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Discovering Thoughts, Inventing Future

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By S. A. Baranov

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I. Introduction

Interest in magnetic micro and nanowires has greatly increased in the last few years mainly due to their technological applications. Glass-coated amorphous magnetic micro- and nanowires (GCAMNWs) [1-4] are attracting particular attention because of their applicability for multifunctional radioabsorbing shielding (important results published, for example in Refs. [2-20]).

Cast GCAMNWs are produced by the Taylor Ulitovsky method (see in Ref. [1-4]) as depicted in Fig.1.

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The alloy is heated, in an inductor, up to the melting point. The portion of the glass tube adjacent to melting metal softens, enveloping the metal droplet.

Under suitable conditions, the molten metal fills the glass capillary and a microwire is thus formed, with the metal core completely covered by a glass shell.

The microstructure of GCAMNWs depend mainly upon the cooling rate, which can be controlled when the metal-filled capillary enters a stream of cooling liquid on its way to the receiving coil. Critical quenching rates (10^5-10^7 K/s) for fabrication of amorphous material may be obtained.

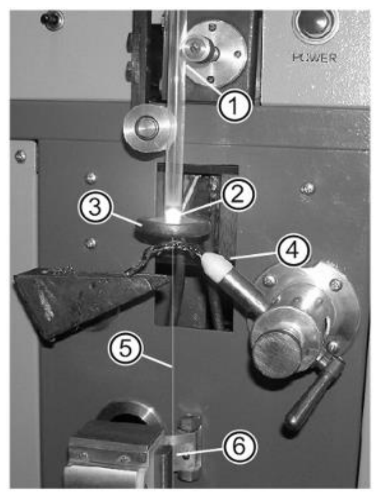


Fig. 1: Process of casting glass-coated amorphous magnetic microwires (see [4] and below).
1. Glass tube. 2. Drop of metal. 3. Inductor. 4. Water. 5. Glass-coated microwire. 6. Rotating support.

The glass coating of the cast GCAMNWs, in addition to protecting the metallic nucleus from corrosion and providing electrical insulation, induces large mechanical stresses in the nucleus. Coupled with its magnetostriction, these determine its magnetoelastic anisotropy, at the origin of a unique magnetic behaviour. The residual stresses are the result of differences in the coefficients of thermal expansion of the metal and of the glass. A simple theory for the distribution of residual stresses was presented in Refs. [2- 8].

The theory of residual stress is presented in Ref. [4]. We will use results of this theory. Coupled with the magnetostriction of the latter, these factors determine the magnetoelastic anisotropy which is at the origin of a unique magnetic behavior.

In cylindrical coordinates, the residual tension is characterized by the axial, σ_z, radial, σ_r, and tangential, σ_φ, components which are independent of the radial coordinate. The value of these stresses depends on the ratio of the radius, R_m, of the metallic kernel to the total microwire radius, R_c:

$$x = \left(\frac{R_C}{R_m}\right)^2 - 1 \quad (1)$$

Using the cylinder–shell model, we then obtain a formula for stresses in the metallic kernel of the cast GCAMNWs:

$$\sigma_r = \sigma_\phi \equiv P_\nu , \quad (2)$$

$$P_\nu = \varepsilon E_1 \frac{kx}{[k(1-2\nu)+1]x + 2(1-\nu)} , \quad (3)$$

where $P_0 \sim \varepsilon E_1 = \sigma_o \sim 2$ GPa is the maximum stress in the metallic kernel; ε is the difference between the thermal expansion of the metallic core and that of the glass shell with the expansion coefficients α_1 and α_2: ($\varepsilon = (\alpha_1 - \alpha_2)(T^* - T)$); E_1 is the Young modulus of the metal core, T^* is the solidification temperature of the composite in the metal / glass contact region ($T^* \sim$ (800…1000) K), T is the room temperature. The technological parameter k is the ratio of Young's modulus of the glass and the metal:

$k = E_2/E_1 \sim 0.3…0.6$;ν is the Poisson ratio.

Let us consider the case where all the Poisson ratios

$$\nu = 1/3$$

in order to obtain

$$P = \varepsilon E_1 \frac{kx}{(k/3+1)x + 4/3} , \quad (4)$$

$$\sigma_z = P\frac{(k+1)x+2}{kx+1} . \quad (5)$$

For materials with positive magnetostriction, the orientation of the microwire magnetization is parallel to the maximal component of the stress tensor, which is directed along the axis of the microwire. Therefore, cast microwires with a positive magnetostriction show a rectangular hysteresis loop with a single large Barkhausen jump between two stable magnetization states and exhibit the phenomenon of NFMR. Equations (1-5) adequately explain the experiments concerning FMR and NFMR (see below).

We suggested a model in which the residual stresses σ_r and σ_z in the GCAMNW monotonically decrease towards the strand axis. This model differs from the models of the standard theory (see Ref. [2-8]).

With additional longitudinal strain, which occurs when the microwire is embedded in a solid matrix that itself deforms under external influence, the following term is added to the expression for the residual axial tension:

$$\sigma_{ez} = \frac{P_o}{S_m(k\ x\ +1)} , \quad (6)$$

where P_o is the force applied to the composite; S_m is the microwire cross-sectional area; k is the ratio of the Young modulus of the shell to that of the microwire; x is the ratio of the cross-sectional area of the shell to that of the microwire.

For materials with positive magnetostriction, the orientation of the microwire magnetization is parallel to the maximal component of the stress tensor, which is directed along the axis of the wire (see [2-8]). Therefore, cast Fe–based microwires with positive magnetostriction constant show a rectangular hysteresis loop with single large Barkhausen jump between two stable magnetization states and exhibit the phenomenon of natural ferromagnetic resonance (NFMR) (see [2-8]).

The ferromagnetic-resonance (FMR) method is often used for investigation of amorphous magnetic materials (ribbons, wires, thin films). Both macroscopic and microscopic heterogeneity of amorphous materials can be investigated by FMR. Residual stress is an important parameters for amorphous materials which can be studied by FMR (see [1-3]).

FMR is also used for diagnostics of the uniformity of amorphous materials. Extrinsic broadening of FMR lines due to fluctuations of the anisotropy, magnetization, and exchange-interaction constant in amorphous materials has also been investigated. Microwave experiments are very useful for investigation of spin-wave effects. In particular, microwave generation and amplification are of great interest. Investigation of structural relaxation of amorphous materials during heat treatment, using FMR is also important. Differential FMR curves combined with hysteresis curves can give important information in this case.

In the present work, cast glass-coated amorphous microwires with metallic cores and diameters between 0.5 and 25 μm are considered. The amorphous structure of the core was investigated by X-ray methods. The thickness of the glass casing varied between 1 and 20 μm. Using microscopy, we have chosen samples with the most ideal form, and with lengths of about 3-5 mm, for investigation. Microwires based on iron, cobalt, and nickel (doped with manganese), with additions of boron, silicon, and carbon, were studied. Microwires made from different materials have diverse magnetostriction. We have studied microwire from the same spool, whose glass casing was removed by etching in hydrofluoric acid.

In almost all cases, standard FMR spectrometers from 2 to 32 GHz were used. The magnetic field was measured using a Hall sensor (with accuracy within 0.1 %). In addition, magnetometer measurements determined the magnetization, needed for calculation.

The basic measurements were made in a longitudinal field configuration (external magnetic field was directed along the microwire axis). In this case, a signal of the correct form was obtained from good samples. This gives the possibility of measuring resonant-curve width.

For thick cores, skin effect must be taken into account. In this case the resonant frequency was described by the Kittel formula for a plane (with longitudinal magnetization). The *g* factor was estimated at two resonant frequencies as ~ 2.08 ÷ 2.1 on average. Our results are in good agreement with literature data on the *g* factor for amorphous materials (see Refs. [2-20]). In a transverse field (when the external field is perpendicular to the microwire axis), the signal was weak or not observed in samples with negative magnetostriction. Obviously, the presence of this signal is associated with non-uniformity of the high-frequency demagnetizing factor.

II. Natural Ferromagnetic Resonance

A microwire was considered to be a ferromagnetic cylinder with small radius r. For its characterization we introduce following parameters:

1. The depth of the skin layer is:

$$\delta \sim [\omega(\mu\mu_0)_e\Sigma]^{-1/2} \sim \delta_0(\mu)_e^{-1/2}, \quad (7)$$

$(\mu\mu_0)_e$ is the effective magnetic permeability, and Σ is the microwire electrical conductivity. In the case of our magnetic microwires, with the relative permeability $|\mu|$ near resonance of the order 10^2, ($\omega\sim$ (8 ÷ 10) GHz) δ changes from1 to 3 μm.

2. The size of the domain wall (according to Landau-Lifshits theory) is:

$$\Delta \sim (A/K)^{1/2} \sim (10 \div 0{,}1)\ \mu\text{m}, \quad (8)$$

where *A* is the exchange constant and *K* is the anisotropy energy of the microwire ($K\sim\lambda\sigma$, where λ is the magnetostriction constant and σ is the effective residual stress from the fabrication procedure (see Refs. [2, 4] and Eqs. (1-6))). The full theory gives $\Delta_f \sim 0{,}1\mu$m for the size of the domain wall of glass-coated microwires (see Ref. [4]).

3. The radius of a single domain (according to Brown theory) is

$$a \sim A^{1/2}/M_S \sim (0{,}1 \div 0{,}01)\ \mu\text{m}, \quad (9)$$

where M_s is the saturation magnetization of microwire.

According to Refs. [2-8] the frequency of the *NFMR*is:

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(H_e + 2\pi M_S\right)^2 - \left(2\pi M_S\right)^2 \exp\{-2\delta/r\}, \quad (10)$$

where γ is the gyromagnetic ratio ($\gamma \sim 3$ MHz/Oe). The anisotropy field is $H_e \sim 3\lambda\sigma/M_s$(for exact calculations of anisotropy field, see below).

If $r<\delta$, we have:

$$\frac{\omega}{\gamma} = H_e + 2\pi M_S \quad (11)$$

If $r>\delta$, the *NFMR* frequency is given by (see Refs. [2 - 8]):

$$\left(\frac{\omega}{\gamma}\right)^2 = H_e\left(H_e + 4\pi M_S\right). \quad (12)$$

The discovery of natural ferromagnetic resonance (*NFMR*) in amorphous microwires [2] was preceded by their study using standard *FMR* methods [4-7]. Then, the shift in the resonant field, due to core deformation of the microwire associated with fusing the glass and core at the temperature of microwire formation, was observed. The *FMR* line width is also of interest because it characterizes, in particular, the structural parameters [2-18].

Since the skin penetration depth of a microwave field in a metallic wire is relatively small in comparison with its diameter the resonant frequency of *FMR* can be determined by means of Kittel formula (Eq. (12)). Taking into account the magnetoelastic stress field [2, 3], for a thin film magnetized parallel to the surface, we can obtain:

$$\left(\frac{\omega}{\gamma}\right)^2 = \left[H + (N'_Z - N'_X)M_S + 4\pi M_S\right]$$

$$\times \left(H + (N'_Z - N'_Y)M_S\right), \quad (13)$$

where *H* is the FMR field; N'_X, N'_Y, N'_Z are components of tensor of effective demagnetizing factors in case of magnetoelastic stress:

$$N'_i = \frac{3|\lambda|\sigma i}{2M_S^2}\left(\cos^2\theta i - \frac{1}{3}\right); \quad (14)$$

where:

$$\theta_1 = \theta_2 = 90^\circ, \theta_3 = 0.$$

Components σ_i (see Eqs. (1-6)) are residual stresses (see in Ref. [2, 3]). Then,

$$N'_X = N'_Y = -\frac{|\lambda| P}{2M_S^2};$$

$$N'_Z = \frac{|\lambda| P}{2M_S^2}\frac{(k+1)x+2}{kx+1}; \qquad (15)$$

Substituting the σ_i values obtained in our previous works [3-8] (see Eqs. (1-6)), and taking into account Eqs. (13), (14), we can calculate conditions for *FMR*:

$$\left(\frac{\omega}{\gamma}\right)^2 = \left[H + \frac{(3|\lambda|P)}{2M_S}\frac{x\left(k+\frac{2}{3}\right)+\frac{5}{3}}{kx+1} + 4\pi M_S\right]$$

$$\times\left[H + \frac{(3|\lambda|P)}{2M_S}\frac{x\left(k+\frac{2}{3}\right)+\frac{5}{3}}{kx+1}\right]. \qquad (16)$$

If the glass is removed, the stress is completely removed. Then the *FMR* resonant field of wire without glass casing, H_0, is determined from:

$$\left(\frac{\omega}{\gamma}\right)^2 = H_0\left(H_0 + 4\pi M_S\right). \qquad (17)$$

We have shown (see Ref. [1-4, 6-8]) that these relations quantitatively explain all of the basic features of *NFMR* and *FMR*. Note that the value of M_S required for the calculations was determined both by standard methods on a vibration magnetometer and with the use of interpolation formulas given here. The error relative to tabular values for the given alloys is not greater than 5%.

For the frequency of *NFMR* under the simple approximation taking $\varepsilon E_1 \sim$ 2GPaand $k\sim$ 0.4 this formula can be written as:

$$\omega(GHz) \approx \omega o\left(\frac{0.4x}{0.4x+1}\right)^{1/2},$$

$$\omega o(GHz) \approx 1{,}5\left(10^6\lambda\right)^{1/2}. \qquad (18)$$

As you can see, dependence for the frequency of NFMR (Eqs. (18), and Figs 2, 3, 4) is determined from two typical values, x, λ.

The basic contribution to the NFMR frequency and NFMR line width is due to the effective magnetostriction and parameter x (Eqs. (18), and Figs. 2-4).The residual stress in the microwire plays the dominant role in the formation of the absorption line width, as it will be shown below(Figs 5, 6).

The NFMR frequency in the distant diagnostics of dangerous deformations of critical infrastructure objects can be written as

$$\omega(GHz) \approx \omega o\left(\frac{0.4x}{0.4x+1} + \frac{\sigma_{ez}}{\sigma_o}\right)^{1/2}, \qquad (18a)$$

where

$$\omega_o(GHz) \approx 1{,}5(10^6\lambda)^{1/2}$$

These formulas allow you to determine stress in the distant diagnostics of dangerous deformations of critical infrastructure objects such as bridges, dams, wind turbine towers, skyscrapers, stack-furnaces, embankments, etc. To this end, fragments of magnetic microwires will be embedded in the bulk of concrete structures or on their surface during construction or after it by means of coating with a special concrete-adhesive plaster.

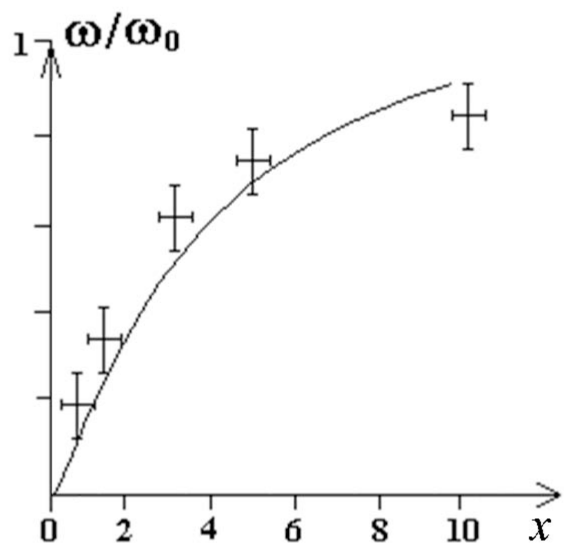


Fig. 2: Theoretical curve of *NFMR* frequency as a function of x (according to Egs. (16) – (18)) and experimental data (see [2])

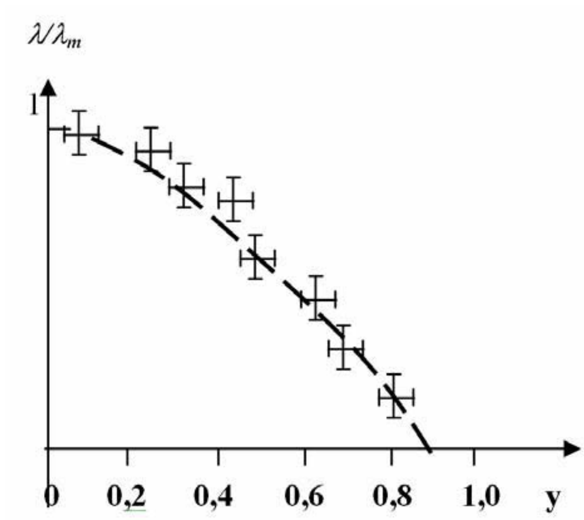


Fig. 3a: Dependence of relative magnetostriction λ/λ_m for alloy composition $(Co_y\ Fe_{1-y})_{75}(BSiC)_{25}$ series cast glass-coated amorphous magnetic microwires according to Eqs. (16) – (18), where y = Co/(Co+Fe) (see [2])

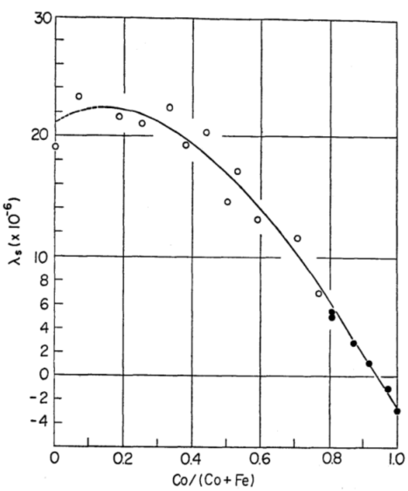


Fig. 3b: Typical saturation magnetostriction λ_s in the amorphous Co –Fe alloys: ○$(Fe_{1-y}Co_y)_{80}(PC)_{20}$;

●$(Fe_{1-y}Co_y)_{75}(SiB)_{25}$(according to Ref. [2]).

When the penetration depth of the microwave field in the metallic wire is small relative to the wire diameter (on account of the skin effect), the resonant frequency in *FMR* and *NFMR* may be determined by means of Eqs. (1-6).(The general theory of residual tension is presented in Refs. [2, 4], but here it is enough to use the simple theory from [3])

Substituting typical values of λand xin Eq. (18) we reach numerical values of *NFMR* in a range from 1 to 12 GHz. A systematic study on the *NFMR* frequency for the alloy series $(Co_yFe_{100-y})_{75}(BSiC)_{25}$ has been performed as a function of the Co content (Fig.3). The magnetostriction has then been evaluated using Eq. (18). The result is plotted in Fig. 3a which shows good agreement with the magnetostriction values as determined through conventional techniques (Fig3b.). Thus, the final theory quantitatively explains all the basic features of *NFMR* and *FMR*. However the area of experimental research in the case of small radius of a metallic nucleus radius of a microwire remains vacant.

III. Radio-Absorption Shielding

The designs of composites from GCAMNWs have following configurations.

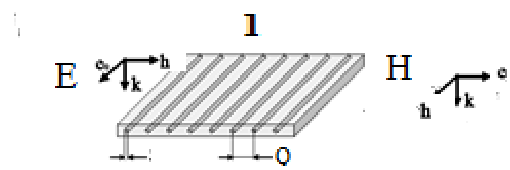


Fig. 4a: Composite shielding for radio absorption protection with GCAMNWs where were maked in grating form. We can consider two types of orientation of a magnetic field: E and H

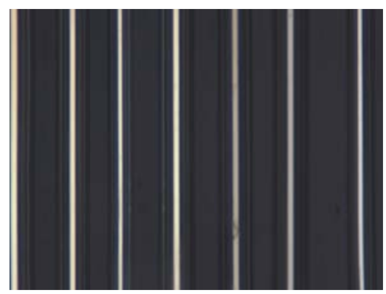


Fig. 4b: Diffraction grating with GCAMNWs

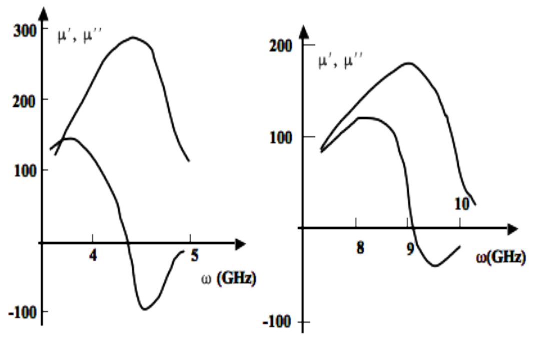
Fig. 4b: Composite shielding for radio absorption protection with GCAMNWs where were made ina stochastic mixture of microwires in the polymeric matrix

Natural ferromagnetic resonance (*NFMR*)occurs when the sample is submitted to a microwave field without application of any biasing field other than the intrinsic anisotropy field of the microwire.

Near the natural ferromagnetic resonance frequency, Ω, the dispersion of permeability μ. given by

$$\mu(\omega) = \mu'(\omega) + i\mu''(\omega), \qquad (19)$$

exhibits a peak in μ" and a zero crossing of μ' .



a b

Fig. 5: Real and imaginary relative permeability components around NFMR for $Co_{59}Fe_{15}B_{16}Si_{10}$ (a) and $Fe_{69}C_5B_{16}Si_{10}$ (b) microwires (r~5μm, x>8 (see [2, 5])

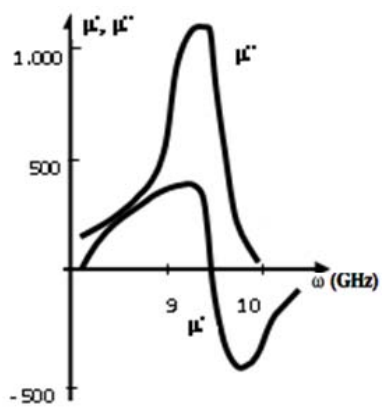


Fig. 5c: Frequency dispersion of the real and imaginary parts of the relative permeability around the NFMR frequency for the $Fe_{68}C_4B_{16}Si_{10}Mn_2$ microwire (R_m ~ 5 μm, x ~ 5)(see [19]).

Figures 5a, 5b and 5c show resonance frequencies of 4.4; 9.0 and 9.5 GHz, and resonance widths of 1,5; 1 and 0.5 GHz. Near resonance, μ" is expected to be described by

$$\mu''/\mu_{dc} \sim \Gamma\Omega / [(\Omega - \omega)^2 + \Gamma^2], \qquad (20)$$

where μ_{dc} is static magnetic permeability and Γ is the width of the resonant curve. Very near resonance,, when $\Gamma > (\Omega - \omega)$, Eq. (20) reduces to

$$\mu''/\mu_{dc} \sim \Omega / \Gamma \sim (10 \div 10^2).$$

Note that in Fig. 5a the imaginary component is rather symmetrically distributed around the resonance frequency. This is due to the symmetric distribution of the circular permeability in the near surface layer, within the penetration depth. In contrast, in Figs. 5b, 5c the imaginary component shows a non-symmetric feature around the resonance frequency. This can be attributed to the inhomogeneous character of the permeability in the region close to the surface of the microwire where metastable phases form, as demonstrated by X-ray studies.

Monitoring the geometry of the microwire (i.e., its diameter) and the magnetostriction through its composition enables one the fabrication of microwires with tailor able permeability dispersion and for creating Radioabsorption materials:

1. Determining the resonant frequency in a range from 1 up to 12 GHz;
2. Controlling the maximum of the imaginary part of magnetic permeability.

High-frequency properties, pieces of microwires have been embedded in planar polymeric matrices to form composite shielding for radio absorption protection. Experiments have been performed employing commercial polymeric rubber around (2 ÷ 3) mm thick. Microwires are spatially randomly distributed within the matrix before its solidification. Concentration is kept below (8 ÷ 10) g of microwire dipoles (1 ÷ 3)mm long) per 100g rubber [2, 6, 7]. A typical result obtained in an anechoic chamber is shown in Fig. 6a for a screen with embedded $Fe_{69}C_5B_{16}Si_{10}$microwires.

As observed, an absorption level of at least 10 dB is obtained in the frequency range from 8 to 12 GHz with a maximum attenuation pick of 30 dB at around 10 GHz. In general, optimal absorption is obtained with microwires with metallic nuclei of diameter $2r$ = (1÷3) μm (2R ~ 20μm (x>10)) and length L = (1÷3) mm. Such pieces of microwires can be treated as dipoles whose length, L, is comparable to the half value of the effective wavelengths, $\Lambda_{eff}/2$, of the absorbed field in the composite material (i.e., in connection to a geometric resonance).

Fig. 6a also shows how the frequency absorption spectrum of shielding with $Fe_{69}C_5B_{16}Si_{10}$ microwires changes when it is rotated (90° each spectrum). We attribute the changing attenuation to the lack of perfect angular distribution of microwires which length not always fit within the shielding thickness.

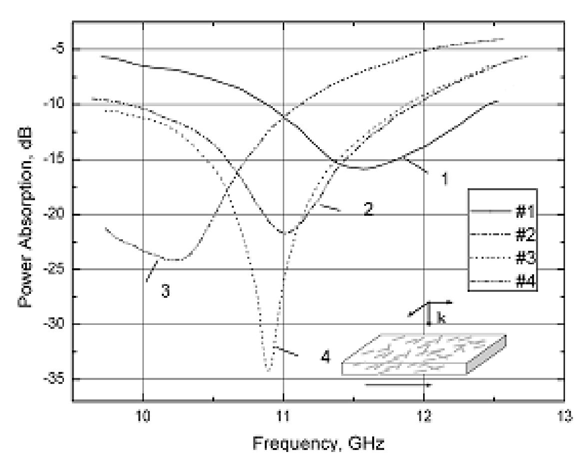


Fig. 6a: Absorption characteristics of shielding by a microwire composite with *NFMR* in the *HF* - field in the range of frequencies 10-12 GHz. Curve 1 represents an initial situation of the screen; then 2, 3, 4: the screen is turned by 90° about a perpendicular axis each time

The effect doesn't even have mirror symmetry.

(The measurement error was less than 10% for the frequency, and while the spread of the attenuation factor was 5 dB).

Small fluctuations in concentration of dipoles at a concentration of dipoles near the percolation threshold can lead to fluctuation of the absorption curve. Similar results were presented in [2].

As observed, both frequency dependences (Fig. 5b and 6) are similar except for the half-width value of the permeability.

Although the design of absorption shielding can be based on disposing the dipolar pieces in a stochastic way, we consider, for simplicity, a theoretical analysis for a diffraction grating (see Fig 6b) with spacing between wires $Q < \Lambda$ (Λ is wavelength of absorbed field).(Another simple example is in Appendix A).

The propagation of an electromagnetic wave through absorption shielding with microwire-based elements is characterized by transmittance, $|T|$, and reflectance, $|R_r|$, coefficients given by:

$$|T| = (\alpha^2 + \beta^2)/[(1+\alpha)^2 + \beta^2];$$

$$|R_r| = 1/[(1+\alpha)^2 + \beta^2], \qquad (21)$$

where $\alpha = 2X_r/Z_0$, and $\beta = 2\,Y/Z_0$, with $Z_0 = 120\pi/Q$, and the complex impedance $Z = X_r + iY$.

The absorption function, G, is correlated with the generalized high-frequency complex conductivity Σ (or high-frequency impedance Z).

Here, we use the analogy between the case of a conductor in a waveguide and that of a diffraction grating. The absorption function, given by:

$$|G| = 1 - |T|^2 - |R_r|^2 = 2\alpha / [(1+\alpha)^2 + \beta^2], \qquad (22)$$

Has a maximum,

$$|G_m| = 0{,}5 \geq |G|,$$

for simultaneous $\alpha = 1$, and $\beta = 0$, for which

$$|T|^2 = |R_r|^2 = 0{,}25.$$

The minimum, $|G| = 0$, occurs at $\alpha = 0$, β any positive number).

Theoretical estimations taking into account only the active résistance of microwires result in attenuation within the range (5 ÷ 15) dB being much lower than experimental results, which for spacing of microwires $Q = 10^{-2}$ m ranges between 18 and 15 dB, while for a spacing $Q = 10^{-3}$ m it increases up to 20 to 40 dB. Thus, it becomes clear that shielding exhibit anomalously high absorption factors, which cannot be explained solely by the resistive properties of microwires.

The high-frequency conductivity, Σ_m, of a stochastic mixture of microwires in the polymeric matrix can be expressed as a function of the conductivities, Σ_i, of non-conducting (polymeric matrix) and conducting (microwire) elements, denoted by sub-indices 1 and 2, respectively, as [2]:

$$\Sigma_m = B + (B^2 + A\Sigma_1\Sigma_2)^{1/2}, \qquad (23)$$

where

$$B = 1/2\{[\Sigma_1(X_1 - AX_2) + \Sigma_2(X_2 - AX_1)];$$

X_i is fractional volume:

$$(X_1 + X_2) = 1;$$

$$A = 1/(J_1 - 1), \text{ with } \quad J_1 \sim (J + Y/X_r)$$

being the fractal dimension of the system (J is the geometrical dimension of the system) and

$$Y/X_r \sim (r/\delta)^2.$$

Fig. 7 shows that in case of a thick microwire ($r > \delta \approx 1\,\mu$m), the conductivity of the system becomes very large, even in case of small microwire concentration, indicating the case of an antenna resonance as reported in [2].

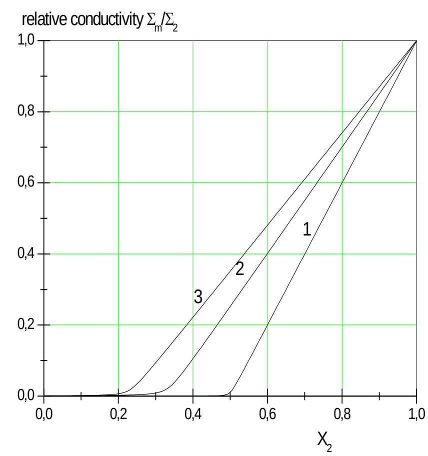


Fig. 7: Generalized conductivity calculated using formula (23) for $\Sigma_2 / \Sigma_1 \sim 10^4$

1 thin microwire ($r < \delta \sim 1\,\mu$m) J = 2, Y = 0
2 thin microwire ($r < \delta \sim 1\,\mu$m) J = 3, Y = 0
3 thick microwire ($r > \delta \sim 1\,\mu$m) $J_1 = 4$, $Y/X_r = 1$

Let us consider the effective absorption function, (as in Eq. (22)):

$$|G_{eff}| \sim \Gamma_{eff}\Omega_{eff} / [(\Omega_{eff} - \Omega)^2 + \Gamma_{eff}^2], \qquad (24)$$

where $\Gamma_{eff} \geq \Gamma$ (see Eq. (12)) and $\Omega \sim \Omega_{eff} = 2\pi c/\Lambda$.

A microwave antenna will resonate when its length, L, satisfies

$$L \sim \Lambda / 2(\mu_{eff})^{1/2}. \qquad (25)$$

The maximum absorption (see Fig 6) occurs for $\Omega_{eff} \sim 10$ GHz ($\Lambda \sim 3$ cm) and $\mu_{eff} \sim 10^2$, (according to Fig. 5). This corresponds to:

$$L \sim (1{,}5 \div 2) \text{ mm}, \qquad (26)$$

when the microwire concentration((see Fig.7) $X_2<0,2$) is much less than the percolation threshold.. A greater concentration of dipoles increases absorption, $|G_{eff}|$, but also increases reflectance, $|R_{r1}|$, which can be simply evaluated to be [2]:

$$|R_{r1}| \sim 1 - 2\sqrt{(\Omega/2\pi\Sigma_m)}, \quad (27)$$

where $\Omega/2\pi \sim 10^{10}$ Hz.

The formula is applicable, and calculation of small reflectance, $|R_{r1}|$ is possible, only if

$$\Sigma_m \sim 10^{11} \text{Hz},$$

for concentration below the percolation threshold (as $\Sigma_2 \sim 10^{15}$Hz).

Thus, for very thin microwires (i.e., thinner than 1 μm diameters) embedded in a composite matrix with concentration larger than the percolation level $X_2 \sim 0,2$ a noticeable absorption effect should be expected.

Atypical result obtained in an anechoic chamber is shown in Fig. 8 for radio-absorbing shielding with embedded $Fe_{68}C_4B_{16}Si_{10}Mn_2$ microwires. As observed, an absorption level of at least 10 dB is obtained in a frequency range of (8…12) GHz with a maximum attenuation peak of 30 dB at about 10 GHz. In general, optimal absorption is obtained for microwires with metallic kernels of diameter $2R_m \sim 10$ μm ($x \sim 5$) and length L = (1…2) mm. These micro wire pieces can be treated as dipoles whose length L is comparable to the half value of the effective wave lengths $\Lambda_{eff}/2$ of the absorbed field in the composite material (i.e., in connection to a geometric resonance). A similar result has been received for radio-absorbing shielding with a different GCAMNW (see Refs. [2-8]).

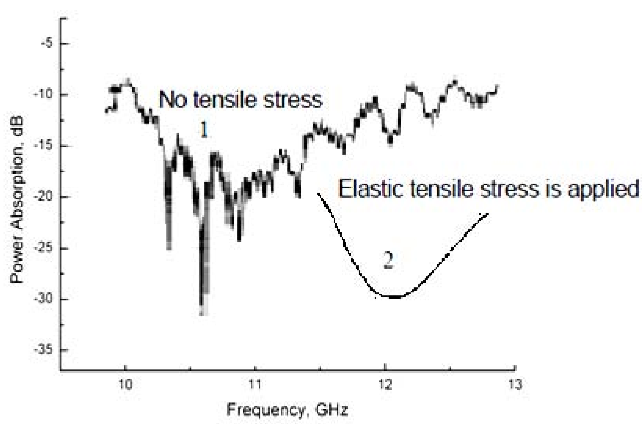


Fig. 8a: 1–Average absorption characteristics of a shielding containing a microwire composite exhibiting NFMR at microwave frequencies ranging from (10…12) GHz for $Fe_{68}C_4B_{16}Si_{10}Mn_2$ microwires ($R_m \sim 5$ μm, $x \sim 5$).

2 – Absorption curve in case of anexternal pressure. (see Ref. [19])

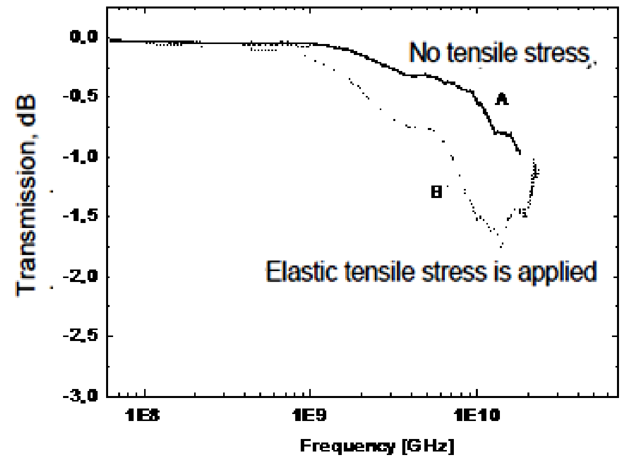


Fig. 8b: A – Average absorption characteristics of diffraction grating with GCAMNWs exhibiting NFMR at microwave frequencies ranging from (1…10) GHz for $Co_{65}Fe_{10}B_{15}Si_{10}$ microwires ($R_m \sim 5$ μm, $x \sim 5$)

B – Absorption curve in case of anexternal pressure.

IV. Final Remarks

One of the important technological features of the GCAMNWsis the high rate of cooling and solidification of the composite fibers drawn from the molten alloy and consisting of a ferromagnetic metal core and a glass coating. Significant differences between the thermal expansion coefficients of the glass and metal alloy lead to the appearance of large residual stresses.

Particular attention has been paid to the parameters determining the anisotropy of cast GCAMNWs. The continuous casting of GCAMNWs (Taylor-Ulitovsky method) has some limitations, determined by the physical properties of metal and glass (see Ref. [2]).

We have presented simple analytical expressions for the residual stresses in the metallic kernel of the microwire, which clearly show their dependence on the ratio of the external radius of the microwire to the radius of the metal kernel and on the ratio of Young's modules of glass and metal. Our modeling based on the theory of thermoelastic relaxation, shows that the residual stresses increase from the axis of the microwire to the surface of its metallic kernel, which is in accordance with the previously obtained experimental data (see Ref. [2]). Thus, in the manufacture of cast microwires with a glass coating by the Taylor-Ulitovsky method, the residual stresses reach maximum values on the surface of the metal core (see Refs. [2-8]).

The cast GCAMNWs exhibit natural ferromagnetic resonance (NFMR), whose frequency depends on the composition, geometrical parametersand deformationof the microwire. The NFMR phenomenon observed in glass-coated magnetic microwires opens up the possibility of developing new radio-absorbing materials with a wide range of properties. An important feature of cast microwires with an amorphous magnetic core is the dependence of the NFMR frequency on the deformation (stress effect). In this regard, the microwave response of a composite consisting of segments of amorphous magnetic microwires with a glass coating in a flat dielectric plate is investigated when the plate is deformed in a microwave field with a frequency in the range from (1...10) GHz. As shown by calculations (see Equations (18), (18a)), the shift of the NFMR frequency as a result of the stress effect can reach 20% before the destruction of the composite. Therefore, this effect can be used for contactless diagnostics of deformations in distant objects (including critical infrastructures) reinforced by cast magnetic microwires with the stress effect of NFMR. To this end, these objects are periodically scanned with a floating-frequency radar to determine the deviation of the initial NFMR frequency due to potentially dangerous deformations of the monitored object.

It is worth mentioning also another principle of detecting mechanical strain which is examined in Ref. [20]. This principle is based on the giant magnetoimpedance (GMI) effect (see Figure 9),so that it is different from that presented in Figure 8a.The GMI effect demands external magnetization of the sample which is not required in the NFMR method (see Refs. [2-8]).

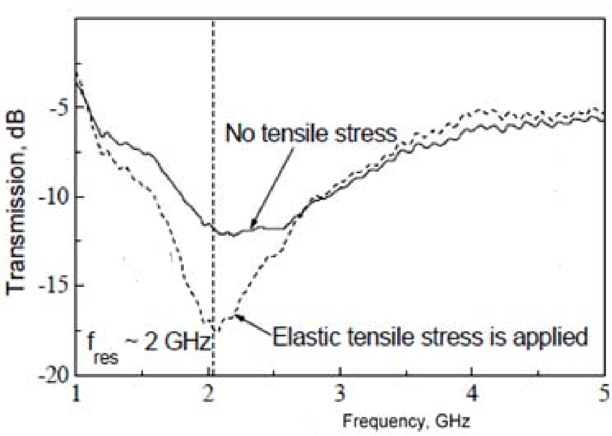


Figure 9: Stress dependence of the transmission coefficient for a single-layer composite sample measured in the free-space near field (see Ref. [20])

According to the above, the proposed method (based on the NFMReffect) is more technologically advanced than the method based on the GMI effect.

V. Conclusion

The microwave electromagnetic response has been analyzed for a composites consisting of dipoles and diffraction grating of amorphous magnetic glass-coated microwires in a dielectric. Thys materials can be employed for radio absorbing screening. The spontaneous NFMR phenomena observed in glass-coated microwires has opened the possibility of developing novel materials with broad-band of radio absorbing materials.

The described studies provide the following basic conclusions.

(A) We have derived simple analytical expressions for residual and mechanical stresses in the metallic core of the microwire, which clearly show their dependence on the ratio of the external radius of the microwire to the radius of the metal core and on the ratio of Young's moduli of the glass and the metal. Our modeling based on the theory of thermoelastic relaxation shows that the residual stresses increase from the axis of the microwire to the surface of the microwire metallic core, which is in accordance with the previously obtained experimental data (see [2]). Thus, in the case of glass-coated cast microwires prepared by the Taylor–Ulitovsky method, the residual stresses achieve maximum values on the surface of the metal core (see [2–8]).

(B) Cast GCAMNWs exhibit NFMR, whose frequency depends on the composition, geometrical parameters, and deformation of the microwire. The NFMR phenomenon observed in glass-coated magnetic microwires opens up the possibility of developing new radio-absorbing materials with a wide range of properties. An important feature of cast microwires with an amorphous magnetic core is the dependence of the NFMR frequency on the deformation (stress effect). The calculations have shown (see (18), (18a)) that the shift of the NFMR frequency caused by the stress effect bc an achieve 20% before the degradation of the composite.

(C) This effect can be used for contactless diagnostics of deformations in distant objects (including critical infrastructures) reinforced by cast magnetic microwires with the stress effect of NFMR. To this end, these objects are periodically scanned with a floating-frequency radar to determine the deviation of the initial NFMR frequency due to potentially dangerous deformations of the monitored object.

(D) The overall technology of magnetic wire composites is cost-effective and is suitable for large-scale applications.

Here we have discussed the electromagnetic properties of composites with magnetic wires showing NFMR phenomena. A striking property of these materials is that the spectra of the effective electromagnetic parameters (permittivity and permeability) can be actively tuned.

Technology of glass coated amorphous and nanocrystalline microwires allows the fabrication of continuous wires

Appendix A.Scheme for measuring the radio-absorbing properties

The material parameters in microwaves frequencies usually are found from the measurement of the reflection and/or transmission coefficients from which the complex permittivity (permeability) are calculated.

The measurement methods can be divided in two categories:

1) Transmission line methods (coaxial lines probes, rectangular waveguides, cavity resonators ((see Ref. [15]). Thus methods (in the first category) require cutting a piece of a sample to be placed inside the transmission line or cavity making a close contact with the probe. The transmission line methods work best for homogeneous materials that can be precisely machined to fit inside the sample holder.
2) Antenna techniques in free space (see below).

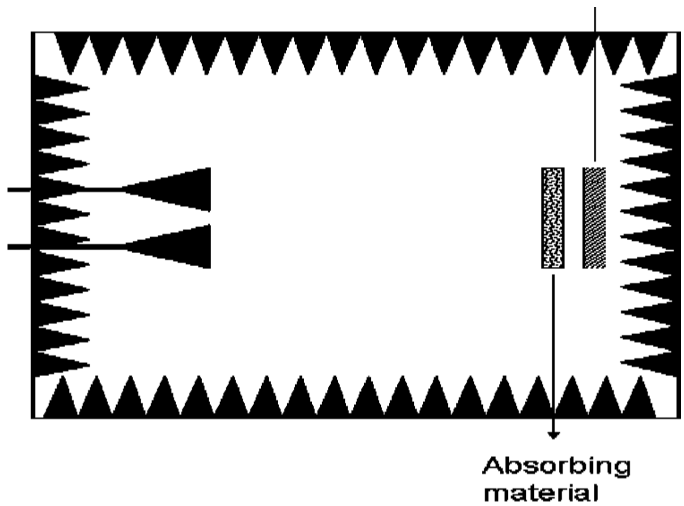


Figure A: A variant of the experimental scheme for microwave measurements

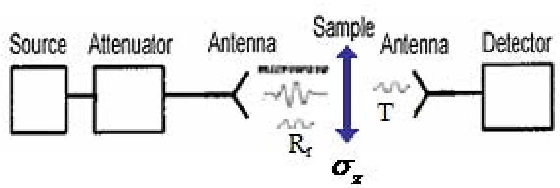


Figure A1: Scheme for measuring the radio-absorbing properties of samples with EFMR (according [2, 6, 19]) under the influence of external mechanical stresses

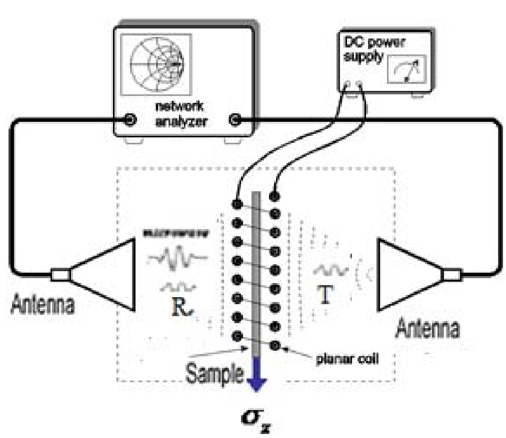


Figure A2: Scheme for measuring the radio-absorbing properties of sampleswith GMI (according [20]) under the influence of external mechanical stresses

Let's examine the simple theory of measurement of radio absorbing composition materials

It is well known, that the simple model of contact of vacuum with the absorbing material gives the following equations [2, 5, 19]

$$1 + R_r = T, \quad \text{(A. 1)}$$

$$(\alpha + i\beta)(1 - R_r) = T;$$

that gives

$$R_r = \frac{1 - \alpha - i\beta}{1 + \alpha + i\beta}, \quad \text{(A. 2)}$$

and at $\beta = 0$, $\alpha = 1$, we find

$$R_r = 0. \quad \text{(A. 3)}$$

From these it is possible to obtain a simple criterion for matching of vacuum with a radio absorbing material:

$$\mu_m \sim \Sigma_m / \Omega, \quad \text{(A. 4)}$$

(where μ_m is effective magnetic permeability of composite). This condition cannot be satisfied for composites containing amorphous magnetic wires. This forces us to use other physical principles for creation of radio absorbing materials presented above).

We note that similar results were obtained in Refs. [2, 7, 8, 19].

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The Preliminary Results of the DC-Toroidal Discharge Plasma with Axial Electric Field in the Batorm

By A. A. Talab & M. M. Masoud
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Abstract- A toroidal plasma system (Batorm) were designed, constructed and developed in Egyptian Atomic Energy Authority (EAEA) since 1998. The plasma parameters of this device have been studied and obtained for the first dimensions during PhD thesis 2003. The word "BATORM" is an abbreviation to "Baby Toroidal of Masoud". The main feature of the system is a low-cost machine which can be operated as a small toroidal plasma device. In this system, the plasma is initiated by linear axial discharge between two plates which create an applied axial field all over the discharge device. This system could be provided rich information on toroidal discharge physics which includes small impurities. The design of the Batorm is upgraded by change its dimensions, according that, the turns number of toroidal, ohmic and vertical coils are increased. The discharge current I_{dis} and total inductance of these three coils were (2 A) and (934.4 μH) respectively. The experimental results have been preliminary taken depending on measurment of electron temperature (KT_e) and ion density (n_i) at each one cm from the outer wall to the inner wall of the chamber. From these results, it is found that the highest values of (KT_e) and (n_i) arrived to 10 eV and 1.52 x10^{-9}cm^{-3} (respectivelely) at distance 7 cm. Besides, it has been seen that there are no plasma arrived to the inner wall according to the toroidal confienment.

Keywords: toroidal plasma/aspect ratio/coils inductance/ electric circuit/ electron temperature and ion density.

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Strictly as per the compliance and regulations of:

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A. A. Talab [α] & M. M. Masoud [σ]

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Keywords: toroidal plasma/aspect ratio/coils inductance/ electric circuit/ electron temperature and ion density.

I. Introduction

BATORM is the first toroidal plasma machine designed and operated in our lab at EAEA [1]. It was designed and built in 1998 depending on an idea to produced plasma in two ranges. In beginning, a low pressure DC glow plasma discharge is produced. After that, the excited magnetic trap in toroidal chamber are used to confine the energy and charged particles of the plasma to produce toroidal plasma confinement (as in the to kamaks devices [2]).

In this work, we are going to study and investigate experimentally the different parameters of the plasma of low pressure toroidal glow discharge in radial magnetic field [3]. The magnetic field has two orthogonal components: one that is created by a system of current-carrying coils around the plasma (toroidal magnetic field (TF)) and the other is created by a current that is induced in the plasma (poloidal magnetic field (PF)). The resulting magnetic field lines spiral around a set of nested toroidal flux surfaces, providing an effective plasma-confinement system, which can heat the plasma [4]. There is a weak vertical field is added by two separate set of external coils parallel to the Ohmic coil [5-7] provides additional stabilizing forces which required to prevent the radial expansion of the plasma column.

The Batorm is considered as one of the small and low cost devices. The first one (from 1998 to 2003) had aspect ratio (R/a) 1.68 with R = 6.125 cm and a = 3.625 cm. The toroidal coil consisted of 48 turns while the Ohmic coil had 22 turns and the vertical coil had 5 turns. In that work, there are four electrical circuits was designed and demonstrated to produce toroidal plasma inside the vacuum and magnetic vessel. The discharge energy was in the range between 45 J to around 384 J. It was during discharge time around 0.68 msec to 1.6 msec. The plasma current was between 3.4 KA to 12.75 KA with electron temperature from 2.25 eV to 9 eV by double electric probe [1].

Recently, updating Batorm configuration is beginning to increase the plasma properties (current, temperature and magnetic field). The new updating one will be explained in details in next section. Our study gives some preliminary results and makes complete survey to the plasma properties from this system. Also, it is considered as the first step to make use this plasma in different industry applications in future plan.

II. Design and Operation of the Batorm

The BATORM consists mainly of vacuum chamber, vacuum system, magnetic vessel and electric circuit [1]. The photographic view of the new Batorm is illustrated in figure (1). The vacuum chamber consists of two parallel aluminum plate electrodes, which is fixed at Pyrex glass discharge chamber of 15 cm length, and 32 cm inner diameter. At the center of it there is another glass tube of 15 cm length, and 10 cm outer diameter. So, the aspect ratio of this device R/a = 2. The working gas pressure of helium is 9 x 10^{-2}Torr.

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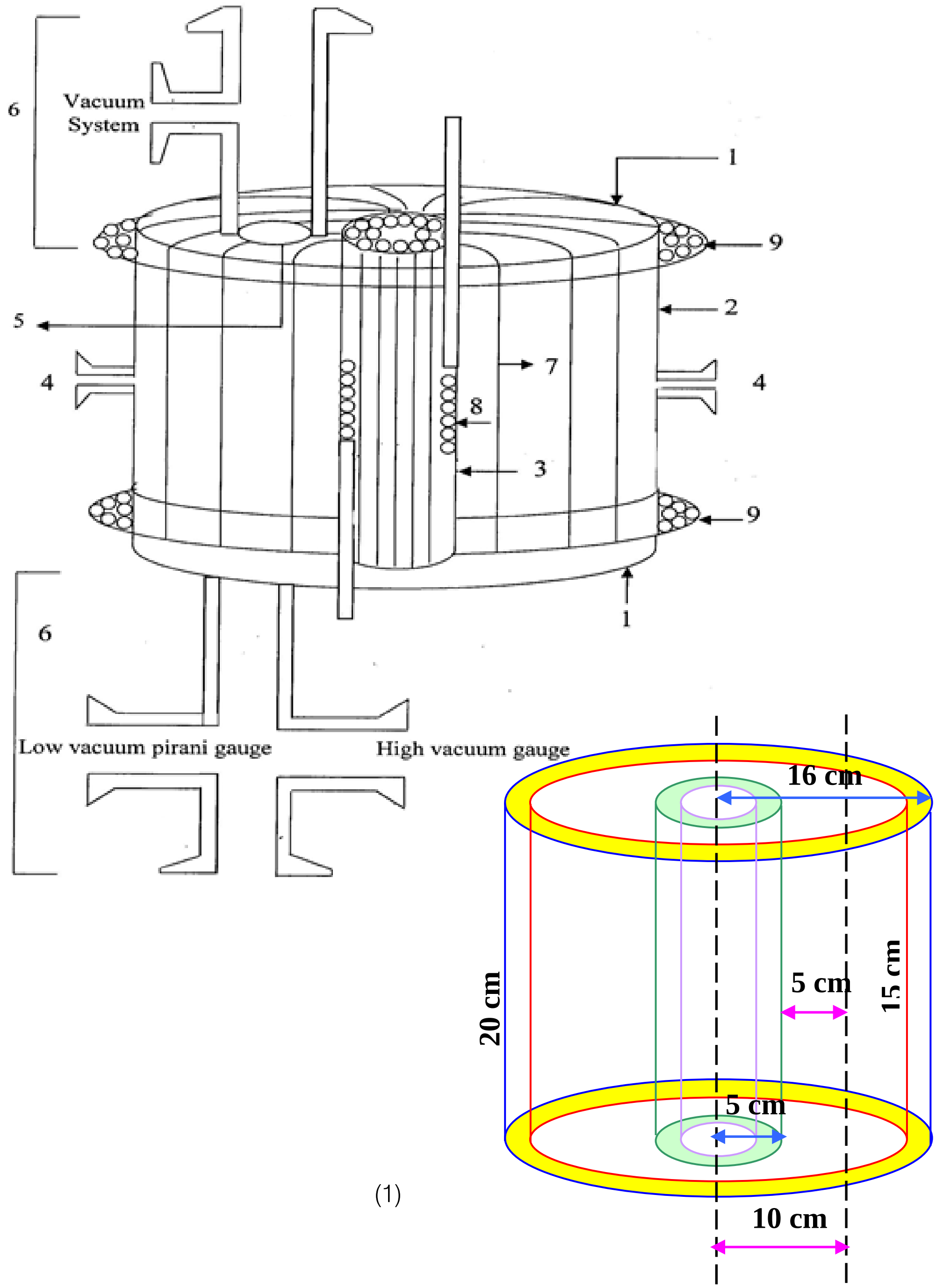
A magnetic vessel is used to contain and stabilize the plasma by Toroidal, Ohmic and Vertical coils. The toroidal magnetic field (TF) is produced by an external coil consisting of 75 turns wound directly on the discharge chamber after insulating the two electrodes. And the poloidal magnetic field is produced by 30 turns which generates large plasma current. These turns form the cylindrical air solenoid for the OH transformer. The weak vertical field is generated by two coils parallel to the OH-coil, each coil has 6 turns. The schematic diagram of the vacuum chamber and magnetic vessel is shown in figure (2).

Fig. 1: Photographic view of the BATORM device

The inductances of the toroidal, ohmic and vertical coils are calculated from the following equations (1) and (2) as [8]:

$$L_{toroidal} = \frac{\mu_o N^2 h}{2\pi} \ln\left(\frac{b}{a}\right) \quad (1)$$

$$L_{Solinoid} = \mu_o n^2 hA \quad (2)$$



(1)

Fig. 2: Schematic diagram of the plasma discharge chamber and magnetic vessel

1- Two parallel aluminum plate electrodes	6- Two (T) connections
2- Pyrex glass discharge chamber	7- Toroidal Coil
3- Glass tube	8- Ohmic coil
4- Three small ports	9- Vertical coil
5- Two large ports	

Where; μ_o is the permeability constant, n is the number of turns per unit length and A is the area of one turn. Therefore,

$$L_{toroidal} = 785\ \mu H,\ L_{Ohmic} = 28.4\ \mu H \text{ and } L_{vertical} = 121\ \mu H$$

So, the total inductance of the all coils: 934.4 μH.

III. Evaluation of Plasma Parameters

a) The Electrical Circuit and Operation Conditions

In this section, the electrical Circuit in our system has been investigated in figure (3). The plasma is initiated by linear axial discharge between the two plates which will apply an axial electric field all over the discharge. So, first make pre ionization by connected the two electrodes in series with DC power supply has 7 kV and 125 mA to get glow discharge plasma at 300 volts. Then to confine the plasma connected the Torodia, Ohmic and Vertical coils with the electric circuit which is illustrated in figure (3). It consists of three phase electric tap has a neutral point, three diodes as rectifier, three heater wire with different power values connected in series with the tree coils To, Oh, V during an overload 150 A.

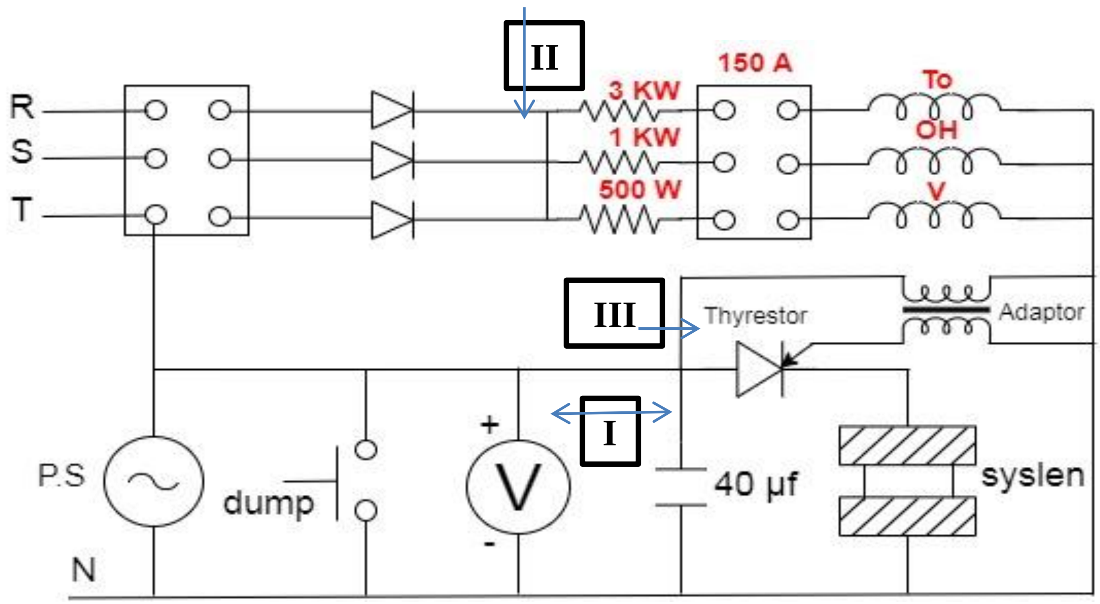


Fig. 3: Schematic diagram of the plasma discharge circuit. It has three parts (I) to produce plasma, (II) to prepare the coils to confinement the plasma and (III) as a trigger circuit to confinement the plasma.

The discharge period τ and the discharge current I_{dis} can be calculated from equations (3) and (4) as follows [1]:

$$\tau = 2\pi\sqrt{L_{total}C} \quad (3)$$

$$I_{dis} = \frac{2\pi CV}{\tau} \quad (4)$$

Where, *C* is the capacity of the capacitor bank.
Therefore, I_{dis} = 2 A.

b) Double Probe Method

The double electric probe is a diagnostic tool to measure the electron plasma temperature and ion density. It consists of two identical probes, normally cylindrical configuration, biased with respect to each other by an external source voltage (V_{probe}) with an associated current (i_d), but the entire system floats with the plasma potential. Its technical data is illustrated in table (1). The double probe has an advantage that it can be used in plasma, with high space potential and it could disturb the plasma only at its location [9, 10].

If the potential difference between the probes is increased, then the relatively positive probe collects more electrons, until at a certain potential it reaches saturation.

Table 1: The technical Data for Probe Designing

Probe material	Tungsten
Tip length	4mm
Tip diameter	1.5 mm
Tip area	9.87 mm^2
Insulating material	Glass
DC power supply	From 0 V to 27 V

IV. The Experimental Results and Discusion

The (I-V) characteristics of double Langmuir probe at each cm of the length of the plasma inside the BATORM are shown in figure (4). In this figure the (I-V) characteristics for glow discharge drown by black points, while for toroidal confinement by red points. This figure illustrated that the confinement success to move the plasma far from the inner wall of the BAtorm (r = 11 cm) more than the outer wall (r = 0 cm).

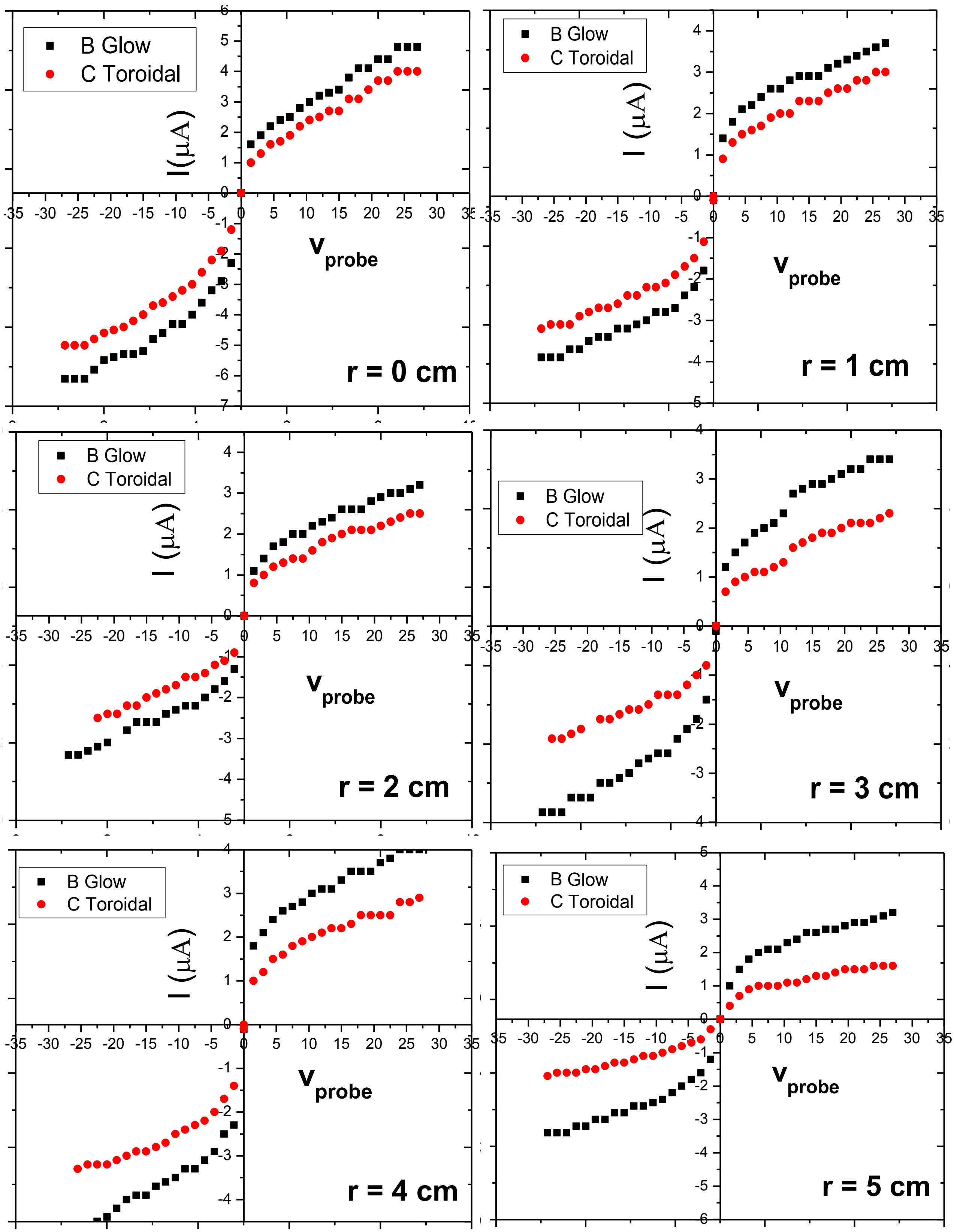
Table 2: The different values of electron temperature (KT_e), ion density (n_i) and energy density at different distances

r (cm)	KT_e (ev)	n_i (cm^{-3}) x 10^9	(KT_e x n_i) x 10^9
0	2.5	1.6	3.99
1	2.5	1.89	4.7
2	3.5	1.28	4.48
3	5.5	1.02	5.61
4	6.25	1.44	8.98
5	6.25	0.72	4.5
6	8.75	0.81	7.1
7	10	1.52	15.2
8	7.5	1.75	13.1
9	2.5	1.06	2.7
10	2.5	0.38	0.947
11	0	0	0

Figure (5) shows the variation of plasma temperature, plasma density and plasma energy density at deferent distances from 0 to 11 cm. From the results, it is clear that, the plasma has maximum temperature and density at distance 7 cm from the outer wall. Plasma is found around the minor axis but there are run away electrons which escape in direction of the outer wall. From knowing the energy density, the biggest value of plasma kinetic pressure is 15.2 x 10^9eV/cm^3; at 7 cm from the outer wall.

V. Conclusion

BATORM is the first toroidal plasma machine designed and operated in our lab at EAEA. This device is the first step to build small and not expensive plasma toroidal devices. The aim from them is to produce plasma in different properties by using available equipment and facilities. On condition, this plasma is in high accuracy to useful in different applications. By this way our lab will open contacts with the universities to explain experimentally and simply the plasma technology for students. Adding to contact with any destination need to use these sources of plasma in different applications such as industry, archeology, petroleum, medicine, agriculture, environment and etc. according to plan will stomach for each application.


B Glow
C Toroidal
I(μA)
V probe
r = 0 cm
B Glow
C Toroidal
I (μA)
V probe
r = 1 cm
B Glow
C Toroidal
I (μA)
V probe
r = 2 cm
B Glow
C Toroidal
I (μA)
V probe
r = 3 cm
B Glow
C Toroidal
I (μA)
V probe
r = 4 cm
B Glow
C Toroidal
I (μA)
V probe
r = 5 cm

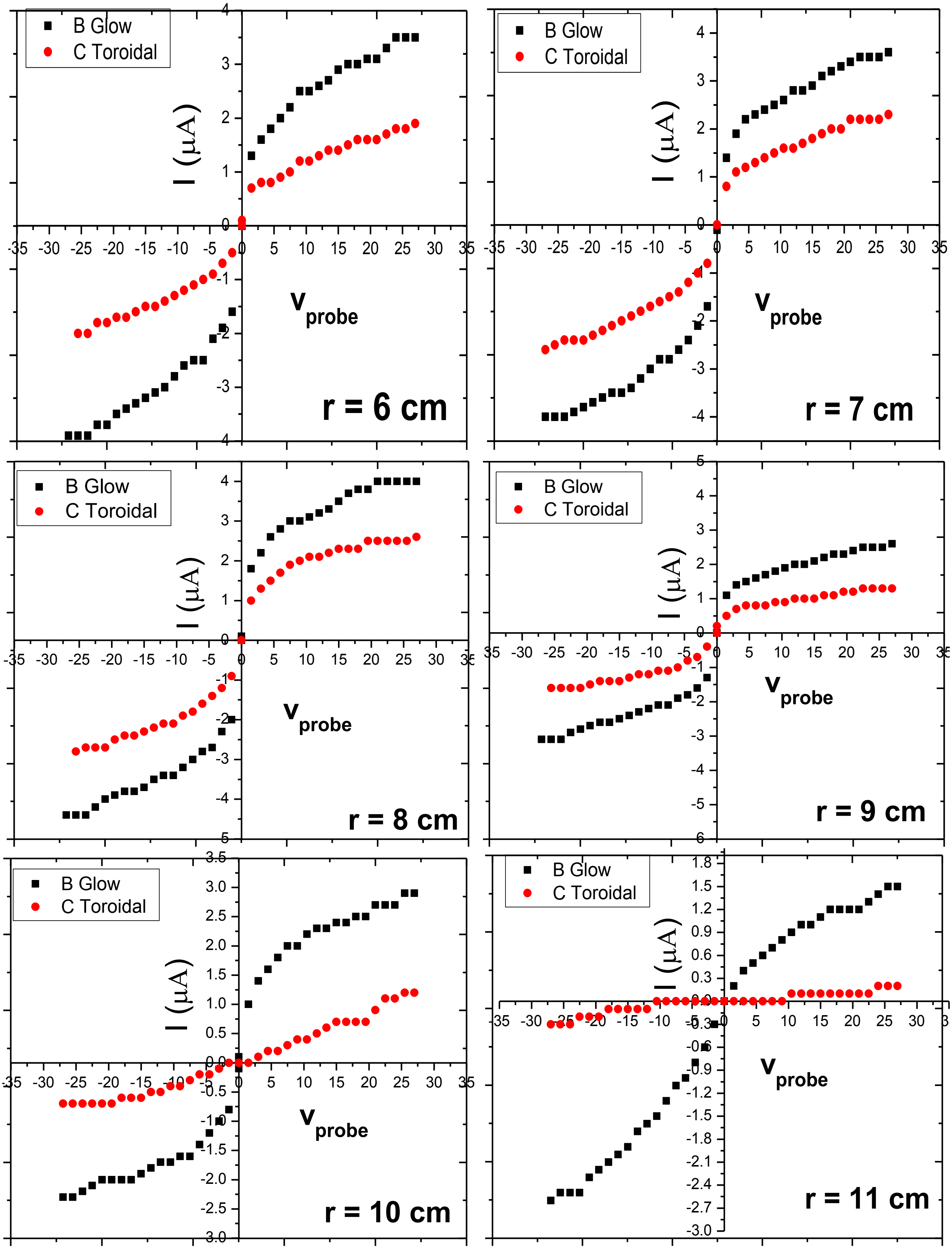


Fig. 4: The variation of plasma temperature, plasma density and plasma energy density at deferent distances from 0 to 11 cm

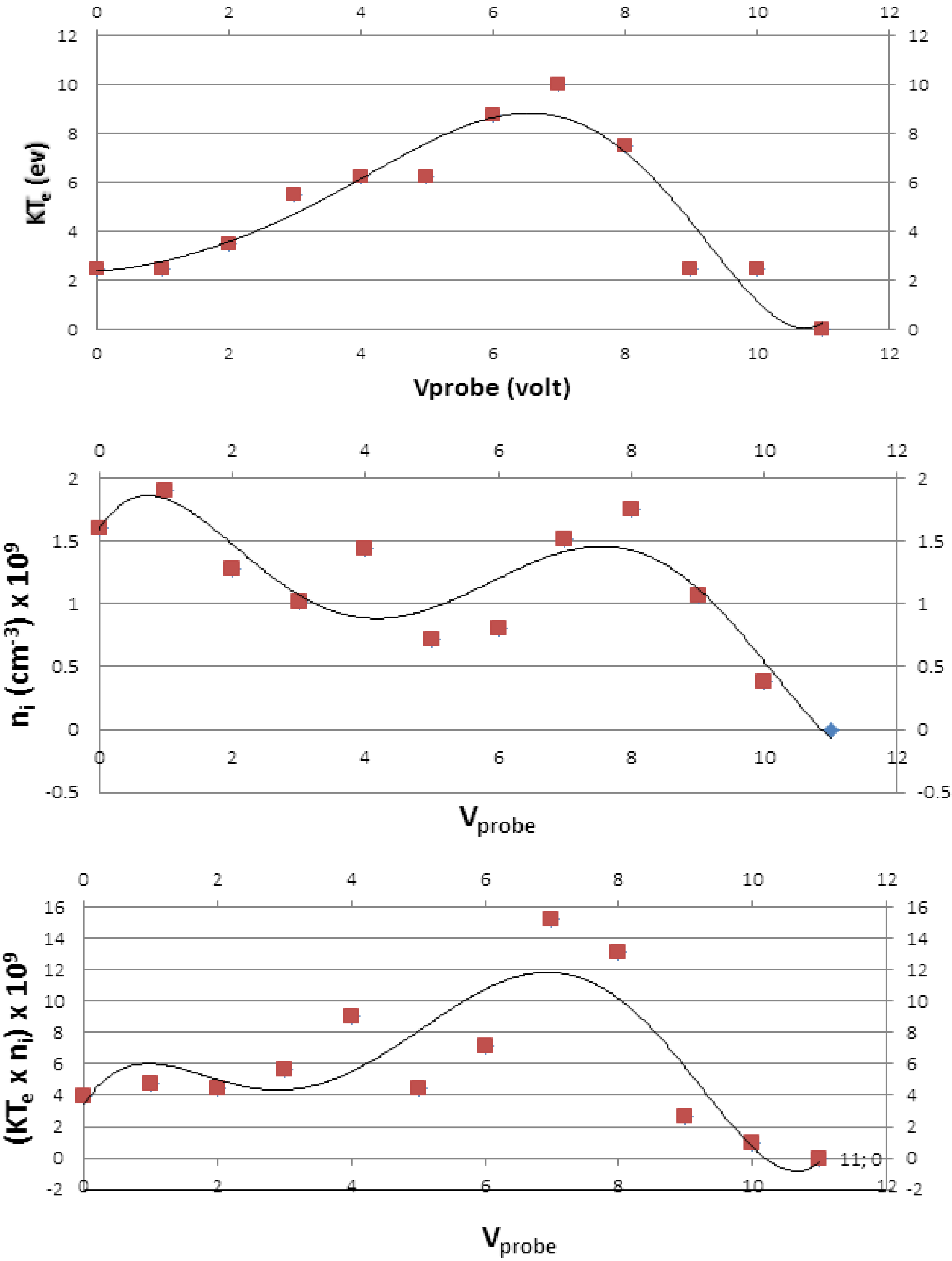


Fig. 5: The variation of plasma temperature, plasma density and plasma energy density at deferent distances from 0 to 11 cm

The BATORM device consists mainly of vacuum chamber & vacuum system, Toroidal, Ohmic & Vertical coils and electrical circuit. The vacuum chamber consists of two parallel aluminum plate electrodes, which is fixed at Pyrex glass discharge chamber of 15 cm length, and 32 cm inner diameter. At the center of this chamber there is another glass tube of 15 cm length, and 10 cm outer diameter. From that, the aspect ratio for this device is R/a = 2.

This is the first attempt to obtain DC plasma in BATORM device. The double electric probe estimates the electron temperature. The peak electron temperature is 10 eV and peak ion density is 1.5 x 10^9 cm^{-3}at 7 cm from the outer wall of the device.

It is clear that from the values of ion density, it could be increased the dc glow discharge voltage in future to give maximum values for n_i. Besides, it is possible to connect more capacitor banks with ohmic coil circuit to increase poloidal magnetic field. This work has be considered as the first step to produce a good view for Batorm system. This will be used as a sourse of toroidal plasma machien which very useful in different applications related to material science.

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Information Mechanics: The Dynamics of “It from Bit”

By Dr. Zhi Gang Sha & Rulin Xiu

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Keywords: *information mechanics, hierarchy problem, cosmological constant problem, dark matter, dark energy, black hole, it from bit, grand unification theory.*

GJSFR-A Classification: *FOR Code: 240201p*

INFORMATIONMECHANICSTHEDYNAMICSOFITFROMBIT

Strictly as per the compliance and regulations of:

Information Mechanics: The Dynamics of "It from Bit"

Dr. Zhi Gang Sha [α] & Rulin Xiu [σ]

Summary- John Wheeler proposed the idea "It from Bit," suggesting that information gives rise to physics.

However, the process of "It from Bit" remains to be clarified. In this work, we propose *Information Mechanics*. We suggest that information determines observed phenomena. The interaction of the two basic elements making up information creates everything we observe. We introduce information space and time to be the basic elements. The interaction of information space and time creates everything we observe. Based on these assumptions, we derive an action, the information action, and the information function. The information action represents the maximum possibilities, i.e., information, in a system. Information action appears to be similar to the string action in string theory and superstring theory but with a different meaning. The information function calculates the possible states in a system. We demonstrate that elementary particles and their wave-particle duality, fundamental forces, dark matter, and dark energy can emerge from the information function. We show that it is possible to derive a value of the cosmological constant consistent with astrophysical observation. We demonstrate that one may derive the hierarchy between the weak scale and the Planck scale. We point out that one may study what is inside a black hole and deduce that the entropy of a black hole to an outside observer is proportional to the area of the event horizon. Based on the various problems Information Mechanics may address, we suggest that it could lead to the grand unification theory we seek.

Keywords: information mechanics, hierarchy problem, cosmological constant problem, dark matter, dark energy, black hole, it from bit, grand unification theory.

I. Introduction

The principles and laws of creation are sought in many disciplines, including sciences, philosophies, and ideologies. Current cosmology suggests that the creation of our universe is through a "big bang." However, the natural law that has led to the big bang waits to be explored further.

John Wheeler proposed [Refs 1, 2, 3] the idea "It from Bit." He suggested that information sits at the core of physics and every "it," whether a particle or field, derives its existence from observations. To show how everything comes to existence through observation, John Wheeler acknowledged [Ref 3] that time played an essential role, but this is not understood well enough.

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The Grand Unification Theory is an attempt to use one mathematical formula to explain everything, including all elementary particles, fundamental forces, dark matter, dark energy, and the macro structure of the universe, and to unify quantum physics with Einstein's general relativity theory about gravity. So far, string theory is the only mathematically consistent theory that can unify everything [Refs 4, 5]. However, current string theory is still limited in its ability to make predictions. Something is still missing.

In this paper, we propose *Information Mechanics*, in which information determines and creates everything we observe. We present the basic principles and formula about how information underlies all observed phenomena, including giving the emergence to elementary particles, fundamental forces, dark matter, dark energy, black holes, and the universe. We proffer two basic laws governing Information Mechanics. The First Law of Information Mechanics comes from quantum physics, indicating the information contained in our measurement determines the observed phenomena. The Second Law of Information Mechanics comes from the ancient Chinese wisdom about yin yang. It proposes that two basic elements, yin and yang, make up everything, including information, and that the interaction of these two elements creates all the observed physical phenomena. We will show that the interaction of two pairs of yin yang elements: space-time and exclusion-exclusion, create all elementary particles, forces, dark matter, dark energy, and the universe we observe.

We show that the laws of Information Mechanics give rise to an information action. Information action represents the maximum possibilities, i.e., information, in a system. Information action appears to be similar to string action. From the information action, one can define information function. Information function calculates the possible states in a system. *Information function* seems to be an extension of wave function in quantum mechanics. With information action and information function, we demonstrate that elementary particles and their wave-particle duality emerge from the Poincaré symmetry, fundamental forces come about due to the diffeomorphism symmetry, and classical equations of motion come from Weyl symmetry. Observation of dark matter and dark energy at the large scale of universe can be explained in Information Mechanics. We find that it is possible to calculate the

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cosmological constant consistent with the current astrophysical observation. The hierarchy problem regarding cosmological constant can be possibly explained and derived. We also demonstrate that it is plausible to deduce the large hierarchy between the weak symmetry breaking scale and Planck scale. We find that information action and function provide a way to study what is inside a black hole and also to derive the entropy of a black hole as seen by an outside observer to be proportional to the area of its event horizon.

II. Basic Principles and Laws of Information Mechanics

a) First Law of Information Mechanics

The observed phenomenon is determined by the information of the measurement.

The First Law of Information Mechanics comes from quantum physics, which indicates that the process of making a measurement determines the observed phenomenon.

For instance, quantum physics shows us that what detector we use and where we place the detector determines what phenomenon we observe.

b) Second Law of Information Mechanics

Everything is made of two basic elements. These two elements are opposite, relative, co-created, inseparable, and co-dependent. The interaction of these two elements creates everything.

The Second Law of Information Mechanics originates from ancient Chinese wisdom about yin yang [Ref 6, 7]. We keep the Chinese words here and call these two basic elements that make up everything: yin and yang.

These two basic elements, yin and yang, make up the "Bit" of information. The Second Law of Information Mechanics suggests the interaction of yin and yang, the essential elements of information, creates everything.

III. Space and Time as the Fundamental Yin-Yang Pair

What are the basic yin yang elements that create the observed elementary particles, forces, dark matter, dark energy, and the universe?

We suggest that space and time are one of the fundamental pairs of yin-yang elements that create everything we observe.

We propose another meaning of space-time, which we call information space and time. Information space and time are related to two basic measurements we conduct. Information time relates to the measurement of movement and change. Information space relates to the measurement of stillness and solidity. For instance, the flow of sand in an hourglass and the movements of the sun and the moon have all been used as measurements of time. The duration of a day is based on the measurement of the rotation of the earth. The measurements of space, such as the length, height, and width of an object, are the measurement of its fixedness and stillness.

Information space and time are a yin-yang pair. They are opposite and relative. Change and stillness are opposites and relative. Space and time are co-created because whenever one measures change, one refers to something unchanged. Whenever one measures something as unchanged, one compares it with something one considers changing. Therefore, information space and time are inseparable and co-dependent.

According to quantum physics, it takes energy and momentum to measure time and space. How accurately time and space can be measured depends on the amount of energy and momentum used in the measurement. More specifically, to measure the time of duration $\Delta\tau$, it takes an energy of $\Delta E \sim \hbar/\Delta\tau$. To measure a space of size $\Delta\sigma$, it takes the momentum $\Delta p \sim \hbar/\Delta\sigma$.

If one takes gravity into consideration, according to general relativity, energy curves space-time. When one measures time interval $\Delta\tau$, the energy ΔE used for a time measurement will curve space-time. It will create a black hole with the horizon on the order of $G\Delta E/c^4$. When $\Delta\sigma$ is smaller than $G\Delta E/c^4$, no information can escape. Therefore, the measurable causal region is:

$$\Delta\sigma \geq G\Delta E/c^4.$$

Therefore, there exists the uncertainty relation between the measurable space $\Delta\sigma$ and measurable time $\Delta\tau$,

$$\Delta\sigma\, \Delta\tau \geq l_p\, t_p. \quad (1)$$

Here l_p is the Planck length and t_p is the Planck time. This uncertainty relation suggests that information space and time affect each other. They are not independent. They are a yin-yang pair.

Next, we propose that inclusion and exclusion are the other basic yin-yang pair in measurement. This is because, when one makes a measurement, one needs to give the information about what is included and what is excluded.

We propound, all measurements are based on these two basic yin-yang pairs: information space-time measurement and inclusive-exclusive measurement. To see this, one can examine all possible measurements, such as measurement of velocity, acceleration, energy, momentum, temperature, spin, electricity, magnetic field, mass, charge, and force. One can see that these measurements are various combinations of space and time measurement and inclusive and exclusive measurement.

IV. Derivation of Information Action

If all measurement is made of two basic measurements: space-time measurement and inclusion-exclusion measurement, according to the first and second laws of Information Mechanics, the interaction of the space and time yin-yang pair and the interaction of the inclusion and exclusion yin-yang pair should create all observed phenomena.

The simplest action created by the interaction of the information space time is:

$$A_1 = \alpha \int \Delta\tau \Delta\sigma. \quad (2)$$

Here the symbol σ represents information space and the symbol τ represents information time. The integral symbol $\int$ represents the summation over information space and time, and α is a constant. From the uncertainty relation between information space and time (1), we get

$$\alpha = 1/(l_p t_p).$$

To introduce the second yin-yang pair into the action, we need to include the inclusion and exclusion elements. To do this, we realize that corresponding to the inclusion and exclusion yin-yang pair, in nature there exist two types of particles, fermions and bosons. Fermions have half (1/2) spin. They repel each other. They cannot be in the same state. Bosons have integer spin. They tend to clump. The normal time and space coordinates τ and σ are of bosonic nature. If we assume each information space or time coordinate has both the fermion (yang, repulsive) and boson (yin, clumping) parts, each information time and space coordinate become two elements:

$$\tau \rightarrow (\tau, \theta_\tau) \quad (3)$$

$$\sigma \rightarrow (\sigma, \theta_\sigma). \quad (4)$$

Here we use θ_σ and θ_τ to represent the fermion part of information space and time coordinates σ and τ. The θ_σ and θ_τ can only take on the value of 0 or 1 because they are repulsive and refuse to stay at the same place. The four elements of the two yin-yang pairs are represented by σ, τ, θ_σ, and θ_τ.

The simplest action created by these two yin-yang pairs is:

$$A_2 = \alpha' \int \Delta\tau \Delta\sigma \Delta\theta_\tau \Delta\theta_\sigma. \quad (5)$$

We will call the action A_1 and A_2 the information action.

One can see that the action A_1 is basically the action of bosonic string and the action A_2 is the action of the super string[Ref 4, 5]. It is interesting to see that from the basic laws of Information Mechanics we can derive string action [Ref 4, 5].

There is a fundamental difference in the meaning and function between information action and string action. For instance, suppose the integration of τ and σ is over (0, T) and (0, L). We can see that T and L correspond to the time and space scale involved in our measurement in Information Mechanics. They are different in different measurements. If our measurement and observation is the whole universe, T and L should be the age and horizon length of the universe. This is different from string theory, in which L is set to be the string length and T is set to be infinite.

The information actions A_1 and A_2 in equations (2) and (5) give the amount of the information in a system with the observation scale (0, T) and (0, L). Information action expresses the possible states that can be observed in a system. Seth Lloyd derived a similar result in his paper [Ref 8], viz. that the maximum observable information in a system is represented by (2).

V. Two Types of Space-Time

To derive the observable phenomena in Information Mechanics, it is necessary to realize that there exist two types of space time. One is the information space time associated with the fundamental yin-yang pair (τ, σ). It is related to the basic elements of information. We call it information space and time. The other is the physical measurement of space distance and time duration or physical location of space time X^μ. Let's call X^μ physical space and time.

The physical space and time X^μ is a projection from the information space and time (τ, σ),

$$X^\mu: (\tau, \sigma) \rightarrow X^\mu (\tau, \sigma).$$

Suppose in this projection, the total information is unchanged. The action A_1 becomes:

$$A_1' = \alpha \int d\tau\, d\sigma\, (-\det h_{ab})^{1/2}. \quad (6)$$

Here, $h_{ab} = \partial_a X^\mu \partial_b X_\mu$.

In the following, for the sake of the simplicity of illustration, we will work with the "bosonic string,"

$$A_1 = \alpha \int_0^T d\tau \int_0^L d\sigma.$$

One can follow and extend the same discussion to the general case of "superstring,"

$$A_2 = \alpha' \int_0^T d\tau \int_0^L d\sigma d\theta_\tau d\theta_\sigma.$$

VI. Definition of Information Function

Now let's define the information function Ψ:

$$\Psi = \exp(i A). \quad (7)$$

Here A is the information action. We can see that the information function Ψ is related to the amount of information I in a system through the formula:

$$I = A = -i\ln\Psi. \quad (8)$$

Suppose the information function at $\tau = 0$ and $\sigma = 0$ is Ψ_0. The information Ψ at $\tau = T$ and $\sigma = L$ is

$$\Psi = \exp(iA_1)\Psi_0. \quad (9)$$

Here $A_1 = \alpha\int_0^T d\tau \int_0^L d\sigma$ or $A_2 = \alpha \int_0^T d\tau \int_0^L d\sigma d\theta_\tau d\theta_\sigma$.

Using the action A_1' in (6), the information function at $\tau = T$ and $\sigma = L$ now becomes

$$\Psi = \exp(iA_1)\Psi_0 = \int \mathcal{D}X \exp(iA_1')\, \Psi_0.$$

Here $A_1' = \alpha\int_0^T d\tau \int_0^L d\sigma\, (-\det h_{ab})^{1/2}$.

Here $\int \mathcal{D}X$ represents summing over all possible X, similar to Feynman's path integral definition [Ref 9].

Compare to the wave function in quantum physics:

$$\Psi(T) = \int \mathcal{D}X \exp(iS)\, \Psi_0.$$

Here $S = \int_0^T dt\, \mathcal{L}(x(t), \dot{x}(t))$.

One can see that the information function is a natural extension of the wave function in quantum physics. The action in Information Mechanics integrates over both time and space while the action in quantum physics integrates over time.

Information Mechanics is also different from quantum field theory. The action of Information Mechanics integrates over two-dimensional information space and time, while in quantum field theory the action integrates over four-dimensional physical space time. The main task of quantum field theory is to calculate the correlation function and scattering cross-sections. In Information Mechanics, the correlation function and, thus, scattering cross-section can be obtained through the information function in the equation (9). Note that the wavelength and frequency in quantum field theory are now replaced with the measurement scales T and L in Information Mechanics. We will discuss the correspondence between the calculations of Information Mechanics with those of quantum field theory in more detail in future work.

VII. Emergence of Elementary Particles and Fundamental Forces

As discovered in particle physics, the basic constituents of everything are elementary particles and fundamental forces. It is found that elementary particles have wave-particle duality, meaning that they behave like a wave but each elementary particle has its own specific mass and spin, no matter where and when one makes the measurement. The wave-particle duality of elementary particles and fundamental forces was proposed by Einstein and Niels Bohr and indicated by experiments [Ref10, 11]. However, it is never derived from the first principle in theoretical physics.

In the following, we show how the wave-particle duality of elementary particles and fundamental force se merge in Information Mechanics.

To do this, first notice, as shown in string theory [Ref 4,5],it is possible to introduce a metric tensor γ^{ab} and rewrite the action A_1' in (6) in the form:

$$A_1''[X, \gamma] = \alpha \int d\tau\, d\sigma\, (-\det\gamma_{ab})^{1/2} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu. \quad (10)$$

In Information Mechanics, the possibility to introduce tensor γ^{ab} is due to the relativity between the yin yang, the information time and space (τ, σ).

The action A_1'' is invariant under the following three transformations:

1. D-dimensional Poincaré transformation

$$X'^\mu(\tau,\sigma) = \Lambda^\mu_\nu X^\mu(\tau,\sigma) + a^\mu$$

$$\gamma'_{ab}(\tau,\sigma) = \gamma_{ab}(\tau,\sigma) \quad (11)$$

2. Diffeomorphism transformation

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma)$$

$$\frac{\partial\sigma'^c}{\partial\sigma'^a}\frac{\partial\sigma'^d}{\partial\sigma'^b}\gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma) \quad (12)$$

3. Two-dimensional Weyl transformation

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma)$$

$$\gamma'_{ab}(\tau,\sigma) = \exp\big(2\omega(\tau,\sigma)\big)\gamma_{ab}(\tau,\sigma). \quad (13)$$

Information action has three symmetries: Poincaré symmetry, diffeomorphism symmetry, and Weyl symmetry.

VIII. Emergence and Observation of Elementary Particles Due to Poincaré Symmetry

In Information Mechanics, the observed world is made of a certain amount of information, which represents different possibilities in a system. The observation of the same elementary particles regardless of physical space and time is due to the Poincare symmetry. The observed elementary constituents should be invariants of Poincaré transformation. From group theory, one knows that mass and spin are the two invariants under Poincaré transformation. Because of this, the basic constituents, elementary particles, are specified by mass and spin.

The wave aspect of elementary particles is represented by the information function. It comes from the basic assumption that everything we observe comes from information. Information represents different possibilities. In this way, the wave-particle duality of elementary particles and fundamental forces emerge in Information Mechanics.

IX. Emergence of Gravity and Gauge Force Due to Diffeomorphism Symmetry

In Information Mechanics, the emergence of gravity and gauge interaction is due to diffeomorphism symmetry (12). Diffeomorphism invariance(12) suggests one can introduce the physical space-time metric tensor $G_{\mu\nu}(X^{\mu})$and anti-symmetric tensor $B_{\mu\nu}(X^{\mu})$ in the action (10):

$$A_1''' \, [X, \gamma, G_{\mu\nu}, B_{\mu\nu}] =$$

$$\alpha \int d\tau \, d\sigma \, (-\det \gamma_{ab})^{\frac{1}{2}} (\gamma^{ab} G_{\mu\nu} + \varepsilon^{ab} B_{\mu\nu}) \partial_a X^{\mu} \partial_b X^{\nu}. \quad (14)$$

The action $A_1''' \, [X, \gamma, G_{\mu\nu}, B_{\mu\nu}]$ *has* the diffeomorphism invariance in physical space-timeX^{μ}:

$$\frac{\partial X'^{\alpha}}{\partial X'^{\mu}} \frac{\partial X'^{\beta}}{\partial X'^{\nu}} G'_{\alpha\beta}(X'^{\mu}) = G_{\mu\nu}(X^{\mu})$$

$$\frac{\partial X'^{\alpha}}{\partial X'^{\mu}} \frac{\partial X'^{\beta}}{\partial X'^{\nu}} B'_{\alpha\beta}(X'^{\mu}) = B_{\mu\nu}(X^{\mu}).$$

The introduction of physical space-time tensor metric $G_{\mu\nu}(X^{\mu})$and anti-symmetric tensor $B_{\mu\nu}(X^{\mu})$can induce gravity and gauge interaction in physical space-time.

The fact that $G_{\mu\nu}(X^{\mu})$ and $B_{\mu\nu}(X^{\mu})$ describe the gravity and gauge interaction in physical space-time can be further confirmed by the equations of motion associated with$G_{\mu\nu}(X^{\mu})$ *and* $B_{\mu\nu}(X^{\mu})$. In the following, we will show that, from the Weyl invariance, one can obtain the equations of motion regarding $G_{\mu\nu}$ *and* $B_{\mu\nu}$, which shows that $G_{\mu\nu}$ *and* $B_{\mu\nu}$ follow the equations of motion associated with gravity and gauge force.

X. Weyl Invariance, Holography, and Classical Equations of Motion

The Weyl invariance (13) is automatically preserved at the first order in information action. However, higher-order corrections could possibly violate it. For instance, in [Refs 4, 5], it is shown that there are the following second-order corrections to the information action:

$$\beta^{G}{}_{\mu\nu} = \alpha R_{\mu\nu} + \alpha/4 \, H_{\mu\lambda\omega} H_{\nu}{}^{\lambda\omega} + O(\alpha^2) \quad (15)$$

$$\beta^{B}{}_{\mu\nu} = -\alpha/4 \, \nabla^{\omega} H_{\mu\lambda\omega} + O(\alpha^2). \quad (16)$$

The preservation of Weyl Invariance at the higher orders requires that:

$$\beta^{G}{}_{\mu\nu} = \beta^{B}{}_{\mu\nu} = 0.$$

Notice that $\beta^{G}{}_{\mu\nu} = 0$ leads to the generalized Einstein's equation with the source terms from the anti-symmetric tensor. The equation $\beta^{B}{}_{\mu\nu}=0$ is the anti symmetric generalization of Maxwell's equations.

We can see that requiring Weyl invariance, one is able to obtain classical equations of motion including Einstein's general relativity and gauge interactions. In this way, Information Mechanics includes classical physics.

In Information Mechanics, all the physical phenomena are projected from a two-dimensional information space time. The two-dimensional information space time has Weyl invariance. This means that the two-dimensional information space time is a hologram from which all observed phenomena, including physical space time, elementary particles, gravity, and gauge interactions emerge. Classical equations of motion come from the holographic nature, the Weyl invariance of information space time.

XI. Emergence of Dark Energy and Dark Matter

The observed accelerated expansion and large structure of our universe [Refs12, 13, 14, 15, 16]indicates an unknown source of energy, dark energy, and an unknown source of matter, dark matter. There are many proposals about the potential candidates for dark matter and dark energy.

In the following, we will show that Information Mechanics may explain the observation of dark energy and dark matter. Dark matter and dark energy can emerge from information function.

Dark energy and dark matter are phenomena observed in the large structure of the universe. To see what matter and energy emerges in the large structure of universe, we need to study the information function:

$$\Psi = \exp(iA_1'') = \int \mathcal{D}X \exp(i A_1'').$$

Here

$$A_1'' = \alpha \int_0^T d\tau \int_0^L d\sigma \, (-\det \gamma_{ab})^{\frac{1}{2}} (\gamma^{ab} G_{\mu\nu} + \varepsilon^{ab} B_{\mu\nu}) \partial_a X^{\mu} \partial_b X^{\nu},$$

where T is the age of the universe and L is the length of the horizon of the universe.

One may notice that in the information function Ψ, there exist vibrations in the energy state (n, m) with the frequency $\nu_n = n/T$ and wavelength $\lambda_m = L/m$. The frequency and wavelength of some of these vibrations have a frequency and wavelength on the order of 1/T and L. These vibrations are almost impossible to detect at this moment. This is because, to observe them, it takes a detector as large as the universe or it takes time as long as the age of the universe. These vibrations are very "dark" due to this innate difficulty to be observed by detectors. They can only be observed on the large structure of the universe. They are natural possible candidates for dark matter and dark energy.

XII. Calculation of Cosmological Constant

The cosmological constant is the simplest possible form of dark energy. The current standard model of cosmology, the Lambda-CDM model, assumes a non-zero cosmological constant as the source of dark energy. It is found that, in terms of the Planck unit, and as a natural dimensionless value, the cosmological constant is calculated to be on the order of 10^{-122} [Ref 15, 16]. The large discrepancy between the natural energy scale, Planck scale, and the observed value of 10^{-122} in terms of Planck scale is the so called cosmology constant problem.

In the following, we will estimate the vacuum energy of the information function. Surprisingly, we are able to obtain a value for the cosmological constant consistent with the observation.

To calculate the vacuum energy of the universe, we use the fact that the energy at state (n, m) with the frequency $\nu_n = n/T$ and wavelength $\lambda_m = L/m$ is:

$$E_{n,m} = (n + ½)\, \hbar/T.$$

The lowest energy of the vacuum fluctuation is $E_{0n,m} = \hbar/2T$.

In Information Mechanics, with the space and time measurement scale at T and L, the total number of possible states N is:

$$N = TL/(l_p t_p).$$

If we assume each of the possible states can contribute to vacuum fluctuation energy of $E_{0n,m} = \hbar/2T$, then the total vacuum energy is:

$$E_{vac} = \hbar /2T \times TL/ l_p^2 = \hbar L/(2\, l_p^2).$$

The vacuum energy density in three-dimensional observed space is:

$$\rho_{vac} = E_{vac}/(4\pi L^3/3) = \rho_p 3 l_p^2/(8\pi L^2).$$

Here ρ_p is the Planck energy density,

$$\rho_p = E_p/l_p^3,$$

and E_p is the Planck energy $E_p = \hbar/t_p = 1.956 \times 10^9$ Joule.

The cosmological constant Λ_c is: $\Lambda_c = 8\pi\, \rho_{vac}$. Therefore

$$\Lambda_c = 8\pi\, \rho_{vac} = 3\rho_p\, l_p^2/L^2.$$

We know that:

$$l_p^2/L^2 = t_p^2/t_u^2 = 10^{-122}.$$

Here t_u is the age of the universe. We use:
t_u = 13.799 billion years = 4.35×10^{17} seconds.
From this, we obtain:

$$\Lambda = 3 \times 10^{-122} \rho_p. \qquad (17)$$

According to results published by the Planck Collaboration in 2018 [Refs 15,16], the cosmological constant is 2.888 x 10^{-122} in Planck units. The result in (17), $\Lambda = 3 \times 10^{-122}\rho_p$, is consistent with the data from the Planck Collaboration in 2018 [Refs 15, 16].

Information Mechanics seems to be able to address the large hierarchy problem regarding the cosmological constant.

XIII. Emergence of the Electroweak Scale and Planck Scale Hierarchy

There are two outstanding hierarchy problems in physics. One is the cosmological constant problem. The other is to derive the large difference between weak force and gravity, or equivalently between Higgs mass and Planck mass [Ref 17, 18, 19, 20]. We have shown above that Information Mechanics may help address the cosmological constant problem; next, we will explore how it may help cope with the second hierarchy problem.

To derive the weak scale and Higgs mass in Information Mechanics, we study the information action A_2 and write *it* in terms of complex coordinates, z, $\bar{z}$, θ, and $\bar{\theta}$[Ref 4, 5]:

$$A_2 = \alpha' \int dz d\bar{z}\, d\theta\, d\bar{\theta} = \alpha \int d^2z\, d^2\theta.$$

We introduce observable space-time $X^\mu(z, \bar{z}, \theta, \bar{\theta})$ in superspace, which includes both bosonic spacetime $X^\mu(Z, \bar{z})$ and its fermionic counterpart $\psi^\mu(Z, \bar{z})$ and $\tilde{\psi}^\mu(z, \bar{z})$:

$$X^\mu(z, \bar{z}, \theta, \bar{\theta}) = X^\mu(Z, \bar{z}) + \theta\psi^\mu(Z, \bar{z}) + \bar{\theta}\tilde{\psi}^\mu(z, \bar{z}) + \theta\bar{\theta}F^\mu.$$

The term F^μ is the auxiliary field, which can usually be eliminated through equations of motion. We suggest that bosonic space time $X^\mu(z, \bar{z})$ corresponds to observable spacetime coordinates and the fermionic spacetime $\psi^\mu(z, \bar{z})$ and $\tilde{\psi}^\mu(z, \bar{z})$ correspond to elementary particles. In superspace $X^\mu(z, \bar{z}, \theta, \bar{\theta})$, there is supersymmetry, which is the invariance under the transformation between space-time bosonic coordinates and fermion coordinates. After integrating over fermion coordinates (θ, $\bar{\theta}$), the action A_2 including metric tensor $G_{\mu\nu}$ and anti symmetric tensor $B_{\mu\nu}$ becomes [Ref 4, 5]:

$$A_2 = \alpha \int d^2z[(G_{\mu\nu} + B_{\mu\nu})\, \partial_z X^\mu \partial_{\bar{z}} X^\nu + G_{\mu\nu}(X)(\psi^\mu D_{\bar{z}}\psi^\nu + \tilde{\psi}^\mu D_z \tilde{\psi}^\nu) + ½\, R_{\mu\nu\rho\sigma}(X)\psi^\mu\psi^\nu\tilde{\psi}^\rho\tilde{\psi}^\sigma]. \qquad (18)$$

Here, covariant derivatives are:

$$D_{\bar{z}}\psi^\nu = \partial_{\bar{z}}\psi^\nu + [\, \Gamma^\nu{}_{\rho\sigma}(X) + \tfrac{1}{2}H^\nu{}_{\rho\sigma}(X)\,]\, \partial_{\bar{z}} X^\rho \psi^\sigma$$

$$D_z\tilde{\psi}^\nu = \partial_z\tilde{\psi}^\nu + [\, \Gamma^\nu{}_{\rho\sigma}(X) - \tfrac{1}{2}H^\nu{}_{\rho\sigma}(X)\,]\, \partial_z X^\rho\, \tilde{\psi}^\sigma.$$

Here $\Gamma^{\nu}{}_{\rho\sigma}(X)$ is the Christoffel connection. It is related to the gravitational interaction. And $H^{\nu}{}_{\rho\sigma}(X)$ is the anti-symmetric tensor field strength.

To see how the hierarchy between Higgs mass and Planck mass emerges, we use the similarity between Higgs mechanism and superconductivity. The Higgs mechanism was originally suggested in 1962 by Philip Anderson when he noticed the similarity between electroweak symmetry breaking and superconductivity [Ref 21, 22]. In the following, we show that in Information Mechanics, one may use superconductor theory, BCS theory, to induce the large hierarchy between Planck mass and Higgs mass.

Notice the observable space-time coordinates $X^{\mu}(Z, \bar{z})$consist of a series of vibrations. Similar to the phonons in a superconductor interacting with fermions through electromagnetic force, $X^{\mu}(Z, \bar{z})$ vibrations interact with fermions, gravity, and gauge interactions, as indicated in the information action (18) through the interaction terms:

$$V_{int} = G_{\mu\nu}(X)\psi^{\mu} [\Gamma^{\nu}{}_{\rho\sigma}(X) +\tfrac{1}{2}H^{\nu}{}_{\rho\sigma}(X)] \partial_{\bar{z}}X^{\rho}\psi^{\sigma}+ G_{\mu\nu}(X) \tilde{\psi}^{\mu}[\Gamma^{\nu}{}_{\rho\sigma}(X) -\tfrac{1}{2}H^{\nu}{}_{\rho\sigma}(X)] \partial_{Z}X^{\rho} \tilde{\psi}^{\sigma} + \tfrac{1}{2} R_{\mu\nu\rho\sigma}(X)\psi^{\mu}\psi^{\nu}\tilde{\psi}^{\rho}\tilde{\psi}^{\sigma}. \quad (19)$$

As discovered in BCS theory, these interactions can add a negative potential energy which leads to a ground state with the formation of coherent fermion pairs. This ground state energy results in a non-zero gauge field, which breaks the gauge symmetry.

In the interaction terms (19), the gauge interaction is proportional to $\partial_{\bar{z}}X^{\rho}$ and $\partial_{Z}X^{\rho}$. This indicates that the ground state with the formation of fermion pairs will not only break gauge symmetry but also space-time translational symmetry. This means that it can lead to space-time compactification and also super symmetry breaking. This may be a natural non-perturbative mechanism for gauge symmetry breaking, space-time compactification, and super symmetry breaking.

According to the BCS mechanism, the non-perturbative ground state potential energy in the weak interaction limit is on the order of [Ref 21, 22]:

$$W = - n_c\hbar\omega\exp[-\tfrac{1}{NV}].$$

Here N is the state density and V is the interaction potential. The energy term$\hbar\omega$corresponds to the energy of space-time vibration. It can be on the order of the space-time compactification scale, which may be of the same energy scale as the super symmetry breaking. One can see that the Higgs mass is exponentially smaller than the space-time compactification scale.

Because of the exponential form of the non-perturbative potential energy in the ground state, the large hierarchy between Higgs mass and compactification scale and thus Planck scale can be easily generated. We will expand the detailed model, calculation, and discussion of this possible scheme in future work.

XIV. Black Holes

Any fundamental physics theory including gravity should be able to address what is happening inside a black hole. Let's take a look at how Information Mechanics could help us study what is inside a black hole.

A black hole is created when the gravity force becomes stronger than the fermionic exclusion force, and matter is collapsed by gravity to the point that all matter including light is confined to a limited space and time [Ref 23, 24]. Thus, we propose that a black hole is related to physical space-time compactification in the observable four-dimensional space-time.

In Information Mechanics, to study black holes, we can start with the action (18):

$$A_2 = \alpha\int d^2z[(G_{\mu\nu}+ B_{\mu\nu}) \partial_z X^{\mu}\partial_{\bar{z}}X^{\nu}+ G_{\mu\nu}(X)(\psi^{\mu} D_{\bar{z}}\psi^{\nu}+ \tilde{\psi}^{\mu}D_z\tilde{\psi}^{\nu}) + \tfrac{1}{2} R_{\mu\nu\rho\sigma}(X)\psi^{\mu}\psi^{\nu}\tilde{\psi}^{\rho}\tilde{\psi}^{\sigma}].$$

In the case of a black hole, $G_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}(X)$ become very large, and we assume this leads to compactification of space-time in the observable four-dimensional space-time. This means that:

$$X^0 \cong X^0 + D^0,$$

$$X^i \cong X^i+ D^i,\ i = 1,2,3.$$

Inaction A_2(18), this space-time compactification will lead to a positive kinetic energy:

$$(G_{\mu\nu}+ B_{\mu\nu}) \partial_z X^{\mu}\partial_{\bar{z}}X^{\nu}\cong (G_{\mu\nu}+ B_{\mu\nu}) \frac{1}{D^{\mu}}\frac{1}{D^{\nu}} \quad (19)$$

This additional kinetic energy term will balance the negative potential energy from the gravity and gauge interaction term:

$$G_{\mu\nu}(X)\psi^{\mu} [\Gamma^{\nu}{}_{\rho\sigma}(X) +\tfrac{1}{2}H^{\nu}{}_{\rho\sigma}(X)] \partial_{\bar{z}}X^{\rho}\psi^{\sigma} + G_{\mu\nu}(X)\tilde{\psi}^{\mu}[\Gamma^{\nu}{}_{\rho\sigma}(X) - \tfrac{1}{2}H^{\nu}{}_{\rho\sigma}(X)] \partial_{Z}X^{\rho} \tilde{\psi}^{\sigma} + \tfrac{1}{2} R_{\mu\nu\rho\sigma}(X)\psi^{\mu}\psi^{\nu}\tilde{\psi}^{\rho}\tilde{\psi}^{\sigma} \quad (20)$$

The balance between the kinetic energy in (19) and the potential energy in (20) could lead to a new stable ground state. It indicates that the internal structure of a black hole is similar to a crystal, or liquid crystal, or some other ordered and coherent state. The action A_2 can enable us to study the detailed dynamics inside a black hole with matter, space-time, gravity, and gauge interactions all present in one formula. We will defer the detailed calculation and discussion to future work.

To study the dynamics outside of the black hole, the measurement scale L is larger than the horizon of the black hole. For an observer outside a black hole, a black hole appears as a particle with specific mass and spin.

Like everything, a black hole carries information. For an outside observer, the entropy of a black hole is the unknown information or possibilities associated with the black hole. Since the outside observer can't receive any information beyond the horizon of a black hole, the information space scale, L, for the observation of a black hole is the black hole's horizon. The information time scale associated with the external observation of a black hole is L/c. According to the equation (2), the maximum amount of unknown information associated with the observation of a black hole for an outside observer is:

$$S = \frac{L^2}{l_p^2}.$$

It's interesting that, in Information Mechanics, one may derive the result that the entropy of a black hole is proportional to the area of the event horizon in units of Planck scale.

The holographic principle [Ref 25, 26, 27]emerges in Information Mechanics, but in a different way. Here the maximum information is proportional to the area covered by the information space-time, not the physical space-time. In three-dimensional space, the area covered by the physical space could be the same as or proportional to the area of information space-time. This coincidence only happens in four-dimensional physical space-time.

XV. Discussion and Conclusion

In this paper we introduce Information Mechanics and its two basic principles and laws. We propose that information determines the observed phenomena. The interaction of basic yin yang elements making up information and everything creates the observed phenomena. We derive the information action and information function. We show that the observed phenomena, such as physical space time, elementary particles and their wave-particle duality, fundamental forces, classical equations of motion, dark matter, and dark energy, may emerge from the information action and information function. We show how classical physics, quantum physics, quantum field theory, and string theory may emerge in information Mechanics. We discover that it is possible to derive a value of the cosmological constant consistent with astrophysical observation. We suggest a plausible scheme to derive the hierarchy between the weak scale and the Planck scale using information action. We indicate that one can study what is inside a black hole and deduce that the entropy of a black hole to an outside observer is proportional to the area of the event horizon.

Information Mechanics appears to be promising to address various challenging problems facing theoretical physics. More detailed calculations and further investigation are still needed. We welcome more people to participate in this project.

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Derivatives Pricing in Non-Arbitrage Market

By N. S. Gonchar

Abstract- The general method is proposed for constructing a family of martingale measures for a wide class of evolution of risky assets. The sufficient conditions are formulated for the evolution of risky assets under which the family of equivalent martingale measures to the original measure is a non-empty set. The set of martingale measures is constructed from a set of strictly nonnegative random variables, satisfying certain conditions. The inequalities are obtained for the non-negative random variables satisfying certain conditions. Using these inequalities, a new simple proof of optional decomposition theorem for the nonnegative super-martingale is proposed. The family of spot measures is introduced and the representation is found for them. The conditions are found under which each martingale measure is an integral over the set of spot measures. On the basis of nonlinear processes such as ARCH and GARCH, the parametric family of random processes is introduced for which the interval of non-arbitrage prices are found. The formula is obtained for the fair price of the contract with option of European type for the considered parametric processes. The parameters of the introduced random processes are estimated and the estimate is found at which the fair price of contract with option is the least.

Keywords: random process; spot set of measures; optional doob decomposition; super-martingale; martingale; assessment of derivatives.

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I. Introduction

The study of non-arbitrage markets was begun for the first time in Bachelier's work [1]. Then, in the famous works of Black F. and Scholes M. [2] and Merton R. S. [3] the formula was found for the fair price of the standard call option of European type. The absence of arbitrage in the financial market has a very transparent economic sense, since it can be considered reasonably arranged. The concept of non arbitrage in financial market is associated with the fact that one cannot earn money without risking, that is, to make money you need to invest in risky or risk-free assets. The exact mathematical substantiation of the concept of non arbitrage was first made in the papers [4], [5] for the finite probability space and in the general case in the paper [6]. In the continuous time evolution of risky asset, the proof of absent of arbitrage possibility see in [7]. The value of the established Theorems is that they make it possible to value assets. They got a special name "The First and The Second Fundamental Asset Pricing Theorems." Generalizations of these Theorems are contained in papers [8], [9], [10].

If the martingale measure is not the only one for a given evolution of a risky asset, then a rather difficult problem of describing all martingale measures arises in order to evaluate, for example, derivatives.

Assessment of risk in various systems was begun in papers [11], [12], [13], [14].

Statistical studies of the time series of the logarithm of the price ratio of risky assets contain heavy tails in distributions with strong elongation in the central region. The temporal behavior of these quantities exhibits the property of clustering and a strong dependence on the past. All this should be taken into account when building models for the evolution of risky assets.

In this paper, we generalize the results of the papers [15], [16], [17] and construct the evolution of risky assets for which we completely describe the set of equivalent martingale measures.

The aim of this study is to describe the family of martingale measures for a wide class of risky asset evolutions. The paper proposes the general concept for constructing the family of martingale measures equivalent to a given measure for a wide class of evolutions of risky assets. In particular, it also contains the description

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of the family of martingale measures for the evolution of risky assets given by the ARCH [18] and GARCH [19], [20] models. In section 2, we formulate the conditions relative to the evolution of risky assets and give the examples of risky asset evolution satisfying these conditions. Section 3 contains the construction of measures by recurrent relations. It is shown that under the conditions relative to the evolution of risky assets such construction is meaningful. It is proved that the constructed set of measures is equivalent to an initial measure. In theorem 1, we are proved that under certain integrability conditions of risky asset evolution the set of constructed measures is a set of martingale measures relative to this evolution of risky asset. In Section 4 we prove the inequalities for the nonnegative random values very useful for the proof of optional decomposition for the non negative super-martingales relative to the set of all martingale measures.

First, we show an integral inequality for a nonnegative random variable under the inequality for this nonnegative random variable with respect to the constructed family of measures. Further, using this integral inequality for the non-negative random variable, a pointwise system of inequalities is obtained for this non-negative random variable for a particular case. After that, the pointwise system of inequalities is obtained for the non-negative random variable in the general case. Then, using the resulting pointwise system of inequalities, an inequality is established for this non-negative random variable whose right-hand side is such that its conditional mathematical expectation is equal to one.

On the basis of the results of Section 4, in Section 5, we prove the optional decomposition for the non negative super-martingales. In Section 6, we introduce the spot measures by the recurrent relations and find the representation for them. Using these facts under certain conditions we prove integral representation for every martingale measure over the set of spot measures.

First, the optional decomposition for diffusion processes super-martingale was opened by by El Karoui N. and Quenez M. C. [21]. After that, Kramkov D. O. and Follmer H. [22], [23] proved the optional decomposition for the nonnegative bounded super-martingales. Folmer H. and Kabanov Yu. M. [24], [25] proved analogous result for an arbitrary super-martingale. Recently, Bouchard B. and Nutz M. [26] considered a class of discrete models and proved the necessary and sufficient conditions for the validity of the optional decomposition.

Section 7 contains applications of the results obtained. A class of random processes is considered, which contains well-known processes of the type ARCH and GARCH ones. Two types of random processes are considered, those for which the price of an asset cannot go down to zero and those for which the price can go down to zero during the period under consideration. The first class of processes describes the evolution of well-managed assets. We will call these assets relatively stable. For the evolution of relatively stable assets in the period under consideration, the family of martingale measures is one and the same. The family of martingale measures for the evolution of risky assets whose price can drop to zero is contained in the family of martingale measures for the evolution of relatively stable assets. Each of the martingale measures for the considered class of evolutions is an integral over the set of spot martingale measures.

The interval of non-arbitrage prices is found for a wide class of payoff functions in the case when evolution describes relatively unstable assets. This range is quite wide for the payoff functions of standard put and call options. The fair price of the super hedge is in this case the starting price of the underlying asset. The estimates are found for the fair price of the super-hedge for the introduced class of evolutions with respect to stable assets. The formulas are found for the fair price of contracts with call and put options for the evolution of assets described by parametric processes.

The same formulas are found for Asian-type put and call options. A characteristic feature of these estimates is that for the evolution of relatively stable assets the fair price of the super hedge is less than the price of the underlying asset.

In Section 8, the estimates of the parameters of risky assets included in the evolution are obtained. The formulas are found for the fair price of contracts with call and put options for the obtained parameter estimates, and the interval of non-

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arbitrage prices for different statistics is found. The same results are obtained for Asian-style call and put options.

II. Evolutions of Risky Assets

Let $\{\Omega_N, \mathcal{F}_N, P_N\}$ be a direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1,N}$, $\Omega_N = \prod_{i=1}^{N} \Omega_i^0$, $P_N = \prod_{i=1}^{N} P_i^0$, $\mathcal{F}_N = \prod_{i=1}^{N} \mathcal{F}_i^0$, where the σ-algebra $\mathcal{F}_N$ is a minimal σ-algebra, generated by the sets $\prod_{i=1}^{N} G_i$, $G_i \in \mathcal{F}_i^0$. On the measurable space $\{\Omega_N, \mathcal{F}_N\}$, under the filtration $\mathcal{F}_n$, $n = \overline{1,N}$, we understand the minimal σ-algebra generated by the sets $\prod_{i=1}^{N} G_i$, $G_i \in \mathcal{F}_i^0$, where $G_i = \Omega_i^0$ for $i > n$. We also introduce the probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1,N}$, where $\Omega_n = \prod_{i=1}^{n} \Omega_i^0$, $\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}_i^0$, $P_n = \prod_{i=1}^{n} P_i^0$. There is a one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n = \prod_{i=1}^{n} \mathcal{F}_i^0$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$. Therefore, we don't introduce new denotation for the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}$, since it always will be clear the difference between the above introduced σ-algebra $\mathcal{F}_n$ of filtration on the measurable space $\{\Omega_N, \mathcal{F}_N\}$ and the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$.

We assume that the evolution of risky asset $\{S_n\}_{n=0}^{N}$, given on the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, is consistent with the filtration $\mathcal{F}_n$, that is, S_n is a $\mathcal{F}_n$-measurable. Due to the above one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$, we give the evolution of risky assets in the form $\{S_n(\omega_1, \dots, \omega_n)\}_{n=0}^{N}$, where $S_n(\omega_1, \dots, \omega_n)$ is an $\mathcal{F}_n$-measurable random variable, given on the measurable space $\{\Omega_n, \mathcal{F}_n\}$. It is evident that such evolution is consistent with the filtration $\mathcal{F}_n$ on the measurable space $\{\Omega_N, \mathcal{F}_N, P_N\}$.

Further, we assume that

$$P_n((\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n > 0) > 0,$$

$$P_n((\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n < 0) > 0, \quad n = \overline{1,N}, \tag{1}$$

where $\Delta S_n = S_n(\omega_1, \dots, \omega_n) - S_{n-1}(\omega_1, \dots, \omega_{n-1}),\ n = \overline{1,N}$.

Let us introduce the denotations

$$\Omega_n^- = \{(\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n \le 0\}, \quad \Omega_n^+ = \{(\omega_1, \dots, \omega_n) \in \Omega_n,\ \Delta S_n > 0\}, \tag{2}$$

$$\Delta S_n^- = -\Delta S_n \chi_{\Omega_n^-}(\omega_1, \dots, \omega_n), \quad \Delta S_n^+ = \Delta S_n \chi_{\Omega_n^+}(\omega_1, \dots, \omega_n), \tag{3}$$

$$V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2) = \Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1) + \Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2),$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

$$(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\in\Omega_n^-,\quad (\omega_1,\ldots,\omega_{n-1},\omega_n^2)\in\Omega_n^+. \tag{4}$$

We use the following denotation $\Omega_n^a,\ n=\overline{1,N}$, where a takes two values $-$ and $+$. Our assumption, in this paper, is that for $\Omega_n^a,\ a=-,+$, the representations

$$\Omega_n^-=\bigcup_{k=1}^{N_n}[A_n^{0,k-}\times V_{n-1}^k],\quad \Omega_n^+=\bigcup_{k=1}^{N_n}[A_n^{0,k+}\times V_{n-1}^k],\quad N_n\le\infty, \tag{5}$$

are true, where

$$\Omega_{n-1}=\bigcup_{k=1}^{N_n}V_{n-1}^k,\ A_n^{0,k-},\ A_n^{0,k+}\in\mathcal{F}_n^0,\quad A_n^{0,k-}\cup A_n^{0,k+}=\Omega_n^0,$$

$$A_n^{0,k-}\cap A_n^{0,k+}=\emptyset,\quad V_{n-1}^k\cap V_{n-1}^j=\emptyset,\quad k\neq j,\quad V_{n-1}^k\in\mathcal{F}_{n-1}. \tag{6}$$

The number N_n may be finite or infinite. Since $\Omega_n^-\cup\Omega_n^+=\Omega_n$, $\Omega_n^-\cap\Omega_n^+=\emptyset$, and $P_n(\Omega_n^-)>0,\ P_n(\Omega_n^+)>0$, we have

$$P_n(\Omega_n^-)=\sum_{k=1}^{N_n}P_n^0(A_n^{0,k-})P_{n-1}(V_{n-1}^k),$$

$$P_n(\Omega_n^+)=\sum_{k=1}^{N_n}P_n^0(A_n^{0,k+})P_{n-1}(V_{n-1}^k),\quad P_n^0(A_n^{0,k-})+P_n^0(A_n^{0,k+})=1. \tag{7}$$

Further, in this paper, we assume that $P_n^0(A_n^{0,k-})>0,\ P_n^0(A_n^{0,k+})>0,\ n=\overline{1,N},\ k=\overline{1,N_n}$. We also assume some technical suppositions: there exist subsets $B_{n,i}^{0,k-}\in\mathcal{F}_n^0,\ i=\overline{1,I_n},\ I_n>1$, and $B_{n,s}^{0,k+}\in\mathcal{F}_n^0,\ s=\overline{1,S_n},\ S_n>1$, satisfying the conditions

$$B_{n,i}^{0,k-}\cap B_{n,j}^{0,k-}=\emptyset,\ i\neq j,\quad B_{n,s}^{0,k+}\cap B_{n,l}^{0,k+}=\emptyset,\ s\neq l,\quad k=\overline{1,N_n},$$

$$P_n^0(B_{n,i}^{0,k-})>0,\ i=\overline{1,I_n},\ P_n^0(B_{n,s}^{0,k+})>0,\ s=\overline{1,S_n},\quad k=\overline{1,N_n},$$

$$A_n^{0,k-}=\bigcup_{i=1}^{I_n}B_{n,i}^{0,k-},\quad A_n^{0,k+}=\bigcup_{s=1}^{S_n}B_{n,s}^{0,k+},\quad k=\overline{1,N_n}. \tag{8}$$

Below, we give the examples of evolutions $\{S_n(\omega_1,\ldots,\omega_n)\}_{n=1}^N$ for which the representations (5) are true.

Suppose that the random values $a_i(\omega_1,\ldots,\omega_i)$, $\eta_i(\omega_i)$ satisfy the inequalities $0<a_i(\omega_1,\ldots,\omega_i)\le 1,\ 1+\eta_i(\omega_i)\ge 0,\ P_i^0(\eta_i(\omega_i)<0)>0,\ P_i^0(\eta_i(\omega_i)>0)>0$, $i=\overline{1,N}$. If $S_n(\omega_1,\ldots,\omega_n)$ is given by the formula

$$S_n(\omega_1,\ldots,\omega_n)=S_0\prod_{i=1}^n(1+a_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i)),\quad n=\overline{1,N}, \tag{9}$$

then

$$\{\omega_i \in \Omega_i^0,\ \eta_i(\omega_i) \leq 0\} = A_i^{0,1-}, \quad \{\omega_i \in \Omega_i^0,\ \eta_i(\omega_i) > 0\} = A_i^{0,1+},$$

$$V_{i-1}^1 = \Omega_{i-1}, \quad \Omega_i^- = A_i^{0,1-} \times \Omega_{i-1}, \quad \Omega_i^+ = A_i^{0,1+} \times \Omega_{i-1}, \quad i = \overline{1,N}. \tag{10}$$

In general case, let us consider the evolution of risky asset $\{S_n(\omega_1,\dots,\omega_n)\}_{n=1}^N$, given by the formula

$$S_n(\omega_1,\dots,\omega_n) =$$

$$S_0 \prod_{i=1}^n (1 + \sum_{k=1}^{N_i} \eta_i^k(\omega_i)\chi_{V_{i-1}^k}(\omega_1,\dots,\omega_{i-1}) a_i^k(\omega_1,\dots,\omega_i)), \quad n = \overline{1,N}, \tag{11}$$

where the random values $a_i^k(\omega_1,\dots,\omega_i)$, $\eta_i^k(\omega_i)$ satisfy the inequalities

$0 < a_i^k(\omega_1,\dots,\omega_i) \leq 1,\ 1 + \eta_i^k(\omega_i) \geq 0,\ P_i^0(\eta_i^k(\omega_i) < 0) > 0,\ P_i^0(\eta_i^k(\omega_i) > 0) > 0,$

$i = \overline{1,N},\ \ k = \overline{1,N_n}$, and $\bigcup_{k=1}^{N_i} V_{i-1}^k = \Omega_{i-1}$, $V_{i-1}^k \cap V_{i-1}^s = \emptyset$, $k \neq s$. Then, if to put

$$\{\omega_i \in \Omega_i^0,\ \eta_i^k(\omega_i) \leq 0\} = A_i^{0,k-}, \quad \{\omega_i \in \Omega_i^0,\ \eta_i^k(\omega_i) > 0\} = A_i^{0,k+},$$

we obtain

$$\Omega_i^- = \bigcup_{k=1}^{N_i} [A_i^{0,k-} \times V_{i-1}^k], \quad \Omega_i^+ = \bigcup_{k=1}^{N_i} [A_i^{0,k+} \times V_{i-1}^k], \quad i = \overline{1,N}. \tag{12}$$

$$\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n) \leq 0, \quad (\omega_1,\dots,\omega_{n-1},\omega_n) \in \Omega_n^-,, \quad n = \overline{1,N},$$

$$\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0, \quad (\omega_1,\dots,\omega_{n-1},\omega_n) \in \Omega_n^+, \quad n = \overline{1,N}. \tag{13}$$

III. Construction of the Set of Martingale Measures

In this section, we present the construction of the set of measures on the basis of evolution of risky assets given by the formulas (9), (11) on the measurable space $\{\Omega_N, \mathcal{F}_N\}$. For this purpose, we use the set of nonnegative random values $\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})$, given on the probability space $\{\Omega_n^- \times \Omega_n^+, \mathcal{F}_n^- \times \mathcal{F}_n^+, P_n^- \times P_n^+\}$, $n = \overline{1,N}$, where $\mathcal{F}_n^- = \mathcal{F}_n \cap \Omega_n^-$, $\mathcal{F}_n^+ = \mathcal{F}_n \cap \Omega_n^+$. The measure P_n^- is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^-$ and the measure P_n^+ is a contraction of the measure P_n on the σ-algebra $\mathcal{F}_n^+$. After that, we prove that this set of measures, defined the above set of random values, is equivalent to the measure P_N. At last, Theorem 1 gives the sufficient conditions under that the constructed set of measures is a set of martingale measures for the considered evolution of risky assets. Sometimes, we use the abbreviated denotations $\{\omega_1^1,\dots,\omega_n^1\} = \{\omega\}_n^1$, $\{\omega_1^2,\dots,\omega_n^2\} = \{\omega\}_n^2$.

We assume that the set of random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\}) = \alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $(\{\omega\}_n^1;\{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+$, $n = \overline{1,N}$, satisfies the following conditions:

Global Journal of Science Frontier Research (A) Volume XIV Issue I Version I Year 2020

$$P_n^- \times P_n^+((\{\omega\}_n^1; \{\omega\}_n^2) \in \Omega_n^- \times \Omega_n^+, \alpha_n(\{\omega\}_n^1; \{\omega\}_n^2) > 0) =$$

$$P_n(\Omega_n^-) \times P_n(\Omega_n^+), \quad n = \overline{1, N}; \tag{14}$$

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1) \chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \times$$

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) < \infty,$$

$$(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\}) \in \Omega_{n-1} \times \Omega_{n-1},$$

$$(\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \quad n = \overline{1, N}; \tag{15}$$

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1) \chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) = 1,$$

$$(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\}) \in \Omega_{n-1} \times \Omega_{n-1}, \quad n = \overline{1, N}. \tag{16}$$

In the next Lemma 1, we give the sufficient conditions under which the conditions (14) - (16) are valid.

Lemma 1. *Suppose that for* $\Omega_n^a, a = -, +, \ n = \overline{1, N},$ *the representations (5) are true. If the conditions*

$$\inf_{1 \le k \le N_n} P_n^0(A_n^{0,k-} \setminus B_{n,i}^{0,k-}) > 0, \quad i = \overline{1, I_n}, \quad I_n > 1, \quad n = \overline{1, N},$$

$$\inf_{1 \le k \le N_n} P_n^0(A_n^{0,k+} \setminus B_{n,s}^{0,k+}) > 0, \quad s = \overline{1, S_n}, \quad S_n > 1, \quad n = \overline{1, N},$$

$$\inf_{1 \le k \le N_n} P_n^0(B_{n,i}^{0,k-}) > 0, \quad i = \overline{1, I_n}, \quad I_n > 1, \quad n = \overline{1, N},$$

$$\inf_{1 \le k \le N_n} P_n^0(B_{n,s}^{0,k+}) > 0, \quad s = \overline{1, S_n}, \quad S_n > 1, \quad n = \overline{1, N},$$

$$\int\limits_{\Omega_N} \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n) dP_N < \infty, \quad n = \overline{1, N}, \tag{17}$$

are true, then the set of bounded random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2)$, *satisfying the conditions (14) - (16), is a nonempty set.*

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

Proof. Let us put

$$\alpha_n^{i-}(\omega_1^1,\dots,\omega_n^1)=\sum_{k=1}^{N_n}\alpha_{n,k,i}^{-}(\omega_n^1)\chi_{A_n^{0,k-}}(\omega_n^1)\chi_{V_{n-1}^k}(\omega_1^1,\dots,\omega_{n-1}^1),$$

$$\alpha_n^{s+}(\omega_1^2,\dots,\omega_n^2)=\sum_{k=1}^{N_n}\alpha_{n,k,s}^{+}(\omega_n^2)\chi_{A_n^{0,k+}}(\omega_n^2)\chi_{V_{n-1}^k}(\omega_1^2,\dots,\omega_{n-1}^2),$$

where

$$\alpha_{n,k,i}^{-}(\omega_n^1)=(1-\delta_i^n)\frac{\chi_{B_{n,i}^{0,k-}}(\omega_n^1)}{P_n^0(B_{n,i}^{0,k-})}+\delta_i^n\frac{\chi_{A_n^{0,k-}\setminus B_{n,i}^{0,k-}}(\omega_n^1)}{P_n^0(A_n^{0,k-}\setminus B_{n,i}^{0,k-})},$$

$$0<\delta_i^n<1,\quad i=\overline{1,I_n},\quad k=\overline{1,N_n}, \tag{18}$$

$$\alpha_{n,k,s}^{+}(\omega_n^2)=(1-\mu_s^n)\frac{\chi_{B_{n,s}^{0,k+}}(\omega_n^2)}{P_n^0(B_{n,s}^{0,k+})}+\mu_s^n\frac{\chi_{A_n^{0,k+}\setminus B_{n,s}^{0,k+}}(\omega_n^2)}{P_n^0(A_n^{0,k+}\setminus B_{n,s}^{0,k+})},$$

$$0<\mu_s^n<1,\quad s=\overline{1,S_n},\quad k=\overline{1,N_n}. \tag{19}$$

If to introduce the nonnegative set of real numbers

$$\gamma_{i,s}\geq 0,\quad i=\overline{1,I_n},\quad s=\overline{1,S_n},\quad \sum_{i,s=1}^{I_n,S_n}\gamma_{i,s}=1,\quad n=\overline{1,N}, \tag{20}$$

then

$$\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})=$$

$$\sum_{i,s=1}^{I_n,S_n}\gamma_{i,s}\alpha_n^{i-}(\omega_1^1,\dots,\omega_n^1)\alpha_n^{s+}(\omega_1^2,\dots,\omega_n^2),\quad n=\overline{1,N}, \tag{21}$$

satisfies the condition (14) - (16).

Really, due to the Lemma 1 conditions, the random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, are strictly positive by construction. Therefore, the conditions (14) are true.

Due to the boundedness of $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)\leq C,\ n=\overline{1,N},\ 0<C<\infty$, the inequalities

$$\int\limits_{\Omega_n^0\times\Omega_n^0}\chi_{\Omega_n^-}(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1)\chi_{\Omega_n^+}(\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\leq$$

$$C\int\limits_{\Omega_n^0}\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)dP_n^0(\omega_n^1)<\infty,\quad n=\overline{1,N},\tag{22}$$

are true for almost everywhere $(\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}$, $n=\overline{1,N}$, relative to the measure P_{n-1}, owing to the inequalities (17) and Foubini Theorem. This proves the inequality (15). The equality (16) is also satisfied due to the construction of $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$. Lemma 1 is proved.

The values, which the random variables $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2\})$, $n=\overline{1,N}$, constructed in Lemma 1, take, are determined by the values at points $\omega_n^1\in\Omega_n^{0-}$ and $\omega_n^2\in\Omega_n^{0+}$ for all $(\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}$.

On the basis of the set of random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, constructed in Lemma 1, let us introduce into consideration the family of measure $\mu_0(A)$ on the measurable space $\{\Omega_N,\mathcal{F}_N\}$ by the recurrent relations

$$\mu_N^{(\omega_1,\dots,\omega_{N-1})}(A)=\int\limits_{\Omega_N^0\times\Omega_N^0}\chi_{\Omega_N^-}(\omega_1,\dots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\dots,\omega_{N-1},\omega_N^2)\times$$

$$\alpha_N(\{\omega_1,\dots,\omega_{N-1},\omega_N^1\};\{\omega_1,\dots,\omega_{N-1},\omega_N^2\})\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\dots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\dots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\dots,\omega_{N-1},\omega_N^1)}(A)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\dots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\dots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\dots,\omega_{N-1},\omega_N^2)}(A)\right]dP_N^0(\omega_N^1)dP_N^0(\omega_N^2),\tag{23}$$

$$\mu_{n-1}^{(\omega_1,\dots,\omega_{n-1})}(A)=\int\limits_{\Omega_n^0\times\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\dots,\omega_{n-1},\omega_n^1)}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\dots,\omega_{n-1},\omega_n^2)}(A)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2),\quad n=\overline{2,N},\tag{24}$$

$$\mu_0(A)=\int\limits_{\Omega_1^0\times\Omega_1^0}\chi_{\Omega_1^-}(\omega_1^1)\chi_{\Omega_1^+}(\omega_1^2)\alpha_1(\omega_1^1;\omega_1^2)\times$$

$$\left[\frac{\Delta S_1^+(\omega_1^2)}{V_1(\omega_1^1,\omega_1^2)}\mu_1^{(\omega_1^1)}(A)+\frac{\Delta S_1^-(\omega_1^1)}{V_1(\omega_1^1,\omega_1^2)}\mu_1^{(\omega_1^2)}(A)\right]dP_1^0(\omega_1^1)dP_1^0(\omega_1^2),\tag{25}$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

where we put

$$\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)=\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N),\quad A\in\mathcal{F}_N. \tag{26}$$

Lemma 2. *Suppose that the conditions of Lemma 1 are true. For the measure $\mu_0(A)$, $A\in\mathcal{F}_N$, constructed by the recurrent relations (23) - (25), the representation*

$$\mu_0(A)=\int\limits_{\Omega_N}\prod_{n=1}^{N}\psi_n(\omega_1,\ldots,\omega_n)\chi_A(\omega_1,\ldots,\omega_N)\prod_{i=1}^{N}dP_i^0(\omega_i) \tag{27}$$

is true and $\mu_0(\Omega_N)=1$, that is, the measure $\mu_0(A)$ is a probability measure being equivalent to the measure P_N, where we put

$$\psi_n(\omega_1,\ldots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\ldots,\omega_n), \tag{28}$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\int\limits_{\Omega_n^0}\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2),\quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \tag{29}$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\int\limits_{\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1),\quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}. \tag{30}$$

Proof. Due to Lemma 1 conditions, the set of the strictly positive bounded random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, satisfying the conditions (14) - (16), is a non empty set. We prove Lemma 2 by induction down. Let us denote

$$\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)=\chi_A(\omega_1,\ldots,\omega_N). \tag{31}$$

Then,

$$\int\limits_{\Omega_N^0}\psi_N(\omega_1,\ldots,\omega_{N-1},\omega_N)\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)dP_N^0(\omega_N)=$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

$$\int\limits_{\Omega_N^0} \chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N)\psi_N^1(\omega_1,\ldots,\omega_{N-1},\omega_N)\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)dP_N^0(\omega_N)+$$

$$\int\limits_{\Omega_N^0} \chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N)\psi_N^2(\omega_1,\ldots,\omega_{N-1},\omega_N)\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)dP_N^0(\omega_N)=$$

$$\int\limits_{\Omega_N^0} \chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\psi_N^1(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}(A)dP_N^0(\omega_N^1)+$$

$$\int\limits_{\Omega_N^0} \chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\psi_N^2(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}(A)dP_N^0(\omega_N^2). \quad (32)$$

Substituting $\psi_N^1(\omega_1,\ldots,\omega_{N-1},\omega_N^1),\ \psi_N^2(\omega_1,\ldots,\omega_{N-1},\omega_N^2)$ into (32), we obtain

$$\int\limits_{\Omega_N^0} \psi_N(\omega_1,\ldots,\omega_{N-1},\omega_N)\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)dP_N^0(\omega_N)=$$

$$\int\limits_{\Omega_N^0\times\Omega_N^0} \chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\times$$

$$\alpha_N(\{\omega_1,\ldots,\omega_{N-1},\omega_N^1\};\{\omega_1,\ldots,\omega_{N-1},\omega_N^2\})\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}(A)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}(A)\right]dP_N^0(\omega_N^1)dP_N^0(\omega_N^2)=$$

$$\mu_{N-1}^{(\omega_1,\ldots,\omega_{N-1})}(A). \quad (33)$$

Suppose that we are proved that

$$\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n)}(A)=$$

$$\int\limits_{\prod_{i=n+1}^{N}\Omega_i^0} \prod_{i=n+1}^{N}\psi_i(\omega_1,\ldots,\omega_i)\chi_A(\omega_1,\ldots,\omega_N)\prod_{i=n+1}^{N}dP_i^0(\omega_i). \quad (34)$$

Let us calculate

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n)}(A)dP_n^0(\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n)\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n)\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n)}(A)dP_n^0(\omega_n)+$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n)\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n)\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n)}(A)dP_n^0(\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}(A)dP_n^0(\omega_n^1)+$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}(A)dP_n^0(\omega_n^2). \qquad (35)$$

Substituting $\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n^1), \psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)$ into (35), we obtain

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n)}(A)dP_n^0(\omega_n) =$$

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)\chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)\times$$

$$\alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}(A)\right] dP_n^0(\omega_n^1)dP_n^0(\omega_n^2). \qquad (36)$$

From the recurrent relations (23) - (25), we have

$$\mu_{n-1}^{(\omega_1, \ldots, \omega_{n-1})}(A) = \int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)\chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)\times$$

$$\alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n^1\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}\mu_n^{(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}(A)+\right.$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\dots,\omega_{n-1},\omega_n^2)}(A)\Big] dP_n^0(\omega_n^1)dP_n^0(\omega_n^2), \quad n=\overline{1,N}. \tag{37}$$

From the last equality, we have

$$\mu_{n-1}^{(\omega_1,\dots,\omega_{n-1})}(A)=\int\limits_{\Omega_n^0}\psi_n(\omega_1,\dots,\omega_{n-1},\omega_n)\mu_n^{(\omega_1,\dots,\omega_{n-1},\omega_n)}(A)dP_n^0(\omega_n), \quad n=\overline{1,N}. \tag{38}$$

Substituting into (38) the induction supposition (34), we obtain

$$\mu_{n-1}^{(\omega_1,\dots,\omega_{n-1})}(A)=$$

$$\int\limits_{\prod_{i=n}^N\Omega_i^0}\prod_{i=n}^N\psi_i(\omega_1,\dots,\omega_i)\chi_A(\omega_1,\dots,\omega_N)\prod_{i=n}^N dP_i^0(\omega_i). \tag{39}$$

To prove that $\mu_0(\Omega_N)=1$, let us prove the equality

$$\int\limits_{\Omega_n^0}\psi_n(\omega_1,\dots,\omega_n)dP_n^0(\omega_n)=1, \quad (\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}, \qquad n=\overline{1,N}. \tag{40}$$

We have

$$\int\limits_{\Omega_n^0}\psi_n(\omega_1,\dots,\omega_n)dP_n^0(\omega_n)=$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n^2\})\times$$

$$\Big[\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\Big] dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n^2\})dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=1. \tag{41}$$

The last equality follows from the fact that the set of random values $\alpha_n(\{\omega_1\}_n^1;\{\omega_1\}_n^2)$, $n=\overline{1,N}$, satisfies the condition (16). The equalities (40) proves that every measure (27), defined by the set of random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, $n=\overline{1,N}$, satisfying the conditions (14), (16), is a probability measure being equivalent to the measure P_N.

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

This proves Lemma 2

Note 1. *Due to the equality (40), the contraction of measure* $\mu_0(A), A \in \mathcal{F}_N$, *on the* σ*-algebra* $\mathcal{F}_n$ *of filtration we denote by* μ_0^n. *If* A *belongs to the* σ*-algebra* $\mathcal{F}_n$ *of filtration, then* $A = B \times \prod_{i=n+1}^{N} \Omega_i^0$, *where* B *belongs to the* σ*-algebra* $\mathcal{F}_n$ *of the measurable space* $\{\Omega_n, \mathcal{F}_n\}$, *therefore, for this contraction we obtain the formula*

$$\mu_0^n(A) = \int_{\Omega_n} \prod_{i=1}^{n} \psi_i(\omega_1, \ldots, \omega_i)\chi_B(\omega_1, \ldots, \omega_n) \prod_{i=1}^{n} dP_i^0(\omega_i), \quad B \in \mathcal{F}_n. \tag{42}$$

Further, we also use the probability spaces $\{\Omega_n, \mathcal{F}_n, \mu_0^n\}$, $n = \overline{1, N}$, *where under the measure* $\mu_0^n(B), B \in \mathcal{F}_n$, *we understand the measure, given by the formula*

$$\mu_0^n(B) = \int_{\Omega_n} \prod_{i=1}^{n} \psi_i(\omega_1, \ldots, \omega_i)\chi_B(\omega_1, \ldots, \omega_n) \prod_{i=1}^{n} dP_i^0(\omega_i), \quad B \in \mathcal{F}_n. \tag{43}$$

Note 2. *Assume that for* $\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\})$, *constructed in Lemma 1, the inequalities*

$$0 < c_n \leq \alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \leq C_n < \infty,$$

are true. Suppose that the conditions

$$\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n) \leq B_n < \infty, \quad n = \overline{1, N}, \tag{44}$$

are valid, where c_n, C_n, B_n *are constant, then the set of equivalent measures to the measure* P_N, *described in Lemma 2, is nonempty one.*

Proof. Due to Lemma 2 conditions, the equality (14) is true. Further,

$$\int_{\Omega_n^0} \int_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1)\chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) \times$$

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) \leq B_n,$$

$$(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\}) \in \Omega_{n-1} \times \Omega_{n-1}, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1},$$

$$\int_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^-}(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1)\chi_{\Omega_n^+}(\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2) \times$$

$$\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\}) dP_n^0(\omega_n^1) dP_n^0(\omega_n^2) = 1,$$

$$(\{\omega_1^1, \ldots, \omega_{n-1}^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2\}) \in \Omega_{n-1} \times \Omega_{n-1}. \tag{45}$$

The last inequality and the equality (45) means that the conditions (14) - (16) are satisfied. Note 2 is proved.

For the nonnegative random value $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, given on the measurable space $\{\Omega_n^-\times\Omega_n^+,\mathcal{F}_n^-\times\mathcal{F}_n^+\}$, $\mathcal{F}_n^-=\mathcal{F}_n\cap\Omega_n^-$, $\mathcal{F}_n^+=\mathcal{F}_n\cap\Omega_n^+, n=\overline{1,N}$, let us define the integral for the nonnegative random value $f_N(\omega_1,\dots,\omega_N)$ relative to the measure $\mu_0(A)$ using the recurrent relations

$$\mu_{n-1}^{f_N}(\omega_1,\dots,\omega_{n-1})=$$

$$\int\limits_{\Omega_n^0\times\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{f_N}(\omega_1,\dots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{f_N}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2),\quad n=\overline{1,N},\quad (46)$$

$$\mu_{N-1}^{f_N}(\omega_1,\dots,\omega_{N-1})=\int\limits_{\Omega_N^0\times\Omega_N^0}\chi_{\Omega_N^-}(\omega_1,\dots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\dots,\omega_{N-1},\omega_N^2)\times$$

$$\alpha_N(\{\omega_1,\dots,\omega_{N-1},\omega_N^1\};\{\omega_1,\dots,\omega_{N-1},\omega_N^2\})\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\dots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\dots,\omega_{N-1},\omega_N^1,\omega_N^2)}f_N(\omega_1,\dots,\omega_{N-1},\omega_N^1)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\dots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\dots,\omega_{N-1},\omega_N^1,\omega_N^2)}f_N(\omega_1,\dots,\omega_{N-1},\omega_N^2)\right]dP_N^0(\omega_N^1)dP_N^0(\omega_N^2).\quad (47)$$

From the formula (27) of Lemma 2, it follows that

$$E^{\mu_0}f_N=\int\limits_{\Omega_N}\prod_{n=1}^N\psi_n(\omega_1,\dots,\omega_n)f_N(\omega_1,\dots,\omega_{N-1},\omega_N)\prod_{i=1}^N dP_i^0(\omega_i)\quad (48)$$

for every nonnegative $\mathcal{F}_N$-measurable random value $f_N(\omega_1,\dots,\omega_{N-1},\omega_N)$.

Theorem 1. *Suppose that the conditions of Lemma 1 are true. Then, the set of nonnegative random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions*

$$E^{\mu_0}|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|=$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i(\omega_1,\dots,\omega_i)|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i)<\infty,\quad n=\overline{1,N},\quad (49)$$

is a nonempty one and the convex linear span of the set of measures (27), defined by the random values $\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})$, $n=\overline{1,N}$, *satisfying the conditions (49), is a set of martingale measures being equivalent to the measure* P_N.

Proof. Taking into account the equality (40), the conditions (49) can be written in the form

$$\int\limits_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^{N}dP_i^0(\omega_i)=$$

$$\int\limits_{\Omega_n}\prod_{i=1}^{n}\psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^{n}dP_i^0(\omega_i)=$$

$$2\int\limits_{\Omega_{n-1}}\prod_{i=1}^{n-1}\psi_i(\omega_1,\ldots,\omega_i)\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\times$$

$$dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\prod_{i=1}^{n-1}dP_i^0(\omega_i),\quad n=\overline{1,N}. \tag{50}$$

Since the conditions of Lemma 1 are true, then the the set of bounded random values $\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})$, $n=\overline{1,N}$, constructed in Lemma 1, satisfy the conditions (14) - (16).

From the equality (50) for the set of bounded random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, satisfying the conditions (14) - (16), we obtain the inequality

$$\int\limits_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^{N}dP_i^0(\omega_i)\leq$$

$$C\int\limits_{\Omega_N}\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)dP_N<\infty,\quad n=\overline{1,N}, \tag{51}$$

for a certain constant $0<C<\infty$. This proves that the set of nonnegative random values $\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})$, $n=\overline{1,N}$, satisfying the conditions (49), is a non empty set.

Let us prove that

$$\int\limits_{\Omega_n^0}\psi_n(\omega_1,\ldots,\omega_n)\Delta S_n(\omega_1,\ldots,\omega_n)dP_n^0(\omega_n)=0,$$

$$(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1},\quad n=\overline{1,N}. \tag{52}$$

Really,

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1,\dots,\omega_n)\Delta S_n(\omega_1,\dots,\omega_n)dP_n^0(\omega_n) =$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n^2\})\times$$

$$\Big[-\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)+$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Big]\,dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=0, \qquad (53)$$

due to the condition (15).

To complete the proof of Theorem 1, let A belongs to the filtration $\mathcal{F}_{n-1}$, then $A = B\times\prod_{i=n}^{N}\Omega_i^0$, where B belongs to the σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1},\mathcal{F}_{n-1}\}$. Taking into account the equality (41), (53), we have, due to Foubini theorem,

$$\int\limits_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\dots,\omega_i)\chi_A(\omega_1,\dots,\omega_N)\Delta S_n(\omega_1,\dots,\omega_n)\prod_{i=1}^{N}dP_i^0(\omega_i) =$$

$$\int\limits_{\Omega_n}\prod_{i=1}^{n}\psi_i(\omega_1,\dots,\omega_i)\chi_B(\omega_1,\dots,\omega_{n-1})\Delta S_n(\omega_1,\dots,\omega_n)\prod_{i=1}^{n}dP_i^0(\omega_i) =$$

$$\int\limits_{\Omega_{n-1}}\prod_{i=1}^{n-1}\psi_i(\omega_1,\dots,\omega_i)\chi_B(\omega_1,\dots,\omega_{n-1})\prod_{i=1}^{n-1}dP_i^0(\omega_i)\times$$

$$\int\limits_{\Omega_n^0}\psi_n(\omega_1,\dots,\omega_n)\Delta S_n(\omega_1,\dots,\omega_n)dP_n^0(\omega_n)=0. \qquad (54)$$

The last means that $E^{\mu_0}\{S_n(\omega_1,\dots,\omega_n)|\mathcal{F}_{n-1}\} = S_{n-1}(\omega_1,\dots,\omega_{n-1})$. Since every measure, belonging to the convex linear span of the measures considered above, is a finite sum of such measures, then it is a martingale measure being equivalent to the measure P_N. Theorem 1 is proved.

Our aim is to describe this convex span of martingale measures in particular cases.

IV. Inequalities for the Nonnegative Random Values

In this section, we prove some inequalities, which will be very useful for to prove optional decomposition for super-martingale relative to all martingale measures. First, we prove an integral inequality for a nonnegative random variable under the fulfillment of the inequality for this nonnegative random variable with respect to the constructed family of measures $\mu_0(A)$. Further, using this integral inequality for the non-negative random variable, a pointwise system of inequalities is obtained for this non-negative random variable for a particular case. After that, the pointwise system of inequalities is obtained for the non-negative random variable in the general case. Then, using the resulting pointwise system of inequalities, the inequality is established for this non-negative random variable whose right-hand side is such that its conditional mathematical expectation is equal to one.

Definition 1. Let $\{\Omega_1, \mathcal{F}_1\}$ be a measurable space. The decomposition $A_{n,k}$, $n, k = \overline{1, \infty}$, of the space Ω_1 we call exhaustive one, if the following conditions are valid:

1) $A_{n,k} \in \mathcal{F}_1$, $A_{n,k} \cap A_{n,s} = \emptyset$, $k \neq s$, $\bigcup\limits_{k=1}^{\infty} A_{n,k} = \Omega_1$, $n = \overline{1, \infty}$;

2) the $(n+1)$-th decomposition is a sub-decomposition of the n-th one, that is, for every j, $A_{n+1,j} \subseteq A_{n,k}$ for a certain $k = k(j)$;

3) the minimal σ-algebra containing all $A_{n,k}$, $n, k = \overline{1, \infty}$, coincides with $\mathcal{F}_1$.

Lemma 3. Let $\{\Omega_1, \mathcal{F}_1\}$ be a measurable space with a complete separable metric space Ω_1 and Borel σ-algebra $\mathcal{F}_1$ on it. Then, $\{\Omega_1, \mathcal{F}_1\}$ has an exhaustive decomposition.

The proof of Lemma 3 see, for example, in [15], [16].

For the proof of integral inequalities, we cannot require the fulfillment for the random values $\alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\})$, $n = \overline{1, N}$, the condition (15) in the Lemma 4.

Lemma 4. Suppose that Ω_n^0 is a complete separable metric space, $\mathcal{F}_n^0$ is a corresponding Borel σ-algebra on Ω_n^0, $n = \overline{1, N}$, and the conditions of Lemma 1 are valid. If, on the probability space $\{\Omega_{n-1}, \mathcal{F}_{n-1}, \mu_0^{n-1}\}$, for each $B \in \mathcal{F}_{n-1}$, $\mu_0^{n-1}(B) > 0$, the nonnegative random value $f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$ satisfies the inequality

$$\frac{1}{\mu_0^{n-1}(B)} \int\limits_B \int\limits_{\Omega_n^0} \prod_{i=1}^n \psi_i(\omega_1, \ldots, \omega_i) f_n(\omega_1, \ldots, \omega_n) \prod_{i=1}^n dP_i^0(\omega_i) \leq 1, \quad B \in \mathcal{F}_{n-1}, \quad (55)$$

then the inequality

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_n) f_n(\omega_1, \ldots, \omega_n) dP_n^0(\omega_n) \leq 1,$$

$$\{\omega_1, \ldots, \omega_{n-1}\} \in \Omega_{n-1}, \; n = \overline{1, N}, \quad (56)$$

is true almost everywhere relative to the measure P_{n-1}.

Proof. The metric space Ω_{n-1} is a complete separable metric space with the metric $\rho(x, y) = \sum\limits_{i=1}^{n-1} \rho_i(x_i, y_i)$, where $x = (x_1, \ldots, x_{n-1}), y = (y_1, \ldots, y_{n-1}) \in \Omega_{n-1}$, $(x_i, y_i) \in \Omega_i^0$, $\rho_i(x_i, y_i)$ is a metric in Ω_i^0. This means that the metric space Ω_{n-1} has an exhaustive decomposition$\{B_{mk}\}_{m,k=1}^{\infty}$. Suppose that $(\omega_1, \ldots, \omega_{n-1}) \in B_{m,k}$ for a certain k, depending on m, and there exists an infinite number of m for which $\mu_0^{n-1}(B_{m,k}) > 0$. On the probability space $\{\Omega_{n-1}, \mathcal{F}_{n-1}, \mu_0^{n-1}\}$, for every integrable finite valued random value $\varphi_{n-1}(\omega_1, \ldots, \omega_{n-1})$ the sequence $E^{\mu_0^{n-1}}\{\varphi_{n-1}(\omega_1, \ldots, \omega_{n-1})|\bar{\mathcal{F}}_m\}$ converges to $\varphi_{n-1}(\omega_1, \ldots, \omega_{n-1})$ with probability one, as $m \to \infty$, since it is a regular martingale. Here, we denoted $\bar{\mathcal{F}}_m$ the σ-algebra, generated by the sets $B_{m,k}, k = \overline{1, \infty}$.

It is evident that for those $B_{m,k}$, for which $\mu_0^{n-1}(B_{m,k}) \neq 0$,

$$E^{\mu_0^{n-1}}\{\varphi_{n-1}(\omega_1, \ldots, \omega_n)|\bar{\mathcal{F}}_m\} =$$

$$\frac{\int\limits_{B_{m,k}} \varphi_{n-1}(\omega_1, \ldots, \omega_{n-1}) d\mu_0^{n-1}}{\mu_0^{n-1}(B_{m,k})}, \quad (\omega_1, \ldots, \omega_n) \in B_{m,k}. \tag{57}$$

Denote $A_m = A_m(\omega_1, \ldots, \omega_{n-1})$ those sets $B_{m,k}$ for which $(\omega_1, \ldots, \omega_n) \in B_{m,k}$ for a certain k, depending on m, and $\mu_0^{n-1}(A_m) > 0$. Then, for every integrable finite valued $\varphi_{n-1}(\omega_1, \ldots, \omega_{n-1})$

$$\lim_{m \to \infty} \frac{\int\limits_{A_m} \varphi_{n-1}(\omega_1, \ldots, \omega_{n-1}) d\mu_0^{n-1}}{\mu_0^{n-1}(A_m)} = \varphi_{n-1}(\omega_1, \ldots, \omega_{n-1}) \tag{58}$$

almost everywhere relative to the measure μ_0^{n-1}. If to put

$$\varphi_{n-1}(\omega_1, \ldots, \omega_{n-1}) =$$

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \ldots, \omega_n) f_n(\omega_1, \ldots, \omega_n) dP_n^0(\omega_n), \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \tag{59}$$

then we obtain the proof of Lemma 4.

In Theorem 2, we assume that for $\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$, $n = \overline{1, N}$, the representation

$$\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$S_{n-1}(\omega_1, \ldots, \omega_{n-1}) a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) \eta_n(\omega_n) =$$

$$d_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) \eta_n(\omega_n), \quad n = \overline{1, N}, \quad S_0 > 0, \tag{60}$$

is true, where the random values $d_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$, $a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$, $\eta_n(\omega_n)$, $n = \overline{1, N}$, given on the probability space $\{\Omega_n, \mathcal{F}_n, P_n\}$, satisfy the conditions

$$0 < a_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) \leq 1, \quad 1 + a_n(\omega_1, \quad \ldots, \omega_{n-1}, \omega_n) \eta_n(\omega_n) > 0,$$

$$d_n(\omega_1, \ldots, \omega_{n-1}, \omega_n) > 0, \quad P_n^0(\eta_n(\omega_n) > 0) > 0, \quad P_n^0(\eta_n(\omega_n) < 0) > 0. \tag{61}$$

From these conditions we obtain $\Omega_n^- = \Omega_n^{0-} \times \Omega_{n-1}$, $\Omega_n^+ = \Omega_n^{0+} \times \Omega_{n-1}$, where $\Omega_n^{0-} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) \leq 0\}$, $\Omega_n^{0+} = \{\omega_n \in \Omega_n^0, \eta_n(\omega_n) > 0\}$.

From the suppositions above, it follows that $P_n^0(\Omega_n^{0-}) > 0$, $P_n^0(\Omega_n^{0+}) > 0$. The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$.

Theorem 2. Let Ω_i^0 be a complete separable metric space and let $\mathcal{F}_i^0$ be a Borell σ-algebra on Ω_i^0, $i=\overline{1,N}$. Suppose that for $\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)$, $n=\overline{1,N}$, the representation (60) is valid and Lemma 4 conditions are true. Then, for the non-negative random value $f_n(\omega_1,\ldots,\omega_{n-1},\omega_n)$ the inequalities

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\chi_{\Omega_n^{0+}}(\omega_n^2)\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]\leq 1,$$

$$(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1},\quad (\omega_n^1,\omega_n^2)\in\Omega_n^{0-}\times\Omega_n^{0+},\quad n=\overline{1,N}, \tag{62}$$

are true almost everywhere relative to the measure $P_{n-1}\times P_n^{0-}\times P_n^{0+}$ on the measurable space $\{\Omega_{n-1}\times\Omega_n^{0-}\times\Omega_n^{0+},\mathcal{F}_{n-1}\times\mathcal{F}_n^{0-}\times\mathcal{F}_n^{0+}\}$.

Proof. Under Theorem 2 conditions, the set of martingale measures is a nonempty one. Due to the equality (40), we obtain

$$\int\limits_{\Omega_N}\prod_{i=1}^N\psi_i(\omega_1,\ldots,\omega_i)f_n(\omega_1,\ldots,\omega_n)\prod_{i=1}^N dP_i^0(\omega_i)=$$

$$\int\limits_{\Omega_n}\prod_{i=1}^n\psi_i(\omega_1,\ldots,\omega_i)f_n(\omega_1,\ldots,\omega_n)\prod_{i=1}^n dP_i^0(\omega_i). \tag{63}$$

Further,

$$\int\limits_{\Omega_n^0}\psi_n(\omega_1,\ldots,\omega_n)f_n(\omega_1,\ldots,\omega_n)dP_n^0(\omega_n)=$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2). \tag{64}$$

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_n^1)=\chi_{\Omega_{n-1}}(\omega_1,\ldots,\omega_{n-1})\chi_{\Omega_n^{0-}}(\omega_n^1),$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_n^2)=\chi_{\Omega_{n-1}}(\omega_1,\ldots,\omega_{n-1})\chi_{\Omega_n^{0+}}(\omega_n^2). \tag{65}$$

Due to Lemma 4, the inequality

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^{0-}}(\omega_n^1)\chi_{\Omega_n^{0+}}(\omega_n^2)\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\leq 1, \tag{66}$$

is true almost everywhere relative to the measure P_{n-1} on the σ-algebra $\mathcal{F}_{n-1}$. Let us put

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})=\alpha_n(\omega_n^1;\omega_n^2), \tag{67}$$

where $\alpha_n(\omega_n^1;\omega_n^2)$ satisfy the condition

$$\int\limits_{\Omega_n^{0-}}\int\limits_{\Omega_n^{0+}}\alpha_n(\omega_n^1;\omega_n^2)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=1. \tag{68}$$

Since, on the probability space $\{\Omega_n^{0-}\times\Omega_n^{0+},\mathcal{F}_n^{0-}\times\mathcal{F}_n^{0+},P_n^{0-}\times P_n^{0+}\}$, there exists an exhaustive decomposition $\{A_{m,k}\}_{m,k=1}^{\infty}$, let us put

$$\alpha_n(\omega_n^1;\omega_n^2)=(1-\varepsilon)\frac{\chi_{A_{m,k}}(\omega_n^1;\omega_n^2)}{\mu_n(A_{m,k})}+\varepsilon\frac{\chi_{\Omega_n^{0-}\times\Omega_n^{0+}\setminus A_{m,k}}(\omega_n^1;\omega_n^2)}{\mu_n(\Omega_n^{0-}\times\Omega_n^{0+}\setminus A_{m,k})}, \tag{69}$$

where $\mu_n(A)=[P_n^{0-}\times P_n^{0+}](A),\ A\in\mathcal{F}_n^{0-}\times\mathcal{F}_n^{0+}$, and we assume that $\mu_n(A_{m,k})>0$, $\mu_n(\Omega_n^{0-}\times\Omega_n^{0+}\setminus A_{m,k})>0$. Suppose that $(\omega_n^1;\omega_n^2)\in A_{m,k}$ and $\mu_n(A_{m,k})>0$ for the infinite number of m and k. Then,

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0}\chi_{\Omega_n^{0-}}(\omega_n^1)\chi_{\Omega_n^{0+}}(\omega_n^2)\left[(1-\varepsilon)\frac{\chi_{A_{m,k}}(\omega_n^1;\omega_n^2)}{\mu_n(A_{m,k})}+\varepsilon\frac{\chi_{\Omega_n^{0-}\times\Omega_n^{0+}\setminus A_{m,k}}(\omega_n^1;\omega_n^2)}{\mu_n(\Omega_n^{0-}\times\Omega_n^{0+}\setminus A_{m,k})}\right]\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\leq 1. \tag{70}$$

Going to the limit as $m,k\to\infty$ and then as $\varepsilon\to 0$, we obtain the inequality

$$\chi_{\Omega_n^{0,-}}(\omega_n^1)\chi_{\Omega_n^{0,+}}(\omega_n^2)\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]\leq 1,\quad(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \tag{71}$$

which is valid almost everywhere relative to the measure μ_n. Theorem 2 is proved.

Lemma 5. Let Ω_n^0 be a complete separable metric space and let $\mathcal{F}_n^0$ be a Borel σ-algebra on Ω_n^0, $n=\overline{1,N}$. If the conditions of Lemma 4 are true, then the inequality

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]\leq 1,\quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1},\quad (72)$$

is valid almost everywhere relative to the measure $P_{n-1}\times[P_n^0\times P_n^0]$ on the measurable space $\{\Omega_{n-1}\times\Omega_n^0\times\Omega_n^0,\mathcal{F}_{n-1}\times\mathcal{F}_n^0\times\mathcal{F}_n^0\}$.

Proof. Due to the conditions for Ω_n^a, $a=-,+$, the representation

$$\Omega_n^a=\bigcup_{k=1}^{N_n}[A_n^{0,ka}\times V_{n-1}^k] \quad (73)$$

is true. Owing to Lemma 5 conditions, there exists an exhaustive decomposition D_{mi}^n, $m,i=\overline{1,\infty}$, such that $\bigcup\limits_{i=1}^{\infty}D_{mi}^n=\Omega_n^0$, $m=\overline{1,\infty}$. Let us denote $A_n^{0,ka}\cap D_{mi}^n=E_{mi}^{nka}$. It is evident that E_{mi}^{nka} forms an exhaustive decomposition of sets $A_n^{0,ka}$, $n=\overline{1,N}$, $k=\overline{1,\infty}$, $a=-,+$, correspondingly. Due to Lemma 4, the inequality

$$\int\limits_{\Omega_n^0}\psi_n(\omega_1,\ldots,\omega_n)f_n(\omega_1,\ldots,\omega_n)dP_n^0(\omega_n)\leq 1,\quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \quad (74)$$

is true almost everywhere relative to the measure P_{n-1}. The equality

$$\int\limits_{\Omega_n^0}\psi_n(\omega_1,\ldots,\omega_n)f_n(\omega_1,\ldots,\omega_n)dP_n^0(\omega_n)=$$

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0}\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) \quad (75)$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

is valid. From the equality (75) and Lemma 4, the inequality

$$\int\limits_{\Omega_n^0}\int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)\leq 1, \qquad (76)$$

is true almost everywhere relative to the measure P_{n-1} on the σ-algebra $\mathcal{F}_{n-1}$. Let us put

$$\alpha_n^{r,s-}(\omega_1^1,\ldots,\omega_n^1)=\sum_{k=1}^{N_n}\alpha_{n,k,r,s}^-(\omega_n^1)\chi_{A_n^{0,k-}}(\omega_n^1)\chi_{V_{n-1}^k}(\omega_1^1,\ldots,\omega_{n-1}^1),$$

$$\alpha_n^{m,i+}(\omega_1^2,\ldots,\omega_n^2)=\sum_{k=1}^{N_n}\alpha_{n,k,m,i}^+(\omega_n^2)\chi_{A_n^{0,k+}}(\omega_n^2)\chi_{V_{n-1}^k}(\omega_1^2,\ldots,\omega_{n-1}^2),$$

$$\alpha_n^{r,s,m,i}(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})=\alpha_n^{r,s-}(\omega_1^1,\ldots,\omega_n^1)\alpha_n^{m,i+}(\omega_1^2,\ldots,\omega_n^2), \qquad (77)$$

where

$$\alpha_{n,k,r,s}^-(\omega_n^1)=\left[(1-\delta)\frac{\chi_{E_{rs}^{nk-}}(\omega_n^1)}{P_n^0(E_{rs}^{nk-})}+\delta\frac{\chi_{A_n^{0k-}\setminus E_{rs}^{nk-}}(\omega_n^1)}{P_n^0(A_n^{0k-}\setminus E_{rs}^{nk-})}\right],$$

$$\alpha_{n,k,m,i}^+(\omega_n^2)=\left[(1-\delta)\frac{\chi_{E_{mi}^{nk+}}(\omega_n^2)}{P_n^0(E_{mi}^{nk+})}+\delta\frac{\chi_{A_n^{0k+}\setminus E_{mi}^{nk+}}(\omega_n^2)}{P_n^0(A_n^{0k+}\setminus E_{mi}^{nk+})}\right], \quad 0<\delta<1. \qquad (78)$$

In the formulas (78), we assume that the inequalities

$$P_n^0(E_{rs}^{nk-})>0,\ P_n^0(A_n^{0k-}\setminus E_{rs}^{nk-})>0,\ P_n^0(E_{mi}^{nk+})>0,\ P_n^0(A_n^{0k+}\setminus E_{mi}^{nk+})>0, \quad (79)$$

are true. Let us consider

$$\alpha_n^{r,s,m,i}(\{\omega_1,\ldots,\omega_{n-1},\omega_{n-1}^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})=$$

$$\alpha_n^{r,s-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\alpha_n^{m,i+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2). \qquad (80)$$

Suppose that $(\omega_1,\ldots,\omega_{n-1})\in V_{n-1}^k$ for a certain k. Then,

$$\alpha_n^{r,s,m,i}(\{\omega_1,\ldots,\omega_{n-1},\omega_{n-1}^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\}) =$$

$$\left[(1-\delta)\frac{\chi_{E_{rs}^{nk-}}(\omega_n^1)}{P_n^0(E_{rs}^{nk-})}+\delta\frac{\chi_{A_n^{0k-}\setminus E_{rs}^{nk-}}(\omega_n^1)}{P_n^0(A_n^{0k-}\setminus E_{rs}^{nk-})}\right]\times$$

$$\left[(1-\delta)\frac{\chi_{E_{mi}^{nk+}}(\omega_n^2)}{P_n^0(E_{mi}^{nk+})}+\delta\frac{\chi_{A_n^{0k+}\setminus E_{mi}^{nk+}}(\omega_n^2)}{P_n^0(A_n^{0k+}\setminus E_{mi}^{nk+})}\right]. \quad (81)$$

We assume that the point $(\omega_n^1,\omega_n^2) \in E_{rs}^{nk-}\times E_{mi}^{nk+}$ for the infinite number of r,s and m,i , where $P_n^0(E_{rs}^{nk-})>0,\ P_n^0(E_{mi}^{nk+})>0$.

Substituting (81) into (76) and going to the limit as $m,k\to\infty$ $r,s\to\infty$ and then as $\delta\to 0$, we obtain the needed inequality. Lemma 5 is proved.

Theorem 3. Suppose that the conditions of Theorem 2 are true. If for a certain $\omega_n^1\in\Omega_n^{0-}$ *and* $\omega_n^2\in\Omega_n^{0+}$ *the inequalities*

$$\sup_{(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}}\frac{1}{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}<\infty,$$

$$\sup_{(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}}\frac{1}{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}<\infty, \quad n=\overline{1,N}, \quad (82)$$

are true, then the nonnegative random values $f_n(\omega_1,\ldots,\omega_{n-1},\omega_n),\ n=\overline{1,N}$, *satisfy the inequalities*

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n)\leq$$

$$(1+\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)), \quad n=\overline{1,N}, \quad (83)$$

where $\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})$ *is a bounded* $\mathcal{F}_{n-1}$*-measurable random value.*

Proof. From the inequality (71), it follows the inequality

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\leq$$

$$1+\frac{1-f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}\Delta S_n^+(\omega_1,\quad\ldots,\omega_{n-1},\omega_n^2),\ \omega_n^1\in\Omega_n^{0-},\ \omega_n^2\in\Omega_n^{0+}. \quad (84)$$

Let us define

$$\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})=\inf_{\{\omega_n^1,\eta_n^-(\omega_n^1)>0\}}\frac{1-f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}, \quad (85)$$

then, taking into account the inequality (84), we obtain the inequality

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\leq 1+\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2). \quad (86)$$

From the definition of $\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})$, we obtain the inequality

$$f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1) \le 1-\gamma_{n-1}(\omega_1,\dots,\omega_{n-1})\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1). \quad (87)$$

The inequalities (86), (87) give the inequality

$$f_n(\omega_1,\dots,\omega_{n-1},\omega_n) \le 1+\gamma_{n-1}(\omega_1,\dots,\omega_{n-1})\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n). \quad (88)$$

Let us prove the boundedness of $\gamma_{n-1}(\omega_1,\dots,\omega_{n-1})$. From the inequalities (86), (87) we obtain

$$\frac{1}{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)} \ge$$

$$\gamma_{n-1}(\omega_1,\dots,\omega_{n-1}) \ge -\frac{1}{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}. \quad (89)$$

Due to Theorem 3 conditions, we obtain the boundedness of $\gamma_{n-1}(\omega_1,\dots,\omega_{n-1})$. The $\mathcal{F}_{n-1}$ measurability of the random value $\gamma_{n-1}(\omega_1,\dots,\omega_{n-1})$ follows from the fact that Ω_n^0 is separable metric space and infimum is reached on the countable set, which is dense in Ω_n^0. Theorem 3 is proved.

Theorem 4. Let the conditions of Lemma 5 be valid. If there exist $\omega_n^1 \in A_n^{0k-}$, $\omega_n^2 \in A_n^{0k+}$, and the real numbers a_k, b_k, $k=\overline{1,N_n}$, such that

$$\sup_{(\omega_1,\dots,\omega_{n-1})\in V_{n-1}^k} \frac{1}{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)} = a_k^n < \infty,$$

$$\sup_{(\omega_1,\dots,\omega_{n-1})\in V_{n-1}^k} \frac{1}{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)} = b_k^n < \infty, \quad k=\overline{1,N_n}, \quad n=\overline{1,N},$$

$$\max_{1\le n\le N} \sup_{1\le k\le N_n} \max\{a_k^n, b_k^n\} < \infty, \quad (90)$$

then there exists a bounded $\mathcal{F}_{n-1}$-measurable random value $\gamma_{n-1}(\omega_1,\dots,\omega_{n-1})$ such that the inequalities

$$f_n(\omega_1,\dots,\omega_{n-1},\omega_n)) \le$$

$$(1+\gamma_{n-1}(\omega_1,\dots,\omega_{n-1})\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)), \quad n=\overline{1,N}, \quad (91)$$

are true.

Proof. For $\omega_n^1 \in A_n^{0k-}$, $\omega_n^2 \in A_n^{0k+}$ and $(\omega_1,\dots,\omega_{n-1}) \in V_{n-1}^k$, we have that $(\omega_1,\dots,\omega_{n-1},\omega_n^1) \in \Omega_n^-$, $(\omega_1,\dots,\omega_{n-1},\omega_n^2) \in \Omega_n^+$. Then, from the inequality (72), we obtain the inequality

$$\left[\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} f_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)} f_n(\omega_1,\dots,\omega_{n-1},\omega_n^2)\right] \le 1. \quad (92)$$

From the inequality (92), it follows the inequality

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2) \leq 1 + \frac{1 - f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2). \quad (93)$$

Let us define

$$\gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1}) =$$

$$\inf_{\{\omega_n^1 \in A_n^{0,k-}\}} \frac{1 - f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in V_{n-1}^k, \quad (94)$$

then, taking into account the inequality (93), we have the inequality

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2) \leq 1 + \gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1})\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2). \quad (95)$$

From the definition of $\gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1})$, we obtain the inequality

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1) \leq 1 - \gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1})\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1). \quad (96)$$

The inequalities (95), (96) give the inequality

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n) \leq 1 + \gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1})\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n). \quad (97)$$

Let us prove the boundedness of $\gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1})$. From the inequalities (95), (96), we obtain the inequalities

$$a_k^n = \sup_{(\omega_1,\ldots,\omega_{n-1})\in V_{n-1}^k} \frac{1}{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)} \geq$$

$$\gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1}) \geq -\sup_{(\omega_1,\ldots,\omega_{n-1})\in V_{n-1}^k} \frac{1}{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)} = -b_k^n. \quad (98)$$

From this, it follows the boundedness of $\gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1})$. The $\mathcal{F}_{n-1}$ measurability of the random value $\gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1})$ follows from the fact that Ω_n^0 is separable metric space and infimum is reached on the countable set, which is dense in Ω_n^0. To complete the proof of Theorem 4, let us put

$$\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1}) = \sum_{k=1}^{N_n} \chi_{V_{n-1}^k}((\omega_1,\ldots,\omega_{n-1})\gamma_{n-1}^k(\omega_1,\ldots,\omega_{n-1}), \quad (99)$$

then for such $\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})$ the inequality (91) are satisfied. Theorem 4 is proved.

V. Optional Decomposition for Super-Martingales

In this section, we give simple proof of optional decomposition for the nonnegative super-martingale relative to the set of equivalent martingale measures. Such a proof first appeared in the paper [16]. First, the optional decomposition for diffusion processes super-martingale was opened by El Karoui N. and Quenez M. C. [21]. After that, Kramkov D. O. and Follmer H. [22], [23] proved the optional decomposition for the nonnegative bounded super-martingales. Folmer H. and Kabanov Yu. M. [24], [25] proved analogous result for an arbitrary super-martingale. Recently, Bouchard B. and Nutz M. [26] considered a class of discrete models and proved the necessary and sufficient conditions for the validity of the optional decomposition.

Theorem 5. *Let Ω_i^0 be a complete separable metric space and let $\mathcal{F}_i^0$ be a Borell σ-algebra on Ω_i^0, $i=\overline{1,N}$. Suppose that the evolution $\{S_n(\omega_1,\ldots,\omega_n)\}_{n=1}^N$ of risky assets satisfies the conditions of Theorems 1, 2, 3, 4, then for every nonnegative super-martingale $\{f_n^1(\omega_1,\ldots,\omega_n)\}_{n=0}^N$ relative to the set of martingale measure M, described in Theorem 1, the optional decomposition is true.*

Proof. Without loss of generality, we assume that $f_n^1(\omega_1,\ldots,\omega_n)\geq a$, where a is a real positive number. If it is not so, then we can come to the super-martingale $f_n^1(\omega_1,\ldots,\omega_n)+a$. Let us consider the set of random values

$$f_n(\omega_1,\ldots,\omega_n)=\frac{f_n^1(\omega_1,\ldots,\omega_n)}{f_{n-1}^1(\omega_1,\ldots,\omega_{n-1})},\quad n=\overline{1,N}. \tag{100}$$

Every random value $f_n(\omega_1,\ldots,\omega_n)$ satisfies the conditions of Lemma 4. Due to Theorems 3, 4, the inequalities

$$\frac{f_n^1(\omega_1,\ldots,\omega_n)}{f_{n-1}^1(\omega_1,\ldots,\omega_{n-1})}\leq 1+\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})\Delta S_n(\omega_1,\ldots,\omega_n),\quad n=\overline{1,N}, \tag{101}$$

are true, where $\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})$ is a bounded $\mathcal{F}_{n-1}$-measurable random value. Since $E^Q|\Delta S_n(\omega_1,\ldots,\omega_n)|<\infty$, $Q\in M$, we have

$$E^Q\{\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})\Delta S_n(\omega_1,\ldots,\omega_n)|\mathcal{F}_{n-1}\}=0,\quad Q\in M,\quad n=\overline{1,N}. \tag{102}$$

Let us denote

$$\xi_n^0(\omega_1,\ldots,\omega_n)=1+\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})\Delta S_n(\omega_1,\ldots,\omega_n),\quad n=\overline{1,N}. \tag{103}$$

Then, from the inequalities (101), we obtain the inequalities

$$f_n^1(\omega_1,\ldots,\omega_n)\leq$$
$$f_{n-1}^1(\omega_1,\ldots,\omega_{n-1})+f_{n-1}^1(\omega_1,\ldots,\omega_{n-1})[\xi_n^0(\omega_1,\ldots,\omega_n)-1],\quad n=\overline{1,N}. \tag{104}$$

Introduce the denotations

$$g_n(\omega_1,\ldots,\omega_n)=$$
$$-f_n^1(\omega_1,\ldots,\omega_n)+f_{n-1}^1(\omega_1,\ldots,\omega_{n-1})\xi_n^0(\omega_1,\ldots,\omega_n),\quad n=\overline{1,N}. \tag{105}$$

Then, $g_n(\omega_1,\ldots,\omega_n)\geq 0$, $n=\overline{1,N}$, and

$$E^Q g_n(\omega_1,\dots,\omega_n) \le E^Q f_n^1(\omega_1,\dots,\omega_n) + E^Q f_n^1(\omega_1,\dots,\omega_{n-1}). \quad (106)$$

The equalities (105) give the equalities

$$f_n^1(\omega_1,\dots,\omega_n) =$$

$$f_0^1 + \sum_{i=1}^n f_{i-1}^1(\omega_1,\dots,\omega_{n-1})[\xi_i^0(\omega_1,\dots,\omega_i) - 1] - \sum_{i=1}^n g_i(\omega_1,\dots,\omega_i), \ n=\overline{1,N}. \quad (107)$$

Let us put

$$M_n(\omega_1,\dots,\omega_n) = f_0^1 + \sum_{i=1}^n f_{i-1}^1(\omega_1,\dots,\omega_{i-1})[\xi_i^0(\omega_1,\dots,\omega_i) - 1], \quad n=\overline{1,N}, \quad (108)$$

then $E^Q\{M_n(\omega_1,\dots,\omega_n)|\mathcal{F}_{n-1}\} = M_{n-1}(\omega_1,\dots,\omega_{n-1})$. Theorem 5 is proved.

VI. Spot Measures and Integral Representation for Martingale Measures

In this section, we introduce the family of spot measures. After that, we obtain the representations for the family of spot measures and define integral over these set of measures. The sufficient conditions are found, under which the integral over these set of measures is a set of martingale measures being equivalent to the initial measure. The introduced family of spot measures is a family of extreme points for these set of equivalent measures.

We give an evident construction of the set of martingale measures for risky assets evolution, given by the formula (9). First of all, to do that we consider a simple case as the measures P_n^0 is concentrated at two points $\omega_n^1, \omega_n^2 \in \Omega_n^0$, where $\omega_n^1 \in A_n^{0k-}, \omega_n^2 \in A_n^{0k+}$ for a certain k, depending on n, for the representation Ω_n^- and Ω_n^+, given by the formula (5). Let us put $P_n^0(\omega_n^1) = p_n^k$, $P_n^0(\omega_n^2) = 1 - p_n^k$, where $0 < p_n^k < 1$. Then, to satisfy the conditions (14) - (16), we need to put

$$\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\}) = \frac{1}{p_n^k(1-p_n^k)}, \quad n=\overline{1,N}, \quad (109)$$

and to require that

$$\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1) < \infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^1) \in \Omega_n^-,$$

$$\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2) < \infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^2) \in \Omega_n^+. \quad (110)$$

Let us denote $\mu_{\{\omega_n^1,\omega_n^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ the measure, generated by the recurrent relations (23) - (25), for the measures P_n^0, $n=\overline{1,N}$, concentrated at two points. For the point $\{\omega_n^1,\omega_n^2\},\dots,\{\omega_N^1,\omega_N^2\} \in \Omega_N \times \Omega_N$, the recurrent relations (23) - (25) is converted relative to the set of measures $\mu^{(\omega_1,\dots,\omega_{n-1})}_{\{\omega_n^1,\omega_n^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ into the recurrent relations

$$\mu^{(\omega_1,\dots,\omega_{N-1})}_{\{\omega_N^1,\omega_N^2\}}(A) = \chi_{\Omega_N^-}(\omega_1,\dots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\dots,\omega_{N-1},\omega_N^2)\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}(A)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}(A)\right], \quad A\in\mathcal{F}_N, \tag{111}$$

$$\mu^{(\omega_1,\ldots,\omega_{n-1})}_{\{\omega_n^1,\omega_n^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu^{(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}_{\{\omega_{n+1}^1,\omega_{n+1}^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu^{(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}_{\{\omega_{n+1}^1,\omega_{n+1}^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)\right], \quad n=\overline{2,N}, \quad A\in\mathcal{F}_N, \tag{112}$$

$$\mu_{\{\omega_n^1,\omega_n^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)=\chi_{\Omega_1^-}(\omega_1^1)\chi_{\Omega_1^+}(\omega_1^2)\times$$

$$\left[\frac{\Delta S_1^+(\omega_n^2)}{V_1(\omega_1^1,\omega_1^2)}\mu^{(\omega_1^1)}_{\{\omega_2^1,\omega_2^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)+\frac{\Delta S_1^-(\omega_1^1)}{V_1(\omega_1^1,\omega_1^2)}\mu^{(\omega_1^2)}_{\{\omega_2^1,\omega_2^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)\right], \tag{113}$$

where we put

$$\mu_N^{(\omega_1,\ldots,\omega_{N-1},\omega_N)}(A)=\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N), \quad A\in\mathcal{F}_N. \tag{114}$$

The recurrent relations (111) - (113) we call the recurrent relations for the spot measures $\mu_{\{\omega_n^1,\omega_n^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$.

Let us consider the random values

$$\psi_n(\omega_1,\ldots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\ldots,\omega_n), \tag{115}$$

where

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n)=\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \tag{116}$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}. \tag{117}$$

Lemma 6. For the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ the representation

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}), \quad A\in\mathcal{F}_N, \tag{118}$$

is true.

Proof. The proof of Lemma 6 we lead by induction down. Let us prove the equality

$$\mu^{(\omega_1,\ldots,\omega_{N-1})}_{\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_N=1}^{2}\psi_N(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N})\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N}). \tag{119}$$

Really,

$$\psi_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^1)+$$

$$\psi_N(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^2) =$$

$$\left[\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}+\right.$$

$$\left.\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\right]\times$$

$$\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^1)+$$

$$\left[\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}+\right.$$

$$\left.\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\right]\times$$

$$\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^2) =$$

$$\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^1)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\right], \quad A\in\mathcal{F}_N. \tag{120}$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

The last prove the needed. Suppose that we proved that the equality

$$\mu^{(\omega_1,\ldots,\omega_{n-1},\omega_n)}_{\{\omega^1_{n+1},\omega^2_{n+1}\},\ldots,\{\omega^1_N,\omega^2_N\}}(A) =$$

$$\sum_{i_{n+1}=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=n+1}^{N}\psi_j(\omega_1,\ldots,\omega_n,\omega^{i_{n+1}}_{n+1},\ldots,\omega^{i_j}_j)\chi_A(\omega_1,\ldots,\omega_n,\omega^{i_{n+1}}_{n+1},\ldots,\omega^{i_N}_N),$$

$$A \in \mathcal{F}_N, \tag{121}$$

is true. By the same way as above, we have

$$\sum_{i_n=1}^{2}\psi_n(\omega_1,\ldots,\omega_{n-1},\omega^{i_n}_n)\mu^{(\omega_1,\ldots,\omega_{n-1},\omega^{i_n}_n)}_{\{\omega^1_{n+1},\omega^2_{n+1}\},\ldots,\{\omega^1_N,\omega^2_N\}}(A) =$$

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega^1_n)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega^2_n)\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega^2_n)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega^1_n,\omega^2_n)}\mu^{(\omega_1,\ldots,\omega_{n-1},\omega^1_n)}_{\{\omega^1_{n+1},\omega^2_{n+1}\},\ldots,\{\omega^1_N,\omega^2_N\}}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega^1_n)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega^1_n,\omega^2_n)}\mu^{(\omega_1,\ldots,\omega_{n-1},\omega^2_n)}_{\{\omega^1_{n+1},\omega^2_{n+1}\},\ldots,\{\omega^1_N,\omega^2_N\}}(A)\right] =$$

$$\mu^{(\omega_1,\ldots,\omega_{n-1})}_{\{\omega^1_n,\omega^2_n\},\ldots,\{\omega^1_N,\omega^2_N\}}(A), \quad A \in \mathcal{F}_N. \tag{122}$$

The last proves Lemma 6.

Let us define the integral for the random value $f_N(\omega_1,\ldots,\omega_{N-1},\omega_N)$ relative to the measure $\mu_{\{\omega^1_1,\omega^2_1\},\ldots,\{\omega^1_N,\omega^2_N\}}(A)$ by the formula

$$\int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_{N-1},\omega_N)d\mu_{\{\omega^1_1,\omega^2_1\},\ldots,\{\omega^1_N,\omega^2_N\}} =$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega^{i_1}_1,\ldots,\omega^{i_j}_j)f_N(\omega^{i_1}_1,\ldots,\omega^{i_N}_N). \tag{123}$$

To describe the convex set of equivalent martingale measures, we introduce the family of α-spot measures, depending on the point $(\{\omega^1_1,\{\omega^2_1\},\ldots,\{\omega^1_N,\{\omega^2_N\})$ belonging to $\Omega_N\times\Omega_N$ and the set of strictly positive random values

$$\alpha_n(\{\omega^1_1,\ldots,\omega^1_{n-1},\omega^1_n\};\{\omega^2_1,\ldots,\omega^2_{n-1},\omega^2_n\}), \quad n=\overline{1,N}, \tag{124}$$

at points $W_n = (\{\omega^1_1,\ldots,\omega^1_n\};\{\omega^2_1,\ldots,\omega^2_n\})$, being constructed by the point $(\{\omega^1_1,\omega^2_1\},\ldots,\{\omega^1_N,\omega^2_N\})$.

Further, in this section, we assume that the evolution of risky asset is given by the formula (9). Therefore, in this case

$$\Omega_n^- = \Omega_n^{0-} \times \Omega_{n-1}, \quad \Omega_n^+ = \Omega_n^{0+} \times \Omega_{n-1}, \quad n = \overline{1,N}, \tag{125}$$

and the condition (16) is formulated, as follows:

$$\int\limits_{\Omega_n^0 \times \Omega_n^0} \chi_{\Omega_n^{0-}}(\omega_n^1)\chi_{\Omega_n^{0+}}(\omega_n^2)\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) = 1, \quad n = \overline{1,N}. \tag{126}$$

Let us determine the random values

$$\psi_n^\alpha(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\ldots,\omega_n), \tag{127}$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \tag{128}$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}. \tag{129}$$

Let us define the set of α-spot measures on the σ-algebra $\mathcal{F}_N$ by the formula

$$\mu_{W_N}^\alpha(A) = \sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j^\alpha(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{130}$$

and the set of the measures

$$\mu_0(A) =$$

$$\int\limits_{\Omega_N \times \Omega_N} \sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j^\alpha(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N})dP_N \times dP_N, \; A \in \mathcal{F}_N. \tag{131}$$

Theorem 6. *Suppose that the conditions of Lemma 1 are true. If the strictly positive random values*

$$\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\}),\ \ n=\overline{1,N}, \tag{132}$$

given on the probability space $\{\Omega_n\times\Omega_n,\mathcal{F}_n\times\mathcal{F}_n,P_n\times P_n\}$, $n=\overline{1,N}$, *satisfy the conditions (126), then for the measure* $\mu_0(A)$, *given by the formula (131), the representation*

$$\mu_0(A)=$$

$$\int\limits_{\Omega_N\times\Omega_N}\prod_{i=1}^N\alpha_i(\{\omega_1^1,\ldots,\omega_i^1\};\{\omega_1^2,\ldots,\omega_i^2\})\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)dP_N\times dP_N \tag{133}$$

is true.

Proof. Due to Lemma 1, the set of random values $\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})$, $n=\overline{1,N}$, satisfying the conditions (126), is a non empty set.

We prove Theorem 6 by induction down. For the spot measure the relation

$$\mu^{(\omega_1,\ldots,\omega_{N-1})}_{\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^1)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\right],\quad A\in\mathcal{F}_N, \tag{134}$$

is true. Multiplying the relation (134) on $\alpha_N(\{\omega_1^1,\ldots,\omega_{N-1}^1,\omega_N^1\};\{\omega_1^2,\ldots,\omega_{N-1}^2,\omega_N^2\})$ and after that, integrating relative to the measure $P_N^0\times P_N^0$ on the set $\Omega_N^0\times\Omega_N^0$, we obtain

$$\int\limits_{\Omega_N^0}\int\limits_{\Omega_N^0}\alpha_N(\{\omega_1^1,\ldots,\omega_{N-1}^1,\omega_N^1\};\{\omega_1^2,\ldots,\omega_{N-1}^2,\omega_N^2\})\times$$

$$\mu^{(\omega_1,\ldots,\omega_{N-1})}_{\{\omega_N^1,\omega_N^2\}}(A)dP_N^0(\omega_N^1)dP_N^0(\omega_N^2)=$$

$$\int\limits_{\Omega_N^0}\int\limits_{\Omega_N^0}\alpha_N(\{\omega_1^1,\ldots,\omega_{N-1}^1,\omega_N^1\};\{\omega_1^2,\ldots,\omega_{N-1}^2,\omega_N^2\})\times$$

$$\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\times$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

$$\left[\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^1)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\right] dP_N^0(\omega_N^1)dP_N^0(\omega_N^2) =$$

$$\mu_{N-1}^{(\omega_1,\ldots,\omega_{N-1})}(A), \quad A \in \mathcal{F}_N. \tag{135}$$

Suppose that we proved the equality

$$\int\limits_{\prod\limits_{i=n+1}^{N}[\Omega_i^0\times\Omega_i^0]} \prod_{i=n+1}^{N} \alpha_i(\{\omega_1^1,\ldots,\omega_n^1,\omega_{n+1}^1,\ldots,\omega_i^1\};\{\omega_1^2,\ldots,\omega_n^2,\omega_{n+1}^2,\ldots,\omega_i^2\})\times$$

$$\mu^{(\omega_1,\ldots,\omega_n)}_{\{\omega_{n+1}^1,\omega_{n+1}^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)\prod_{i=n+1}^{N} dP_i^0(\omega_i^1)dP_i^0(\omega_i^2) = \mu_n^{(\omega_1,\ldots,\omega_n)}(A). \tag{136}$$

Then, using the induction supposition (136), the relation for the spot measure

$$\mu^{(\omega_1,\ldots,\omega_{n-1})}_{\{\omega_n^1,\omega_n^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu^{(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}_{\{\omega_{n+1}^1,\omega_{n+1}^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu^{(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}_{\{\omega_{n+1}^1,\omega_{n+1}^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)\right], \quad A \in \mathcal{F}_N, \tag{137}$$

and multiplying it on $\prod\limits_{i=n}^{N}\alpha_i(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1,\ldots,\omega_i^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2,\ldots,\omega_i^2\})$ and then integrating relative to the measure $\prod\limits_{i=n}^{N}[P_i^0\times P_i^0]$ on the set $\prod\limits_{i=n}^{N}[\Omega_i^0\times\Omega_i^0]$, we obtain the equality

$$\int\limits_{\Omega_n^0\times\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}(A)+\right.$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}(A)\Big] dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) =$$

$$\mu_{n-1}^{(\omega_1,\ldots,\omega_{n-1})}(A), \quad n=\overline{1,N}. \tag{138}$$

Thus, we proved the following recurrent relations

$$\mu_{n-1}^{(\omega_1,\ldots,\omega_{n-1})}(A) = \int\limits_{\Omega_n^0\times\Omega_n^0} \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\})\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}(A)\right] dP_n^0(\omega_n^1)dP_n^0(\omega_n^2), \quad n=\overline{1,N}. \tag{139}$$

To finish the proof of Theorem 6, let us calculate

$$\int\limits_{\Omega_N^0\times\Omega_N^0}\sum_{i_N=1}^{2}\psi_N^{\alpha}(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N})\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N})dP_N^0(\omega_N^1)dP_N^0(\omega_N^2). \tag{140}$$

Calculating the expression

$$\sum_{i_N=1}^{2}\psi_N^{\alpha}(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N})\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N}) =$$

$$\psi_N^{\alpha}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^1)+$$

$$\psi_N^{\alpha}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^2) =$$

$$\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})\times$$

$$\chi_{\Omega_N^-}(\omega_1,\ldots,\omega_{N-1},\omega_N^1)\chi_{\Omega_N^+}(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\times$$

$$\left[\frac{\Delta S_N^+(\omega_1,\ldots,\omega_{N-1},\omega_N^2)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^1)+\right.$$

$$\left.\frac{\Delta S_N^-(\omega_1,\ldots,\omega_{N-1},\omega_N^1)}{V_N(\omega_1,\ldots,\omega_{N-1},\omega_N^1,\omega_N^2)}\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^2)\right], \quad A\in\mathcal{F}_N, \tag{141}$$

and substituting (141) into (140), we obtain the equality

$$\int\limits_{\Omega_N^0\times\Omega_N^0}\sum_{i_N=1}^{2}\psi_N^{\alpha}(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N})\chi_A(\omega_1,\ldots,\omega_{N-1},\omega_N^{i_N})dP_N^0(\omega_N^1)dP_N^0(\omega_N^2) =$$

$$\mu_{N-1}^{(\omega_1,\ldots,\omega_{N-1})}(A). \tag{142}$$

Suppose that we already proved the equality

$$\int\limits_{\prod\limits_{i=n+1}^{N}\Omega_i^0\times\Omega_i^0} \sum_{i_{n+1}=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j^{\alpha}(\omega_1,\ldots,\omega_n,\omega_{n+1}^{i_{n+1}}\ldots,\omega_j^{i_j})\prod_{i=n+1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)=$$

$$\mu_n^{(\omega_1,\ldots,\omega_n)}(A). \tag{143}$$

Then, acting as above, we obtain the equalities

$$\int\limits_{\Omega_n^0\times\Omega_n^0}\sum_{i_n=1}^{2}\psi_n^{\alpha}(\omega_1,\ldots,\omega_{n-1},\omega_n^{i_n})\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^{i_n})}(A)dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=$$

$$\int\limits_{\Omega_n^0\times\Omega_n^0}\alpha_n(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})\times$$

$$\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\times$$

$$\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}(A)+\right.$$

$$\left.\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}\mu_n^{(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}(A)\right]dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)=$$

$$\mu_{n-1}^{(\omega_1,\ldots,\omega_{n-1})}(A), \quad A\in\mathcal{F}_N. \tag{144}$$

We proved that the recurrent relations (144) are the same as the recurrent relations (139). This proves Theorem 6.

Let us introduce the denotations

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(\Omega_N)=\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}),$$

$$W_N=\{\omega_1^1,\ldots,\omega_N^1;\omega_1^2,\ldots,\omega_N^2\}=\{\{\omega\}_N^1,\{\omega\}_N^2\}. \tag{145}$$

Further, only those points $(\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\})\in\Omega_N\times\Omega_N$ play important role for which $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(\Omega_N)\neq 0$.

Below, in the next two Theorems, we assume that the random value

$$\alpha_n(\{\omega_1^1,\ldots,\omega_n^1\};\{\omega_1^2,\ldots,\omega_n^2\}) \tag{146}$$

given on the probability space $\{\Omega_n\times\Omega_n,\mathcal{F}_n\times\mathcal{F}_n,P_n\times P_n\}$, $n=\overline{1,N}$, satisfy the conditions (126).

Under the above conditions, for the measure $\mu_0(A)$, given by the formula (133), the representation

$$\mu_0(A) = \int\limits_{\Omega_N} \prod_{n=1}^{N} \psi_n(\omega_1, \dots, \omega_n)\chi_A(\omega_1, \dots, \omega_N) \prod_{i=1}^{N} dP_i^0(\omega_i) \tag{147}$$

is true, where

$$\psi_n(\omega_1, \dots, \omega_n) = \chi_{\Omega_n^-}(\omega_1, \dots, \omega_{n-1}, \omega_n)\psi_n^1(\omega_1, \dots, \omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1, \dots, \omega_{n-1}, \omega_n)\psi_n^2(\omega_1, \dots, \omega_n), \tag{148}$$

$$\psi_n^1(\omega_1, \dots, \omega_{n-1}, \omega_n) = \int\limits_{\Omega_n^0} \chi_{\Omega_n^+}(\omega_1, \dots, \omega_{n-1}, \omega_n^2)\alpha_n(\{\omega_1^1, \dots, \omega_n^1\}; \{\omega_1^2, \dots, \omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^2), \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \tag{149}$$

$$\psi_n^2(\omega_1, \dots, \omega_{n-1}, \omega_n) = \int\limits_{\Omega_n^0} \chi_{\Omega_n^-}(\omega_1, \dots, \omega_{n-1}, \omega_n^1)\alpha_n(\{\omega_1^1, \dots, \omega_n^1\}; \{\omega_1^2, \dots, \omega_n^2\})\times$$

$$\frac{\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)} dP_n^0(\omega_n^1), \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}. \tag{150}$$

Due to the conditions (126) relative to the random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2)$, we have

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1, \dots, \omega_n) dP_n^0(\omega_n) = 1, \quad n = \overline{1, N}. \tag{151}$$

for $\psi_n(\omega_1, \dots, \omega_n)$, given by the formula (148). The proof of the equalities (151) is the same as in Theorem 1.

Theorem 7. *Suppose that the conditions of Lemma 1 are true. Then, the set of strictly positive random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2), n = \overline{1, N}$, satisfying the conditions*

$$E^{\mu_0}|\Delta S_n(\omega_1, \dots, \omega_{n-1}, \omega_n)| =$$

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i(\omega_1, \dots, \omega_i)|\Delta S_n(\omega_1, \dots, \omega_{n-1}, \omega_n)| \prod_{i=1}^{N} dP_i^0(\omega_i) < \infty, \quad n = \overline{1, N}, \tag{152}$$

is a non empty set for the measures $\mu_0(A)$, given by the formula (133). The measure $\mu_0(A)$, constructed by the strictly positive random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2), n = \overline{1, N}$, satisfying the conditions (126), (152) is a martingale measure for the evolution of risky asset, given by the formula (9). Every measure, belonging to the convex linear span of such measures, is also martingale measure for the evolution of risky asset, given by the formula (9). They are equivalent to the measure P_N. The set of spot measures $\mu_{\{\omega_1^1, \omega_1^2\}, \dots, \{\omega_N^1, \omega_N^2\}}(A)$ is a set of martingale measures for the evolution of risky asset, given by the formula (9).

Proof. The first fact, that the set of random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2), n=\overline{1,N}$, satisfying the conditions (126), (152) is a non empty one, follows from Lemma 1. From the representation (147) for the set of measures $\mu_0(A)$, given by the formula (133), as in the proof of Theorem 1, it is proved that this set of measures is a set of martingale measures being equivalent to the measure P_N.

Let us prove the last statement of Theorem 7. Since for the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ the representation

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}), \quad A\in\mathcal{F}_N, \tag{153}$$

is true, let us calculate

$$\sum_{i_j=1}^{2}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}) = \psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+\psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2) =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_n^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\psi_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2)+$$

$$\chi_{\Omega_n^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\psi_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2) =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}+$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} =$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}=$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)=\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)=$$

$$\begin{cases}1, & \omega_j^1\in\Omega_j^{0-} \quad \omega_j^2\in\Omega_j^{0+},\\ 0, & \text{otherwise},\end{cases}, \quad j=\overline{1,N}. \tag{154}$$

Further,

$$\sum_{i_j=1}^{2}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\Delta S_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})=$$

$$\psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\Delta S_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+$$

$$\psi_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\Delta S_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)=$$

$$\chi_{\Omega_j^-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\chi_{\Omega_j^+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\times$$

$$\left[-\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+\right.$$

$$\left.\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\right]=0, \quad j=\overline{1,N}. \tag{155}$$

Let us prove that the set of measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ is a set of martingale measures. Really, for A, belonging to the σ-algebra $\mathcal{F}_{n-1}$ of the filtration we have $A=B\times\prod_{i=n}^{N}\Omega_i^0$, where B belongs to σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1},\mathcal{F}_{n-1}\}$. Then,

$$\int_A\Delta S_n(\omega_1,\ldots,\omega_n)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_n=1}^{2}\prod_{j=1}^{n}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_{n-1}=1}^{2}\prod_{j=1}^{n-1}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}})\times$$

$$\sum_{i_n=1}^{2}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\Delta S_n(\omega_1^{i_1},\dots,\omega_n^{i_n})=0, \quad A\in\mathcal{F}_{n-1}. \tag{156}$$

The last means the needed statement. Theorem 7 is proved.

Below, in Theorem 8, we present the consequence of Theorems 6, 7.

Theorem 8. Let the evolution of risky asset be given by the formula (9) and let Lemma 1 conditions be true. Suppose that the random value $\alpha_N(\{\omega\}_N^1;\{\omega\}_N^2)$, *given on the probability space* $\{\Omega_N^-\times\Omega_N^+,\mathcal{F}_N^-\times\mathcal{F}_N^+,P_N^-\times P_N^+\}$, *satisfy the conditions*

$$P_N^-\times P_N^+((\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\}),\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})>0)=$$

$$\prod_{n=1}^{N}P_n^0(\Omega_n^{0-})\times P_n^0(\Omega_n^{0+}); \tag{157}$$

$$\int\limits_{\Omega_n^{0-}\times\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)<\infty,$$

$$(\omega_1,\dots,\omega_{n-1})\in\Omega_{n-1}; \tag{158}$$

$$\int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod_{i=1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)=1, \tag{159}$$

where

$$\alpha_n(\{\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\dots,\omega_{n-1}^2,\omega_n^2\})= \tag{160}$$

$$\frac{\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod_{i=n+1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\alpha_N(\{\omega_1^1,\dots,\omega_N^1\};\{\omega_1^2,\dots,\omega_N^2\})\prod_{i=n}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)},\ n=\overline{1,N}.$$

If the set of strictly positive random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2),n=\overline{1,N}$, *given by the formula (160), satisfies the condition*

$$E^{\mu_0}|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|=$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020
71

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i(\omega_1, \ldots, \omega_i)|\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)| \prod_{i=1}^{N} dP_i^0(\omega_i) < \infty, \quad n = \overline{1, N}, \quad (161)$$

then, for the martingale measure $\mu_0(A)$ the representation

$$\mu_0(A) =$$

$$\int\limits_{\Omega_N \times \Omega_N} \alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A) dP_N \times dP_N \quad (162)$$

is true.

Proof. The random values $\alpha_n(\{\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_{n-1}^2, \omega_n^2\})$, $n = \overline{1, N}$, satisfy the conditions (14) - (16), due to the conditions of Theorem 8. It is evident that

$$\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) = \prod_{n=1}^{N} \alpha_n(\{\omega_1^1, \ldots, \omega_n^1\}; \{\omega_1^2, \ldots, \omega_n^2\}). \quad (163)$$

Due to Theorem 7, $\mu_0(A)$, given by the formula (162), is a martingale measure being equivalent to the measure P_N.

Let us indicate how to construct the random values $\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)$, since these random values determine the set of all martingale measures. Suppose that the random value $\alpha_i^k(\omega_i^1, \omega_i^2)$, $k = \overline{1, K}$, is a bounded strictly positive random value, given on the measurable space $\{\Omega_i^{0-} \times \Omega_i^{0+}, \mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}\}$, $i = \overline{1, N}$, and satisfying the conditions

$$\int\limits_{\Omega_i^{0-} \times \Omega_i^{0+}} \alpha_i^k(\omega_i^1, \omega_i^2) dP_i^0(\omega_i^1) dP_i^0(\omega_i^2) = 1, \quad i = \overline{1, N}, \quad k = \overline{1, K}. \quad (164)$$

Let us denote

$$\alpha_N^k(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) = \prod_{i=1}^{N} \alpha_i^k(\omega_i^1, \omega_i^2), \quad k = \overline{1, K}, \quad (165)$$

where K runs natural numbers. If γ_k, $k = \overline{1, K}$, are strictly positive real numbers such that $\sum_{k=1}^{K} \gamma_k = 1$, then

$$\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) = \sum_{k=1}^{K} \gamma_k \alpha_N^k(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \quad (166)$$

satisfy the conditions of Theorem 8. The set of random values (166) is dense in the set of random values $\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\})$, satisfying the condition (157) - (159). Theorem 8 is proved.

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

Another way to construct $\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})$ is to use the equalities (126). The set of $\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})$ can construct as follows: suppose that $\alpha_n^1(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})$ satisfies the inequalities

$$0 < h_n \leq \alpha_n^1(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\}) \leq H_n < \infty \tag{167}$$

for a certain real positive numbers h_n, H_n. If to put

$$\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\}) =$$

$$\frac{\alpha_n^1(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})}{\int\limits_{\Omega_n^{0-}\times\Omega_n^{0+}}\alpha_n^1(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)}, \tag{168}$$

then the set of random values $\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\}),\ n=\overline{1,N}$, is bounded and satisfy the conditions (14) - (16) under the conditions of Theorem 7. We can put

$$\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\}) =$$

$$\prod_{n=1}^{N}\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\}). \tag{169}$$

It is evident that $\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\}),\ n=\overline{1,N}$, must satisfy the conditions (161).

VII. Derivatives Assessment

In the papers [27], [28], the range of non arbitrage prices are established. In the paper [27], for the Levy exponential model, the price of super-hedge for call option coincides with the price of the underlying asset under the assumption that the Levy process has unlimited variation, does not contain a Brownian component, with negative jumps of arbitrary magnitude. The same result is true, obtained in the paper [28], if the process describing the evolution of the underlying asset is a diffusion process with the jumps described by Poisson jump process. In these papers the evolution is described by continuous processes. Below, we consider the discrete evolution of risky assets that is more realistic from the practical point of view. Two types of risky asset evolutions are considered: 1) the price of an asset can take any non negative value; 2) the price of the risky asset may not fall below a given positive value for finite time of evolution. For each of these types of evolutions of risky assets, the bounds of non-arbitrage prices for a wide class of contingent liabilities are found, among which are the payoff functions of standard call and put options.

Below, on the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, where $\Omega_N=\prod_{i=1}^{N}\Omega_i^0$, $\mathcal{F}_N=\prod_{i=1}^{N}\mathcal{F}_i^0$, $P_N=\prod_{i=1}^{N}P_i^0$, Ω_i^0 is a complete separable metric space, $\mathcal{F}_i^0$ is a Borel σ-algebra on Ω_i^0, P_i^0 is a probability measure on $\mathcal{F}_i^0,\ i=\overline{1,N}$, we consider the evolution of risky asset given by the formula

$$S_n(\omega_1,\ldots,\omega_n) =$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

$$S_0 \prod_{i=1}^{n}(1 + a_i(\omega_1, \dots, \omega_{i-1})(e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)} - 1)), \quad n = \overline{1, N}, \tag{170}$$

where $a_i(\omega_1, \dots, \omega_{i-1}), \sigma_i(\omega_1, \dots, \omega_{i-1})$ are $\mathcal{F}_{i-1}$-measurable random values, satisfying the conditions $0 < a_i(\omega_1, \dots, \omega_{i-1}) \leq 1$, $\sigma_i(\omega_1, \dots, \omega_{i-1}) \geq \sigma_i > 0$, where σ_i, $i = \overline{1, N}$, are real positive numbers. Further, we assume that the random value $\varepsilon_i(\omega_i)$ satisfy the conditions: there exists $\omega_i^1 \in \Omega_i^0$ such that $\varepsilon_i(\omega_i^1) = 0$, $i = \overline{1, N}$, and for every real number $t > 0$, $P_i^0(\varepsilon_i(\omega_i) < -t) > 0$, $P_i^0(\varepsilon_i(\omega_i) > t) > 0$, $i = \overline{1, N}$.

For the evolution of risky asset (170), we have

$$\Delta S_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$S_{n-1}(\omega_1, \dots, \omega_{n-1}) a_n(\omega_1, \dots, \omega_{n-1})(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)} - 1) = \tag{171}$$

$$d_n(\omega_1, \dots, \omega_{n-1}, \omega_n)(e^{\sigma_n \varepsilon_n(\omega_n)} - 1),$$

where

$$d_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$S_{n-1}(\omega_1, \dots, \omega_{n-1}) a_n(\omega_1, \dots, \omega_{n-1}) \frac{(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)} - 1)}{(e^{\sigma_n \varepsilon_n(\omega_n)} - 1)}. \tag{172}$$

It is evident that $d_n(\omega_1, \dots, \omega_{n-1}, \omega_n) > 0$, therefore for $\Delta S_n(\omega_1, \dots, \omega_{n-1}, \omega_n)$ the representation (60) is true with $\eta_n(\omega_n) = (e^{\sigma_n \varepsilon_n(\omega_n)} - 1)$. Therefore,

$$\frac{\Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^2)} - 1}{e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^2)} - e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^1)}}, \quad \omega_n^2 \in \Omega_n^{0+}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \tag{173}$$

$$\frac{\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_{n^1})}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{1 - e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^1)}}{e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^2)} - e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^1)}}, \quad \omega_n^1 \in \Omega_n^{0-}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \tag{174}$$

where we denoted

$$\Omega_n^{0-} = \{\omega_n \in \Omega_n^0, \varepsilon_n(\omega_n) \leq 0\}, \quad \Omega_n^{0+} = \{\omega_n \in \Omega_n^0, \varepsilon_n(\omega_n) > 0\},$$

$$\Omega_n^- = \Omega_n^{0-} \times \Omega_{n-1}, \quad \Omega_n^+ = \Omega_n^{0+} \times \Omega_{n-1}. \tag{175}$$

From the formulas (173), (174) and Theorem 1, it follows that the set of martingale measures M do not depend on the random values $a_i(\omega_1, \dots, \omega_{i-1})$, $i = \overline{1, N}$. If

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

to put $a_i(\omega_1,\dots,\omega_{i-1})=1,\ i=\overline{1,N}$, in the formula (170), then for the risky asset evolution we obtain the formula

$$S_n(\omega_1,\dots,\omega_{n-1},\omega_n)=S_0\prod_{i=1}^{n}e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)},\quad n=\overline{1,N}. \tag{176}$$

The evolution of risky assets, given by the formula (176), includes a wide class of evolutions of risky assets, used in economics. For example, under an appropriate choice of probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$ and a choice of sequence of independent random values $\varepsilon_i(\omega_i)$ with the normal distribution $N(0,1)$, we obtain ARCH model (Autoregressive Conditional Heteroskedastic Model) introduced by Engle in [18] and GARCH model (Generalized Autoregressive Conditional Heteroskedastic Model) introduced later by Bollerslev in [19]. In these models, the random variables $\sigma_i(\omega_1,\dots,\omega_{i-1}),\ i=\overline{1,N}$, are called the volatilities which satisfy the nonlinear set of equations.

The very important case of evolution of risky assets (170) is when $a_i(\omega_1,\dots,\omega_{i-1})=a_i,\ i=\overline{1,N}$, are constants, that is,

$$S_n(\omega_1,\dots,\omega_{n-1},\omega_n)=S_0\prod_{i=1}^{n}(1+a_i(e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}-1)),\quad n=\overline{1,N}, \tag{177}$$

where $0\le a_i\le 1$.

If $0<a_i<1, i=\overline{1,N}$, then the evolution of risky asset, given by the formula (177), we call the evolution of relatively stable asset.

Further, we assume that the evolution of risky asset given by the formulas (170), (176), (177) satisfy the conditions

$$\int\limits_{\Omega_N}S_n(\omega_1,\dots,\omega_{n-1},\omega_n)dP_N<\infty,\quad n=\overline{1,N}. \tag{178}$$

From the conditions (178), it follows the inequalities

$$\int\limits_{\Omega_N}\Delta S_n^-(\omega_1,\dots,\omega_n)dP_N<\infty,\quad n=\overline{1,N}. \tag{179}$$

Taking into account that

$$\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)=$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})a_n(\omega_1,\dots,\omega_{n-1})(1-e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^1)}),\quad \omega_n^1\in\Omega_n^{0-}, \tag{180}$$

$$\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)=$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})a_n(\omega_1,\dots,\omega_{n-1})(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n^2)}-1),\quad \omega_n^2\in\Omega_n^{0+}, \tag{181}$$

we have

$$\frac{1}{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}\le\frac{1}{\prod\limits_{i=1}^{n-1}(1-a_i^1)a_n^0(1-e^{\sigma_n\varepsilon_n(\omega_n^1)})}<\infty,\quad \varepsilon_n(\omega_n^1)<0, \tag{182}$$

$$\frac{1}{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)} \leq \frac{1}{\prod\limits_{i=1}^{n-1}(1-a_i^1)a_n^0(e^{\sigma_n \varepsilon_n(\omega_n^2)}-1)} < \infty, \quad \varepsilon_n(\omega_n^2) > 0, \quad (183)$$

under the conditions that

$$0 < a_n^0 \leq a_n(\omega_1, \ldots, \omega_{n-1}) \leq a_n^1 < 1, \quad n = \overline{1, N}. \quad (184)$$

Theorem 9. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by one of the formula (170), (176), (177) that satisfies the conditions (178).*

If the inequalities $0 < a_n^0 \leq a_n(\omega_1, \ldots, \omega_{n-1}) \leq a_n^1 < 1$, $0 < a_i < 1$, $i = \overline{1, N}$, *are true, then the set of martingale measures* M *is the same for every evolution of risky assets, given by the formulas (170), (177). For every non-negative super-martingale relative to the set of martingale measures* M *the optional decomposition is valid. Every measure of* M *is an integral over the spot measures. The fair price* f_0 *of super-hedge for the nonnegative payoff function* $f(x)$ *is given by the formula*

$$f_0 = \sup_{P \in M} E^P f(S_N) = \sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}. \quad (185)$$

The set of martingale measures M_1 *for the evolution of risky asset, given by the formula (176), is contained in the set* M.

Proof. From the equalities (173) - (174) and the inequalities (178), it follows that the set M is a nonempty one and every martingale measure constructed by the set of random values $\alpha_n(\omega_1^1, \ldots, \omega_n^1; \omega_1^2, \ldots, \omega_n^2)$, $n = \overline{1, N}$, belongs to the set M, if the inequalities (49) are true. To prove that the set of martingale measures, defined by the evolutions (170), (177), coincide it is necessary to prove the inequalities

$$0 < A_n^1 \leq \frac{S_n^1(\omega_1, \ldots, \omega_n)}{S_n^2(\omega_1, \ldots, \omega_n)} \leq B_n^1 < \infty, \quad n = \overline{1, N}, \quad (186)$$

where we denoted by $S_n^1(\omega_1, \ldots, \omega_n)$ the evolution, given by the formula (170), and by $S_n^2(\omega_1, \ldots, \omega_n)$ the evolution, given by the formula (177). Under the conditions of Theorem 9, we have

$$\frac{S_n^1(\omega_1, \ldots, \omega_n)}{S_n^2(\omega_1, \ldots, \omega_n)} =$$

$$\frac{S_0 \prod\limits_{i=1}^{n}(1 + a_i(\omega_1, \ldots, \omega_{i-1})(e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)} - 1))}{S_0 \prod\limits_{i=1}^{n}(1 + a_i(e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)} - 1))}, \quad n = \overline{1, N}. \quad (187)$$

Since

$$\frac{1 + a_i(\omega_1, \ldots, \omega_{i-1})(e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)} - 1)}{1 + a_i(e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)} - 1)} =$$

$$\frac{1 - a_i(\omega_1, \ldots, \omega_{i-1}) + a_i(\omega_1, \ldots, \omega_{i-1})e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}}{1 - a_i + a_i e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}} = D_i, \quad i = \overline{1, N}, \quad (188)$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

we have

$$\frac{1-a_i^1+a_i^0 e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}{1-a_i+a_i e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}} \leq D_i \leq$$

$$\frac{1-a_i^0+a_i^1 e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}{1-a_i+a_i e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}, \quad i=\overline{1,N}. \tag{189}$$

Let us denote

$$A_i = \inf_{(\omega_1,\dots,\omega_i)\in\Omega_i} \frac{1-a_i^1+a_i^0 e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}{1-a_i+a_i e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}, \quad i=\overline{1,N},$$

$$B_i = \sup_{(\omega_1,\dots,\omega_i)\in\Omega_i} \frac{1-a_i^0+a_i^1 e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}{1-a_i+a_i e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}, \quad i=\overline{1,N}. \tag{190}$$

It is evident that $0 < A_i, B_i < \infty, \ i=\overline{1,N}$, and

$$A_i \leq D_i \leq B_i, \quad i=\overline{1,N}, \tag{191}$$

therefore

$$A_n^1 = \prod_{i=1}^n A_i \leq \frac{S_n^1(\omega_1,\dots,\omega_n)}{S_n^2(\omega_1,\dots,\omega_n)} \leq \prod_{i=1}^n B_i = B_n^1, \quad n=\overline{1,N}. \tag{192}$$

So,

$$A_N^2 \leq \frac{S_n^1(\omega_1,\dots,\omega_n)}{S_n^2(\omega_1,\dots,\omega_n)} \leq B_N^2, \quad n=\overline{1,N}, \tag{193}$$

where we put $A_N^2 = \min_{1\leq n\leq N} A_n^1, \ B_N^2 = \max_{1\leq n\leq N} B_n^1$. Since

$$|\Delta S_n^1(\omega_1,\dots,\omega_{n-1},\omega_n)| =$$

$$S_{n-1}^1(\omega_1,\dots,\omega_{n-1}) a_n(\omega_1,\dots,\omega_{n-1}) |(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)}-1)|, \tag{194}$$

$$|\Delta S_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)| =$$

$$S_{n-1}^2(\omega_1,\dots,\omega_{n-1}) a_n |(e^{\sigma_n(\omega_1,\dots,\omega_{n-1})\varepsilon_n(\omega_n)}-1)|, \tag{195}$$

we have

$$\frac{|\Delta S_n^1(\omega_1,\dots,\omega_{n-1},\omega_n)|}{|\Delta S_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)|} =$$

$$\frac{S_{n-1}^1(\omega_1,\dots,\omega_{n-1}) a_n(\omega_1,\dots,\omega_{n-1})}{S_{n-1}^2(\omega_1,\dots,\omega_{n-1}) a_n}. \tag{196}$$

Taking into account the obtained inequalities, we have the inequalities

$$A_N^2 \frac{\min\limits_{1\le n\le N} a_n^0}{\max\limits_{1\le n\le N} a_n} \le \frac{|\Delta S_n^1(\omega_1,\dots,\omega_{n-1},\omega_n)|}{|\Delta S_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)|} \le B_N^2 \frac{\max\limits_{1\le n\le N} a_n^1}{\min\limits_{1\le n\le N} a_n}, \quad n=\overline{1,N}. \tag{197}$$

The inequalities (197) proves that the set of martingale measures for the evolutions of risky assets given by the formulas (170), (177) are the same, since the inequalities (49) for the evolutions of risky assets, given by formulas (170), (177), are fulfilled simultaneously.

For the evolution of risky assets (177), satisfying the conditions (184), the inequalities (182), (183) are true. From this, it follows that the conditions of Theorem 5 are valid. This proves the optional decomposition for every nonnegative supermartingale relative to the family of martingale measures M. From [17], it follows the formula for the fair price f_0 of super-hedge

$$f_0 = \sup_{P\in M} E^P f(S_N). \tag{198}$$

Further, the conditions of Theorem 8 is also true. Therefore, the formula

$$\sup_{P\in M} E^P f(S_N) = \sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} \tag{199}$$

is valid.

To complete the proof of Theorem 9, it needs to show that the set $M_1 \subseteq M$. Let us denote $S_n^3(\omega_1,\dots,\omega_n)$ the evolution of risky asset, given by the formula (176). Then, as above

$$\frac{S_n^3(\omega_1,\dots,\omega_n)}{S_n^2(\omega_1,\dots,\omega_n)} \le \prod_{i=1}^{n}\frac{1}{a_i} = C_n, \quad n=\overline{1,N}. \tag{200}$$

Therefore,

$$\frac{|\Delta S_n^3(\omega_1,\dots,\omega_{n-1},\omega_n)|}{|\Delta S_n^2(\omega_1,\dots,\omega_{n-1},\omega_n)|} =$$

$$\frac{S_{n-1}^3(\omega_1,\dots,\omega_{n-1})}{S_{n-1}^2(\omega_1,\dots,\omega_{n-1})a_n} \le \frac{\max\limits_{1\le n\le N} C_n}{\min\limits_{1\le n\le N} a_n}, \quad n=\overline{1,N}. \tag{201}$$

The inequality (201) proves the needed statement. Theorem 9 is proved.

Theorem 10. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, *let the evolution of risky asset be given by the formula (170). Suppose that* $0 \le a_i(\omega_1,\dots,\omega_{i-1}) \le 1$, $\sigma_i(\omega_1,\dots,\omega_{i-1}) > \sigma_i > 0$, $i=\overline{1,N}$, *and* $a_n = 1$ *for a certain* $1\le n\le N$. *If the nonnegative payoff function* $f(x)$, $x\in[0,\infty)$, *satisfies the conditions:*
1) $f(0)=0,\ f(x)\le ax,\ \lim\limits_{x\to\infty}\frac{f(x)}{x} = a,\ a>0,$ *then*

$$\sup_{P\in M} E^P f(S_N) = aS_0. \tag{202}$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \tag{203}$$

where M is a set of equivalent martingale measures for the evolution of risky asset, given by the formula (170). The interval of non-arbitrage prices of contingent liability $f(S_N)$ lies in the set $[f(S_0), aS_0]$.

Proof. Since the conditions of Theorem 9 are satisfied, then the formula

$$\sup_{Q \in M} \int\limits_{\Omega_N} f(S_N) dQ = \sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \tag{204}$$

is true, where for the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ the representation

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \chi_A(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{205}$$

is valid, and

$$\sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{206}$$

where we denoted $\Omega_s^{0-} = \{\omega_s \in \Omega_s^0,\ \varepsilon_s(\omega_s) \leq 0\}$, $\Omega_s^{0+} = \{\omega_s \in \Omega_s^0,\ \varepsilon_s(\omega_s) > 0\}$. From the inequality, $f(S_N) \leq aS_N$, we have

$$\sup_{Q \in M} \int\limits_{\Omega} f(S_N) dQ \leq aS_0. \tag{207}$$

To prove the inverse inequality, we use the inequality

$$\sup_{Q \in M} \int\limits_{\Omega} f(S_N) dQ \geq$$

$$\sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right). \tag{208}$$

In the right hand side of the last inequality, let us put $\varepsilon_s(\omega_s^1) = 0, \ s \neq n$. Such elementary events ω_s^1 exist, due to the conditions relative to the random values $\varepsilon_s(\omega_s), \ s = \overline{1,N}$. We obtain

$$\sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) =$$

$$\sum_{i_n=1}^{2} \psi_n(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^{i_n}) f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^{i_n})}\right). \tag{209}$$

Therefore,

$$\sup_{Q\in M} \int_{\Omega} f(S_N)dQ \geq$$

$$\sup_{\omega_n^1\in\Omega_n^{0-},\omega_n^2\in\Omega_n^{0+}} \sum_{i_n=1}^{2} \psi_n(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^{i_n}) f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^{i_n})}\right). \tag{210}$$

Further,

$$\sup_{\omega_n^1\in\Omega_n^{0-},\omega_n^2\in\Omega_n^{0+}} \sum_{i_n=1}^{2} \psi_n(\omega_1^1,\dots,\omega_n^{i_n}) \times$$

$$f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^{i_n})}\right) =$$

$$\sup_{\omega_n^1\in\Omega_n^{0-},\omega_n^2\in\Omega_n^{0+}} \left[\frac{\Delta S_n^+(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^2)}{V_n(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1,\omega_n^2)} f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}\right) + \right.$$

$$\left. \frac{\Delta S_n^-(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1)}{V_n(\omega_1^1,\dots,\omega_{n-1}^1,\omega_n^1,\omega_n^2)} f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)}\right)\right] \geq$$

$$\lim_{\varepsilon_n(\omega_n^2)\to\infty} \lim_{\varepsilon_n(\omega_n^1)\to-\infty} \left[\frac{e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)} - 1}{e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)} - e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}} \times \right.$$

$$f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}\right) +$$

$$\left. \frac{1 - e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}}{e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)} - e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}} f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)}\right)\right] =$$

$$\lim_{\varepsilon_n(\omega_n^2)\to\infty} \frac{1}{e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)}} f\left(S_0 e^{\sigma_n(\omega_1^1,\dots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)}\right) = aS_0. \tag{211}$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

Substituting the inequality (211) into the inequality (209), we obtain the needed inequality.

Let us prove the equality (203). Using the Jensen inequality, we obtain

$$\inf_{P\in M} E^P f(S_N) \geq f(E^P S_N) = f(S_0). \tag{212}$$

Let us prove the inverse inequality. It is evident that

$$\sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq$$

$$\inf_{P\in M} E^P f(S_N). \tag{213}$$

Putting in this inequality $\varepsilon_i(\omega_i^1)=0,\ i=\overline{1,N}$, we obtain the needed. The last statement about the interval of non-arbitrage prices follows from [7] and [6]. Theorem 10 is proved.

Theorem 11. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, *let the evolution of risky asset be given by the formula (170). Suppose that* $0\leq a_i(\omega_1,\ldots,\omega_{i-1})\leq 1$, $\sigma_i(\omega_1,\ldots,\omega_{i-1})>\sigma_i>0,\ i=\overline{1,N}$, *and* $a_n=1$ *for a certain* $1\leq n\leq N$. *If the nonnegative payoff function* $f(x),\ x\in[0,\infty)$, *satisfies the conditions:*
1) $f(0)=K,\ f(x)\leq K$, *then*

$$\sup_{P\in M} E^P f(S_N) = K. \tag{214}$$

If, in addition, the nonnegative payoff function $f(x)$ *is a convex down one, then*

$$\inf_{P\in M} E^P f(S_N) = f(S_0), \tag{215}$$

where M *is a set of equivalent maqtingale measures for the evolution of risky asset, given by the formula (170). The interval of non-arbitrage prices of contingent liability* $f(S_N)$ *coincides with the set* $[f(S_0),K]$.

Proof. Due to Theorem 9, the equality

$$\sup_{Q\in M}\int_{\Omega_N} f(S_N)dQ = \sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \tag{216}$$

is valid, where for the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ the representation

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N, \tag{217}$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

is true, and

$$\sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int_{\Omega_N} f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}=$$

$$\sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right). \tag{218}$$

It is evident that

$$\sup_{P\in M} E^P f(S_N)\leq K. \tag{219}$$

Further,

$$\sup_{Q\in M}\int_{\Omega} f(S_N)dQ\geq$$

$$\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right). \tag{220}$$

In the right hand side of the last inequality, let us put $\varepsilon_s(\omega_s^1)=0,\ s\neq n$. We obtain

$$\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)=$$

$$\sum_{i_n=1}^{2}\psi_n(\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^{i_n})f\left(S_0e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^{i_n})}\right). \tag{221}$$

From the last equality, we obtain

$$\sup_{Q\in M}\int_{\Omega} f(S_N)dQ\geq$$

$$\sup_{\omega_n^1\in\Omega_n^{0-},\omega_n^2\in\Omega_n^{0+}}\sum_{i_n=1}^{2}\psi_n(\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^{i_n})f\left(S_0e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^{i_n})}\right). \tag{222}$$

Further,

$$\sup_{\omega_n^1 \in \Omega_n^{0-}, \omega_n^2 \in \Omega_n^{0+}} \sum_{i_n=1}^{2} \psi_n(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^{i_n}) f\left(S_0 e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^{i_n})}\right) =$$

$$\sup_{\omega_n^1 \in \Omega_n^{0-}, \omega_n^2 \in \Omega_n^{0+}} \left[\frac{\Delta S_n^+(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^2)}{V_n(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1, \omega_n^2)} f\left(S_0 e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}\right) + \right.$$

$$\left. \frac{\Delta S_n^-(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1)}{V_n(\omega_1^1, \ldots, \omega_{n-1}^1, \omega_n^1, \omega_n^2)} f\left(S_0 e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)}\right) \right] \geq$$

$$\lim_{\varepsilon(\omega_n^2) \to \infty} \lim_{\varepsilon(\omega_n^1) \to -\infty} \left[\frac{e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)} - 1}{e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)} - e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}} f\left(S_0 e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}\right) + \right.$$

$$\left. \frac{1 - e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}}{e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)} - e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^1)}} f\left(S_0 e^{\sigma_n(\omega_1^1,\ldots,\omega_{n-1}^1)\varepsilon_n(\omega_n^2)}\right) \right] =$$

$$f(0) = K. \tag{223}$$

Substituting the inequality (223) into the inequality (221), we obtain the needed inequality.

Let us prove the equality (215). Due to the convexity of the payoff function $f(x)$, using the Jensen inequality, we obtain

$$\inf_{P \in M} E^P f(S_N) \geq f(E^P S_N) = f(S_0). \tag{224}$$

Let us prove the inverse inequality. It is evident that

$$\sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) \geq$$

$$\inf_{P \in M} E^P f(S_N). \tag{225}$$

Putting in this inequality $\varepsilon_i(\omega_i^1) = 0,\ i = \overline{1, N}$, we obtain the needed. The last statement about the interval of non-arbitrage prices follows from [7] and [6]. Theorem 11 is proved.

Theorem 12. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177). Suppose that* $0 \leq a_i \leq 1$, $\sigma_i(\omega_1, \ldots, \omega_{i-1}) > \sigma_i > 0$, $i = \overline{1, N}$. *If the nonnegative payoff function* $f(x)$, $x \in [0, \infty)$, *satisfies the conditions: 1)* $f(0) = 0,\ f(x) \leq ax,\ \lim\limits_{x \to \infty} \frac{f(x)}{x} = a,\ a > 0,$ *then the inequalities*

$$f\left(S_0\prod_{i=1}^{N}(1-a_i)\right)+aS_0\left(1-\prod_{i=1}^{N}(1-a_i)\right)\leq \sup_{P\in M}E^Pf(S_N)\leq aS_0 \qquad (226)$$

are true. If, in addition, the nonnegative payoff function $f(x)$ is a convex down one, then

$$\inf_{P\in M}E^Pf(S_N)=f(S_0), \qquad (227)$$

where M is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (177).

Proof. As before,

$$aS_0\geq \sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\int\limits_{\Omega_N}f(S_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}=$$

$$\sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right). \qquad (228)$$

$$\sup_{\omega_N^1\in\Omega_N^{0-},\omega_N^2\in\Omega_N^{0+}}\sum_{i_N=1}^{2}\psi_N(\omega_1^{i_1},\ldots,\omega_N^{i_N})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)=$$

$$\sup_{\omega_N^1\in\Omega_N^{0-},\omega_N^2\in\Omega_N^{0+}}\left[\frac{\Delta S_N^+(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}\times\right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\left.\frac{\Delta S_N^-(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\right]\geq$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\left[\frac{e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1}{e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}}\times\right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\frac{1-e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}}{e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}}\times$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\Bigg]=$$

$$f(S_{N-1}(1-a_N))+aa_NS_{N-1}, \tag{229}$$

where we put

$$S_{N-1}=S_0\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right). \tag{230}$$

Substituting the inequality (229) into (228), we obtain the inequality

$$\sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq$$

$$\sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N-1}}\sum_{i_1=1,\ldots,i_{N-1}=1}^{2}\prod_{j=1}^{N-1}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0(1-a_N)\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)+aa_NS_0. \tag{231}$$

Applying $(N-1)$ times the inequality (231), we obtain the inequality

$$\sup_{Q\in M}\int_\Omega f(S_N)dQ\geq f(S_0\prod_{i=1}^{N}(1-a_i))+aS_0\sum_{i=1}^{N}a_i\prod_{s=i+1}^{N}(1-a_s)=$$

$$f\left(S_0\prod_{i=1}^{N}(1-a_i)\right)+aS_0\left(1-\prod_{i=1}^{N}(1-a_i)\right). \tag{232}$$

Let us prove the equality (227). Using the Jensen inequality, we obtain

$$\inf_{P\in M}E^Pf(S_N)\geq f(S_0). \tag{233}$$

Let us prove the inverse inequality. It is evident that

$$\sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) \geq$$

$$\inf_{P \in M} E^P f(S_N). \quad (234)$$

Putting in the inequality (234) $\varepsilon_n(\omega_n) = 0, n = \overline{1, N}$, we obtain the inverse inequality.

Theorem 13. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177). Suppose that* $0 \leq a_i \leq 1$, $\sigma_i(\omega_1,\dots,\omega_{i-1}) > \sigma_i > 0$, $i = \overline{1, N}$. *If the nonnegative payoff function* $f(x)$, $x \in [0, \infty)$, *satisfies the conditions: 1)* $f(0) = K$, $f(x) \leq K$, *then*

$$f\left(S_0 \prod_{i=1}^{N} (1 - a_i)\right) \leq \sup_{P \in M} E^P f(S_N) \leq K. \quad (235)$$

If, in addition, the nonnegative payoff function $f(x)$ *is a convex down one, then*

$$\inf_{P \in M} E^P f(S_N) = f(S_0), \quad (236)$$

where M *is the set of equivalent martingale measures for the evolution of risky asset, given by the formula (177).*

Proof. Let us obtain the estimate from below. Really,

$$K \geq \sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right). \quad (237)$$

Further,

$$\sup_{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}} \sum_{i_N=1}^{2} \psi_N(\omega_1^{i_1},\dots,\omega_N^{i_N}) f\left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) =$$

$$\sup_{\omega_N^1 \in \Omega_N^{0-}, \omega_N^2 \in \Omega_N^{0+}} \left[\frac{\Delta S_N^+(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^2)}{V_N(\omega_1^{i_1},\dots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)} \times \right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\frac{\Delta S_N^-(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}},\omega_N^1)}{V_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}},\omega_N^1,\omega_N^2)}f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\Bigg]\geq$$

$$\lim_{\varepsilon_N(\omega_N^2)\to\infty}\lim_{\varepsilon_N(\omega_N^1)\to-\infty}\left[\frac{e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1}{e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}}\times\right.$$

$$f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}-1\right)\right)\right)+$$

$$\frac{1-e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}}{e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^1)}}\times$$

$$\left.f\left(S_{N-1}\left(1+a_N\left(e^{\sigma_N(\omega_1^{i_1},\ldots,\omega_{N-1}^{i_{N-1}})\varepsilon_N(\omega_N^2)}-1\right)\right)\right)\right]=$$

$$f(S_{N-1}(1-a_N)), \tag{238}$$

where we put

$$S_{N-1}=S_0\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right). \tag{239}$$

Substituting the inequality (238) into (237), we obtain the inequality

$$\sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0\prod_{s=1}^{N}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right)\geq$$

$$\sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N-1}}\sum_{i_1=1,\ldots,i_{N-1}=1}^{2}\prod_{j=1}^{N-1}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f\left(S_0(1-a_N)\prod_{s=1}^{N-1}\left(1+a_s\left(e^{\sigma_s(\omega_1^{i_1},\ldots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})}-1\right)\right)\right). \tag{240}$$

Applying $(N-1)$ times the inequality (240), we obtain the inequality

$$\sup_{Q\in M}\int_\Omega f(S_N)dQ\geq f(S_0\prod_{i=1}^{N}(1-a_i)). \tag{241}$$

Let us prove the equality (236). Using the Jensen inequality we obtain

$$\inf_{P \in M} E^P f(S_N) \geq f(S_0). \tag{242}$$

Let us prove the inverse inequality. It is evident that

$$\sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \dots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) \geq$$

$$\inf_{P \in M} E^P f(S_N). \tag{243}$$

Putting in the inequality (243) $\varepsilon_n(\omega_n) = 0, \ n = \overline{1, N}$, we obtain the inverse inequality.

Theorem 14. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177). Suppose that* $0 \leq a_i \leq 1$, $\sigma_i(\omega_1, \dots, \omega_{i-1}) > \sigma_i > 0$, $i = \overline{1, N}$. *For the payoff function* $f(x) = (x - K)^+$, $x \in (0, \infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i)) \geq K, \\ S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right), & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i) < K. \end{cases} \tag{244}$$

If $S_0 \prod_{i=1}^{N}(1 - a_i)) \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $S_0 \prod_{i=1}^{N}(1 - a_i) < K$ *the set of non arbitrage prices coincides with the set* $\left[(S_0 - K)^+, S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right)\right]$.

Proof. Let us introduce the denotations

$$I_N = \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \dots, \omega_j^{i_j}) \times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \tag{245}$$

$$I_N^1 = \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f_1\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \qquad (246)$$

$$I_N^0 = \sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \sum_{i_1=1,\dots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1},\dots,\omega_j^{i_j})\times$$

$$f\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right), \qquad (247)$$

where we put $f_1(x) = (K - x)^+$. Let us estimate from above the value I_N. For this we use the equality

$$I_N = I_N^1 + S_0 - K, \qquad (248)$$

which follows from the identity: $f(x) = f_1(x) + x - K, \ x \geq 0$. Since

$$f_1\left(S_0 \prod_{s=1}^{N}\left(1 + a_s\left(e^{\sigma_s(\omega_1^{i_1},\dots,\omega_{s-1}^{i_{s-1}})\varepsilon_s(\omega_s^{i_s})} - 1\right)\right)\right) \leq f_1\left(S_0 \prod_{s=1}^{N}(1 - a_s)\right), \qquad (249)$$

we obtain the inequality

$$I_N \leq S_0 - K + f_1\left(S_0 \prod_{s=1}^{N}(1 - a_s)\right). \qquad (250)$$

From the inequality (250), we have

$$I_N^0 \leq S_0 - K + f_1\left(S_0 \prod_{s=1}^{N}(1 - a_s))\right) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i)) \geq K, \\ S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right), & \text{if} \quad S_0 \prod_{i=1}^{N}(1 - a_i) < K. \end{cases} \qquad (251)$$

Due to the inequality (226) of Theorem 12,

$$I_N^0 \geq f\left(S_0 \prod_{i=1}^{N}(1 - a_i)\right) + S_0\left(1 - \prod_{i=1}^{N}(1 - a_i)\right) \qquad (252)$$

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

and the inequality

$$I_N^0 \geq (S_0 - K)^+, \tag{253}$$

which follows from the Jensen inequality, we have

$$I_N^0 \geq \max \left\{ S_0 - K)^+, f\left(S_0 \prod_{i=1}^{N}(1-a_i) \right) + S_0 \left(1 - \prod_{i=1}^{N}(1-a_i) \right) \right\} =$$

$$\begin{cases} (S_0 - K)^+, & \text{if} \quad S_0 \prod_{i=1}^{N}(1-a_i)) \geq K, \\ S_0 \left(1 - \prod_{i=1}^{N}(1-a_i) \right), & \text{if} \quad S_0 \prod_{i=1}^{N}(1-a_i) < K. \end{cases} \tag{254}$$

This proves Theorem 14.

Theorem 15. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177). Suppose that* $0 \leq a_i \leq 1$, $\sigma_i(\omega_1, \ldots, \omega_{i-1}) > \sigma_i > 0$, $i = \overline{1, N}$. *For the payoff function* $f_1(x) = (K - x)^+$, $x \in (0, \infty)$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1 \left(S_0 \prod_{i=1}^{N}(1-a_i) \right). \tag{255}$$

The set of non arbitrage prices coincides with the interval

$$\left[(K - S_0)^+, f_1 \left(S_0 \prod_{i=1}^{N}(1-a_i) \right) \right].$$

Proof. The inequality

$$I_N^1 = \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f_1 \left(S_0 \prod_{s=1}^{N} \left(1 + a_s \left(e^{\sigma_s(\omega_1^{i_1}, \ldots, \omega_{s-1}^{i_{s-1}}) \varepsilon_s(\omega_s^{i_s})} - 1 \right) \right) \right) \leq f_1 \left(S_0 \prod_{i=1}^{N}(1-a_i) \right) \tag{256}$$

is true. Taking into account the inequality (235) of Theorem 13, we prove Theorem 15.

Theorem 16. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177). Suppose that* $0 \leq a_i \leq 1$, $\sigma_i(\omega_1, \ldots, \omega_{i-1}) > \sigma_i > 0$, $i = \overline{1, N}$. *For the payoff function* $f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1} \right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q\in M} E^Q f_1(S_0,S_1,\ldots,S_N) = \left(K - \frac{S_0\sum\limits_{i=0}^{N}\prod\limits_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \qquad (257)$$

The set of non arbitrage prices coincides with the interval

$$\left[(K-S_0)^+, \left(K - \frac{S_0\sum\limits_{i=0}^{N}\prod\limits_{s=1}^{i}(1-a_s)}{N+1}\right)^+\right], \text{ if } K > \frac{S_0\sum\limits_{i=0}^{N}\prod\limits_{s=1}^{i}(1-a_s)}{N+1}.$$

For $K \leq \frac{S_0\sum\limits_{i=0}^{N}\prod\limits_{s=1}^{i}(1-a_s)}{N+1}$ *the set of non arbitrage prices coincides with the point* 0.

Proof. Let us denote

$$S_n(\omega_1^1,\ldots,\omega_n^1) = S_0\prod_{s=1}^{n}\left(1 + a_s\left(e^{\sigma_s(\omega_1^1,\ldots,\omega_{s-1}^1)\varepsilon_s(\omega_s^1)} - 1\right)\right), \quad n=\overline{1,N},$$

$$t_N(\omega_1^1,\ldots,\omega_N^1) = \prod_{s=1}^{N}\frac{e^{\sigma_s(\omega_1^1,\ldots,\omega_{s-1}^1)\varepsilon_s(\omega_s^2)} - 1}{e^{\sigma_s(\omega_1^1,\ldots,\omega_{s-1}^1)\varepsilon_s(\omega_s^2)} - e^{\sigma_s(\omega_1^1,\ldots,\omega_{s-1}^1)\varepsilon_s(\omega_s^1)}}. \qquad (258)$$

It is evident that

$$I_N^2 = \sup_{\omega_i^1\in\Omega_i^{0-},\omega_i^2\in\Omega_i^{0+},i=\overline{1,N}}\sum_{i_1=1,\ldots,i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\times$$

$$f_1\left(S_0, S_1(\omega_1^{i_1}),\ldots,S_N(\omega_1^{i_1},\ldots,\omega_N^{i_N})\right) \geq$$

$$\lim_{\varepsilon_s(\omega_s^1)=-\infty,\ \varepsilon_s(\omega_s^2)\to\infty, s=\overline{1,N}} f_1\left(S_0, S_1(\omega_1^1),\ldots,S_N(\omega_1^1,\ldots,\omega_N^1)\right)\times$$

$$t_N(\omega_1^1,\ldots,\omega_N^1) = f_1\left(S_0, S_0(1-a_1),\ldots,S_0\prod_{s=1}^{N}(1-a_s)\right), \qquad (259)$$

$$I_N^2 \geq f_1\left(S_0, S_0(1-a_1),\ldots,S_0\prod_{s=1}^{N}(1-a_s)\right) = \left(K - \frac{S_0\sum\limits_{i=0}^{N}\prod\limits_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \qquad (260)$$

Let us prove the inverse inequality. We have

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

$$I_N^2 \leq \sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f_1\left(S_0, S_0(1-a_1), \ldots, S_0 \prod_{s=1}^{N}(1-a_s)\right) =$$

$$f_1\left(S_0, S_0(1-a_1), \ldots, S_0 \prod_{s=1}^{N}(1-a_s)\right) = \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{N}(1-a_s)}{N+1}\right)^+. \quad (261)$$

Therefore,

$$I_N^2 \leq \left(K - \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right)^+. \quad (262)$$

The inequalities (260), (262) prove Theorem 16.

Theorem 17. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177). Suppose that* $0 \leq a_i \leq 1$, $\sigma_i(\omega_1, \ldots, \omega_{i-1}) > \sigma_i > 0$, $i = \overline{1,N}$. *For the payoff function* $f(S_0, S_1, \ldots, S_N) = \left(\frac{\sum_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \ldots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_i)}{N+1} \geq K, \\ S_0\left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right), & \text{if } S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1} < K. \end{cases} \quad (263)$$

If $\frac{S_0 \sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_i)}{N+1} \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $S_0 \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1} < K$ *the set of non arbitrage prices coincides with the interval* $\left[(S_0 - K)^+, S_0\left(1 - \frac{\sum_{i=0}^{N} \prod_{s=1}^{i}(1-a_s)}{N+1}\right)\right]$.

Proof. Let us introduce the denotation

$$V_N = \sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f\left(S_0, S_1(\omega_1^{i_1}), \ldots, S_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N})\right). \tag{264}$$

Then, we have

$$V_N = \sup_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \sum_{i_1=1,\ldots,i_N=1}^{2} \prod_{j=1}^{N} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) \times$$

$$f_1\left(S_0, S_1(\omega_1^{i_1}), \ldots, S_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N})\right) + S_0 - K. \tag{265}$$

Due to Theorem 16,

$$V_N = (S_0 - K) + \left(K - \frac{S_0 \sum\limits_{i=0}^{N} \prod\limits_{s=1}^{i}(1-a_s)}{N+1}\right)^+ =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 \sum\limits_{i=0}^{N} \prod\limits_{s=1}^{i}(1-a_i)}{N+1} \geq K, \\ S_0\left(1 - \frac{\sum\limits_{i=0}^{N} \prod\limits_{s=1}^{i}(1-a_s)}{N+1}\right), & \text{if } S_0 \frac{\sum\limits_{i=0}^{N} \prod\limits_{s=1}^{i}(1-a_s)}{N+1} < K. \end{cases} \tag{266}$$

In the formula (265) we used the denotation

$$f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum\limits_{i=0}^{N} S_i}{N+1}\right)^+. \tag{267}$$

Theorem 17 is proved.

VIII. Estimation of Parameters

Suppose that $\{g_i(X_N)\}_{i=1}^N$ is a mapping from the set $[0,1]^N$ into itself, where $X_N = \{x_1, \ldots, x_N\}$, $0 \leq x_i \leq 1$, $i = \overline{1,N}$. If $S_0, S_1, \ldots, S_N$ is a sample of the process (177), let us denote the order statistic $S_{(0)}, S_{(1)}, \ldots, S_{(N)}$ of this sample. Introduce also the denotation $g_i([S]_N) = g_i\left(\frac{S_{(0)}}{S_{(N)}}, \ldots, \frac{S_{(N-1)}}{S_{(N)}}\right)$, $i = \overline{1,N}$.

Theorem 18. *Suppose that $S_0, S_1, \ldots, S_N$ is a sample of the random process (177). Then, for the parameters $a_1, \ldots, a_N$ the estimation*

$$a_1 = 1 - \tau_0 \frac{S_{(0)}}{S_0} g_1([S]_N), \quad 0 < \tau_0 \leq 1,$$

$$a_i = 1 - \frac{g_i([S]_N)}{g_{i-1}([S]_N)}, \quad i = \overline{2,N}, \tag{268}$$

is valid, if for $g_N([S]_N) > 0,\ [S]_N \in [0,1]^N$, *the inequalities* $g_1([S]_N) \geq g_2([S]_N) \geq \ldots \geq g_N([S]_N)$ *are true. If* $\tau_0 = 0$, *then* $a_i = 1,\ i = \overline{1,N}$.

Proof. The estimation of the parameters $a_1, \ldots, a_N$ we do using the representation of random process $S_n,\ n = \overline{1,N}$. The smallest value of the random variable S_n is equal $S_0 \prod_{i=1}^{n}(1-a_i),\ n = \overline{1,N}$. Let us determine the parameters a_i from the relations

$$S_0 \prod_{i=1}^{N}(1-a_i) = \tau g_N\left([S]_N\right), \ldots, S_0 \prod_{i=1}^{N-k}(1-a_i) = \tau g_{N-k}\left([S]_N\right), \ldots,$$

$$S_0 \prod_{i=1}^{N-k-1}(1-a_i) = \tau g_{N-k-1}\left([S]_N\right), \ldots, S_0(1-a_1) = \tau g_1\left([S]_N\right), \tag{269}$$

where $\tau > 0$. Taking into account the relations (269), we obtain

$$S_0(1-a_1) = \tau g_1\left([S]_N\right),$$

$$\tau g_{N-k-1}\left([S]_N\right)(1-a_{N-k}) = \tau g_{N-k}\left([S]_N\right), \quad k = \overline{2,N}. \tag{270}$$

Solving the relations (270), we have

$$a_1 = 1 - \frac{\tau}{S_0} g_1\left([S]_N\right), \quad a_{N-k} = 1 - \frac{g_{N-k}\left([S]_N\right)}{g_{N-k-1}\left([S]_N\right)}, \quad k = \overline{2,N}. \tag{271}$$

It is evident that $a_{N-k} \geq 0,\ k = \overline{2,N}$. To provide the positiveness of a_1 and the inequalities $\tau g_{N-n}\left([S]_N\right) \leq S_{N-n},\ n = \overline{0,N-1},\ S_0 \geq S_{(0)}$, meaning that the random process (177) takes all the values from the sample $S_n,\ n = \overline{0,N}$, we must to put $\tau = \tau_0 S_{(0)},\ 0 < \tau_0 \leq 1$. It is evident that, if $\tau_0 = 0$, then $a_i = 1,\ i = \overline{1,N}$ Theorem 18 is proved.

Remark 1. *It is evident that*

$$a_i = 1, \quad i = \overline{N-k,N},\ 1 < k \leq N-1,\ a_i = 1 - \frac{g_i([S]_N)}{g_{i-1}([S]_N)},\ i = \overline{2,N-k-1},$$

$$a_1 = 1 - \frac{\tau_0 S_{(0)}}{S_0} g_1([S]_N), \quad 0 < \tau_0 \leq 1, \tag{272}$$

is also estimation of the parameters $a_1, \ldots, a_N$ *if*

$$0 < g_{N-k-1}([S]_N) \leq g_{N-k-2}([S]_N) \ldots \leq g_1([S]_N),\ [S]_N \in [0,1]^N.$$

Such estimation is not interesting since

$$\prod_{i=1}^{N-i}(1-a_i)=0, \quad i=\overline{0,k}.$$

Remark 2. If

$$g(x)=\begin{cases}\frac{S_0}{S_{(0)}}x, & if \quad 0\leq x\leq \frac{S_{(0)}}{S_0},\\ 1, & if \quad \frac{S_{(0)}}{S_0}<x\leq 1,\end{cases} \tag{273}$$

$$g_i([S]_N)=g\left(\frac{S_{(N-i)}}{S_{(N)}}\right), \quad i=\overline{1,N}, \ \tau_0=1,$$

then for the parameters $a_1,\ldots,a_N$ *the estimation*

$$a_i=\begin{cases}1-\frac{S_{(N-i)}}{S_{(N-i+1)}}, & if \quad \frac{S_{(N-i+1)}}{S_{(N)}}\leq\frac{S_{(0)}}{S_0},\\ 1-\frac{S_{(N-i)}}{S_{(N)}}\frac{S_0}{S_{(0)}}, & if \quad \frac{S_{(N-i+1)}}{S_{(N)}}>\frac{S_{(0)}}{S_0}, \ \frac{S_{(N-i)}}{S_{(N)}}\leq\frac{S_{(0)}}{S_0},\\ 0, & if \quad \frac{S_{(N-i)}}{S_{(N)}}>\frac{S_{(0)}}{S_0}.\end{cases} \quad i=\overline{2,N}, \tag{274}$$

$$a_1=\begin{cases}1-\frac{S_{(N-1)}}{S_{(N)}}, & if \quad \frac{S_{(N-1)}}{S_{(N)}}\leq\frac{S_{(0)}}{S_0},\\ 1-\frac{S_{(0)}}{S_0}, & if \quad \frac{S_{(N-1)}}{S_{(N)}}>\frac{S_{(0)}}{S_0}\end{cases} \tag{275}$$

is true. The following equalities

$$\prod_{i=1}^{N}(1-a_i)=\frac{S_{(0)}}{S_0}g\left(\frac{S_{(0)}}{S_{(N)}}\right)=\frac{S_{(0)}}{S_{(N)}},$$

$$\prod_{i=1}^{N-k}(1-a_i)=\begin{cases}\frac{S_{(k)}}{S_{(N)}}, & if \quad \frac{S_{(k)}}{S_{(N)}}\leq\frac{S_{(0)}}{S_0},\\ \frac{S_{(0)}}{S_0}, & if \quad \frac{S_{(k)}}{S_{(N)}}>\frac{S_{(0)}}{S_0},\end{cases} \quad k=\overline{1,N-1}, \tag{276}$$

are valid.

Remark 3. Suppose that $g(x)=x, \ x\in[0,1]$*. Let us put* $g_{N-i}([S]_N)=g(\frac{S_{(i)}}{S_{(N)}})=\frac{S_{(i)}}{S_{(N)}}, \ i=\overline{0,k}, \ g_{N-i}([S]_N)=1, \ i=\overline{k+1,N-1}$*. Then,*

$$a_1=1-\tau_0\frac{S_{(0)}}{S_0}, \quad 0<\tau_0\leq 1, \quad a_i=0, \quad i=\overline{2,N-k-1},$$

$$a_i=1-\frac{g_i([S]_N)}{g_{i-1}([S]_N)}, \quad i=\overline{N-k,N}, \tag{277}$$

is an estimation for the parameters $a_1,\ldots,a_N$*.*

In the next Theorems we put $\tau_0=1$. This corresponds to the fact that fair price of super-hedge is minimal for the considered statistic.

Theorem 19. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (177), with parameters a_i, $i=\overline{1,N}$, given by the formula (268). For the payoff function $f(x)=(x-K)^+$, $x \in (0,\infty)$, $K>0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_N) =$$

$$\begin{cases} (S_0-K)^+, & \text{if} \quad S_{(0)} g_N([S]_N) \geq K, \\ S_0\left(1-\frac{S_{(0)} g_N([S]_N)}{S_0}\right), & \text{if} \quad S_{(0)} g_N([S]_N) < K. \end{cases} \tag{278}$$

If $S_{(0)} g_N([S]_N) \geq K$, then the set of non arbitrage prices coincides with the point $(S_0-K)^+$, in case if $S_{(0)} g_N([S]_N) < K$ the set of non arbitrage prices coincides with the closed set $\left[(S_0-K)^+, S_0\left(1-\frac{S_{(0)} g_N([S]_N)}{S_0}\right)\right]$.

The fair price of super-hedge for the statistic (274), (275) is given by the formula

$$\sup_{Q \in M} E^Q f(S_N) = \begin{cases} (S_0-K)^+, & \text{if} \quad S_0 \frac{S_{(0)}}{S_{(N)}} \geq K, \\ S_0\left(1-\frac{S_{(0)}}{S_{(N)}}\right), & \text{if} \quad S_0 \frac{S_{(0)}}{S_{(N)}} < K. \end{cases} \tag{279}$$

If $S_0 \frac{S_{(0)}}{S_{(N)}} \geq K$, then the set of non arbitrage prices coincides with the point $(S_0-K)^+$, in case if $S_0 \frac{S_{(0)}}{S_{(N)}} < K$ the set of non arbitrage prices coincides with the closed set $\left[(S_0-K)^+, S_0\left(1-\frac{S_{(0)}}{S_{(N)}}\right)\right]$.

The fair price of super-hedge is minimal one for the statistic (268) with $g_i(X_N) = g_N(X_N) = 1$, $i=\overline{1,N-1}$, and is given by the formula

$$\sup_{Q \in M} E^Q f(S_N) = \begin{cases} (S_0-K)^+, & \text{if} \quad S_{(0)} \geq K, \\ S_0 - S_{(0)}, & \text{if} \quad S_{(0)} < K. \end{cases} \tag{280}$$

If $S_{(0)} \geq K$, then the set of non arbitrage prices coincides with the point $(S_0-K)^+$, in case if $S_{(0)} < K$ the set of non arbitrage prices coincides with the closed set $[(S_0-K)^+, S_0 - S_{(0)}]$.

Theorem 20. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let the evolution of risky asset be given by the formula (177) with the parameters a_i, $i=\overline{1,N}$, given by the formula (268). For the payoff function $f_1(x)=(K-x)^+$, $x \in (0,\infty)$, $K>0$, the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1\left(S_{(0)} g_N([S]_N)\right). \tag{281}$$

The set of non arbitrage prices coincides with the closed interval $\left[(K-S_0)^+, f_1\left(S_{(0)} g_N([S]_N)\right)\right]$.

The fair price of super-hedge for the statistic (274), (275) is given by the formula

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1\left(S_0 \frac{S_{(0)}}{S_{(N)}}\right). \tag{282}$$

The set of non arbitrage prices coincides with the closed interval $\left[(K-S_0)^+, f_1\left(S_0 \frac{S_{(0)}}{S_{(N)}}\right)\right]$.

The fair price of super-hedge is minimal one for the statistic (268) with $g_i(X_N) = g_N(X_N) = 1,\ i = \overline{1, N-1}$, *and is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_N) = f_1\left(S_{(0)}\right). \tag{283}$$

The set of non arbitrage prices coincides with the closed interval $\left[(K-S_0)^+, f_1\left(S_{(0)}\right)\right]$.

Theorem 21. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177) with the parameters* $a_i,\ i = \overline{1, N}$, *given by the formula (268). For the payoff function* $f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{\sum_{i=0}^{N} S_i}{N+1}\right)^+$, $K > 0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i\left([S]_N\right)}{(N+1)}\right)^+. \tag{284}$$

The set of non arbitrage prices coincides with the closed interval

$$\left[(K-S_0)^+, \left(K - \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)}\right)^+\right],\ \textit{if } K > \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)}.$$

For $K \le \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)}$ *the set of non arbitrage prices coincides with the point* 0.

The fair price of super-hedge is minimal one for the statistic (268) with $g_i(X_N) = g_N(X_N) = 1,\ i = \overline{1, N-1}$, *and is given by the formula*

$$\sup_{Q \in M} E^Q f_1(S_0, S_1, \ldots, S_N) = \left(K - \frac{S_0 + S_{(0)} N}{(N+1)}\right)^+. \tag{285}$$

The set of non arbitrage prices coincides with the closed interval

$\left[(K-S_0)^+, \left(K - \frac{S_0 + S_{(0)} N}{(N+1)}\right)^+\right]$, *if* $K > \frac{S_0 + S_{(0)} N}{(N+1)}$. *For* $K \le \frac{S_0 + S_{(0)} N}{(N+1)}$ *the set of non arbitrage prices coincides with the point* 0.

Theorem 22. *On the probability space* $\{\Omega_N, \mathcal{F}_N, P_N\}$, *let the evolution of risky asset be given by the formula (177) with the parameters* $a_i,\ i=\overline{1,N}$, *given by the formula (268). For the payoff function* $f(S_0, S_1, \dots, S_N) = \left(\frac{\sum_{i=0}^{N} S_i}{N+1} - K\right)^+$, $K>0$, *the fair price of super-hedge is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \dots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)} \geq K, \\ \left(S_0 - \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)}\right), & \text{if } \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)} < K. \end{cases} \tag{286}$$

If $\frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)} \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $\frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)} < K$ *the set of non arbitrage prices coincides with the closed interval* $\left[(S_0 - K)^+, \left(S_0 - \frac{S_0 + S_{(0)} \sum_{i=1}^{N} g_i([S]_N)}{(N+1)}\right)\right]$.

The fair price of super-hedge is minimal one for the statistic (268) with $g_i(X_N) = g_N(X_N) = 1,\ i = \overline{1, N-1}$, *and is given by the formula*

$$\sup_{Q \in M} E^Q f(S_0, S_1, \dots, S_N) =$$

$$\begin{cases} (S_0 - K)^+, & \text{if } \frac{S_0 + S_{(0)} N}{(N+1)} \geq K, \\ \left(S_0 - \frac{S_0 + S_{(0)} N}{(N+1)}\right), & \text{if } \frac{S_0 + S_{(0)} N}{(N+1)} < K. \end{cases} \tag{287}$$

If $\frac{S_0 + S_{(0)} N}{(N+1)} \geq K$, *then the set of non arbitrage prices coincides with the point* $(S_0 - K)^+$, *in case if* $\frac{S_0 + S_{(0)} N}{(N+1)} < K$ *the set of non arbitrage prices coincides with the closed interval* $\left[(S_0 - K)^+, \left(S_0 - \frac{S_0 + S_{(0)} N}{(N+1)}\right)\right]$.

IX. Conclusions

Section 1 provides an overview of the achievements and formulates the main problem that has been solved. Section 2 contains the formulation of conditions which must satisfy the evolution of risky assets. In Section 3, conditions (14) - (16) are formulated for the set of nonnegative random variables with the help of which a family of measures is constructed in a recurrent way. In Lemma 1, conditions were found for the existence of bounded nonnegative random variables satisfying the conditions

(14) - (16). In Lemma 2, it was proved that the family of measures introduced in the recurrent way is equivalent to the original measure.

Theorem 1 gives sufficient conditions under which the introduced family of measures is the set of martingale measures equivalent to the original measure for the evolution of risky assets considered in Section 1.

In Section 4, relying on the concept of an exhaustive decomposition of a measurable space, in Lemma 4, we prove an integral inequality for a nonnegative random variable for the constructed family of martingale measures.

In Theorem 2, for a special class of evolutions of risky assets for the nonnegative random variable satisfying the integral inequality, obtained in Lemma 4, a pointwise system of inequalities is obtained.

In Lemma 5, on the basis of Lemma 4, we obtained a pointwise system of inequalities for a nonnegative random variable for the general case of the evolution of risky assets.

Theorem 3 contains sufficient conditions under the fulfillment of which the resulting system of inequalities with respect to the nonnegative random variable has a solution whose right-hand side satisfies the condition: the conditional expectation of the right-hand side of the inequality with respect to the filtration is equal to 1.

Theorem 4 solves the same problem as in Theorem 5 for the general case of the evolution of risky assets.

In Section 5, based on the inequalities obtained in Theorems 3 and 4, we prove a theorem on the optional decomposition of nonnegative super-martingales with respect to the family of equivalent martingale measures.

The description of the family of equivalent martingale measures given in Theorem 1 is rather general, therefore, in Section 6, a spot set of measures is introduced. In Lemma 6, the representation is obtained for the family of spot measures.

Based on the concept of the spot family of measures, the family of α-spot measures based on a set of positive random variables is introduced. Theorem 6 provides sufficient conditions for the integral over the set of α-spot measures to be an integral over the set of spot measures.

In Theorem 7, sufficient conditions are given when the family of spot measures is a family of martingale measures and the constructed family of measures, that is an integral over the set of α-spot measures, is a family of martingale measures being equivalent to the original measure.

Theorem 8 describes the class of evolutions of risky assets for which the family of equivalent martingale measures is such that each martingale measure is an integral over the set of spot measures.

Section 7 is devoted to the application of the results obtained in the previous sections. A class of random processes is considered, which contains well-known processes of the type ARCH and GARCH ones. Two types of random processes are considered, those for which the price of an asset cannot go down to zero and those for which the price can go down to zero during the period under consideration. The first class of processes describes the evolution of well-managed assets. We will call these assets relatively stable.

Theorem 9 asserts that for the evolution of relatively stable assets in the period under consideration, the family of martingale measures is one and the same. The family of martingale measures for the evolution of risky assets whose price can drop to zero is contained in the family of martingale measures for the evolution of relatively stable assets. Each of the martingale measures for the considered class of evolutions is an integral over the set of spot martingale measures. On this basis, the fair price of the super hedge is given by the formula (185). In Theorems 10 and 11, an interval of non-arbitrage prices is found for a wide class of payoff functions in the case when evolution describes relatively unstable assets. This range is quite wide for the payment functions of standard put and call options. The fair price of the super hedge is in this case the starting price of the underlying asset. In Theorems 12, 13 estimates are found for the fair price of the super-hedge for the introduced

Global Journal of Science Frontier Research (A) Volume XX Issue XIV Version I Year 2020

class of evolutions with respect to stable assets. In Theorems 14 and 15, formulas are found for the fair price of contracts with call and put options for the evolution of assets described by parametric processes.

In Theorems 16 and 17, the same formulas are found for Asian-type put and call options. A characteristic feature of these estimates is that for the evolution of relatively stable assets, the fair price of the super hedge is less than the price of the initial price of the asset.

In Section 8, the estimates of the parameters of risky assets included in the evolution are obtained. This result is contained in Theorem 18. In Theorems 19 and 20, formulas are found for the fair price of contracts with call and put options for the obtained parameter estimates, and the interval of non-arbitrage prices for different statistics is found. The same results are contained in Theorems 21, 22 for Asian-style call and put options.

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Move the Spacecraft at 99% Speed of Light by Rotation Technique

By Deepak Kamalta

Abstract- In this research, a rectangular box is rotated in two direction at the same time. Which creates a rotating path to travel from one place in space to another. By which we can get 99% speed of light using today`s rocket. But the speed of today`s rockets is only 11,000 m per second. The speed of the rocket increases in two stages in the spacecraft created by the technology of rotation. To understand this, one has to read the method given below.

Keywords: rotation technique, light speed spacecraft.

GJSFR-A Classification: FOR Code: 020109

MOVETHESPACECRAFTAT99SPEEDOFLIGHTBYROTATIONTECHNIQUE

Move the Spacecraft at 99% Speed of Light by Rotation Technique

Deepak kamalta

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Keywords: rotation technique, light speed spacecraft.

I. Introduction

Currently, spacecraft made of propulsion system are used to travel in deep space. But the speed of rocket is very low compared to the speed of light. There are some hypotheses that describe how to travel at the speed of light. Dr. Harold "Sonny" White [1] It appear that the warp drive model has nearly all the desirable mathematical characteristics of true interstellar space drive, the metric has one less appealing characteristic – it violates all 3 energy conditions (strong, weak, and dominant) because of the need for negative energy density. Kevin L. G. Parkin [2] Breakthrough Starshot is an initiative to prove ultra-fast light-propelled Nano craft.

Our technology has not yet been developed enough to make the spacecraft described in these hypotheses. But I can move the spacecraft at the 99% speed of light at the present time using the technique of rotation.

II. Method

Before understanding the rotation technique, you need to know how it becomes a spacecraft.

First, make a rectangular box. Take two rockets and connect them both with a circular shaft. Now connect the circular shaft to the rectangular box at the same point as shown in fig 1.1.

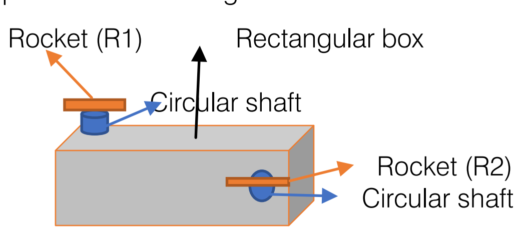

Fig. 1.1

Author: e-mail: deepak1kamalta@gmail.com

Work of rotations in spacecraft:

When we start the rockets engine it rotates the entire

Rectangular box through a circular shaft in two direction at same time.

a) *I will try to explain the rotation of rectangular box with the help of fig 1.2*

In this fig 1.2 it is shown that when the rectangular box is rotated in two directions, what will be the position of the rectangular box at 90°, 180°, 270° and 360°.

When we look at the rotation path of a rectangular box in this fig 1.2, we find that it travels two direction at a time. In this fig 1.2, one path of the rectangular box is shown in green and the other in yellow. When the rectangular box rotates 360degrees it travels 3 times on the green route and 2 times more than its size on the yellow route. We can use the green path shown in the fig 1.2 to run at 99% light speed.

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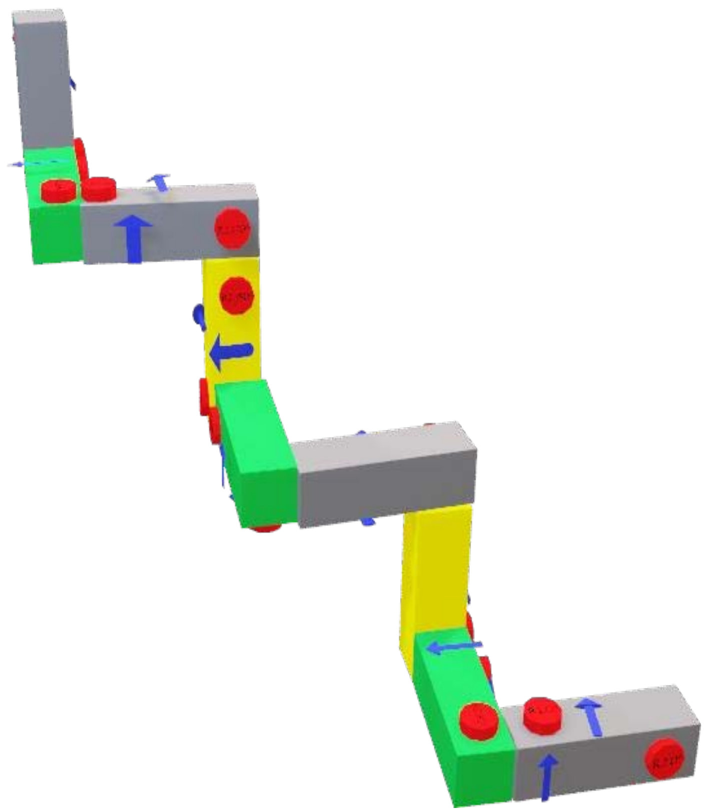

Fig. 1.2

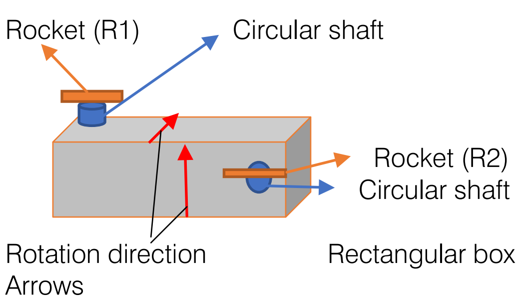

Fig. 1.3

With the help of the direction arrow shown above the rectangular box in fig 1.3, we can know in which direction R1 and R2 will rotate the rectangular box.

The rotation speed of rectangular box will depend on the size of the spacecraft. If you want to build a small sized spacecraft, the rotation speed of the rectangular box must be increased.

Since the speed of the rocket is only 11,000m/s, therefore, we must reduce the size of the circumference of the circular shaft. So that we can increase the rotation speed of circular shaft. Which rotate the rectangular box.

There are two side of circular shaft whose circumference will be different. I named both side of the circular shaft as C1 and C2. C1 will always connect with the rockets and C2 will connect with the rectangular box. If you want to increase the speed of circular shaft the size of C1 circumference will always be 1meter. And the size of C2 circumference will always be less than 1 meter.

I have prepared a formula by which we can know the speed of the spacecraft created by the rotation technique.

b) *3(Rotation speed of rectangular box × Length of rectangular box)*

To use this formula, you must first decide the size of your rectangular box which you can take as your need. I`m taking the rectangular box length 200 meter. To get the speed of light we first need to know what will be the rotation speed of a rectangular box when the rectangular box is 200 meter in length.

To know this, we can use the formula given below:

Rotation speed of rectangular box = speed of light ÷ 3(length of rectangular box)

Rotation speed of rectangular box = 300,000Km/s ÷ 3 (0.2Km) = 500,000/s

Now, we can know by using the driven pulley formula that if we want the rotation speed of C2 to be 124,887.083/s then what will be the diameter of C2.
To calculate the diameter of C2 by driven pulley method:

RPM1 ÷ RPM2 = Diameter1 ÷ Diameter2

We know RPM1 is C1 which is equal to speed of rocket speed. Rocket on earth can accelerated at a speed of 90m/s. But there is no gravity in space. So the rocket`s acceleration increases slightly in space.
We can calculate the acceleration of rocket by using formula

Acceleration = resultant force divided by mass and the resultant force is the thrust – weight

But in space weight is always zero. So the resultant force in space is always equal to the thrust.
If we accelerate the rocket to a speed of 90m/s in space, we will still achieve the speed of 35,730m/s in 397s.

So here we can take the speed of C1 is 35730m/s.

And RPM2 is C2 = 500000/s

Now Diameter1 of C1 = 0.32meter

So, the diameter of C2 is

35730m/s ÷ 500000/s = Diameter 2 ÷ 0.32m

C2 = 0.32m ÷ 13.99m

Now C2 = 0.0228734811

1m = 39.38inch

So C2 = 0.0228734811 × 39.38inch

C2 = 0.900757686inch.

Because the entire spaceship will be rotated by C2 shaft, it is very important for C2 shaft to be strong.

High quality graphene is the only material with ultimate tensile strength of 130 gig pascal. That can easily handle the weight of a spaceship.

III. Controlling of Spacecraft

First know that the rectangular box is only one engine of the spacecraft. We have to cover the rectangular box with a spherical ball. So that the rectangular box rotates easily inside it and we will find a place to place the payload above the spherical ball. The rectangular box inside the spherical ball can rotate in any direction. But after starting the rocket, it will move only in one direction. Which can be any direction of the spherical ball. So we have to put rocket booster on 6 direction of spherical ball. Which will help us in the direction control of the spacecraft. It will also be very strong due to the spherical shape of the spacecraft.

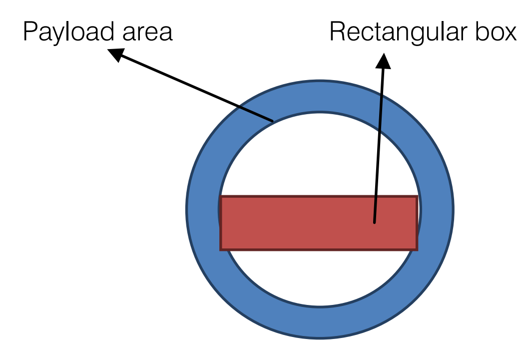

Fig. 1.4: Full diagram of light speed spacecraft

IV. Result and Discussion

Speed of spacecraft = 3(rotation speed of rectangular × length of rectangular box)

The rotation speed of rectangular box is equal to the rotation speed of C2.
So the speed of space craft = 3(500,000/s × 0.2km)

= 300,000km/s

Currently, there is no Spacecraft that can travel at the 99% speed of light.

V. Conclusion

If we have to launch a spacecraft from earth, we have to consider other ways of rotating a rectangular box.

Because rocket engines can move spacecraft made by rotation technology at a speed of 299,729.009km/s only in space.

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c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- o You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- o Give details of all of your remarks as much as possible, focusing on mechanisms.
- o Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- o One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- o Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

Index

A

B

D

E

F

I

M

P

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