of Process Capability Papers. *Quality and Reliability Engineering International*, 19(5), pp. 445-460.
16. Spiring, F.A. (1995). Process Capability: A Total Quality Management Tool. *Total Quality Management & Business Excellence*, 6(1) pp. 21–34.
17. Vännman, K and Albing, M. (2007). Process Capability Indices for One-Sided Specification Intervals and Skewed Distributions, *Quality and Reliability Engineering International*, 23(6) pp.755–765.
18. Wu, C. W, Pearn, W. L and Kotz, S (2009). An Overview of Theory and Practice on Process Capability Indices for Quality Assurance, *International Journal of Production Economics*, 117(2), pp.338–359.
19. Wu, H. H and Swain, J. J (2009). A Monte Carlo Comparison of Capability Indices when Processes are Non-normally Distributed. *Quality and Reliability Engineering International*, 17(3), pp. 219–231.
20. Yum, B.J.and Kim, K.W (2010). A Bibliography of the Literature on Process Capability Indices: 2000–2009, *Quality and Reliability Engineering International*, 27(3), pp.251-268.
21. Yitzhaki, S (2010). Gini's mean difference: A superior measure of variability for non-normal distributions, *Metron*, 61(2), pp. 285-316.

Notes

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Functional Calculus for the Series of Semigroup Generators via Transference

By Shawgy Hussein, Simon Joseph, Ahmed Sufyan, Murtada Amin, Ranya Tahire & Hala Taha

Upper Nile University

Abstract- In this paper, apply an established transference principle to obtain the boundedness of certain functional calculi for the sequence of semigroup generators. It is proved thatif $-A_j$ be the sequence generates C_0- semigroups on a Hilbert space, then for each $\varepsilon > -1$ the sequence of operators A_j has bounded calculus for the closed ideal of bounded holomorphic functions on right half–plane. The bounded of this calculus grows at most logarithmically as$(1+\varepsilon) \searrow 0$. As a consequence decay at ∞. Then showed that each sequence of semigroup generator has a so-called (strong) m-bounded calculus for all m ∈ ℕ, and that this property characterizes the sequence of semigroup generators. Similar results are obtained if the underlying Banach space is a UMD space. Upon restriction to so-called $\gamma_j - bounded$ semigroups, the Hilbert space results actually hold in general Banach spaces.

Keywords: functional calculus, transference, operator semigroup, fourier multiplier, γ-boundedness.

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2. C.Batty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.

Functional Calculus for the Series of Semigroup Generators via Transference

Shawgy Hussein [α], Simon Joseph [σ], Ahmed Sufyan [ρ], Murtada Amin [ω], Ranya Tahire [¥] & Hala Taha [§]

Abstract- In this paper, apply an established transference principle to obtain the boundedness of certain functional calculi for the sequence of semigroup generators. It is proved thatif $-A_j$ be the sequence generates C_0- semigroups on a Hilbert space, then for each $\varepsilon > -1$ the sequence of operators A_j has bounded calculus for the closed ideal of bounded holomorphic functions on right half–plane. The bounded of this calculus grows at most logarithmically as$(1+\varepsilon) \searrow 0$. As a consequence decay at ∞. Then showed that each sequence of semigroup generator has a so-called (strong) m-bounded calculus for all $m \in \mathbb{N}$, and that this property characterizes the sequence of semigroup generators. Similar results are obtained if the underlying Banach space is a UMD space. Upon restriction to so-called$\gamma_j - bounded$ semigroups, the Hilbert space results actually hold in general Banach spaces.

Keywords: *functional calculus, transference, operator semigroup, fourier multiplier, γ-boundedness.*

I. Introduction

Functional calculus for thesequence of operators A_j on a Banach space X is a "method" of associating a closed sequence of operators $f_j(A_j)$ to all $f_j = f_j(z_j)$ taken from a Set of functions in such a way that formulae valid for the functions turn into valid formulae for the operators upon replacing the independent variables Z_j by A_j. A common way to establish such a calculus is to start with an algebra of "good" functions f_j where definitions of $f_j(A_j)$ as bounded sequence of operators are more or less straightforward, and then extend this "primary" or "elementary calculus" by means of multiplicative in[1,Chapter 1] and [2]. It is then natural to ask which of the so constructed closed sequence of operators $f_j(A_j)$ are actually bounded, a question particularly relevant in applications, e.g., to evolution equations,see, [3,4].

The latter question links functional calculus theory to the theory of vector-valued singular integrals, best seen in the theory of sectorial operators with a bounded H^∞ - calculus, see, [5]. It appears there that in order to obtain nontrivial results the

Author α: Sudan University of Science and Technology, College of Science, Department of Mathematics, Sudan. e-mail: shawgy2020@gmail.com
Author σ: Upper Nile University, Faculty of Education, Department of Mathematics, South Sudan. e-mail: s.j.u.khafiir@gmail.com
Author ρ: Ministry of Education, Department of Mathematics, Sultanate Oman. e-mail: shibphdoman@gmail.com
Author ω: Ministry of Education and Higher Education, Department of Mathematics, Qatar. e-mail: mlgandi2@gmail.com
Author ¥: Jazan University, Faculty of Science, Department of Mathematics, Kingdom of Saudi Arabia. e-mail: rayya2014@hotmail.com
Author §: Princess Nourah bint Abdulrahman University, College of Science, Department of Mathematics, Kingdom of Saudi Arabia. e-mail: hala_taha2011@hotmail.com

underlying Banach space must allow for singular integrals to converge, i.e., be a UMD space. Furthermore, even if the Banach space is a Hilbert space, it turns out that simple resolvent estimates are not enough for the boundedness of an H^∞ -calculus.

However, some of the central positive results in that theory — show that the presence of a C_0-group of operators does warrant the boundedness of certain H^∞-calculi. In [6], the underlying structure of these results was brought to light, namely a transference principle, a factorization of the sequence of operators $f_j(A_j)$ in terms of vector-valued Fourier multiplier operators. Finally, in [7], it was shown that C_0 -semigroups also allow for such transference principles.

Markus Haase and Jan Rozendaal [8] developed this approach further. They apply the general form of the transference principle for semigroups given in [9] to obtain bounded functional calculi for the sequence of generators of C_0 -semigroups. These results, in theorems 3.3, 3.7, and 4.3,are proved for general Banach spaces. However, they make use of the analytic $L^{1+\varepsilon}(\mathbb{R};\ X)$ Fourier multiplier algebra, and hence are useful only if the underlying Banach space has a geometry that allows for nontrivial Fourier multiplier operators. In case X = H is a Hilbert space one, obtains particularly nice results, which want to summarize here.

Theorem 1.1: Let $-A_j$ be the sequence of generators of bounded C_0–semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ on a Hilbert space H with M:= $sup_{t\in\mathbb{R}_+}$. Then the following assertions hold.

(a) For $\omega_j<0$ and $f_j\in H^\infty(R_{\omega_j})$ one has $f_j\ (A_j)T^j(1+\varepsilon)\in\mathcal{L}(\mathrm{H})$ with

$$\left\|\sum_j f_j(A_j)T^j\ (1+\varepsilon)\right\| \le c(1+\varepsilon)\mathrm{M}^2\sum_{\mathrm{j}}\|f_j\|_{\mathrm{H}^\infty\left(\mathrm{R}_{\omega_j}\right)} \tag{1}$$

where $c(1+\varepsilon) = O(|\log(1+\varepsilon)|)$ as$(1+\varepsilon)\searrow 0$, and $c(1+\varepsilon) = O(1)$ as $(1+\varepsilon)\to\infty$.

(b) For $\omega_j<0<\beta+\varepsilon$ and $\lambda_j\in\mathbb{C}$ with Re $\lambda_j<0$ there is $\varepsilon\ge$ -1 such that

$$\left\|\sum_j f_j(A_j)\,(A_j-\lambda_j)^{-(\beta+\varepsilon)}\right\| \le (1+\varepsilon)M^2\sum_j\|f_j\|_{H^\infty\left(R_{\omega_j}\right)} \tag{2}$$

For all $f_j\in H^\infty(R_{\omega_j})$. In particular, $\mathrm{dom}(A_j^{\beta+\varepsilon})\subseteq\mathrm{dom}(f_j(A_j))$.

(c) A_j has strong m-bounded H^∞ -calculus of type 0 for each $m\in\mathbb{N}$.

When X is a UMD space, one can derive similar results, we extend the Hilbert space results to general Banach spaces by replacing the assumption of boundedness of the semigroup by its γ_j -boundedness, a concept strongly put forward by Kalton and Weis [9]. In particular, Theorem 1.1 holds true for γ_j -bounded semigroups on arbitrary Banach spaces with M being the γ_j -bound of the semigroups.

Stress the fact that in contrast to [1], where sectorial operators and, accordingly, functional calculi on sectors, were considered, deals with general sequence of semigroup generators and with functional calculi on half-planes. The abstract theory of (holomorphic) functional calculi on half-planes can be found in [2 corollaries 6.5and 7.1]

Ref

6. M.Haase: A transference principle for general groups and functional calculus on UMD spaces. Math.Ann.Ann.345(2)(2009) 245-265.

The starting point of the present work was the article [10] by Hans Zwart. There is shown that one has an estimate (1) with $c(1+\varepsilon) = O((1+\varepsilon)^{-1/2})$ as $(1+\varepsilon) \searrow 0$. (The case $\beta + \varepsilon > 1/2$) in (2) is an immediate consequence, however, that case is essentially trivial)

In [7] and its sequal paper [11], the functional calculus for a semigroup generator is constructed in a rather unconventional way using ideas from systems theory. However, a closer inspection reveals that transference is present there as well, hidden in the very construction of the functional calculus.

a) Notation and terminology

Write $\mathbb{N}:= \{1, 2, \ldots\}$ for the natural numbers and $\mathbb{R}_+ := [0, \infty)$ for the nonnegative reals.The letters X and Y are used to denote Banach spaces over the complex number field. The space of bounded linear operators on X is denoted by $\mathcal{L}(X)$. For a closed sequence of operatorsA_j on X their domainsare denoted by $\mathrm{dom}(A_j)$ and their ranges by $\mathrm{ran}(A_j)$. The spectrums of A_j are $\sigma(A_j)$ and the resolvent sets $\rho(A_j) := \mathbb{C} \text{ “ } \sigma(A_j)$. For all $z_j \in \rho(A_j)$ the operators$R(Z_j, A_j) := (z_j - A_j)^{-1} \in \mathcal{L}(X)$ is the resolvents of A_j at z_j.

For $\varepsilon > 1$, $L^{1+\varepsilon}(\mathbb{R}; X)$ is the Bochner space of equivalence classes of X - valued $(1+\varepsilon)$ –Lebesgue integrable functions on $\mathbb{R}$.The Hölder conjugate of $(1+\varepsilon)$ is $\left(\frac{1+\varepsilon}{\varepsilon}\right)$. The norm on $L^{1+\varepsilon}(\mathbb{R}, X)$is usually denoted by$\|\cdot\|_{1+\varepsilon}$.

For $\omega_j \in \mathbb{R}$ and $z_j \in \mathbb{C}$, let $e_{\omega_j}(z_j) := e^{\omega_j z_j}$. By $M(\mathbb{R})$ (resp. $M(\mathbb{R}_+)$), denote the space of complex-valued Borel measures on $\mathbb{R}$ (resp. $\mathbb{R}_+$) with the total variation norm, and write$M_{\omega_j}(\mathbb{R}_+)$ for the distributions μ^j on $\mathbb{R}_+$ of the form $\mu^j(\mathrm{ds}) = e^{\omega_j s}\nu^j(\mathrm{ds})$ for some $\nu^j \in M(\mathbb{R}_+)$. Then $M_{\omega_j}(\mathbb{R}_+)$ is a Banach algebra under convolution with the series of norms

$$\sum_j \|\mu^j\|_{M_{\omega_j}(\mathbb{R}_+)} = \sum_j \left\|e_{-\omega_j}\mu^j\right\|_{M(\mathbb{R}_+)}$$

For $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$, let $\mathrm{supp}(\mu^j)$ be the topological support of $e_{-\omega_j}\mu^j$, functions g^j such that $e_{-\omega_j} g^j \in L^1(\mathbb{R}_+)$ are usually identified with its associated measures $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$ given by $\mu^j(\mathrm{ds}) = g^j(s)\,\mathrm{ds}$. Functions and measures defined on $\mathbb{R}_+$ are identified with their extensions to $\mathbb{R}$ by setting them equal to zero outside $\mathbb{R}_+$.

For an open subset $\Omega \neq \emptyset$ of $\mathbb{C}$, let $H^\infty(\Omega)$ be the space of bounded holomorphic functions on Ω, until Banach algebra concerning to the series of norms

$$\sum_j \|f_j\|_\infty = \sum_j \|f_j\|_{H^\infty(\Omega)} = sup_{z_j \in \Omega} \sum_j |f_j(z_j)| \quad (f_j \in H^\infty(\Omega))$$

Consider the case where Ω is equal to a right half-planes

$$R_{\omega_j} = \{z_j \in \mathbb{C} | Re(z_j) > \omega_j\}$$

for some $\omega_j \in \mathbb{R}$ (we write $\mathbb{C}_+$ for R_0).

11. F.L.Schwenninger, H.Zwart: Weakly admissible $\mathcal{H}_\infty^-$-calculus on reflexive Banach spaces, Indag. Math. (N.S) 23 (4) (2012) 796-815.

For convenience abbreviate the coordinate functions $Z_j \mapsto z_j$ simply by the letters z_j. Under this convention, $f_j = f_j(z_j)$ for functions f_j defined on some domain $\Omega \subseteq C$.

The Fourier transform of an X-valued tempered distribution Φ on $\mathbb{R}$ is denoted by $\mathcal{F}\Phi$. If $\mu^j \in M(\mathbb{R})$ then $\mathcal{F}_{\mu^j} \in L^\infty(\mathbb{R})$ are given by

$$\sum_j \mathcal{F}_{\mu^j}(\xi) = \int_{\mathbb{R}} \sum_j e^{-i\xi s}\, \mu^j(ds) \qquad (\xi \in \mathbb{R})$$

For $\omega_j \in \mathbb{R}$ and $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$, let $\widehat{\mu^j} \in H^\infty(R_{\omega_j}) \cap C(\overline{R_{\omega_j}})$,

$$\sum_j \widehat{\mu^j}(z_j) = \int_0^\infty \sum_j e^{-z_j s}\, \mu^j(ds) \qquad (z_j \in R_{\omega_j})$$

Be the Laplace–Stieltjes transforms of μ^j.

II. Fourier Multipliers and Functional Calculus

Discuss some of the concepts that will be used in what follows (see,e.g., [8]).

a) Fourier multipliers

Fix a Banach space X and let $m \in L^\infty(\mathbb{R}; \mathcal{L}(X))$ and $\varepsilon \geq 0$. Then m is a bounded $L^{1+\varepsilon}(\mathbb{R}; X)$-Fourier multiplier if there exists $\varepsilon \geq -1$ such that

$$T_m^j(\varphi_j) = \mathcal{F}^{-1}(m.\mathcal{F}\varphi_j) \in L^{1+\varepsilon}(\mathbb{R}; X) \text{ and } \left\| \sum_j T_m^j(\varphi_j) \right\|_{1+\varepsilon} \leq (1+\varepsilon) \sum_j \|\varphi_j\|_{1+\varepsilon}$$

for each X-valued Schwartz functions φ_j. In this case, the mappings T_m^j extends uniquely to bounded sequence of operators on $L^{1+\varepsilon}(\mathbb{R}; X)$ if $\varepsilon < \infty$ and on $C_0(\mathbb{R}; X)$ if $\varepsilon = \infty$. Let $\|m\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$ be the norms of the operators T_m^j and let $\mathcal{M}_{1+\varepsilon}(X)$ be the unital Banach algebra of all bounded $L^{1+\varepsilon}(\mathbb{R}; X)$-Fourier multipliers, endowed with the norm $\|\cdot\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$.

For $\omega_j \in \mathbb{R}$ and $\varepsilon \geq 0$, we let

$$A_j M_{1+\varepsilon}^X(R_{\omega_j}) = \left\{ f_j \in H^\infty(R_{\omega_j}) \,|\, f_j(\omega_j + i\,\cdot) \in \mathcal{M}_{1+\varepsilon}(X) \right\} \tag{3}$$

be the analytic L^(1+ε)(R; X)-Fourier multiplier algebras on R·(ω·j), endowed the series of norms

$$\sum_j \|f_j\|_{A_j M_{1+\varepsilon}^X} = \sum_j \|f_j\|_{A_j M_{(1+\varepsilon)}^X(R_{\omega_j})} = \sum_j \|f_j(\omega_j + i\,\cdot)\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

Here $f_j(\omega_j + i\,\cdot) \in L^\infty(\mathbb{R})$ denotes the trace of the holomorphic functions f_j on the boundary $\partial R_{\omega_j} = \omega_j + i\mathbb{R}$. By classical Hardy space theories,

Ref
8. M.Haase, J.Rozendaal: Functional calculus for the of semigroup generators via transference.

$$f_j(\omega_j + is) = lim_{\acute{\omega}_j \searrow \omega_j} f_j(\acute{\omega}_j + is) \tag{4}$$

Exists for almost all s $\in \mathbb{R}$, with $\sum_j \left\| f_j(\omega_j + \mathrm{i}\cdot) \right\|_{L^\infty(\mathbb{R})} = \sum_j \left\| f_j \right\|_{H^\infty(R_{\omega_j})}$.

Remark 2.1: (Important!). To simplify notation sometimes omit the reference to the Banach space X and write $A_j M_1\ (R_{\omega_j})$ instead of $A_j M_1^X(R_{\omega_j})$, whenever it is convenient.

Notes

The spaces $A_j M_{1+\varepsilon}^X(R_{\omega_j})$ are until Banach algebra, constructively embedded in $H^\infty\left(R_{\omega_j}\right)$, and $A_j M_1^X\ (R_{\omega_j}) = A_j M_\infty^X\left(R_{\omega_j}\right)$ are contractively embedded in $A_j M_{1+\varepsilon}^X\left(R_{\omega_j}\right)$ for all $\varepsilon > 0$,

Need two lemmas about the analytic multiplier algebra.

Lemma 2.2: For every Banach space X, all $(0 \le \varepsilon \le \infty)$,

$$\sum_j A_j M_{1+\varepsilon}^X\left(R_{\omega_j}\right) = \left\{ f_j \in H^\infty\left(R_{\omega_j}\right) \mid sup_{\acute{\omega}_j > \omega_j} \sum_j \left\| f_j(\acute{\omega}_j + i\cdot) \right\|_{\mathcal{M}_{1+\varepsilon}(X)} < \infty \right\}$$

With

$$\sum_j \left\| f_j \right\|_{A_j M_{1+\varepsilon}^X\left(R_{\omega_j}\right)} = sup_{\acute{\omega}_j > \omega_j} \sum_j \left\| f_j(\acute{\omega}_j + i\cdot) \right\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

for all $f_j \in A_j M_{1+\varepsilon}^X(R_{\omega_j})$

Proof. Let $\omega_j \in \mathbb{R}$, $f_j \in A_j M_{1+\varepsilon}\ (R_{\omega_j})$. For all $\acute{\omega}_j > \omega_j$ and s $\in \mathbb{R}$,

$$\sum_j f_j(\acute{\omega}_j + is) = \sum_j \frac{\acute{\omega}_j - \omega_j}{\pi} \int_{\mathbb{R}} \frac{f_j(\omega_j - ir)}{(s-r)^2 + (\acute{\omega}_j - \omega_j)^2}\, dr$$

The right-hand side is the series of the convolutions of $f_j(\omega_j - i\cdot)$ and the Poisson kernel

$$P_{\acute{\omega}_j - \omega_j}(r) = \frac{\acute{\omega}_j - \omega_j}{\pi(r^2 + (\acute{\omega}_j - \omega_j)^2)}$$

Since $\sum_j \left\| P_{(\acute{\omega}_j - \omega_j)} \right\|_{L^1(\mathbb{R})} = 1$,

$$\left\| \sum_j f_j(\acute{\omega}_j + i\cdot) \right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} \le \sum_j \left\| f_j(\omega_j - i\cdot) \right\|_{\mathcal{M}_{1+\varepsilon}(X)} = \sum_j \left\| f_j \right\|_{A_j M_{(1+\varepsilon)}^X(R_{\omega_j})}$$

The converse follows from (4) ■

For $\mu^j \in$M($\mathbb{R}$) and $\varepsilon \ge 0$, let $L_{\mu^j} \in \mathcal{L}(L^{1+\varepsilon}\ (\mathbb{R};\ \mathrm{X}))$,

$$L_{\mu^j}(f_j) := \mu^j * f_j, \quad \left(f_j \in L^{1+\varepsilon}(\mathbb{R};\ \mathrm{X}) \right), \tag{5}$$

be the convolution sequence of operators associated with μ^j.

Lemma 2.3: For each $\omega_j \in \mathbb{R}$ the Laplace transform induces an isometric algebra isomorphism from $M_{\omega_j}(\mathbb{R}_+)$ onto $A_j M_1^{\mathbb{C}}(R_{\omega_j}) = A_j M_1^X(R_{\omega_j})$. Moreover,

$$\sum_j \|\widehat{\mu^j}\|_{A_j M_{1+\varepsilon}^X(R_{\omega_j})} = \sum_j \left\| L_{e_{-\omega_j}\mu^j} \right\|_{\mathcal{L}(L^{(1+\varepsilon)}(X))}$$

for all $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$, $\varepsilon \geq 0$

Proof: The mappings $\mu^j \mapsto e_{-\omega_j}\mu^j$ and $f_j \mapsto f_j(\cdot + \omega_j)$ are isometric algebra isomorphisms $M_{\omega_j}(\mathbb{R}_+) \to M(\mathbb{R}_+)$ and $A_j M_{1+\varepsilon}(R_{\omega_j}) \to A_j M_{1+\varepsilon}(\mathbb{C}_+)$, respectively. Hence it suffices to let$\omega_j = 0$.The Fourier transform induces an isometric isomorphism from M($\mathbb{R}$) onto $\mathcal{M}_1$(X). If $\mu^j \in$M($\mathbb{R}_+$) and $f_j = \widehat{\mu^j} \in H^\infty(\mathbb{C}_+)$ then $f_j(i\cdot) = \mathcal{F}\mu^j \in \mathcal{M}_1$(X) with $\sum_j \|f_j(i\cdot)\|_{\mathcal{M}_1(X)} = \sum_j \|\mu^j\|_{M(\mathbb{R}_+)}$ Moreover, for $\varepsilon \geq 0$,

$$\sum_j \|f_j(i\cdot)\|_{\mathcal{M}_{1+\varepsilon}(X)} = \sum_j sup_{\|g^j\|_{1+\varepsilon}\leq 1} \|\mathcal{F}^{-1}(f_j(i\cdot)\mathcal{F}\, g^j)\|_{1+\varepsilon} = sup_{\|g^j\|_{1+\varepsilon}\leq 1} \sum_j \|\mu^j * g^j\|_{1+\varepsilon} = \sum_j \|L_{\mu^j}\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

If $f_j \in A_j M_1(\mathbb{C}_+)$ then $f_j(i\cdot) = \mathcal{F}\mu^j$ for some $\mu^j \in$M($\mathbb{R}$). An application of Liouville's theorem shows that supp$(\mu^j) \subseteq \mathbb{R}_+$, hence $f_j = \widehat{\mu^j}$. ■

b) Functional Calculus

Assume that we are familiar with the basic notions and results of the theory of C_0-semigroups as developed, e.g., in [5]

All C_0–semigroups $T^j = (T^j(t))_{t\in\mathbb{R}_+}$ on a Banach space X has the type (M, ω_j) for some M $\geq$ 1 and $\omega_j \in \mathbb{R}$, which means that $\|\sum_j T^j(t)\| \leq M\sum_j e^{\omega_j t}$ for all t $\geq$ 0. The generators of T^j are the uniqueclosed sequence of operators $-A_j$ such that

$$\sum_j (\lambda_j + A_j)^{-1}x = \int_0^\infty \sum_j e^{-\lambda_j t}\, T^j(t)x\,dt \quad (x \in X)$$

for Re(λ_j) large. The Hille–Phillips (functional) calculus for A_j are defined as follows. Fix M $\geq$ 0 and $(\omega_j)_0 \in \mathbb{R}$ such that T^j has types$\left(M, -(\omega_j)_0\right)$. For $\mu^j \in M_{(\omega_j)_0}(\mathbb{R}_+)$ defines $T^j_{\mu^j} \in \mathcal{L}$(X) by

$$\sum_j T^j_{\mu^j}x = \int_0^\infty \sum_j T^j(t)x\mu^j(dt), \quad (x \in X) \tag{6}$$

For $f_j = \widehat{\mu^j} \in A_j M_j\left(R_{(\omega_j)_0}\right)$ sets $f_j(A_j) := T^j_{\mu^j}$. The mappings $f_j \mapsto f_j(A_j)$ is an algebra homomorphism. In a second stepthe definitions of $f_j(A_j)$ is extended to a larger class of functions via regularization, i.e.,

$$f_j(A_j) := e(A_j)^{-1}(ef_j)(A_j)$$

Ref
5. P. Kunstmann, L. Weis: Maximal L_p–regularity for parabolic equations, Fourier multiplier theorems and H^∞ – functional calculus, vol.1855, Springer BBerlin, 2004, pp.65-312.

If there exists $e \in A_j M_1\left(R_{(\omega_j)_0}\right)$ such that $e(A_j)$ is injective and $ef_j \in A_j M_1\left(R_{(\omega_j)_0}\right)$. Then $f_j(A_j)$ is closed and unbounded operator on X and the definition of $f_j(A_j)$ are independents of the choice of regularize. The following lemma shows in particular that for $\omega_j < (\omega_j)_0$ the sequence of operators $f_j(A_j)$ are defined for all $f_j \in H^\infty(R_{\omega_j})$ by virtue of the regularizes $e(z_j) = (Z_j - \lambda_j)^{-1}$,
where $\mathrm{Re}(\lambda_j) < \omega_j$.

Lemma 2.4: Let $\beta + \varepsilon > \frac{1}{2}$, $\lambda_j \in \mathbb{C}$ *and* ω_j , $(\omega_j)_0 \in \mathbb{R}$, $\varepsilon \geq 0$. Then

$$f_j\ (z_j)(z_j\ -\ \lambda_j)^{-(\beta+\varepsilon)} \in A_j \mathrm{M}_1(R_{\omega_j})_0 \text{ . for all } f_j \in H^\infty R_{(\omega_j)}$$

Proof: After shifting suppose that $\omega_j = 0$. Sets $h_j(z_j) := f_j\ (z_j)(z_j\ -\ \lambda_j)^{-(\beta+\varepsilon)}$ for $z_j \in \mathbb{C}_+$. Then $h_j(i\,\cdot) \in L^2(\mathbb{R})$ with

$$\left\| \sum_j h_j\ (i\,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq \int_{\mathbb{R}} \sum_j \frac{|f_j(is)|^2}{|is - \lambda_j|^{2(\beta+\varepsilon)}} ds \leq \int_{\mathbb{R}} \sum_j \frac{\|f_j\|^2_{M^\infty(\mathbb{C}_+)} ds}{|is - \lambda_j|^{2(\beta+\varepsilon)}}$$

Hence $h_j = \widehat{\mathrm{g}^j}$ for some $\mathrm{g}^j \in L^2(\mathbb{R}_+)$. Then $e_{-(\omega_j)_0}\mathrm{g}^j \in L^1(\mathbb{R}_+)$ and $\widehat{e_{-\omega_0}\mathrm{g}^j}\ (z_j) = h_j\ (z_j\ +\ (\omega_j)_0\ )$ for $z_j \in \mathbb{C}_+$. Lemma 2.3 yields $h_j \in A_j \mathrm{M}_1 R_{(\omega_j)_0}$ with

$$\sum_j \|h_j\|_{A_j M_1 R_{(\omega_j)_0}} = \sum_j \|h_j(\cdot + (\omega_j)_0)\|_{A_j M_1(\mathbb{C}_+)} = \sum_j \left\| e_{-(\omega_j)_0}\mathrm{g}^j \right\|_{L^1(\mathbb{R}_+)} \blacksquare$$

The Hille–Phillips calculus is an extension of the holomorphic functional calculus for the sequence of operators of half-plane type discussed in [2]. The sequence operators of A_j are of the half-plane types $(\omega_j)_0 \in \mathbb{R}$ if $\sigma(A_j) \subseteq \overline{R_{(\omega_j)_0}}$ with

$$sup_{\lambda_j \in \mathbb{C} \setminus R_{(\omega_j)}} \sum_j \|R(\lambda_j, A_j)\| < \infty\,,$$

for all $\varepsilon > 0$

One can associate the sequence of operators $f_j(A_j) \in \mathcal{L}(\mathrm{X})$ to certain elementary functions via Cauchy integrals and regularize as above to extend the definitions to all $f_j \in H^\infty\ (R_{\omega j})$. If $-A_j$ generates C_0 -semigroups of types $(\mathrm{M}, -(\omega_j)_0\ )$ then A_j are of half-plane types $(\omega_j)_0$, for $\omega_j < (\omega_j)_0$, $\varepsilon > 0$ and $f_j \in H^\infty(R_{\omega_j})$ the definitions of $f_j(A_j)$ via the Hille–Phillips calculus and the half-plane calculus coincide.

Lemma 2.5: (Convergence Lemma). Let A_j be densely defined sequence of operators of half-plane types $(\omega_j)_0 \in \mathbb{R}$ on a Banach space X. Let $\omega_j < (\omega_j)_0$ and $(f_j)_{j \in \mathrm{J}} \subseteq H^\infty(R_{\omega_j})$ be satisfying the following conditions:
(1) $\sup\{|(f_j)_j(z_j)|\ |\ z_j \in R_{\omega_j}, \mathrm{j} \in \mathrm{J}\ \} < \infty$;

Ref

2. C.Batty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.

(2) $(f_j)_j(A_j) \in \mathcal{L}(X)$ for all j ∈ J and $sup_{j\in J}\|(f_j)_j(A_j)\| < \infty$;
(3) $f_j(z_j) := Lim_{j\in J} f_j(z_j)$ exists for all $z_j \in R_{\omega_j}$.
Then $f_j \in H^\infty(R_{\omega_j})$, $f_j(A_j) \in \mathcal{L}(X)$, $(f_j)_j(A_j) \to f_j(A_j)$ strongly and

$$\left\|\sum_j f_j(A_j)\right\| \leq \lim\sup_{j\in J} \sum_j \|(f_j)_j(A_j)\|$$

Let A_j be the sequence of operators of half-plane types $(\omega_j)_0$ and $\omega_j < (\omega_j)_0$. For a Banach algebra F of functions continuously embedded in $H^\infty(R_{\omega_j})$, say that A_j has bounded F -calculus if there exists a constant $\varepsilon \geq -1$ such that $f_j(A_j) \in \mathcal{L}(X)$ with

$$\left\|\sum_j f_j(A_j)\right\|_{\mathcal{L}(X)} \leq (1+\varepsilon)\sum_j \|f_j\|_F \text{ for all } f_j \in F \tag{7}$$

The sequence of operators $-A_j$ generates a C_0-semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of types (M, ω_j) if and only if - $(A_j + \omega_j)$ generates the semigroups sequence of $(e^{-\omega_j t}T^j(t))_{t\in\mathbb{R}_+}$ of types (M, 0). The functional calculi for A_j and $A_j + \omega_j$ are linked by the simple composition rules "$f_j(A_j + \omega_j) = f_j(\omega_j + z_j)(A_j)$". Henceforth we shall mainly consider bounded semigroups; all results carry over to general semigroups by shifting.

III. Functinal Calculus for Semigroup Generators

Define the function $\eta : (0, \infty) \times (0, \infty) \times [1, \infty] \to \mathbb{R}_+$ by

$$\eta(\beta+\varepsilon, t, 1+\varepsilon) = inf\left\{\|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} | \psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)} \text{ on } [t, \infty)\right\} \tag{8}$$

The set on the right-hand side is not empty: choose for instance $\psi_j := 1_{[0,t]}e_{-(\beta+\varepsilon)}$ and $\varphi_j = \frac{1}{t}e_{-(\beta+\varepsilon)}$. By Lemma A.1,

$$\eta(\beta+\varepsilon, t, 1+\varepsilon) = O(|\log((\beta+\varepsilon)t)|) \text{as } (\beta+\varepsilon)t \to 0, \text{for } \varepsilon > 0.$$

For the following result recall the definitions of the operators L_{μ^j} from (5) and $T^j_{\mu^j}$ from (6).

Proposition 3.1: Let $(T^j(t))_{t\in\mathbb{R}_+}$ be C_0-semigroup of type (M, 0) on a Banach space X. Let $\varepsilon \geq 0$, $1+\varepsilon$, $\omega_j > 0$ and $\mu^j \in M_{-\omega_j}(\mathbb{R}_+)$ with $supp(\mu^j) \subseteq [1+\varepsilon, \infty)$. Then

$$\left\|\sum_j T^j_{\mu^j}\right\|_{\mathcal{L}(X)} \leq M^2\eta\sum_j(\omega_j, 1+\varepsilon, 1+\varepsilon)\left\|L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}\left(L^{1+\varepsilon}(X)\right)} \tag{9}$$

Proof: Factorizes $T^j_{\mu^j}$ as $T^j_{\mu^j} = P \circ L_{e_{\omega_j}\mu^j} \circ \imath$, where

a) $\imath : X \to L^{1+\varepsilon}(\mathbb{R}; X)$ is given by

$$\iota(x)(s) = \begin{cases} \psi_j(-s)T^j(-s)x & if\ \ s \le 0\,, \\ 0 & if\ s > 0\,, \end{cases} \qquad (x \in X)$$

b) $P : L^{1+\varepsilon}(\mathbb{R}; X) \to X$ is given by

$$\sum_j P(f_j) = \int \sum_j \varphi_j(t)T^j(t)f_j(t)\,dt \quad (f_j \in L^{1+\varepsilon}(\mathbb{R}, X))$$

c) $\psi_j \in L^{1+\varepsilon}(\mathbb{R}_+)$ and $\varphi_j \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}_+)$ are such that $\psi_j * \varphi_j \equiv e_{-\omega_j}$ on $[1+\varepsilon, \infty)$. This is deduced that $\mu^j = (\psi_j * \varphi_j)e_{\omega_j}\,\mu^j$. Hölder's inequality then implies

$$\left\| \sum_j T^j_{\mu^j} \right\| \le M^2 \sum_j \|\psi_j\|_{1+\varepsilon} \left\| L_{e_{\omega_j}\mu^j} \right\|_{\mathcal{L}(L^{1+\varepsilon}(X))} \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

and taking the infimum over all such ψ_j and φ_j yields (9). ■

Define, for a Banach space X, $\omega_j \in \mathbb{R}$, and $\varepsilon > -1$, the spaces

$$A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}) = \left\{ f_j \in A_j M^X_{(1+\varepsilon)}(R_{\omega_j}) \,|\, f_j(z_j) = O\left(e^{-(1+\varepsilon)Re(z_j)}\right) \text{ as } |z_j| \to \infty \right\}$$

end owed with the norms of $A_j M^X_{1+\varepsilon}(R_{\omega_j})$.

Lemma 3.2: For every Banach space X, $\omega_j \in \mathbb{R}$, $1 \le \varepsilon \le \infty$, and , ε ¿ -1

$$A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j}) = A_j M^X_{(1+\varepsilon)}(R_{\omega_j}) \cap e_{-(1+\varepsilon)}H^\infty(R_{\omega_j}) = e_{-(1+\varepsilon)}A_j M^X_{(1+\varepsilon)}(R_{\omega_j}) \qquad (10)$$

In particular, $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ are closed ideal in $A_j M^X_{(1+\varepsilon)}(R_{\omega_j})$.

Proof: The first equality in (10) is clear, and so are the inclusions $e_{-(1+\varepsilon)}A_j M_{(1+\varepsilon)}(R_{\omega_j}) \subseteq A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$. Conversely, if $f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j}) \cap e_{-(1+\varepsilon)}H^\infty(R_{\omega_j})$ then $e_{(1+\varepsilon)}f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j})$, since

$$\sum_j \left\| e^{(1+\varepsilon)(\omega_j + i\cdot)} f_j(\omega_j + i\,\cdot) \right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} = \sum_j e^{(1+\varepsilon)\omega_j} \left\| f_j\,(\omega_j + \mathrm{i}\,\cdot) \right\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

Suppose that $((f_j)_n)_{n\in\mathbb{N}} \subseteq A_j M_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ converges to $f_j \in A_j M_{(1+\varepsilon)}(R_{\omega_j})$. The Maximum Principle implies

$$\sum_j \left\| e_{(1+\varepsilon)}(f_j)_n \right\|_{H^\infty(R_{\omega_j})} = \sum_j e^{(1+\varepsilon)\omega_j} \left\| (f_j)_n \right\|_{H^\infty(R_{\omega_j})},$$

hence $(e_{(1+\varepsilon)}(f_j)_n)_{n\in\mathbb{N}}$ is Cauchy in $H^\infty(R_{\omega_j})$. Since it converges pointwise to $e_{(1+\varepsilon)}f_j$, (10) implies $f_j \in A_j M_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$. ■

To prove the main result [8] of this section. Note that the union of the ideals $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ for ε> -1 is densest in $A_j M^X_{(1+\varepsilon)}(R_{\omega_j})$ with respect to pointwise and bounded convergence of sequences. If there was a single constant independent of $\varepsilon > -1$

bounding the $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ - calculus for all, the Convergence Lemma would imply that A_j has bounded $A_j M^X_{(1+\varepsilon)}(R_{\omega_j})$ -calculus, but this is known to be false in general [1, Corollary 9.1.8].

Theorem 3.3: For each $0 < \varepsilon < \infty$, there exists a constant $c_{1+\varepsilon} \geq 0$ such that the following holds. Let $-A_j$ the sequence of generates C_0 -semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of type (M, 0) on a Banach space X and let$(1+\varepsilon)$, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}(X)$ and

$$\left\|\sum_j f_j(A_j)\right\| \leq \begin{cases} c_{(1+\varepsilon)} M^2 \sum_j \left|\log(\omega_j(1+\varepsilon))\right| \|f_j\|_{A_j M^X_{(1+\varepsilon)}} & \text{if } \omega_j(1+\varepsilon) \leq \min\left(\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right) \\ 2M^2 \sum_j e^{-\omega_j(1+\varepsilon)} \|f_j\|_{A_j M^X_{(1+\varepsilon)}}, & if\ \omega_j(1+\varepsilon) > min\left(\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right) \end{cases}$$

for all $f_j \in A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{-\omega_j})$. In particular, A_j has bounded $A_j M^X_{(1+\varepsilon),(1+\varepsilon)}(R_{-\omega_j})$ -calculus.

Proof: First consider $f_j \in A_j M_{1,(1+\varepsilon)}(R_{-\omega_j})$. Let $\delta_{(1+\varepsilon)} \in M_{-\omega_j}(\mathbb{R}_+)$ be the unit point mass at $\varepsilon > -1$. By Lemmas 3.2 and 2.3 there exists $\mu^j \in M_{-\omega_j}(\mathbb{R}_+)$ such that $f_j = e_{-(1+\varepsilon)}\widehat{\mu^j} = \widehat{\delta_{(1+\varepsilon)} * \mu^j}$. Since $\delta_{(1+\varepsilon)} * \mu^j \in M_{-\omega_j}(\mathbb{R}_+)$ with supp $(\delta_{(1+\varepsilon)} * \mu^j) \subseteq [1+\varepsilon, \infty)$, Proposition 3.1 and Lemma 2.3 yield

$$\left\|\sum_j f_j(A_j)\right\| \leq M^2 \eta \sum_j \left(\omega_j, (1+\varepsilon), (1+\varepsilon)\right) \|f_j\|_{A_j M^X_{1+\varepsilon}} \tag{11}$$

Suppose $f_j \in A_j M_{(1+\varepsilon),(1+\varepsilon)}(R_{\omega_j})$ are arbitrary. For ε ¿ 0, k $\in \mathbb{N}$ and $z_j \in R_{-\omega_j}$

Set $s\, g^j_{k,}(z_j) := \frac{k}{z_j - \omega_j + k}$ and $(f_j)_{k,\epsilon}(z_j) = f_j(z_j + \epsilon) g^j_k(z_j + \epsilon)$. Lemma 2.4 yields $(f_j)_{k,\epsilon} \in A_j M_{1,(1+\varepsilon)}(R_{-\omega_j})$, hence, by what have shown,

$$\left\|\sum_j (f_j)_{k,\epsilon}(A_j)\right\| \leq M^2 \eta \sum_j (\omega_j, 1+\varepsilon, 1+\varepsilon) \|(f_j)_{k,\epsilon}\|_{A_j M^X_{1+\varepsilon}}$$

The inclusions $A_j M_1(R_{-\omega_j}) \subseteq A_j M_{1+\varepsilon}(R_{-\omega_j})$ are contractive, so Lemma 2.3 implies that $g^j_k \in A_j M_{1+\varepsilon}(R_{-\omega_j})$ with

$$\left\|\sum_j g^j_k\right\|_{A_j M^X_{(1+\varepsilon)}} \leq \sum_j \|g^j_k\|_{A_j M_1} = k\|e_{-k}\|_{L^1(\mathbb{R}_+)} = 1$$

Combining this with Lemma 2.2 yields

$$\left\|\sum_j (f_j)_{k,\epsilon}\right\|_{A_j M^X_{(1+\varepsilon)}} \leq \sum_j \|f_j(\cdot+\epsilon)\|_{A_j M^X_{(1+\varepsilon)}} \|g^j_k(\cdot+\epsilon)\|_{A_j M^X_{(1+\varepsilon)}} \leq \sum_j \|f_j\|_{A_j M^X_{(1+\varepsilon)}}$$

In particular, $sup_{k,\epsilon}\left\|\sum_j (f_j)_{k,\epsilon}\right\|_\infty < \infty$ and $sup_{k,\epsilon}\left\|\sum_j (f_j)_{k,\epsilon}(A_j)\right\| < \infty$. The Convergence Lemma 2.5 implies that $f_j(A_j) \in \mathcal{L}(X)$ satisfies (11). Lemma A.1 concludes the proof. ■

Remark 3.4: Because $A_j M_1(R_{-\omega_j}) = A_j M_\infty(R_{-\omega_j})$ are contructively embedded in $A_j M_{(1+\varepsilon)}(R_{-\omega_j})$ Theorem 3.3 also holds for $\varepsilon \geq 0$. However, A_j trivially has abounded $A_j M_1$-calculus bylimma 2.3 and the Hille-Phillips calculus.

Note that the exponential decays of $\sum_j |f_j(z_j)|$ are only required as the real parts of z_j tends to infinity. If $\sum_j |f_j(z_j)|$ decays exponentially as $|z_j| \to \infty$ the result is not interesting by lemma 2.4.

Equivalently formulate Theorem 3.3 as a statement about composition with sequence semigroupoperators.

Corollary 3.5: Under the assumptions of Theorem 3.3, $f_j(A_j)T^j(1+\varepsilon) \in \mathcal{L}(X)$ and

$$\left\|\sum_j f_j(A_j)T^j(1+\varepsilon)\right\| \leq \begin{cases} c_{1+\varepsilon}M^2 \sum_j |\log(\omega_j(1+\varepsilon))| e^{\omega_j(1+\varepsilon)} \|f_j\|_{A_j M^X_{1+\varepsilon}}, & if\ \omega_j(1+\varepsilon) \leq \min\left(\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right) \\ 2M^2 \sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}}, & if\ \omega_j(1+\varepsilon) > \min\left(\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right) \end{cases}$$

For all $f_j \in A_j M^X_{1+\varepsilon}(R_{-\omega_j})$.

Proof. Note that $\sum_j f_j(A_j)T^j(1+\varepsilon) = \sum_j (e_{-(1+\varepsilon)}f_j)(A_j)$ and $\sum_j \|e_{-(1+\varepsilon)}f_j\|_{A_j M^X_{\varepsilon+1}}$

$$= \sum_j e^{\omega_j(1+\varepsilon)} \|f_j\|_{A_j M^X_{1+\varepsilon}} \quad ■$$

a) Additional results

As the first corollary of Theorem 3.3 we obtain a sufficient condition for a semigroupgenerator to have a bounded $A_j M_{1+\varepsilon}$- calculus (see,e.g.,[8]).

Corollary 3.6: Let $-A_j$ be the sequence of generates bounded C_0 -semigroups $(T^j(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ with

$$\bigcup_{\varepsilon > -1} \sum_j ran(T^j(1+\varepsilon)) = X$$

Then A_j has bounded $A_j M^X_{1+\varepsilon}(R_{\omega_j})$-calculus for all $\omega_j ¡ 0$, $\varepsilon \geq 0$.

Proof: Using Corollary 3.5 note that $f_j(A_j)T^j(1+\varepsilon) \in \mathcal{L}(X)$ implies ran $(T^j(1+\varepsilon)) \subseteq$ dom$(f_j(A_j))$. An application of the Closed Graph Theorem (using the Convergence Lemma) yields (7). ■

Theorem 3.7: Let $0 < \varepsilon < \infty$, , $\omega_j > 0$ and $\beta + \varepsilon$, $\lambda_j \in \mathbb{C}$ with $\mathrm{Re}(\lambda_j) < 0 < \mathrm{Re}(\beta + \varepsilon)$. There exists a constant $C = C(1+\varepsilon, \beta + \varepsilon, \lambda_j, \omega_j) \geq 0$ such that the following holds. Let $-A_j$ be the sequence of generates C_0- semigroups$(T^j(t))_{t\in\mathbb{R}_+}$of type (M,0) on a Banach space X. Then dom $\left((A_j - \lambda_j)^{(\beta+\varepsilon)}\right) \subseteq$dom $(f_j(A_j))$ and

$$\left\| \sum_j f_j(A_j)\,(A_j - \lambda_j)^{-(\beta+\varepsilon)} \right\| \leq (1+\varepsilon) M^2 \sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}}$$

for all $f_j \in A_j M^X_{1+\varepsilon}(R_{-\omega_j})$.

Proof: First note that $-(A_j - \lambda_j)$ generates the exponentially stable semigroups $(e^{\lambda_j t}(T^j(t))_{t\in\mathbb{R}_+}$. Hence to write

$$\sum_j (A_j - \lambda_j)^{-(\beta+\varepsilon)} x = \frac{1}{\Gamma(\beta+\varepsilon)} \int_0^\infty t^{(\beta+\varepsilon)-1} \sum_j e^{\lambda_j t} T^j(t)\, x\, dt \qquad (x \in X)$$

Fix $f_j \in A_j M_{1+\varepsilon}(R_{-\omega_j})$ and set a:$= \frac{1}{\omega_j} \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\}$. By Corollary 3.5,

$$\int_0^\infty t^{Re(\beta+\varepsilon)-1} e^{Re(\lambda_j)t} \left\| \sum_j f_j(A_j) T^j(t)(x) \right\| dt \leq (1+\varepsilon) M^2 \sum_j \|f_j\|_{A_j M^X_{1+\varepsilon}} \|x\| < \infty$$

for all x $\in$X, where

$$C = c_{1+\varepsilon} \int_0^a t^{Re(\beta+\varepsilon)-1} \sum_j \left|\log(\omega_j t)\right| e^{\left(Re(\lambda_j)+\omega_j\right)t} dt + 2\int_a^\infty t^{Re(\beta+\varepsilon)-1} \sum_j e^{\left(Re(\lambda_j)\right)t} dt$$

are independents of f_j, M, and x. Since $f_j(A_j)$ are closed operators, this implies that$(A_j - \lambda_j)^{-(\beta+\varepsilon)}$maps into dom $f_j(A_j)$ with

$$\sum_j f_j(A_j)\,(A_j - \lambda_j)^{-(\beta+\varepsilon)} = \frac{1}{\Gamma(\beta+\varepsilon)} \int_0^\infty t^{(\beta+\varepsilon)-1} \sum_j e^{\lambda_j t} f_j(A_j) T^j(t) dt$$

as a strong integral. ■

Remark 3.8: Theorem 3.7 shows that for all analytic multiplier functions f_j the domainsdom $(f_j(A_j))$ are relatively large, it contains the real interpolation spaces $(X, \mathrm{dom}(A_j))_{(\theta,1+\varepsilon)}$ and the complex interpolation spaces $[X, \mathrm{dom}(A_j)]_\theta$ for all $\theta \in (0, 1)$ and $\varepsilon \geq 0$.

Remark 3.9: Describe the ranges of $f_j(A_j)(A_j - \lambda_j)^{-(\beta+\varepsilon)}$ in Theorem 3.7. More explicitly. In fact

$$\mathrm{ran}\left(f_j(A_j)(A_j - \lambda_j)^{-(\beta+\varepsilon)}\right) \subsetneq dom(A_j - \lambda_j)^\beta$$

for all Re$(\beta) <$ $Re(\beta + \varepsilon)$. Indeed, this follows if show that

$\text{ran}\left((A_j-\lambda_j)^{-(\beta+\varepsilon)}\right) \subsetneq dom\left((A_j-\lambda_j)^{\beta} f_j(A_j)\right)$ implies

$$dom\left((A_j-\lambda_j)^{\beta} f_j(A_j)\right) = dom(f_j(A_j)) \cap dom\left([(z_j-\lambda_j)^{\beta} f_j(z_j)](A_j)\right)$$

The inclusion $\text{ran}\left((A_j-\lambda_j)^{-(\beta+\varepsilon)}\right) \subsetneq dom(f_j(A_j))$ follows from Theorem 3.7. Since

$$\left[(z_j-\lambda_j)^{\beta} f_j(z_j)\right](A_j)(A_j-\lambda_j)^{-(\beta+\varepsilon)} = [(z_j-\lambda_j)^{-\varepsilon} f_j(z_j)](A_j) = f_j(A_j)(A_j-\lambda_j)^{-\varepsilon}$$

The same holds for the inclusion $\text{ran}\left((A_j-\lambda_j)^{-(\beta+\varepsilon)}\right) \subseteq dom\left([(z_j-\lambda_j)^{\beta} f_j(z_j)](A_j)\right)$

b) Semigroups on Hilbert spaces

If X = H is a Hilbert space, Plancherel's Theorem implies $A_j M_2^H = H^\infty$ with equality of norms. Hence the theory above specializes to the following result, implying (a) and (b) of Theorem (1.1),

Corollary 3.10: Let $-A_j$ be the sequence ofgenerates bounded C_0-semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of type (M, 0) on aHilbert space H. Then the following assertions hold.

(a) There exists a universal constant $c \geq 0$ such that the following holds. Let $1+\varepsilon$, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}(H)$ and

$$\left\|\sum_j f_j(A_j)\right\| \leq \begin{cases} cM^2 \sum_j \left|\log(\omega_j(1+\varepsilon))\right| \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) \leq \frac{1}{2} \\ 2M^2 \sum_j e^{-\omega_j(1+\varepsilon)} \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) > \frac{1}{2} \end{cases}$$

for all $f_j \in e_{-(1+\varepsilon)} H^\infty(R_{-\omega_j})$. Moreover, $f_j(A_j)T^j(1+\varepsilon) \in \mathcal{L}(H)$ with

$$\left\|\sum_j f_j(A_j)\, T^j(1+\varepsilon)\right\| \leq \begin{cases} cM^2 \sum_j \left|\log(\omega_j(1+\varepsilon))\right| e^{\omega_j(1+\varepsilon)} \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) \leq \frac{1}{2} \\ 2M^2 \sum_j \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) > \frac{1}{2} \end{cases}$$

for all $f_j \in H^\infty(R_{-\omega_j})$.

(b) If

$$\bigcup_{\varepsilon>-1} \sum_j ran\left(T^j(1+\varepsilon)\right) = H$$

then A_j has bounded $H^\infty(R_{\omega_j})$-calculus for all $\omega_j < 0$.

(c) For $\omega_j < 0$ and $\beta+\varepsilon$, $\lambda_j \in \mathbb{C}$ with $\text{Re}(\lambda_j) < 0 < \text{Re}(\beta+\varepsilon)$ there is $C = C(\beta+\varepsilon, \lambda_j, \omega_j)$ such that

$$\left\|\sum_j f_j(A_j)(A_j - \lambda_j)^{-(\beta+\varepsilon)}\right\| \leq CM^2 \sum_j \|f_j\|_\infty$$

for all $f_j \in H^\infty(R_{\omega_j})$. In particular, $\mathrm{dom}(A_j^{\beta+\varepsilon}) \subseteq \mathrm{dom}(f_j(A_j))$.

Note: We can deduce that:

$$C\sum_j \|f_j\|_\infty \leq \frac{(1+\varepsilon)}{C}\sum_j \|f_j\|_{A_j M_{1+\varepsilon}^X},$$

From Theorem 3.7 and Corollary 3.10 Part (c).

Part (c) shows that, even though the sequence of semigroup generators on Hilbert spaces do not have abounded H^∞-calculus in general, each functions f_j that decays with polynomial rate $\varepsilon > 0$ at infinity yields bounded sequence of operators $f_j(A_j)$. For $\beta+\varepsilon > \frac{1}{2}$ this is already covered by Lemma 2.4, but for $\beta+\varepsilon \in (0, \frac{1}{2}]$ it appears to be new.

Remark 3.11: Part (c) of Corollary 3.10 yields a statement about stability of numerical methods. Let $-A_j$ be the sequence generates an exponentially stable semigroups $(T^j(t))_{t\geq 0}$ on a Hilbert space,

Let r $\in H^\infty(\mathbb{C}_+)$ be such that $\|r\|_{H^\infty(\mathbb{C}_+)} \leq 1$, and let $\beta+\varepsilon$, $h_j > 0$. Then

$$sup\left\{\left\|r(h_j A_j)^n x\right\| | n \in \mathbb{N}, x \in dom(A_j^{\beta+\varepsilon})\right\} < \infty \qquad (12)$$

Follows from (c) in Corollary 3.10 after shifting the generator. Elements of the form $r(h_j A_j)^n x$ are often used in numerical methods to approximate the solution of the abstract Cauchy problem associated to $-A_j$ with initial value x, and (12) shows that such approximations are stable whenever x $\in \mathrm{dom}(A_j^{\beta+\varepsilon})$.

IV. M-Bounded Functional Calculus

Describe another transference principle for semigroups, one that provides estimates for the norms of the sequence ofoperators of the form $f_j^{(m)}(A_j)$ for f_j analytic multiplier functions and $f_j^{(m)}$its m-th derivatives, m $\in \mathbb{N}$. Moreover, recall our notational simplifications $A_j M_{1+\varepsilon}(R_{\omega_j}) := A_j M_{1+\varepsilon}^X(R_{\omega_j})$ (Remark 2.1).

Let $\omega_j < (\omega_j)_0$ be real numbers. The sequence operators of A_j of half-plane types $(\omega_j)_0$ a Banach space X, has an m-bounded $A_j M_{1+\varepsilon}^X(R_{\omega_j})$-calculus if there exists $\varepsilon \geq -1$, such that $f_j^{(m)}(A_j) \in \mathcal{L}(X)$ with

$$\left\|\sum_j f_j^{(m)}(A_j)\right\| \leq (1+\varepsilon)\sum_j \|f_j\|_{A_j M_{1+\varepsilon}^X} \qquad for\ all\ f_j \in A_j M_{1+\varepsilon}^X(R_{\omega_j})$$

This is well defined since the Cauchy integral formula implies that $f_j^{(m)}$is bounded on every half-planes $R_{\acute{\omega}_j}$with $\acute{\omega}_j > \omega_j$.

Say that A_j has a strongm-bounded $A_j M_{1+\varepsilon}^X$ -calculus of types $(\omega_j)_0$ if A_j has an m-bounded $A_j M_{1+\varepsilon}^X$ (R_{ω_j}) -calculus for every $\omega_j < (\omega_j)_0$ such that for some $\varepsilon \geq 0$ one has

$$\left\| \sum_j f_j^{(m)} (A_j) \right\| \leq (1+\varepsilon) \sum_j \frac{1}{\left((\omega_j)_0 - \omega_j\right)^m} \|f_j\|_{A_j M_{1+\varepsilon}^X (R_{\omega_j})} \tag{13}$$

for all $f_j \in A_j M_{1+\varepsilon}^X \left(R_{\omega_j}\right)$and $\omega_j ¡(\omega_j)_0$.

Lemma 4.1: Let A_j be the sequence of operators of half-plane types $(\omega_j)_0 \in \mathbb{R}$ on a Banach space X, and let $0 \leq \varepsilon \leq \infty$, and m$\in \mathbb{N}$. If A_j has a strong m-bounded $A_j M_{1+\varepsilon}^X$ -calculus of types$(\omega_j)_0$, then A_j has a strong n-bounded $A_j M_{1+\varepsilon}^X$ -calculus of types$(\omega_j)_0$ for all n, $\varepsilon > 0$,

Proof: Let $\omega_j < \beta + \varepsilon < (\omega_j)_0$,$f_j \in A_j M_{1+\varepsilon}(R_{\omega_j})$ and $n \in \mathbb{N}$. Then

$$\sum_j f_j^{(n)} (\beta + is) = \frac{(n)!}{2\pi!} \int_{\mathbb{R}} \sum_j \frac{f_j\left((\beta+\varepsilon)+ir\right)}{\left((\beta+\varepsilon)+ir\right) - (\beta+is)^{n+1}} dr$$

$$= \frac{(n)!}{2\pi!} \sum_j \left(f_j\left((\beta+\varepsilon)+i\cdot\right) * \left((\varepsilon - i\cdot)^{-n-1}\right)\right)(s)$$

For some s $\in \mathbb{R}$, by the Cauchy Integral formula. Hence, using lemma 2.2,

$$\left\| \sum_j f_j^{(n)} (\beta + i\cdot) \right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} \leq \frac{(n)!}{2\pi!} \|(\varepsilon - i\cdot)^{-n-1}\|_{L^1(\mathbb{R})} \sum_j \left\| f_j\left((\beta+\varepsilon)+i\cdot\right)\right\|_{\mathcal{M}_{(1+\varepsilon)}(X)}$$

$$\leq \frac{C}{(-\varepsilon)^n} \sum_j \|f_j\|_{A_j M_{1+\varepsilon}(R_{\omega_j})}$$

for some C = C$(n) \geq 0$ independents of f_j , β, $\beta + \varepsilon$ and ω_j. Letting $\beta + \varepsilon$ tend to ω_j yields

$$\left\| \sum_j f_j^{(n)} \right\|_{A_j M_{(1+\varepsilon)}(R_\beta)} = \left\| \sum_j f_j^{(n)} (\beta + i\cdot) \right\|_{\mathcal{M}_{(1+\varepsilon)}(X)} \leq C \sum_j \frac{1}{\left(\beta - \omega_j\right)^n} \|f_j\|_{A_j M_{(1+\varepsilon)}\left(R_{\omega_j}\right)} \tag{14}$$

Let $\varepsilon \geq 0$. Applying (14) with $n - m$ in place of n shows that $f_j^{(n-m)} \in A_j M_{1+\varepsilon}(R_\beta)$ with

$$\left\| \sum_j f_j^{(n)} (A_j) \right\| \leq C' \sum_j \frac{1}{((\omega_j)_0 - \beta)^m} \left\| f_j^{(n-m)} \right\|_{A_j M_{1+\varepsilon}(R_\beta)}$$

$$\leq CC' \sum_j \frac{1}{((\omega_j)_0 - \beta)^m (\beta - \omega_j)^{n-m}} \|f_j\|_{A_j M_{(1+\varepsilon)}(R_{\omega_j})}$$

Finally, letting $\beta + \varepsilon = \frac{1}{2}((\omega_j + (\omega_j)_0)$,

$$\left\|\sum_j f_j^{(n)}(A_j)\right\| \leq C'' \sum_j \frac{1}{((\omega_j)_0 - \omega_j)^{(n)}} \|f_j\|_{A_j M_{(1+\varepsilon)}(R_{\omega_j})}$$

for some $C'' \geq 0$ independents of f_j and ω_j.■

For the transference principle in Proposition 3.1 it is essential that the support of $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$ are contained in some interval $[1+\varepsilon, \infty)$ with $\varepsilon > -1$. One cannot expect to find such a transference principle for arbitraries μ^j, as this would allow one to prove that the sequence of semigroup generators has a bounded analytic multiplier calculus. However, if we let $t\mu^j$ be given by $(t\mu^j)(dt) := t\,\mu^j(dt)$ then we can deduce the following transference principle. Use the conventions $1/\infty := 0$, $\infty^0 := 1$.

Proposition 4.2: Let $-A_j$ be the sequence of generators of a C_0 –semigroups $(T^j(t))_{t\in\mathbb{R}+}$ of type (M, 0) On a Banach space X. Let $0 \leq \varepsilon \leq \infty$, $\omega_j \downarrow 0$ and $\mu^j \in M_{\omega_j}(\mathbb{R}_+)$. Then

$$\left\|\sum_j T^j_{t\mu^j}\right\| \leq M^2 \sum_j \frac{1}{|\omega_j|}(1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)}\left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\left\|L_{e_{-\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

Proof: Factorizes $T^j_{t\mu^j}$ as $T^j_{t\mu^j} = P \circ L_{e_{-\omega_j}\mu^j} \circ \iota$, where ι and P are as in the proof of Proposition 3.1 with $\psi_j, \varphi_j := \mathbf{1}_{\mathbb{R}_+} e_{\omega_j}$. Since $(\psi_j * \varphi_j)e_{-\omega_j}\mu^j = t\,\mu^j$. Then

$$\left\|\sum_j T^j_{t\mu^j}\right\| \leq M^2 \sum_j \|e_{\omega_j}\|_{\frac{1+\varepsilon}{\varepsilon}} \left\|L_{e_{-\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))} \|e_{\omega_j}\|_{1+\varepsilon}$$

$$= M^2 \sum_j \frac{1}{|\omega_j|}(1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)}\left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\left\|L_{e_{-\omega_j}\mu^j}\right\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

by Holder's inequality. ■

To prove our main result m - bounded functional calculus, a generalization of theorem 7.1 in [2] to arbitrary Banach spaces.

Theorem 4.3: Let A_j be densely defined sequence of operators of half-plane type 0 on a Banach space X.Then the following assertions are equivalent:

(i) $-A_j$ is the sequence of generators of bounded C_0 -semigroup on X.

(ii) A_j has a strong m-bounded $A_j M^X_{1+\varepsilon}$ -calculus of type 0 for some/all $\varepsilon \geq 0$ and some/allm $\in \mathbb{N}$.

If $-A_j$ be the sequence of generates bounded C_0 -semigroup then A_j has an m-bounded $A_j M^X_{1+\varepsilon}(R_{\omega_j})$-calculus for all $\omega_j < 0$, $\varepsilon \geq 0$ and m$\in \mathbb{N}$.

Proof. (i) ⇒ (ii) By Lemma 4.1 it suffices to let m = 1. Proceed along the same lines as the proof of Theorem 3.3. Let $(T^j(t))_{t\in\mathbb{R}_+} \subseteq \mathcal{L}(X)$ be the sequence semigroups generated by $-A_j$ and fix $\omega_j < 0, \varepsilon \geq 0$ and $f_j \in A_j M_{1+\varepsilon}(R_{\omega_j})$. Define the functions $(f_j)_{k,\epsilon} := f_j(\cdot + \epsilon)g^j_k(\cdot + \epsilon)$ for k $\in \mathbb{N}$ and $\varepsilon > 0$, where $g^j_k(z_j) := \frac{k}{z_j - \omega_j + k}$ for $z_j \in R_{\omega_j}$. Then

Ref
2. C.Batty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.

$(f_j)_{k,\epsilon} \in A_j M_1(R_{\omega_j})$ by Lemma 2.4, and Lemma 2.3 yields $(\mu^j)_{k,\epsilon} \in M_{\omega_j}(\mathbb{R}_+)$ with $(f_j)_{k,\epsilon} = \widehat{\mu^j_{k,\epsilon}}$. Then

$$\sum_j (\acute{f_j})_{k,\epsilon}(z_j) = lim_{h_j \to 0} \sum_j \frac{(f_j)_{k,\epsilon}(z_j + h_j) - (f_j)_{k,\epsilon}(z_j)}{h_j}$$

$$= lim_{h_j \to 0} \int_0^\infty \sum_j \frac{e^{-(z_j+h_j)t} - e^{-z_j t}}{h_j} (\mu^j)_{k,\epsilon}(dt) = -\int_0^\infty \sum_j t e^{-z_j t}\, \mu^j_{k,\epsilon}(dt)$$

$$= -\sum_j \widehat{t\mu^j_{k,\epsilon}}(z_j)$$

for $z_j \in R_{\omega_j}$, by the Dominated Convergence Theorem. Hence $\left(\acute{f_j}\right)_{k,\epsilon}(A_j) = -T^j_{t\mu^j_{k,\epsilon}}$, and Proposition 4.2 and Lemma 2.3 imply

$$\left\| \sum_j (\acute{f_j})_{k,\epsilon}(A_j) \right\| \leq (1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)} \left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} M^2 \sum_j \frac{\|(f_j)_{k,\epsilon}\|_{A_j M^X_{1+\varepsilon}}}{|\omega_j|}$$

Furthermore, $sup_{k,\epsilon} \|\sum_j (f_j)_{k,\epsilon}\|_{A_j M^X_{1+\varepsilon}}$. the $(f_j)_{k,\epsilon}$ are uniformly bounded. By the Cauchy Cauchy integral formula, so are the derivatives $(\acute{f_j})_{k,\epsilon}$ on every smaller half-plane. Since $(\acute{f_j})_{k,\epsilon}(z_j) \to (\acute{f_j})(z_j)$ for all $z_j \in R_{\omega_j}$ as k $\to \infty$, $\epsilon \to 0$, the Convergence Lemma yields $\acute{f_j}(A_j) \in \mathcal{L}(X)$ with

$$\left\| \sum_j \acute{f_j}(A_j) \right\| \leq (1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)} \left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} M^2 \sum_j \frac{\|f_j\|_{A_j M^X_{1+\varepsilon}}}{|\omega_j|}$$

which is (4.1) for m = 1.

For (ii) ⇒ (i) assume that A_j has a strong m-bounded $A_j M_{1+\varepsilon}$-calculus of type 0 for some $\varepsilon \geq 0$ and some m ∈ ℕ. Then

$$e_{-t} \in A_j M_1\ (R_{\omega_j}) \subseteq A_j M_{1+\varepsilon}(R_{\omega_j})$$

for all t > 0 and $\omega_j < 0$, with

$$\left\| \sum_j e_{-t} \right\|_{A_j M_{1+\varepsilon}(R_{\omega_j})} \leq \sum_j \|e_{-t}\|_{A_j M_1(R_{\omega_j})} = \sum_j e^{-t\omega_j}$$

Then, $(e_{-t})^{(m)} = (-t)^m\ e_{-t}$ implies

$$t^m \left\| \sum_j e^{-tA_j} \right\| \leq C \sum_j \frac{1}{|\omega_j|^m} e^{-t\omega_j}$$

Letting $\omega_j := -\frac{1}{t}$] yields the required statement ■

If X = H is a Hilbert space then Plancherel's theorem yields the following result.

Corollary 4.4: Let A_j be densely defined sequence of operators of half-plane type 0 on a Hilbert space H. Then the following assertions are equivalent:

(i) $-A_j$ is the sequence of generators of a bounded C_0 -semigroup on H.

(ii) A_j has strong m-bounded H^∞ -calculus of type 0 for some/all m ∈ ℕ.

In particular, if $-A_j$ be the sequence of generates bounded C_0-semigroup then A_j has m-bounded $H^\infty(R_{\omega_j})$-calculus for all $\omega_j < 0$ and m ∈ ℕ.

V. Semigroups on UMD Spaces

A Banach space X is a UMD space if the function t ⟼ sgn(t) is a bounded L^2(X)-Fourier multiplier. For $\omega_j \in \mathbb{R}$, let

$$H_1^\infty\left(R_{\omega_j}\right) = \left\{f_j \in H^\infty(R_{\omega_j}) \middle| (Z_j - \omega_j)\acute{f}_j(z_j) \in H^\infty(R_{\omega_j})\right\}$$

be the analytic Mikhlin algebras on R_{ω_j}, a Banach algebra endowed with the series of norms

$$\sum_j \|f_j\|_{H_1^\infty} = \sum_j \|f_j\|_{H_1^\infty\left(R_{\omega_j}\right)} = sup_{z_j \in R_{\omega_j}} \sum_j |f_j(z_j)| + \sum_j |(Z_j - \omega_j)\acute{f}_j(z_j)| \left(f_j \in H_1^\infty\left(R_{\omega_j}\right)\right)$$

Lemma 2.2 yield the continuous inclusion

$$H_1^\infty\left(R_{\omega_j}\right) \hookrightarrow A_j M_{1+\varepsilon}^X\left(R_{\omega_j}\right)$$

For each $\varepsilon > 0$, if X is a UMD space. Combining this with Theorems 3.3 and 4.3 and Corollaries 3.5 and 3.6 proves the following theorem (see ,e.g., [8]).

Theorem 5.1: Let $-A_j$ be the sequence of generates C_0-semigroups $(T^j(t))_{t\in\mathbb{R}_+}$ of type (M, 0) on a UMD space X. Then the following assertions hold.

(a) For each $\varepsilon > 0$, there exists a constant $c_{\varepsilon+1} = c(1+\varepsilon, X) \geq 0$ such that the following holds.

Let $1+\varepsilon$, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}(X)$ with

$$\left\|\sum_j f_j(A_j)\right\| \leq \begin{cases} c_{\varepsilon+1}M^2 \sum_j \left|\log\left(\omega_j(1+\varepsilon)\right)\right| \|f_j\|_{H_1^\infty} \ if\ \omega_j(1+\varepsilon) \leq \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \\ 2c_{\varepsilon+1}M^2 \sum_j e^{-\omega_j(1+\varepsilon)} \|f_j\|_{H_1^\infty} \ if\ \omega_j(1+\varepsilon) > \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \end{cases}$$

for all $f_j \in H_1^\infty(R_{-\omega_j}) \cap e_{-(1+\varepsilon)}H^\infty(R_{-\omega_j})$, and $f_j(A_j)T^j(1+\varepsilon) \in \mathcal{L}(X)$ with

8. M.Haase, J.Rozendaal: Functional calculus for the of semigroup generators via transference.

$$\left\|\sum_j f_j(A_j)T^j\ (1+\varepsilon)\right\|$$

$$\leq \begin{cases} c_{\varepsilon+1}M^2\sum_j \left|\log\left(\omega_j(1+\varepsilon)\right)\right| e^{\omega_j(1+\varepsilon)}\left\|f_j\right\|_{H_1^\infty} & if\ \omega_j(1+\varepsilon) \leq \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \\ 2c_{\varepsilon+1}M^2\sum_j \left\|f_j\right\|_{H_1^\infty} & if\ \omega_j(1+\varepsilon) > \min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\} \end{cases}$$

for all $f_j \in H_1^\infty\ (R_{-\omega_j})$.

(b) If

$$\bigcup_{\varepsilon>-1}\sum_j ran\left(T^j(1+\varepsilon)\right) = X$$

then A_j has bounded $H_1^\infty(R_{\omega_j})$ -calculus for all $\omega_j < 0$.

(c) A_j has a strong m-bounded H_1^∞ -calculus of type 0 for all m $\in \mathbb{N}$.

Remark 5.2: Theorem 3.7 yields the domain inclusions dom $(A_j^{\beta+\varepsilon}) \subseteq \text{dom}(f_j(A_j))$ for all $\beta+\varepsilon \in \mathbb{C}_+, \omega_j < 0$ and $f_j \in H_1^\infty\ (R_{\omega_j})$, on a UMD space X. However, this inclusion in fact, holds true on a general Banach space X. Indeed, for $\lambda_j \in \mathbb{C}$ with $\text{Re}(\lambda_j) < 0$, Bernstein's Lemma [12, Proposition 8.2.3] implies $\frac{f_j(z_j)}{(\lambda_j - z_j)^{\beta+\varepsilon}} \in A_j \text{M}_1\ (\mathbb{C}_+)$, hence $f_j(A_j)(\lambda_j - A_j)^{-(\beta+\varepsilon)} \in \mathcal{L}(\text{X})$ and $\text{dom}(A_j^{\beta+\varepsilon}) \subseteq \text{dom}(f_j\ (A_j))$. Series estimates

$$\left\|\sum_j f_j(A_j)\ (\lambda_j - A_j)^{-(\beta+\varepsilon)}\right\| \leq (1+\varepsilon)\sum_j \left\|f_j\right\|_{H_1^\infty(R_{\omega_j})}$$

then follows from an application of the Closed Graph Theorem and the Convergence Lemma.

Remark 5.3: To apply Theorem 5.1 one can use the continuous inclusion

$$H^\infty\left(R_{\omega_j} \cup (S_{\varphi_j} + a)\right) \subseteq H_1^\infty\left(R_{\acute{\omega}_j}\right) \tag{15}$$

For $\acute{\omega}_j > \omega_j$, a $\in \mathbb{R}$ and $\varphi_j \in (\frac{\pi}{2}, \pi]$. Here $R_{\omega_j} \cup \left(S_{\varphi_j} + \text{a}\right)$ are the union of R_{ω_j} and the translated sectors $S_{\varphi_j} + \text{a}$, where

$$S_{\varphi_j} = \left\{z_j \in \mathbb{C}\ \middle|\ \left|\arg(z_j)\right| < \varphi_j\right\}$$

Indeed, to derive (15) it suffices to let a = 0, yields the desired result.

VI. γ_j – Bounded Semigroups

The geometry of the underlying Banach space X played an essential role in the results of properties of the analytic multiplier algebras $A_j M_{1+\varepsilon}^X$. To wit, in to identify

nontrivial functions in $A_j M_{1+\varepsilon}^X$ one needs a geometric assumption on X, for instance that it is a Hilbert or a UMD space. Take a different approach and make additional assumptions on the semigroup instead of the underlying space. Show that if the semigroups in questions are γ_j -bounded then one can recover the Hilbert space results on an arbitrary Banach space X.

Assume to be familiar with the basics of the theory of γ_j -radonifying sequence of operators and γ_j -boundedness as collected in the survey article by van Neerven[13].

Let H be a Hilbert space and X a Banach space. The linear sequence of operators $T^j : H \to X$ is γ_j –summing if

$$\sum_j \|T^j\|_{\gamma_j} = sup_F \sum_j \left(\mathbb{E} \left\| \sum_{h_j \in F} (\gamma_j)_{h_j} T^j h_j \right\|_X^2 \right)^{1/2} < \infty ,$$

Where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and $((\gamma_j)_{h_j})_{h_j \in F}$ is an independent collection of complex-valued standard Gaussian random variables on some probability space. Endow

$$(\gamma_j)_\infty (H; X) := \{T^j : H \to X \mid T^j \text{ are } \gamma_j \text{ -summing}\}$$

with the norms $\|\cdot\|_{\gamma_j}$ and let the spaces $\gamma_j (H ; X)$ of all γ_j -radonifying sequence of operators be the closure in $(\gamma_j)_\infty (H ; X)$ of the finite-rank sequence of operators $H \otimes X$.

For a measure spaces (Ω, μ^j) let $\gamma_j(\Omega; X)$ (resp. $(\gamma_j)_\infty(\Omega; X)$) be the space of all weakly L^2–functions $f_j : \Omega \to X$ for which the integration sequence of operators of $(J)_{f_j} : L^2(\Omega) \to X$,

$$\sum_j (J)_{f_j}(g^j) = \int_\Omega \sum_j g^j . f_j \, d\mu^j \qquad (g^j \in L^2(\Omega))$$

Is γ_j -radonifying (γ_j–summing), and endow it with the norms $\|f_j\|_{\gamma_j} = \|(J)_{f_j}\|_{\gamma_j}$. Collections $\mathcal{T}^j \subseteq \mathcal{L}(X)$ are γ_j -bounded if there exists a constant $C \geq 0$ such that

$$\left(\mathbb{E} \left\| \sum_j \sum_{T^j \in \acute{\mathcal{T}}^j} (\gamma_j)_{T^j} T^j x_{T^j} \right\|^2 \right)^{1/2} \leq C \sum_j \left(\mathbb{E} \left\| \sum_{T^j \in \acute{\mathcal{T}}^j} (\gamma_j)_{T^j} x_{T^j} \right\|^2 \right)^{1/2}$$

for all finite subsets $\acute{\mathcal{T}}^j \subseteq \mathcal{T}^j$, sequences $((x)_{T^j})_{T^j \in \acute{\mathcal{T}}^j} \subseteq X$ and independent complex-valued standard Gaussian random variables $((\gamma_j)_{T^j})_{T^j \in \acute{\mathcal{T}}^j}$. The smallest such C is the γ_j -bound of $\mathcal{T}^j$ and is denoted by. $[\![T^j]\!]^{\gamma_j}$ Every γ_j -bounded collections are uniformly bounded with supremum boundless than orequal to the γ_j -bound, and the converse holds if X is a Hilbert space.

An important result involving γj -boundedness is the multiplier theorem. State a version that is tailored to the purposes. Given a Banach space Y, a function $g^j : \mathbb{R} \to Y$

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is piecewise $W^{1,\infty}$ if $g^j \in W^{1,\infty}(\mathbb{R} \text{ “ } \{a_1 , . . . , a_n\}; Y)$ for some finite set $\{a_1 , . . . , a_n \} \subseteq \mathbb{R}$.

Theorem 6.1 (Multiplier Theorem): Let X and Y be Banach spaces and $T^j : \mathbb{R} \to \mathcal{L}(X, Y)$ a strongly measurable mappings such that

$$\mathcal{T}^j = -T^j (s) \mid s \in \mathbb{R}\}$$

Notes

are γ_j -bounded. Suppose furthermore that there exists a dense subset $D \subseteq X$ such that $s \longmapsto T^j(s)x$ is piecewise $W^{1,\infty}$ for all $x \in D$. Then the multiplication sequence of operators

$$\mathcal{M}_{T^j} : L^2(\mathbb{R}) \otimes X \to L^2(\mathbb{R}; Y), \mathcal{M}_{T^j} (f_j \otimes x) = f_j (\cdot)T^j (\cdot)x$$

Extends uniquely to boundedsequence of operators

$$\mathcal{M}_{T^j} : \gamma_j(L^2(\mathbb{R}); X) \to \gamma_j(L^2(\mathbb{R}); Y)$$

with

$$\left\| \sum_j \mathcal{M}_{T^j} \right\| \leq \sum_j [\![\mathcal{T}^j]\!]^{\gamma_j}$$

Proof: That $\mathcal{M}_{T^j}$ extends uniquely to bounded sequence of operators into $(\gamma_j)_\infty(L^2(\mathbb{R});Y)$ with $\|\sum_j \mathcal{M}_{T^j}\| \leq \sum_j [\![\mathcal{T}^j]\!]^{\gamma_j}$. To see that in facts $\text{ran}(\mathcal{M}_{T^j}) \subseteq \gamma_j(\mathbb{R}; Y)$, employ a density argument. For $x \in D$ let $a_1 , . . . , a_n \in \mathbb{R}$ be such that $s \longmapsto T^j (s)x \in W^{1,\infty}(\mathbb{R} \text{ “}\{a_1 , . . . , a_n \}; Y)$, and set $a_0 := -\infty$, $a_{n+1} := \infty$. Let $f_j \in C_c (\mathbb{R})$ be given and note that

$$\sum_j \int_{a_j}^{a_{j+1}} \|f_j\|_{L^2(s,a_{j+1})} \|T^j(s)' x\| ds < \infty$$

for all $j \in \{1, . . . , n\}$. Furthermore,

$$\int_{-\infty}^{a_1} \sum_j \|f_j\|_{L^2(-\infty,s)} \|T^j(s)' x\| ds < \infty$$

yields $(\mathbf{1}_{(a_j,a_{j+1})} f_j)(\cdot)T^j(\cdot)x \in \gamma_j(\mathbb{R}; Y)$ for all $0 \leq j \leq n$, hence $f_j(\cdot)T^j (\cdot)x \in \gamma_j(\mathbb{R}; Y)$. Since $C_c(\mathbb{R}) \otimes D$ is dense in $L^2(\mathbb{R}) \otimes X$, which in turn is dense in $\gamma_j(L^2(\mathbb{R}); X)$, theresult follows. ■

To prove a generalization of part (a) of Corollary 3.10, recall that

$$e_{-(1+\varepsilon)}H^\infty (R_{\omega_j}) = \{ f_j \in H^\infty (R_{\omega_j}) \mid f_j (z_j) = O (e^{-(1+\varepsilon) R(z_j)} \text{as } |z_j| \to \infty\}$$

for $\varepsilon > -1, \omega_j \in \mathbb{R}$.

Theorem 6.2: There exists a universal constant c≥ 0 such that the following holds. Let $-A_j$ be sequence of generates γ_j- bounded C_0-semigroups $(T^j(t))_{t\epsilon\mathbb{R}_+}$ with M:=$[\![T^j]\!]^{\gamma_j}$ on Banach space X, and let 1+ ε, $\omega_j > 0$. Then $f_j(A_j) \in \mathcal{L}(X)$ with

$$\left\|\sum_j f_j(A_j)\right\| \leq \begin{cases} cM^2 \sum_j |\log(\omega_j(1+\varepsilon))| \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) \leq \frac{1}{2} \\ 2M^2 \sum_j e^{-\omega_j(1+\varepsilon)} \|f_j\|_\infty & if\ \omega_j(1+\varepsilon) > \frac{1}{2} \end{cases}$$

for all $f_j \in e_{-(1+\varepsilon)}H^\infty(R_{-\omega_j})$.

In particular, A_j has a bounded $e_{-(1+\varepsilon)}H^\infty(R_{-\omega_j})$-calculus.

Proof: Only need to show that the estimate (9) in Proposition 3.1 can be refined to

$$\left\|\sum_j T^j_{\mu^j}\right\| \leq M^2\eta \sum_j (\omega_j, 1+\varepsilon, 2) \left\|L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}(\gamma_j(\mathbb{R},X))} \tag{16}$$

for $\mu^j \in M_{-\omega_j}(\mathbb{R}_+)$ with supp$\mu^j \subseteq [1+\varepsilon, \infty)$. Then one uses that

$$\left\|\sum_j L_{e_{\omega_j}\mu^j}\right\|_{\mathcal{L}(\gamma_j(\mathbb{R},X))} \leq \sum_j \left\|\widehat{e_{\omega_j}\mu^j}\right\|_{H^\infty(\mathbb{C}_+)} = \sum_j \|\widehat{\mu^j}\|_{H^\infty(R_{-\omega_j})}$$

by the ideal properties of $\gamma_j(L^2(\mathbb{R}); X)$ [13, Theorem 6.2], and proceeds as in the proof of Theorem 3.3 to deduce the desired result.

To obtain (16) we factorizes $T^j_{\mu^j}$ as $T^j_{\mu^j} = P\circ L_{e_{-\omega_j}\mu^j}\circ\iota$, where $\iota: X \to \gamma_j(\mathbb{R}; X)$ and $P: \gamma_j(\mathbb{R}; X) \to X$ are given by

$$\iota x(s) := \psi_j(-s)T^j(-s)x \quad (x \in X, s \in \mathbb{R}),$$

$$\sum_j Pg^j = \int_0^\infty \sum_j \varphi_j(t)T^j(t)\, g^j(t)dt \qquad (g^j \in \gamma_j(\mathbb{R}, X))$$

for $\psi_j, \varphi_j \in L^2(\mathbb{R}_+)$ such that $\psi_j * \varphi_j \equiv e_{-\omega_j}$ on $[1+\varepsilon, \infty)$. Show that the maps ι and P are well-defined and bounded. To this end, first note that $s \mapsto T^j(-s)x$ is piecewise $W^{1,\infty}$ for all x in the dense subset dom$(A_j) \subseteq X$ and that

$$\psi_j(-\cdot) \otimes x \in L^2(-\infty, 0) \otimes X \subseteq \gamma_j(L^2(\mathbb{R}); X).$$

Therefore Theorem 6.1 yields $\iota x \in \gamma_j(\mathbb{R}, X)$ with

$$\left\|\sum_j l\, x\right\|_{\gamma_j} = \left\|\sum_j J_l\, x\right\|_{\gamma_j} \leq M \sum_j \|\psi_j(-\cdot)\otimes x\|_{\gamma_j} = M \sum_j \|\psi_j\|_{L^2(\mathbb{R}_+)} \|x\|_X$$

As for P, write

$$\sum_j Pg^j = \int_0^\infty \sum_j \varphi_j(t) T^j(t)\, g^j(t)\, dt = \sum_j J_{T^j g^j}(\varphi_j)$$

And use Theorem 6.1 once again to see that $T^j g^j \in \gamma_j(\mathbb{R}; X)$. Hence

$$\left\| \sum_j P g^j \right\|_X \leq \sum_j \left\| J_{T^j g^j} \right\|_{\gamma_j} \|\varphi_j\|_{L^2(\mathbb{R}_+)} \leq M \sum_j \|\varphi_j\|_{L^2(\mathbb{R}_+)} \|g^j\|_{\gamma_j}$$

Finally, estimating the norms of $T^j_{\mu^j}$ through this factorization and taking the infimum over all ψ_j and φ_j yields (16).■

Note: In putting μ^j by $t\,\mu^j$ in the proof of Theorem 6.2 we have,

$$\sum_j (\omega_j, 1+\varepsilon, 2) \left\| L_{e_{\omega_j} \mu^j} \right\|_{\mathcal{L}(\gamma_j(\mathbb{R},X))} \leq \frac{1}{n} \sum_j \frac{1}{|\omega_j|} (1+\varepsilon)^{-\left(\frac{1}{1+\varepsilon}\right)} \left(\frac{1+\varepsilon}{\varepsilon}\right)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} \left\| L_{e_{\omega_j} \mu^j} \right\|_{\mathcal{L}(L^{1+\varepsilon}(X))}$$

Corollary 6.3: Corollary 3.10 generalizes to γ_j –bounded semigroups on arbitrary Banach spaces upon replacing the uniform bound M of T^j by $[\![T^j]\!]^{\gamma_j}$.

Theorem 4.3 can be extended in an almost identical manner to γ_j–versions (see,e.g., [8]).

Theorem 6.4: Let $-A_j$ be the sequence generates γ_j -bounded C_0-semigroup on a Banach X. Then A_j has a strongm-bounded H^∞-calculus of type 0 for all m $\in \mathbb{N}$.

Appendix A. Growth estimates

In this appendix we examine the function η: (0, ∞) × (0, ∞) × [1, ∞] → $\mathbb{R}_+$ from (3.1):

$$\eta(\beta+\varepsilon, t, 1+\varepsilon) = inf\left\{ \|\psi_j\|_{1+\varepsilon} \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} | \psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)} \text{ on } [t, \infty) \right\}$$

Use the notation $f_j \lesssim g^j$ for real-valued functions $f_j, g^j : Z \to \mathbb{R}$ on some set Z to indicate that there exists a constant c ≥ 0 such that $f_j(z_j) \leq c g^j(z_j)$ for all $z_j \in Z$.

Lemma A.1: For each $\varepsilon > 0$ there exist constants $c_{1+\varepsilon}, d_{1+\varepsilon} \geq 0$ such that

$$d_{1+\varepsilon} |\log((\beta+\varepsilon)t)| \leq \eta(\beta+\varepsilon, t, 1+\varepsilon) \leq c_{1+\varepsilon} |\log((\beta+\varepsilon)t)| \qquad (A.1)$$

If $(\beta+\varepsilon)t \leq min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\}$ If $(\beta+\varepsilon)t > min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\}$ $then$

$$e^{-(\beta+\varepsilon)t} \leq \eta(\beta+\varepsilon, t, 1+\varepsilon) \leq 2e^{-(\beta+\varepsilon)t} \qquad (A.2)$$

Proof: First note that $\eta(\beta+\varepsilon, t, 1+\varepsilon) = \eta((\beta+\varepsilon)t, 1, 1+\varepsilon) = \eta(1, (\beta+\varepsilon)t, 1+\varepsilon)$, for all $\beta+\varepsilon$, t and $1+\varepsilon$. Indeed, for $\psi_j \in L^{1+\varepsilon}(\mathbb{R}_+)$, $\varphi_j \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}_+)$ with $\psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)}$ on [1, ∞) defines $(\psi_j)_t(s) := t^{-\left(\frac{1}{\varepsilon+1}\right)} \psi_j\left(\frac{s}{t}\right)$ and $(\varphi_j)_t(s) := t^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)} \psi_j(s/t)$ for some s ≥ 0. Then

$$\sum_j (\psi_j)_t * (\varphi_j)_t(r) = \int_0^\infty \sum_j \psi_j\left(\frac{r-s}{t}\right)\varphi_j\left(\frac{s}{t}\right)\frac{ds}{t} = \sum_j \psi_j * \varphi_j\left(\frac{r}{t}\right)$$

for all r ≥ 0, so $(\psi_j)_t * (\varphi_j)_t \equiv e_{-(\beta+\varepsilon)}$ on [t, ∞). Moreover,

$$\sum_j \|(\psi_j)_t\|_{1+\varepsilon}^{1+\varepsilon} = \int_0^\infty \sum_j \left|\psi_j\left(\frac{s}{t}\right)\right|^{1+\varepsilon}\frac{ds}{t} = \int_0^\infty \sum_j |\psi_j(s)|^{1+\varepsilon}\,ds = \sum_j \|\psi_j\|_{1+\varepsilon}^{1+\varepsilon}$$

and similarly $\sum_j \|(\varphi_j)_t\|_{\frac{1+\varepsilon}{\varepsilon}} = \sum_j \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$. Hence $\eta(\beta+\varepsilon, t, 1+\varepsilon) \le \eta((\beta+\varepsilon)t, 1, 1+\varepsilon))$. Considering $(\psi_j)_{(1/t)}$ and $(\varphi_j)_{(1/t)}$ yields $\eta(\beta+\varepsilon, t, 1+\varepsilon) = \eta((\beta+\varepsilon)t, 1, 1+\varepsilon)$. The other equality follows immediately. Hence, to prove all of the inequalities in (A.1) or (A.2), assume either that $\beta+\varepsilon = 1$ or that t = 1 (but not both).

For the left-hand inequalities, assume that $\beta+\varepsilon = 1$ and first consider the left-hand inequality of (A.1). Let t < 1 and $\psi_j \in L^{1+\varepsilon}(\mathbb{R}_+)$, $\varphi_j \in L^{\frac{1+\varepsilon}{\varepsilon}}(\mathbb{R}_+)$ such that $\psi_j * \varphi_j \equiv e_{-1}$ on [t, ∞). Then

$$|\log(t)| = -\log(t) = \int_t^1 \frac{ds}{s} \le e\int_t^1 e^{-s}\frac{ds}{s} = e\int_t^1 \sum_j |\psi_j * \varphi_j(s)|\frac{ds}{s}$$

$$\le e\int_t^1\int_0^s \sum_j |\psi_j(s-r)|\cdot|\varphi_j(r)|\,dr\frac{ds}{s}$$

$$\le e\int_0^\infty\int_r^\infty \sum_j \frac{|\varphi_j(s-r)|}{s}\,ds|\psi_j(r)|dr$$

$$= e\int_0^\infty\int_0^\infty \sum_j \frac{|\psi_j(r)||\varphi_j(r)|}{s+r}\,dsdr \le \frac{e\pi}{\sin(\pi/1+\varepsilon)}\sum_j \|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

where used Hilbert's absolute inequality [14, Theorem 5.10.1]. It follows that

$$\eta(1, t, 1+\varepsilon) \ge \frac{\sin(\pi/1+\varepsilon)}{e\pi}|\log(t)|$$

For the left-hand inequality of (A.2), assume that $\beta+\varepsilon = 1$ and let t > 0 be arbitrary. Then

$$e^{-t} = \sum_j (\psi_j * \varphi_j)(t) \le \int_0^t \sum_j |\psi_j(t-s)||\varphi_j(s)|\,ds \le \sum_j \|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

By Hölder's inequality, hence $e^{-t} \le \eta(1, t, 1+\varepsilon)$.

For the right-hand inequalities in (A.1) and (A.2), assume that t = 1 and first consider the right-hand inequality in (A.1) for $\beta+\varepsilon \le min\left\{\frac{1}{1+\varepsilon}, \frac{\varepsilon}{1+\varepsilon}\right\}$. In the proof of Lemma A.1, it is shown that

$$\big((\psi_j)_0 * (\varphi_j)_0\big)(s) = \begin{cases} s, & s \in [0,1) \\ 1, & s \ge 1 \end{cases}$$

for

$$\sum_j (\psi_j)_0 = \sum_{j=0}^{\infty} \beta_j \, \mathbf{1}_{(j,j+1)} \;\; and \;\; (\varphi_j)_0 = \sum_{j=0}^{\infty} \acute{\beta}_j \, \mathbf{1}_{(j,j+1)}$$

where $(\beta_j)_j$ and $(\acute{\beta}_j)_j$ are sequences of positive scalars such that $\beta_j = \mathrm{O}\left((1+j)^{-\left(\frac{1}{1+\varepsilon}\right)}\right)$ and $\beta_j^{'} = \mathrm{O}\left((1+j)^{-\left(\frac{\varepsilon}{1+\varepsilon}\right)}\right)$ as j $\to \infty$. Let $\psi_j := e_{-(\beta+\varepsilon)}(\psi_j)_0$ and $\varphi_j := e_{-(\beta+\varepsilon)}(\varphi_j)_0$. Then $\psi_j * \varphi_j \equiv e_{-(\beta+\varepsilon)}$ on [1,∞) and

$$\left\| \sum_j \psi_j \right\|_{1+\varepsilon}^{1+\varepsilon} = \left\| \sum_j e_{-(\beta+\varepsilon)} (\psi_j)_0 \right\|_{1+\varepsilon}^{1+\varepsilon} = \sum_{j=0}^{\infty} \beta_j^{1+\varepsilon} \int_j^{j+1} e^{-(\beta+\varepsilon)(1+\varepsilon)s} \, ds \lesssim \sum_{j=0}^{\infty} \frac{e^{-(\beta^2+\beta(1+\varepsilon)+\varepsilon)j}}{1+j}$$

$$\leq 1 + \int_0^{\infty} \frac{e^{-(\beta^2+\beta(1+\varepsilon)+\varepsilon)s}}{1+s} \, ds = 1 + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)} \int_{\alpha q}^{\infty} \frac{e^{-s}}{s} \, ds$$

The constant in the first inequality depends only on $1+\varepsilon$. Since $(\beta^2 + \beta(\varepsilon+1) + \varepsilon) \leq 1$,

$$\left\| \sum_j \psi_j \right\|_{1+\varepsilon}^{1+\varepsilon} \lesssim 1 + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)} \left(\int_{(\beta+\varepsilon)(1+\varepsilon)}^{1} \frac{e^{-s}}{s} \, ds + \int_1^{\infty} \frac{e^{-s}}{s} \, ds \right)$$

$$\leq 1 + \int_{(\beta+\varepsilon)(1+\varepsilon)}^{1} \frac{1}{s} \, ds + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)} \int_1^{\infty} e^{-s} \, ds$$

$$= 1 - \log(\beta^2 + \beta(1+\varepsilon) + \varepsilon) + e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)-1} \leq log\left(\frac{1}{(\beta+\varepsilon)}\right) + 2$$

Moreover, $\frac{1}{(\beta+\varepsilon)} \geq 1 + \varepsilon > 1 \; hence \; \log\left(\frac{1}{\beta+\varepsilon}\right) \geq \log(1+\varepsilon) > 0$

and

$$\log\left(\frac{1}{\beta+\varepsilon}\right) + 2 \leq \left(1 + \frac{2}{\log(1+\varepsilon)}\right) \log\left(\frac{1}{\beta+\varepsilon}\right)$$

Therefore

$$\left\| \sum_j \psi_j \right\|_{1+\varepsilon} \lesssim \log\left(\frac{1}{\beta+\varepsilon}\right)^{\frac{1}{1+\varepsilon}} = |\log(\beta+\varepsilon)|^{\frac{1}{1+\varepsilon}}$$

For a constant depending only on $1+\varepsilon$. Similarly deduce

$$\left\| \sum_j \varphi_j \right\|_{\frac{1+\varepsilon}{\varepsilon}} \lesssim |\log(\beta+\varepsilon)|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}$$

for a constant depending only on $\frac{1+\varepsilon}{\varepsilon}$ (and thus on $1+\varepsilon$). This yields (A.1).

For the right-hand side of (A.2) we assume that t = 1 and, without loss of generality (Since$(\beta+\varepsilon,t,1+\varepsilon)=\eta(\beta+\varepsilon,t,\frac{1+\varepsilon}{\varepsilon})$), that $\beta+\varepsilon>\frac{1}{1+\varepsilon}$ let $\varphi_j = \mathbf{1}_{[0,1]}e_{(\beta+\varepsilon)(\varepsilon)}$ and $\psi_j = \frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\mathbf{1}_{\mathbb{R}_+}e_{-(\beta+\varepsilon)}$. Then

$$\sum_j \psi_j * \varphi_j(r) = \frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\int_0^1 e^{(\beta+\varepsilon)(\varepsilon)s}e^{-(\beta+\varepsilon)(r-s)}\,ds = e^{-(\beta+\varepsilon)r}$$

For $r \geq 1$. Hence

$$\eta(\beta+\varepsilon,1,1+\varepsilon) \leq \sum_j \|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

$$= \frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\left(\int_0^\infty e^{-(\beta^2+\beta(1+\varepsilon)+\varepsilon)s}\,ds\right)^{\left(\frac{1}{1+\varepsilon}\right)}\left(\int_0^1 e^{\left(\beta+\varepsilon\right)((\varepsilon)\left(\frac{1+\varepsilon}{\varepsilon}\right)s)}ds\right)^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}$$

$$= \frac{(\beta^2+\beta(1+\varepsilon)+\varepsilon)^{\left(\frac{\varepsilon}{1+\varepsilon}\right)}}{e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1}\left(\int_0^1 e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)s}\,ds\right)^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} = (e^{(\beta^2+\beta(1+\varepsilon)+\varepsilon)}-1)^{-\left(\frac{1}{1+\varepsilon}\right)}$$

$$\leq 2^{\left(\frac{1}{1+\varepsilon}\right)}e^{-(\beta+\varepsilon)} \leq 2e^{-(\beta+\varepsilon)}$$

Where haveused the assumption $(\beta^2+\beta(1+\varepsilon)+\varepsilon)>1$ in the penultimate inequality. ■

Note: Deduce that:

(1) $\|\sum_j \psi_j\|_{1+\varepsilon} \leq M_{\beta,\varepsilon}\sum_j\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$

(2) $e^{-t} \leq \|\psi_j\|_{1+\varepsilon}\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} \leq 2e^{-(\beta+\varepsilon)}$

When $\beta+\varepsilon=1$, $t>0$

Proof. (1) Since

$$\left\|\sum_j \psi_j\right\|_{1+\varepsilon} \leq |\log(\beta+\varepsilon)|^{\frac{1}{1+\varepsilon}} \qquad (a)$$

And

$$\left\|\sum_j \varphi_j\right\|_{\frac{1+\varepsilon}{\varepsilon}} \leq |\log(\beta+\varepsilon)|^{\left(\frac{\varepsilon}{1+\varepsilon}\right)} \qquad (b)$$

Divide we have

$$\left\|\sum_j \psi_j\right\|_{1+\varepsilon} \leq M_{\beta,\varepsilon}\sum_j\|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}}$$

Where we have $M_{\beta,\varepsilon} = |\log(\beta+\varepsilon)|^{\left(\frac{1-\varepsilon}{1+\varepsilon}\right)}$
(2)From (A.2), we can get

$$e^{-t} \leq \sum_j \|\psi_j\|_{1+\varepsilon} \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} \qquad (c)$$

$$\leq \left(\sum_j \|\psi_j\|^2_{1+\varepsilon}\right)^{\frac{1}{2}} \left(\sum_j \|\varphi_j\|^2\right)^{\frac{1}{2}} = \|\psi_j\|_{1+\varepsilon} \|\varphi_j\|_{\frac{1+\varepsilon}{\varepsilon}} \leq 2e^{-(\beta+\varepsilon)} \qquad \blacksquare.$$

Conflict of Interests
The authors declare that there is no conflict of interests.

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References Références Referencias

1. Markus Haase: The Functional Calculus for Sectorial Operators. Oper. Theory Adv.Appl., vol.169, Birkhauser Verlag, 2006.
2. C.Batty.M.Haase, J.Mubeen: The Holomorphic functional calculus approach to operator semigroups. Acta Sci.Math. (Szeged)79 (2013) 289-323.
3. W.Arendt: Semigroups and evolutions: Functional calculus, regularity and kernel estimates, Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2004.
4. N.J.Kalton, L.Weis: The H^∞-calculus and sums of closed operators, Math. Ann.321 (2) (2001) 391-345.
5. P. Kunstmann, L. Weis: Maximal L_p–regularity for parabolic equations, Fourier multiplier theorems and $H^\infty -$ functional calculus, vol.1855, Springer BBerlin, 2004, pp.65-312.
6. M.Haase: A transference principle for general groups and functional calculus on UMD spaces. Math.Ann.Ann.345(2)(2009) 245-265.
7. Simon Joseph, Ahmed Syfyan, Hala Taha: and Ranya Tahir: TransferencePrincipales for the series of Semigroups with a Theorem of Peller. Scientific Research Publishing, Advance of Pure Mathematics, 9(2) (2019). http://doi.org/10.4236/apm.2019.92009
8. M.Haase, J.Rozendaal: Functional calculus for the of semigroup generators via transference. J. Funct. Anal. 265 (12) (2013) 3345-3368. http://dx.doi.org/10.1016/j.jfa.2013.08.019
9. Nigel Kalton: Lutz Weis, The H^∞- functional calculus and square function estimate. un published manuscript, 2004.
10. H.Zwart: Toeplitz operators and $\mathcal{H}^\infty$-calculus, J.Funct.Anal.263 (1) (2012) 167-182 .
11. F.L.Schwenninger, H.Zwart: Weakly admissible $\mathcal{H}^-_\infty$-calculus on reflexive Banach spaces, Indag. Math. (N.S) 23 (4) (2012) 796-815.
12. W. Arendt, CharlesJ.K.Batty, Matthias Hieber, and Frank Neubrander: Vector Valued - LaplaceTransforms and Cauchy Problems. Springer Monogr. Math., vol. 96, Birkhauser/Springer Basel AG, Basel, 2011.

13. Jan van Neervenγ-Radoniifying operators – A Survey in: The AMSI – ANU Workshop on spectral Theory and Harmonic Analysis, in: Proc. Centre Math. Appl. Austral. Nat.Univ., vol.44, Austral. Nat. Univ, Canberra, 2010, pp.1-61.
14. D.I.H.Garling: Inequilities: A journey in to linear Analysis. Combridge University Press, Combridge, 2007.

Notes

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Fellows

FELLOW OF ASSOCIATION OF RESEARCH SOCIETY IN SCIENCE (FARSS)

Global Journals Incorporate (USA) is accredited by Open Association of Research Society (OARS), U.S.A and in turn, awards "FARSS" title to individuals. The 'FARSS' title is accorded to a selected professional after the approval of the Editor-in-Chief/Editorial Board Members/Dean.

- The "FARSS" is a dignified title which is accorded to a person's name viz. Dr. John E. Hall, Ph.D., FARSS or William Walldroff, M.S., FARSS.

FARSS accrediting is an honor. It authenticates your research activities. After recognition as FARSB, you can add 'FARSS' title with your name as you use this recognition as additional suffix to your status. This will definitely enhance and add more value and repute to your name. You may use it on your professional Counseling Materials such as CV, Resume, and Visiting Card etc.

The following benefits can be availed by you only for next three years from the date of certification:

FARSS designated members are entitled to avail a 40% discount while publishing their research papers (of a single author) with Global Journals Incorporation (USA), if the same is accepted by Editorial Board/Peer Reviewers. If you are a main author or co-author in case of multiple authors, you will be entitled to avail discount of 10%.

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We shall provide you intimation regarding launching of e-version of journal of your stream time to time.This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.

The MARSS member can apply for approval, grading and certification of standards of their educational and Institutional Degrees to Open Association of Research, Society U.S.A.

Once you are designated as MARSS, you may send us a scanned copy of all of your credentials. OARS will verify, grade and certify them. This will be based on your academic records, quality of research papers published by you, and some more criteria.

It is mandatory to read all terms and conditions carefully.

AUXILIARY MEMBERSHIPS

Institutional Fellow of Global Journals Incorporation (USA)-OARS (USA)

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The IFOARS institution is entitled to form a Board comprised of one Chairperson and three to five board members preferably from different streams. The Board will be recognized as "Institutional Board of Open Association of Research Society"-(IBOARS).

The Institute will be entitled to following benefits:

The IBOARS can initially review research papers of their institute and recommend them to publish with respective journal of Global Journals. It can also review the papers of other institutions after obtaining our consent. The second review will be done by peer reviewer of Global Journals Incorporation (USA) The Board is at liberty to appoint a peer reviewer with the approval of chairperson after consulting us.

The author fees of such paper may be waived off up to 40%.

The Global Journals Incorporation (USA) at its discretion can also refer double blind peer reviewed paper at their end to the board for the verification and to get recommendation for final stage of acceptance of publication.

The IBOARS can organize symposium/seminar/conference in their country on behalf of Global Journals Incorporation (USA)-OARS (USA). The terms and conditions can be discussed separately.

The Board can also play vital role by exploring and giving valuable suggestions regarding the Standards of "Open Association of Research Society, U.S.A (OARS)" so that proper amendment can take place for the benefit of entire research community. We shall provide details of particular standard only on receipt of request from the Board.

The board members can also join us as Individual Fellow with 40% discount on total fees applicable to Individual Fellow. They will be entitled to avail all the benefits as declared. Please visit Individual Fellow-sub menu of GlobalJournals.org to have more relevant details.

We shall provide you intimation regarding launching of e-version of journal of your stream time to time. This may be utilized in your library for the enrichment of knowledge of your students as well as it can also be helpful for the concerned faculty members.

After nomination of your institution as "Institutional Fellow" and constantly functioning successfully for one year, we can consider giving recognition to your institute to function as Regional/Zonal office on our behalf.
The board can also take up the additional allied activities for betterment after our consultation.

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Open Association of Research Society, U.S.A (OARS) By-laws states that an individual Fellow may use the designations as applicable, or the corresponding initials. The Credentials of individual Fellow and Associate designations signify that the individual has gained knowledge of the fundamental concepts. One is magnanimous and proficient in an expertise course covering the professional code of conduct, and follows recognized standards of practice.

Open Association of Research Society (US)/ Global Journals Incorporation (USA), as described in Corporate Statements, are educational, research publishing and professional membership organizations. Achieving our individual Fellow or Associate status is based mainly on meeting stated educational research requirements.

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We shall provide print version of 12 issues of any three journals [as per your requirement] out of our 38 journals worth $ 2376 USD.

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The individual Fellow and Associate designations accredited by Open Association of Research Society (US) credentials signify guarantees following achievements:

- The professional accredited with Fellow honor, is entitled to various benefits viz. name, fame, honor, regular flow of income, secured bright future, social status etc.

- In addition to above, if one is single author, then entitled to 40% discount on publishing research paper and can get 10%discount if one is co-author or main author among group of authors.
- The Fellow can organize symposium/seminar/conference on behalf of Global Journals Incorporation (USA) and he/she can also attend the same organized by other institutes on behalf of Global Journals.
- The Fellow can become member of Editorial Board Member after completing 3yrs.
- The Fellow can earn 60% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.
- Fellow can also join as paid peer reviewer and earn 15% remuneration of author charges and can also get an opportunity to join as member of the Editorial Board of Global Journals Incorporation (USA)
- • This individual has learned the basic methods of applying those concepts and techniques to common challenging situations. This individual has further demonstrated an in–depth understanding of the application of suitable techniques to a particular area of research practice.

Note :

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- In future, if the board feels the necessity to change any board member, the same can be done with the consent of the chairperson along with anyone board member without our approval.
- In case, the chairperson needs to be replaced then consent of 2/3rd board members are required and they are also required to jointly pass the resolution copy of which should be sent to us. In such case, it will be compulsory to obtain our approval before replacement.
- In case of "Difference of Opinion [if any]" among the Board members, our decision will be final and binding to everyone.

"

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We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

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2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
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Acknowledgments

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Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
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- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

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The full postal address of any related author(s) must be specified.

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The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

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1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

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3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

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12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

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Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- Single section and succinct.
- An outline of the job done is always written in past tense.
- Concentrate on shortening results—limit background information to a verdict or two.
- Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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