Charged Particle in a Flat Box with Static Electromagnetic field and Landau's Levels

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Abstract- We study the quantization of the motion of a charged particle without spin inside a flat box under a static electromagnetic field. Contrary to Landau's solution with constant magnetic field transverse to the box, we found a non separable variables solution for the wave function, and this fact remains when static electric field is added. However, the Landau's Levels appear in all cases.

Keywords: landau's levels, quantum hall effect.

I. Introduction

Landau' solution [1] of a charged particle in a flat surface with magnetic field has become of great importance in understanding integer hall effect [2–6], fractional Hall effect [6–9], and topological insulators [10–16]. This last elements promise to become essential for future nanotechnology devices [17–19]. Due to this considerable application of the Landau's levels, it is worth to re-study this problem and its variations with an static electric field. In this paper, we show that there exists a non separable solution for this type of quantum problems, but having the same Landau's levels. In our cases, instead of having a flat surface, we consider to have a flat box with lengths L_x, L_y, and L_z such that $L_z \ll L_x, L_y$

II. Analytical Approach for the Case B = (0, 0, B)

Let us consider a charged particle "q" with mass "m" in a flat box with a constant magnetic field orthogonal to the flat surface, $\mathbf{B} = (0, 0, B)$, as shown in the next figure.
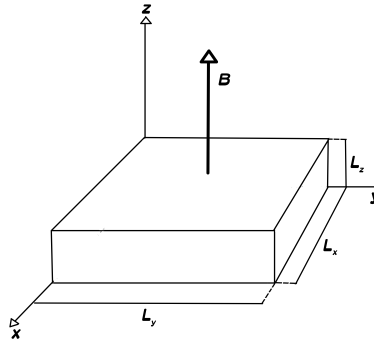

Figure 1: Electric charged in a at box with magnetic field

For a non relativistic charged particle, the Hamiltonian of the system (units CGS) is

$$H = \frac{(\mathbf{p} - q\mathbf{A}/c)^2}{2m}, \tag{1}$$

where $\mathbf{p}$ is the generalized linear momentum, $\mathbf{A}$ is the magnetic potential such that $\mathbf{B} = \nabla \times \mathbf{A}$, and "c" is the speed of light. We can choose the Landau's gauge to have the vector potential of the form $\mathbf{A} = (-By, 0, 0)$. Therefore, the Hamiltonian has the following form

$$H = \frac{(p_x + qBy/c)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}. \tag{2}$$

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To quantize the system, we need to solve the Schrödinger's equation [20]

$$i\hbar\frac{\partial\Psi}{\partial t} = \left\{\frac{(\hat{p}_x + qBy/c)^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m}\right\}\Psi. \tag{3}$$

where $\Psi = \Psi(\mathbf{x},t)$ is the wave function, $\hbar$ is the Plank's constant divided by 2π, $\hat{p}_i$ are the momentum operators such that $[x_i, \hat{p}_j] = i\hbar\delta_{ij}$. Now, the argument used by Landau is that due to commutation relation $[\hat{p}_x, \hat{H}] = 0$, between the operators $\hat{p}_x$ and the Hamiltonian $\hat{H}$ (implying that $\hat{p}_x$ is a constant of motion), it is possible to replace this component of the momentum by $\hbar k_x$, having a solution for the eigenvalue problem of separable variable type, $f_1(t)f_2(x)f_3(y)f_4(z)$. However, we will see that this type of commutation does not imply necessarily separability of the solution. Since the Hamiltonian $\hat{H}$ does not depend explicitly on time, the proposition

$$\Psi(\mathbf{x},t) = e^{-iEt/\hbar}\Phi(\mathbf{x}) \tag{4}$$

reduces the equation to an eigenvalue problem

$$\widehat{H}\Phi = E\Phi. \tag{5}$$

Then, this equation is written as

$$\left\{\frac{1}{2m}\left(\hat{p}_x^2 + \frac{2qB}{c}y\hat{p}_x + \frac{q^2B^2}{c^2}y^2\right) + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m}\right\}\Phi = E\Phi. \tag{6}$$

The variable "z" is separable through the proposition

$$\Phi(\mathbf{x}) = \phi(x,y)e^{-ik_z z}, \qquad k_z \in \Re, \tag{7}$$

resulting the following equation

$$\left\{\frac{1}{2m}\left(\hat{p}_x^2 + \frac{2qB}{c}y\hat{p}_x + \frac{q^2B^2}{c^2}y^2\right) + \frac{\hat{p}_y^2}{2m}\right\}\phi = E'\phi, \tag{8}$$

where E' is

$$E' = E - \frac{\hbar^2 k_x^2}{2m}. \tag{9}$$

That is, the resulting partial differential equation is of the form

$$\frac{1}{2m}\left\{-\hbar^2\frac{\partial^2\phi}{\partial x^2} - i\frac{2qB\hbar}{c}y\frac{\partial\phi}{\partial x} + \frac{q^2B2}{c^2}y^2\phi\right\} - \frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial y^2} = E'\phi. \tag{10}$$

This equation does not admit a separable variable solution ($\phi(x,y) = f(x)g(y)$) as Landau' solution is, but we can use Fourier transformation [21] on the variable "x",

$$\hat{\phi}(k,y) = \mathcal{F}[\phi] = \frac{1}{\sqrt{2\pi}}\int_{\Re} e^{ikx}\phi(x,y)dx, \tag{11}$$

to solve this equation. Applying Fourier transformation to this equation, knowing its property $\mathcal{F}[\partial\phi/\partial x] = (-ik)\hat{\phi}$, we get the ordinary differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2\hat{\phi}}{dy^2} + \frac{m}{2}\omega_c^2(y-y_0)^2\hat{\phi} = E'\hat{\phi}, \tag{12}$$

where ω_c is the cyclotron frequency

$$\omega_c = \frac{qB}{mc} \tag{13a}$$

and y_0 is the displacement parameter

$$y_0 = \frac{\hbar c}{qB}k. \tag{13b}$$

This equation is just the quantum harmonic oscillator in the "y" direction displaced by a amount y_0. So, the solution is

$$\hat{\phi}_n(k,y) = \quad_n(\xi), \quad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(y - y_0), \quad \psi_n(\xi) = A_n e^{-\xi^2} H_n(\xi), \tag{14}$$

being $H_n(\xi)$ the Hermit polynomials, and A_n is a constant of normalization, $An = (m\omega_c/\pi\hbar)^{1/4}/\sqrt{2^n n!}$. and

$$E'_n = \hbar\omega_c(n + 1/2). \tag{15}$$

Now, the solution in the real space $\phi_n(x,y)$ is gotten by using the inverse Fourier transformation,

$$\phi_n(x,y) = \mathcal{F}^{-1}[\phi_n(k,y)] = \frac{1}{\sqrt{2\pi}} \int_{\Re} e^{-ikx} \psi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}(y - \hbar ck/qB)\right) dk. \tag{16}$$

Making the change of variable $\sigma = \sqrt{m\omega_c/\hbar}(y - \hbar ck/qB)$, and knowing that the Fourier transformation of the harmonic oscillator solution is another harmonic oscillator solution, we get

$$\phi_n(x,y) = \frac{-qB}{\sqrt{mc^2\hbar\omega_c}} e^{-i\frac{qB}{\hbar c}xy} \psi_n\left(\frac{qB\ x}{\sqrt{mc^2\hbar\omega_c}}\right). \tag{17}$$

This is indeed the non separable solution of (8). Therefore, the normalized eigenfunctions of the eigenvalue problem (5) are (ignoring the sign)

$$\Phi_{n,k_z}(\mathbf{x},t) = \frac{\sqrt{qB}}{(mc^2\hbar\omega_c)^{1/4}} e^{-i(\frac{qB}{\hbar c}xy - k_z z)} \psi_n\left(\frac{qB\ x}{\sqrt{mc^2\hbar\omega_c}}\right). \tag{18a}$$

and

$$E_{n,k_z} = \hbar\omega_c(n + \frac{1}{2}) + \frac{\hbar^2 k_z^2}{2m}. \tag{18b}$$

These eigenvalues represent just the Landau's levels , but its solution (18a) is totally different to that given by Landau since it is of non separable type. Note that there is not displacement at all in the harmonic oscillation solution. Now, assuming a periodicity in the z-direction, $\Phi_{n,k_z}(\mathbf{x},t) = \Phi_{n,k_z}(x,y,z+L_z,t)$, the usual condition $k_z L_z = 2\pi n'$, $n' \in \mathcal{Z}$ makes the eigenvalues to be written as and the general solution of Schrödinger's equation (3) can be written as

$$E_{n,n'} = \hbar\omega_c(n + 1/2) + \frac{\hbar^2 2\pi^2}{mL_z^2}n'^2. \tag{19}$$

We must observed that this quantum numbers correspond to the degree of freedom in the "y (n)" and "z(n')" directions. The quantization conditions of the magnetic flux appears rather naturally since by asking periodicity in the y direction $\Psi(\mathbf{x},t) = \Psi(x,y+L_y,z,t)$, this one must be satisfied for any $x \in [0,L_x]$. So, in particular for $x = L_x$. Thus, it follows from the phase term that

$$\frac{qBL_xL_y}{\hbar c} = 2\pi j, \quad j \in \mathcal{Z}, \tag{20}$$

where BL_xL_y is the magnetic flux crossing the surface with area L_xL_y, and $\hbar c/q$ is the so called quantum flux [22]. Then, equation (18a) is

$$\Phi_{nn'j}(\mathbf{x},t) = \frac{\sqrt{qB}}{(mc^2\hbar\omega_c)^{1/4}} e^{-i(\frac{2\pi j}{L_xL_y}xy - \frac{2\pi n'}{L_z}z)} \psi_n\left(\frac{qB\ x}{\sqrt{mc^2\hbar\omega_c}}\right). \tag{21}$$

The degeneration of the eigenvalues (19) comes from the degree of freedom in "x" and can be obtained by making use the following quasi-classical argument: given the energy of the harmonic oscillator $E_o = \hbar\omega_c(n + 1/2)$, we know the the maximum displacement of the particle (classically) is given by $x_{max} = \pm\sqrt{2E_o/m\omega_c^2}$, and since the periodicity in the variable 'y" mentioned before is valid for any "x" value, we must have that the maximum value of the quantum number "j" must be

$$\Delta j = \frac{qBL_y}{\pi\hbar c}x_{max} = \frac{qBL_y}{\pi\hbar c}\sqrt{\frac{2\hbar(n+1/2)}{m\omega_c}}, \tag{22}$$

and this represents the degeneration, $D(n)$, we have in the system

$$D(n) = \left[\frac{qBL_y}{\pi\sqrt{mc^2\hbar\omega_c}}\sqrt{2n+1}\right]. \tag{23}$$

where $[\xi]$ means the integer part of the number ξ. Therefore, the general solution (absorbing the sign in the constants) is

$$\Psi(\mathbf{x},t) = \sum_{n,'n}\sum_{j=0}^{D(n)}C_{nn'j}\sqrt{\frac{2\pi j}{L_xL_y}}\left(\frac{\hbar}{m\omega_c}\right)^{1/4}e^{-i(\frac{2\pi j}{L_xL_y}xy-\frac{2\pi n'}{L_z}z)}e^{-i\frac{E_{n,n'}}{\hbar}t}\psi_n\left(\sqrt{\frac{\hbar}{m\omega_c}}\left(\frac{2\pi j}{L_xL_y}\right)x\right), \tag{24}$$

where the constants $C_{nn'j}$ must satisfy that $\sum_{n,n',j}|C_{nn'j}|^2 = 1$. The Landau's levels $E_{n,n'}$ are given by expression (19).

III. ANALYTICAL APPROACH FOR THE CASE B ⊥ E

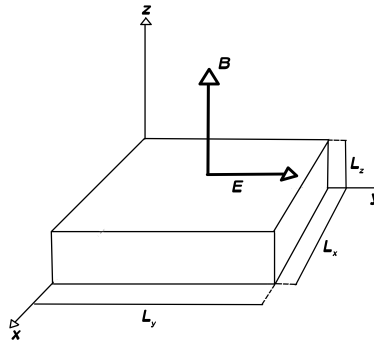This case is illustrated on the next figure,

Figure 2: Electric charged in a flat box with magnetic and electric fields

where the magnetic and electric constant fields are given by $\mathbf{B} = (0,0,B)$ and $\mathbf{E} = (0,\mathcal{E},0)$. We select Landau's gauge for the magnetic field such that the vector and scalar potentials are $\mathbf{A} = (-By,0,0)$ and $\phi = -\mathcal{E}y$. Then, our Hamiltonian is [23–25]

$$\hat{H} = \frac{(\hat{\mathbf{p}} - \frac{q}{c}\mathbf{A})^2}{2m} + q\phi(\mathbf{x},) \tag{25}$$

and the Schrödinger's equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi, \tag{26}$$

is written as

$$i\hbar\frac{\partial\Psi}{\partial t} = \left\{\frac{1}{2m}\left(\hat{p}_x + \frac{qB}{c}y\right)^2 + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m}\right\}\Psi - q\mathcal{E}y\Psi. \tag{27}$$

Using the definition $\hat{p}_j = -i\hbar\partial/\partial x_j$ and the commutation relation $[x_k, \hat{p}_j] = i\hbar\delta_j k$, the above expression is written as the following partial differential equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} - i\frac{qB\hbar}{mc}\frac{\partial\Psi}{\partial x} + \frac{q^2B^2}{2mc^2}y^2\Psi - \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial z^2} - q\mathcal{E}y\Psi. \tag{28}$$

Taking the Fourier transformation with respect the x-variable, $\hat{\Psi}(k, y, z, t) = \mathcal{F}_x[\Psi(\mathbf{x}, t)]$, the resulting expression is

$$i\hbar\frac{\partial\hat{\Psi}}{\partial t} = \left[\frac{\hbar^2k^2}{2m} - \left(\frac{qB\hbar k}{mc} + q\mathcal{E}\right)y + \frac{q^2B^2}{2mc^2}y^2\right]\hat{\Psi} - \frac{\hbar^2}{2m}\frac{\partial^2\hat{\Psi}}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2\hat{\Psi}}{\partial z^2}. \tag{29}$$

By proposing a solution of the form

$$\hat{\Psi}(k, yz, t) = e^{-iEt/\hbar + ik_z z}\Phi(k, y) \tag{30}$$

and after some rearrangements, the resulting equation for Φ is

$$-\frac{\hbar^2}{2m}\frac{d^2\Phi}{dy^2} + \frac{1}{2}m\omega_c^2(y - y_0)^2\Phi = E'\Phi, \tag{31}$$

where ω_c is the cyclotron frequency (13a), and we have made the definitions

$$y_0 = \frac{\hbar c}{qB}k + \frac{mc^2\mathcal{E}}{qB^2} \tag{32}$$

and

$$E' = E - \frac{\hbar^2k^2}{2m} - \frac{\hbar^2k_z^2}{2m} + \frac{1}{2m}\left(\hbar k + \frac{mc\mathcal{E}}{B}\right)^2. \tag{33}$$

This equation is again the quantum harmonic oscillator on the variable "y" with a cyclotron frequency ω_c and displaced by a quantity y_0. Therefore, the solution (14) is

$$\Phi(k, y) = \psi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}(y - y_0)\right) \tag{34}$$

and

$$E_n' = \hbar\omega_c(n + 1/2). \tag{35}$$

Thus, the solution in the Fourier space is

$$\hat{\Psi}(k, y, z, t) = e^{-iE_{n,k_z}t/\hbar + ik_z z}\psi_n\left(\sqrt{\frac{m\omega_c}{\hbar}}(y - y_0)\right) \tag{36}$$

with the energies E_{n,k_z} given by

$$E_{n,k_z} = \hbar\omega_c(n + 1/2) + \frac{\hbar^2k_z^2}{2m} - \frac{mc^2\mathcal{E}^2}{2B^2} - \frac{c\mathcal{E}\hbar}{B}k. \tag{37}$$

The solution in the space-time is obtained by applying the inverse Fourier transformation,

$$\Psi_{n,k_z}(\mathbf{x}, t) = \mathcal{F}[\hat{\Psi}_{n,k_z}(k, y, z, t)] = \frac{1}{\sqrt{2\pi}}\int_{\Re}e^{-ixk}\hat{\Psi}_{n,k_z}(k, y, z, t)dk, \tag{38}$$

which after a proper change of variable and rearrangement, we get the normalized function (ignoring the sign)

$$\Psi_{n,k_z}(\mathbf{x}, t) = \frac{\sqrt{qB}}{(mc^2\hbar\omega_c)^{1/4}}e^{-i\phi_{n,k_z}(\mathbf{x}, t)}\psi_n\left(\frac{qB}{\sqrt{mc^2\hbar\omega_c}}\left(x - \frac{c\mathcal{E}t}{B}\right)\right), \tag{39}$$

where the phase $\phi_{n,k_z}(\mathbf{x},t)$ has been defined as

$$\phi_{n,k_z}(\mathbf{x},t) = \left[\hbar\omega_c(n+1/2) + \frac{\hbar^2 k_z^2}{2m} - \frac{mc^2\mathcal{E}^2}{2B^2}\right]\frac{t}{\hbar} - k_z z + \frac{qB}{\hbar c}\left(x - \frac{c\mathcal{E}t}{B}\right)\left(y - \frac{mc^2\mathcal{E}}{qB^2}\right). \qquad (40)$$

asking for the periodicity with respect the variable "z", $\Psi_{n,k_z}(\mathbf{x},t) = \Psi_{n,k_z}(z,y,z+L_z,t)$, it follows that $k_z L_z = 2\pi n'$ where n' is an integer number , and the above phase is now written as

$$\phi_{nn'}(\mathbf{x},t) = \left[\hbar\omega_c(n+1/2) + \frac{\hbar^2 2\pi^2 n'^2}{mL_z^2} - \frac{mc^2\mathcal{E}^2}{2B^2}\right]\frac{t}{\hbar} - \frac{2\pi n'}{L_z}z + \frac{qB}{\hbar c}\left(x - \frac{c\mathcal{E}t}{B}\right)\left(y - \frac{mc^2\mathcal{E}}{qB^2}\right). \qquad (41)$$

Note from this expression that the term $e^{-i\phi(\mathbf{x},t)}$ contains the element $e^{i\frac{qB}{\hbar c}xy}$, and by assuming the periodic condition $\Psi(\mathbf{x},t) = \Psi(x,y+L_y,z,t)$, will imply that $\Psi(\mathbf{x},t)$ will be periodic with respect the variable "y", for any "x" at any time "t." In particular, this will be true for $x = L_x$. This bring about the quantization of the magnetic flux of the form

$$\frac{qBL_xL_y}{\hbar c} = 2\pi j, \quad J \in \mathcal{Z} , \qquad (42)$$

obtaining the same expression as (20), and this phase is now depending of the quantum number "j"

$$\phi_{nn'j}(\mathbf{x},t) = e_{nn'}t/\hbar - \frac{2\pi n'}{L_z}z + \frac{2\pi j}{L_xL_y}xy - \frac{2\pi j}{L_xL_y}\left[\frac{mc^2\mathcal{E}}{qB^2}x + \frac{c\mathcal{E}}{B}ty\right]. \qquad (43)$$

where $e_{nn'}$ is the energy associated to the system,

$$e_{n,n'} = \hbar\omega_c(n+1/2) + \frac{2\pi^2\hbar^2}{mL_z^2}n'^2 + \frac{mc^2\mathcal{E}^2}{2B^2}. \qquad (44)$$

In this way, from these relations and the expression (39) we have a family of solutions $\{\Psi_{nn'j}(\mathbf{x},t)\}_{n,n',j\in\mathcal{Z}}$ of the Schrödinger equation (27),

$$\Psi_{nn'j}(\mathbf{x},t) = \sqrt{\frac{2\pi j}{L_xL_y}}\left(\frac{\hbar}{m\omega_c}\right)^{1/4}e^{-i\phi_{nn'j}(\mathbf{x},t)}\,_n\left(\sqrt{\frac{\hbar}{m\omega_c}}\left(\frac{2\pi j}{L_xL_y}\right)\left(x - \frac{c\mathcal{E}t}{B}\right)\right), \qquad (45)$$

Now, by the same arguments we did in the previous case, the degeneration of the systems would be given by (23), and the general solution would be of the form

$$\Psi(\mathbf{x},t) = \sum_{n,n'}\sum_{j=0}^{D(n)}\widetilde{C}_{nn'j}\Psi_{nn'j}(\mathbf{x},t). \qquad (46)$$

IV. ANALYTICAL APPROACH FOR THE CASE B || E
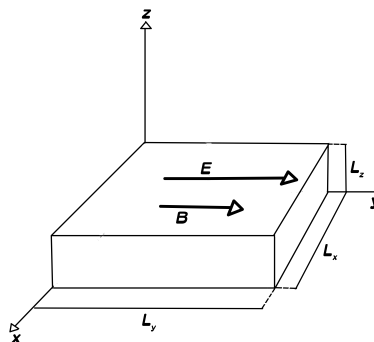
The following figure shows this case.

Figure 3: Electric charged in a flat box with parallel electric and magnetic fields

The fields are of the form $\mathbf{B} = (0, B, 0)$ and $\mathbf{E} = (0, \mathcal{E}, 0)$. The scalar and vector potentials are chosen as $\mathbf{A} = (Bz, 0, 0)$ and $\phi = -\mathcal{E}y$. The Shrödinger equation is for this case as

$$i\hbar\frac{\partial\Psi}{\partial t} = \left\{\frac{(\hat{p}_x - qBz/c)^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} - q\mathcal{E}y\right\}\Psi, \tag{47}$$

which defines the following partial differential equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + i\frac{qB\hbar z}{mc}\frac{\partial\Psi}{\partial x} + \frac{q^2B^2}{2mc^2}z^2\Psi - \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial z^2} - q\mathcal{E}y\Psi. \tag{48}$$

Proposing a solution of the form $\Psi(\mathbf{x}, t) = e^{-iEt/\hbar}\Phi(\mathbf{x})$, we get the following eigenvalue problem

$$E\Phi = -\frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial x^2} + i\frac{qB\hbar z}{mc}\frac{\partial\Phi}{\partial x} + \frac{q^2B^2}{2mc^2}z^2\Phi - \frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial z^2} - q\mathcal{E}y\Phi. \tag{49}$$

Applying the Fourier transformation over the x-variable, $\hat{\Phi}(k, y, z) = \mathcal{F}_x[\Phi(\mathbf{x})]$, the following equation arises after some rearrangements

$$E\hat{\Phi} = \frac{(\hbar k + qBz/c)^2}{2m}\hat{\Phi} - \frac{\hbar^2}{2m}\frac{\partial^2\hat{\phi}}{\partial z^2} - \frac{\hbar^2}{2m}\frac{\partial^2\hat{\Phi}}{\partial y^2} - q\mathcal{E}y\hat{\Phi}, \tag{50}$$

which can be written as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\hat{\Phi}}{\partial z^2} + \frac{1}{2}m\omega_c(z + z_0)^2\hat{\Phi} - \frac{\hbar^2}{2m}\frac{\partial^2\hat{\Phi}}{\partial y^2} - q\mathcal{E}y\hat{\Phi}, \tag{51a}$$

where ω_c is the cyclotron frequency (13a), and z_0 has been defined as

$$z_0 = \frac{\hbar c}{qB}k. \tag{51b}$$

This equation admits a variable separable approach since by the proposition $\hat{\Phi}(k, y, z) = f(k, z)g(y)$, the following equations are bringing about

$$-\frac{\hbar^2}{2m}\frac{d^2f}{dz^2} + \frac{1}{2}m\omega_c^2(z + z_0)^2 = E^{(1)}f \tag{52a}$$

and

$$-\frac{\hbar^2}{2m}\frac{d^2g}{dy^2} - g\mathcal{E}yg = E^{(2)}g, \tag{52b}$$

where $E = E^{(1)} + E^{(2)}$. The solutions of these equations are, of course, the quantum harmonic oscillator and the quantum bouncer, which are given by

$$f_n(k, z) = A_n e^{-\xi^2/2}H_n(\xi), \qquad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(z + z_0), \qquad E_n^{(1)} = \hbar\omega_c(n + 1/2). \tag{53a}$$

and

$$g_{n'}(y) = \frac{Ai(\tilde{\xi} - \tilde{\xi}_{n'})}{|Ai'(-\tilde{\xi}_{n'})|}, \qquad \tilde{\xi} = y/l, \qquad E_n^{(2)} = -q\mathcal{E}l\tilde{\xi}_{n'}, \tag{53b}$$

where $A_n = (m\omega_c/\pi\hbar)^{1/4}/\sqrt{2^n n!}$, $l = (\hbar^2/(-2mq\mathcal{E}))^{1/3}$, $Ai(-\tilde{\xi}_{n'}) = 0$, and $Ai'(\xi)$ is the differentiation of the Airy function. In this way, we have

$$\hat{\Phi}_{n,n'}(k, y, z) = a_{n'}\ _n\left(\sqrt{\frac{m\omega_c}{\hbar}}(z + z_0)\right)Ai(l^{-1}(y - y_{n'})), \qquad E_{n,n'} = \hbar\omega_c(n + 1/2) - q\mathcal{E}y_{n'}, \tag{54}$$

where we have defined $a_{n'}$ as $a_{n'} = 1/|Ai'(-l^{-1}y_{n'})|$. Now, the inverse Fourier transformation will affect only the quantum harmonic oscillator function ψ_n through the k-dependence on the parameter z_0, and the resulting expression is

$$\Phi_{n,n'}(\mathbf{x}) = \frac{a_{n'}qB}{\sqrt{mc^2\hbar\omega_c}}e^{i\frac{qB}{\hbar c}xz}\psi_n\left(\frac{qBx}{\sqrt{mc^2\hbar\omega_c}}\right)Ai\left(l^{-1}(y - y_{n'})\right). \tag{55}$$

Now, asking for the periodicity condition of the above solution with respect the z-variable, $\Psi(\mathbf{x}, t) = \Psi(x, y, z + L_z, t)$, the periodicity must satisfy for any x-values, and in particular for $x = L_x$. Thus it follows the quantization expression for the magnetic flux

$$\frac{qBL_xL_z}{\hbar c} = 2\pi j, \qquad j \in \mathcal{Z}. \tag{56}$$

Using the same arguments shown above for the degeneration of the system, we have the same expression (23) for the degeneration of the system and the function (55) is given by (normalized)

$$\Phi_{nn'j}(\mathbf{x}) = a_{n'}\sqrt{\frac{2\pi j}{L_xL_y}}\left(\frac{\hbar}{m\omega_c}\right)^{1/4}e^{i\frac{2\pi j}{L_xL_z}xz}\psi_n\left(\sqrt{\frac{\hbar}{m\omega_c}}\left(\frac{2\pi j}{L_xL_y}\right)x\right)Ai\left(l^{-1}(y - y_{n'})\right). \tag{57}$$

Then, we have obtained a family of solution of the Schrödinger equation (48),

$$\Psi_{n,n'}(\mathbf{x}, t) = e^{-iE_{n,n'}t/\hbar}\Phi_{nn'j}(\mathbf{x}), \tag{58}$$

where the energies $E_{n,n'}$ are given by the expression (54). The general solution of (48) can be written as

$$\Psi(\mathbf{x}, t) = \sum_{n,n'}\sum_{j=0}^{D(n)}C_{n,n'}^*e^{-iE_{n,n'}t/\hbar}e^{i\frac{2\pi j}{L_xL_z}xz}\tilde{u}_{n,n'}(x, y), \tag{59}$$

with the condition $\sum_{n,n'}|C_{n,n'}^*|^2 = 1$, and where it has been defined the functions $\tilde{u}_{n,n'}$ as

$$\tilde{u}_{n,n'}(x, y) = a_{n'}\sqrt{\frac{2\pi j}{L_xL_y}}\left(\frac{\hbar}{m\omega_c}\right)^{1/4}\psi_n\left(\sqrt{\frac{\hbar}{m\omega_c}}\left(\frac{2\pi j}{L_xL_y}\right)x\right)Ai\left(l^{-1}(y - y_{n'})\right). \tag{60}$$

a) Same system but with new magnetic gauge

Let us consider the magnetic gauge given such that the vector potential is of the form $\mathbf{A} = (0, 0, -Bx)$, and the potential is the same $\phi = -\mathcal{E}y$. Passing directly to the eigenvalue problem for the Schrödinger equation when we select the wave function of the form $\Psi(\mathbf{x}, t) = e^{-iEt/\hbar}\Phi(\mathbf{x})$, the resulting equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial y^2} - \frac{\hbar^2}{2m}\frac{\partial^2\Phi}{\partial z^2} - i\frac{qB\hbar}{mc}x\frac{\partial\Phi}{\partial z} + \frac{q^2B^2}{2mc^2}x^2\Phi - q\mathcal{E}y\Phi = E\Phi. \tag{61}$$

Taking the Fourier transformation with respect the z-variable, $\hat{\Phi}(x, y, k) = \mathcal{F}_z[\Phi(\mathbf{x})]$, and making some rearrangements, it follows that

$$-\frac{\hbar^2}{2m}\frac{\partial^2\hat{\Phi}}{\partial x^2} + \frac{1}{2m}\left(\hbar k - \frac{qB}{c}x\right)^2\hat{\Phi} - \frac{\hbar^2}{2m}\frac{\partial^2\hat{\Phi}}{\partial y^2} - q\mathcal{E}y\hat{\Phi} = E\hat{\Phi}. \tag{62}$$

This equation admits a variable separable solution of the form $\hat{\Phi}(x, y, k) = \phi_1(k, x)\phi_2(y)$, where the functions ϕ_1 and ϕ_2 satisfy the equations

$$-\frac{\hbar^2}{2m}\frac{d^2\phi_1}{dx^2} + \frac{(\hbar k - \frac{qB}{c}x)^2}{2m}\phi_1 = E^{(1)}\phi_1 \tag{63}$$

and

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi_2}{\partial y^2} - q\mathcal{E}y\phi_2 = E^{(2)}\phi_2, \tag{64}$$

where $E = E^{(1)} + E^{(2)}$. The solution of these equations are

$$\phi_{1n}(k,x) = \; _n(\xi) = A_n e^{-\xi^2/2} H_n(\xi), \quad \xi = \sqrt{\frac{m\omega_c}{\hbar}}(x - x_0), \qquad E_n^{(1)} = \hbar\omega_c(n + 1/2) \tag{65}$$

and

$$\phi_{2n'}(y) = a_{n'} Ai(l^{-1}(y - y_{n'})), \quad l = \left(\frac{\hbar^2}{-2mq\mathcal{E}}\right)^{1/3}, \qquad E_{n'}^{(2)} = -q\mathcal{E}y_{n'}, \tag{66}$$

where ω_c is the cyclotron frequency (13a), x_0 is the displacement $x_0 = \hbar ck/qB$, $a_{n'} = 1/|Ai'(l^{-1}y_{n'})|$ is a constant, and A_n the constant associated to the quantum harmonic oscillator solution. The inverse Fourier transformation affect only the function ϕ_1, and we have

$$\phi_{1n}(z,x) = \mathcal{F}^{-1}[\phi_{1n}(k,x)] = \frac{-qB}{\sqrt{mc^2\hbar\omega_c}} e^{-i\frac{qB}{\hbar c}xz} \psi_n\left(\frac{qBz}{\sqrt{mc^2\hbar\omega_c}}\right). \tag{67}$$

The periodic condition on the variable "x", $\Psi(\mathbf{x},t) = \Psi(x + L_x, y, z, t)$, for any value of the other variables, implies that this will happen in particular for the value of $z = L_z$. So, we get the quantization of the magnetic flux (BL_xL_y),

$$\frac{qBL_xL_z}{\hbar c} = 2\pi j, \qquad j \in \mathcal{Z}. \tag{68}$$

Thus, we have a family of solutions $\{\Psi_{nn'j}(\mathbf{x},t)\}$ of the Shcrödinger equation of the form

$$\Psi_{nn'j}(\mathbf{x},t) = e^{-iE_{n,n'}t/\hbar}\Phi_{nn'j}(\mathbf{x}), \tag{69}$$

or (normalized and ignoring the sign)

$$\Psi_{nn'j}(\mathbf{x},t) = a_{n'}\sqrt{\frac{2\pi j}{L_xL_y}}\left(\frac{\hbar}{m\omega_c}\right)^{1/4}e^{-i(E_{n,n'}\frac{t}{\hbar} + \frac{2\pi j}{L_xL_z}xz)}\psi_n\left(\sqrt{\frac{\hbar}{m\omega_c}}\left(\frac{2\pi j}{L_xL_y}\right)z\right)Ai(l^{-1}(y - y_{n'})). \tag{70}$$

By the same arguments about the degenerationn of the systems, the general solution is just a combination of all of these,

$$\Psi(\mathbf{x},t) = \sum_{n,n'} A_{nn'j}e^{-i(E_{n,n'}\frac{t}{\hbar} + \frac{2\pi j}{L_xL_z}xz)}v_{nn'j}(y,z), \tag{71}$$

where the condition $\sum_{n,n'}|A_{nn'j}|^2 = 1$ must be satisfied, and the function $v_{nn'j}$ is given by

$$v_{nn'j}(y,z) = a_{n'}\sqrt{\frac{2\pi j}{L_xL_y}}\left(\frac{\hbar}{m\omega_c}\right)^{1/4}\psi_n\left(\sqrt{\frac{\hbar}{m\omega_c}}\left(\frac{2\pi j}{L_xL_y}\right)z\right)Ai(l^{-1}(y - y_{n'})). \tag{72}$$

V. CONCLUSIONS AND COMMENTS

We have studied the quantization of a charged particle in a flat box and under constants magnetic and electric fields for several cases and have shown that a full separation of variable solution is not admitted in these cases (contrary to Landau's solution in one of these cases). This situation arises since the commutation of a component of the generalized linear momentum operator with the Hamiltonian of the system does not imply necessarily that a variable separation of its associated variable must exist in the Schrödinger equation. However, using the Fourier transformation, we were be able to find the full solution of the problems. As expected, Landau's level appears in all these cases, and a characteristic phase which help us to find the quantization of the magnetic flux in a natural way. We consider that the approach given here maybe very useful to understand quantum Hall effect and related phenomena.

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