Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 20 Issue 7 Version 1.0 Year 2020
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Composite Multiplication Pre-Frame Operatorson the Space of Vector-Valued Weakly Measurable Functions

By S. Senthil, M. Nithya & D. C. Kumar
Mother Teresa Women's University

Abstract- In this paper,we first characterize the boundedness of the condition under which composite multiplication pre-frame operators on L^2 (μ) -space, namely $M_{u,T,f}$ and its adjoint. Then, we identify the relation between the adjoint of $M_{u,T,f}$ and the composite multiplication frame operators which is denoted by $S_{u,T,f}$ all the results have been obtained in terms of radonnikodymderivative h_T.

Keywords: *composite multiplication operator, expectation, composite multiplication preframe operator.*

GJSFR-F Classification: *MSC 2010: 47B33, 47B20, 46C05*

COMPOSITEMULTIPLICATIONPREFRAMEOPERATORSONTHESPACEOFVECTORVALUEDWEAKLYMEASURABLEFUNCTIONS

Strictly as per the compliance and regulations of:

Ref

12. Duffin RJ & Shaeffer AC, A class of non-harmonic Fourier series, Trans. Amer. Math. Soc, vol.72, pp.341-366 (1952).

Composite Multiplication Pre-Frame Operatorson the Space of Vector-Valued Weakly Measurable Functions

S. Senthil [α], M. Nithya [σ] & D. C. Kumar [ρ]

Abstract- In this paper,we first characterize the boundedness of the condition under which composite multiplication pre-frame operators on L^2 **(μ)** -space, namely $M_{u,T,f}$ and its adjoint. Then, we identify the relation between the adjoint of $M_{u,T,f}$ and the composite multiplication frame operators which is denoted by $S_{u,T,f}$ all the results have been obtained in terms of radonnikodymderivative h_T.

Keywords: composite multiplication operator, expectation, composite multiplication preframe operator.

I. Introduction

Frames were developed as a powerful tool in signal processing. the frame in a Hilbert space was defined by Duffin and Schaeffer [12] for investigating non-harmonic Fourier series. A discrete frame is a countable family of elements in a separable Hilbert space, which allows stable and not necessarily unique decomposition of arbitrary elements in an expansion of frame elements. In this paper, H refers to a Hilbert space over C and the closed unit ball of H is denoted by H_1.

Let (X, Σ, μ) be a σ-finite measure space. Then a mapping T from X into X is said to be a measurable transformation if $T^{-1}(E) \in \Sigma$ forevery $E \in \Sigma$. A measurable transformation T is said to be non-singular if $\mu(T^{-1}(E)) = 0$ whenever $\mu(E) = 0$. If T is non-singular then the measure μT^{-1} defined as $\mu T^{-1}(E) = \mu(T^{-1}(E))$ for every E in Σ, is an absolutely continuous measure on Σ with respect to μ.Since μ is a σ-finite measure, then by the Radon-Nikodym theorem, there exists a non-negative function h_T in $L^1(\mu)$ such that $\mu T^{-1}(E) = \int_E h_T \, d\mu$ for every $E \in \Sigma$. The function h_T is called the Radon-Nikodym derivative of μT^{-1} with respect to μ.

Every non- singular measurable transformation T from X into itself induces a linear transformation C_T on $L^p(\mu)$ defined as $C_T f = f \circ T$ for every f in $L^p(\mu)$. In case C_T is continuous from $L^p(\mu)$ into itself, then it is called a composition operator on $L^p(\mu)$ induced by T. We restrict our study of the composition operators on $L^2(\mu)$ which has Hilbert space structure. If u is an essentially bounded complex-valued measurable function on X, then the

Author α: Department of Economics and Statistics, Government of Tamilnadu, DRDA, Dindigul, India.
Author σ: Department of Mathematics, Vickram College of Engineering, Sivagangai, Enathi, Tamilnadu, India.
Author ρ: Department of Mathematics, Mother Teresa Women's University, Kodaikanal, Tamilnadu, India.
e-mail: senthilsnc83@gmail.com

mapping M_u on $L^2(\mu)$ defined by $M_u f = u \cdot f$, is a continuous operator with range in $L^2(\mu)$. The operator M_u is known as the multiplication operator induced by u.

A composite multiplication operator is linear transformation acting on a set of complex valued Σ measurable functions f of the form

$$M_{u,T}(f) = C_T M_u(f) = (u \circ T)\ (f \circ T)$$

where u is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator, denoted by C_T.

In the study considered is the using conditional expectation of composite multiplication operator on L^2-spaces. For each $f \in L^p(X, \Sigma, \mu)$, $1 \le p \le \infty$, there exists an unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A g f\, d\mu = \int_A g E(f)\, d\mu$$

for every $T^{-1}(\Sigma)$-measurable function g, for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of $L^p(\mu)$, E is the projection onto the closure of range of T and E is the identity on $L^p(\mu)$, $p \ge 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [1, 2, 3, 4].

The study of weighted composition operators on L^2 spaces was initiated by R.K.Singh and D.C.Kumar [5]. During the last thirty years, several authors have studied the properties of various classes of weighted composition operator. Boundedness of the composition operators in $L^p(\Sigma)$, $(1 \le p < \infty)$ spaces, where the measure spaces are σ-finite, appeared already in [6]. Also boundedness of weighted operators on $C(X,E)$ has been studied in [7]. Recently S.Senthil, P.Thangaraju and D.C.Kumar have proved several theorems on n-normal, n-quasi-normal, k-paranormal, and (n,k) paranormal of composite multiplication operators on L^2 spaces [8, 9, 10, 11,17].

The theory of weighted translation pre-frame operators is the generalizations of the theory of c-frames and c-Bessel mappings. The properties of c-frames and c-Bessel mappings have been studied in [13]. The change of variable formula will be frequently used throughout this paper and we remind it here as follows:

$$\int_{T^{-1}(B)} f \circ T\, d\mu = \int_{T^{-1}(B)} f\, d\mu \circ T^{-1} = \int_{T^{-1}(B)} f \frac{d\mu \circ T^{-1}}{d\mu} d\mu = \int_B h_T f\, d\mu, \quad B \in \Sigma\ , f \in L^1(\Sigma).$$

In this paper we investigate composite multiplication pre-frame operators on $L^2(\mu)$-spaces.

1.1 Let $L^2(X,H)$ be the class of all measurable mappings $f : X \to H$ such that

$$\| f \|_2^2 = \int_X \| f(x) \|^2\, d\mu < \infty$$

Global Journal of Science Frontier Research (F) Volume XX Issue VII Version I Year 2020

Ref

7. Takagi, H & Yokouchi, K, Multiplication and Composition operators between two pL - spaces, Contem. Math., vol.232, pp.321-338 (1999).

For any $f, g \in L^2(X, H)$, based on the polar identity, we may conclude that the mapping $x \to \langle f(x), g(x) \rangle$ of X to C, is measurable and it can be seen that $L^2(X, H)$ is a Hilbert space with the inner product defined by

$$\langle f, g \rangle_{L^2} = \int_X \langle f(x), g(x) \rangle d\mu.$$

Notes

We shall write $L^2(X)$ when $H = C$

II. Composite Multiplication Pre-Frame Operator

2.1 Let $f: X \to H$ be a mapping. We say that f is weakly measurable if for each $h \in H$, the mapping $x \to \langle h, f(x) \rangle$ of X to C is measurable.

2.2 Let $f: X \to H$ be weakly measurable. We say that f is a c-frame for H, if there exist $0 < A \le B < \infty$ such that

$$A \| h \|^2 \le \int_X \left| \langle h, f(x) \rangle \right|^2 d\mu \le B \| h \|^2, \quad h \in H.$$

If only the right hand inequality is satisfied, then we say that f is a c-Bessel mapping for H. Let $f: X \to H$ be a c-Bessel for H. Let $M_{u,T,f} : L^2(X) \to H$ be defined by

$$\langle M_{u,T,f}(g), h \rangle = \int_X (u \circ T)(x)(g \circ T)(x) \langle f(x), h \rangle d\mu(x), \quad h \in H, \ g \in L^2(X).$$

It is obvious that $M_{u,T,f}$ is well-defined and linear. For each $g \in L^2(x)$ and $h \in H$, we have

$$\left\| M_{u,T,f}(g) \right\| = \sup_{h \in H} \left| \langle M_{u,T,f}(g), h \rangle \right|$$

$$= \sup_{h \in H} \left| \int_X (u \circ T)(x)(g \circ T)(x) \langle f(x), h \rangle d\mu \right|$$

$$= \sup_{h \in H} \left| \int_X ((ug) \circ T)(x) \langle f(x), h \rangle d\mu \right|$$

$$\le \left(\int_X \left| ((ug) \circ T) \right|^2 d\mu \right)^{\frac{1}{2}} \sup_{h \in H_1} \left(\int_X \left| \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$= \left(\int_X E \left| ((ug) \circ T) \right|^2 d\mu \right)^{\frac{1}{2}} \sup_{h \in H_1} \left(\int_X \left| \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$= \left(\int_X h_T |u|^2 |g|^2 d\mu \right)^{\frac{1}{2}} \sup_{h \in H_1} \left(\int_X \left| \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\le B^{\frac{1}{2}} \| g \|_2 \| J \|_{\infty}^{\frac{1}{2}}$$

Consequently, $M_{u,T,f}$ is bounded. We shall denoted

Global Journal of Science Frontier Research (F) Volume XX Issue VII Version I Year 2020

$M_{u,T,f} : L^2(X) \to H$, by $M_{u,T,f}(g) = \int_X (u \circ T)(g \circ T) f \, d\mu, \quad g \in L^2(X)$ is called the composite multiplication pre-frame operator of f.

For each $g \in L^2(X)$ and $h \in H$ by an application of the conditional expectation properties and the change of variable formula,

$$\left\langle g, M^*_{u,T,f}(h) \right\rangle = \left\langle M_{u,T,f}(g), h \right\rangle$$

$$= \int_X (u \circ T)(x)(g \circ T)(x) \left\langle f(x), h \right\rangle d\mu$$

$$= \int_X ((ug) \circ T)(x) \left\langle f(x), h \right\rangle d\mu$$

$$= \int_X E\left(((ug) \circ T)(x) \left\langle f(x), h \right\rangle \right) d\mu$$

$$= \int_X h_T \, u(x) g(x) E\left(\left\langle f(x), h \right\rangle \right) \circ T^{-1} \, d\mu$$

$$= \left\langle g, \; h_T \, u E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1} \right\rangle$$

Thus, $M^*_{u,T,f}(h) = h_T \, u E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1}$

Also, for each $h \in H$, we have

$$\left\| M^*_{u,T,f}(h) \right\|^2 = \left\langle M^*_{u,T,f}(h), \; M^*_{u,T,f}(h) \right\rangle$$

$$= \int_X \left\langle M_{u,T,f} \, M^*_{u,T,f}(h), \; h \right\rangle d\mu$$

$$= \int_X \left| u h_T \, E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1} \right|^2 d\mu$$

The mapping $M^*_{u,T,f} : H \to L^2(X)$ is called the composite multiplication analysis operator of f.

We define, $S_{u,T,f} : H \to H$ by $S_{u,T,f}(h) = M_{u,T,f} \, M^*_{u,T,f}(h)$

$$= M_{u,T,f}\left(h_T \, u E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1} \right)$$

$$= \int_X u^2 \circ T \; h_T \circ T \; E\left(\overline{\left\langle f, h \right\rangle} \right) f \, d\mu$$

and it is called the composite multiplication frame operator of f.

Theorem 2.1. Let $S_{u,T,f}$ is composite multiplication frame operator of f. The mapping $S_{u,T,f} : H \to H$ and For each c-Bessel mapping $f : X \to H$, Then $S_{u,T,f}$ is invertible if and only if $M_{u,T,f}$ is surjective.

Notes

Notes

Proof. Since $S_{u,T,f}$ is a self-adjoint operator on H then by [14, Theorem 9.2.1], we have

$\inf_{h\in H_1}\langle S_{u,T,f}\, h, h\rangle = \inf_{h\in H_1}\left\| M^*_{u,T,f}(h)\right\|^2 \in \text{Sepc } S_{u,T,f}$, the spectrum of $S_{u,T,f}$.

By hypothesis $0\notin \text{Spec } S_{u,T,f}$, Hence, $\inf_{h\in H_1}\left\| M^*_{u,T,f}\right\| > 0$. It follows that

$$\inf_{h\in H_1}\left\| M^*_{u,T,f}\right\|\left\| h\right\| \le \left\| M^*_{u,T,f}\right\| \text{and so } M_{u,T,f} \text{ is surjective.}$$

Conversely, Let $M_{u,T,f}$ is surjective. Then there exists $K>0$ such that for each $h\in H$

$\left\| M^*_{u,T,f}\right\|^2 \ge K\left\| h\right\|^2$

$$\text{So, } \langle S_{u,T,f}(h), h\rangle = \langle M_{u,T,f}\, M^*_{u,T,f}(h), h\rangle = \left\| M^*_{u,T,f}\right\|^2 \ge K\left\| h\right\|^2$$

For each $h\in H$, we have

$$\langle S_{u,T,f}(h), h\rangle = \langle M_{u,T,f}\, M^*_{u,T,f}(h), h\rangle$$

$$= \int_X \langle f,h\rangle \left(u\, M^*_{u,T,f}\right)\circ T\, d\mu$$

$$= \int_X u^2\circ T\, h_T\circ T\, E\left(\overline{\langle f,h\rangle}\right)\langle f,h\rangle\, d\mu$$

$$= \int_X E\,(\,u^2\circ T\, h_T\circ T\, E\left(\overline{\langle f,h\rangle}\right)\langle f,h\rangle)\, d\mu$$

$$= \int_X u^2\circ T\, h_T\circ T\, E\left(\overline{\langle f,h\rangle}\right) E\left(\langle f,h\rangle\right)\, d\mu$$

$$\le \int_X u^2\circ T\, h_T\circ T\, E\left(\left|\langle f,h\rangle\right|^2\right)\, d\mu$$

$$= \int_X (u^2 h_T)\circ T\, \left|\langle f,h\rangle\right|^2\, d\mu \le 1$$

$$\le \left\|(u^2 h_T)\circ T\right\|_\infty B\left\| h\right\|^2 \text{ for some } B>0$$

Therefore $K\le S_{u,T,f}\le \left\|(u^2 h_T)\circ T\right\|_\infty B$, $S_{u,T,f}$ is invertible.

Theorem 2.2. Let $M_{u,T,f}$ is composite multiplication pre-frame operator of f. For each $x\in X$, the map $x\to\langle f(x), h\rangle$ is $T^{-1}(\Sigma\ )$measurable. Then $f: X\to H$, is a c-frame for H if and only if the operator $M_{u,T,f}$ is a bounded and onto operator.

Proof. Let f be c-frame by definition 2.2, it is clear that $M_{u,T,f}$ is bounded. We have to prove only that $M_{u,T,f}$ is onto.

Since $(u^2 h_T)\circ T>0$ almost everywhere, Now we assume that $(u^2 h_T)\circ T>\delta$ for some $\delta>0$. Then, by using the change of variable formula, we get

$$\left\| M^*_{u,T,f}(h) \right\|^2 = \int_X \left| u\, h_T\, E\left(\overline{\langle f, h \rangle} \right) \circ T^{-1} \right|^2 d\mu$$

$$= \int_X |u|^2 \, |h_T|^2 \left| E\left(\overline{\langle f, h \rangle} \right) \circ T^{-1} \right|^2 d\mu$$

$$= \int_X |u|^2 \, |h_T| \left| E\left(\overline{\langle f, h \rangle} \right) \circ T^{-1} \right|^2 d\mu \circ T^{-1}$$

$$= \int_X |u|^2 \circ T \, |h_T| \circ T \left| E\left(\overline{\langle f, h \rangle} \right) \right|^2 d\mu$$

$$= \int_X \left| u^2 h_T \right| \circ T \left| E\left(\overline{\langle f, h \rangle} \right) \right|^2 d\mu$$

$$\geq \delta \int_X \left| E\left(\overline{\langle f, h \rangle} \right) \right|^2 d\mu = \delta \int_X \left| \overline{\langle f, h \rangle} \right|^2 d\mu = \delta \int_X \left| \langle h, f \rangle \right|^2 d\mu$$

$$\geq \delta A \|h\|^2$$

Therefore, by [15, lemma 2.4.1], $M_{u,T,f}$ is onto.

Conversely, let $M_{u,T,f}$ is bounded and onto operator, by [15,Lemma 2.4.1], there exists a constant $c > 0$ such that for each $h \in H$, $c\|h\|^2 \leq \left\| M^*_{u,T,f}(h) \right\|^2$.

On the other hand, by the change of variable formula, we get

$$c\|h\|^2 \leq \left\| M^*_{u,T,f}(h) \right\|^2 = \int_X \left| u\, h_T\, E\left(\overline{\langle f, h \rangle} \right) \circ T^{-1} \right|^2 d\mu$$

$$= \int_X \left| u^2 h_T \right| \circ T \left| E\left(\overline{\langle f, h \rangle} \right) \right|^2 d\mu$$

$$\leq \left\| (u^2 h_T) \circ T \right\|_\infty \int_X \left| \langle h, f \rangle \right|^2 d\mu$$

Since $\left\| (u^2 h_T) \circ T \right\|_\infty > 0$, we get $A\,\|h\|^2 \leq \int_X \left| \langle h, f \rangle \right|^2 d\mu$ for some constant $A > 0$.

To proved is that f is c-Bessel, For this the change of variable formula and the properties of the conditional expectation are essentially used to obtain by

$$\delta \int_X \left| \langle h, f \rangle \right|^2 d\mu \leq \delta \int_X E\left| \langle h, f \rangle \right|^2 d\mu$$

$$\leq \int_X (u^2 h_T) \circ T \; E\left| \langle h, f \rangle \right|^2 d\mu$$

$$= \int_X (u^2 h_T) \; E\left| \langle h, f \rangle \circ T^{-1} \right|^2 d\mu \circ T^{-1}$$

$$= \int_X (u^2 h_T) \; E\left| \langle h, f \rangle \circ T^{-1} \right|^2 h_T \; d\mu$$

Notes

Global Journal of Science Frontier Research (F) Volume XX Issue VII Version I Year 2020

$$= \int_X \left| u h_T \; E\langle h, f \rangle \circ T^{-1} \right|^2 d\mu$$

$$= \left\| M^*_{u,T,f}(h) \right\|^2 \le \left\| M^*_{u,T,f} \right\|^2 \| h \|^2$$

Hence $\int_X \left| \langle h, f \rangle \right|^2 d\mu \le B \| h \|^2$ for some $B > 0$

Notes

Theorem 2.3. Let K be a Hilbert space, $f: X \to H$ be a c-Bessel mapping for H and $v: H \to K$ be a bounded linear mapping. Then

(i) The mapping $vf: X \to K$ is a c-Bessel mapping for K and $vM_{u,T,f} = M_{u,T,vf}$

(ii) For each $x \in X$ the map, $x \to \langle h, f(x) \rangle$ is $T^{-1}(\Sigma\ )-$measurable. Let f be a c-frame for H. Then vf is

Proof. (i).Since $\sup_{h\in H_1} \int_X \left| \langle h, v(f(x)) \rangle \right|^2 d\mu \le \| v \|^2 \sup_{h\in H_1} \int_X \left| \langle h, f(x) \rangle \right|^2 d\mu$, vf is a c-Bessel mapping for K.

For each $g \in L^2(X)$, we have $\langle M_{u,T,vf}(g), k \rangle = \int_X u \circ T(x) \; g \circ T(x) \langle v(f(x)), k \rangle d\mu$

$$= \int_X (ug) \circ T(x) \left\langle f(x), v^*(k) \right\rangle d\mu$$

$$= \left\langle M_{u,T,f}(g), \; v^*(k) \right\rangle = \left\langle v M_{u,T,f}(g), \; k \right\rangle$$

Hence $M_{u,T,vf} = v M_{u,T,f}$.

(ii). Suppose that v is surjective, by (i) it is clear that $M_{u,T,vf}$ is also surjective.

Hence by Theorem 2.2, vf is a c-frame for K.

Conversely, suppose that vf is a c-frame for K, then by Theorem 2.2, $M_{u,T,vf}$ is surjective and again by (i) v is clearly surjective.

III. Dual of C-Bessel Mapping

3.1 Let f, g be c-Bessel mappings for $h \in H$ we say that f equals weakly to g whenever $M^*_{u,T,f} = M^*_{u,T,g}$, which is equivalent with $\langle h, f \rangle = \langle h, g \rangle$ almost everywhere, for all $h \in H$.

Theorem 3.1. Let f, g be c-Bessel mappings for H. Then the following assertions are equivalent,

(1). For each $h \in H$, $h = M_{u,T,f}\left(\left\langle h, \; g \circ T^{-1} \right\rangle\right)$

(2). For each $k \in H$, $k = M_{\overline{u \circ T},T,g}\left(\left\langle k, \; f \circ T^{-1} \right\rangle\right)$

(3). For each $h, \; k \in H$, $\langle h, k \rangle = \int_X u \circ T(x) \langle h, g(x) \rangle \langle f(x), k \rangle d\mu$

(4). For each $h \in H$, $\| h \|^2 = \int_X u \circ T(x) \langle h, g(x) \rangle \langle f(x), h \rangle d\mu$

(5). For each orthonormal bases $\{ e_i \}_{i\in I}$ for H

$$\langle e_i, \gamma_j \rangle = \int_X u\circ T(x) \langle e_i, g(x) \rangle \langle f(x), \gamma_j \rangle d\mu, \quad i\in I, \; j\in J$$

(6). For each orthonormal bases $\{ \gamma_j \}_{j\in J}$ and $\{ e_i \}_{i\in I}$ for H

$$\langle e_i, e_j \rangle = \int_X u\circ T(x) \langle e_i, g(x) \rangle \langle f(x), e_j \rangle d\mu, \quad i\in I, \; j\in J$$

Proof. $(1)\to(2)$, choose $h, k\in H$ arbitrarily then

$$\langle h,k \rangle = \left\langle M_{u,T,f}\left(\left\langle h, g\circ T^{-1} \right\rangle\right), k \right\rangle$$

$$= \int_X u\circ T(x) \left(\left\langle h, g\circ T^{-1} \right\rangle \right)\circ T(x) \; \langle f(x), k \rangle d\mu$$

$$= \int_X u\circ T(x) \langle h, g(x) \rangle \; \langle f(x), k \rangle d\mu$$

$$= \int_X \overline{\overline{u\circ T(x)} \; \langle k, f(x) \rangle \; \langle g(x), h \rangle d\mu}$$

$$= \overline{\left\langle M_{\overline{u\circ T},T,g}\left(\left\langle k, f\circ T^{-1} \right\rangle\right), \; h \right\rangle}$$

$$= \left\langle h, \; M_{\overline{u\circ T},T,g}\left(\left\langle k, f\circ T^{-1} \right\rangle\right) \right\rangle$$

Hence $k = M_{\overline{u\circ T},T,g}\left(\left\langle k, f\circ T^{-1} \right\rangle\right)$

$(2)\to(3)$ is proved in a similar way and proof of the other implications refer[16, Theorem 3.4].

3.2 Let f, g be c-Bessel mappings for H. we say that f, g is a dual pair if one of the assertions of Theorem 3.1 is satisfied.

Note that:

$$\| h \|^2 = \int_X u\circ T(x) \langle h, g(x) \rangle \langle f(x), h \rangle d\mu$$

$$\leq \int_X \left| u\circ T(x) \langle h, g(x) \rangle \langle f(x), h \rangle \right| d\mu$$

$$\leq \left(\int_X \left| \langle h, g(x) \rangle \right|^2 d\mu \right)^{\frac{1}{2}} \left(\int_X \left| u\circ T(x) \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\leq \left(\int_X \left| \langle h, g(x) \rangle \right|^2 d\mu \right)^{\frac{1}{2}} \| u\circ T \|_\infty B^{\frac{1}{2}} \| h \|$$

Notes

Notes

Hence g is a c-frame for H.

Theorem 3.2. For each $x \in X$ and $h \in H$, the map $x \to \langle h, f(x) \rangle$ is $T^{-1}(\Sigma)$ – measurable. Let f be a c-frame for H. Then the following arguments hold.

(1). For each $h \in H$, we find the following formulas $h = M_{u,T,S^{-1}_{u,T,f}}\left(u h_T E(\langle h, f \rangle) \circ T^{-1}\right)$ and $h = M_{u,T,f}\left(u h_T E\left(\left\langle S^{-1}_{u,T,f}(h), f \right\rangle\right) \circ T^{-1}\right)$

(2). In the formula $h = M_{u,T,f}\left(u h_T E\left(\left\langle S^{-1}_{u,T,f}(h), f \right\rangle\right) \circ T^{-1}\right)$,

$h = M_{u,T,f}\left(u h_T E\left(\left\langle h, S^{-1}_{u,T,f}(f) \right\rangle\right) \circ T^{-1}\right)$ has the least norm among all of the retrieval formulas.

(3). For each $h \in H$, $h = M_{u,T,f}\left\langle h, g \circ T^{-1}\right\rangle$ if and only if there exists a c-Bessel mapping $l \in H$ Such that $g \circ T^{-1} = S^{-1}_{u,T,f}\, f + l$, where for each $k \in H, \langle k, l \rangle \in Ker(M_{u,T,f})$.

(4). The map f has just one dual if and only if $R(M^*_{u,T,f}) = L^2(X)$.

Proof.(1).Since f is c-frame, then by Theorem 2.2 , $M_{u,T,f}$ is onto and hence $S_{u,T,f}$ is an invertible operator. Consequently, for each $h \in H$, we obtain that

$$h = S^{-1}_{u,T,f}\, S_{u,T,f}(h) = S^{-1}_{u,T,f}\, M_{u,T,f}\, M^*_{u,T,f}(h)$$

$$= M_{u,T,S^{-1}_{u,T,f}}\left(u h_T E(\langle h, f \rangle) \circ T^{-1}\right)$$

Now, we have $h = S^{-1}_{u,T,f}\, S_{u,T,f}(h) = M_{u,T,f}\, M^*_{u,T,f}(h)\left(S^{-1}_{u,T,f}\right)$

$$= M_{u,T,f}\left(u h_T E\left(\left\langle S^{-1}_{u,T,f}(h), f \right\rangle\right) \circ T^{-1}\right).$$

(2). Choose $\phi \in L^2(X)$ and $h = M_{u,T,f}(\phi)$. Then for each $g \in H$, we have

$$\langle h, g \rangle = \left\langle M_{u,T,f}\left(u h_T E\left(\left\langle S^{-1}_{u,T,f}(h), f \right\rangle\right) \circ T^{-1}\right), g \right\rangle$$

$$= \int_X u \circ T(x)\left(u h_T E\left(\left\langle S^{-1}_{u,T,f}(h), f \right\rangle\right) \circ T^{-1}\right) \circ T(x) \langle f, g \rangle d\mu$$

$$= \int_X u^2 \circ T(x)\, h_T \circ T(x)\, E\left(\left\langle S^{-1}_{u,T,f}(h), f \right\rangle\right) \langle f(x), g \rangle d\mu$$

Similarly, we have

$$\langle h, g \rangle = \langle M_{u,T,f}(\phi), g \rangle = \int_X u \circ T(x)\, \phi \circ T(x) \langle f(x), g \rangle d\mu$$

Global Journal of Science Frontier Research (F) Volume XX Issue VII Version I Year 2020

Therefore $\langle h,g\rangle-\langle h,g\rangle=\left\langle M_{u,T,f}\left[\left(u\,h_T\,E\left(\left\langle S_{u,T,f}^{-1}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\right],\,g\right\rangle$

$$=\int_X u\circ T(x)\left(\left(u\,h_T\,E\left(\left\langle S_{u,T,f}^{-1}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi(x)\right)\circ T(x)\langle f(x),\,g\rangle d\mu=0$$

$$M_{u,T,f}\left(\left(u\,h_T\,E\left(\left\langle S_{u,T,f}^{-1}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\right)=0$$

Implies that $\left(u\,h_T\,E\left(\left\langle S_{u,T,f}^{-1}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\in Ker(M_{u,T,f})$

Since f is a c-Bessel mapping for H, we obtain that $\left(u\,h_T\,E\left(\left\langle S_{u,T,f}^{-1}(h),f\right\rangle\right)\circ T^{-1}\right)\in R(M_{u,T,f}^*)$

But $L^2(X)=ker(M_{u,T,f})\oplus R(M_{u,T,f}^*)$

Consequently,

$$\|\phi\|^2=\left\|\left(u\,h_T\,E\left(\left\langle S_{u,T,f}^{-1}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\right\|^2+\left\|\left(u\,h_T\,E\left(\left\langle S_{u,T,f}^{-1}(h),f\right\rangle\right)\circ T^{-1}\right)\right\|^2$$

and (2) is proved.

(3). Let g be a c-Bessel mapping for H. For each $h\in H$, assume that $h=M_{u,T,f}\left\langle h,\,g\circ T^{-1}\right\rangle$

Let $g\circ T^{-1}-S_{u,T,f}^{-1}f=l$ by Theorem 3.1, for each $h,k\in H$ we have

$$\left\langle M_{u,T,f}\langle k,l\rangle,h\right\rangle=\left\langle M_{u,T,f}\left\langle k,g\circ T^{-1}\right\rangle,h\right\rangle-\left\langle M_{u,T,f}\left\langle k,\,S_{u,T,f}^{-1}f\right\rangle,\,h\right\rangle$$

$$=\int_X u\circ T\left\langle k,g\circ T^{-1}\right\rangle\circ T\langle f,h\rangle d\mu-\int_X u\circ T\left\langle k,\,S_{u,T,f}^{-1}f\right\rangle\circ T\langle f,h\rangle d\mu$$

$$=\int_X u\circ T\langle k,g\rangle\langle f,h\rangle d\mu-\int_X u\circ T\left\langle k,\,S_{u,T,f}^{-1}f\circ T\right\rangle\langle f,h\rangle d\mu$$

$$=\langle k,h\rangle-\langle k,h\rangle=0$$

Hence, for each $k\in H$, $\langle k,l\rangle\in R\left(M_{u,T,f}^*\right)^{\perp}=ker\left(M_{u,T,f}\right)$.

Now, let $g\circ T^{-1}=S_{u,T,f}^{-1}f+l$, Then for each $h\in H$,we have

$$\int_X u\circ T\langle f,h\rangle\langle k,g\rangle d\mu=\int_X u\circ T\langle f,h\rangle\left\langle k,\,\left(S_{u,T,f}^{-1}f+l\right)\circ T\right\rangle d\mu$$

$$=\int_X u\circ T(x)\,\langle f(x),h\rangle\left\langle k,\,\left(S_{u,T,f}^{-1}f\right)\circ T\right\rangle d\mu+\int_X u\circ T(x)\,\langle f(x),h\rangle\langle k,\,l\circ T\rangle d\mu$$

$$=\langle k,h\rangle+\left\langle M_{u,T,f}\langle k,l\rangle,\,h\right\rangle=\langle k,h\rangle.$$

By Theorem 3.1, $h=M_{u,T,f}\left\langle h,\,g\circ T^{-1}\right\rangle$.

(4). Let $R\left(M_{u,T,f}^*\right)\neq L^2(X)$ and Let $l\in R\left(M_{u,T,f}^*\right)^{\perp}$ with $\|l\|=1$

Notes

Consider the map $k : X \to L^2(X)$ defined by $k(x) = l \circ T(x) l$. For each $t \in L^2(X)$, the map $X \to C$, defined by $x \to \langle t, k(x) \rangle$ is Σ -measurable and $\int_X |\langle t, k(x)\rangle|^2 d\mu = \int_X |\langle t, l \circ T(x) l\rangle|^2 d\mu$

$$= \int_X |\langle t, l\rangle|^2 |l \circ T(x)|^2 d\mu = |\langle t, l\rangle|^2 \le \| t \|^2$$

Notes

Thus, k is a c-Bessel mapping for $L^2(X)$. let $v : L^2(X) \to H$ be a mapping such that $v(l) \neq 0$. Then vk is c-Bessel mapping for H and $S^{-1}_{u,T,f} f + vk$ is a c-Bessel mapping for H.

Let $h \in H$, $\int_X u \circ T(x) \left\langle h, S^{-1}_{u,T,f} f(x) + vk(x) \right\rangle \langle f(x), h \rangle d\mu$

$$= \int_X u \circ T(x) \left\langle h, S^{-1}_{u,T,f} f(x) \right\rangle \langle f(x), h \rangle d\mu + \int_X u \circ T(x) \langle h, vk(x) \rangle \langle f(x), h \rangle d\mu$$

$$= \| h \|^2 + \left\langle v^*(h), l \right\rangle \int_X u \circ T(x) \overline{l \circ T(x)} \langle f(x), h \rangle d\mu$$

$$= \| h \|^2 + \left\langle v^*(h), l \right\rangle \langle M_{u,T,f}(l), h \rangle = \| h \|^2$$

Therefore $S^{-1}_{u,T,f} f + vk$ is the dual of f.

The equation $\langle v(l), vk(x) \rangle = \langle v(l), l \circ T(x) v(l) \rangle = \overline{l \circ T(x)} \langle v(l), v(l) \rangle$

This implies that, $S^{-1}_{u,T,f} f + vk$ is not weakly equal to $S^{-1}_{u,T,f} f$

Conversely, Assume that $L^2(X) = R(M^*_{u,T,f})$, Now, $g \circ T^{-1} = S^{-1}_{u,T,f} f + l$ where for each $k \in H$ $\langle k, l \rangle \in ker(M_{u,T,f}) = R(M^*_{u,T,f})^{\perp} = \{0\}$, $l = 0$ weakly, so f has a dual.

Acknowledgements

We would like to thank the reviewers for carefully reading manuscript and for their constructive comments. I want to thank **Ms.K.Kavitha, Project Director**, District Rural Development Agency (DRDA),Collectorate,Dindigulfor his support and encouragement during preparation of the paper.

References Références Referencias

1. Campbell, J & Jamison, J, On some classes of weighted composition operators, Glasgow Math.J.vol.32, pp.82-94, (1990).
2. Embry Wardrop, M & Lambert, A, Measurable transformations and centred composition operators, Proc. Royal Irish Acad, vol.2(1), pp.23-25 (2009).
3. Herron, J, Weighted conditional expectation operators on L^p -spaces, UNC charlotte doctoral dissertation.
4. Thomas Hoover, Alan Lambert and Joseph Quinn, The Markov process determined by a weighted composition operator, Studia Mathematica, vol. XXII (1982).
5. Singh, RK & Kumar, DC, Weighted composition operators, Ph.D.thesis, Univ. of Jammu (1985).

6. Singh, RK Composition operators induced by rational functions, Proc. Amer. Math. Soc., vol.59, pp.329-333(1976).
7. Takagi, H & Yokouchi, K, Multiplication and Composition operators between two pL - spaces, Contem. Math., vol.232, pp.321-338 (1999).
8. Panaiyappan, S & Senthilkumar, D, Parahyponormal and M^* −parahyponormal composition operators, Acta CienciaIndica, Vol. XXVIII (4) (2002).
9. Senthil, S, Thangaraju, P & Kumar, DC, n-normal and n-quasi-normal composite multiplication operator on L^2-spaces, Journal of Scientific Research & Reports,8(4),1-9 (2015).
10. Senthil, S, Thangaraju, P & Kumar, DC, k-*paranormal, k-quasi-*paranormal and (n,k)-quasi-*paranormal composite multiplication operator on 2 L -spaces, British Journal of mathematics and computer science, BJMCS20166 (2015).
11. Senthil, S, Thangaraju, P & Kumar, DC, Composite multiplication operator on L^2 -spaces of vector valued functions, International research Journal of Mathematical Sciences, vol.4 (2), pp.1 (2015).
12. Duffin RJ & Shaeffer AC, A class of non-harmonic Fourier series, Trans. Amer. Math. Soc, vol.72, pp.341-366 (1952).
13. Faroughi MH & Osgooei E, c-frame and c-Bessel mappings, Bull. Iranian Math. Soc, vol. 38(1), pp.203-222 (2012).
14. Harrington D & Whitly R, Seminormal composition operators, J.Operator theory, vol.38(1), pp.125-135 (1984).
15. Christensen O, Frames and Bases, An Introductory Course, Kegs. Lyngby, Denmark November (2007).
16. Moayyerizadeh Z & Emamalipour H, Weighted composition operator valued integral, Mathematiche(Catania), vol 71(2), pp.161-172 (2016).
17. Senthil & DC kumar et al,), (α,β)-normal and skew normal composite multiplication operators on Hilbert Spaces, In. journal of Discrete Mathematics, doi: 10.11648/ XXXX2019XXXX.XX.

Notes