Global Journals Inc.

Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 20 Issue 2 Version 1.0 Year 2020
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Solution of Integral Equation in Two-Dimensional using Spectral Relationships

By F. M. Alharbi
Umm Al-Quraa University

Abstract- This paper concerned using spectral relationships in the solution of the integral equation (IE) in two-dimensional. To discuss that, the (IE) in two-dimensional under certain conditions was considered. The existence of at least one solution of the (IE) was discussed by proving the continuity and compactness of an integral operators. Chebyshev polynomials of the first kind were used to transform the (IE) to a linear algebraic system. Many numerical results and estimating errors were calculated and plotted by the Maple program in different cases.

Keywords: *Integral equation (IE) in two-dimensional. Schauder fixed point theorem. Continuity and compactness of an integral operators. Spectral relationships .Linear algebraic system.*

GJSFR-F Classification: *MSC 2010: 31A10*

SOLUTIONOFINTEGRALEQUATIONINTWODIMENSIONALUSINGSPECTRALRELATIONSHIPS

Strictly as per the compliance and regulations of:

Solution of Integral Equation in Two-Dimensional using Spectral Relationships

F. M. Alharbi

Abstract- This paper concerned using spectral relationships in the solution of the integral equation (IE) in two-dimensional. To discuss that, the (IE) in two-dimensional under certain conditions was considered. The existence of at least one solution of the (IE) was discussed by proving the continuity and compactness of an integral operators. Chebyshev polynomials of the first kind were used to transform the (IE) to a linear algebraic system. Many numerical results and estimating errors were calculated and plotted by the Maple program in different cases.

Keywords: Integral equation (IE) in two-dimensional. Schauder fixed point theorem. Continuity and compactness of an integral operators. Spectral relationships .Linear algebraic system.

I. Introduction

In recent years, the main theorem of spectral relationships plays an important role in many applications in various areas, including models of nanotech nology particles, genetic engineering, medicine, mathematical chemistry, heat condition and physical phenomena, see [1-4].

On the other side, many problems of mathematical physics, contact problems in elastic media, many applications in electronic engineering applications, and mathematical biology models lead to an **(IEs)** in smooth or singular forms. The references [5-7] discuss the methods for solving the **(IEs)** in smooth form antically. While the singular forms take a large aria in types of researches , see [8-11]. In[10], a spectral technique for solving two-dimensional fractional integral equations with singular kernel was discussed by using Legendre and Chebyshev polynomials. Abdou and Salama in [12] obtained the spectral relationships for the Volterra-Fredholm integral equation **(V-FIE)** of the first kind. Abdou [13] discussed the spectral relationships that have many important applications in astrophysics, for the **(F-VIE)** of the first kind, when the kernel of position takes a generalized potential form. The relation between the contact problems and the **(F-VIE)** in three dimensions were obtained by Abdou and Moustafa in [14]. Abdou and Nasr in [15] used Chebyshev polynomial to obtain the solution of **(F-VIE)** when the kernel takes a logarithmic form. The application of Orthogonal polynomials in spectral relationships of some kinds of singular contact problems is discussed by Alharbi in [16].Abdou and Basseem in [17] used Chyebyshev polynomial, the main theorem of spectral relationships of **(FIE)** to obtain the solution of **(F-VIE)** of the second

Ref

10. F.M.Alharbi, *Boundary and Initial Value problems and Integral operator,* Computational and Applied Mathematics, 19(4)(2018), 391-404.

Author: *Common First Year Deanship, Umm Al– Qura University, Makah, Saudi Arabia. e-mail: fmharbi@uqu.edu.sa*

kind numerically. Abdou and Alharbi in [18] derived a general main theorem of spectral relationships for a mixed integral equation of the first kind **(MIE)** in the space $L_2[-1,1] \times C[0,T]$.

Consider the integral equation,

$$\phi(x,y) = f(x,y) + \lambda \int_{-1}^{1}\int_{-1}^{1} k(x,u,y,v)\phi(u,v)dudv, \qquad (1)$$

Under the following conditions:

i. $\left[\int_{-1}^{1}\int_{-1}^{1} k^2(x,u,y,v)\, dxdudydv\right]^{1/2} \leq c, \;\; c \; is\; constant.$

ii. The function $f(x,y)$ with its partial derivatives with respect to x and y are continuous in $L_2[-1,1] \times [-1,1]$ and its norm can be defined as,

$$\|f(x,y)\| = \left[\int_{-1}^{1}\int_{-1}^{1} f^2(x,y)\, dxdy\right]^{1/2} \leq M \;\; , \;\; M \; is\; constant.$$

iii. The unknown function $\phi(x,y)$ in the space $L_2[-1,1] \times [-1,1]$ behaves as the given function $f(x,y)$.

II. The Existence of at Least one Solution of a Two Dimensional IE

Theorem 1

The integral equation (1) has at least one solution under the previous condition(i)-(iii).
Define the integral operator forms:

$$W\phi = \lambda \int_{-1}^{1}\int_{-1}^{1} k(x,u,y,v)\phi(u,v)du\, dv \qquad (2)$$

Eq. (1) can be written in operator form as:

$$\bar{W}\phi = f + W\phi. \qquad (3)$$

The proof of **theorem1** can be obtained automatically from the proofs of following lemmas.

Lemma1

The integral operator (3) under the conditions (i)-(iii) is bounded in the space $L_2[-1,1] \times [-1,1]$.

Ref

18. M.A. Abdou, F.M. Alharbi, *Generalized main theorem of spectral relationships for Logarithmic kernel and its applications*, Journal of Computational and Theoretical Nanoscience, 16(2019),1-8.

Notes

Proof:

Taking the norm of Eq.(3) we get

$$\|\bar{W}\phi\| \leq \|f(x,y)\| + |\lambda| \left\| \int_{-1}^{1}\int_{-1}^{1} k(x,u,y,v)\ \phi(u,v)dudv \right\| \quad (4)$$

Applying the Cauchy-Schwarz inequality, then using conditions (i)-(iii) to have

$$\|\bar{W}\phi(x,y)\| \leq M + |\lambda|C. \quad (5)$$

Eq.(5) means that the integral operator $\bar{W}$ maps the ball S_α into itself, where $\alpha = M + |\lambda|C$. Also, in the second term of inequality (5), we deduce that the integral operator $W\phi(x,y)$ is bounded in the space $L_2([-1,1] \times [-1,1])$. Therefore, $\bar{W}\phi(x,y)$ is also bounded.

Lemma 2

The integral operator (3) under the conditions (i)-(iii) is continuous in the space $L_2[-1,1] \times [-1,1]$.

Proof:

Let $\phi_1(x,y), \phi_2(x,y)$ be any two functions in the space $L_2[-1,1] \times [-1,1]$ then,

$$\|\bar{W}\phi_1(x,y) - \bar{W}\phi_2(x,y)\| \leq |\lambda| \left\| \int_{-1}^{1}\int_{-1}^{1} \quad |k(x,u,y,v)|\ |\phi_1(x,y) - \phi_2(x,y)| \right\|$$

After applying Cauchy-Shwarz inequality then using the conditions(i)-(iii), the previous inequality becomes,

$$\|\bar{W}\phi_1(x,y) - \bar{W}\phi_2(x,y)\| \leq |\lambda|c\|\phi_1(x,y) - \phi_2(x,y)\|$$

So,

$$\|\bar{W}\| \leq |\lambda|c \quad (6)$$

Inequality (6) implies the continuity of $\bar{W}$ in the space $L_2[-1,1] \times [-1,1]$.

Lemma 3

Suppose that a sequence of continuous functions $\{k_{n,m}(x,u,y,v)\}$ such that,

$$\lim_{n,m\to\infty} \left[\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left|k_{n,m}(x,u,y,v) - k(x,u,y,v)\right|^2 dxdudydv\right]^{1/2} = 0. \quad (7)$$

Then, there exist positive integers n_0, m_0, such that, for $n > n_0, m > m_0$, in general $n \neq m$, after neglecting the very small constants, we have

$$\left[\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|k_{n,m}(x,u,y,v)\right|^{2}dxdudydv\right]^{1/2} \leq c \qquad (8)$$

Proof:

$$\left[\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|k_{n,m}(x,u,y,v)\right|^{2}dxdudydv\right]^{1/2}$$

$$\leq \left[\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|k_{n,m}(x,u,y,v)-k(x,u,y,v)\right|^{2}\right.$$

$$+2\left|k_{n,m}(x,u,y,v)-k(x,u,y,v)\right|\left|k(x,u,y,v)\right|$$

$$\left.+\left|k(x,u,y,v)\right|^{2}\right)dxdudydv\Bigg]^{1/2}$$

Hence, for each $n > n_0, m > m_0$ using Eq.(7) and conditions(i), formula (8) is verified after neglected a small constant.

Lemma 4

Suppose that $\{\overline{W}_{n,m}(x,y)\}$ is a sequence of integral operators where the conditions(i)-(iii) are satisfied, then the sequence of operators

$$\overline{W}_{n,m}(x,y) = f(x,y) + \lambda\int_{-1}^{1}\int_{-1}^{1}k_{n,m}(x,u,y,v)\phi(u,v)dudv \qquad (9)$$

is the bounded and continuous sequence.

Proof:

Taking the norm of Eq.(9) then using the conditions (i)-(iii) to get

$$\left\|\overline{W}_{n,m}(x,y)\right\| \leq \alpha\,, \qquad \alpha = M + |\lambda|c \qquad (10)$$

Therefore $\overline{W}_{n,m}$ maps the largest ball S_α into itself.

Also to prove the continuity of $\overline{W}_{n,m}$ we choose any two functions $\phi_1(x,y), \phi_2(x,y)$ in S_α then applying Cauchy-Schwarz inequality and the conditions (i)-(iii) we get

$$\left\|\overline{W}_{n,m}\phi_1(x,y) - \overline{W}_{n,m}\phi_2(x,y)\right\| \leq |\lambda|c \quad \forall\, n > n_0\,, m > m_0 \qquad (11)$$

Notes

Notes

Lemma 5

If conditions (i)-(iii) are verified then the $\bar{W}(S_\alpha)$ is compact.

Proof:

$$\|\bar{W}_{n,m}\phi(x,y)-\bar{W}\phi(x,y)\| =$$

$$|\lambda|\left\|\int_{-1}^{1}\int_{-1}^{1}\left(k_{n,m}(x,u,y,v)-k(x,u,y,v)\right)\phi(u,v)dudv\right\|.$$

Hence, using condition(iii) yields

$$\|\bar{W}_{n,m}\phi(x,y)-\bar{W}\phi(x,y)\|$$

$$\leq |\lambda|E\left[\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left|k_{n,m}(x,u,y,v)-k(x,u,y,v)\right|^2 dxdudydv\right]^{1/2}$$

Also, from Eq.(7) we have the following condition:

$$\|\bar{W}_{n,m}\phi(x,y)-\bar{W}\phi(x,y)\| = 0, \qquad as\ n,m\to\infty. \tag{12}$$

To prove the compactness of $\bar{W}$, we let $\{\phi_{n,m}(x,y)\}$ be any sequence in S_α. Then we can choose a subsequence$\{\phi_{n1,m}(x,y)\}$ such that $\{\bar{W}_{n1,m}\phi_{n1,m}(x,y)\}$ converges. From that subsequence, we can extract a new subsequence $\{\phi_{n1,m1}(x,y)\}$ in which $\{\bar{W}_{n1,m1}\phi_{n1,m1}(x,y)\}$converges, and so on.Thus, we obtain a chain of subsequences,

$$\{\phi_{n,m}(x,y)\}\supset\{\phi_{n1,m}(x,y)\}\supset\{\phi_{n1,m1}(x,y)\}\supset\cdots\supset\{\phi_{ni,mj}(x,y)\}\supset\cdots$$

Such that the sequence $\{\bar{W}_{ni,mk}\phi_{nj,ml}(x,y)\}$ converges for all $i=1,2,\ldots,j\ and\ k=1,2,\ldots,l$. Finally, we pick the sequence $\{\phi_{n_n,m_m}(x,y)\}$ which is a subsequence of every ϕ_{n_i,m_k} except for a finite number of elements, and clearly $\{\bar{W}_{ni,mk}\phi_{nn,mm}(x,y)\}$ converges for every i,k. Now, since

$$\left\|\bar{W}_{n_i,m_k}\phi_{n_n,m_m}-\bar{W}_{n_i,m_k}\phi_{p_p,q_q}\right\|\to 0\ \ as\ m,n,p,q\to 0.$$

For large j,k, and from( ),we get

$$\left\|\bar{W}\phi_{n_n,m_m}-\bar{W}\phi_{p_p,q_q}\right\|\leq 2\sigma,\quad \forall n,p>n_0(\sigma),m,q>m_0(\sigma).$$

Hence, $\{\bar{W}\phi_{n,m}\}$ is a Cauchy sequence, so $\bar{W}(S_\alpha)$ is compact.

According to the previous lemmas, by Schauder fixed point theorem, see [19-20], $\bar{W}$ has at least one fixed point in S_α , and Theorem 1 is proved.

III. Chebyshev Polynomials and the System of the Integral Equation

Suppose the approximate kernel $k_{n,m}(x,u,y,v)$ in the continuous case as,

$$k_{n,m}(x,u,y,v)=\sum_{n=0}^{N}\sum_{m=0}^{M}\psi_n(x)\chi_n(u)\omega_m(y)\eta_m(v) \tag{12}$$

where it satisfies the condition in Eq.(7).

Therefore Eq.(1) reduce to the algebraic system form as,

$$\phi_{n,m}(x,y)=f_{n,m}(x,y)-\lambda\int_{-1}^{1}\int_{-1}^{1}k_{n,m}(x,u,y,v)\phi_{n,m}(u,v)dudv+R_{n,m} \tag{13}$$

where,

$$R_{n,m}=\left|\phi-\phi_{n,m}\right|\to 0 \quad as\ n,m\to\infty \tag{14}$$

is the approximate error.

To use the spectral relationships, we write the kernel of Eq.(12) in the form

$$k_{n,m}(x,u,y,v)=\sum_{n=0}^{N}\sum_{m=0}^{M}T_n(x)T_n(u)T_m(y)T_m(v) \tag{15}$$

where $T_l(z)$ is the Chybeshev polynomials of first kind and degree l .

Then Eq.(13) reduces to

$$\phi_{n,m}(x,y)-\lambda\sum_{n=0}^{N}\sum_{m=0}^{M}T_n(x)T_m(y)\int_{-1}^{1}\int_{-1}^{1}T_n(u)T_m(v)\phi(u,v)dudv=f_{n,m}(x,y) \tag{16}$$

Such that,

$$\phi_{n,m}(x,y)=\sum_{n=0}^{N}\sum_{m=0}^{M}a_{n,m}T_n(x)T_m(y) \tag{17}$$

Since,

$$\int_{-1}^{1}\int_{-1}^{1}k_{n,m}(x,u,y,v)\phi_{n,m}(u,v)dudv=T_n(x)T_m(y)\int_{-1}^{1}\int_{-1}^{1}T_n(u)T_m(v)T_i(u)T_j(v)dudv \tag{18}$$

then by using the relation, see [21]

$$T_n(u)T_i(u)=\frac{1}{2}\left[T_{n+i}(u)+T_{|n-i|}(u)\right] \tag{19}$$

Ref

21. I. M. Gradshteyn, I. M, Ryzhik, Tables of Integrals, Series and Products Academic Press, Inc. New York, 1994.

Notes

and,

$$\int_{-1}^{1} T_n(u)du = \begin{cases} \frac{2}{1-n^2} & , \quad n=0,2,4,\ldots \\ 0\,, & \quad n=1,3,5\,,\ldots \end{cases} \tag{20}$$

Eq.(18) reduce to

$$T_n(x)T_m(y)\int_{-1}^{1}\int_{-1}^{1} T_n(u)T_m(v)T_i(u)T_j(v)dudv$$

$$= \left[\frac{1}{1-(n+i)^2} + \frac{1}{1-|n-i|^2}\right]\left[\frac{1}{1-(m+j)^2} + \frac{1}{1-|m-j|^2}\right]T_n(x)T_m(y) \tag{21}$$

Then, Eq.(13) reduce to obtaining the following algebraic system,

$$\left(I - \lambda\mu_{n,m,i,j}\right)a_{n,m} = b_{n,m} \tag{22}$$

where,

$$\mu_{n,m,i,j} = \left[\frac{1}{1-(n+i)^2} + \frac{1}{1-|n-i|^2}\right]\left[\frac{1}{1-(m+j)^2} + \frac{1}{1-|m-j|^2}\right] \tag{23}$$

$$b_{n,m} = \int_{-1}^{1}\int_{-1}^{1} f(x,y)T_n(x)T_m(y)dxdy \tag{24}$$

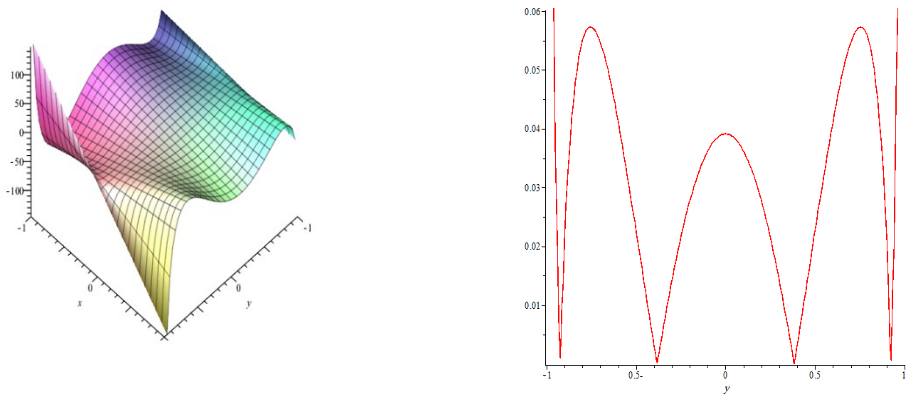
IV. Application and Numerical Discussion

Consider the IE

$$\phi(x,y) = f(x,y) + \lambda\int_{-1}^{1}\int_{-1}^{1}(x+u^2)y^2v\,\phi(u,v)dudv \tag{25}$$

For fixed values of $N = M = 100$, we can graph the solution $\phi(x,y)$ at different values of λ, and different shapes of surfaces $f(x,y)$, then we can graph the estimating errors

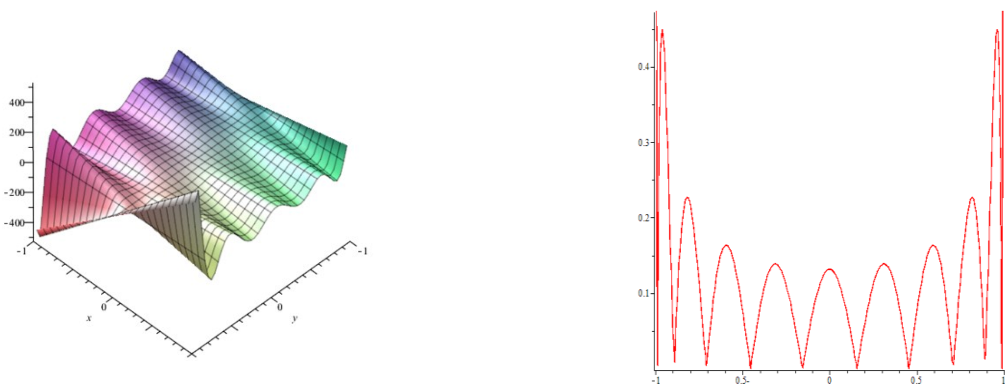
$$f(x,y) = xy,\ \ \lambda = 0.04$$



The solution $\phi_{n,m}(x,y)$ The estimating errorE_{100}

Figure(1)

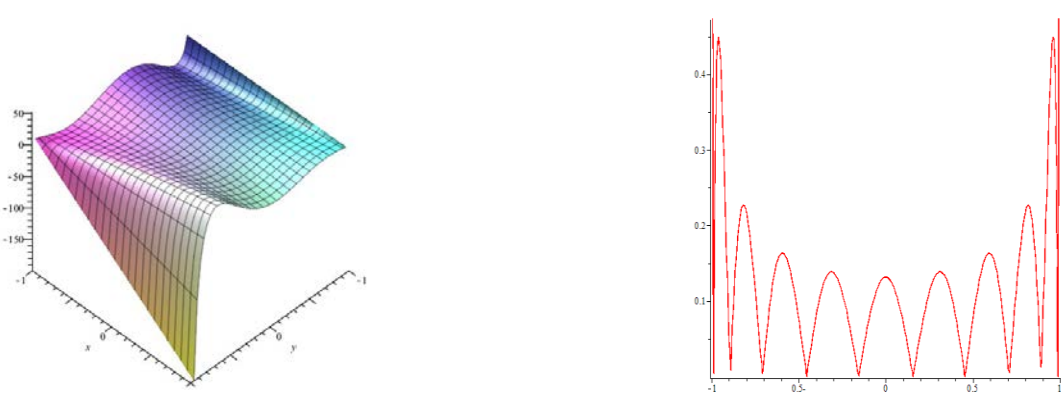
$$f(x,y) = xy \ ,\lambda = 0.1$$



The solution $\phi_{n,m}(x,y)$ The estimating error E_{100}

Figure(2)

$$f(x,y) = x + y, \lambda = 0.04$$



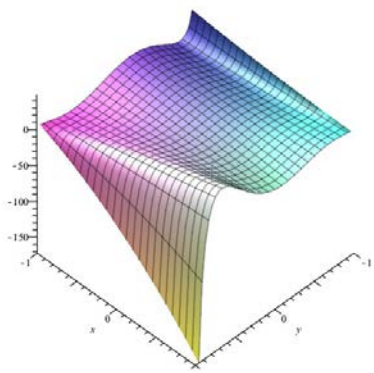
The solution $\phi_{n,m}(x,y)$ The estimating errorE_{100}

Figure(3)

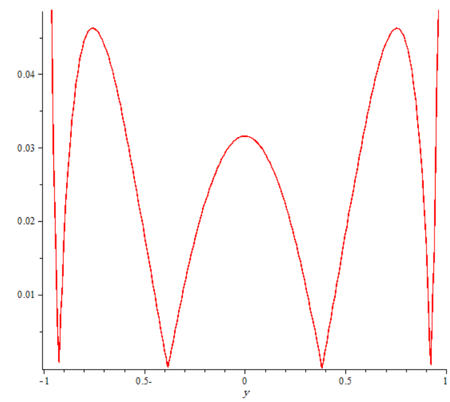
Notes

Notes

$$f(x,y) = x + y, \lambda = 0.1$$



The solution $\phi_{n,m}(x,y)$



The estimating error E_{100}

Figure(4)

References Références Referencias

1. A.M.Wazwaz, Linear and Nonlinear Integral Equations Methods and Applications, Springer, Heidelberg, Dordrecht, London, New York, 2011.
2. R.L.Magin, Fractional Calculus in Bioengineering, Begell House Publishers, 2006.
3. R.Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
4. M. Kolev, *Mathematical modelling of the competition between tumors and immune system considering the role of the antibodies*, Mathematical and Computer Modelling, 37(11)(2003), 1121-1252.
5. F.G. Tricomi, Integral equations, N.Y. 1985.
6. H. Hochstadt, Integral equations, N.Y. London, 1971.
7. G. A. Anastasias, A. Aral, *Generalized Picard singular integrals*, Computers & Mathematics with Applications, 57 (2009), 821-830.
8. R. Kangro, P. Oja, *Convergence of spline collection for Volterra integral equation*, Applied Numerical Mathematics, 58(2008), 1434-1447.
9. T. Diogo, P. Lima, *Super convergence of collocation methods for class of weakly singular integral equations*, Journal of Computational and Applied Mathematics, 218(2008) 307-316.
10. F.M.Alharbi, *Boundary and Initial Value problems and Integral operator*, Computational and Applied Mathematics, 19(4)(2018), 391-404.
11. F.M.Alharbi, *Numerical Solutions of an integro-differentail equation with smooth and singular kernels*, Journal of Research and Reviews in Applied Sciences,13(12)(2019), 573 – 586.
12. M.A.Abdou, F.A.Salama, *Volterra Fredholm integral equation of the first kind and spectral relationships*, Applied Mathematics and Computation, 153(1) (2004), 141-153.
13. M.A.Abdou, *Fredholm Volterra integral equation and generalized potential kernel*, Journal of Computational and Applied Mathematics, 131(1) (2002), 81-94.
14. M.A.Abdou, A.A.Nasr, *On the numerical treatment of the singular integral equation of the second kind,* Journal of Computational and Applied Mathematics. 146 (2)(2003), 373-380.

15. M.A.Abdou, O.L.Moustafa, *Fredholm Volterra integral equation in contact problem*, Journal of Computational and Applied Mathematics, 138(2) (2003), 199-215.
16. F.M.Alharbi, *Application of Orthogonal Polynomial in Spectral Relationships of Contact Problem with Singular Kernel*, Far East Journal of Mathematical Sciences, 102(9)(2017),1881-1905.
17. M. A. Abdou, M. Basseem, *Solution of mixed integral equation in position and time using spectral relationships,* Journal of the Association of Arab Universities for Basic and Applied Sciences23(C), January (2017),1-5.
18. M.A. Abdou, F.M. Alharbi, *Generalized main theorem of spectral relationships for Logarithmic kernel and its applications,* Journal of Computational and Theoretical Nanoscience, 16(2019),1-8.
19. S. Brzychczy and R. R. Poznanski, Mathematical Neuroscience, Copyright Φ 2013 Elsevier Inc. All rights reserved.
20. G. Kassay and V. D. Rădulescu, Equilibrium Problems and Applications, Copyright Φ 2019 Elsevier Inc. All rights reserved.
21. I. M. Gradshteyn, I. M, Ryzhik, Tables of Integrals, Series and Products Academic Press, Inc. New York, 1994.

Notes