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I. Introduction

Modern quantum field theory (QFT) encounters several difficulties, of which we note the problem of divergences and the problem of quantization of the gravitational field. The first one is that the values for the observable quantities in the framework of the methods adopted in the QFT contain diverging expressions, which requires the use of particular efforts for calculating them. For this purpose, renormalization methods have been developed, the essence of which is to replace the "naked" or "seed" quantities in the final expressions with effective or physical ones. Moreover, all infinities are attributed to the seed (and unobservable) quantities. According to this purely mathematical procedure, all field theories are divided into renormalizable, i.e., those for which this program leads to an acceptable result, and non-renormalizable [1].

The second problem is that the theory of gravity, which is understood as A. Einstein's general theory of relativity (GR), does not allow the renormalization procedure to be applied to it, i.e., GR refers to non-renormalizable ones [1]. As a result, there was a belief that the quantization of gravity requires a specific treatment so that GR can be included in the overall picture of QFT.

This work does not aim to quantize the gravitational field. Instead, a different picture of the combination of QFT with the classical theory of gravity is proposed, which is based on the calculation of formally divergent expressions of QFT in a curved metric as a result of which they obtain finite values. For this, the mass of objects for calculating in the QFT is considered from general relativity as a source of curvature of the space-time metric, and not as a simple parameter of the theory. As a result, the final expressions do not contain divergences, and the renormalization procedure is unnecessary.

The work has the following structure. The Introduction describes the problem and methods for overcoming it. Sections 2 and 3 discuss the calculation of the Casimir force between two parallel plates using traditional renormalization methods using the cut-off factor method and the zeta-regularization method. Section 4 presents the results of the proposed method for constructing a QFT that does not contain divergence. Section 5 is devoted to a discussion of the new vision.

II. Calculation of the Casimir Force between Two Plates

Initially, H. Casimir considered the appearance of an interaction between two conducting parallel endless plates of zero thickness located at a distance d from each other [2]. The force arises due to the shift of the energy of the zero-point oscillations of the vacuum due to the presence of plates. In the following presentation, we will follow [1], according to which we replace the electromagnetic field with a massless scalar field, and instead of the 3 + 1-dimensional problem, we consider the 1 + 1-dimensional one. This does not affect the consideration generality. The plates are located perpendicular to the axis OX at points $x = 0$ and $x = d$. According to [1], the change in energy is equal to

$$\Delta E = \frac{\pi\hbar c}{2d}\sum_{n=1}^{\infty} n \qquad (1)$$

Since the modes are given by the expressions $sin(n\pi x/d)$ (n is an integer), and the energies corresponding to them are equal $\hbar\omega_n = n\,\pi c\hbar/d$, $\hbar$ – Plank constant, c – light speed. Following the methods of QFT [1] instead of (1) consider $\Delta E(d) + \Delta E(L\text{-}d)$, introducing an additional third plate located at $x = L$, $L >> d$. Also, a cut-off factor is introduced into the sum (1) $\exp(-\alpha\omega_n)$, $\alpha\omega_n >> 1$ for $n >> 1$ considering that for high frequencies the plates are not visible. In other words, instead of (1), we have

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$$\Delta E(d)=\frac{\pi\hbar c}{2d}\sum_{n=1}^{\infty}n\exp(-\alpha\omega_n)=\frac{\pi\hbar c}{2d}\frac{\exp\left(\frac{\pi c\alpha}{d}\right)}{\left[\exp\left(\frac{\pi c\alpha}{d}\right)-1\right]^2} \tag{2}$$

In the limit of small α, an expression for $\Delta E(d)$ looks as follows

$$\Delta E(d)\approx\frac{d\hbar}{2\pi\alpha^2 c}-\frac{\pi\hbar c}{24d} \tag{3}$$

The final expression for the Casimir force has the form

$$\frac{\partial}{\partial d}\left[\Delta E(d)-\Delta E(L-d)\right]\xrightarrow[\alpha\to 0]{}\frac{\pi\hbar c}{24}\left[\frac{1}{d^2}-\frac{1}{(L-d)^2}\right]=\frac{\pi\hbar c}{24d^2} \tag{4}$$

conclusion is somewhat complicated, but it allows one to understand the procedure for eliminating of divergence by subtracting expressions of the analogous structure - α was not included in the final formula.

III. Zeta Regularization Method [3]

Let's consider the 1 + 1-dimensional Casimir problem for a massive scalar field Φ *(t, x)* obeying the Klein-Gordon equation [1]

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}+m^2\right)\Phi(t,x)=0 \tag{5}$$

m – is a mass of the field [4]. Stationary states have the form $\Phi(t,x)=e^{i\omega t}\varphi(x)$, and the spatial part of the field satisfies the equation

$$\left(-\frac{\partial^2}{\partial x^2}+m^2\right)\varphi(x)=\omega^2\varphi(x) \tag{6}$$

ω – is an energy of the stationary state of the field. For the Dirichlet boundary conditions, the wave function has the form $\varphi(x)=\sin k_n x/\sqrt{\omega_n d}$, where $\omega_n=\sqrt{k_n^2+m^2}, k_n=\pi n/d, n=1,2,3...$ In the frame of the zeta regularization method, the diverging expression for the energy of zero vibrations has the form

$$E(s)=\frac{1}{2}\mu^{2s}\sum_{\omega}\left(\omega^2\right)^{\frac{1}{2}-s}=\frac{1}{2}\mu^{2s}\zeta\left(s-\frac{1}{2}\right) \tag{7}$$

Where $\zeta(s)=\sum_{\omega}\left(\omega^2\right)^{-s}$ - the zeta function of the Laplace-type operator on the left-hand side of (6) and the parameter μ with the dimension of mass is introduced so that the quantity *E* (*s*) has the dimension of energy.

By some mathematical transformations, which we omit, we represent *E* (*s*) in the form ($\beta = md/\pi$) [4]

$$E(s)=E^{div}(s)+E^{ren}(s)$$

$$E^{div}(s)=\frac{m}{2}\left(\frac{\mu}{m}\right)^{2s}\left[-\frac{1}{2}+\frac{\sqrt{\pi}}{2}\frac{\Gamma(s-1)}{\Gamma\left(s-\frac{1}{2}\right)}\beta\right]$$

$$E^{ren}(s)=-m\beta\int_1^{\infty}\frac{\sqrt{x^2-1}}{e^{2\pi\beta x}-1}dx \tag{8}$$

where the diverging and renormalized energy parts $E^{div}(s)$ and $E^{ren}(s)$ are explicitly pointed. The diverging fraction includes the part of the total energy, which is conserved in the classical limit to which corresponds $m \to \infty$. It contains the same divergences as the total energy. Following the renormalization rules, it must be subtracted from the total energy. At small distances between the plates

$$E^{ren}\big|_{md\to 0}=-\frac{m}{24\beta}=-\frac{\pi}{24d} \tag{9}$$

which is the same as the result of the previous calculation using the cut-off factor.

[1] This section uses the Heaviside unit system in which c = $\hbar$ = 1

IV. Zeta Function Calculation

Note that if we substitute in (1) the expression for the Dirichlet series for $\zeta(-1)$ ($\zeta(s)=\sum_{s=1}^{\infty} n^{-s}$ – Riemann zeta-function)

$$\zeta(-1)=\sum_{n=1}^{\infty} n=-\frac{1}{12} \tag{10}$$

Then we immediately get the final expression for the Casimir force (4). The sums of divergent series, in particular series (10), are calculated using the Euler-Maclaurin formula [5]. Like other methods of summing divergent series, this one, although it leads to the correct result, leaves some questions unanswered. The same applies to the above methods of renormalization. In particular, it remains unclear why the answer looks that way, and not otherwise. In fairness, one must say that mathematics does not set such a goal. Nevertheless, the answer to this question allows us to find out some details that had not been paid attention to before. Therefore, later in this section, we will try to answer it by turning to the calculation of the zeta function $\zeta(-1)$.

The sum of the Dirichlet series of the zeta function ζ (–1) was calculated in [6]. Series (10) are divergent and, as is commonly believed, have no sum in the usual sense applicable to converging ones [5]. However, this is true only when calculating is performed in a flat space-time metric. Calculation in the curved metric leads to the value $\zeta(-1)\approx -0.08035$, which coincides with the exact one $-1/12 = -0.0833(3)$ within the relative error of $\sim$ 3.576%. The calculation is performed in the metric (s is the interval, t is the coordinate time, K is the gravitational constant).

$$ds^2=g_{00}c^2dt^2-g_{11}dx^2$$
$$g_{00}=1+\frac{x}{x_c},\, g_{11}=g_{00}^{-1},\, x_c=\frac{c^2}{4\pi\sigma K} \tag{11}$$

which is created by the gravitational field of an infinite massive plane perpendicular to the OX axis with a constant mass density σ.

In other words, in the initial Casimir problem, divergences can be avoided if massive plates are considered from the very beginning. The metric created by two plates has the form [7, 8]

$$ds^2=\left(1+\frac{2x}{x_c}\right)c^2dt^2-\left(1+\frac{2x}{x_c}\right)^{-1}dx^2,\, x<-\frac{x_c}{2}$$
$$ds^2=\left(1-\frac{2x}{x_c}\right)c^2dt^2-\left(1-\frac{2x}{x_c}\right)^{-1}dx^2,\, x>\frac{x_c}{2}$$
$$ds^2=c^2dt^2-dx^2,\, -\frac{x_c}{2}<x<\frac{x_c}{2} \tag{12}$$

Expressions for metrics (11) and (12) are obtained from solutions of Einstein's equations [9]

$$R_{ik}-\frac{1}{2}g_{ik}R=\frac{8\pi K}{c^2}T_{ik}$$
$$ds^2=g_{ik}dx^idx^k \tag{13}$$

g_{ik} – metric tensor, R_{ik} , T_{ik} – Ricci tensor and momentum-energy tensor, $R = R^i_i$, $i,k = 0,1$. Einstein's equations reduce to a condition imposed on the Ricci tensor $R_{ik} = 0$. The solution of (13) looks as follows[2] $g_{00} = g_{11}^{-1} = C_1x + C_2$, $C_{1,2}$ – constants whose values are found from the relation $g_{00} = 1 + 2\varphi/c^2$ in the region where the gravitational field of the plate is weak, $\varphi(x)=2\pi\sigma Kx$ – Newtonian gravitational potential of the plate. As a result, expression (11) is obtained for the space-time interval. The calculation of ζ (–1) itself requires solving the relativistic equations of motion of the material point

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}=0 \tag{14}$$

Γ^i_{kl} - Christoffel symbols [9], using metric (11) with initial conditions determined by the form of partial sums (10) (integer m plays the role of time)

$$S_m=\sum_{n=1}^{m} n=\frac{m(m+1)}{2}=S_0+\frac{m}{2}+\frac{m^2}{2} \tag{15}$$
$$S_0=0$$

which coincide with the distances traveled by the material point moving along a straight line with constant acceleration $w = ½$ and having in the initial position the speed $v_0 = ½$ [6].

The value of the Casimir force can be obtained by performing calculations with the metric (12), too, since the solutions of Eqs (14) can be considered on a half-line OX limited by a singularity. Let us embed a number line with metric (11) in a manifold of greater dimension - the plane. This allows one to state that [10]

[2] The rest components of the metric tensor $g_{ik} = 0$.

1. The embedding is possible only for the half-line, limited on one side by the singularity of the metric
2. The embedding plane has hyperbolic geometry

Let us return to the solution of the equation of motion (14). Its solution shows that on the trajectory the condition $v_0 = ½$ cannot be satisfied, so the original task has to be changed. Instead of calculating S_m (15) we calculate S'_m

$$S'_m = \sum_{n=1}^{m}(n-1) = S'_0 - \frac{m}{2} + \frac{m^2}{2} \qquad (16)$$

$$S'_0 = 0$$

Using the Ramanujan summation method, one can show that $S_\infty = S'_\infty + 0{,}5$ [11]. At the starting point on the trajectory $v'_0 = -½$, which is satisfied when $q_0 = x_0/x_c = -1{,}41965$. Strictly speaking, the point x_0 should not be considered the initial, but the final point of the trajectory starting in the singularity $q_c = -x_c/x_c = -1$, thereby placing the initial moment in $-\infty$. The distance traveled by a point from q_c to q_0, which we consider as[3] $-S_\infty = -\zeta(-1)$ is equal $q_0 - q_c + 0{,}5 = 0{,}08035$, which leads to the value $\zeta(-1)$ coinciding with the exact value $-0{,}0833(3)$ within the relative error 3,576%.

V. Discussion

As noted earlier, the calculation result of $\zeta\,(-1)$ can be considered as yet another confirmation of Einstein's general theory of relativity (GR), obtained along with other experimental ones [8]. Its feature is low accuracy compared to the experimental ones. This is due to the approximate nature of metric (11) because when it was obtained, the dependence of the acceleration w in a fixed system on the velocity of the point v was not taken into account

$$w = \left(1 - \frac{v^2}{c^2}\right)^{3/2} w' \qquad (17)$$

Metric (11) ensures the constancy of acceleration in the point's frame of reference w'. The relative smallness of the error is because in the considered section of the trajectory $v/c << 1$ [6].

Despite this, the main goal of the work is achieved - it is shown that taking into account the mass of plates and the resulting space-time curvature allows us to obtain a solution to the Casimir problem without divergences.

It can be assumed that this result can be generalized to other problems of QFT, whose solution in a flat space-time leads to divergences.

[3] Since the point moves in the direction of decreasing the coordinate q, series' sum $S_\infty = -\Delta q$

As already noted above, the final values for sums of divergent series similar to the series appearing in the Casimir problem are obtained using the Euler – Maclaurin or Abel – Plan formulas [12]. The author's approach allows one to connect these results with the geometric properties of the numerical continuum and speaks of a deep connection between different points of view on the nature of the calculations.

A historical analogy can be drawn with the interpretation of the Lorentz transformations as properties of Maxwell's equations, adopted in the pre-Einstein era and currently as properties of space-time as a whole.

The effect of gravity on the Casimir effect has been considered previously in many works. In [15, 16], the Casimir effect was studied in a weak background gravitational field quadratic in curvature invariants. In [17], the so-called dynamic Casimir effect was investigated associated with the influence of the movement of plates in curved space-time on quantum fluctuations. The smallness of corrections to the classical Casimir effect will require additional experiments on gravitational interferometry to observe them.

The interest in constructing quantum field theory in curved space-time is due to the existence of regimes when the influence of the space-time curvature is significant, and the effects of quantum gravity can still be neglected [18].

VI. Conclusion

The paper constructs a version of quantum field theory in a curved space-time that does not contain divergences. As an example, the calculation of the vacuum energy for a massive scalar field with boundary conditions on two parallel infinite massive plates (a variant of the Casimir problem) is considered. A standard solution to this problem containing divergences in flat space-time is given. Traditional methods for their elimination are considered - the cut-off factor method and the zeta-regularization method. It is shown that the divergences are due to the solution of the problem in neglecting the mass of plates and, as a consequence, the curvature of space-time. It is also shown that the curvature of space-time by taking into account the gravitational effect of the plates leads to the disappearance of divergences.

The result of the work allows us to conclude that the real reason for the appearance of divergences in the QFT is the consideration of problems within the framework of the concept that is called "physics in a box" in the literature, when for the sake of convenience or simplification in the problem to be solved only some features of the phenomenon are taken into account that is important for obtaining an early decisions and features that seem insignificant are discarded [19].

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