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Static Mantle Density Distribution 3 Dimpling and Bucking of Spherical Crust

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Abstract- This paper is the third step of project “Static mantle distribution, Equation, Solution and Application”. It consists of < Static Mantle Distribution 1 Equation>, < Static Mantle Density Distribution 2 Improved Equation and Solution>, and this paper. Our result on shape of core is a “X type”, which differs from the traditional view that core is a sphere. Which one is correct? or, both are not correct? The aim of this paper is to study dimpling and bucking of the spherical crust under mantle loading. Dimpling analysis depends on the outer solution of non-homogeneous non-linear D. E., while bucking analysis depends on non-linear Eigen value of the homogeneous D. E The results based on two models and governing equations show that crust dimpled at poles is proved theoretically and numerical result well consists with pole radius, while the non-linear bucking Eigen value boundary problem is solved by decomposition method. The results show that bucking can occur, and the un-continuity of internal force per unit length causes un-continuity of masses by mantle material emitting to crust at turning point of “X”. The growing of Tibet high-land might be viewed as an evidence of the mass $m_s(\theta_0)$ increasing due to mantle emission. Both poles radius and equatorial radius have been used to support our analysis. Question: how the nature makes cold at poles?

Keywords: mantle distribution, shell theory, shallow spherical shell, shell bucking/dimple, non-linear differential equations (d.e.), segmental non-linear eigen value boundary problem, operator and eigen value decomposition.

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STATICMANTLEDENSITYDISTRIBUTION3DIMPLINGANDBUCKINGOFSPHERICALCRUST

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I. Introduction

This paper is the third step of project "Static mantle distribution, Equation, Solution and Application", consists of < Static Mantle Distribution 1 Equation> [1], < Static Mantle Density Distribution 2 Improved Equation and Solution> [2] and this paper. The major result on static mantle distribution inside the Earth is a "X type" in cross-section. There are three zones: the sink zone located in the waist of "X" including equator plane, where no negative mass exists, while the positive mass is uniformly distributed in it; the buoyed zone located in the head of "X" including poles, where on positive mass exists, while negative mass uniformly distributed in it; the neural zone is the boundary between sink zone and buoyed zone. Our results conflicts to the traditional view of the shape of core, where it is considered as a sphere[3], like an egg. Which one is correct ? or both are in-correct? Until now, human has no ability to observe the status of mantle directly. The knowledge or sources of information about interior of earth is obtained by indirect methods such as analysis of rocks from mining area, volcanic eruption, and earthquake etc. In such a situation, no directly observation and no experiment can be done, how to prove the accuracy of our results?

The target of this paper is to study the spherical crust under mantle loading. Two models, model of thin elastic spherical shell with resisting bending moment and model of membrane, and two governing equations, one in English and the other in Russian, have been used to study dimpling and bucking. Dimpling analysis depends on the outer solution of non-homogeneous D.E., while bucking analysis depends on Eigen value of the homogeneous D. E. The results show that dimpled crust at poles is proved theoretically and numerical result well consistent with poles' radius, while the high-order non-linear bucking Eigen value boundary problem is solved by decomposition method. The results show that non-zero Eigen value exists, i.e., bucking can occur, and the un-continuity of internal force per unit length causes un-continuity of masses by emitting of mantle material at turning point of "X". Both poles radius and equatorial radius have been used to support our analysis.

a) *Previous study on shell*

Shell has been studied widely in various cases[4]: The pure theory from shells with ideal form; shapes and materials; filled with liquid, gas; plastic deformation; variable thickness; loadings on part of space; dynamic loading; multilayer shell; mathematical analysis; energetic analysis. The pure theory from shells with initial imperfections; Combined theoretical-experimental studies from shell with ideal form and imperfections, etc.

Spherical shell has been studied for various cases. There are researches on dimpled and bucking shallow spherical shells or spherical shells. For examples, Polar dimpling of completed spherical shell [5], Shallow spherical caps under axis-symmetric sub-bucking pressure distributions [6]; Axis-symmetric behavior of elastic spherical shell compressed between rigid plated [7], Asymmetrical bucking of shallow shells under asymmetric concentrated and uniformly distributed loads had been studied by numerical analysis [8-11], experiments [12-13], and dynamic instability of

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asymmetric dimpled shallow spherical shells [14-15], In-extensional bending thin spherical shell compressed by two parallel rigid plates [16], etc.

b) *What is the feature of our study differed from previous study?*

Bucking and dimpling behavior have been studied previous for spherical shells created by mankind, like structure, container and nature bio-cell etc., but dimpling and bucking of spherical crust shell subjected to mantle loading and segmental non-linear Eigen value boundary value problem for crust have not been found. Furthermore, high-order non-linear Eigen value problem is solved by decomposition of operator and Eigen-values are also the feature of our study.

II. Mathematical Model of the Spherical Shell of Crust

a) *Basic Hypotheses*

(1). The Earth is considered as a thin sphere shell with thickness varied in 8 – 40 km, the mean radius 6,371.032 km[17]. The ratio of thickness/radius is less than and can be considered as a very thin shell.

(2) Material of the crust

The crust is considered as homogenous, linear isotropic elastic material.

(3) Axis-symmetric loading --- static loading from mantle; inertia centrifugal force.

(4) Thermal stress and mantle-crust co-reaction are neglect.

Since we consider that the Earth is in a stable equilibrium status, therefore no changing on thermal stress field and no changing on coherence between mantle and crust occur.

Coordinates:

Cylindrical coordinates: Let (r,ϑ,z) be the cylindrical coordinates of the geometric center of the Earth. The relation between (x, y) and (r,ϑ) is:

$$\begin{cases} x = r\cos\vartheta, \\ y = r\sin\vartheta, \end{cases} \quad (0 \le \vartheta \le 2\pi,\ 0 \le r < \infty, -\infty < z < \infty) \tag{2.1-1}$$

$(\mathbf{i},\ \mathbf{j},\ \mathbf{k})$ and $(\mathbf{e}_r, \mathbf{e}\ , \mathbf{k})$ denote the unit vectors of Cartesian and cylindrical coordinates respectively. In the following, for asymmetry loading, a point $f(r,\vartheta,z)$ independents to ϑ and can be simplified by $f(r,z)$, and discussion only focus on semi-sphere.

b) *Crust model*

(1) Recall of [2].

1. According to [2]: The mantle is divided into sink zone, neural zone and buoyed zone. The sink zone is located in a region with boundaries of a inclined line, angle $\alpha_0 = 35°15$, with apex at $O(0,0)$ revolving around the z-axis, inside the crust involving the equator. Where no negative mass exists, while positive mass is uniformly distributed. The buoyed zone is located in the remainder part, inside the crust involving poles. Where no positive mass exists, while negative mass is uniformly distributed. The neural zone is the boundary between the buoyed and sink zones. The substance of negative mass must be liquid, while the substance of positive mass could be gas, liquid or solid.
2. In SIN zone, $(0 \le \alpha \le 35°15')$, there is no negative mass (liquid), $m_n = 0$, while positive mass (solid) is uniformly distributed, i.e.,

$$\rho_{ps} = C_1 == \frac{6\sqrt{3}}{2\pi}\frac{\omega_c^2}{G}\left(\frac{z_0}{z_{real1}}\right)^3.$$

Two necessary conditions for stable equilibrium system are:

$$r_0^2 = 2z_0^2, \text{ or } \tan\alpha_0 = 1/\sqrt{2} = z_0/r_0,\ \alpha_0 = 35°15\ , \quad \text{(3.2-20) of [2]}$$

$$M_p^- = 6\sqrt{3}\frac{\omega_c^2}{G}z_0^3 = 2M_n^- \quad \text{(3.2-24) of [2]}$$

In BUO zone, $(35°15' \leq \alpha \leq \pi/2)$, there is no positive mass, $m_p = 0$, while negative mass is uniformly distributed, i.e., $\rho_{nb} = C_4 =$

$$-\frac{3\sqrt{3}}{2\pi}\frac{\omega_c^2}{G}\left(\frac{z_0}{z_{real2}}\right)^3 \quad \text{(4.6-4) of [2]}$$

c) Mantle loading to the crust shell

On the boundary,

$$(r_s, z_s) \text{or } (r_b, z_b) = (R\cos\alpha, \mp R\sin\alpha) = (R\sin\theta, R\cos\theta); \quad H_s = (r_s^2 + z_s^2)^{3/2} = R^3;$$

where α is the meridional angle, i.e., the angle between vector R and the equator plane; θ is the angle between vector R and the z-axis.

$$\theta + \alpha = \pi/2, \tag{2.2-1}$$

In SIN zone, by $\sum F_z = 0$ of an cylindrical element, we have

$$F_z(r_s, z_s) = G\frac{m_p M_p^-}{H_s} z_s = 6\sqrt{3} m_p \omega_c^2 z_0^3 H_s^{-1} z_s = 2 m_p \omega_c^2 z_s =$$

$$\sigma_z^0(r_s, z_s) dA_z,$$

$$\sigma_z^0(r_s, z_s) = 2\frac{dv}{dA_z} m_p \omega_c^2 z_s = 2\rho_p \omega_c^2 R^2 \cos\theta \sin\theta \tag{2.2-2}$$

Where $\sigma_z^0(r_s, z_s)$ is the principle stress exerted only by Newton's attraction force applied at $m_p(r_s, z_s)$; $M_p^- = 6\sqrt{3}\frac{\omega_c^2}{G} z_0^3$ is the mass group located at central $O(0,0)$ (due to hypotheses 2 of [2]), represented total positive mass, except (r_s, z_s) itself; gravitational constant $G = 6.674 \times 10^{-11}, N.(\frac{m}{kg})^2$; $H_s = H_b = H_0 = (r_s^2 + z_s^2)^{3/2}$. $r_s^2 = r_b^2 = r_0^2 = 2z_0^2 = 2z_b^2 = 2z_s^2$.

By $\sum F_r = 0$ of an cylindrical element, we have

$$F_r(r_s, z_s) = \sigma_r^0(r_s, z_s) dA_r = m_p\left[-G M_p^- H_s^{-1} + \omega_c^2\right] r_s = -m_p \omega_c^2 r_s,$$

$$\sigma_r^0(r_s, z_s) = -m_p \omega_c^2 r_s \frac{dv}{dA_r} = \rho_p \omega_c^2 R^2 \sin\theta \cos\theta, \tag{2.2-3}$$

Similarly, in BUO zone, we have

$$F_z(r_b, z_b) = G\frac{m_n M_n^-}{H_b} z_b = 3\sqrt{3} m_n \omega_c^2 z_0^3 H_b^{-1} z_b = m_n \omega_c^2 z_b,$$

$$\sigma_z^0(r_b, z_b) = m_n \omega_c^2 z_b \frac{dv}{dA_z} = \rho_n \omega_c^2 R^2 \cos\theta \sin\theta, \tag{2.2-4}$$

$$\sigma_r^0(r_b, z_b) = m_n\left[G M_n^- H_b^{-1} - \omega_c^2\right] r_b \frac{dv}{dA_r} = 0, \tag{2.2-5}$$

Where $m_n(r_b,z_b)$ is negative mass, $M_n^-=3\sqrt{3}\frac{\omega_c^2}{G}z_0^3$ is the mass group located at central $O(0,0)$ (due to hypotheses 2 of [2]), represented total negative mass, except (r_s,z_s) itself. (2.2-5) shows the forces exerted by attraction force and by centrifugal force are in equilibrium.

$$\theta_0 = \frac{\pi}{2} - \alpha_0 = 54°45'$$

Summary: In SIN zone, $(\theta_0 \le \theta \le \pi/2)$, $m_n = 0$, $\sigma_z^0(r_s,z_s) = 2\rho_p\omega_c^2R^2\cos\theta\sin\theta$ exerted by attraction force, and is uniformly distributed attraction to crust; $\sigma_r^0(r_s,z_s) = \rho_p\omega_c^2R^2\sin\theta\cos\theta$ exerted by attraction force and centrifugal force and is uniformly distributed and against to crust.

In BUO zone, $(0 \le \theta \le \theta_0)$, $m_p = 0$, $\sigma_z^0(r_b,z_b) = \rho_n\omega_c^2R^2\cos\theta\sin\theta$ exerted by attraction of negative masses, and is uniformly distributed against to crust; $\sigma_r^0(r_b,z_b) = 0$ exerted by attraction of negative masses, and centrifugal force, they are in equilibrium, neither against nor attraction to crust.

III. Dimpling of Shallow Spherical Shell

The governing equations

Using the Marguerre type elastic -dynamic shell theory, the behavior of a thin shell can be described by the non-dimensional governing equations [3,4,12,13]:

$$\left.\begin{array}{l}\nabla^4 w - L[z+w,f] = q(x) - bw_{,tt} \\ \nabla^4 f + L\left[z+\frac{w}{2},w\right] = 0,\end{array}\right\} (0 < x < 1), \quad (3\text{-}1),\ (3\text{-}2)$$

Where x and θ are polar coordinates in the basic plane (parallel to the equator plane),

$$()' = \partial()/\partial x,\ ()'' = \partial^2()/\partial x^2,\ (\dot{}) = \partial()/\partial\theta,\ (\ddot{}) = \partial^2()/\partial\theta^2,$$

$$()_{,tt} = \partial^2()/\partial t^2,$$

$$L[g,s] = g''(s'/x + \ddot{s}/x^2) + s''(g'/x + \ddot{g}/x^2) - 2(\dot{g}/x)'(\dot{s}/x)',$$

$$\nabla^4() = \nabla^2\nabla^2(),\ \nabla^2() = ()'' + ()'/x + (\ddot{}),$$

The non-dimensional radius coordinate x, the un-deformed meridian curve z, the vertical deflection w, the stress function f, the loading q and inertia term b, are related in the corresponding physical variables by

$$x = r/r_0,\ z = (Z/r_0)(1/\varepsilon^2\varphi_0), \quad w = (W/r_0)(1/\varepsilon^2\xi_0),\ f = F/D,$$

$$q(x) = (r_0^3/D\xi_0\varepsilon^2)p(x),\ b = \rho r_0^3h/D, \quad D = Eh^3/[12(1-\nu^2)],$$

$$\varepsilon^2 = h/\{[12(1-\nu^2)]^{1/2}r_0\xi_0\},\ \xi_0 = 2Z(r_0)/r_0,$$

r_0 – radius of basic plane, h – thickness of the shell, E – elastic modulus, ν- Poisson's ratio, $Z = Z(r)$ – un-deformed meridian curve with Z(0)=0, $p(x)$ – non-dimension loading (stress/E), ρ mass density.

For shallow spherical shell, $Z(r) = r^2/2R$, R– the radius of the spherical shell. $H = Z(r_0)$ - the apex rise. $\xi_0 = r_0/R$.

The boundary conditions for a shell with completely clamped basic plane are the regular conditions at apex and at the outer edge:

$$x = 0,\ w = w' = f = f' = 0, \quad (3\text{-}3)$$

$$x = 1, w = w' = 0,\ f'' - \nu\ (f' + \ddot{f}) = 0,\ f''' - f'(1-\nu) + (2+\nu)\dot{f}' - 3\ddot{f} = 0, \quad (3\text{-}4)$$

For the cases of static axis-symmetric deflection $(w_{,tt} = 0)$, the above equations (3-1) – (3-4) are reduced to:

$$\varepsilon^2 x(\Phi_1'' + \Phi'_1/x - \Phi_1/x^2) - \Phi_1\Psi = 4kxP(x) + \varepsilon^2 xQ(x),\ (0 < x < 1) \quad (3\text{-}5)$$

$$\varepsilon^2 x(\Psi'' + \Psi'/x + \Psi/x^2) + (\Phi_1^2 - \Phi_0^2)/2 = 0,\ (0 < x < 1) \quad (3\text{-}6)$$

$$x = 0,\ \Phi_1 = 0,\ \Psi = 0, \quad (3\text{-}7)$$

$$x = 1,\ \Phi_1 = 1,\ \Psi' - \nu\ \Psi = 0, \quad (3\text{-}8)$$

where $\Phi_1 = \Phi_1(x) = \Phi\ (x) + \Phi_0(x)$, $\varepsilon^2\xi_0 dZ/dr \equiv \Phi_0(x)$, $\varepsilon^2\xi_0 dW/dr \equiv \Phi\ (x)$, $z' = \Phi_0/\varepsilon^2$, $w' = \Phi\ (x)/\varepsilon^2$, $\Psi = \Psi(x) = \varepsilon^2 f'$, $Q(x) = \Phi''_0 + \Phi'_0/x - \Phi_0/x^2$, $k = p_e/p_c$, p_e the peak value of p(t), $P_c = \frac{2Eh^2}{\sqrt{3(1-\nu^2)}R^2}$, classical bucking load, a uniformly distributed compressed load [3].

For shallow spherical shells, $\Phi(x) = x$, $p_e = 2Eh^2/\left[R^2\left(3(1-\nu^2)\right)^{1/2}\right]$,

$$P(x) = (1/x)\int_0^x tp(t)\,dt, \quad (3\text{-}9)$$

If $\varepsilon^2 \ll 1$, from (3-5), (3-6), the leading term of outer solution for the

$$\text{case } \Phi_0(x) = x \text{ and } p(x) = p_e(1 - c^2x^2), (c^2 > 1) \quad (3\text{-}10)$$

is [5,6,14,15]:

$$w'_d \sim -\frac{2x}{\varepsilon^2},\quad f'_d \sim kx(2 - c^2x^2)/\varepsilon^2,\quad (0 \lesssim x < x_T) \quad (3\text{-}11)$$

$$w'_d \sim 0,\quad f'_d \sim -kx(2 - c^2x^2)/\varepsilon^2,\quad (x_T < x < 1), \quad (3\text{-}12)$$

Where x_T is the dimensionless dimple base radius and is known to be $x_T = \sqrt{2}/c$ from $\Psi(x_T) = 0$. One can see from (3.11), (3.12) that the meridional slope of the deformed polar dimpling remains unchanged in $(x_T, 1)$ while it changes only the sign of the unchanged meridional slope is changed in $(0, x_T)$. According to [5], (3-11) and (3-12) hold for a wide range values of k, e.g., $\varepsilon^2 \ll k\varepsilon = O(1)$.

The above is repeated from [15].

a) For our case

Our analysis is all the same as above except the loading term $P(x)$.

Note that from (3-5), we have $-\Phi\Psi = 4kxP(x)$. or $\Psi = P(x)$.

$$\Phi(x) = 4kx.$$

Calculating P(x):

$$P(x) = (1/x)\int_0^x tp(t)\,dt,\quad (0 \le t \le x \le 1) \quad (3\text{-}9)$$

$P(x)$ is an equivalent non-dimensional loading. Where t is a non-dimension distance, between 0 and p(t), $p(t) \perp \mathbf{t}$. $p(t) = p_e \cos\theta = p_e\sqrt{1-t^2}$ is a non- dimension force, and $tp(t)$ forms a non-dimension bending moment. $tp(t) = +$ is defined that the bending moment tp(t) causes the curvature of the shell segment increasing. Otherwise, $tp(t) = -$. $p_e = (m_n\omega_c^2)/E$ is the peak value of $p(t)$.

Determination of shell dimpling

Like the analysis of a cantilever beam or a tall building, the cantilever end or the upper part of the building is used as an isolated segment, the upper part of the spherical shell, i.e., in the BUO zone $(0 \le \theta \le \theta_0)$, is used as an isolated segment for analyze. The dimple of shallow spherical shell is characterized by the non-dimensional equivalent load $P(\theta)$ has multiple values at dimple θ_d, i.e.,

$$P(\theta_d) = \Psi(x_d) = 0 = \begin{cases} P(\theta_d - \delta), & \theta \le \theta_d \\ P(\theta_d + \delta), & \theta \ge \theta_d \end{cases} \quad (\delta > 0), \tag{3.1-1}$$

Before dimpling, $P(\theta) = P(\theta_d - \delta)$, after dimpling $P(\theta) = P(\theta_d + \delta)$.

In BUO zone $(0 \le x = \sin\theta \le x_0)$, $m_p = 0$, $p_1(t) = p_e\sqrt{1-t^2}$, against to the shell, $p_2(t) = F_r/E = 0$.

$$P(\theta) = \left(\frac{1}{x}\right)\int_0^x tp_1(t)dt = \left(\frac{p_e}{x}\right)\int_0^x t\sqrt{1-t^2}\,dt = -\frac{p_e}{3x}\left[\sqrt{(1-x^2)^3} - 1\right],$$

$$(0 \le x \le x_0) \tag{3.1-2}$$ [18]

To find $\boldsymbol{\theta_d}$ or $\mathbf{x_d}$ such that $P(\theta_d) = \Psi(x_d) = 0$. Let

$$P(\theta_d) = 0, \tag{3.1-3}$$

By (3.1-2), we have:

$$\sqrt{\left(1-x_d^2\right)^3} - 1 = 0, \quad x_d = 0,\ 0r\ \ \theta_d = 0 \tag{3.1-4}$$

x_d is the dimple base radius. When $x_d = 0$, $P(0) = 0/0$, Using L'Hospital rule we have $P(0) = O(1)$, That is $P(0-) = 0$, $P(0+) = \delta$. Comparing the loading $P(\theta)$ of (3-10) and (3.1-2), we have

$$w'_d \sim -\frac{x}{\varepsilon^2}, \quad f'_d \sim \left(\frac{p_e}{3}\right)\left[1 - \sqrt{\left(1-x_d^2\right)^3}\right]/\varepsilon^2, \quad (0 \le x < x_d), \tag{3.1-5}$$

$$w'_d \sim 0, \quad f'_d \sim -\left(\frac{p_e}{3}\right)\left[1 - \sqrt{\left(1-x_d^2\right)^3}\right]/\varepsilon^2, \quad (x_d < x < 1) \ \ (3.1\text{-}6) \tag{3.1-6}$$

Eq.(3.1-5) shows that the changed meridional slope $w'_d(x) = \Phi(x) = x$ is changed only in a minus sign in $(0, x_d)$ while (3.1-6) shows that it 264 remains unchanged in $(x_d, 1)$.

b) *Comparing the highness of the dimpled shell with poles' radius.*

The dimpling occurs at apex (pole) with dimple radius $x_d = 0$ for $k\varepsilon = \varepsilon p_e/p_c = O(1)$ (loading approaches critical load) theoretically, However, one can not calculate the dimple for $x_d = 0$ numerically. If a perturbation with small parameter δ adding to the stable equilibrium system, then the dimpling occurs at apex with $x_d = \sin\theta_d = \delta$. The highness of dimple h_d is:

$$h_d = \delta \sin\theta_d, \tag{3.2-1}$$

$$\delta = R\tan\theta_d, \tag{3.2-2}$$

$$2h_d = 2R\sin\theta_d \tan\theta_d \quad (3.2\text{-}3)$$

$$R_{pd} = R - 2h_d = R(1 - 2\sin\theta_d \tan\theta_d) \quad (3.2\text{-}4)$$

Where the mean radius $R = 6{,}371.032\,km$[17]. The dimple highness h_d is the raise of the un-deformed shell at apex. R_{pd} is the highness of the dimpled pole's radius.

Now, we choose $x_d = \delta = 0.00116$, or $\theta_d = 0°0'4''$, by (3.2-4), we have:

$$R_{pd} = 6{,}371{,}032 \times (1 - 2 \times 0.00116 \times 0.00116) = 6{,}371{,}014(km)$$

Comparing with poles' radius $R_p = 6{,}365{,}777km$[17], $error = \frac{R_{pd} - R_p}{R_{pd}} = \frac{6{,}371{,}014 - 6{,}365{,}777}{6{,}371{,}014} =$ 0.001292887 (3.2-5)

IV. Bucking Analysis

Dimpling analysis only uses local information of shallow spherical cap. Other information, e,g., loading and structure of the lower part, have not been included. Therefore, whole information must be used for analysis of whole behavior. That is that bucking analysis needs information of the whole spherical shell.

a) *Bucking of segmental spherical shell*

Since the mantle loading on crust of SIN zone is differed from the BUO zone, therefore, segmental governing equations with boundary conditions are set for these zones.

In SIN zone:

The governing equation, for axis-symmetry deformation, of a thin spherical shell, is[4]

$$D\nabla^4 w_s - R\nabla^2 F_s - L(F_s)w^0_{s\theta\theta} - \cot\theta\, w^0_{s\theta} F_{s\theta\theta} - R^2[T^0_{s1} w_{s\theta\theta} + T^0_{s2} L(w_s)] = 0, \quad (\theta_0 \le \theta \le \pi/2) \quad (4.\text{-}1)^{**}$$

$$\nabla^4 F_s + Eh\left[R\nabla^2 w_s + L(w_s)w^0_{s\theta\theta} + \cot\theta\, w^0_{s\theta} w_{s\theta\theta}\right] = 0, \ (\theta_0 \le \theta \le \pi/2) \quad (4.\text{-}2)^{**}$$

Where $\nabla^4 = \nabla^2\nabla^2$; $\nabla^2(\,) = (\,)_{\theta\theta} + \cot\theta(\,)_\theta$; $()_\theta = \partial()/\partial\theta$; $L(\,) = \cot\theta(\,)_\theta$; F --- stress function, w --- displacement component.

$$F_{\theta\theta} = T_2 = k(\varepsilon_{22} + \nu\varepsilon_{11}),\ F_{\varphi\varphi} = T_1 = k(\varepsilon_{11} + \nu\varepsilon_{22}), \quad (4\text{-}3)$$

$k = \frac{Eh}{1-\nu^2}$; $D = \frac{Eh^3}{12(1-\nu^2)}$; E -- elastic modulus; v-- Poisson's ratio; h – hickness; w^0, T^0_1, T^0_2 are mid-plane w, inner forces per unit length in θ, φ (the latitude and longitude) directions respectively.

Dimension check: dimensions of each term of (4.-1) and (4.-2) should be the same. But Dim (i = 1,2,3,4,5,6) $:= (Eh^4, Eh^2, Eh^2, Eh^2, Eh^4, Eh^4)$. Dim $(i = 1,2,3,4) := (Eh, Eh^3, Eh^3, Eh^3)$ If given (4-1), (4-2) a little modified, like

$$(D/R^2)\nabla^4 w_s - R\nabla^2 F_s - L(F_s)w^0_{s\theta\theta} - \cot\theta\, w^0_\theta F_{s\theta\theta} - [T^0_{s1} w_{s\theta\theta} + T^0_{s2} L(w_s)] = 0, \ (\theta_0 \le \theta \le \pi/2) \quad (4.1\text{-}1)$$

$$\nabla^4 F_s + Eh\left[R\nabla^2 w_s + L(w_s)w^0_{s\theta\theta} + \cot\theta\, w^0_{s\theta} w_{s\theta\theta}\right]/R^2 = 0, \ (\theta_0 \le \theta \le \pi/2) \quad (4.1\text{-}2)$$

then, the dimensions of (4.1-1) and (4.1-2) are reduced to:

$$\mathrm{Dim}\ (i = 1,2,3,4,5,6)\ := (Eh^2, Eh^2, Eh^2, Eh^2, Eh^2, Eh^2)$$

Dim (i = 1,2,3,4) $:= (Eh, Eh, Eh, Eh)$ Therefore (4 -1) and (4-2) are in printing error, and (4.1-1) and (4.1-2) are corrected.

Similarly, In BUO zone: we have:

$$(D/R^2)\nabla^4 w_b - R\nabla^2 F_b - L(F_b)w^0_{b\theta\theta} - \cot\theta\, w^0_{b\theta} F_{b\theta\theta} - [T^0_{b1}$$

$$w_{b\theta\theta} + T^0_{b2} L(w_b)] = 0,\ \ (0 \le \theta \le \theta_0) \tag{4.1-4}$$

$$\nabla^4 F_b + Eh\left[R\nabla^2 w_b + L(w_b)w^0_{b\theta\theta} + \cot\theta\, w^0_{b\theta} w_{b\theta\theta}\right]/R^2 = 0,$$

$$(0 \le \theta \le \theta_0) \tag{4.1-5}$$

The above governing equations are coupled high order non-linear D.E. with unknown functions w and F. How to solve these equations?

First, combining (4.1-1) and (4.1-2) into one equation.

Substituting (4.1-2) into $\nabla^2\cdot\ (4.1-1)$, we have

$$\nabla^2 (D/R^2)\nabla^4 w_s + Eh\left[R\nabla^2 w_s + L(w_s)w^0_{s\theta\theta} + \cot\theta\, w^0_{s\theta} w_{s\theta\theta}\right]/R -$$

$$\nabla^2 L(F_s)w^0_{s\theta\theta} - \nabla^2 \cot\theta\, F_{s\theta\theta} w^0_{s\theta} - \nabla^2 [T^0_{s1} w_{s\theta\theta} + T^0_{s2} \cot\theta\, w_{s\theta}] = 0, \tag{4.1-6}$$

Similarly, in BUO zone, we have

$$\nabla^2 (D/R^2)\nabla^4 w_b + Eh\left[R\nabla^2 w_b + L(w_s)w^0_{b\theta\theta} + \cot\theta\, w^0_{b\theta} w_{b\theta\theta}\right]/R -$$

$$\nabla^2 L(F_b)w^0_{b\theta\theta} - \nabla^2 \cot\theta\, F_{b\theta\theta} w^0_{s\theta} - \nabla^2 \left[T^0_{b1} w_{b\theta\theta} + T^0_{b2} \cot\theta\, w_{b\theta}\right] = 0, \tag{4.1-7}$$

Secondly, transforming the D.E. into Eigen value problem.

Eigen value problem related to vibration and dimpling or bucking problems have been widely studied. Introducing parameter λ, such that for some λ, the corresponding homogeneous D.E. have non-zero solution,

$$\nabla^2 w_s = -\lambda_s^2 w_s,\ (\theta_0 \le \theta \le \pi/2) \tag{4.1-8}$$

$$\nabla^2 F_s = -\lambda_s^2 F_s,\quad (\theta_0 \le \theta \le \pi/2) \tag{4.1-9}$$

$$\nabla^2 w_b = -\lambda_b^2 w_b,\ (0 \le \theta \le \theta_0) \tag{4.1-10}$$

$$\nabla^2 F_b = -\lambda_b^2 F_b,\quad (0 \le \theta \le \theta_0) \tag{4.1-11}$$

Boundary conditions of the crust shell

$$\theta = 0,\ \ w_b = R - R_p + \epsilon_1,\ \ F_{b\theta\theta} = T^0_{b1} = 0, \tag{4.1-12}$$

$$\theta = \theta_0,\quad w_s(\theta_0) = w_b(\theta_0),\ \ F_{s\theta\theta}(\theta_0) - F_{b\theta\theta}(\theta_0) = \delta, \tag{4.1-13}$$

$$\theta = \pi/2,\ w_s\left(\frac{\pi}{2}\right) = R_e - R + \epsilon_2,\ F_{s\theta\theta}(\pi/2) = \left(\sum(m_p + m_n\right) / 4\pi Re{=}0, \text{[2]} \quad (4.1\text{-}4)$$

b) Calculation of $\mathbf{T^0_{s1}}, \mathbf{T^0_{s2}}, \mathbf{T^0_{b1}}, \mathbf{T^0_{b2}}$*, due to mantle loading:*

In BUO zone: $(0 \le \theta \le \theta_0),\ m_p = 0,$

$\sigma^0_z(r_b, z_b) = \rho_n \omega_c^2 R^2 \cos\theta \sin\theta$, against to crust.

$\sigma^0_r(r_b, z_b) = 0$, attraction to crust.

Let the upper part $(0 \le \theta \le \theta_0)$ of the shell be an isolated system, by the equilibrium equation $\sum F_z = 0$ of the system, we have:

$$2\pi R \sin\theta\, T^0_{b2}(\theta) h \sin\theta = \pi(R\sin\theta)^2 \rho_n \omega_c^2 R^2 \cos\theta \sin\theta \qquad (4.2\text{-}1)$$

$$T^0_{b2}(\theta) = \frac{R^3}{2h} \rho_n \omega_c^2 \cos\theta \sin\theta, \quad (0 \le \theta \le \theta_0) \qquad (4.2\text{-}2)$$

Where stress is uniformly distributed along the thickness in T^0_{b1} and T^0_{b2}.
Let the half of the upper part of the shell, cut by cross-section perpendicular to equatorial plane, be an isolated body, by the equilibrium equation $\sum F_r = 0$ of the body, we have:

$$\int_0^\theta 2hT^0_{b1}(\theta)\, d\theta = \int_0^\theta 2R \sin t\, \sigma^0_r(r_b, t) dt = 0 \qquad (4.3\text{-}3)$$

$$T^0_{b1}(\theta) = 0, \qquad (0 \le \theta \le \theta_0) \qquad (4.2\text{-}4)$$

In SIN zone, $(\theta_0 \le \theta \le \pi/2),\ m_n = 0,$

$\sigma^0_z(r_s, z_s) = 2\rho_p \omega_c^2 R^2 \cos\theta \sin\theta$ attraction to crust.

$\sigma^0_r(r_s, z_s) = \rho^p \omega_c^2 R^2 \sin\theta \cos\theta$, against to crust,

Let the upper part $(\theta_0 \le \theta \le \pi/2)$ of the shell be an isolated system, by the equilibrium equation $\sum F_z = 0$ of the system, we have:

$$2\pi R \sin\theta\, T^0_{s2}(\theta) h \sin\theta =$$

$$-\pi(R\sin\theta)^2 \rho_n \omega_c^2 R^2 \cos\theta \sin\theta + \pi(R\sin\theta)^2 2\rho_p \omega_c^2 R^2 \cos\theta \sin\theta,$$

$$T^0_{s2}(\theta) = 2\frac{R^3}{h} \omega_c^2 (2\rho_p - \rho_n) \cos\theta \sin\theta, \quad (\theta_0 \le \theta \le \pi/2), \qquad (4.2\text{-}5)$$

Let the half of the upper part of the shell, cut by cross-section perpendicular to equatorial plane, be an isolated body, by the equilibrium equation $\sum F_r = 0$ of the body, we have:

$$\int_0^{\theta_0} 2hT^0_{b1}(t)\, dt + \int_{\theta_0}^{\theta} 2hT^0_{s1}(t) dt = 0 + \int_{\theta_0}^{\theta} 2hT^0_{s1}(t)\, dt =$$

$$\int_{\theta_0}^{\theta} 2R \sin t\, \sigma^0_r(r_s, t) dt = 2R^3 \rho_p \omega_c^2 \int_{\theta_0}^{\theta} (\sin t)^2 \cos t\, dt, \qquad (4.2\text{-}6)$$

$$T^0_{s1}(\theta) = \frac{1}{3}\frac{R^3}{h} \rho_p \omega_c^2 [(\sin\theta)^3 - (\sin\theta_0)^3], \quad (\theta_0 \le \theta \le \pi/2), \qquad (4.2\text{-}7)$$

c) *Calculation of* $F_{s\theta\theta}$ *and* $F_{b\theta\theta}$.

By (4-3), $F_{\theta\theta} = T_2 = k(\varepsilon_{22} + \nu\varepsilon_{11}),\ F_{\varphi\varphi} = T_1 = k(\varepsilon_{11} + \nu\varepsilon_{22}).$

In BUO zone, $(0 \le \theta \le \theta_0)$ by (4.2-2),

$$T_{b2} = \frac{R^3}{2h}\rho_n\omega_c^2\cos\theta\sin\theta = F_{b\theta\theta} = k(\varepsilon_{22} + \nu\varepsilon_{11}), \tag{4.3-1}$$

By (4.2-4),

$$T_{b1} = 0 = F_{b\varphi\varphi} = k(\varepsilon_{11} + \nu\varepsilon_{22}) \tag{4.3-2}$$

we have $\varepsilon_{11} = -\nu\ \varepsilon_{22}$, then (4.3 -1) becomes to:

$$F_{b\theta\theta} = k\varepsilon_{22}(1 - \nu^2) = \frac{R^3}{2h}\rho_n\omega_c^2\cos\theta\sin\theta \tag{4.3-3}$$

In SIN zone, $(\theta_0 \le \theta \le \pi/2)$ by (4.2-5),

$$F_{s\theta\theta} = k(\varepsilon_{22} + \nu\varepsilon_{11}) = T_{s2}^0 = 2\frac{R^3}{h}(2\rho_p - \rho_n)\omega_c^2\cos\theta\sin\theta \tag{4.3-4}$$

d) *Solution of non-linear Eigen value equation by decomposition method.*

The non-linear Eigen value equation is decomposed, where the operator and Eigen value are decomposed.

The general form of second order non-linear Eigen value equation

$$\nabla^2(X) = A\frac{d^2}{d\theta^2}(X) + B\frac{d}{d\theta}(X) = -\lambda_0^2(X), \tag{4.4-1}$$

Where $A = A(\theta), B = B(\theta)$ are known functions, X is an unknown function, λ_0^2 is an Eigen value, subscript 0 can be s (SIN zone) or b (BUO zone).

Decomposition of operator ∇^2 *into* d^2 *and* d *Eigen values* λ_0^2 *into* λ_{0A}^2 *and* λ_{0B}^2. *That is:*

$$A(\theta)\frac{d^2}{d\theta^2}(X_A) = -\lambda_{0A}^2(X_A) \tag{4.4-2}$$

Integration both sides of (4.4-2) twice, we have:
In BUO zone,

$$\frac{d}{d\theta}\int_0^\theta \frac{1}{X_A}dX_A = -\lambda_{bA}^2\int_0^\theta A^{-1}\,d\theta,$$

$$\int_0^\theta d\left[\frac{\ln X_A(\theta)}{\ln X_A(0)}\right] = d\int_0^\theta\left[\frac{\ln X_A(\theta)}{\ln X_A(0)}\right] = \left[\frac{\ln X_A(\theta)}{\ln X_A(0)}\right] = -\lambda_{bA}^2\int_0^\theta d\theta\int_0^\theta A^{-1}\,d\theta,$$

$$\frac{X_A(\theta)}{X_A(0)} = \exp\left[-\lambda_{bA}^2\int_0^\theta d\theta\int_0^\theta A^{-1}\,d\theta\right],\ (0 \le \theta \le \theta_0) \tag{4.4-3}$$

In SIN zone,

$$\frac{X_A(\theta)}{X_A(\theta_0)} = \exp\left[-\lambda_{sA}^2\int_{\theta_0}^\theta d\theta\int_{\theta_0}^\theta A^{-1}\,d\theta\right],\ (\theta_0 \le \theta \le \pi/2) \tag{4.4-4}$$

$$B(\theta)\frac{d}{d\theta}(X_B) = -\lambda_{0B}^2(X_B) \tag{4.4-5}$$

Integration both side of (4.4-5), we have:
In BUO zone:

$$\frac{\ln X_B(\theta)}{\ln X_B(0)} = \int_0^\theta \frac{dX_B}{X_B} = -\lambda_{0A}^2 \int_0^\theta B^{-1}\, d\theta,$$

$$\frac{X_B(\theta)}{X_B(0)} = \exp\left[-\lambda_{bB}^2 \int_0^\theta B^{-1} d\theta\right],\ (0 \le \theta \le \theta_0) \qquad (4.4\text{-}6)$$

In SIN zone,

$$\frac{X_B(\theta)}{X_B(\theta_0)} = \exp\left[-\lambda_{sB}^2 \int_{\theta_0}^\theta B^{-1} d\theta\right], (\theta_0 \le \theta \le \pi/2) \qquad (4.4\text{-}7)$$

$$\nabla^2(X) = Ad^2(X_A) + Bd(X_B) = -\lambda_0^2(X), \qquad (4.4\text{-}8)$$

$$\nabla^4(X) = \nabla^2\nabla^2(X) = [Ad^2(X_A) + Bd(X_B)][Ad^2(X_A) + Bd(X_B)] = \lambda_0^4(X), \qquad (4.4\text{-}9)$$

$$\nabla^2(X) = -\lambda_0^2(X) = Ad^2(X_A) + Bd(X_B) = -\lambda_{0A}^2(X_A) - \lambda_{0B}^2(X_B) \qquad (4.4\text{-}10)$$

$$\lambda_0^2(X) = \lambda_{0A}^2(X_A) + \lambda_{0B}^2(X_B) \qquad (4.4\text{-}11)$$

$$\nabla^4(X) = \lambda_0^4(X) = Ad^4(X_A) + 2Ad^2(X_A)Bd(X_B) + Bd^2(X_B) = \lambda_A^4(X_A) + 2(-\lambda_A^2)(-\lambda_B^2)(X_A)(X_B) + \lambda_B^4(X_B) \qquad (4.4\text{-}12)$$

$$\lambda_0^4(X) = \lambda_A^4(X_A) + 2\lambda_A^2\lambda_B^2(X_A)(X_B) + \lambda_B^4(X_B), \qquad (4.4\text{-}13)$$

For our case, the particular form of A and B of (4.4-1).

$$\nabla^2(X) = \frac{d^2}{d\theta^2}(X_A) + \cot\theta\,\frac{d}{d\theta}(X_B) = -\lambda_0^2(X) \qquad (4.4\text{-}14)$$

$$\frac{d^2}{d\theta^2}(X_A) = -\lambda_{0A=1}^2(X_A), \qquad (A = 1) \qquad (4.4\text{-}15)$$

By (4.4-3), we have

$$\frac{X_A(\theta)}{X_A(0)} = \exp\left[-\frac{1}{2}\lambda_{bA=1}^2\theta^2\right],\ (0 \le \theta \le \theta_0) \qquad (4.4\text{-}16)$$

By (4.4-4), we have

$$\frac{X_A(\theta)}{X_A(\theta_0)} = \exp\left[-\frac{1}{2}\lambda_{sA=1}^2(\theta-\theta_0)^2\right],\ (\theta_0 \le \theta \le \pi/2) \qquad (4.4\text{-}17)$$

$$\cot\theta\,\frac{d}{d\theta}(X_B) = -\lambda_{0B}^2(X_B),\ (B = \cot\theta) \qquad (4.4\text{-}18)$$

By (4.4-6), we have

$$\frac{X_B(\theta)}{X_B(0)} = \exp\left[-\lambda_{bB}^2\int_0^\theta \tan\theta\, d\theta\right] = \exp\left[\lambda_{bB}^2\frac{\ln\cos\theta}{\ln\cos\theta_0}\right] = \exp[\lambda_{bB}^2]\frac{\cos\theta}{1},\ (0 \le \theta \le \theta_0) \qquad (4.4\text{-}19)$$

By (4.4-7), we have

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$$\frac{X_B(\theta)}{X_B(\theta_0)} = \exp\,[\lambda_{sB}^2]\frac{\cos\theta}{\cos\theta_0},\ (\theta_0 \le \theta \le \pi/2) \qquad , (4.4\text{-}20)$$

e) *Calculating* $T_{s1}^0 w_{s\theta\theta}, T_{s2}^0 \cot\theta w_{s\theta}, T_{b1}^0 w_{b\theta}, T_{b2}^0 \cot\theta w_{b\theta}$ *by Eigen value.*

$$w_{s\theta\theta} = \frac{d^2}{d\theta^2} w_s = -\lambda_{sA=1}^2 w_s,\ (\theta_0 \le \theta \le \pi/2) \qquad (4.5\text{-}1)$$

$$T_{s1}^0 w_{s\theta\theta} = -\lambda_{sA=1}^2 \frac{1}{3}\frac{R^3}{h}\rho_n\omega_c^2[(\sin\theta)^3 - (\sin\theta_0)^3]w_s, \qquad (4.5\text{-}2)$$

$$w_{b\theta\theta} = \frac{d^2}{d\theta^2} w_b = -\lambda_{bA=1}^2 w_b,\ (0 \le \theta \le \theta_0) \qquad (4.5\text{-}3)$$

$$T_{b1}^0 w_{b\theta\theta} = 0, \qquad (4.5\text{-}4)$$

$$\cot\theta\, w_{s\theta} = \cot\theta \frac{d}{d\theta} w_s = -\lambda_{sB}^2 w_s,\ (\theta_0 \le \theta \le \pi/2) \qquad (4.5\text{-}5)$$

$$T_{s2}^0 \cot\theta\, w_{s\theta\theta} = -\lambda_{sB}^2 \frac{2R^3}{h}\omega_c^2(2\rho_p - \rho_n)\cos\theta\sin\theta\, w_s \qquad (4.5\text{-}6)$$

$$\cot\theta\, w_{b\theta} = \cot\theta \frac{d}{d\theta} w_b = -\lambda_{bB}^2 w_b,\ (0 \le \theta \le \theta_0) \qquad (4.5\text{-}7)$$

$$T_{b2}^0 \cot\theta\, w_{b\theta} = -\lambda_{bB}^2 \frac{R^3}{h}\rho_n\omega_c^2\cos\theta\sin\theta\, w_b \qquad (4.5\text{-}8)$$

f) *Calculating* $F_s w_{s\theta\theta}^0, F_{s\theta} w_{s\theta}^0, F_b w_{b\theta\theta}^0, F_{b\theta} w_{b\theta}^0,\ w_s w_{s\theta}^0,\ w_{s\theta} w_{s\theta\theta}^0, w_b w_{b\theta}^0,\ w_{b\theta} w_{b\theta\theta}^0$ *by Eigen value.*

By (4.5-1) and (4.5-3), we have

$$F_s w_{s\theta\theta}^0 = -\lambda_{sA=1}^2 w_s F_s,\ (\theta_0 \le \theta \le \pi/2) \qquad (4.6\text{-}1)$$

$$F_b w_{b\theta\theta}^0 = -\lambda_{bA=1}^2 w_b F_b,\ (0 \le \theta \le \theta_0), \qquad (4.6\text{-}2)$$

$$F_{s\theta} w_{s\theta}^0 = \frac{d}{d\theta}F_s \frac{d}{d\theta} w_s = \lambda_{sB=1}^2 \lambda_{sB=1}^2 F_s w_s,\ (\theta_0 \le \theta \le \pi/2) \qquad (4.6\text{-}3)$$

$$F_{b\theta} w_{b\theta}^0 = \frac{d}{d\theta}F_b \frac{d}{d\theta} w_b = \lambda_{bB=1}^2 \lambda_{bB=1}^2 F_b w_b,\ (0 \le \theta \le \theta_0) \qquad (4.6\text{-}4)$$

$$w_s w_{s\theta}^0 = w_s \frac{d}{d\theta} w_s = -\lambda_{sB=1}^2 w_s^2,\ (\theta_0 \le \theta \le \pi/2) \qquad (4.6\text{-}5)$$

$$w_b w_{b\theta}^0 = w_b \frac{d}{d\theta} w_b = -\lambda_{bB=1}^2 w_b^2,\ (0 \le \theta \le \theta_0) \qquad (4.6\text{-}6)$$

$$w_{s\theta} w_{s\theta\theta}^0 = \frac{d}{d\theta} w_s \frac{d^2}{d\theta^2} w_s = \lambda_{sB=1}^2 \lambda_{sA=1}^2 w_s^2,\ (\theta_0 \le \theta \le \pi/2) \qquad (4.6\text{-}7)$$

$$w_{b\theta} w_{b\theta\theta}^0 = \frac{d}{d\theta} w_b \frac{d^2}{d\theta^2} w_b = \lambda_{bB=1}^2 \lambda_{bA=1}^2 w_b^2,\ (0 \le \theta \le \theta_0) \qquad (4.6\text{-}8)$$

g) Application of decomposition method ---Putting down the order of non-linear Eigen value equation

Using (4.1-8) --- (4.1-11), (4.4-1) --- (4.6-8), one can put down the order of Eigen value equation. For example, (4.1-6) can be reduced to:

$$(D/R^2)\lambda_s^4 w_s - Eh\lambda_s^2 w_s - \frac{[Eh\lambda_{sB}^2 w_s w_{s\theta}^0 + Eh\lambda_{sB}^2 w_{s\theta} w_{s\theta\theta}^0]}{R} +$$

$$\nabla^2\lambda_{sB}^2(-\lambda_{sA=1}^2 F_s w_s) + \nabla^2\lambda_{sB}^2\lambda_{sB=1}^4 F_s w_s^0 -$$

$$\nabla^2[T_{s1}^0 w_{s\theta\theta} + T_{s2}^0 \cot\theta\, w_{s\theta}] = 0,$$

Or, again use (4.1-8) --- (4.6-8) to reduce the $\nabla^2$and other terms, we have:

$$(\frac{D}{R^2})\lambda_s^4 w_s - Eh\lambda_s^2 w_s - \frac{[-Eh\lambda_{sB}^2\lambda_{sB=1}^2 w_s^2 + Eh\lambda_{sB}^2\lambda_{sB=1}^2\lambda_{sA=1}^2 w_s]}{R} +$$

$$\lambda_{sB}^2\lambda_s^2\lambda_{sA=1}^2 F_s w_s^0 - \lambda_{sB}^2\lambda_s^2\lambda_{sB=1}^4 F_s w_s^0 + \nabla^2[T_{s1}^0 w_{s\theta\theta} + T_{s2}^0 \cot\theta\, w_{s\theta}] = 0,$$

$$(\theta_0 \le \theta \le \pi/2) \qquad (4.7\text{-}1)$$

Calculating $\nabla^2[T_{s1}^0 w_{s\theta\theta} + T_{s2}^0 \cot\theta\, w_{s\theta}]$

Let $X = T_{s1}^0 w_{s\theta\theta}$, by (4.4-14), then, we have

$$\nabla^2(X) = \frac{d^2}{d\theta^2}(X_A) + \cot\theta\frac{d}{d\theta}(X_B) = -\lambda_{sA=1}^2(X_A) - \lambda_{sB}^2(X_B) = -\lambda_s^2(X),$$

$$(\theta_0 \le \theta \le \pi/2) \qquad (4.7\text{-}2)$$

Where by (4.5-2), (4.4-17) and (4.4-20), we have:

$$X = T_{s1}^0 w_{s\theta\theta} = -\lambda_{sA=1}^2\frac{R^3}{3h}\rho_n\omega_c^2[(\sin\theta)^3 - (\sin\theta_0)^3]w_s, \qquad (4.7\text{-}3)$$

$$\frac{X_A(\theta)}{X_A(\theta_0)} = \exp\,[\tfrac{1}{2}\lambda_{sA=1}^2(\theta-\theta_0)^2],\ (\theta_0 \le \theta \le \pi/2) \qquad (4.7\text{-}4)$$

$$\frac{X_B(\theta)}{X_B(\theta_0)} = \exp(\lambda_{sB}^2)\frac{\cos\theta}{\cos\theta_0},\ (\theta_0 \le \theta \le \pi/2) \qquad (4.7\text{-}5)$$

Let $Y = T_{s2}^0 \cot\theta\, w_{s\theta}$, by (4.4-14), then, we have:

$$\nabla^2(Y) = \frac{d^2}{d\theta^2}(Y_A) + \cot\theta\frac{d}{d\theta}(Y_B) = -\lambda_{sA=1}^2(Y_A) - \lambda_{sB}^2(Y_B) = -\lambda_s^2(Y),$$

$$(\theta_0 \le \theta \le \pi/2) \qquad (4.7\text{-}6)$$

Where by (4.5-6), (4.4-17) and (4.4-20), we have:

$$Y = T_{s2}^0 \cot\theta\, w_{s\theta\theta} = -\lambda_{sB}^2\frac{2R^3}{h}\omega_c^2(2\rho_p - \rho_n)\cos\theta\sin\theta\, w_s,\ (\theta_0 \le \theta \le \pi/2) \qquad (4.7\text{-}7)$$

$$\frac{Y_A(\theta)}{Y_A(\theta_0)} = \exp\left[\frac{1}{2}\lambda_{sA-1}^2(\theta-\theta_0)^2\right],\ (\theta_0 \le \theta \le \pi/2) \qquad (4\text{-}7\text{-}8)$$

$$\frac{Y_B(\theta)}{Y_B(\theta_0)} = \exp(\lambda_{sB}^2)\frac{\cos\theta}{\cos\theta_0},\ (\theta_0 \le \theta \le \pi/2), \tag{4.7-9}$$

Substituting (4.7-2) --- (4.7-9) into (4.7-1), we have:

$$\left(\frac{D}{R^2}\right)\lambda_s^4 w_s - Eh\lambda_s^2 w_s - \frac{[-Eh\lambda_{sB}^2\lambda_{sB=1}^2 w_s^2 + Eh\lambda_{sB}^2\lambda_{sB=1}^2\lambda_{sA=1}^2 w_s]}{R}$$

$$+ (\lambda_{sB}^2\lambda_s^2\lambda_{sA=1}^2 - \lambda_{sB}^2\lambda_s^2\lambda_{sB=1}^4)F_s w_s^0$$

$$-(\lambda_s^2 + \lambda_{sA=1}^2)\left(\frac{R^3}{3h}\rho_n\omega_c^2[(\sin\theta)^3 - (\sin\theta_0)^3]w_s\right)$$

$$-(\lambda_s^2 + \lambda_{sB}^2)\left[\frac{2R^3}{h}(2\rho_p - \rho_n)\omega_c^2\cos\theta\sin\theta\, w_s\right] = 0, \tag{4.7-10}$$

By (4.7-5) or (4.7-9), for $\theta = \theta_0$, we have

$$\lambda_{sB}^2 = 0 \tag{4.7-11}$$

Substituting (4.7-11) into (4.7-10), we have

$$\left(\frac{D}{R^2}\right)\lambda_s^4 w_s - Eh\lambda_s^2 w_s$$

$$-(\lambda_s^2 + \lambda_{sA=1}^2)\left(\frac{R^3}{3h}\rho_n\omega_c^2[(\sin\theta)^3 - (\sin\theta_0)^3]w_s\right) - \lambda_s^2[\frac{2R^3}{h}\omega_c^2$$

$$(2\rho_p - \rho_n)\cos\theta\sin\theta\, w_s] = 0, \tag{4.7-12}$$

Similarly, in BUO zone, we have

$$\left(\frac{D}{R^2}\right)\lambda_b^4 w_b - Eh\lambda_b^2 w_b$$

$$-(\lambda_b^2 + \lambda_{bA=1}^2)\left(\frac{R^3}{3h}\rho_n\omega_c^2[(\sin\theta)^3 - (\sin\theta_0)^3]w_b\right) - \lambda_b^2$$

$$[\frac{2R^3}{h}\omega_c^2(2\rho_p - \rho_n)\cos\theta\sin\theta\, w_b] = 0, \tag{4.7-13}$$

Eqs. (4.7-12) and (4.7-13) are lowest order Eigen value equations, obtained from original high order non-linear D.E. (4.1-6) and (4.1-7) by decomposition method.

h) The membrane model.

Like a balloon, the membrane is so thin, that it can not bear or resist bending and twisting moment, but it can only resist tensile stress. For axis-symmetry deformation, there is no shearing stress on the cross-sections of a cylindrical element, while tensile stress is uniformly distributed along thickness. That is: the resisting bending rigidity D = 0,

$$D/R^2 = \frac{Eh^3}{12(1-\nu^2)R^2} = 0 \quad (4.8\text{-}1)$$

$$T_1 = T_1^0,\ T_2 = T_2^0,\ w = w^0 \quad (4.8\text{-}2)$$

Substituting (4.8-1) into (4.7-12) and (4.7-13), we have:

$$Eh\lambda_s^2 w_s + (\lambda_s^2 + \lambda_{sA=1}^2)\left(\frac{R^3}{3h}\rho_n\omega_c^2[(\sin\theta)^3 - (\sin\theta_0)^3]w_s\right) +$$

$$\lambda_s^2(\frac{2R^3}{h}\omega_c^2\left(2\rho_p - \rho_n\right)\cos\theta\sin\theta\, w_s) = 0,\ (\theta_0 \le \theta \le \pi/2) \quad (4.8\text{-}3)$$

$$Eh\lambda_b^2 w_b + \left(\lambda_b^2 + \lambda_{bA=1}^2\right)\left(\frac{R^3}{3h}\rho_n\omega_c^2[(\sin\theta)^3 - (\sin\theta_0)^3]w_b\right) +$$

$$\lambda_b^2(\frac{2R^3}{h}\omega_c^2\left(2\rho_p - \rho_n\right)\cos\theta\sin\theta\, w_b) = 0,\ (0 \le \theta \le \theta_0) \quad (4.8\text{-}4)$$

i) The boundary conditions (4.1-12) --- (4.1-14) and boundary Eigen values problem

There are 6 basic unknowns $w_s, F_s, w_b, F_b, \lambda_s^2, \lambda_b^2$ in boundary Eigen value problem (4.8-3) and (4.8-4), we need 6 boundary conditions to determine 6 unknowns. That is constraints shown in (4.1-12) --- (4.1-14). Eqs. (4.8-3) and (4.8-4) are linked by boundary condition (4.1-13).
The boundary conditions (4.1-12), by (4.3-3), show that

$$w_b(0) = R - R_p + \epsilon_1 \quad (4.9\text{-}1)$$

$$F_{b\theta\theta}(0) = T_{b2}(0) = 0 \quad (4.9\text{-}2)$$

By (4.4-3), $F_{b\theta\theta}(0) = -\lambda_{bA=1}^2 F_s(0) = 0$, we have

$$\lambda_{bA=1}^2 = 0 \quad (4.9\text{-}3)$$

The boundary conditions (4.1-14) , by (4.3-4), show that

$$w_s(\pi/2) = R_e - R + \epsilon_2, \quad (4.9\text{-}4)$$

$$F_{s\theta\theta}(\pi/2) = \left(\sum(m_p + m_n)/(4\pi R_e)\right) = 0,^{[2]} \quad (4.9\text{-}5)$$

Substituting $\theta = \theta_0$, (4.9-3) into (4.8-3) and (4.8-4), by

$$w_s(\theta_0) = w_b(\theta_0), \quad (4.9\text{-}6)$$

we have:

$$\lambda_s^2 = \lambda_b^2, \quad (4.9\text{-}7)$$

By (4.1-13), (4.3-3) and (4.3-4), we have

$$F_{s\theta\theta}(\theta_0) - F_{b\theta\theta}(\theta_0) = T_{s2}(\theta_0) - T_{b2}(\theta_0) = \delta \quad (4.9\text{-}8)$$

$$\delta = \left[2(2\rho_p - \rho_n) - \frac{1}{2}\rho_n\right]\frac{R^3}{h}\omega_c^2 \cos\theta_0 \sin\theta_0 \quad (4.9\text{-}9)$$

Let

$$m_s(\theta_0) := \frac{hT_{s2}(\theta_0)}{\omega_c^2} = 2(2\rho_p - \rho_n)R^3 \cos\theta_0 \sin\theta_0 \quad (4.9\text{-}10)$$

$$m_b(\theta_0) := \frac{hT_{b2}(\theta_0)}{\omega_c^2} = \frac{1}{2}\rho_n R^3 \cos\theta_0 \sin\theta_0, \quad (4.9\text{-}11)$$

be the masses of crust at θ_0 of SIN zone and BUO zone respectively. Then, (4.9-9) means that the internal force per unit length $T_{s2}(\theta_0) \neq T_{b2}(\theta_0)$, the un-continuity of T_2 due to mantle loading can be viewed as masses $m_s(\theta_0) \neq m_b(\theta_0)$ un-continuity at θ_0.

$$m_s(\theta_0) - m_b(\theta_0) = \frac{h\delta}{\omega_c^2} = \left[2(2\rho_p - \rho_n) - \frac{1}{2}\rho_n\right]R^3 \cos\theta_0 \sin\theta_0 \quad , (4.9\text{-}12)$$

Where the masses are formed from mantle emission. Since the static steady process is concerned, which independents of time. The growing of Tibet high-land might be viewed as an evidence of the mass $m_s(\theta_0)$ increasing due to mantle emission in the process.

Substituting (4.9-3), (4.8-3) into (4.9-4), we have:

$$w_s(\pi/2) = R_p - R + \epsilon_2 = \lambda_s^2 C_s w_s(\pi/2) \quad (4.9\text{-}13)$$

$$\text{Or } \lambda_s^2 = \lambda_b^2 = C_s^{-1} = \left[Eh - \frac{R^3}{2h}\rho_n\omega_c^2(\sin\theta_0)^3\right]^{-1} \neq 0 \quad (4.9\text{-}14)$$

Eq. (4.9-14) shows that non-zero Eigen value exists. The corresponding Eigen function $w_s(\theta)$, by (4.8-3), is:

$$w_s(\theta) =$$

$$Eh + \frac{R^3}{2h}\rho_n\omega_c^2((\sin\theta)^3 - (\sin\theta_0)^3) + \frac{2R^3}{h}(2\rho_p - \rho_n)\omega_c^2 \cos\theta \sin\theta,$$

$$(\theta_0 \leq \theta \leq \pi/2) \quad (4.9\text{-}15)$$

Substituting (4.9-15) into (4.9-4), we have

$$\epsilon_2 = |R_e - R| = 6378160 - 6371032 = 7128 \text{ (km)}, \quad (4.9\text{-}16)$$ [17]

The related error $\epsilon = \frac{\epsilon_2}{R} = \frac{7128}{6371032} = 0.0011188.$

Similarly,

$$w_b(\theta) =$$

$$Eh + \frac{R^3}{2h}\rho_n\omega_c^2((\sin\theta)^3 - (\sin\theta_0)^3) + \frac{2R^3}{h}(2\rho_p - \rho_n)\omega_c^2 \cos\theta \sin\theta,$$

$$(0 \leq \theta \leq \theta_0) \quad (4.9\text{-}17)$$

Substituting (4.9-17) into (4.9-1), we have

$$\epsilon_1 = \left|R - R_p\right| = 6371032 - 6356777 = 14255\ (km),^{[17]} \qquad (4.9\text{-}18)$$

The related error $\epsilon = \frac{\epsilon_2}{R} = \frac{14255}{6371032} = 0.002237.$

Now, the governing high-order non-linear coupled D.E. (4.1-1) ---(4.1-5) is transferred to Eigen value problem and solved by decomposition method.

V. Results and Conclusion

Analysis spherical crust is a complex problem, its governing equations, even for the simpler elastic spherical shell, involves high-order non-linear D.E. with coupled unknown functions w and F. We use two models, one with resisting bending moment, the other is a membrane model without resisting bending moment, two governing equations, one in English, the other in Russian, to study dimpling and bucking. Dimpling analysis follows the previous work but instead the loading by mantle loading. Dimpling study depends on outer solution of the non-homogeneous D.E., while bucking analysis depends on the non-zero solution of homogeneous D.E. Dimpling occurs at apex (poles), like an apple. Its analysis can just use local information, like a shallow spherical cap, but the bucking analysis needs the total information of the whole structure.

VI. Results

Dimpling occurring at poles, is proved theoretically and numerical analysis well consists with poles' radius.

Bucking analysis uses transformation to Eigen value problem and solves by decomposition method. The results show that non- zero Eigen value exits. That means bucking can occur under mantle loading. An other feature is that the un-continuity of internal force per unit length due to mantle loading causes masses un-continuity by mantle material emitting out to crust at the turning point of the "X".

Both poles radius and equatorial radius have been used to support our analysis.

Question: how the nature makes cold at poles?

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