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New Integrals for Horn Hypergeometric Functions in Two Variables

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NEWINTEGRALSFORHORNHYPERGEOMETRICFUNCTIONSINTWOVARIABLES

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9. A. Hasanov, M.G. Bin Saad and A. Ryskan, *Some properties of Horn type second order double hypergeometric series*, Bulletin KRASEC. Phys. Math. Sci., 21 (2018), 32-47.

New Integrals for Horn Hypergeometric Functions in Two Variables

Jihad Ahmed Younis

Abstract- In this paper, we establish several new integral representations of Euler-type involving Horn's functions $G_1, G_2, G_3, H_1, H_2, H_5, H_6$ and H_7. Some corollaries have also been obtained as special cases of our main results.

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I. Introduction

Integral representations of hypergeometric functions have found applications in divers fields such as mathematics, physics, statistics, and engineering. Hasanov et al. [9] studied some of the properties of the Horn type second-order double hypergeometric function H_2^* involving integral representations, differential equations, and generating functions. Choi et al. [6] introduced certain integral representations for Srivastava's triple hypergeometric functions H_A, H_B and H_C. Younis and Bin-Saad [19, 20] establish several integral representations and operational relations involving quadruple hypergeometric functions $X_i^{(4)}(i = 38, 40, 45, 48, 50)$. Younis and Nisar [21] introduce new integral representations of Euler-type for Exton's hypergeometric functions of four variables D_1, D_2, D_3, D_4 and D_5. Also, in [2-5], authors introduced many integral representations for certain hypergeometric functions in four variables.

Let us recall the Gauss hypergeometric function ${}_2F_1$ is defined as (see, e.g., [14] and [16, Section 1.5])

$$ {}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}, (|x| < 1), \tag{1.1}$$

where $(a)_m$ is the well known Pochhammer symbol given by (see, e.g., [16, p. 2 and pp. 4-6])

$$ (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1 & (m = 0), \\ a(a+1)...(a+m-1) & (m \in \mathbb{N} := \{1, 2, ...\}). \end{cases} \tag{1.2}$$

Euler's integral representation of ${}_2F_1$ is defined by (see, e.g., [14, p. 85] and [16, p. 65])

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$$ {}_2F_1\left(a,b;c;x\right)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \alpha^{a-1}\left(1-\alpha\right)^{c-a-1}\left(1-x\alpha\right)^{-b}d\alpha, $$

$$ \left(Re(a)>0, Re(c-a)>0\right). $$

Appell hypergeometric functions of two variables F_1, F_2 and F_3 are respectively defined by (see [17, p. 53, Eq. (4) - (6)])

$$ F_1\left(a,b,c;d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.3} $$

$$ F_2\left(a,b,c;d,e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.4} $$

and

$$ F_3\left(a,b,c,d;e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{1.5} $$

Integral representations of Euler type for the functions F_1, F_2, F_3 were already given by Appell [1, Chap. III]. For various integral representations of hypergeometric functions, the interested reader may refer to [8-10, 12, 13, 15, 18].

Other hypergeometric functions of two variables are the following Horn's functions $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 defined by (cf. [7], [8], [11])

$$ G_1\left(a,b,c;x,y\right)=\sum_{m,n=0}^{\infty}(a)_{m+n}(b)_{n-m}(c)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.6} $$

$$ G_2\left(a,b,c,d;x,y\right)=\sum_{m,n=0}^{\infty}(a)_m(b)_n(c)_{n-m}(d)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.7} $$

$$ G_3\left(a,b;x,y\right)=\sum_{m,n=0}^{\infty}(a)_{2n-m}(b)_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.8} $$

$$ H_1\left(a,b,c;d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.9} $$

$$ H_2\left(a,b,c,d;e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.10} $$

$$ H_3\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.11} $$

$$ H_4\left(a,b;c,d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.12} $$

$$ H_5\left(a,b;c;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_{n-m}}{(c)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.13} $$

$$ H_6\left(a,b,c;x,y\right)=\sum_{m,n=0}^{\infty}(a)_{2m-n}(b)_{n-m}(c)_n\frac{x^m}{m!}\frac{y^n}{n!}, \tag{1.14} $$

$$ H_7\left(a,b,c;d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{1.15} $$

Ref

7. A. Erdjelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.

Notes

In this paper, we aim to establish the further integral representation of Euler type for Horn double hypergeometric functions G_1, G_2, G_3, H_1, H_2, H_5, H_6 and H_7

II. Main Results

Theorem 2.1. The following integral representations hold:

$$G_1\left(a,b,c;x,y\right)=\frac{\Gamma(b+b')}{\Gamma(b)\Gamma(b')}\int_0^{\infty}\alpha^{b-1}\left(1+\alpha\right)^{-(b+b')}H_1\left(c,a,b+b';b';\frac{x}{\alpha},\frac{\alpha y}{(1+\alpha)}\right)d\alpha,$$

$$\left(Re(b)>0,Re(b')>0\right), \qquad (1)$$

$$G_1\left(a,b,c;x,y\right)=\frac{\Gamma(b+b')\Gamma(c+c')}{2^{b+b'+c+c'-2}\Gamma(b)\Gamma(b')\Gamma(c)\Gamma(c')}\int_{-1}^{1}\int_{-1}^{1}\left(1+\alpha\right)^{b'-1}\left(1-\alpha\right)^{b-1}\left(1+\beta\right)^{c'-1}$$

$$\times\left(1-\beta\right)^{c-1}F_2\left(a,c+c',b+b';b',c';\frac{\left(1+\alpha\right)\left(1-\beta\right)x}{2\left(1-\alpha\right)},\frac{\left(1-\alpha\right)\left(1+\beta\right)y}{2\left(1-\beta\right)}\right)d\alpha d\beta,$$

$$\left(Re(b)>0,Re(b')>0,Re(c)>0,Re(c')>0\right), \qquad (2)$$

$$G_1\left(a,b,c;x,y\right)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)(S-R)^{a+b-1}}\int_R^S\left(\alpha-R\right)^{a-1}\left(S-\alpha\right)^{b+c-1}\left[\left(S-\alpha\right)-\left(\alpha-R\right)x\right]^{-c}$$

$$\times{}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};1-c;\frac{-4\left[\left(\alpha-R\right)\left(S-\alpha\right)-\left(\alpha-R\right)^2x\right]y}{(S-R)^2}\right)d\alpha,$$

$$\left(Re(a)>0,Re(b)>0,R<S\right), \qquad (3)$$

$$G_1\left(a,b,c;x,y\right)=\frac{2M^{a+b}\Gamma(b+c)}{\Gamma(b)\Gamma(c)}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{b-\frac{1}{2}}\left(cos^2\alpha\right)^{c-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{-(b+c)}$$

$$\times\left[M-xcot^2\alpha-M^2ytan^2\alpha\right]^{-a}d\alpha,$$

$$\left(Re(b)>0,Re(c)>0,M>0\right). \qquad (4)$$

Proof. To prove the result in equality (1) asserted in Theorem 2.1, let $\mho$ denote the right-hand side of the equality (1). Then from the definition of Horn's function H_1 in (1.9), we obtain

$$\mho=\frac{\Gamma(b+b')}{\Gamma(b)\Gamma(b')}\sum_{m,n=0}^{\infty}\frac{(c)_{m-n}(a)_{m+n}(b+b')_n}{(b')_m}\int_0^{\infty}\frac{\alpha^{b+n-m-1}}{(1+\alpha)^{b+b'+n}}d\alpha. \qquad (5)$$

Employing the integral representation of the Beta function (see, e.g., [7, p. 9, Eq. (2)])

$$B(a,b)=\int_0^{\infty}\frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}d\alpha,\left(Re(a)>0,Re(b)>0\right),$$

in (5), we have

$$\mho=\frac{\Gamma(b+b')}{\Gamma(b)\Gamma(b')}\sum_{m,n=0}^{\infty}\frac{(c)_{m-n}(a)_{m+n}(b+b')_nB(b+n-m,b'+m)}{(b')_m}. \qquad (6)$$

Now applying well known beta function (see, e.g., [16, Section 1.1])

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

in (6), we are led to the desired result. Then, the similar way we can easily get (2)-(4).

The following theorem can be proved, like Theorem 2.1. So the details are omitted.

Theorem 2.2. The following integral representations holds:

$$G_2(a,b,c,d;x,y)=\frac{\Gamma(a+b)\Gamma(c+c')\Gamma(d+d')}{2^{a+b+c+c'+d+d'-6}\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c')\Gamma(d)\Gamma(d')}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{a-\frac{1}{2}}$$

$$\times\left[(1-\alpha)^2\right]^{b-\frac{1}{2}}\left(1+\alpha^2\right)^{-(a+b)}\left[(1+\beta)^2\right]^{c'-\frac{1}{2}}\left[(1-\beta)^2\right]^{c-\frac{1}{2}}\left(1+\beta^2\right)^{-(c+c')}\left[(1+\gamma)^2\right]^{d'-\frac{1}{2}}$$

$$\times\left[(1-\gamma)^2\right]^{d-\frac{1}{2}}\left(1+\gamma^2\right)^{-(d+d')}$$

$$\times F_2\left(a+b,d+d',c+c';c',d';\frac{(1+\alpha)^2(1+\beta)^2(1-\gamma)^2x}{4(1+\alpha^2)(1-\beta)^2(1+\gamma^2)},\frac{(1-\alpha)^2(1-\beta)^2(1+\gamma)^2y}{4(1+\alpha^2)(1+\beta^2)(1-\gamma)^2}\right)d\alpha d\beta d\gamma,$$

$$\left(Re(a)>0,Re(b)>0,Re(c)>0,Re(c')>0,Re(d)>0,Re(d')>0\right), \tag{7}$$

$$G_2(a,b,c,d;x,y)=\frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\int_0^{\infty}\int_0^{\infty}\left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{b-1}\left(e^{-\beta}\right)^c\left(1-e^{-\beta}\right)^{a+b+d-1}$$

$$\times\left[\left(1-e^{-\beta}\right)-e^{-\alpha+\beta}\left(1-e^{-\beta}\right)^2x-e^{-\beta}\left(1-e^{-\alpha}\right)y\right]^{-(a+b)}d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(c)>0,Re(d)>0\right), \tag{8}$$

$$G_2(a,b,c,d;x,y)=\frac{\Gamma(a+c)\Gamma(b+d)\left(S_1-T_1\right)^a\left(R_1-T_1\right)^{b+c+d}\left(S_2-T_2\right)^b\left(R_2-T_2\right)^{a+c+d}}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}$$

$$\times\int_{R_1}^{S_1}\int_{R_2}^{S_2}\left(\alpha-R_1\right)^{a-1}\left(S_1-\alpha\right)^{b+c+d-1}\left(\beta-R_2\right)^{b-1}\left(S_2-\beta\right)^{a+c+d-1}$$

$$\times\left[\left(R_1-T_1\right)\left(S_1-\alpha\right)\left(S_2-R_2\right)\left(\beta-T_2\right)-\left(S_1-T_1\right)\left(\alpha-R_1\right)\left(R_2-T_2\right)\left(S_2-\beta\right)x\right]^{-(b+d)}$$

$$\times\left[\left(S_1-R_1\right)\left(\alpha-T_1\right)\left(R_2-T_2\right)\left(S_2-\beta\right)-\left(R_1-T_1\right)\left(S_1-\alpha\right)\left(S_2-T_2\right)\left(\beta-R_2\right)y\right]^{-(a+c)}d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(c)>0,Re(d)>0\right), \tag{9}$$

$$G_2(a,b,c,d;x,y)=\frac{2M^{a+c}\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{a+d-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{-(c+d)}$$

$$\times\left(Msinh^2\alpha-x\right)^{-a}\left(1-Mysinh^2\alpha\right)^{-b}d\alpha,$$

$$\left(Re(c)>0,Re(d)>0,M>0\right). \tag{10}$$

Notes

Notes

Theorem 2.3. The following integral representations hold:

$$G_3\left(a,b;x,y\right)=\frac{2M^a\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a-\frac{1}{2}}(cos^2\alpha)^{a'-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{-(a+a')}$$

$$\times H_6\left(b,a+a',1-a';\left(1+\frac{1}{M}cot^2\alpha\right)x,-\frac{M^2ysin^2\alpha tan^2\alpha}{cos^2\alpha+Msin^2\alpha}\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0,M>0\right), \quad (11)$$

$$G_3\left(a,b;x,y\right)=\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a'-1}\left(\frac{1}{2}-\alpha\right)^{a-1}$$

$$\times H_7\left(b,\frac{a+a'}{2},\frac{a+a'+1}{2};a';\frac{(1+2\alpha)\,x}{(1-2\alpha)},(1-2\alpha)^2\,y\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right), \quad (12)$$

$$G_3\left(a,b;x,y\right)=\frac{(1+M_1)^a(1+M_2)^b\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}\int_0^1\int_0^1\alpha^{a-1}\left(1-\alpha\right)^{a'-1}\left(1+M_1\alpha\right)^{-(a+a')}$$

$$\times\beta^{b-1}\left(1-\beta\right)^{b'-1}\left(1+M_2\beta\right)^{-(b+b')}{}_2F_1\left(\frac{b+b'}{2},\frac{b+b'+1}{2};a';\frac{4(1+M_2)^2\left(1-\alpha\right)\beta^2x}{(1+M_1)\alpha\left(1+M_2\beta\right)^2}\right)$$

$$\times{}_2F_1\left(\frac{a+a'}{2},\frac{a+a'+1}{2};b';\frac{4(1+M_1)^2\alpha^2\left(1-\beta\right)y}{(1+M_2)\left(1+M_1\alpha\right)^2\beta}\right)d\alpha d\beta,$$

$$\left(Re(a)>0,Re(a')>0,Re(b)>0,Re(b')>0,M_1>-1,M_2>-1\right), \quad (13)$$

$$G_3\left(a,b;x,y\right)=\frac{2(1+M)^{a+2b}\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{b-\frac{1}{2}}(cos^2\alpha)^{a-\frac{1}{2}}$$

$$\times\left[(1+M)\left(1+Msin^2\alpha\right)-(1+M)^3xsin^2\alpha tan^2\alpha-ycos^2\alpha cot^2\alpha\right]^{-(a+b)}d\alpha,$$

$$\left(Re(a)>0,Re(b)>0,M>-1\right), \quad (14)$$

Theorem 2.4. The following integral representations hold:

$$H_1\left(a,b,c;d;x,y\right)=\frac{\Gamma(2d)}{\Gamma(a)\Gamma(2d-a)(S-R)^{2d-1}}\int_R^S\left(\alpha-R\right)^{2d-a-1}\left(S-\alpha\right)^{a-1}$$

$$\times F_1\left(b,d+\frac{1}{2},c;2d-a;\frac{4\left(\alpha-R\right)\left(S-\alpha\right)x}{(S-R)^2},\frac{\left(\alpha-R\right)y}{\left(S-\alpha\right)}\right)d\alpha,$$

$$\left(Re(a)>0,Re(2d-a)>0,R<S\right), \quad (15)$$

$$H_1\left(a,b,c;d;x,y\right)=\frac{2\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a-\frac{1}{2}}(cos^2\alpha)^{a'-\frac{1}{2}}$$

$$\times F_2\left(b,a+a',c;d,a';xsin^2\alpha,ycot^2\alpha\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right), \quad (16)$$

$$H_1(a,b,c;d;x,y) = \frac{\Gamma(d)(S-T)^b(R-T)^{d-b}}{\Gamma(b)\Gamma(d-b)(S-R)^{d-a-1}}\int_R^S (\alpha-R)^{b-1}(S-\alpha)^{d-b-1}(\alpha-T)^{a-d}$$

$$\times\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)x\right]^{-a}$$

$$\times {}_2F_1\left(c,1+b-d;1-a;\frac{(S-T)(\alpha-R)\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)x\right]y}{(R-T)(S-R)(S-\alpha)(\alpha-T)}\right)d\alpha,$$

$$\left(Re(b)>0, Re(d-b)>0, T<R<S\right), \tag{17}$$

$$H_1(a,b,c;d;x,y) = \frac{(1+M)^{a+c}\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}(1+M\alpha)^{-(a+b)}$$

$$\times\left[(1+M)\alpha-(1-\alpha)y\right]^{-c}{}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};d;\frac{4(1+M)\alpha(1-\alpha)x}{(1+M\alpha)^2}\right)d\alpha,$$

$$\left(Re(a)>0, Re(b)>0, M>-1\right). \tag{18}$$

Corollary 2.5. Let $y=0$ in (18). Then the following result holds true:

$${}_2F_1(a,b;d;x) = \frac{(1+M)^a\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}(1+M\alpha)^{-(a+b)}$$

$$\times\,{}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};d;\frac{4(1+M)\alpha(1-\alpha)x}{(1+M\alpha)^2}\right)d\alpha,$$

$$\left(Re(a)>0, Re(b)>0, M>-1\right). \tag{19}$$

Theorem 2.6. The following integral representations hold:

$$H_2(a,b,c,d;e;x,y) = \frac{\Gamma(b+c)\Gamma(2e)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(2e-a)(S_1-R_1)^{b+c-1}(S_2-R_2)^{2e-1}}\int_{R_1}^{S_1}\int_{R_2}^{S_2}$$

$$\times(\alpha-R_1)^{b-1}(S_1-\alpha)^{c-1}(\beta-R_2)^{a-1}(S_2-\beta)^{2e-a-1}$$

$$\times F_1\left(b+c,e+\frac{1}{2},d;2e-a;\frac{4(\alpha-R_1)(\beta-R_2)(S_2-\beta)x}{(S_1-R_1)(S_2-R_2)^2},\frac{(S_1-\alpha)(S_2-\beta)y}{(S_1-R_1)(\beta-R_2)}\right)d\alpha d\beta,$$

$$\left(Re(a)>0, Re(b)>0, Re(c)>0, Re(2e-a)>0, R_1<S_1, R_2<S_2\right), \tag{20}$$

$$H_2(a,b,c,d;e;x,y) = \frac{\Gamma(2e)}{2^{2(e-1)}\Gamma(a)\Gamma(2e-a)}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{2e-a-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a-\frac{1}{2}}$$

$$\times\left(1+\alpha^2\right)^{-e}F_3\left(b,c,e+\frac{1}{2},d;2e-a;\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^2x,\left(\frac{1+\alpha}{1-\alpha}\right)^2y\right)d\alpha,$$

$$\left(Re(a)>0, Re(2e-a)>0\right), \tag{21}$$

$$H_2(a,b,c,d;e;x,y) = \frac{(1+M)^a\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}(1+M\alpha)^{-(a+b)}$$

$$\times H_7\left(a+b,c,d;e;\frac{(1+M)\alpha(1-\alpha)x}{(1+M\alpha)^2},\frac{(1+M\alpha)y}{(1+M)\alpha}\right)d\alpha,$$

$$\left(Re(a)>0, Re(b)>0, M>-1\right), \tag{22}$$

Notes

$$H_2\left(a,b,c,d;e;x,y\right)=\frac{2M^{e-b}\Gamma(e)}{\Gamma(b)\Gamma(e-b)}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{e-b-\frac{1}{2}}\left(cos^2\alpha\right)^{b-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{a-e}$$

$$\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-xcos^2\alpha\right]^{-a}{}_2F_1\left(c,d;1-a;\left[\frac{xcos^2\alpha}{(cos^2\alpha+Msin^2\alpha)}-1\right]y\right)d\alpha,$$

$$\left(Re(b)>0,Re(e-b)>0,M>0\right).\tag{23}$$

Theorem 2.7. The following integral representations hold:

$$H_5\left(a,b;c;x,y\right)=\frac{\Gamma(2c)}{\Gamma(b)\Gamma(2c-b)}\int_0^{\infty}\left(e^{-\alpha}\right)^b\left(1-e^{-\alpha}\right)^{2c-b-1}$$

$$\times H_3\left(a,c+\frac{1}{2};2c-b;\left(e^{\alpha}-1\right)x,4e^{-\alpha}\left(1-e^{-\alpha}\right)y\right)d\alpha,$$

$$\left(Re(b)>0,Re(2c-b)>0\right),\tag{24}$$

$$H_5\left(a,b;c;x,y\right)=\frac{\Gamma(b+b')}{2^{b+b'-1}\Gamma(b)\Gamma(b')}\int_{-1}^{1}\left(1+\alpha\right)^{b'-1}\left(1-\alpha\right)^{b-1}$$

$$\times H_4\left(a,b+b';b',c;\left(\frac{1+\alpha}{1-\alpha}\right)x,\frac{(1-\alpha)}{2}y\right)d\alpha,$$

$$\left(Re(b)>0,Re(b')>0\right),\tag{25}$$

$$H_5\left(a,b;c;x,y\right)=\frac{2M^{a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{a-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{b-c}$$

$$\times\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]^{-b}$$

$$\times{}_2F_1\left(\frac{1+a-c}{2},\frac{a-c}{2}+1;1-b;-\frac{4M^2xsinh^4\alpha\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]}{(1+Msinh^2\alpha)}\right)d\alpha,$$

$$\left(Re(a)>0,Re(c)>0,M>0\right),\tag{26}$$

$$H_5\left(a,b;c;x,y\right)=\frac{2M^{c-a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{c-a-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{b-c}$$

$$\times\left[\left(1+Msinh^2\alpha\right)-y\right]^{-b}$$

$$\times{}_2F_1\left(\frac{1+a-c}{2},\frac{a-c}{2}+1;1-b;-\frac{4x\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]}{M^2sinh^4\alpha\,(1+Msinh^2\alpha)}\right)d\alpha,$$

$$\left(Re(a)>0,Re(c)>0,M>0\right).\tag{27}$$

Corollary 2.8. Let $x=0$ in (26). Then the following result holds true:

$${}_2F_1\left(a,b;c;y\right)=\frac{2M^{a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{a-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{b-c}$$

$$\times\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]^{-b}d\alpha,$$

$$\left(Re(a)>0,Re(c)>0,M>0\right).\tag{28}$$

Theorem 2.9. The following integral representations hold:

$$H_6\left(a,b,c;x,y\right)=\frac{\Gamma(a+c)\Gamma(b+b')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{c-1}\left(\frac{1}{2}-\alpha\right)^{a-1}$$

$$\times\left(\frac{1}{2}+\beta\right)^{b'-1}\left(\frac{1}{2}-\beta\right)^{b-1}$$

$$F_3\left(\frac{a+c}{2},\frac{b+b'}{2},\frac{a+c+1}{2},\frac{b+b'+1}{2};b';\frac{(1-2\alpha)^2(1+2\beta)x}{(1-2\beta)},\frac{(1+2\alpha)(1-4\beta^2)y}{(1-2\alpha)}\right)d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(b')>0,Re(c)>0\right),\tag{29}$$

$$H_6\left(a,b,c;x,y\right)=\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\infty}\alpha^{a-1}(1+\alpha)^{-(a+a')}G_2\left(a+a',c,b,1-a';\frac{-\alpha^2x}{(1+\alpha)},\frac{-y}{\alpha}\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right),\tag{30}$$

$$H_6\left(a,b,c;x,y\right)=\frac{\Gamma(b+b')}{2^{b+b'-2}\Gamma(b)\Gamma(b')}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{b'-\frac{1}{2}}\left[(1-\alpha)^2\right]^{b-\frac{1}{2}}\left(1+\alpha^2\right)^{-(b+b')}$$

$$\times H_6\left(a,1-b',c;-\left(\frac{1+\alpha}{1-\alpha}\right)^2x,-\left(\frac{1-\alpha}{1+\alpha}\right)^2y\right)d\alpha,$$

$$\left(Re(b)>0,Re(b')>0\right),\tag{31}$$

$$H_6\left(a,b,c;x,y\right)=\frac{4(1+M_1)^{a+b+b'}(1+M_2)^{b'}\Gamma(a+c)\Gamma(b+b')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a-\frac{1}{2}}$$

$$\times\left(cos^2\alpha\right)^{c-\frac{1}{2}}\left(sin^2\beta\right)^{b'-\frac{1}{2}}\left(cos^2\beta\right)^{b-\frac{1}{2}}\left(1+M_1sin^2\alpha\right)^{-(a+c)}$$

$$\times\left[(1+M_1)(1+M_2sin^2\beta)-ycot^2\alpha cos^2\beta\right]^{-(b+b')}$$

$$\times{}_2F_1\left(\frac{a+c}{2},\frac{a+c+1}{2};b';\frac{4(1+M_1)^2(1+M_2)xsin^4\alpha tan^2\beta}{(1+M_1sin^2\alpha)^2}\right)d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(b')>0,Re(c)>0,M_1>-1,M_2>-1\right).\tag{32}$$

Theorem 2.10. The following integral representations hold:

$$H_7\left(a,b,c;d;x,y\right)=\frac{\Gamma(a+a')(S-T)^a(R-T)^{a'}}{\Gamma(a)\Gamma(a')(S-R)^{a+a'-1}}\int_R^S(\alpha-R)^{a-1}(S-\alpha)^{a'-1}(\alpha-T)^{-(a+a')}$$

$$\times{}_2F_1\left(\frac{a+a'}{2},\frac{a+a'+1}{2};d;\left(\frac{2(S-T)(\alpha-R)}{(S-R)(\alpha-T)}\right)^2x\right){}_2F_1\left(b,c;a';\frac{(R-T)(S-\alpha)y}{(S-T)(\alpha-R)}\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0,T<R<S\right),\tag{33}$$

Notes

$$H_7\left(a,b,c;d;x,y\right)=\frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)}\int_0^{\infty}\left(e^{-\alpha}\right)^b\left(1-e^{-\alpha}\right)^{d-b-1}$$

$$\times H_6\left(a,1+b-d,c;\left(e^{-\alpha}-1\right)x,\frac{y}{(1-e^{\alpha})}\right)d\alpha,$$

$$\left(Re(b)>0,Re(d-b)>0\right),\qquad(34)$$

Notes

$$H_7\left(a,b,c;d;x,y\right)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)(S-R)^{a+b-1}}\int_R^S\left(\alpha-R\right)^{a+c-1}\left(S-\alpha\right)^{b-1}$$

$$\times\left[\left(\alpha-R\right)-\left(S-\alpha\right)y\right]^{-c}{}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};d;4\left(\frac{\alpha-R}{S-R}\right)^2x\right)d\alpha,$$

$$\left(Re(a)>0,Re(b)>0,R<S\right),\qquad(35)$$

$$H_7\left(a,b,c;d;x,y\right)=\frac{2\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a'-\frac{1}{2}}(cos^2\alpha)^{a-\frac{1}{2}}$$

$$\times H_2\left(1-a',a+a',b,c;d;-xcot^2\alpha cos^2\alpha,-ytan^2\alpha\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right).\qquad(36)$$

Corollary 2.11. Let $y=0$ in (32). Then the following result holds true:

$${}_2F_1\left(\frac{a}{2},\frac{a+1}{2};d;4x\right)=\frac{\Gamma(a+a')(S-T)^a(R-T)^{a'}}{\Gamma(a)\Gamma(a')(S-R)^{a+a'-1}}\int_R^S(\alpha-R)^{a-1}(S-\alpha)^{a'-1}$$

$$\times(\alpha-T)^{-(a+a')}{}_2F_1\left(\frac{a+a'}{2},\frac{a+a'+1}{2};d;4\left(\frac{(S-T)(\alpha-R)}{(S-R)(\alpha-T)}\right)^2x\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0,T<R<S\right).\qquad(37)$$

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Notes