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A New Approach to Quantum Gravity

By Gang Lee

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I. Introduction

To solve the quantization in the gravitational field, we introduce a different approach to the theory of the gravitational field. This method can give the semiclassical graviton directly. We discuss the dynamics and quantization of graviton and obtain the field equation of graviton. The quantum field theory constructed in this paper is classically equivalent to the general theory of relativity. We obtain the Green's function of graviton; it can solve the difficulty of the Feynman integral divergence over large virtual momenta.

In section 2, there are some mathematical preparations. We give a brief description of noncommutative lattices. And then we introduce a very different approach to the theory of the noncommutative gravitational field. In section 3, we discuss the dynamics and quantization of the gravitons. In section 4, we compute the energy-momentum tensor of the gravitational field itself in two completely different ways: the method of the quantum field theory and the method of the general theory of relativity. In section 5, we obtain the Green's function of the graviton, the resulting Feynman rule can solve the difficulty of divergence of the Feynman integral over large virtual momenta.

II. Gravity on Noncommutative Lattices

The traditional arena of geometry is a set of points with some particular structure that, for want of a better name, we call space. In noncommutative geometry, under the influence of quantum physics, this general idea of replacing sets of points by classes of functions is taken further. In this way, we find that noncommutative lattices are the best tool to solve the contradiction between the general theory of relativity and the uncertainty principle. The most important result of this approach is to solve the difficulty of divergence of Feynman integral over large virtual momenta of the graviton.

Let's first introduce a simple model for future reference while referring to [1] and [2] for more details. Let $S^1 = \{0 \leq \phi \leq 2\pi, \mod 2\pi\}$, the detectors are U_1, U_2, U_3 as follows

$$U_1 = (-\frac{1}{3}\pi, \frac{2}{3}\pi),\ U_2 = (\frac{1}{3}\pi, \frac{4}{3}\pi),\ U_3 = (\pi, 2\pi) \tag{2.1}$$

The detectors can't distinguish any two points in their area. If two detectors, U_1 and U_2 say, are on, we will know that the particles are in the intersection $U_1 \cap U_2$, although we will be unable to distinguish any two points in this intersection. So we are forced to identify the points which cannot be distinguished and S^1 will be represented by a collection of six points $P = \{\alpha, \beta, \gamma, a, b, c\}$ which correspond to the following identifications

Author: *e-mail: ganglee69@msn.com*

$$\begin{aligned}
\alpha &\rightarrow U_1 \cap U_3 = \{\frac{5}{3}\pi < \phi < 2\pi\} \\
\beta &\rightarrow U_1 \cap U_2 = \{\frac{1}{3}\pi < \phi < \frac{2}{3}\pi\} \\
\gamma &\rightarrow U_2 \cap U_3 = \{\pi < \phi < \frac{4}{3}\pi\} \\
a &\rightarrow U_1 \setminus \{(U_1 \cap U_2) \cup (U_1 \cap U_3)\} = \{0 \leq \phi \leq \frac{1}{3}\pi\} \\
b &\rightarrow U_2 \setminus \{(U_2 \cap U_1) \cup (U_2 \cap U_3)\} = \{\frac{2}{3}\pi \leq \phi \leq \pi\} \\
c &\rightarrow U_3 \setminus \{(U_3 \cap U_2) \cup (U_3 \cap U_1)\} = \{\frac{4}{3}\pi \leq \phi \leq \frac{5}{3}\pi\}
\end{aligned} \tag{2.2}$$

By these six points we get the noncommutative lattices $P_6(S^1)$. For the 1-dimensional real axis $\mathbb{R}$, the noncommutative lattices is $P_\infty(\mathbb{R})$. The unit of the noncommutative lattices is the $\vee$ poset. Please refer to section 3 of [2] for details of noncommutative lattices such as the $\vee$ poset and Hasse diagram.

The uncertainty principle is $\triangle r \triangle p \sim \hbar$, $\triangle t \triangle E \sim \hbar$; it gives an area with radius $(\triangle r, \triangle t)$. Due to the limitation of the uncertainty principle, the detector can only say in or out of this area. If in, the detector can't say at which point in this area. In mathematics, the detector can't distinguish different points in this area.

From the vierbein formalism of the gravitational field in the general theory of relativity[4], we know that the locally inertial coordinate system spans the cotangent space of the gravitational field, which means that the local inertial coordinate system is related to energy-momentum from the operator representation in quantum theory. Therefore, we assume that the limit of measurement for any spacetime point in the inertial coordinate system is limited by the uncertainty principle, so that in the inertial coordinate system we will be unable to distinguish different points in the area with the radius $(\triangle r, \triangle t)$. Considering the quantum fluctuation of spacetime in the Planck scale, let's try to take this radius equal to the Planck length l_P and the Planck time t_P: $(\triangle r, \triangle t) = (l_P, t_P)$.

Based on the above assumptions, let's first study the 1-dimensional case. Let x be the coordinate of the 1-dimensional spacetime, a and b are two points of the spacetime. Let spacetime be flat; then we can choose an inertial coordinate system, write it as $\xi(x)$. Similar to the method in quantum theory, we denote the operator of the inertial coordinate system of the 1-dimensional spacetime as $\hat{\xi}$. In the absence of the influence of the uncertainty principle, the operator $\hat{\xi}$ act on the spacetime point. At any point X, the value $\xi(X)$ is the eigenvalue of $\hat{\xi}$

$$\begin{aligned}
\lambda(\hat{\xi})\big|_X &= \xi(x)\big|_{x=X} \\
&= \xi(X)
\end{aligned} \tag{2.3}$$

Let's take the uncertainty principle into the inertial coordinate system. The inertial coordinate system spans the cotangent space, then it relates to energy-momentum. After introducing the uncertainty principle, the measurement for any point in the inertial coordinate system $\xi(x)$ will be limited by the uncertainty principle. Then the value $\xi(a)$ can't correspond accurately to the point a; it corresponds to the entire line segment with length l_P. The analogy with the mathematical model discussed earlier, $\xi(a)$ is similar to the detectors U_1, U_2, U_3. Therefore, due to the effect of the uncertainty principle, the 1-dimensional space spanned by the inertial coordinate system becomes the noncommutative lattices.

The unit of the noncommutative lattices is the $\vee$ poset. Therefore the inertial coordinate system must be based on each $\vee$ poset. It means that the operator

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$\hat{\xi}$ act on $\vee$ posets, the eigenvalue of $\hat{\xi}$ corresponds to $\vee$ posets. We can attach $\xi(a)$ and $\xi(b)$ to each arm of the $\vee$ poset, then on the $\vee$ poset, the eigenvalue of $\hat{\xi}$ is

$$\lambda(\hat{\xi})\big|_{\vee} = \xi(a) \text{ or } \xi(b) \tag{2.4}$$

This relation can be interpreted as: in the interval (a, b), two detectors $\xi(a)$ and $\xi(b)$ say, are on.

In this 1-dimensional spacetime limited by the uncertainty principle, given a point a, for another point b, if the distance between a and b less than l_P, we will be unable to distinguish these two points in the inertial coordinate system $\xi(x)$, or it could be interpreted that we will be unable to distinguish $\xi(a)$ and $\xi(b)$ if the distance between a and b less than l_P. Therefore the distance between a and b is equal to l_P if $\xi(a)$ and $\xi(b)$ are neighboring. Note that the Planck length l_P is the quantity in the laboratory coordinate x. Write l_P as l'_P in the inertial coordinate system $\xi(x)$. According to the scale transformation between different coordinate systems, we have

$$\frac{l'_P}{l_P} = \frac{d\xi(x)}{dx} \tag{2.5}$$

In the laboratory coordinate x, we have

$$b = a \pm l_P \tag{2.6}$$

Then in the inertial coordinate system $\xi(x)$, we have

$$\begin{aligned} \xi(b) &= \xi(a) \pm l'_P \\ &= \xi(a) \pm l_P \cdot \frac{d\xi(x)}{dx}\bigg|_{x=a} \end{aligned} \tag{2.7}$$

The operator $\hat{\xi}$ act on $\vee$ posets, by Eq.(2.4) and Eq.(2.7), at any point X, we can obtain the general expression of the eigenvalue of $\hat{\xi}$ as follows

$$\lambda(\hat{\xi}) = \xi(x)\big|_{x=X} \pm l_P \cdot \frac{d\xi(x)}{dx}\bigg|_{x=X} \tag{2.8}$$

where l_P is equal to 0 or l_P.

$\hat{\xi}$ is the operator of the inertial coordinate system, the eigenvalue $\xi(x)$ is the inertial coordinate system of spacetime without the uncertainty principle. Of course, there is no gravitational effect in the inertial coordinate system $\xi(x)$. After introducing the uncertainty principle, spacetime becomes noncommutative lattices, the operator $\hat{\xi}$ act on $\vee$ posets, then the eigenvalue should also be base on $\vee$ posets instead of points, thus from Eq.(2.8) we can see that the eigenvalue of $\hat{\xi}$ is no longer the former inertial coordinate system $\xi(x)$. It means that the former inertial coordinate system $\xi(x)$ which is not influenced by the uncertainty principle is no longer the inertial coordinate system of the noncommutative lattices and we know that there is no gravitational effect in $\xi(x)$, therefore the coordinate system $\xi(x)$ can only be interpreted as the inertial coordinate system at point X, that is, the locally inertial coordinate system $\xi_X(x)$.

Eq.(2.8) can be understood in this way: After introducing the uncertainty principle, space spanned by the inertial coordinate system becomes noncommutative lattices, spacetime as noncommutative lattices is flat. The operator $\hat{\xi}$ no longer acts on points of spacetime; it act on $\vee$ posets. The eigenvalue of $\hat{\xi}$, that is, the inertial coordinate system should also be interpreted as that it is based on $\vee$ posets, then the argument of the global inertial coordinate system of noncommutative lattices must be $\vee$ poset. Therefore the former inertial coordinate system $\xi(x)$ is no longer the global inertial coordinate system of the noncommutative lattices, it becomes the locally inertial coordinate system $\xi_X(x)$, so that the gravitational forces appear in the sense of the principle of equivalence. The space made up of $\vee$ posets is flat, the space made up of points like X can be interpreted as curved.

Now let's discuss the 4-dimensional case. After introducing the uncertainty principle, spacetime becomes noncommutative lattices. Let this noncommutative lattices be Minkowski space. In the 4-dimensional Cartesian rectangular coordinate system x^μ, the Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(+,+,+,-)$.

It is convenient to replace the Cartesian coordinate system x^μ with the spherical polar coordinate system $r^i = (r, \theta, \phi, t)$, defined as usual by

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta \\ x^4 &= t \end{aligned} \tag{2.9}$$

where $r \in [0, +\infty),\ \theta \in [0, \pi],\ \phi \in [0, 2\pi),\ t \in (-\infty, +\infty)$.

In the spherical polar coordinate system r^i, the Minkowski metric is

$$\eta_{ij} = \begin{pmatrix} 1 & & & \\ & r^2 & & \\ & & r^2\sin^2\theta & \\ & & & -1 \end{pmatrix} \tag{2.10}$$

Denote $\hat{\xi}$ as the operator of the inertial coordinate system of Minkowski space. In the coordinate system r^i, the components of $\hat{\xi}$ are

$$\lambda(\hat{\xi}) = \left(\lambda(\hat{\xi}^r), \lambda(\hat{\xi}^\theta), \lambda(\hat{\xi}^\phi), \lambda(\hat{\xi}^t)\right) \tag{2.11}$$

After introducing the uncertainty principle, by Eq.(2.8), we can get the eigenvalue $\lambda(\hat{\xi})$ as follows

$$\begin{aligned} \lambda(\hat{\xi}^r) &= \xi^r + l_P \cdot \frac{\partial \xi^r}{\partial r} \\ \lambda(\hat{\xi}^\theta) &= \xi^\theta \\ \lambda(\hat{\xi}^\phi) &= \xi^\phi \\ \lambda(\hat{\xi}^t) &= \xi^t \pm t_P \cdot \frac{\partial \xi^t}{\partial t} \end{aligned} \tag{2.12}$$

where $(\xi^r, \xi^\theta, \xi^\phi, \xi^t)$ are the components of the locally inertial coordinate system at the origin of r^i.

Because we let the spacetime as noncommutative lattices be Minkowski space, we get the following equations

$$\begin{cases} \lambda(\hat{\xi^r}) = \xi^r + l_P \cdot \dfrac{\partial \xi^r}{\partial r} = r \\ \lambda(\hat{\xi^\theta}) = \xi^\theta = \theta \\ \lambda(\hat{\xi^\phi}) = \xi^\phi = \phi \\ \lambda(\hat{\xi^t}) = \xi^t \pm t_P \cdot \dfrac{\partial \xi^t}{\partial t} = t \end{cases} \tag{2.13}$$

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The solution is

$$\xi^i = \begin{cases} \xi^r = (r - l_P) + C_r \exp(-\dfrac{r}{l_P}) + C'_r \\ \\ \xi^\theta = \theta + C'_\theta \\ \\ \xi^\phi = \phi + C'_\phi \\ \\ \xi^t = (t - t_P) + C_t \exp(-\dfrac{|t|}{t_P}) + C'_t \end{cases} \tag{2.14}$$

where C_r, C_t are the integral constants, $C'_r, C'_\theta, C'_\phi, C'_t$ are the arbitrary constants.

Because we always take the derivative of ξ^i, we can omit the constants $C'_r, C'_\theta, C'_\phi, C'_t$ for brevity. Then the solution (2.14) can be written as follows

$$\xi^i = \begin{cases} \xi^r = r + C_r \exp(-\dfrac{r}{l_P}) \\ \\ \xi^\theta = \theta \\ \\ \xi^\phi = \phi \\ \\ \xi^t = t + C_t \exp(-\dfrac{|t|}{t_P}) \end{cases} \tag{2.15}$$

We will see in the next section that since the integral constants C_r, C_t as the variables of motion must satisfy the field equation, in this paper, the decomposition of spacetime in 3+1 dimensional does not break Lorentz invariance.

If $C = \dfrac{1}{2\triangle x}$, the function $C\exp(-\dfrac{|x|}{\triangle x})$ be the Dirac δ-function while $\triangle x \to 0$. Because the Planck length and the Planck time (l_P, t_P) are very small quantities, the solution (2.15) approximate to the Dirac δ-function, therefore it can be explained as a particle. Eq.(2.15) means the locally inertial coordinate system at the origin of r^i, so that this particle can be interpreted as a semiclassical graviton at the origin of r^i.

This section can be understood in this way: The spacetime made up of physical points limited by the uncertainty principle is noncommutative lattices, it is the space of the gravitons, and it is flat. If we depend on traditional differential geometry to explain the spacetime limited by the uncertainty principle, the 4-dimensional space made up of mathematical points can be interpreted as curved, it is the space of gravitational effect because the gravitational forces appear in the sense of the principle of equivalence.

III. Dynamics and Quantization of Graviton

Let's discuss the dynamics of a graviton. The spherical polar coordinate system r^i is the local coordinate system of the graviton, denote τ as the parameter of the trajectory of the origin of r^i, then the moving graviton can be written as $\xi^i(\tau, r)$. On every point of the path τ, we set the coordinate r^i_τ, where the index τ in r^i_τ indicates that the origin of r^i is at the point τ. In the case of no doubt, we can omit the index τ. The variable of motion are the integral constants and the phase angles: $\big(C_r, \theta, \phi, C_t\big)$. On the path τ, they become the functions of τ: $\big(C_r, \theta, \phi, C_t\big) \to \big(C_r(\tau), \theta(\tau), \phi(\tau), C_t(\tau)\big)$.

If a graviton is excited at the point τ, the locally inertial coordinate system ξ^i can be written as follows

$$\xi^i(\tau, r) = \begin{cases} \xi^r = r + C_r(\tau)\exp(-\dfrac{r}{l_P}) \\ \xi^\theta = \theta(\tau) \\ \xi^\phi = \phi(\tau) \\ \xi^t = t + C_t(\tau)\exp(-\dfrac{|t|}{t_P}) \end{cases} \tag{3.1}$$

The natural action relate to the path of graviton as it propagates through spacetime, defined by[3]

$$\begin{aligned} S &= m \int ds \\ &= m \int d\tau \sqrt{-\eta_{ij}\frac{\partial \xi^i}{\partial \tau}\frac{\partial \xi^j}{\partial \tau}} \\ &\equiv m \int d\tau \sqrt{-\eta_{ij}\dot{\xi}^i\dot{\xi}^j} \end{aligned} \tag{3.2}$$

The Lagrangian can be written as follows

$$\begin{aligned} \mathscr{L} &= m \cdot \sqrt{-\eta_{ij}\dot{\xi}^i\dot{\xi}^j} \\ &= m \cdot \sqrt{-\dot{\xi}^i\dot{\xi}_i} \\ &\equiv m \cdot \sqrt{-\dot{\xi}^2} \end{aligned} \tag{3.3}$$

The Lagrange equation is

$$\partial_\tau \left(\frac{\partial \mathscr{L}}{\partial \dot{\xi}^i}\right) = \frac{\partial \mathscr{L}}{\partial \xi^i} \tag{3.4}$$

For this Lagrangian,

$$\frac{\partial \mathscr{L}}{\partial \xi^i} = 0 \tag{3.5}$$

Then the equation of motion can be written as follows

$$\partial_\tau \left(\frac{\partial \mathscr{L}}{\partial \dot{\xi}^i}\right) = \partial_\tau \left(m \cdot \frac{\dot{\xi}^i}{\sqrt{-\dot{\xi}^2}}\right) = 0 \tag{3.6}$$

The action is dimensionless. Although a graviton has the 4-dimensional extension structure in the coordinates r^i, τ is the trajectory of the origin of r^i, which is 1-dimensional in spacetime. In this sense, the graviton is similar to the vector particle. So the dimensional analysis of the action (3.2) shows that the factor m has the dimension of mass, which can be interpreted as the mass. It is very different from string theory. In string theory, the motion of a string related to the motion of each particle on the string and thus relate to the surface area of the "world-sheet" swept by the string, then the factor m has the dimension of the square of mass. Note that in the case of one graviton, we can use only one coordinate system coordinate system r^i, but in the case of more than one graviton or in quantum field theory, it is necessary to describe gravitons by using two coordinate systems r^i and x^μ at the same time, so that the field of graviton has one more index than the field of vector particle, more like the field of tensor particle.

In the case of the graviton, we know that $m = 0$. The Lagrangian (3.3) is ill-defined for the massless particle, so that we should choose another action which is classical equal to the action (3.2) for the graviton.

Take the auxiliary variable $e(\tau)$ as an einbein on the path τ. The associated metric is $g_{\tau\tau} = e^2$, $g^{\tau\tau} = e^{-2}$. Then we can take an equivalent action as follows

$$\begin{aligned} S &= -\frac{1}{2}\int d\tau \sqrt{g_{\tau\tau}}\left(g^{\tau\tau}\dot{\xi}^i\dot{\xi}^j\eta_{ij} - m^2\right) \\ &= -\frac{1}{2}\int d\tau\, e\left(\frac{\dot{\xi}^2}{e^2} - m^2\right) \end{aligned} \tag{3.7}$$

Varying $e(\tau)$ in action (3.7)

$$\delta S = \frac{1}{2}\int d\tau \left(\frac{\dot{\xi}^2}{e^2} + m^2\right)\delta e \tag{3.8}$$

Setting $\delta S = 0$, we obtain the equation of motion for $e(\tau)$

$$\frac{\dot{\xi}^2}{e^2} + m^2 = 0 \tag{3.9}$$

By this equation of motion, we can obtain

$$e = \frac{\sqrt{-\dot{\xi}^2}}{m} \tag{3.10}$$

Varying ξ^i in the action (3.7)

$$\delta S = \frac{1}{2}\int d\tau \left(\frac{2\dot{\xi}^i}{e}\right)\partial_\tau \delta\xi^i \tag{3.11}$$

After partial integration, we obtain the equation of motion

$$\partial_\tau\left(e^{-1}\dot{\xi}^i\right) = 0 \tag{3.12}$$

Substituting Eq.(3.10) into Eq.(3.7), we obtain Eq.(3.2), Substituting Eq.(3.10) into Eq.(3.12), we obtain Eq.(3.6). Then we have proved that the action (3.7) is classical equal to the action (3.2).

The equation of motion (3.12) can be written as follows

$$\partial_\tau\dot{\xi}^i - e^{-1}\dot{e}\dot{\xi}^i = 0 \tag{3.13}$$

It is the equation of the geodetic line. Because the space of graviton is noncommutative lattices and we let the noncommutative lattices be Minkowski space, then this equation can be written as follows

$$\partial^\tau\partial_\tau\xi^i = 0 \tag{3.14}$$

It is a wave equation.

The parameter τ is the parameter of a free graviton's trajectory. For the quantum field theory, we can set the orthogonal coordinate system x^μ to indicate each excited graviton, so that the parameter τ should be replaced by the position x: $\tau \to x$, and the coordinate system becomes $r^i_\tau \to r^i_x$, where the index x in r^i_x indicates that the origin of r^i is at the point x. In case of no doubt, we can omit the index x. Then $\xi^i(x)$ can be written as follows

$$\xi^i(x,r) = \begin{cases} \xi^r = r + C_r(x)\exp(-\frac{r}{l_P}) \\ \xi^\theta = \theta(x) \\ \xi^\phi = \phi(x) \\ \xi^t = t + C_t(x)\exp(-\frac{|t|}{t_P}) \end{cases} \tag{3.15}$$

The free field equation is

$$\partial^\mu \partial_\mu \xi^i = 0 \tag{3.16}$$

The solution is

$$\xi^i(x,r) = \begin{cases} \xi^r = r + \langle x|C_r\rangle\exp(-\frac{r}{l_P}) \\ \xi^\theta = \langle x|\theta\rangle \\ \xi^\phi = \langle x|\phi\rangle \\ \xi^t = t + \langle x|C_t\rangle\exp(-\frac{|t|}{t_P}) \end{cases} \tag{3.17}$$

where

$$\begin{aligned} \langle x|C_r\rangle &= \int d^4k \Big(C_r(k)\exp(ikx) + C_r^*(k)\exp(-ikx)\Big) \\ \langle x|\theta\rangle &= \int d^4k \Big(\theta(k)\exp(ikx) + \theta^*(k)\exp(-ikx)\Big) \\ \langle x|\phi\rangle &= \int d^4k \Big(\phi(k)\exp(ikx) + \phi^*(k)\exp(-ikx)\Big) \\ \langle x|C_t\rangle &= \int d^4k \Big(C_t(k)\exp(ikx) + C_t^*(k)\exp(-ikx)\Big) \end{aligned} \tag{3.18}$$

The variables of motion should be quantized. The commutators are as follows

$$\begin{aligned} &\big[C_r(k), C_r^*(k')\big] = \delta(k-k') \\ &\big[\theta(k), \theta^*(k')\big] = \delta(k-k') \\ &\big[\phi(k), \phi^*(k')\big] = \delta(k-k') \\ &\big[C_t(k), C_t^*(k')\big] = \delta(k-k') \end{aligned} \tag{3.19}$$

All other commutators are equal to 0.

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IV. Energy-Momentum of Gravitational Field Itself

In the function of $\xi^i(x,r)$, the variable of motion is x, not r, then the Lagrangian density of gravitational field is

$$\mathscr{L} = -\frac{\eta^{\mu\nu}}{2}\frac{\partial\xi^i(x,r)}{\partial x^\mu}\frac{\partial\xi^j(x,r)}{\partial x^\nu}\eta_{ij} \tag{4.1}$$

Note that in the case of graviton, the mass term vanishing.

The canonical momentum conjugate to ξ^i is

$$\Pi_i = -\frac{\partial\mathscr{L}}{\partial\dot{\xi}^i} = \dot{\xi}_i \tag{4.2}$$

The momentum is

$$P = \Pi_i \cdot \bigtriangledown\xi^i = \dot{\xi}_i \cdot \bigtriangledown\xi^i \tag{4.3}$$

The canonical Hamiltonian is given by

$$\mathscr{H} = \Pi_i\dot{\xi}^i - \mathscr{L} = \dot{\xi}_i\dot{\xi}^i - \mathscr{L} \tag{4.4}$$

The energy-momentum tensor is

$$\begin{aligned} T_{\mu\nu} &= \eta_{\mu\nu}\mathscr{L} - \frac{\partial\mathscr{L}}{\partial(\partial^\mu\xi^i)}\partial_\nu\xi^i \\ &= -\frac{\eta_{\mu\nu}}{2}\partial^\lambda\xi^i\partial_\lambda\xi^j\eta_{ij} + \partial_\mu\xi^i\partial_\nu\xi^j\eta_{ij} \end{aligned} \tag{4.5}$$

It is the energy-momentum tensor of the gravitational field itself in the quantum field theory.

The 4-dimensional momentum is

$$\begin{aligned} p_\nu = T_{4\nu} &= -\frac{\eta_{4\nu}}{2}\partial^\lambda\xi^i\partial_\lambda\xi_i + \dot{\xi}^i\partial_\nu\xi_i \\ &= (P, \mathscr{H}) \end{aligned} \tag{4.6}$$

Now we compute the energy-momentum tensor of the gravitational field itself in the general theory of relativity[4].

The metric is

$$g_{\mu\nu} = \frac{\partial\xi^i}{\partial x^\mu}\frac{\partial\xi_i}{\partial x^\nu} \tag{4.7}$$

We write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.8}$$

By Eq.(3.15) we can see that $h_{\mu\nu}$ vanishes at infinity.

The part of the Ricci tensor $R_{\mu\nu}$ linear in $h_{\mu\nu}$ is[5]

$$R_{\mu\nu}^{(1)} \equiv \frac{1}{2}\left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\mu \partial x^\lambda} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda}\right) \tag{4.9}$$

where we adopting the convenient convention that the indices on $h_{\mu\nu}$, $R_{\mu\nu}^{(1)}$ and $\partial/\partial x^\lambda$ are raised and lowered with η's, for example, $h^\lambda{}_\lambda \equiv \eta^{\lambda\kappa} h_{\lambda\kappa}$, $\partial/\partial x_\lambda \equiv \eta^{\lambda\kappa}\partial/\partial x^\kappa$, $R^{(1)} \equiv \eta^{\mu\nu} R_{\mu\nu}^{(1)}$, whereas indices on true tensors such as $g_{\mu\nu}$ are raised and lowered with g's as usual.

Then $R_{\mu\nu}^{(1)}$ can be written as follows

$$\begin{aligned}
R_{\mu\nu}^{(1)} \equiv& \frac{1}{2}\left(\frac{\partial^2 h^\lambda{}_\lambda}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h^\lambda{}_\mu}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h^\lambda{}_\nu}{\partial x^\mu \partial x^\lambda} + \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x_\lambda}\right) \\
=& \frac{1}{2}\left(\frac{\partial^2\left(\frac{\partial \xi^i}{\partial x^\lambda}\frac{\partial \xi_i}{\partial x_\lambda}\right)}{\partial x^\mu \partial x^\nu} - \frac{\partial^2\left(\frac{\partial \xi^i}{\partial x^\mu}\frac{\partial \xi_i}{\partial x_\lambda}\right)}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2\left(\frac{\partial \xi^i}{\partial x^\nu}\frac{\partial \xi_i}{\partial x_\lambda}\right)}{\partial x^\mu \partial x^\lambda} + \frac{\partial^2\left(\frac{\partial \xi^i}{\partial x^\mu}\frac{\partial \xi_i}{\partial x^\nu}\right)}{\partial x^\lambda \partial x_\lambda}\right) \\
=& \frac{1}{2}\left(\frac{\partial^3 \xi^i}{\partial x^\mu \partial x^\nu \partial x^\lambda}\frac{\partial \xi_i}{\partial x_\lambda} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\lambda}\frac{\partial^2 \xi_i}{\partial x^\nu \partial x_\lambda} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x_\lambda}\frac{\partial^2 \xi_i}{\partial x^\nu \partial x^\lambda} + \frac{\partial \xi^i}{\partial x^\lambda}\frac{\partial^3 \xi_i}{\partial x^\mu \partial x^\nu \partial x_\lambda}\right) \\
&- \frac{1}{2}\left(\frac{\partial^3 \xi^i}{\partial x^\mu \partial x^\nu \partial x^\lambda}\frac{\partial \xi_i}{\partial x_\lambda} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\nu}\frac{\partial^2 \xi_i}{\partial x^\lambda \partial x_\lambda} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\lambda}\frac{\partial^2 \xi_i}{\partial x^\nu \partial x_\lambda} + \frac{\partial \xi^i}{\partial x^\mu}\frac{\partial^3 \xi_i}{\partial x^\nu \partial x^\lambda \partial x_\lambda}\right) \\
&- \frac{1}{2}\left(\frac{\partial^3 \xi^i}{\partial x^\mu \partial x^\nu \partial x^\lambda}\frac{\partial \xi_i}{\partial x_\lambda} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\nu}\frac{\partial^2 \xi_i}{\partial x^\lambda \partial x_\lambda} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x_\lambda}\frac{\partial^2 \xi_i}{\partial x^\nu \partial x^\lambda} + \frac{\partial \xi^i}{\partial x^\nu}\frac{\partial^3 \xi_i}{\partial x^\mu \partial x^\lambda \partial x_\lambda}\right) \\
&+ \frac{1}{2}\left(\frac{\partial^3 \xi^i}{\partial x^\mu \partial x^\lambda \partial x_\lambda}\frac{\partial \xi_i}{\partial x^\nu} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\lambda}\frac{\partial^2 \xi_i}{\partial x^\nu \partial x_\lambda} + \frac{\partial^2 \xi^i}{\partial x^\mu \partial x_\lambda}\frac{\partial^2 \xi_i}{\partial x^\nu \partial x^\lambda} + \frac{\partial \xi^i}{\partial x^\mu}\frac{\partial^3 \xi_i}{\partial x^\nu \partial x^\lambda \partial x_\lambda}\right)
\end{aligned} \tag{4.10}$$

According to the free field equation (3.16), some of the terms vanishing, and most of the terms cancel, leaving us with

$$R_{\mu\nu}^{(1)} = \frac{\partial^2 \xi^i}{\partial x^\mu \partial x^\lambda}\frac{\partial^2 \xi_i}{\partial x^\nu \partial x_\lambda} \tag{4.11}$$

From the mass-shell constraint $p_\mu p^\mu = 0$, we have

$$\begin{aligned}
p_\mu p^\mu &= PP - \mathscr{H}\mathscr{H} \\
&= \left(\dot{\xi}_i \cdot \triangledown \xi^i\right)\cdot\left(\dot{\xi}_i \cdot \triangledown \xi^i\right) - \left(\dot{\xi}_i\dot{\xi}^i - \mathscr{L}\right)\cdot\left(\dot{\xi}_i\dot{\xi}^i - \mathscr{L}\right) \\
&= 0
\end{aligned} \tag{4.12}$$

Then we can get the mass-shell condition

$$\partial^\lambda \xi^i \partial_\lambda \xi_i = 0 \tag{4.13}$$

By the mass-shell condition (4.13) we have

$$\int d\xi^i d\xi_i = C \tag{4.14}$$

where C is the integral constant.

By Eq.(4.14) we can obtain

$$\frac{\partial\xi^i}{\partial x^\mu}\frac{\partial\xi_i}{\partial x^\nu} = \int \frac{\partial^2\xi^i}{\partial x^\mu\partial x^\lambda}\frac{\partial^2\xi_i}{\partial x^\nu\partial x_\lambda}\,d\xi^i d\xi_i$$
$$= C\cdot\frac{\partial^2\xi^i}{\partial x^\mu\partial x^\lambda}\frac{\partial^2\xi_i}{\partial x^\nu\partial x_\lambda} \tag{4.15}$$

Then Eq.(4.11) can be written as

$$R^{(1)}_{\mu\nu} = \frac{\partial^2\xi^i}{\partial x^\mu\partial x^\lambda}\frac{\partial^2\xi_i}{\partial x^\nu\partial x_\lambda}$$
$$= \frac{1}{C}\cdot\frac{\partial\xi^i}{\partial x^\mu}\frac{\partial\xi_i}{\partial x^\nu} \tag{4.16}$$

and

$$R^{(1)} = \eta_{\mu\nu}R^{(1)}_{\mu\nu} = \frac{1}{C}\cdot\frac{\partial\xi^i}{\partial x^\kappa}\frac{\partial\xi_i}{\partial x_\kappa} \tag{4.17}$$

For empty space, Ricci tensor is vanishing $R_{\mu\nu}=0$. Then in the general theory of relativity, the energy-momentum tensor of gravitational field itself can be written as follows[5]

$$t_{\mu\nu} = \frac{1}{8\pi G}\left(\frac{1}{2}\eta_{\mu\nu}R^{(1)} - R^{(1)}_{\mu\nu}\right)$$
$$= \frac{1}{8\pi G\cdot C}\left(\frac{1}{2}\eta_{\mu\nu}\frac{\partial\xi^i}{\partial x^\kappa}\frac{\partial\xi_i}{\partial x_\kappa} - \frac{\partial\xi^i}{\partial x^\mu}\frac{\partial\xi_i}{\partial x^\nu}\right) \tag{4.18}$$

Finally, we obtain, up to a constant factor, the energy-momentum tensor (4.18) is equal to the energy-momentum tensor (4.5). It is a strong evidence to prove that the quantum field theory constructed in this paper is classically equivalent to the general theory of relativity.

V. The Green's Function of Gravitational Field

According to the above discussion, the fields $C_i(x) = \big(C_r(x), \theta(x), \phi(x), C_t(x)\big)$ can be interpreted as the equivalent field of the gravitational field. Recall Eq.(2.15), the space of the gravitational effect is described by the coordinate system r^i, then the metric and the connection of the space of gravitational effect are the derivatives concerning the variable r rather than x, so that the metric and the connection contains no derivatives of the fields $C_i(x)$. Therefore the gravitational interaction can be regarded as the perturbation from the fields $C_i(x)$.

In the preceding discussion, when we discuss the dynamics and quantization of graviton, we use the orthogonal coordinate system x^μ and the spherical polar coordinate system r^i at the same time. The coordinate system x^μ conjugate to the energy-momentum of the graviton, so that x^μ describe the dynamics of graviton itself. But it is not clear how r^i affects gravitons, to make this clear, we try to use the same coordinate system. Let's set the orthogonal coordinate system X^μ and let it be the same as x^μ, then transform the spherical polar coordinate system r^i to the orthogonal coordinate system X^μ: $r^i \to X^\mu$. To distinguish variables in X^μ and x^μ, we use a capital letter "X" and lowercase letter "x". By Eq.(2.15), we can see that graviton has the ductility in the coordinate system r^i (or X^μ). Therefore a graviton is not a point particle; it is a wave packet approximate to the Dirac δ-function. It is a important property because the ductility of graviton enables us to solve the difficulty of the Feynman integral divergence by the Fourier transform.

Let's discuss the Feynman rules. The purpose is to obtain the Green's function. In the present case, we should obtain the Green's function of the fields $C_i(x)$ in the orthogonal coordinate system x^μ.

In the coordinate system r_x^i, the components of the Planck length and the Planck time at the point x are $\big(l_P, \theta(x), \phi(x), t_P\big)$. In the orthogonal coordinate system x^μ, it can be written as $L_P^\mu(x) = \big(L_P^1(x), L_P^2(x), L_P^3(x), L_P^4(x)\big)$.

where

$$\begin{aligned} L_P^1(x) &= l_P|\sin\theta(x)\cos\phi(x)| \\ L_P^2(x) &= l_P|\sin\theta(x)\sin\phi(x)| \\ L_P^3(x) &= l_P|\cos\theta(x)| \\ L_P^4(x) &= t_P \end{aligned} \tag{5.1}$$

In the orthogonal coordinate system x^μ, the integral constants $C_r(x), C_t(x)$ can be written as $C^\mu(x) = \big(C^1(x), C^2(x), C^3(x), C^4(x)\big)$.

where

$$\begin{aligned} C^1(x) &= C_r(x)\sin\theta(x)\cos\phi(x) \\ C^2(x) &= C_r(x)\sin\theta(x)\sin\phi(x) \\ C^3(x) &= C_r(x)\cos\theta(x) \\ C^4(x) &= C_t(x) \end{aligned} \tag{5.2}$$

Then in the orthogonal coordinate system x^μ and X^μ, the graviton can be written as

$$\xi^\mu(x, X) = X^\mu + C^\mu(x)\exp\!\Big(-\frac{|X^\mu|}{L_P^\mu(x)}\Big) \tag{5.3}$$

Let $j^\mu(x)$ be the source flow in the orthogonal coordinate system x^μ. We can get the field equation in the orthogonal coordinate system x^μ by Eq.(3.16)

$$\Box^2\,\xi^\mu(x, X) = j^\mu(x) \tag{5.4}$$

The D'Alembertian operator acts on the variable x.

Denote $\dfrac{|X^\mu|}{L_P^\mu(x)}$ as $\overline{L}(X)$ for short. The function $e^{-\overline{L}(X)}$ can be interpreted as a function of the variable X because $L_P^\mu(x)$ is just the component-wise manner of the Planck length (l_P, t_P) in the orthogonal coordinate system x^μ. Later we will show that there is no problem with this approach.

Then the field equation (5.4) can be written as

$$e^{-\overline{L}(X)}\Box^2\,C^\mu(x) = j^\mu(x) \tag{5.5}$$

In the scattering theory, Eq.(5.5) can be written as

$$e^{-\overline{L}(X)}\Box^2\,G^\mu(x) = \delta^4(x) \tag{5.6}$$

where $G^\mu(x)$ is the Green's function of the field $C^\mu(x)$

$$C^\mu(x) = C^{(0)}(x) + \int d^4X'\,G^\mu(x - x')j^\mu(x') \tag{5.7}$$

where $C^{(0)}(x)$ satisfying the homogenous equation.
Let $G^\mu(x)$ be

$$G^\mu(x) = \frac{1}{(2\pi)^4}\int d^4k\,\tilde{G}^\mu(k)e^{-ikx}$$
$$\tilde{G}^\mu(k) = \int d^4x\,G^\mu(x)e^{ikx} \tag{5.8}$$

It is the transformation between coordinate space and momentum space

$$|x\rangle = \frac{1}{(2\pi)^4}\int d^4k\,e^{-ikx}\,|k\rangle$$
$$|k\rangle = \int d^4x\,e^{ikx}\,|x\rangle \tag{5.9}$$

Eq.(5.6) can be rewritten as

$$\Box^2\,G^\mu(x) = \delta^4(x)e^{\overline{L}(X)} \tag{5.10}$$

Denote

$$F = F_1 \cdot F_2$$
$$F_1(x) = \delta^4(x), \quad F_2(X) = e^{\overline{L}(X)} \tag{5.11}$$

Then Eq.(5.10) can be rewritten as

$$\Box^2\,G^\mu(x) = F \tag{5.12}$$

The Fourier transform of F is

$$\tilde{F} = \int d^4x\,F_1 \cdot e^{ikx} \cdot \int d^4X\,F_2 \cdot e^{iKX}$$
$$= \delta\Big(K - \frac{i}{L_P^\mu(x)}\Big) \tag{5.13}$$

By the Fourier transform, Eq.(5.12) can be written as follows

$$\Box^2\,\frac{1}{(2\pi)^4}\int d^4k\,\tilde{G}^\mu(k)e^{-ikx} = \frac{1}{(2\pi)^4}\int d^4k\,\tilde{F}\cdot e^{-ikx} \tag{5.14}$$

Since X^μ and x^μ are the same coordinate system, the solution of Eq.(5.14) is

$$\tilde{G}^\mu(k) = -\frac{1}{k^2}\cdot\delta\Big(k - \frac{i}{L^\mu\ \ x}\Big) \tag{5.15}$$

Then

$$G^{\mu}(x) = -\frac{1}{(2\pi)^4}\int d^4k\,\frac{1}{k^2}\cdot\delta\Big(k-\frac{i}{L_P^{\mu}(x)}\Big)\cdot e^{-ikx} \tag{5.16}$$

Retransform the orthogonal coordinate system X^{μ} back to the spherical polar coordinate system r^i, the Green's function in momentum space can be written as

$$\tilde{G}^i(k) = \begin{cases} \tilde{G}^r(k) = -\dfrac{1}{(k^r)^2}\cdot\delta\Big(k^r-\dfrac{i}{l_P}\Big) \\ \tilde{G}^{\theta}(k) = -\dfrac{1}{(k^{\theta})^2} \\ \tilde{G}^{\phi}(k) = -\dfrac{1}{(k^{\phi})^2} \\ \tilde{G}^t(k) = -\dfrac{1}{\omega^2}\cdot\delta\Big(\omega-\dfrac{i}{t_P}\Big) \end{cases} \tag{5.17}$$

where $k^r, k^{\theta}, k^{\phi}, \omega$ are the expressions of the energy-momentum in the spherical polar coordinate system r^i.

To make the physical image intuitive and clear, we use the orthogonal coordinate system x^{μ} and X^{μ}. We can also use the spherical polar coordinate system r^i and R^i, where the coordinate system R^i is the same as r^i. Replace x^{μ} with R^i, replace X^{μ} with r^i. Then the graviton can be written as follows

$$\xi^i(R,r) = \begin{cases} \xi^r = r + C_r(R)\exp(-\dfrac{r}{l_P}) \\ \xi^{\theta} = \theta(R) \\ \xi^{\phi} = \phi(R) \\ \xi^t = t + C_t(R)\exp(-\dfrac{|t|}{t_P}) \end{cases} \tag{5.18}$$

For the source flow $j^{\mu}(R)$, the field equation is

$$\Box^2\,\xi^{\mu}(R,r) = j^{\mu}(R) \tag{5.19}$$

The D'Alembertian operator acts on the variable R.

The scattering equation is

$$\begin{cases} \exp(-\dfrac{r}{l_P})\Box^2\,G^r(R) = \delta^r(R) \\ \Box^2\,G^{\theta}(R) = \delta^{\theta}(R) \\ \Box^2\,G^{\phi}(R) = \delta^{\phi}(R) \\ \exp(-\dfrac{|t|}{t_P})\Box^2\,G^t(R) = \delta^t(R) \end{cases} \tag{5.20}$$

By the Fourier transform we can also obtain the Green's function (5.17). It shows that there is no problem in treating $e^{-\overline{L}(X)}$ as a function of X.

Compare the Green's function (5.17) with the usual Feynman propagator, we can see that the generalized functions $\delta\Big(k^r - \dfrac{i}{l_P}\Big)$ and $\delta\Big(\omega - \dfrac{i}{t_P}\Big)$ give the regularization to the integral over k^r and ω of the usual Feynman propagator.

The Green's function (5.17) is the generalized function, it must be interpreted in the sense of integral. The bounds of integral over k^r is $[0, +\infty)$, the bounds of integral over ω is $(-\infty, +\infty)$, but the integral paths deviate from the real axis into the complex plane and the integral paths pass through singularities $k^r = \dfrac{i}{l_P}$ and $\omega = \dfrac{i}{t_P}$, respectively. According to the properties of the Dirac δ-function, we just need to give singularities on the integral paths without calculating specific integrals when calculating the Feynman diagrams, so that the difficulty of divergence of Feynman integral over large virtual momenta of graviton has been solved.

The bounds of integral over k^θ is $[0, \pi]$, the bounds of integral over k^ϕ is $[0, 2\pi)$. The Feynman diagram's integral over k^θ and k^ϕ is limited.

Note that the variables k^r and ω always appear in the form of squares in Feynman rule, therefore the introduction of the imaginary number i in the generalized function $\delta\Big(k^r - \dfrac{i}{l_P}\Big)$ and $\delta\Big(\omega - \dfrac{i}{t_P}\Big)$ will change sign.

VI. Conclusion

Since the introduction of the uncertainty principle, space spanned by the inertial coordinate system becomes noncommutative lattices, then the inertial coordinate system of the noncommutative lattices becomes the locally inertial coordinate system on the point of spacetime, therefore the gravitational forces appear in the sense of the principle of equivalence. The space made up of physical points limited by the uncertainty principle is the noncommutative lattices, it is the space of graviton and it is flat. If we depend on traditional differential geometry to explain the spacetime limited by the uncertainty principle, the 4-dimensional space made up of mathematical points can be interpreted as curved, it is the space of the gravitational effect. The gravitational field can be interpreted as coming from energy-momentum limited by the uncertainty principle. By introducing gravitational interactions in this way, we can get a wave packet approximate to the Dirac δ-function; it can be explained as a semi-classical graviton. From compute the energy-momentum tensor of gravitational field itself, it is proved that the quantum field theory constructed in this paper is classically equivalent to the general theory of relativity. Because the graviton is a wave packet approximate to the Dirac δ-function, the propagator of graviton can solve the divergence difficulty caused by the integration over the large virtual momentum of the graviton in the Feynman rule. So we think we have a new approach to quantum gravity, it can be calculated and verified.

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