Global Journal of Science Frontier Research: F
Mathematics & Decision Science
Volume 20 Issue 4 Version 1.0 Year 2020
Type : Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

On a New Conformal Euler-Lagrangian Equations on Para-Quaternionic *Kähler* Manifolds

By Gebreel Mohammed Khur Baba Gebreel,
Ibrahim Yousif Ibrahim Abad Alrhman & Younous Atim Idris Younous
West Kordufan University

Abstract- In this paper we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G\}$ of V that they defined on a generalized quaternionic *Kähler* manifold (M, g, V).

GJSFR-F Classification: MSC 2010: 11R52

Strictly as per the compliance and regulations of:

On a New Conformal Euler-Lagrangian Equations on Para-Quaternionic *Kähler* Manifolds

Gebreel Mohammed Khur Baba Gebreel [α], Ibrahim Yousif Ibrahim Abad Alrhman [σ] & Younous Atim Idris Younous [ρ]

Abstract- In this paper we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{\boldsymbol{F}, \boldsymbol{G}\}$ of V that they defined on a generalized quaternionic ***Kähler*** manifold (M, g, V).

I. Introduction

Today, many branches of science are into our lives. One the branches is mathematics that has multiple applications. In particular , differential geometry and mathematical physics have a lots of different applications. One of them are on geodesics. Geodesics are known the shortest route between two points. Time-dependent equations of geodesics can be easily found with the help of the Euler-Lagrange equations. We can say that differential geometry provides a good working area for studying Lagrangians of classical mechanics and field theory. The dynamic equation for moving bodies is obtained for Lagrangian mechanic. These dynamic equation is illustrated as follows:

Lagrange Dynamics Equation [1,2,3]: let M be an n-dimensional manifold and TM its tangent bundle with canonical projection $TM\colon TM \to M$ is called the phase space of velocities of the base manifold M.

Let $L\colon TM \to R$ be differentiable function on TM called the Lagrangian function. We consider the closed 2-form on TM given by $\Phi_L = dd_J L$ (if $J^2 = -I$, J is a complex structure and if $J^2 = I$, J is a paracomplex structures, $T_r(J) = 0$) Consider the equation:

$$i_X \Phi_L = dE_L \qquad \rightarrow \qquad (1)$$

Then X is a vector field, we shall see that (1) under a certain condition on X is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as Lagrange dynamical equation. We shall see that for motion in potential, $E_L = V(L) - L$ is an energy function and $V = J(X)$ a Liouville vector field. Here dE_L denotes the differential of E. The triple (TM, Φ_L, X) is known as Lagrangian system on

Author α σ ρ: Department of Mathematics, Faculty of Education, West Kordufan University, Khartoum-Sudan.
e-mail: gebreel6600@gmail.com

the tangent bundle TM. If it is continued the operations on (1) for any coordinate system $(q^i(t), p_i(t))$, infinite dimension Lagrange's equation is obtained the form below:

$$\frac{dq^i}{dt} = \dot{q}^i \ , \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} \ , \quad i = 1, \ldots, n \qquad \rightarrow \qquad (2)$$

There are many studies about Lagrangian dynamics, mechanics, formalisms, systems and equations (see detail [4]). There are real, complex, paracomplex and other analogues. It is well-known that Lagrangian analogues are very important tools. They have a simple method to describe the model for mechanical systems. The models about mechanical systems are given as follows.

Some examples of the Lagrangian is applied to model the problems include harmonic oscillator, charge Q in electromagnetic fields, Kepler problem of the earth in orbit around the sun, pendulum, molecular and fluid dynamics. LC networks, Atwood's machine, symmetric to etc. Let's remember some work done. Vries shown that the Lagrangian motion equations have a very simple interpretation in relativistic quantum mechanics [5]. Paracomplex analogues of the Euler-Lagrange equations was obtained in the framework of Para-*Kählerian* manifold and the geometric results on a paracomplex mechanical systems were found by Tekkoyun [6]. Electronic origins, molecular dynamics simulations, computational nanomechanics, multiscale modeling of materials fields were contributed by Liu [7]. Bi-paracomplex analogue of Lagrangian systems was shown on Lagrangian distributions by Tekkoyun and sari [8]. Tekkoyun and Yayli presented generalized-quaternionic *Kählerian* analogue of Lagrangian and Hamiltonian mechanical systems. Eventually, the geometric-physical results related to generalized-quaternionic *Kählerian* mechanical systems are provided [9].

Nowadays, there are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations [2, 4, 10, 11, 12] and there in. There are real, complex, paracomplex and other analogues. As known it is possible to produce different analogous in different spaces. Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = -1 \, , \ ijk = -1.$$

Generalized quaternions are defined as

$$i^2 = -a \, , j^2 = -b \, , k^2 = -ab \, , ijk = -ab.$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. Lots of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra [13, 14, 15, 16, 17, 18, 19].

In the present paper, we present equations related to Lagrangian mechanical systems on generalized-quaternionic *Kähler* manifold.

II. Preliminaries

In this study, all the manifolds and geometric objects are C^∞ and the Einstein summation convention is in use. Also, $A, F(TM), \chi(TM)$ and $\Lambda^1(TM)$ denote the set of

Ref

6. M. Tekkoyun, On para-Euler Lagrange and para-Hamiltonian equations, Physics Letters A, Vol. 340, 7-11, (2005).

Global Journal of Science Frontier Research (F) Volume XX Issue IV Version I Year 2020

paracomplex numbers, the set of (para)-complex functions on TM, the set of (para)-complex vector fields on TM and the set of (para)-complex 1-forms on TM, respectively. The definitions and geometric structures on the differential manifold M given in [20] may be extended to TM as follows:

a) Theorem

Let f be differentiable ϕ, ψ are 1-form , then [21]

- $d(f\phi) = df \wedge \phi + f d\phi$
- $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

b) Definition (Kronecker's delta)

Kornecker's delta denote by δ and defined as follows [22]:

$$\delta_i^j = \begin{cases} 1 & , \ if \quad i = j \\ 0 & , \ if \quad i \neq j \end{cases}$$

III. Conformal Geometry

In mathematics, conformal map is a function which preserves angles. In the most common case the function is between domains in the complex plane. Conformal maps can be defined between domains in higher dimensional Euclidean spaces, and more generally on a Riemann or semi-Riemann manifold. Conformal geometry is the study of the set of angle-preserving (conformal) transformations on space. In two real dimensions, conformal geometry is precisely the geometry of Riemann surfaces. In more than two dimensions, conformal geometry may refer either to the study of conformal transformations of " flat" spaces (such as Euclidean spaces or spheres), or more commonly, to the study of conformal manifolds which are Riemann or pseudo-Riemann manifolds with a class of metrics defined up to scale. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics $\acute{g}$ *and* g are equivalent if and only if:

$$\acute{g} = \lambda^2 g \qquad \rightarrow \qquad (3)$$

Where $\lambda > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class [23].

IV. Conformal Structure

The linear distance between two points can be found easily by Riemann metric, which is very useful and is defined inner product. Many scientists have used the Riemann metric. Einstein was one of the first studies in this field. Einstein discovered which the Riemannian geometry and successfully used it to describe General Relativity in the 1910 that is actually a classical theory for gravitation. However, the universe is really completely not like Riemannian geometry. Each path between two points is not always linear. Also, orbits of move objects may change during movement. So, each two points in space may not be linear geodesic and need not to be. Therefore, new metric is needed for non-linear distances like spherical surface. Then, a method is required for converting non-linear distance to linear. Wey1 introduced a metric with a conformal transformation in 1918.

Ref

20. V. Cruceanu, P.A. Gadea, J.M. Masque, Para-Hermitian and Para-*Kähler* manifolds (http://digital.csic.es/bitstream/10261/15773/1/Rocky CFG.PDF).

Definition 4.1: Let M an n-dimensional smooth manifold. A conformal structure on M is an equivalence class G of Riemann metrics on M. A manifold with a conformal structure is called a conformal manifold

(i) Two Riemann metrics g *and* $\acute{g}$ on M are said to be equivalent if and only if

$$\acute{g} = e^{\lambda} g \qquad \rightarrow \qquad (4)$$

Where λ is a smooth function on M. the equation given by (4) is called a conformal structure

(ii) A Wey1 structure on M is a map $F: G \rightarrow \Lambda^1 M$ satisfying

$$F(e^{\lambda} g) = F(g) - d\lambda \qquad \rightarrow \qquad (5)$$

Where G is a conformal structure. Note that a Riemann metric g and a one-form φ determine a Wey1 structure, namely $F: G \rightarrow \Lambda^1 M$ where G is the equivalence class of g and

$$F(e^{\lambda} g) = \varphi - d\lambda.$$

Theorem 4.1: A connection on the metric bundle φ of a conformal manifold M naturally induces a map $F: G \rightarrow \Lambda^1 M$ and (5), and conversely. Parallel translation of points in φ by the connection is the same as their translation by F [24].

V. Generalized- Quaternionic Kähler Manifolds

A generalized almost quaternion structure on the manifold M is a sub bundle of the bundle of endomorphism's of the tangent bundle M, whose standard fiber is the algebra of quaternions. A generalized almost quaternion structure on a pseudo-Riemannian manifold is called a generalized quaternion-Hermitian if the following conditions hold:

i) The endomorphism's $F, G \text{ and } H \text{ of } T_x M$ satisfy

$$F^2 = -aI, G^2 = -bI, H^2 = -abI, FG = H, GH = bF, HF = aG, \qquad \rightarrow \qquad (6)$$

ii) The compatibility equations are given by for $X, Y \in T_x M$,

$$g(FX, FY) = ag(X, Y), g(GX, GY) = bg(X, Y), g(HX, HY) = abg(X, Y) \qquad \rightarrow \qquad (7)$$

Where I denotes the identity tensor of type (1, 1) in M. In particular, 2-form Q defined by $Q(X, Y) = (X, FY) = (X, GY) = (X, HY)$ on M is called the $Kähler$ form Q on M is closed, i.e. $dQ = 0$, the manifold M is called a generalized-quaternionic $Kähler$ manifold [25].

If $a = b = 1$, M is quaternion manifold. If $a = 1, b = -1$, M is Para-quaternion manifold. The bundle V is a set that locally admits basis $\{F, G, H\}$ satisfying (6) and (7) in any coordinate neighborhood $U \subset M \text{ such that } M = \cup U$ [14]. Then V is called a generalized-quaternionic structure in M. The pair (M, V) denotes a generalized-quaternionic manifold with V. The structure V with such a Riemannian metric g is called a generalized-quaternionic metric structure. The triple (M, g, V) denotes a generalized-quaternionic metric manifold. Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}, i = \overline{1, n}$ be a real coordinate system on a neighborhood U of M, and let $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}\}$ and

Ref

24. G. B. Folland, Wey1.manifolds, J. Differential Geometry, 4, 145-153, (1970).

Global Journal of Science Frontier Research (F) Volume XX Issue IV Version I Year 2020

$\{dx_i\,, dx_{n+i}\,, dx_{2n+i}\,, dx_{3n+i}\}$ be natural bases over R of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M, respectively. Taking into consideration (6), then we can obtain the expressions an follows:

$$F\left(\frac{\partial}{\partial x_i}\right) = a\frac{\partial}{\partial x_{n+i}}\,, F\left(\frac{\partial}{\partial x_{n+i}}\right) = -a\frac{\partial}{\partial x_i}\,, F\left(\frac{\partial}{\partial x_{2n+i}}\right) = a\frac{\partial}{\partial x_{3n+i}}\,, F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -a\frac{\partial}{\partial x_{2n+i}}$$

$$G\left(\frac{\partial}{\partial x_i}\right) = -b\frac{\partial}{\partial x_{2n+i}}, G\left(\frac{\partial}{\partial x_{n+i}}\right) = b\frac{\partial}{\partial x_{3n+i}}, G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -b\frac{\partial}{\partial x_i}, G\left(\frac{\partial}{\partial x_{3n+i}}\right) = b\frac{\partial}{\partial x_{n+i}} \rightarrow \quad (8)$$

$$H\left(\frac{\partial}{\partial x_i}\right) = -ab\frac{\partial}{\partial x_{3n+i}}, H\left(\frac{\partial}{\partial x_{n+i}}\right) = -ab\frac{\partial}{\partial x_{2n+i}}, H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -ab\frac{\partial}{\partial x_{n+i}}, H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -ab\frac{\partial}{\partial x_i}$$

VI. Generalized-Quaternionic Conformal Kähler Manifolds

Definition 6.1: Let $(M\,, g\,, \nabla\,, J\pm)$ be an almost para/pseudo-Hermitian Wey1 manifold. If $\nabla(J\pm) = 0$, then one says that is a (para)-$K\ddot{a}hler$ Wey1 manifold. Note that necessarily $J\pm$ is integrable in this setting.

Theorem 6.1: If $(M\,, g\,, \nabla\,, J\pm)$ is a (para)-$K\ddot{a}hler$ Wey1 manifold with dimension $n \geq 6$ *and with* $H^1(M; R) = 0$ then the underlying wey1 structure on M is trivial.

Theorem 6.2: If $(M\,, g\,, \nabla\,, J\pm)$ is a curvature (para)-$K\ddot{a}hler$ Wey1 manifold with dimension $n \geq 6$ *and with* $H^1(M; R) = 0$ then the underlying wey1 structure on M is trivial.

Theorem 6.3: Let $n \geq 6$. If $(M\,, g\,, \nabla\,, J\pm)$ is a $K\ddot{a}hler - wey1$ structure, then the associated wey1 structure is trivial, i.e. there is a conformally equivalent metric $\tilde{g} = e^{2f} g$ *so that* $(M, \tilde{g}, J\pm)$ *is* $K\ddot{a}hler$ and so that $\nabla = \nabla^{\tilde{g}}$ [26,27,28].

After this part W will be used instead of J. A manifold with a wey1 structure is known as wey1 manifold. λ second structure was chosen the minus sign. Because the condition of the structure required to provide. $W_{\pm}^2 = \pm Id$ [29]. If we rewrite (8) equation with conformal structure, we obtain the following equations:

$$W_F\left(\frac{\partial}{\partial x_i}\right) = ae^{\lambda}\frac{\partial}{\partial x_{n+i}}\,, W_F\left(\frac{\partial}{\partial x_{n+i}}\right) = -ae^{-\lambda}\frac{\partial}{\partial x_i}$$

$$W_F\left(\frac{\partial}{\partial x2n+_i}\right) = ae^{\lambda}\frac{\partial}{\partial x_{3n+i}}\,, W_F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -ae^{-\lambda}\frac{\partial}{\partial x_{2n+i}}$$

$$W_G\left(\frac{\partial}{\partial x_i}\right) = -be^{\lambda}\frac{\partial}{\partial x_{2n+i}}\,, W_G\left(\frac{\partial}{\partial x_{n+i}}\right) = be^{\lambda}\frac{\partial}{\partial x_{3n+i}} \qquad \rightarrow \qquad (9)$$

$$W_G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -be^{-\lambda}\frac{\partial}{\partial x_i}\,, W_G\left(\frac{\partial}{\partial x_{3n+i}}\right) = be^{-\lambda}\frac{\partial}{\partial x_{n+i}}$$

$$W_H\left(\frac{\partial}{\partial x_i}\right) = -abe^{\lambda}\frac{\partial}{\partial x_{3n+i}}\,, W_H\left(\frac{\partial}{\partial x_{n+i}}\right) = -abe^{\lambda}\frac{\partial}{\partial x_{2n+i}}$$

$$W_H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -abe^{-\lambda}\frac{\partial}{\partial x_{n+i}}\,, W_H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -abe^{-\lambda}\frac{\partial}{\partial x_i}$$

Ref

29. R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabau, The geometry of Hamilton and Lagrange spaces, eBook ISBN:0-306-47135-3, Kluwer Academic Publishers, New York, (2002).

We continue our studies thinking of the $(M, g, \nabla, W_{\pm})$ instead of the almost para/pseudo- $Kähler - wey1$ manifolds $(M, g, \nabla, J\pm)$.

VII. Conformal Euler-Lagrangian Mechanical System

Here, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}\, of\, V$ that they defined on a generalized-quternionic $Kähler$ manifold (M, g, V).

First:

$$a\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{n+i}}=0\ , a\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_i}=0$$

$$a\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_{3n+i}}=0\ , a\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_{2n+i}}=0$$

Second: Let G take a local basis element on the generalized-quternionic $Kähler$ manifold (M, g, V), and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field Y determined by

$$Y = Y^i\frac{\partial}{\partial x_i}+Y^{n+i}\frac{\partial}{\partial x_{n+i}}+Y^{2n+i}\frac{\partial}{\partial x_{2n+i}}+Y^{3n+i}\frac{\partial}{\partial x_{3n+i}} \qquad \rightarrow \qquad (10)$$

Where $Y^i = \dot{x}_i, Y^{n+i} = \dot{x}_{n+i}, Y^{2n+i} = \dot{x}_{2n+i}, Y^{3n+i} = \dot{x}_{3n+i}$ and the dot indicates the derivative with respect to time t.

The vector field defined by:

$$V_G(L) = G(Y) = -bY^ie^{\lambda}\frac{\partial L}{\partial x_{2n+i}}+bY^{n+i}e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}-bY^{2n+i}e^{-\lambda}\frac{\partial L}{\partial x_i}+bY^{3n+i}e^{-\lambda}\frac{\partial L}{\partial x_{n+i}}.$$

is named a conformal Liouville vector field on the generalized-quternionic $Kähler$ manifold(M, g, V). For G the closed generalized-quternionic $Kähler$ form is the closed 2-form given by $\Phi_L^G = -dd_GL$ such that

$$d_GL = -be^{\lambda}\frac{\partial L}{\partial x_{2n+i}}dx_i + be^{\lambda}\frac{\partial L}{\partial x_{3n+i}}dx_{n+i} - be^{-\lambda}\frac{\partial L}{\partial x_i}dx_{2n+i}$$

$$+be^{-\lambda}\frac{\partial L}{\partial x_{n+i}}dx_{3n+i} : \ \mathcal{F}(M) \ \rightarrow \Lambda^1 M \qquad \rightarrow \qquad (11)$$

Then we have

$$\Phi_L^G = be^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j\wedge dx_i + be^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j\wedge dx_i - be^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_j\wedge dx_{n+i} -$$

$$be^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j\wedge dx_{n+i} + be^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j\wedge dx_{2n+i} + be^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j\wedge dx_{2n+i} -$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_j\wedge dx_{3n+i} - be^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j\wedge dx_{3n+i} + be^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{n+j}\wedge dx_i +$$

Notes

Notes

$$be^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j}\wedge dx_i - be^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j}\wedge dx_{n+i} - be^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j}\wedge dx_{n+i} +$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{n+j}\wedge dx_{2n+i} + be^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j}\wedge dx_{2n+i} - be^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{n+j}\wedge dx_{3n+i}$$

$$--be^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j}\wedge dx_{3n+i} + be^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{2n+j}\wedge dx_i + be^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j}\wedge dx_i$$

$$be^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{2n+j}\wedge dx_{n+i} - be^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j}\wedge dx_{n+i} +$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j}\wedge dx_{2n+i} +$$

$$be^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j}\wedge dx_{2n+i} - be^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{2n+j}\wedge dx_{3n+i} -$$

$$be^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j}\wedge dx_{3n+i}$$

$$+ be^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{3n+j}\wedge dx_i + be^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j}\wedge dx_i -$$

$$be^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{3n+j}\wedge dx_{n+i} -$$

$$be^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j}\wedge dx_{n+i} + be^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}dx_{3n+j}\wedge dx_{2n+i} +$$

$$be^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j}\wedge dx_{2n+i} -$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j}\wedge dx_{3n+i} - be^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j}\wedge dx_{3n+i} \qquad \rightarrow \qquad (12)$$

Then we calculate

$$i_Y\Phi_L^G = bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}\delta_i^j dx_i - bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_i - bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j$$

$$-bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}\delta_i^j dx_{n+i} + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_j - bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_{n+i} +$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j + bY^i e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}\delta_i^j dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j +$$

$$bY^i e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}\delta_i^j dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j - bY^i e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}\delta_i^j dx_{3n+1} +$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_j - bY^i e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{3n+i} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j +$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_i - bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{n+j} + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_i -$$

$$bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} - bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{n+i} + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{n+i} + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} + bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}\delta_{n+i}^{n+j}dx_{2n+i}$$

$$-bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{n+j} + bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}\delta_{n+i}^{n+j}dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j} -$$

$$bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{3n+1} + bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{n+j} -$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{3n+i}$$

$$+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} + bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_i -$$

$$bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{2n+j} +$$

$$bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_i - bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} -$$

$$bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{n+i} +$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{2n+j} - bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{n+i} +$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} +$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}\delta_{2n+i}^{2n+j}dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} +$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}\delta_{2n+i}^{2n+j}dx_{2n+i}$$

$$-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i} +$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{2n+j}\cdot$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_i$$

$$-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{3n+j} + bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_i - bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} -$$

Notes

Notes

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{n+i}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{3n+j}-$$

$$bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{n+i}$$

$$+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i}-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}dx_{3n+j}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i}-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j}-$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+1}$$

$$+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j}-bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} \quad \rightarrow \quad (13)$$

Energy function is:

$$E_L^G=V_G(L)-L=-bY^{i}e^{\lambda}\frac{\partial L}{\partial x_{2n+i}}+bY^{n+i}e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}-bY^{2n+i}e^{-\lambda}\frac{\partial L}{\partial x_i}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial L}{\partial x_{n+i}}-L \quad \rightarrow \quad (14)$$

And hence

$$dE_L^G=-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j-bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_j+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j-bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_j+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j-$$

$$bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{n+j}-bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j}+$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j}-bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{n+j}-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j}-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{2n+j}-$$

$$bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{2n+j}+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j}-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j}-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{2n+j}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j}-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{3n+j}-bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j}+$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{3n+j}+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j}-bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{i}}dx_{3n+j}-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}dx_{3n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j}-$$

$$\frac{\partial L}{\partial x_j}dx_j-\frac{\partial L}{\partial x_{n+j}}dx_{n+j}-\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}-\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}\quad\rightarrow\qquad(15)$$

By means of Eq(1), we calculate the following expressions:

$$bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}\delta_i^j dx_i+bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_i-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}\delta_i^j dx_{n+i}-bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_{n+i}$$

$$+\ bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}\delta_i^j dx_{2n+i}+bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}\delta_i^j dx_{2n+i}-bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}\delta_i^j dx_{3n+i}-$$

$$bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{3n+i}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_i+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_i-$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{n+i}-bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{n+i}+$$

$$bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}\delta_{n+i}^{n+j}dx_{2n+i}+$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}\delta_{n+i}^{n+j}dx_{2n+i}-bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{3n+1}-$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{3n+i}$$

$$+\ bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_i+bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_i-$$

$$bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{n+i}-bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{n+i}+$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}\delta_{2n+i}^{2n+j}dx_{2n+i}+bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}\delta_{2n+i}^{2n+j}dx_{2n+i}-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i}-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i}+$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_i+bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_i-$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{n+i}-bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{n+i}+$$

Notes

Notes

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i} -$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+1} - bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i} + \frac{\partial L}{\partial x_j}dx_j +$$

$$\frac{\partial L}{\partial x_{n+j}}dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} = 0 \qquad \rightarrow \qquad (16)$$

And thus

$$bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_j +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_j +$$

$$bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_j + bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_j - bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} - bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} - bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_{n+j} - bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} - bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{n+j} + bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_{2n+j} + bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} +$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} + bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} + bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_{2n+j} +$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{2n+j} + bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{2n+j} -$$

$$bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} - bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} -$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} - bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_{3n+j} - bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{3n+j} -$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{3n+j} - bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} + \frac{\partial L}{\partial x_j}dx_j + \frac{\partial L}{\partial x_{n+j}}dx_{n+j} +$$

$$\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} = 0 \qquad \rightarrow \qquad (17)$$

OR

$$\left[bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}+bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}+\right.$$

$$\left.bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}\right]dx_j+$$

$$\left[bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}+bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}+\right.$$

$$\left.bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\right]dx_j+\frac{\partial L}{\partial x_j}dx_j$$

$$-\left[bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}+bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}+\right.$$

$$\left.bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}\right]dx_{n+j}-$$

$$\left[bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}+bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}+\right.$$

$$\left.bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\right]dx_{n+j}$$

$$+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}+$$

$$\left[bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}+bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}+bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}\right]dx_{2n+j}+$$

$$\left[bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}+bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}+bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\right]dx_{2n+j}$$

$$+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}$$

$$-\left[bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}+bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}+bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}+\right.$$

$$\left.bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}\right]dx_{3n+j}$$

$$-\left[bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}+bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}+bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}+\right.$$

$$\left.bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\right]dx_{3n+j}$$

$$+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}=0$$

Notes

In this equation can be concise manner

$$be^{\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{2n+i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{2n+i}}\right]dx_j+\frac{\partial L}{\partial x_j}dx_j$$

$$-be^{\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{3n+i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{3n+i}}\right]dx_{n+j}+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}$$

$$+be^{-\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{i}}\right]dx_{2n+i}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}$$

$$-be^{-\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{n+i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{n+i}}\right]dx_{3n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}=0 \quad \rightarrow \quad (18)$$

Then we have the equations:

$$b\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{2n+i}}=0, \qquad b\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0$$

$$b\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_i}=0, \qquad b\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_{n+i}}=0 \quad \rightarrow \quad (19)$$

Such that the equations calculated in equation(19) are named Euler-Lagrange equations constructed on a generalized-quternionic $K\ddot{a}hler$ manifold (M,g,V) by means of Φ_L^G and thus the triple(M,Φ_L^G,X) is called a mechanical system on generalized-quternionic $K\ddot{a}hler$ manifold (M,g,V).

References Références Referencias

1. J. Klein, Escapes variationnals et Mecanique, Ann. Inst. Fourier, Grenoble, 12 , pp. 1-124,(1962).
2. M. De Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies, Vol. 152, (Elsevier, Amsterdam, 1989).
3. R. Abraham, J.E. Marsden, T. Ratiu, Manifolds, tensor analysis and applications, Springer, pp: 483-542, (2001).
4. M. De Leon, P.R. Rodrigues, Second-Order-Differential Equations and Non-Conservative Lagrangian Mechanics, J. Phys. A: Math. Gen. 20 (1987), 5393-5396.
5. H. de Vries, Understanding relativistic quantum field theory, The Hamiltonian and Lagrangian densities, Chapter 22, (http://www.physics-quest.org/book-Chapter-Lagrangian.pdf), (2009).
6. M. Tekkoyun, On para-Euler Lagrange and para-Hamiltonian equations, Physics Letters A, Vol. 340, 7-11, (2005).
7. W.K. Liu, S. Jun, Computational nanomechanics of materials, American Scientific Publishers, Stevenson Ranch, CA, (2005).
8. M. Tekkoyun, M.Sari., Bi-Para-mechanical systems on the bi-Lagrangian manifold, Physics B-condensed Matter, Vol. 405, Issue10,2390-2393,(2010).
9. M. Tekkoyun, Y. Yayli, Mechanical systems on Generalized-Quaternionic *Kähler* Manifolds, International Journal of Geometric Methods in Modern Physics, Vol. 8, No. 7, 1-13, (2011).

Notes

10. M. Zambine, Hamiltonian Perspeetive on Generalized complex Structure, Communications in Mechanical Physics. 263v(2006), 711-722.
11. M. Tekkoyun, On Para-Euler-Lagrange and Para-Hamiltonian Equations, Phys. Lett. A, 340 (2005), 7-11.
12. M. Tekkoyun, Mechanical Systems on Para-Quaternionic *Kähler* Manifolds, XVI. Geometrical Seminar, 20-25 September, Vrnjacka Banja, Serbia, 2009.
13. D. Stahlke, Quaternions in Classical Mechanics, Phys. 621. http://www.stahlke.org/dan/phys-papers/quaternion-paper.
14. A. S. Dancer- H. R. Jorgensen – A. F. Swann, Metric Geometries over the Split Quaternions, arXiv: 0412.215.
15. L. Kula L. and Y. Yayli, Split Quaternions and Rotations in Semi-Euclidean Space E_2^4, J. Korean Math. Soc. 44 (2007), 1313-1327.
16. L. Kula and Y. Yayli, Dual Split Quaternions and Serew Motion in Minkowski 3-Space Iranian Journal of Sci.&Tech.,Transaction A. 30 (2006).
17. E. Ata and Y. Yayli, Split Quaternions and Semi-Euclidean projective Spaces, Chaos, Solitons & Fractals. 41 (2009), 1910-1915.
18. E. Ata and Yayli, A Global Condition for the Triviality of an Almost Split Quaternionic Structure on Split Complex Manifolds, International Journal of Computational and Mathematical Sciences 2:1 (2008).
19. M. Jafari, Y. Yayli, Generalized Quaternions and their Algebraic properties, Submitted for publication.
20. V. Cruceanu, P.A. Gadea, J.M. Masque, Para-Hermitian and Para-*Kähler* manifolds (http://digital.csic.es/bitrstream/10261/15773/1/Rocky CFG.PDF).
21. Gebreel Mohammed Khur Baba Gebreel, M.A. Bashir. On a new dynamics on *Kähler* manifold using local canonical basis. International Journal of information research and review. Vol.4. Issue.2.February 2017.
22. Gebreel Mohammed Khur Baba, On a New Mechanics Systems on the Standard Cliffordian *Kähler* Manifolds Using A Canonical Local Basis, Global Journal of Science frontier research , Volume 17 Issue 4. 2017.
23. http://en.wikipedia.org/wiki/conformal-class.
24. G. B. Folland, Wey1.manifolds, J. Differential Geometry, 4, 145-153, (1970).
25. O E Arsen'eva, on the Geometry of Quaternion-Hermitan Manifolds, Russian Mathematical Surveys, 48(5):159 (1993).
26. P. Gilkey, S. $Nik\check{e}evi\acute{e}, K\ddot{a}hler\ and\ Para - K\ddot{a}hler$ curvature Wey1 manifolds, arXiv:1011.4844v1, (2010).
27. P. Gilkey, S. Nikeevie, $K\ddot{a}hler - Wey1$ manifolds dimensional 4, http://arxiv.org/abs/1109.4532v1, (2011).
28. H. Pedersen, Y.S. Poon, A. Swann, The Einstein-Wey1 equations in complex and quaternionic geometry. Differential Geom. Appl. 3, no.4, 309321, (1993).
29. R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabau, The geometry of Hamilton and Lagrange spaces, eBook ISBN:0-306-47135-3, Kluwer Academic Publishers, New York, (2002).
30. H. Wey1, Space-Time-Matter, Dover Publ. 1922. Translated from the 4th German edition by H. Brose. London: Methuen. Reprint New York: Dover (1952).

Notes