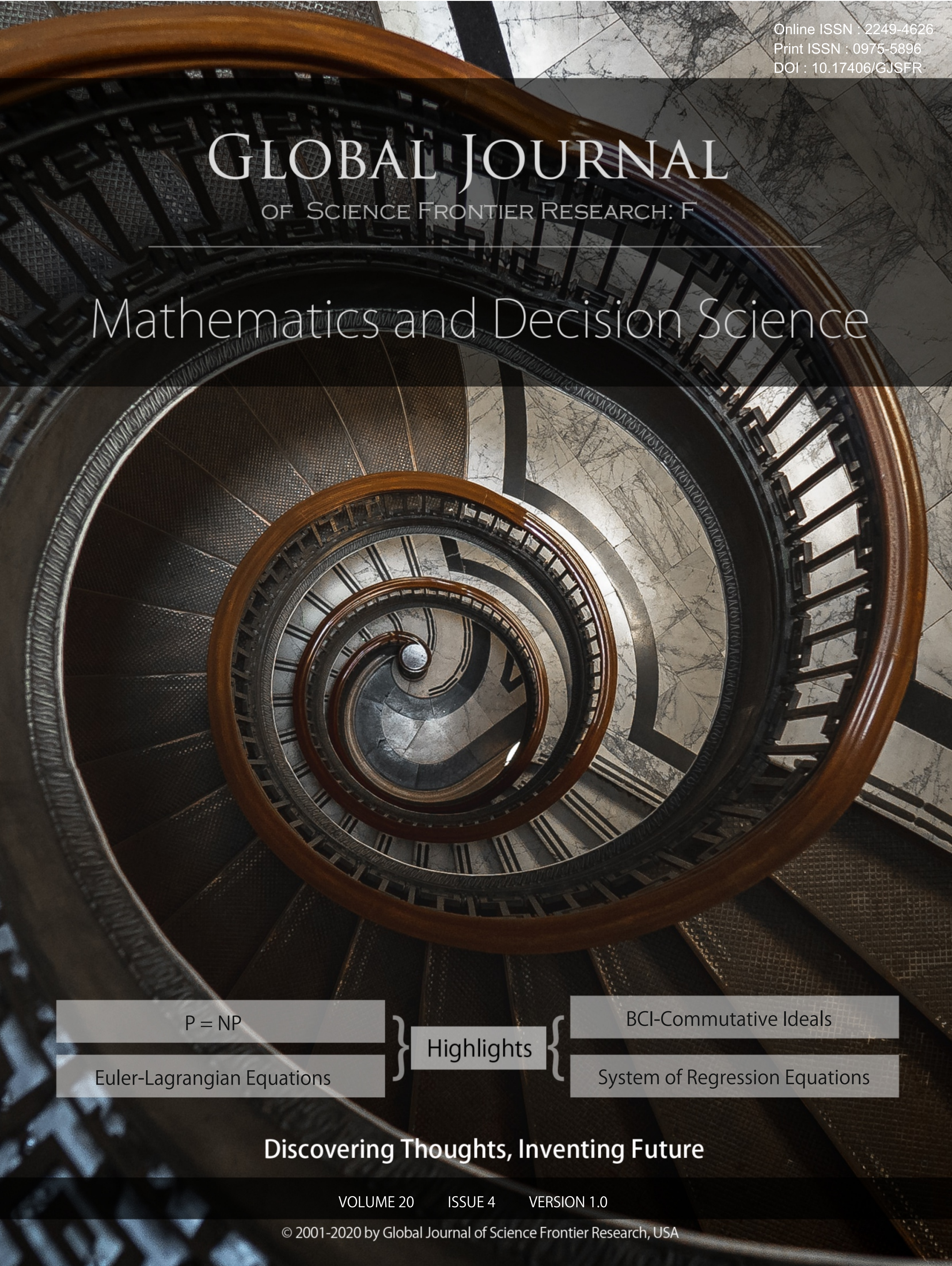

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Mathematics and Decision Science

P = NP

Euler-Lagrangian Equations

Highlights

BCI-Commutative Ideals

System of Regression Equations

Discovering Thoughts, Inventing Future

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P = NP

By B. Litow

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3. K. Manders and L. Adleman. NP-complete decision problems for binary quadratics. JCSS, 16:168–184, 1978.

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I. Introduction

We exhibit a polynomial time algorithm for the **NP** complete problem SBQR, size-bounded quadratic residues. This establishes the equality of the complexity classes **P** and **NP**. Proof of **NP** completeness was given in [3]. SBQR is the set of triples of the binary representations of the positive integers a, b, c such that there exists a positive integer x satisfying $x^2 \equiv a \pmod{b}$ and $x \leq c$. W.L.O.G. we impose $a, c < b$. Polynomial time means determinisic Turing machine time $\log^{O(1)} b$. We follow standard complexity class terminology [1].

II. A Sieve for SBQR

We reserve some notation.

- Unless otherwise indicated $O()$ notation indicates an absolute constant.
- $(x, y), [x, y]$, etc. denote real intervals, with a rounded bracket indicating the endpoint is not included.
- $[x..y]$ is the set of integers z satisfying $x \leq z \leq y$.
- ℓ is the least integer satisfying $b^2 < 2^\ell$.
- $c_* = \lfloor (c^2 - a)/b \rfloor$. Note: $0 \leq c_* < b$.
- ι is the positive branch of $\sqrt{-1}$.
- $\tau = \iota/2^\ell + t$, where $t \in [0, 1]$ is a real variable. Note that any function of τ is obviously a function of t.
- $\mathbf{e}(z) = \exp(\pi \iota z)$. We regard π as represented by a rational but do not carry out the associated error analysis.
- $\Im(z)$ and $\Re(z)$ are the imaginary and real parts of complex z. Where brackets are unnecessary we will write $\Re z$ and $\Im z$.
- $T_{(m:f)}(z)$ is the sum of the first m terms of the Taylor series for $f(z)$.

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We will frequently work with an integral of the form

$$g(t) = \int_u^v f(t,x)dx \ ,$$

where $t \in [0,1]$ and always $g(t)$ is continuous. This means that $\max_{t\in[0,1]} |g(t)|$ exists. Since $|g(t)| \le \int_u^v |f(t,x)|dx$ an upper bound $O(\mu)$ on $\int_u^v |f(t,x)|dx$ is an upper bound on $\max_{t\in[0,1]} |g(t)|$. We will write $|g(t)| = O(\mu)$ rather than $\max_{t\in[0,1]} |g(t)| = O(\mu)$.

We define Ω to be

$$\int_0^1 \sum_{n=1}^{\infty} \mathbf{e}(n^2\tau) \cdot \sum_{j=1}^{c_*} \mathbf{e}(-(a+bj)t)dt \ . \tag{1}$$

The infinite summation exists because $\Im(\tau) > 0$.

The next lemma justifies calling Ω a sieve for SBQR.

Lemma 1 *If there exists a positive integer n satisfying $n \le c$ and $n^2 \equiv a \pmod b$, then $\Omega > \exp(-\pi)$, else $\Omega = 0$.*

Proof: For integer k,

$$\int_0^1 \mathbf{e}(kt)dt = \begin{cases} 0 \text{ if } k \neq 0 \\ 1 \text{ if } k = 0 \end{cases} \tag{2}$$

Eq. 1 can be written as

$$\sum_{j=1}^{c_*} \sum_{n=1}^{\infty} \exp(-\pi n^2/2^\ell) \cdot \int_0^1 \mathbf{e}((n^2 - a - bj)t)dt \ . \tag{3}$$

All summands of Eq. 3 are nonnegative. The SBQR condition is equivalent to the existence of positive integers $n \le c$ and $j \le c_*$ such that $n^2 = a + bj$. The lemma follows from this equivalence, the value of ℓ, Eq. 2 and Eq. 3.

Our polynomial time algorithm for SBQR amounts to computing $\hat{\Omega}$ in polynomial time such that

$$|\Omega - \hat{\Omega}| < \exp(-\pi)/2 \ . \tag{4}$$

By Lemma 1 this solves SBQR in polynomial time.

III. Ω in Terms of a Theta Function

Define the Theta function $\vartheta(\tau)$ to be

$$1 + 2\sum_{n=1}^{\infty} (-1)^n \mathbf{e}(n^2\tau) \ . \tag{5}$$

From Eq. 5 we get

$$\sum_{n=1}^{\infty} (-1)^n \mathbf{e}(n^2\tau) = \frac{\vartheta(\tau) - 1}{2} \ . \tag{6}$$

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Lemma 2 Ω *equals*

$$\sum_{j=1}^{c_*} \int_0^1 (-1)^{a+bj} \frac{\vartheta(\tau)-1}{2} \mathbf{e}(-(a+bj)t)dt \, .$$

Proof: The expression for Ω matches Eq. 1 term by term except that each term of the infinite sum in Eq. 1 is multiplied by $(-1)^n(-1)^{a+bj}$. If $n^2 = a+bj$, then since $n \equiv n^2 \pmod 2$, $(-1)^n(-1)^{a+bj} = 1$. Terms with $n^2 \neq a+bj$ contribute 0 under integration so the sign of $(-1)^n(-1)^{a+bj}$ does not matter.

Lemma 2 suggests that the key to producing $\hat{\Omega}$ is a suitable polynomial time approximation of $\vartheta(\tau)$. Our approximation of $\vartheta(\tau)$ is based on

$$\vartheta(\tau) = -\iota \int_{\iota-\infty}^{\iota+\infty} \mathbf{e}(u^2\tau) \frac{1}{\sin(\pi u)} du \, . \tag{7}$$

A derivation of Eq. 7 due to R. Puzio [4] is included in section ?.

Before proceeding to approximate $\vartheta(\tau)$ we make an observation about a related Theta function, namely

$$\theta(\tau) = 1 + 2\sum_{n=1}^{\infty} \mathbf{e}(n^2\tau) \, ,$$

which is defined for $\Im\tau > 0$. Clearly, $\theta(\tau)$ is very similar to $\vartheta(\tau)$. Let γ be the matrix

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} ,$$

where x, y, z, w are integers and $xw - yz = 1$. The action $\gamma \cdot \tau$ of γ on τ is defined by

$$\gamma \cdot \tau = \frac{x\tau + y}{z\tau + w} \, .$$

For given $t \in [1, 3]$ and corresponding τ there exists γ such that

$$\Im(\gamma \cdot \tau) \geq \sqrt{3}/2 \, .$$

Theorem 4.3 in Chap. III.4 [2] shows that if $z \equiv 0 \pmod 4$, then $\theta(\tau)$ can be expressed directly in terms of $\theta(\gamma \cdot \tau)$. Now, if $\Im(\gamma \cdot \tau) \geq \sqrt{3}/2$, then the series for $\theta(\gamma \cdot \tau)$ converges very rapidly and series truncation leads to very good approximation of $\theta(\tau)$. However, we do not know of an analogue of Eq. 7 for $\theta(\tau)$ and in our situation τ depends on t which ranges over $[0, 2]$. The choice of γ for which $\Im(\gamma \cdot \tau) \geq \sqrt{3}/2$ depends on the order of approximation of each value of t by rationals. The constraint $z \equiv 0 \pmod 4$ further complicates matters. These observations and Eq. 7 led us to work with $\vartheta(\tau)$.

IV. Approximating $\vartheta(\tau)$

The approximation of $\vartheta(\tau)$ will be carried out in three large steps. At some points in these steps additional results will be used: the recovery method and technical auxiliary lemmas. Proofs of the recovery method and auxiliary lemmas are in sections 5 and 6, respectively. Before proceeding we introduce some new parameters. For the rest of the paper the index j has range $[0..3]$ and the index i has range $[1..3]$. We introduce four roots of unity: $\omega_0, \ldots, \omega_3$ as $1, \mathbf{e}(9/16), \mathbf{e}(2/3), \mathbf{e}(4/3)$, respectively.

Ref

2. N. Koblitz. Introduction to Elliptic Curves and Modular Forms. Springer, 1984.

Initial approximations will be carried out in Step 1. Much more detailed approximations are presented in Step 2. The final approximation of $\vartheta(\tau)$ is derived in Step 3 using the recovery method.

a) Step 1

Let $\kappa(t) = -\iota \exp(\pi/2^\ell)\exp(-t)$. Making the change of variable $x = u - \iota$ and noting $\tau = \iota/2^\ell + t$, Eq. 7 becomes

$$\kappa(t)\int_{-\infty}^{\infty} \frac{\exp(-\pi x^2/2^\ell)\mathbf{e}(x^2 t)\exp(-2\pi x t)\mathbf{e}(x/2^{\ell-1})}{\sin(\pi(x+\iota))}dx\ . \tag{8}$$

Define $B(t)$ to be

$$\kappa(t)\int_{-2^{\ell+4}}^{2^{\ell+4}} \frac{\exp(-\pi x^2/2^\ell)\mathbf{e}(x^2 t)\exp(-2\pi x t)\mathbf{e}(x/2^{\ell-1})}{\sin(\pi(x+\iota))}dx\ . \tag{9}$$

Lemma 3 $|\vartheta(\tau) - B(t)| = O(\exp(-2^\ell))$.

Proof: We will show that the sum of the absolute values of the integral of Eq. 8 over $(-\infty, -2^{\ell+4}]$ and $[2^{\ell+4}, \infty)$ is bounded above by $O(\exp(-2^\ell))$. We do this by bounding the absolute values of the integrand of Eq. 8 over these two half infinite ranges.

Now,

$$|\mathbf{e}(u^2\tau)| = |\exp(\pi/2^\ell)\exp(-\pi x^2/2^\ell)\exp(-2\pi x t)|\ .$$

Since $|\exp(\pi/2^\ell)| = O(1)$ it suffices to examine

$$|\exp(-\pi x^2/2^\ell)\exp(-2\pi x t)|\ . \tag{10}$$

The behavior of Eq. 10 depends on the behavior of $-\pi x^2/2^\ell - 2\pi x t$. By calculation, if $|x| \geq 2^{\ell+4}$, then $-\pi x^2/2^\ell - 2\pi x t \leq -|x|$. From this we see that if $|x| \geq 2^{\ell+4}$, then

$$|\mathbf{e}(u^2\tau)| = O(\exp(-|x|))\ . \tag{11}$$

The lemma follows from Eq. 9, Eq. 11 and Eq. 64 of Lemma 8, section 6.

Define $B_+(t)$ to be

$$\kappa(t)\int_0^{2^{\ell+4}} \exp(-\pi x^2/2^\ell)\mathbf{e}(x^2 t)\exp(-2\pi x t)\mathbf{e}(x/2^{\ell-1})\frac{1}{\sin(\pi(\iota+x))}dx$$

and $B_-(t)$ to be

$$\kappa(t)\int_0^{2^{\ell+4}} \exp(-\pi x^2/2^\ell)\mathbf{e}(x^2 t)\exp(2\pi x t)\mathbf{e}(-x/2^{\ell-1})\frac{1}{\sin(\pi(\iota-x))}dx$$

Clearly, $B(t) = B_+(t) + B_-(t)$.

Notes

By Eq. 65 of Lemma 8 of section 6 we can express $B_+(t)$ as

$$-2\iota\kappa(t)\int_0^{2^{\ell+4}} \exp(-\pi x^2/2^\ell)\mathbf{e}(x^2t)\exp(-2\pi xt)\mathbf{e}(x/2^{\ell-1})\sum_{k=0}^{\infty}\exp(-2\pi k)\mathbf{e}(2(k+1)x)dx \tag{12}$$

and $B_-(t)$ as

Notes

$$-2\iota\kappa(t)\int_0^{2^{\ell+4}} \exp(-\pi x^2/2^\ell)\mathbf{e}(x^2t)\exp(2\pi xt)\mathbf{e}(-x/2^{\ell-1})\sum_{k=0}^{\infty}\exp(-2\pi k)\mathbf{e}(-2(k+1)x)dx\ . \tag{13}$$

$B_-(\omega_i t)$ is defined from Eq. 13 under the substitution $t \to \omega_i t$. Absorbing the -2ι factor redefine $\kappa(t) = -2\exp(\pi/2^\ell\mathbf{e}(-t)$.

Now we truncate the infinite sums in Eq. 12 and Eq. 13 to the first $\ell+1$ terms. Noting Eq. 12 define $B_{+,k}(t)$ to be

$$\kappa(t)\int_0^{2^{\ell+4}} \exp(-\pi x^2/2^\ell)\mathbf{e}(x^2t)\exp(-2\pi xt)\mathbf{e}(x/2^{\ell-1})\exp(-2\pi k)\mathbf{e}(2(k+1)x)dx \tag{14}$$

and noting Eq. 13 define $B_{-,k}(t)$ to be

$$\kappa(t)\int_0^{2^{\ell+4}} \exp(-\pi x^2/2^\ell)\mathbf{e}(x^2t)\exp(2\pi xt)\mathbf{e}(-x/2^{\ell-1})\exp(-2\pi k)\mathbf{e}(-2(k+1)x)dx\ . \tag{15}$$

$B_{-,k}(\omega_i t)$ is defined by Eq. 15 in the obvious way.

Now we introduce the truncated versions of $B_+(t), B_-(t)$ and $B_-(\omega_i t)$. Define $C_+(t)$ to be

$$\sum_{k=0}^{\ell} B_{+,k}(t) \tag{16}$$

and $C_-(t)$ to be

$$\sum_{k=0}^{\ell} B_{-,k}(t) \tag{17}$$

and $C_-(\omega_i t)$ to be

$$\sum_{k=0}^{\ell} B_{-,k}(\omega_i t) \tag{18}$$

A useful upper bound on the absolute value of the integrand of the integral defining $B_+(t)$ can be obtained but this does not apply to the integrand of the integral defining $B_-(t)$. It is possible to get useful bounds on the absolute value of the integrand of the integral defining $B_-(\omega_i t)$. Of course, $B_-(\omega_i t)$ is quite different to $B_-(t)$. We will use the recovery method of section 5 to overcome this difficulty.

The following definitions are of great importance.

- $\alpha_j = -\pi/2^\ell + \iota\pi t\omega_j$.
- $\beta_{0,k} = -2\pi t + \iota\pi(1/2^{\ell-1} + 2(k+1))$, where $k \in \mathbb{N}$.
- $\beta_{i,k} = 2\pi t\omega_i + \iota\pi(-1/2^{\ell-1} - 2(k+1))$, where $k \in \mathbb{N}$.

Using these definitions and Eq. 12 we can express $B_{+,k}(t)$ as

$$\kappa(t)\int_0^{\ell+4} \exp(-2\pi k)\exp(\alpha_0 x^2 + \beta_{0,k}x)dx \tag{19}$$

and using Eq. 13 we can express $B_{-,k}(\omega_i t)$ as

$$\kappa(t)\int_0^{\ell+4} \exp(-2\pi k)\exp(\alpha_i x^2 + \beta_{i,k}x)dx \; . \tag{20}$$

Lemma 4 $|\exp(\alpha_j x^2 + \beta_{j,k}x)| = O(1)$.

Proof: It suffices to show for $x \geq 0$ that

$$\Re(\alpha_j x^2 + \beta_{j,k}x) < 0 \; .$$

This follows by inspection since $\Re(\alpha_j) < 0$ and $\Re(\beta_{j,k}) = 2\pi t\Re(2\pi t\omega_j) \leq 0$.

Note that

$$\Re(\beta_{i,k}) = 2\pi t\Re(2\pi t\omega_i) \leq 0$$

is the reason for introducing $\omega_1, \omega_2, \omega_3$. No satisfactory upper bound on

$$|\exp(\alpha_i x^2 + \beta_{i,k}x)|$$

exists if we set the ω_i to 1.

By Eq. 16, noting the factor $\exp(-2k\pi)$ in Eq. 19 and Lemma 4 we get

$$|B_+(t) - C_+(t)| = O(\ell 2^{\ell+4}\exp(-2\pi\ell)) \tag{21}$$

and similarly using Eq. 17 and Eq. 20,

$$|B_-(\omega_i t) - C_-(\omega_i t)| = O(\ell 2^{\ell+4}\exp(-2\pi\ell)) \; . \tag{22}$$

b) Step 2

We break up the integration range $[0..2^{\ell+4}]$ into 'octaves', O_g. Let r be the least integer satisfying $\ell^2 < r$.

- $O_0 = [0, 2^r]$.
- for $g \in [1..\ell + 4 - r]$, $O_g = [2^{r+g-1}, 2^{r+g}]$.

Notes

From this point we reserve the symbols g and r. $O_{g,-}$ and $O_{g,+}$ denote the lower and upper endpoints of O_g. Integration restricted to $t \in O_g$ is denotes by $\int_{O_g}$. $B_{+,k,g}(t)$ is defined by Eq. 14 with integration restricted to $x \in O_g$ and $B_{-,k,g}(t)$ is defined by Eq. 15 with integration restricted to $x \in O_g$.

ith $x \in O_0$ we have by calculation

Notes

$$|\alpha_j x^2 + \beta_{j,k} x| = O(\ell^3) \ . \tag{23}$$

Define $D_{+,k,0}(t)$ to be

$$\int_{O_0} T_{(\ell^4:\exp)}(\alpha_0 x^2 + \beta_{0,k} x) dx$$

and define $D_{-,k,0}(\omega_i t)$ to be

$$\int_{O_0} T_{(\ell^4:\exp)}(\alpha_i x^2 + \beta_{i,k} x) dx$$

Using $O_0 = [0..2^r]$, the truncation error for the Taylor series for exp and Eq. 23 we get for $\ell > 2\exp(1)$ that

$$|D_{+,k,0}(t) - B_{+,k,0}(t| \text{ and } |D_{-,k,0}(\omega_i t) - B_{-,k,0}(\omega_i t| = O(\ell^2/2^{\ell^4}) \ . \tag{24}$$

Now, $x \in O_g$ for $g > 0$. Define $v_{j,k}$ to be

$$v_{j,k} = \alpha_j x^2 + \beta_{j,k} x \ . \tag{25}$$

Clearly,

$$|v_{j,k}| = O(2^{2\ell}) \ . \tag{26}$$

From Eq. 25 we get

$$dx = \frac{dv_{j,k}}{2\alpha_j x + \beta_{j,k}} \ . \tag{27}$$

We have, using $|\beta_{j,k}| = O(\ell)$ (reason for defining octaves):

$$1 \le \frac{\max_{x \in O_g} |2\alpha_j x + \beta_{j,k}|}{\min_{x \in O_g} |2\alpha_j x + \beta_{j,k}|} = \frac{O_{g,+}}{O_{g,-}} \frac{1 \pm O(1/\ell)}{1 \pm O(1/\ell)} = 2 \pm O(1/\ell) < 3 \ . \tag{28}$$

Notice that the lower and upper bounds are independent of k. Using Eq. 28 and Lemma 9 of section 6, a polynomial $R_j(x)$ can be computed in polynomial time in m such that

$$|\frac{1}{2\alpha_j x + \beta_{j,k}} - R_j(x)| < 1/2^m \ . \tag{29}$$

Next, we want to express $R_j(x)$ by expressing x in terms of $v_{j,k}$. We do this by solving Eq. 25 for x. The solutions are

$$x = \frac{\beta_{j,k} \pm \sqrt{\beta_{j,k}^2 - 4\alpha_j v_{j,k}}}{2\alpha_j} \ .$$

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By Eq. 25, at $x = 0$, $v_{j,k} = 0$ so we take the negative branch,

$$x = \frac{\beta_{j,k} - \sqrt{\beta_{j,k}^2 - 4\alpha_j v_{j,k}}}{2\alpha_j} \ . \tag{30}$$

Using Eq. 30 we can write $R_j(x)$ as

$$R_j(\frac{\beta_{j,k} - \sqrt{\beta_{j,k}^2 - 4\alpha_j v_{j,k}}}{2\alpha_j}) \ . \tag{31}$$

We can write Eq. 31 as

$$R_{j,1}(v_{j,k}) + R_{j,2}(v_{j,k}\sqrt{\beta_{j,k}^2 - 4\alpha_j v_{j,k}}) \ , \tag{32}$$

where $R_{1,j}(z)$ and $R_{j,2}(z)$ are polynomials.

Next, we approximate $\sqrt{\beta_{j,k}^2 - 4\alpha_j v_{j,k}}$ by a polynomial $R_{j,3}(v_{j,k})$. We first examine $\beta_{j,k}^2 - 4\alpha_j v_{j,k}$. Let $z_{j,k} = \beta_{j,k}^2 - 4\alpha_j v_{j,k}$. By inspection we get

$$|\frac{\Im(\iota z_{j,k})}{\Re(\iota z_{j,k})}| = O(1/\ell^2) \tag{33}$$

and

$$\frac{\max|\iota z_{j,k}|}{\min|\iota z_{j,k}|} = 4 \pm O(1/\ell^2) \ . \tag{34}$$

From the definition of $\beta_{j,k}$ one has

$$\Re(\iota z_{0,k}) < 0 \text{ and } \Re(\iota z_{i,k}) > 0 \ . \tag{35}$$

Let $\mu_j = \max_{x \in O_g} |z_{j,k}|$. Note that

$$|\mu_j| < 2^{O(\ell)} \ . \tag{36}$$

From Eq. 33, Eq. 34 and Eq. 35 one has

$$|\frac{\mu_0 + \iota z_{0,k}}{\mu_0}| = 3/4 \pm O(1/\ell^2) < 4/5 \text{ and } |\frac{\mu_i - \iota z_{i,k}}{\mu_i}| = 3/4 \pm O(1/\ell^2) < 4/5 \ . \tag{37}$$

Now,

$$\sqrt{\iota z_{0,k}} = \sqrt{\mu_0}\sqrt{1 - \frac{\mu_0 + \iota z_{0,k}}{\mu_0}}$$

and

$$\sqrt{\iota z_{i,k}} = \sqrt{\mu_i}\sqrt{1 - \frac{\mu_i - \iota z_{i,k}}{\mu_i}}$$

From these, Eq. 37 and the Taylor series for $\sqrt{1-\zeta}$ with

$$\zeta = 1 - \frac{\mu_0 + \iota z_{0,k}}{\mu_0}$$

Notes

Notes

and

$$\zeta = 1 - \frac{\mu_i - \iota z_{i,k}}{\mu_i}$$

we get

$$|\sqrt{\iota z_{0,k}} - \sqrt{\mu_0}(T_{(h:\sqrt{})}(\frac{\mu_0 + \iota z_{0,k}}{\mu_0}))| < \sqrt{\mu_0} \cdot (4/5)^h \tag{38}$$

and

$$|\sqrt{\iota z_{i,k}} - \sqrt{\mu_i}(T_{(h:\sqrt{})}(\frac{\mu_i - \iota z_{i,k}}{\mu_i}))| < \sqrt{\mu_i} \cdot (4/5)^h \ , \tag{39}$$

respectively.

By Eq. 36 if $h = 2\ell^2$ for ℓ sufficiently large the upper bounds in Eq. 38 and Eq. 39 can be replaced by $2^{-\ell^2}$. Eq. 38 and Eq. 39 extend to approximating $\sqrt{z_{j,k}}$ by using $\sqrt{\iota z_{j,k}} = \sqrt{\iota}\sqrt{z_{j,k}}$. Denote the resulting approximation polynomials as $P_0(v_{0,k})$ and $P_i(v_{i,k})$. By Eq. 39, a single polynomial works for all i but we retain the index so that we can write $P^j(v^{j,k})$ to cover all cases.

Recalling Eq. 32, define $R'_{j,k}(v_{j,k})$ to be

$$R_{j,2}(P_j(v_{j,k})) \ .$$

Using $h = 2\ell^2$ and the corresponding upper bound $2^{-\ell^2}$ in Eq. 38 and Eq. 39 and standard error estimations we obtain

$$|R_{j,2}(\sqrt{\beta_{j,k}^2 - 4\alpha_j v_{j,k}}) - R'_{j,k}(v_{j,k})| < 2^{-\ell^2/2} \ . \tag{40}$$

Recalling Eq. 31 define $Q_j(v_{j,k})$ to be

$$R_{j,1}(v_{j,k}) + R'_{j,k}(v_{j,k}) \tag{41}$$

With $j = 0$, $E_{+,k,g}(t)$ and with $j \in [1..3]$, $E_{-,k,g}(\omega_j t)$ is defined by

$$\int_{\tilde{O}_g} \exp(v_{j,k}) Q_j(v_{j,k}) dv_{j,k} \ , \tag{42}$$

where $\tilde{O}_g$ arises from O_g under the change of variable x to $v_{j,k}$.

Define $D_{+,k,g}(t)$ and $D_{-,k,g}(\omega_i t)$ by restricting the integrations to O_g in Eq. 19 and Eq. 20, respectively. Note that

$$D_{+,k}(t) = \sum_g D_{+,k,g}(t) \text{ and } D_{-,k}(\omega_i t) = \sum_g D_{-,k,g}(\omega_i t)$$

and similarly for $E_+(t)$ and $E_-(\omega_i t)$. From Lemma 4 and Eq. 42 we get

$$|E_{+,k,g}(t) - D_{+,k,g}(t)|, |E_{-,k,g}(\omega_i t) - D_{-,k,g}(\omega_i t)| < 2^{-2\ell} \ . \tag{43}$$

From the summations $\sum_k$ and $\sum_g$, the triangle inequality and Eq. 43 we get

$$|E_+(t) - D_+(t)| = O(\ell 2^{-2\ell}) \text{ and } |E_-(\omega_i t) - D_-(\omega_i t)| = O(\ell^2 2^{-2\ell}) \ . \tag{44}$$

By Lemma 10 the integration in Eq. 42 can be carried out exactly in polynomial time so that the evaluation of the integral in Eq. 42 can be expressed as

$$\exp(\gamma_{j,k,g,+}\omega_j t)U_{j,k,g,+}(\omega_j t) - \exp(\gamma_{j,k,g,-}\omega_j t)U_{k,g,-}(\omega_j t)\ , \tag{45}$$

where $\gamma_{j,k,g,+}$ and $\gamma_{j,k,g,-}$ are complex constants derived from $\tilde{O}_{g,+}$ and $\tilde{O}_{g,-}$, respectively and $U_{j,k,g,+}(\omega_j t)$ and $U_{k,g,-}(\omega_j t)$ are corresponding complex coefficient polynomials. For $j = 0$, Eq. 45 gives $E_{+,k,g}(t)$ explicitly and for $j \in [1..3]$ it gives $E_{-,k,g}(\omega_j t)$ explicitly.

Define

$$E_+(t) = \sum_k \sum_g E_{+.k,g}(t)$$

and

$$E_-(\omega_i t) = \sum_k \sum_g E_{-.k,g}(\omega_i t)\ .$$

c) Step 3

Via recovery, described in section 5 we will produce $E_-(t)$ from the $E_-(\omega_i t)$ and $\hat{\vartheta}(\tau) = E_+(t) + E_-(t)$ will be our approximation of $\vartheta(\tau)$. From Eq. 45, the comments immediately following, linearity of the recovery operator Υ and Eq. 63 of section 5 we can compute in polynomial time $E_-(t)$ given by

$$\Upsilon(E_{-,R}(\omega_1 t), E_{-,R}(\omega_2 t), E_{-,R}(\omega_3 t)) + \iota\Upsilon(E_{-,I}(\omega_1 t), E_{-,I}(\omega_2 t), E_{-,I}(\omega_3 t))\ .E_{-,R}(\omega_i t) + \iota E_{-,I}(\omega_i t)\ . \tag{46}$$

It is clear that $\hat{\vartheta}(\tau)$ can be computed in polynomial time since $E_+(t)$ can be computed in polynomial time. Next, we determine an upper bound on $|\vartheta(\tau) - \hat{\vartheta}(\tau)|$.

Lemma 5 $|\vartheta(\tau) - \hat{\vartheta}(\tau)| = O(\ell^2 2^{-2\ell})$.

Proof: By Eq. 44,

$$|E_+(t) - D_+(t)| = O(\ell^2 2^{-2\ell})\ . \tag{47}$$

By Eq. 21 and repeated triangle inequality using the fact that the summation range for k is $O(\ell)$,

$$|D_+(t) - C_+(t)| = O(\ell\delta)\ . \tag{48}$$

From Eq. 47 and Eq. 48,

$$|E_+(t) - C_+(t) = O(\ell^2 2^{-2\ell})\ . \tag{49}$$

Again by Eq. 44,

$$|E_-(\omega_i t) - D_-(\omega_i t)| = O(\ell^2 2^{-2\ell})\ . \tag{50}$$

By Eq. 22 and repeated triangle inequality using the fact that the summation range for k is $O(\ell)$,

$$|D_-(\omega_i t) - C_-(\omega_i t)| = O(\ell 2^{-2\ell})\ . \tag{51}$$

From Eq. 50 and Eq. 51,

$$|E_-(\omega_i t) - C_-(\omega_i t)| = O(\ell^2 2^{-2\ell})\ . \tag{52}$$

Notes

From Eq. 52, linearity of Υ and Lemma 7 of section 5,

$$|E_-(t) - C_-(t)| = O(\ell^2 2^{-2\ell}) \ .$$

Thus, we get, using $\hat{\vartheta}(\tau) = E_+(t) + E_-(t)$,

$$|\hat{\vartheta}(\tau) - (C_+(t) + C_-(t)| = O(\ell^2 2^{-2\ell}) \ .$$

Notes

The lemma follows from this, $B(t) = C_+(t) + C_-(t)$ and Lemma 3.

V. Computing $\hat{\Omega}$

Using $\hat{\vartheta}(\tau)$ we are ready to compute $\hat{\Omega}$ and verify Eq. 4. As observed in section 1, Eq. 4 and polynomial time computability of $\hat{\vartheta}(\tau)$ establishes that SBQR is in **P**.

Lemma 6

$$|\Omega - \hat{\Omega}| = O(\ell^2 2^{-\ell}) \ . \tag{53}$$

Proof: Noting Lemma. 2, define $\hat{\Omega}_* = \sum_{j=1}^{c_*} \hat{\Omega}_{*,j}$, where

$$\hat{\Omega}_{*,j} = \int_0^1 (-1)^{a+bj} \frac{\hat{\vartheta}(\tau) - 1}{2} \mathbf{e}(-(a+bj)t)dt \ . \tag{54}$$

By Lemma 5 and $|\mathbf{e}(-(a+bj)t)| = 1$,

$$|\int_0^1 (-1)^{a+bj} \frac{\vartheta(\tau) - 1}{2} \mathbf{e}(-(a+bj)t)dt - \hat{\Omega}_{*,j}| = O(\ell^2 2^{-2\ell}) \ . \tag{55}$$

By Eq. 45 $\hat{\Omega}_{*,j}$ can be exactly evaluated as an expression given by

$$\frac{(-1)^{a+bj} \exp(\zeta_+(a+bj))}{\zeta'_+(a+bj)^2} - \frac{(-1)^{a+bj} \exp(\zeta_-(a+bj))}{\zeta'_-(a+bj)^2} \ , \tag{56}$$

where ζ_+, ζ'_+ and ζ_-, ζ'_- are constants arising from evaluations at the integration endpoints 1 and 0, respectively. Clearly, Eq. 56 can be written as

$$\frac{\exp(\zeta''_+(a+bj))}{\zeta'_+(a+bj)^2} - \frac{\exp(\zeta''_-(a+bj))}{\zeta'_-(a+bj)^2} \ , \tag{57}$$

where $\zeta''_\pm = \zeta_\pm + \pi$.

Define $\hat{\Omega}$ to be

$$\sum_{j=1}^{c_*} \hat{\Omega}_{*,j} \ .$$

Lemma 11 of section 6 (adjusted for endpoints other than powers of 2), the summation range c_*, Eq. 55 and Eq. 57 give

$$|\Omega - \hat{\Omega}| = O(\ell^2 2^{-\ell}) \ ,$$

which is Eq. 53

For ℓ sufficiently large the bound $O(\ell^2 2^{-\ell})$ is less tha $\exp(-\pi)/2$, which satisfies Eq. 4.

VI. Recovery Method

We describe the recovery method. Let $f(t) = \sum_{n=0}^{\infty} f_n t^n$, where the f_n and t are real. Define $\sum_i$ to be

$$\sum_{n \equiv i \pmod 3} f_n t^n \,.$$

We have, using the reality of f_n,

$$\begin{aligned} f(\omega_1 t) &= \textstyle\sum_0 + \omega_1 \sum_1 + \omega_1^2 \sum_2 \\ f(\omega_2 t) &= \textstyle\sum_0 + \omega_2 \sum_1 + \omega_3 \sum_2 \\ f(\omega_3 t) &= \textstyle\sum_0 + \omega_3 \sum_1 + \omega_2 \sum_2 \end{aligned} \,. \tag{58}$$

Let $\mu_i = \Re(\omega_i)$ and $\nu_i = \Im(\omega_i)$ and $\mu_* = \Re(\omega_1^2)$. From Eq. 58 we get

$$\begin{aligned} \Re(f(\omega_1 t)) &= \textstyle\sum_0 + \mu_1 \sum_1 + \mu_* \sum_2 \\ \Im(f(\omega_2 t)) &= \textstyle\nu_2 \sum_1 + \nu_3 \sum_2 \\ \Re(f(\omega_3 t)) &= \textstyle\sum_0 + \mu_3 \sum_1 + \mu_2 \sum_2 \end{aligned} \,. \tag{59}$$

Define X to be

$$\begin{pmatrix} 1 & \mu_1 & \mu_* \\ 0 & \nu_2 & \nu_3 \\ 1 & \mu_3 & \mu_2 \end{pmatrix} .$$

It is a calculation that

$$\mathrm{DET}(X) = -\sin(\pi/3)(2\cos(\pi/3) - \cos(9\pi/16) - \cos(9\pi/8)) \neq 0$$

so that X^{-1} exists. From Eq. 59 we get

$$\begin{pmatrix} \sum_0 \\ \sum_1 \\ \sum_2 \end{pmatrix} = X^{-1} \cdot \begin{pmatrix} \Re(f(\omega_1 t)) \\ \Im(f(\omega_2 t)) \\ \Re(f(\omega_3 t)) \end{pmatrix} . \tag{60}$$

We recover $f(t)$ through

$$f(t) = (1,1,1) \cdot X^{-1} \cdot \begin{pmatrix} \Re(f(\omega_1 t)) \\ \Im(f(\omega_2 t)) \\ \Re(f(\omega_3 t)) \end{pmatrix} . \tag{61}$$

For any 3×1 matrices u, v we have

$$(1,1,1) \cdot X^{-1} \cdot (u+v) = (1,1,1) \cdot X^{-1} \cdot u + (1,1,1) \cdot X^{-1} \cdot v \,. \tag{62}$$

We refer to $(1,1,1) \cdot X^{-1}$ as the recovery operator Υ and write its effect on the column vector of Eq. 61 as $\Upsilon(f(\omega_1 t)), (f(\omega_2 t)), (f(\omega_3 t)))$.

We give an extension to recovery and an error analysis. We need notation here. Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ where both the g_n and z may be complex. Define

$$g_R(z) = \sum_{n=0}^{\infty} \Re(g_n) z^n \text{ and } g_I(z) = \sum_{n=0}^{\infty} \Im(g_n) z^n \,.$$

Notes

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Let t be real and let $f(t)$ have complex coefficients f_n. Given $f_R(\omega_1 t), f_R(\omega_2 t), f_R(\omega_3 t)$ and $f_I(\omega_1 t), f_I(\omega_2 t), f_I(\omega_3 t)$ it is clear that we can recover $f(t)$ as

$$f(t) = \Upsilon(f_R(\omega_1 t), f_R(\omega_2 t), f_R(\omega_3 t)) + \iota \Upsilon(f_I(\omega_1 t), f_I(\omega_2 t), f_I(\omega_3 t)) \ . \tag{63}$$

The decomposition $f(\omega_i t) = f_R(\omega_i t) + \iota f_I(\omega_i t)$ is always possible if $f(t)$ is given as a finite sum where the decomposition can be applied term by term and also holds for absolutely convergent infinite sums. This observation will apply to recovery applied to functions in this paper.

Lemma 7 *Assume t is real. If for $\omega \in \{\omega_1, \omega_2, \omega_3\}$, $|f(\omega t)| < \delta$, then $|f(t)| = O(\delta)$.*

Proof: For $\omega \in \{\omega_1, \omega_2, \omega_3\}$ assume $|f(\omega t)| < \delta$. Since $f(\omega t) = f_R(\omega t) + \iota f_I(\omega t)$ it follows that

$$|f_R(\omega t)| < \delta \text{ and } |f_I(\omega t)| < \delta \ .$$

From these inequalities, Eq. 62 and Eq. 63 we get

$$|f(t)| = |\Upsilon(f_R(\omega_1 t), f_R(\omega_2 t), f_R(\omega_3 t)) + \iota \cdot \Upsilon(f_I(\omega_1 t), f_I(\omega_2 t), f_I(\omega_3 t))| \leq O(\delta) \ .$$

Here the O notation reflects the $O(1)$ size of the entries of X^{-1}.

VII. Auxiliary Lemmas

Lemma 8 *For real x,*

$$\frac{1}{|\sin(\pi(\iota + x))|} = O(1) \tag{64}$$

and

$$\frac{1}{\sin(\pi(\iota + x))} = -2\iota \sum_{k=0}^{\infty} \exp(-2\pi k)\mathbf{e}(2(k+1)x) \ . \tag{65}$$

Proof: Let $z = \pi(1 - \iota x)$. Note that

$$\iota z = \pi(\iota + x).$$

Using

$$\exp(\iota \cdot \iota z) = \cos(\iota z) + \iota \sin(\iota z)$$

and

$$\exp(-\iota \cdot \iota z) = \cos(\iota z) - \iota \sin(\iota z)$$

we get

$$\sin(\pi(\iota + x)) = \frac{\exp(-z) - \exp(z)}{2\iota} \ .$$

Item 1 follows from this last equation.

It also follows that

$$\frac{1}{\sin(\pi(\iota + x))} = \frac{2\iota}{\exp(-z) - \exp(z)} \ . \tag{66}$$

The RHS of Eq. 66 can be written as

$$\frac{-2\iota \exp(-z)}{1-\exp(-2z)} .$$

Now

$$\exp(-2z) = \exp(-2\pi)\exp(2\pi\iota x) .$$

Thus, we can expand the RHS of Eq. 66 in geometric series as

$$-2\iota \sum_{k=0}^{\infty} \exp(-2\pi k)\exp(2(k+1)\pi\iota x) ,$$

which establishes item 2.

Lemma 9 *Assume* $0 < a < b$ *and* $a \leq |z|^2 \leq b$, *where* $z \in \mathbb{C}$.

$$|\frac{1}{z} - (\bar{z}/b)\sum_{k=0}^{h}((b-|z|^2)/b)^k| \leq (b/a)((b-\alpha)/b)^{h+1} ,$$

Proof:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{b-(b-|z|^2)} .$$

From this we get

$$\frac{1}{z} = (\bar{z}/b)\frac{1}{1-(b-|z|^2)/b} .$$

Now,

$$0 \leq (b-|z|^2)/b \leq (b-\alpha)/b < 1 .$$

From this and summing a geometric series, we get

$$|\frac{1}{z} - (\bar{z}/b)\sum_{k=0}^{h}((b-|z|^2)/b)^k| \leq (b/a)((b-\alpha)/b)^{h+1} ,$$

Lemma 9 assumes a simpler form when z is real.

Lemma 10 *If* γ *and* ν *are real and* $\sigma(t)$ *is either* $\cos(\nu t)$ *or* $\sin(\nu t)$, *then for* $h \in \mathbb{N}$,

$$U_h = \int_0^1 t^h \exp(\gamma t)\sigma(t)dt$$

can be computed in $h^{O(1)}$ *time. If* $\gamma = \nu = 0$ *this is trivial, otherwise* U_h *is a polynomial with general term*

$$\frac{Q_d(\gamma,\nu)}{(\gamma^2+\nu^2)^d} ,$$

where $d \in [0..h+1]$ *and* $Q_d(x,y)$ *is a bivariate polynomial.*

Proof: Proof is by straightforward integration by parts.

Notes

Lemma 11 Assume $\sigma \geq 0$. *Let* $A = \sum_{f=2^p}^{2^q} \frac{1}{\sigma+f^2}$, *where* $0 \leq p < q$ *are integers. A can be computed in polynomial time in terms of* σ *and* q.

Proof:

$$A = \sum_{j=0}^{q-p-1} \sum_{f=2^{p+j}}^{2^{p+j+1}-1} \frac{1}{\sigma + f^2} .$$

Notes

Next,

$$1 < \frac{\sigma + (2^{p+j+1}-1)^2}{\sigma + (2^{p+j})^2} < 4 .$$

By Lemma 9,

$$\sum_{f=2^{p+j}}^{2^{p+j+1}-1} \frac{1}{\sigma + f^2}$$

can be computed in polynomial time in terms of σ and q. The lemma follows since $j \in [0..q-p-1]$.

VIII. Derivation of Eq. 7

The following derivation of Eq. 7 is for a function denoted by $\vartheta_4(z|\tau)$. Our $\vartheta(\tau)$ is a special case of $\vartheta(z|\tau)$ with $z = 0$ and $\tau = 2t + \iota/2^\ell$. The identity is only needed in a compact domain of τ for our purpose.

The derivation begins by rearranging the Fourier series of $\cos(ux)$, one obtains the series

$$\frac{\pi \cos(ux)}{2u \sin(\pi u)} = \frac{1}{2u^2} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{u^2 - n^2}$$

This equation which is valid for all real values of x such that $-\pi \leq x \leq \pi$ and all non-integral complex values of u. By comparison with the convergent series $\sum_{n=0}^{\infty} 1/n^2$, it follows that this series is absolutely convergent. Note that this series may be viewed as a Mittag-Leffler partial fraction expansion.

Let y be a positive real number. Multiply both sides by $2ue^{-yu^2}$ and integrate.

$$\int_{i-\infty}^{i+\infty} \frac{\pi \cos(ux) e^{-yu^2}}{\sin(\pi u)} dv = 2 \int_{i-\infty}^{i+\infty} e^{-yu^2} \left[\frac{1}{2u^2} + \sum_{n=0}^{\infty} (-1)^n \frac{\cos(nx)}{u^2 - n^2} \right] u\, du$$

Because of the exponential, the integrand decays rapidly as $u \to i \pm \infty$ provided that $\Re u > 0$, and hence the integral converges absolutely. Make a change of variables $v = u^2$

$$= \int_P e^{-yv} \left[\frac{1}{2v} + \sum_{n=1}^{\infty} (-1)^n \frac{\cos(nx)}{v - n^2} \right] dv$$

The contour of integration P is a parabola in the complex v-plane, symmetric about the real axis with vertex at $v = -1$, which encloses the real axis. Its equation is $\Re v + 1 = 2(\Im v)^2$

Let S_m (m is an integer) be the straight line segment joining the points $v = (i+m+1/2)^2$ and $v = (i-m-1/2)^2$. Along this line segment, we may bound the integrand in absolute value as follows:

$$\left|\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{v-n^2}\right| \le \sum_{n=1}^{\infty}\frac{(-1)^n}{|v-n^2|} \le \sum_{n=1}^{\infty}\frac{(-1)^n}{|v_m-n^2|}$$

where $v_m = m^2+m-3/4$ is the point of intersection of S_m with the real axis. To proceed further, we break up the last summation into two parts.

Since the squares closest in absolute value to v_m are m^2 and $(m+1)^2 = m^2+2m+1$, it follows that $|v_m - n^2| \ge |m-3/4|$ for all m, n. Hence, we have

$$\sum_{i=1}^{2m}\frac{1}{|v_m-n^2|} \le \frac{2m}{m-3/4} \le 8$$

When $n > 2m$, we have $n^2 \ge (2m+1)^2 = 4m^2+4m+1 > 4m^2+4m-3 = 4v_m$. Hence, $|n^2 - v_m| > 3n^2/4$ and

$$\sum_{n=2m+1}^{\infty}\frac{1}{|v_m-n^2|} < \frac{4}{3}\sum_{n=2m+1}^{\infty}\frac{1}{n^2} < \frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi}{9}$$

Finally $1/(2v_m) < 1/2$ since $v_m > 1$ when $m \ge 1$. Also, $|e^{-yv}| = e^{-y\Re v} - e^{-yv_m} < e^{-ym^2}$. From these observations, we conclude that

$$\left|\int_{S_m} e^{-yv}\left[\frac{1}{2v}+\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{v-n^2}\right]dv\right| < e^{-ym^2}\left(1+8+\frac{2\pi}{9}\right)\int_{S_m}dv = (4m+2)\left(9+\frac{2\pi}{9}\right)e^{-ym^2}$$

Note that this quantity approaches 0 in the limit $m\to\infty$.

Let P_m be the arc of the parabola P bounded by the endpoints of S_m. Together, S_m and P_m form a closed contour which encloses poles of the integrand. Hence, by the residue theorem , we have

$$\int_{P_m} e^{-yv}\left[\frac{1}{2v}+\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{v-n^2}\right]dv + \int_{S_m} e^{-yv}\left[\frac{1}{2v}+\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{v-n^2}\right]dv =$$

$$2\pi i\sum_{n=1}^{m}(-1)^n\cos(nx)e^{-n^2y}$$

Taking the limit $m\to\infty$ we obtain

$$\int_{P} e^{-yv}\left[\frac{1}{2v}+\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{v-n^2}\right]dv = 2\pi i\left(\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^n\cos(nx)e^{-n^2y}\right)$$

Going back to the beginning of the proof, where the integral on the left hand side was expressed as an integral with respect to u, we obtain

$$\int_{i-\infty}^{i+\infty}\frac{\pi\cos(ux)e^{-yu^2}}{\sin(\pi u)}dv = 2\pi i\left(\frac{1}{2}+\sum_{n=1}^{\infty}(-1)^n\cos(nx)e^{-n^2y}\right)$$

Making a change of variables $x = 2z, y = -i\pi\tau$ and tidying up some, we obtain

$$\int_{i-\infty}^{i+\infty}\frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}dv = i\left(1+2\sum_{n=1}^{\infty}(-1)^n e^{i\pi n^2\tau}\cos(2nz)\right) = i\vartheta_4(z|\tau)$$

Notes

Because of the initial assumption about the Fourier series, we only know that this formula is valid when τ is purely imaginary with strictly positive imaginary part and z is real and $\pi/2 < z < \pi/2$. However, we can use analytic continuation to extend the domain of its validity. On the one hand, the theta function on the right-hand side is analytic for all z and all τ such that $\Im\tau > 0$.

On the other hand, I claim that the integral on the left hand side is also an analytic function of z and τ whenever $\Im\tau > 0$. To validate this claim, we need to examine the behaviour of the integrand as $u \to i\pm\infty$. The contribution of the denominator is bounded;

$$\left|\frac{1}{\sin \pi u}\right| < c$$

for some constant c whenever $\Im u = 1$. The absolute value of the cosine in the numerator is easy to bound:

$$|\cos(2uz)| \leq e^{2|u|\,|z|}$$

To bound the remaining term, let us examine the argument of the exponential carefully:

$$\Im(\tau u^2) = 2\Re\tau\,\Re u + \Im\tau(\Re u)^2 - \Im\tau = \Im\tau\left(\left(\Re u + \frac{\Re\tau}{\Im\tau}\right)^2 - 1 - \left(\frac{\Re\tau}{\Im\tau}\right)^2\right)$$

Therefore, if $|\Re u| > 1 + 3|\Re\tau|/(\Im\tau)$, it will be the case that $\Im(\tau u^2) \geq \Im\tau\,(\Re u)^2/9$, and so

$$\left|e^{i\pi\tau u^2}\right| = e^{-\pi\Im(\tau u^2)} \leq e^{-\pi\Im\tau\,(\Re u)^2/9}$$

Taken together, the estimates of the last paragraph imply that

$$\left|\int_{i+R}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\right| < c\int_{i+R}^{i+\infty} e^{2|u||z| - \pi\Im\tau\,(\Re u)^2/9}$$

when $R > 1 + 3|\Re\tau|/(\Im\tau)$. If we impose the further conditions

$$R > \frac{180|z|}{\pi\,\Im\tau} \qquad R^2 > \frac{180|z|}{\pi\,\Im\tau}\ ,$$

it will be the case that

$$2|u||z| - \pi\Im\tau\,(\Re u)^2/9 < 2\Re u\,|z| + 2|z| - \pi\Im\tau\,(\Re u)^2/9 <$$
$$\left(2\Re u\,|z| - \pi\Im\tau\,(\Re u)^2/180\right) + \left(2|z| - \pi\Im\tau\,(\Re u)^2/180\right) - \pi\Im\tau\,(\Re u)^2/10 <$$
$$-\pi\Im\tau\,(\Re u)^2/10\,,$$

and hence

$$\left|\int_{i+R}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\,du\right| < c\int_{i+R}^{i+\infty} e^{-\pi\Im\tau\,(\Re u)^2/10}\,du < \frac{5c}{\pi\,\Im\tau}Re^{-\pi\Im\tau\,R^2/10}.$$

Likewise, under the same restriction on R,

$$\left|\int_{i-\infty}^{i-R} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\,du\right| < c\int_{i+R}^{i+\infty} e^{-\pi\Im\tau\,(\Re u)^2/10}\,du < \frac{5c}{\pi\,\Im\tau}Re^{-\pi\Im\tau\,R^2/10}.$$

Notes

Since the contour of integration is compact and the integrand is analytic in a neighborhood of the contour,

$$\int_{i-R}^{i+R} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\, du$$

will be an analytic function of z and τ. Suppose that z and τ are restricted to bounded regions of the complex plane and that, furthermore, $Im\tau$ is positive and bounded away from zero. Then the inequalities of the last paragraph imply that the integral converges uniformly as $R \to \infty$, and hence

$$\int_{i-\infty}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\, du$$

is an analytic function of u and z in the domain $\Im\tau > 0$.

Thus, by the fundamental theorem of analytic continuation, we may conclude that

$$\int_{i-\infty}^{i+\infty} \frac{\cos(2uz)e^{i\pi\tau u^2}}{\sin(\pi u)}\, dv = i\left(1 + 2\sum_{n=1}^{\infty}(-1)^n e^{i\pi n^2\tau}\cos(2nz)\right) = i\vartheta_4(z|\tau)$$

throughout this domain.

References Références Referencias

1. T. Cormen, C. Leiserson, R. Rivest, and C. Stein. Introduction to Algorithms. MIT, 2009, 3rd ed.
2. N. Koblitz. Introduction to Elliptic Curves and Modular Forms. Springer, 1984.
3. K. Manders and L. Adleman. NP-complete decision problems for binary quadratics. JCSS, 16:168–184, 1978.
4. planet math CC-BY-SA 3.0. integral form of Theta. http://planetmath.org/derivationofintegralrepresentationsofjacobivarthetafunctions, 2013, 03/21.

Notes

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On a New Conformal Euler-Lagrangian Equations on Para-Quaternionic *Kähler* Manifolds

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On a New Conformal Euler-Lagrangian Equations on Para-Quaternionic $K\ddot{a}hler$ Manifolds

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I. Introduction

Today, many branches of science are into our lives. One the branches is mathematics that has multiple applications. In particular , differential geometry and mathematical physics have a lots of different applications. One of them are on geodesics. Geodesics are known the shortest route between two points. Time-dependent equations of geodesics can be easily found with the help of the Euler-Lagrange equations. We can say that differential geometry provides a good working area for studying Lagrangians of classical mechanics and field theory. The dynamic equation for moving bodies is obtained for Lagrangian mechanic. These dynamic equation is illustrated as follows:

Lagrange Dynamics Equation [1,2,3]: let M be an n-dimensional manifold and TM its tangent bundle with canonical projection $TM: TM \to M$ is called the phase space of velocities of the base manifold M.

Let $L: TM \to R$ be differentiable function on TM called the Lagrangian function. We consider the closed 2-form on TM given by $\Phi_L = dd_J L$ (if $J^2 = -I$, J is a complex structure and if $J^2 = I$, J is a paracomplex structures, $T_r(J) = 0$) Consider the equation:

$$i_X \Phi_L = dE_L \qquad \to \qquad (1)$$

Then X is a vector field, we shall see that (1) under a certain condition on X is the intrinsical expression of the Euler-Lagrange equations of motion. This equation is named as Lagrange dynamical equation. We shall see that for motion in potential, $E_L = V(L) - L$ is an energy function and $V = J(X)$ a Liouville vector field. Here dE_L denotes the differential of E. The triple (TM, Φ_L, X) is known as Lagrangian system on

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the tangent bundle TM. If it is continued the operations on (1) for any coordinate system $(q^i(t), p_i(t))$, infinite dimension Lagrange's equation is obtained the form below:

$$\frac{dq^i}{dt} = \dot{q}^i \;,\; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} \;,\quad i = 1, \dots, n \qquad \rightarrow \qquad (2)$$

There are many studies about Lagrangian dynamics, mechanics, formalisms, systems and equations (see detail [4]). There are real, complex, paracomplex and other analogues. It is well-known that Lagrangian analogues are very important tools. They have a simple method to describe the model for mechanical systems. The models about mechanical systems are given as follows.

Some examples of the Lagrangian is applied to model the problems include harmonic oscillator, charge Q in electromagnetic fields, Kepler problem of the earth in orbit around the sun, pendulum, molecular and fluid dynamics. LC networks, Atwood's machine, symmetric to etc. Let's remember some work done. Vries shown that the Lagrangian motion equations have a very simple interpretation in relativistic quantum mechanics [5]. Paracomplex analogues of the Euler-Lagrange equations was obtained in the framework of Para-*Kählerian* manifold and the geometric results on a paracomplex mechanical systems were found by Tekkoyun [6]. Electronic origins, molecular dynamics simulations, computational nanomechanics, multiscale modeling of materials fields were contributed by Liu [7]. Bi-paracomplex analogue of Lagrangian systems was shown on Lagrangian distributions by Tekkoyun and sari [8]. Tekkoyun and Yayli presented generalized-quaternionic *Kählerian* analogue of Lagrangian and Hamiltonian mechanical systems. Eventually, the geometric-physical results related to generalized-quaternionic *Kählerian* mechanical systems are provided [9].

Nowadays, there are many studies about Euler-Lagrangian dynamics, mechanics, formalisms, systems and equations [2, 4, 10, 11, 12] and there in. There are real, complex, paracomplex and other analogues. As known it is possible to produce different analogous in different spaces. Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian's defining relation is most succinctly written as:

$$i^2 = j^2 = k^2 = -1\,,\;\; ijk = -1.$$

Generalized quaternions are defined as

$$i^2 = -a\,, j^2 = -b\,, k^2 = -ab\,, ijk = -ab.$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. Lots of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra [13, 14, 15, 16, 17, 18, 19].

In the present paper, we present equations related to Lagrangian mechanical systems on generalized-quaternionic *Kähler* manifold.

II. Preliminaries

In this study, all the manifolds and geometric objects are C^∞ and the Einstein summation convention is in use. Also, $A\,, F(TM), \chi(TM)$ and $\Lambda^1(TM)$ denote the set of

Ref

6. M. Tekkoyun, On para-Euler Lagrange and para-Hamiltonian equations, Physics Letters A, Vol. 340, 7-11, (2005).

paracomplex numbers, the set of (para)-complex functions on TM, the set of (para)-complex vector fields on TM and the set of (para)-complex 1-forms on TM, respectively. The definitions and geometric structures on the differential manifold M given in [20] may be extended to TM as follows:

a) Theorem

Let f be differentiable $\phi\,,\psi$ are 1-form , then [21]

- $d(f\phi) = df\wedge\phi + fd\phi$
- $d(\phi\wedge\psi) = d\phi\wedge\psi - \phi\wedge d\psi$

b) Definition (Kronecker's delta)

Kornecker's delta denote by δ and defined as follows [22]:

$$\delta_i^j = \begin{cases} 1 & , \; if \quad i = j \\ 0 & , \; if \quad i \neq j \end{cases}$$

III. Conformal Geometry

In mathematics, conformal map is a function which preserves angles. In the most common case the function is between domains in the complex plane. Conformal maps can be defined between domains in higher dimensional Euclidean spaces, and more generally on a Riemann or semi-Riemann manifold. Conformal geometry is the study of the set of angle-preserving (conformal) transformations on space. In two real dimensions, conformal geometry is precisely the geometry of Riemann surfaces. In more than two dimensions, conformal geometry may refer either to the study of conformal transformations of " flat" spaces (such as Euclidean spaces or spheres), or more commonly, to the study of conformal manifolds which are Riemann or pseudo-Riemann manifolds with a class of metrics defined up to scale. A conformal manifold is a differentiable manifold equipped with an equivalence class of (pseudo) Riemann metric tensors, in which two metrics $\acute{g}$ *and* g are equivalent if and only if:

$$\acute{g} = \lambda^2 g \qquad \rightarrow \qquad (3)$$

Where $\lambda > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class [23].

IV. Conformal Structure

The linear distance between two points can be found easily by Riemann metric, which is very useful and is defined inner product. Many scientists have used the Riemann metric. Einstein was one of the first studies in this field. Einstein discovered which the Riemannian geometry and successfully used it to describe General Relativity in the 1910 that is actually a classical theory for gravitation. However, the universe is really completely not like Riemannian geometry. Each path between two points is not always linear. Also, orbits of move objects may change during movement. So, each two points in space may not be linear geodesic and need not to be. Therefore, new metric is needed for non-linear distances like spherical surface. Then, a method is required for converting non-linear distance to linear. Wey1 introduced a metric with a conformal transformation in 1918.

Ref

20. V. Cruceanu, P.A. Gadea, J.M. Masque, Para-Hermitian and Para-*Kähler* manifolds (http://digital.csic.es/bitstream/10261/15773/1/Rocky CFG.PDF).

Definition 4.1: Let M an n-dimensional smooth manifold. A conformal structure on M is an equivalence class G of Riemann metrics on M. A manifold with a conformal structure is called a conformal manifold

(i) Two Riemann metrics g *and* $\acute{g}$ on M are said to be equivalent if and only if

$$\acute{g} = e^{\lambda} g \qquad \rightarrow \qquad (4)$$

Where λ is a smooth function on M. the equation given by (4) is called a conformal structure

(ii) A Wey1 structure on M is a map $F\colon G \rightarrow \Lambda^1 M$ satisfying

$$F\left(e^{\lambda} g\right) = F(g) - d\lambda \qquad \rightarrow \qquad (5)$$

Where G is a conformal structure. Note that a Riemann metric g and a one-form φ determine a Wey1 structure, namely $F\colon G \rightarrow \Lambda^1 M$ where G is the equivalence class of g and

$$F\left(e^{\lambda} g\right) = \varphi - d\lambda.$$

Theorem 4.1: A connection on the metric bundle φ of a conformal manifold M naturally induces a map $F\colon G \rightarrow \Lambda^1 M$ and (5), and conversely. Parallel translation of points in φ by the connection is the same as their translation by F [24].

V. Generalized- Quaternionic Kähler Manifolds

A generalized almost quaternion structure on the manifold M is a sub bundle of the bundle of endomorphism's of the tangent bundle M, whose standard fiber is the algebra of quaternions. A generalized almost quaternion structure on a pseudo-Riemannian manifold is called a generalized quaternion-Hermitian if the following conditions hold:

i) The endomorphism's F, G *and* H *of* $T_x M$ satisfy

$$F^2 = -aI\,, G^2 = -bI\,, H^2 = -abI, FG = H, GH = bF, HF = aG, \qquad \rightarrow \qquad (6)$$

ii) The compatibility equations are given by for $X, Y \in T_x M$,

$$g(FX, FY) = ag(X, Y), g(GX, GY) = bg(X, Y), g(HX, HY) = abg(X, Y) \qquad \rightarrow \qquad (7)$$

Where I denotes the identity tensor of type (1, 1) in M. In particular, 2-form Q defined by $Q(X, Y) = (X, FY) = (X, GY) = (X, HY)$ on M is called the *Kähler* form Q on M is closed, i.e. $dQ = 0$, the manifold M is called a generalized-quaternionic *Kähler* manifold [25].

If $a = b = 1$, M is quaternion manifold. If $a = 1\,, b = -1$, M is Para-quaternion manifold. The bundle V is a set that locally admits basis $\{F, G, H\}$ satisfying (6) and (7) in any coordinate neighborhood $U \subset M$ *such that* $M = \cup\, U$ [14]. Then V is called a generalized-quaternionic structure in M. The pair (M, V) denotes a generalized-quaternionic manifold with V. The structure V with such a Riemannian metric g is called a generalized-quaternionic metric structure. The triple (M, g, V) denotes a generalized-quaternionic metric manifold. Let $\{x_i\,, x_{n+i}\,, x_{2n+i}\,, x_{3n+i}\}, i = \overline{1\,, n}$ be a real coordinate system on a neighborhood U of M, and let $\{\frac{\partial}{\partial x_i}\,, \frac{\partial}{\partial x_{n+i}}\,, \frac{\partial}{\partial x_{2n+i}}\,, \frac{\partial}{\partial x_{3n+i}}\}$ and

Ref

24. G. B. Folland, Wey1.manifolds, J. Differential Geometry, 4, 145-153, (1970).

$\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ be natural bases over R of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M, respectively. Taking into consideration (6), then we can obtain the expressions an follows:

$$F\left(\frac{\partial}{\partial x_i}\right) = a\frac{\partial}{\partial x_{n+i}}, F\left(\frac{\partial}{\partial x_{n+i}}\right) = -a\frac{\partial}{\partial x_i}, F\left(\frac{\partial}{\partial x_{2n+i}}\right) = a\frac{\partial}{\partial x_{3n+i}}, F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -a\frac{\partial}{\partial x_{2n+i}}$$

$$G\left(\frac{\partial}{\partial x_i}\right) = -b\frac{\partial}{\partial x_{2n+i}}, G\left(\frac{\partial}{\partial x_{n+i}}\right) = b\frac{\partial}{\partial x_{3n+i}}, G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -b\frac{\partial}{\partial x_i}, G\left(\frac{\partial}{\partial x_{3n+i}}\right) = b\frac{\partial}{\partial x_{n+i}} \rightarrow \quad (8)$$

$$H\left(\frac{\partial}{\partial x_i}\right) = -ab\frac{\partial}{\partial x_{3n+i}}, H\left(\frac{\partial}{\partial x_{n+i}}\right) = -ab\frac{\partial}{\partial x_{2n+i}}, H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -ab\frac{\partial}{\partial x_{n+i}}, H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -ab\frac{\partial}{\partial x_i}$$

VI. Generalized-Quaternionic Conformal Kähler Manifolds

Definition 6.1: Let $(M, g, \nabla, J\pm)$ be an almost para/pseudo-Hermitian Wey1 manifold. If $\nabla(J\pm) = 0$, then one says that is a (para)-$K\ddot{a}hler$ Wey1 manifold. Note that necessarily $J\pm$ is integrable in this setting.

Theorem 6.1: If $(M, g, \nabla, J\pm)$ is a (para)-$K\ddot{a}hler$ Wey1 manifold with dimension $n \geq 6$ *and with* $H^1(M;R) = 0$ then the underlying wey1 structure on M is trivial.

Theorem 6.2: If $(M, g, \nabla, J\pm)$ is a curvature (para)-$K\ddot{a}hler$ Wey1 manifold with dimension $n \geq 6$ *and with* $H^1(M;R) = 0$ then the underlying wey1 structure on M is trivial.

Theorem 6.3: Let $n \geq 6$. If $(M, g, \nabla, J\pm)$ is a $K\ddot{a}hler - wey1$ structure, then the associated wey1 structure is trivial, i.e. there is a conformally equivalent metric $\tilde{g} = e^{2f}g$ *so that* $(M, \tilde{g}, J\pm)$ *is* $K\ddot{a}hler$ and so that $\nabla = \nabla^{\tilde{g}}$ [26,27,28].

After this part W will be used instead of J. A manifold with a wey1 structure is known as wey1 manifold. λ second structure was chosen the minus sign. Because the condition of the structure required to provide. $W_\pm^2 = \pm Id$ [29]. If we rewrite (8) equation with conformal structure, we obtain the following equations:

$$W_F\left(\frac{\partial}{\partial x_i}\right) = ae^{\lambda}\frac{\partial}{\partial x_{n+i}}, W_F\left(\frac{\partial}{\partial x_{n+i}}\right) = -ae^{-\lambda}\frac{\partial}{\partial x_i}$$

$$W_F\left(\frac{\partial}{\partial x2n+_i}\right) = ae^{\lambda}\frac{\partial}{\partial x_{3n+i}}, W_F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -ae^{-\lambda}\frac{\partial}{\partial x_{2n+i}}$$

$$W_G\left(\frac{\partial}{\partial x_i}\right) = -be^{\lambda}\frac{\partial}{\partial x_{2n+i}}, W_G\left(\frac{\partial}{\partial x_{n+i}}\right) = be^{\lambda}\frac{\partial}{\partial x_{3n+i}} \quad \rightarrow \quad (9)$$

$$W_G\left(\frac{\partial}{\partial x_{2n+i}}\right) = -be^{-\lambda}\frac{\partial}{\partial x_i}, W_G\left(\frac{\partial}{\partial x_{3n+i}}\right) = be^{-\lambda}\frac{\partial}{\partial x_{n+i}}$$

$$W_H\left(\frac{\partial}{\partial x_i}\right) = -abe^{\lambda}\frac{\partial}{\partial x_{3n+i}}, W_H\left(\frac{\partial}{\partial x_{n+i}}\right) = -abe^{\lambda}\frac{\partial}{\partial x_{2n+i}}$$

$$W_H\left(\frac{\partial}{\partial x_{2n+i}}\right) = -abe^{-\lambda}\frac{\partial}{\partial x_{n+i}}, W_H\left(\frac{\partial}{\partial x_{3n+i}}\right) = -abe^{-\lambda}\frac{\partial}{\partial x_i}$$

Ref

29. R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabau, The geometry of Hamilton and Lagrange spaces, eBook ISBN:0-306-47135-3, Kluwer Academic Publishers, New York, (2002).

We continue our studies thinking of the $(M, g, \nabla, W_{\pm})$ instead of the almost para/pseudo- $Kähler - wey1$ manifolds $(M, g, \nabla, J\pm)$.

VII. Conformal Euler-Lagrangian Mechanical System

Here, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}$ of V that they defined on a generalized-quternionic $Kähler$ manifold (M, g, V).
First:

$$a\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{n+i}}=0\ ,a\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_i}=0$$

$$a\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_{3n+i}}=0\ ,a\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_{2n+i}}=0$$

Second: Let G take a local basis element on the generalized-quternionic $Kähler$ manifold (M, g, V), and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field Y determined by

$$Y=Y^i\frac{\partial}{\partial x_i}+Y^{n+i}\frac{\partial}{\partial x_{n+i}}+Y^{2n+i}\frac{\partial}{\partial x_{2n+i}}+Y^{3n+i}\frac{\partial}{\partial x_{3n+i}} \qquad \rightarrow \qquad (10)$$

Where $Y^i=\dot{x}_i, Y^{n+i}=\dot{x}_{n+i}, Y^{2n+i}=\dot{x}_{2n+i}, Y^{3n+i}=\dot{x}_{3n+i}$ and the dot indicates the derivative with respect to time t.
The vector field defined by:

$$V_G(L)=G(Y)=-bY^ie^{\lambda}\frac{\partial L}{\partial x_{2n+i}}+bY^{n+i}e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}-bY^{2n+i}e^{-\lambda}\frac{\partial L}{\partial x_i}+bY^{3n+i}e^{-\lambda}\frac{\partial L}{\partial x_{n+i}}.$$

is named a conformal Liouville vector field on the generalized-quternionic $Kähler$ manifold(M, g, V). For G the closed generalized-quternionic $Kähler$ form is the closed 2-form given by $\Phi_L^G=-dd_GL$ such that

$$d_GL=-be^{\lambda}\frac{\partial L}{\partial x_{2n+i}}dx_i+be^{\lambda}\frac{\partial L}{\partial x_{3n+i}}dx_{n+i}-be^{-\lambda}\frac{\partial L}{\partial x_i}dx_{2n+i}$$

$$+be^{-\lambda}\frac{\partial L}{\partial x_{n+i}}dx_{3n+i}:\ \mathcal{F}(M)\ \rightarrow \Lambda^1M \qquad \rightarrow \qquad (11)$$

Then we have

$$\Phi_L^G=be^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j\wedge dx_i+be^{\lambda}\frac{\partial^2L}{\partial x_j\partial x_{2n+i}}dx_j\wedge dx_i-be^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_j\wedge dx_{n+i}-$$

$$be^{\lambda}\frac{\partial^2L}{\partial x_j\partial x_{3n+i}}dx_j\wedge dx_{n+i}+be^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j\wedge dx_{2n+i}+be^{-\lambda}\frac{\partial^2L}{\partial x_j\partial x_i}dx_j\wedge dx_{2n+i}-$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_j\wedge dx_{3n+i}-be^{-\lambda}\frac{\partial^2L}{\partial x_j\partial x_{n+i}}dx_j\wedge dx_{3n+i}+be^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{n+j}\wedge dx_i+$$

Notes

Notes

$$be^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j}\wedge dx_i - be^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j}\wedge dx_{n+i} - be^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j}\wedge dx_{n+i} +$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{n+j}\wedge dx_{2n+i} + be^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j}\wedge dx_{2n+i} - be^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{n+j}\wedge dx_{3n+i}$$

$$--be^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j}\wedge dx_{3n+i} + be^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{2n+j}\wedge dx_i + be^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j}\wedge dx_i$$

$$be^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{2n+j}\wedge dx_{n+i} - be^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j}\wedge dx_{n+i} +$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j}\wedge dx_{2n+i} +$$

$$be^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j}\wedge dx_{2n+i} - be^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{2n+j}\wedge dx_{3n+i} -$$

$$be^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j}\wedge dx_{3n+i}$$

$$+ be^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{3n+j}\wedge dx_i + be^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j}\wedge dx_i -$$

$$be^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{3n+j}\wedge dx_{n+i} -$$

$$be^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j}\wedge dx_{n+i} + be^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}dx_{3n+j}\wedge dx_{2n+i} +$$

$$be^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j}\wedge dx_{2n+i} -$$

$$be^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j}\wedge dx_{3n+i} - be^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j}\wedge dx_{3n+i} \qquad \rightarrow \qquad (12)$$

Then we calculate

$$i_Y\Phi_L^G = bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}\delta_i^j dx_i - bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_i - bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j$$

$$-bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}\delta_i^j dx_{n+i} + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_j - bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_{n+i} +$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j + bY^i e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}\delta_i^j dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j +$$

$$bY^i e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}\delta_i^j dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_j - bY^i e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}\delta_i^j dx_{3n+1} +$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_j - bY^i e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{3n+i} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j +$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_i - bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{n+j} + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_i -$$

$$bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j} - bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{n+i} + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j}dx_{n+i} + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} + bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}\delta_{n+i}^{n+j}dx_{2n+i}$$

$$-bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{n+j} + bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}\delta_{n+i}^{n+j}dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j} -$$

$$bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{3n+1} + bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{n+j} -$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{3n+i}$$

$$+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j} + bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_i -$$

$$bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{2n+j} +$$

$$bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_i - bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j} -$$

$$bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{n+i} +$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{2n+j} - bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}dx_{n+i} +$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j} +$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}\delta_{2n+i}^{2n+j}dx_{2n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} +$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}\delta_{2n+i}^{2n+j}dx_{2n+i}$$

$$-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i} +$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{2n+j}\cdot$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_i$$

$$-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{3n+j} + bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_i - bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} -$$

Notes

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{n+i}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{3n+j}-$$

$$bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}dx_{n+i}$$

$$+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i}-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}dx_{3n+j}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i}-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{3n+j}-$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+1}$$

$$+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j}-bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j}\qquad\rightarrow\qquad(13)$$

Energy function is:

$$E_L^G=V_G(L)-L=-bY^{i}e^{\lambda}\frac{\partial L}{\partial x_{2n+i}}+bY^{n+i}e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}-bY^{2n+i}e^{-\lambda}\frac{\partial L}{\partial x_i}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial L}{\partial x_{n+i}}-L\qquad\rightarrow\qquad(14)$$

And hence

$$dE_L^G=-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j-bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_j+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_j-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j-bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_j+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_j+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j-$$

$$bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{n+j}-bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{n+j}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j}+$$

$$bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j}-bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{n+j}-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j}+$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{n+j}-bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{2n+j}-$$

$$bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j}+bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{2n+j}+bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{2n+j}-$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j}-bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j}+bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{2n+j}+$$

Notes

$$bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j} - bY^{i}e^{\lambda}\frac{\partial \lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_{3n+j} - bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j} +$$

$$bY^{n+i}e^{\lambda}\frac{\partial \lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{3n+j} + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{3n+j} - bY^{2n+i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{i}}dx_{3n+j} -$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}dx_{3n+j} + bY^{3n+i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} -$$

$$\frac{\partial L}{\partial x_{j}}dx_j - \frac{\partial L}{\partial x_{n+j}}dx_{n+j} - \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} - \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} \quad \rightarrow \tag{15}$$

Notes

By means of Eq(1), we calculate the following expressions:

$$bY^{i}e^{\lambda}\frac{\partial \lambda}{\partial x_{j}}\frac{\partial L}{\partial x_{2n+i}}\delta_i^j\,dx_i + bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{j}\partial x_{2n+i}}\delta_i^j\,dx_i - bY^{i}e^{\lambda}\frac{\partial \lambda}{\partial x_{j}}\frac{\partial L}{\partial x_{3n+i}}\delta_i^j\,dx_{n+i} - bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_{j}\partial x_{3n+i}}\delta_i^j\,dx_{n+i}$$

$$+ bY^{i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{j}}\frac{\partial L}{\partial x_{i}}\delta_i^j\,dx_{2n+i} + bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{j}\partial x_{i}}\delta_i^j\,dx_{2n+i} - bY^{i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{j}}\frac{\partial L}{\partial x_{n+i}}\delta_i^j\,dx_{3n+i} -$$

$$bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{j}\partial x_{n+i}}\delta_i^j\,dx_{3n+i} + bY^{n+i}e^{\lambda}\frac{\partial \lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{n+i}^{n+j}\,dx_i + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}\,dx_i -$$

$$bY^{n+i}e^{\lambda}\frac{\partial \lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{n+i}^{n+j}\,dx_{n+i} - bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}\delta_{n+i}^{n+j}\,dx_{n+i} +$$

$$bY^{n+i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{i}}\delta_{n+i}^{n+j}\,dx_{2n+i} +$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{i}}\delta_{n+i}^{n+j}\,dx_{2n+i} - bY^{n+i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{n+i}^{n+j}\,dx_{3n+1} -$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}\,dx_{3n+i}$$

$$+ bY^{2n+i}e^{\lambda}\frac{\partial \lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{2n+i}^{2n+j}\,dx_i + bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}\,dx_i -$$

$$bY^{2n+i}e^{\lambda}\frac{\partial \lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{2n+i}^{2n+j}\,dx_{n+i} - bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}\delta_{2n+i}^{2n+j}\,dx_{n+i} +$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{i}}\delta_{2n+i}^{2n+j}\,dx_{2n+i} + bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}\delta_{2n+i}^{2n+j}\,dx_{2n+i} -$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial \lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{2n+i}^{2n+j}\,dx_{3n+i} - bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}\,dx_{3n+i} +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial \lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}\delta_{3n+i}^{3n+j}\,dx_i + bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}\,dx_i -$$

$$bY^{3n+i}e^{\lambda}\frac{\partial \lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}\delta_{3n+i}^{3n+j}\,dx_{n+i} - bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\delta_{3n+i}^{3n+j}\,dx_{n+i} +$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j}dx_{2n+i} -$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+1} - bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i} + \frac{\partial L}{\partial x_j}dx_j +$$

Notes

$$\frac{\partial L}{\partial x_{n+j}}dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} = 0 \qquad \rightarrow \qquad (16)$$

And thus

$$bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_j +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}dx_j + bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_j + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_j +$$

$$bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_j + bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_j - bY^{i}e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} - bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} - bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{i}e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_{n+j} - bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{n+j} - bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{n+j} -$$

$$bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{n+j} + bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i}dx_{2n+j} + bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} +$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} + bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}dx_{2n+j} + bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i}dx_{2n+j} +$$

$$bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{2n+j} + bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{2n+j} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{2n+j} -$$

$$bY^{i}e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} - bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} - bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} -$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}dx_{3n+j} - bY^{i}e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_{3n+j} - bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_{3n+j} -$$

$$bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{3n+j} - bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j} + \frac{\partial L}{\partial x_j}dx_j + \frac{\partial L}{\partial x_{n+j}}dx_{n+j} +$$

$$\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} = 0 \qquad \rightarrow \qquad (17)$$

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OR

$$\Big[bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{2n+i}} + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{2n+i}} + bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{2n+i}} +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{2n+i}}\Big]dx_j +$$

$$\Big[bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}} + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}} + bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}} +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\Big]dx_j + \frac{\partial L}{\partial x_j}dx_j$$

$$-\Big[bY^i e^{\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{3n+i}} + bY^{n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{3n+i}} + bY^{2n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{3n+i}} +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{3n+i}}\Big]dx_{n+j} -$$

$$\Big[bY^i e^{\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}} + bY^{n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}} + bY^{2n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}} +$$

$$bY^{3n+i}e^{\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}\Big]dx_{n+j}$$

$$+\frac{\partial L}{\partial x_{n+j}}dx_{n+j} +$$

$$\Big[bY^i e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_i} + bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_i} + bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_i} + bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_i}\Big]dx_{2n+j} +$$

$$\Big[bY^i e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_i} + bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i} + bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i} + bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\Big]dx_{2n+j}$$

$$+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}$$

$$-\Big[bY^i e^{-\lambda}\frac{\partial\lambda}{\partial x_j}\frac{\partial L}{\partial x_{n+i}} + bY^{n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{n+j}}\frac{\partial L}{\partial x_{n+i}} + bY^{2n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{2n+j}}\frac{\partial L}{\partial x_{n+i}} +$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial\lambda}{\partial x_{3n+j}}\frac{\partial L}{\partial x_{n+i}}\Big]dx_{3n+j}$$

$$-\Big[bY^i e^{-\lambda}\frac{\partial^2 L}{\partial x_j\partial x_{n+i}} + bY^{n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}} + bY^{2n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}} +$$

$$bY^{3n+i}e^{-\lambda}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\Big]dx_{3n+j}$$

$$+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j} = 0$$

Notes

Notes

In this equation can be concise manner

$$be^{\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{2n+i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{2n+i}}\right]dx_j+\frac{\partial L}{\partial x_j}dx_j$$

$$-be^{\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{3n+i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{3n+i}}\right]dx_{n+j}+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}$$

$$+be^{-\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{i}}\right]dx_{2n+i}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}$$

$$-be^{-\lambda}\sum_{a=0}^{3}Y^{an+i}\left[\frac{\partial\lambda}{\partial x_{an+j}}\frac{\partial L}{\partial x_{n+i}}+\frac{\partial^2 L}{\partial x_{an+j}\partial x_{n+i}}\right]dx_{3n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}=0 \qquad \rightarrow \qquad (18)$$

Then we have the equations:

$$b\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{2n+i}}=0, \qquad b\frac{\partial}{\partial t}\left(e^{-\lambda}\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0$$

$$b\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_i}=0, \qquad b\frac{\partial}{\partial t}\left(e^{\lambda}\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_{n+i}}=0 \qquad \rightarrow \qquad (19)$$

Such that the equations calculated in equation(19) are named Euler-Lagrange equations constructed on a generalized-quternionic $K\ddot{a}hler$ manifold (M,g,V) by means of Φ_L^G and thus the triple(M,Φ_L^G,X) is called a mechanical system on generalized-quternionic $K\ddot{a}hler$ manifold (M,g,V).

References Références Referencias

1. J. Klein, Escapes variationnals et Mecanique, Ann. Inst. Fourier, Grenoble, 12 , pp. 1-124,(1962).
2. M. De Leon, P.R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, North-Holland Mathematics Studies, Vol. 152, (Elsevier, Amsterdam, 1989).
3. R. Abraham, J.E. Marsden, T. Ratiu, Manifolds, tensor analysis and applications, Springer, pp: 483-542, (2001).
4. M. De Leon, P.R. Rodrigues, Second-Order-Differential Equations and Non-Conservative Lagrangian Mechanics, J. Phys. A: Math. Gen. 20 (1987), 5393-5396.
5. H. de Vries, Understanding relativistic quantum field theory, The Hamiltonian and Lagrangian densities, Chapter 22, (http://www.physics-quest.org/book-Chapter-Lagrangian.pdf), (2009).
6. M. Tekkoyun, On para-Euler Lagrange and para-Hamiltonian equations, Physics Letters A, Vol. 340, 7-11, (2005).
7. W.K. Liu, S. Jun, Computational nanomechanics of materials, American Scientific Publishers, Stevenson Ranch, CA, (2005).
8. M. Tekkoyun, M.Sari., Bi-Para-mechanical systems on the bi-Lagrangian manifold, Physics B-condensed Matter, Vol. 405, Issue10,2390-2393,(2010).
9. M. Tekkoyun, Y. Yayli, Mechanical systems on Generalized-Quaternionic $K\ddot{a}hler$ Manifolds, International Journal of Geometric Methods in Modern Physics, Vol. 8, No. 7, 1-13, (2011).

10. M. Zambine, Hamiltonian Perspeetive on Generalized complex Structure, Communications in Mechanical Physics. 263v(2006), 711-722.
11. M. Tekkoyun, On Para-Euler-Lagrange and Para-Hamiltonian Equations, Phys. Lett. A, 340 (2005), 7-11.
12. M. Tekkoyun, Mechanical Systems on Para-Quaternionic *Kähler* Manifolds, XVI. Geometrical Seminar, 20-25 September, Vrnjacka Banja, Serbia, 2009.
13. D. Stahlke, Quaternions in Classical Mechanics, Phys. 621. http://www.stahlke.org/dan/phys-papers/quaternion-paper.
14. A. S. Dancer- H. R. Jorgensen – A. F. Swann, Metric Geometries over the Split Quaternions, arXiv: 0412.215.
15. L. Kula L. and Y. Yayli, Split Quaternions and Rotations in Semi-Euclidean Space E_2^4, J. Korean Math. Soc. 44 (2007), 1313-1327.
16. L. Kula and Y. Yayli, Dual Split Quaternions and Serew Motion in Minkowski 3-Space Iranian Journal of Sci.𝔈Tech.,Transaction A. 30 (2006).
17. E. Ata and Y. Yayli, Split Quaternions and Semi-Euclidean projective Spaces, Chaos, Solitons 𝔈 Fractals. 41 (2009), 1910-1915.
18. E. Ata and Yayli, A Global Condition for the Triviality of an Almost Split Quaternionic Structure on Split Complex Manifolds, International Journal of Computational and Mathematical Sciences 2:1 (2008).
19. M. Jafari, Y. Yayli, Generalized Quaternions and their Algebraic properties, Submitted for publication.
20. V. Cruceanu, P.A. Gadea, J.M. Masque, Para-Hermitian and Para-*Kähler* manifolds (http://digital.csic.es/bitrstream/10261/15773/1/Rocky CFG.PDF).
21. Gebreel Mohammed Khur Baba Gebreel, M.A. Bashir. On a new dynamics on *Kähler* manifold using local canonical basis. International Journal of information research and review. Vol.4. Issue.2.February 2017.
22. Gebreel Mohammed Khur Baba, On a New Mechanics Systems on the Standard Cliffordian *Kähler* Manifolds Using A Canonical Local Basis, Global Journal of Science frontier research , Volume 17 Issue 4. 2017.
23. http://en.wikipedia.org/wiki/conformal-class.
24. G. B. Folland, Wey1.manifolds, J. Differential Geometry, 4, 145-153, (1970).
25. O E Arsen'eva, on the Geometry of Quaternion-Hermitan Manifolds, Russian Mathematical Surveys, 48(5):159 (1993).
26. P. Gilkey, S. *Nikěevié, Kähler and Para − Kähler* curvature Wey1 manifolds, arXiv:1011.4844v1, (2010).
27. P. Gilkey, S. Nikeevie, *Kähler − Wey*1 manifolds dimensional 4, http://arxiv.org/abs/1109.4532v1, (2011).
28. H. Pedersen, Y.S. Poon, A. Swann, The Einstein-Wey1 equations in complex and quaternionic geometry. Differential Geom. Appl. 3, no.4, 309321, (1993).
29. R. Miron, D. Hrimiuc, H. Shimada, S. V. Sabau, The geometry of Hamilton and Lagrange spaces, eBook ISBN:0-306-47135-3, Kluwer Academic Publishers, New York, (2002).
30. H. Wey1, Space-Time-Matter, Dover Publ. 1922. Translated from the 4th German edition by H. Brose. London: Methuen. Reprint New York: Dover (1952).

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Fuzzy Foldness of BCI-Commutative Ideals in BCI -Algebras

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Abstract- This paper aims to introduce new notions of (fuzzy) *n*-fold BCI- commutative ideals, and (fuzzy) *n*-fold weak BCI- commutative ideals in BCI –algebras, and investigate several properties of foldness theory of BCI- commutative ideals in BCI -algebras. Finally, we construct some algorithms for studying the foldness theory of BCI- commutative ideals in BCI -algebras.

Keywords: BCK/BCI algebras, BCI – commutative ideals of BCI-algebras, Fuzzy BCI – a commutative ideal of BCI –algebra, Fuzzy point, (fuzzy) n-fold BCI- commutative ideals, (fuzzy) n-fold weak positive implicative ideals.

GJSFR-F Classification: MSC 2010: 08A72

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I. Introduction

The notions of BCK/BCI-algebras were introduced by Iséki [3] and were investigated by many types of research. The concept of fuzzy sets was introduced by Zadeh [12] In 1991, Xi [11] applied the concept to BCK-algebras. From then on, Jun, Meng et al. [10] applied the concept to the ideals.

The notions of *n*-fold implicative ideal and *n*-fold weak commutative ideals were introduced by Huang and Chen [1]. Y. B. Jun [4] discussed the fuzzification of *n*-fold positive implicative, commutative, and implicative ideal of BCK-algebra.

In this paper, we redefined a BCI – commutative ideals of BCI-algebra and studied the foldness theory of fuzzy BCI – commutative ideals, BCI – commutative weak ideals, fuzzy weak BCI – commutative ideals and weak BCI – commutative weak ideals in BCI-algebras. This theory can be considered as a natural generalization of BCI – commutative ideals. Indeed, given any BCI -algebra X, we use the concept of fuzzy point to characterize *n*-fold BCI -commutative ideals in X. Finally, we construct some algorithms for studying foldness theory of BCI – commutative ideals in BCI -algebra.

Ref

3. K. Iséki and S. Tanaka, An Introduction to the theory of BCK-algebra, Math. Japonica, 23(1978), pp. 935-942.

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II. Preliminaries

Here we include some elementary aspects of BCI that are necessary for this paper, and for more detail, we refer to [1, 3].

An algebra $(X ; *, 0)$ of type (2,0) is called BCI-algebra 1f

$\forall x, y, z \in X$ the following conditions hold:

BCI-1. $((x*y)*(x*z))*(z*y)=0$;

BCI-2. $(x*(x*y))*y=0$;

BCI-3. $x*x=0$;

BCI-4. $x*y=0$ and $y*x=0 \Rightarrow x=y$

A binary relation $\leq$ can be defined by

BCI-5. $x \leq y \Leftrightarrow x*y=0$,

then $(X, \leq)$ is a partially ordered set with least element 0.

The following properties also hold in any BCI-algebra ([5], [10]):

1. $x*0=x$;
2. $x*y=0$ and $y*z=0 \Rightarrow x*z=0$;
3. $x*y=0 \Rightarrow (x*z)*(y*z)=0$ and $(z*y)*(z*x)=0$;
4. $(x*y)*z=(x*z)*y$;
5. $(x*y)*x=0$;
6. $x*(x*(x*y))=x*y$; let $(X, *, 0)$ be a BCI-algebra.

Definition 2.1 (Zadeh [12]). A fuzzy subset of a BCI-algebra X is a function $\mu : X \rightarrow [0,1]$.

Definition 2.2 (C. Lele [6]). Let ξ be the family of all fuzzy sets in X. For $x \in X$ and $\lambda \in (0,1], x_\lambda \in \xi$ is a fuzzy point iff

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$$x_{\lambda}(y)=\begin{cases}\lambda & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

We denote by $\tilde{X}=\{x_{\lambda}: x\in X, \lambda\in(0,1]\}$ the set of all fuzzy points on X and we define a binary operation on $\tilde{X}$ as follows :

$$x_{\lambda}*y_{\mu}=(x*y)_{\min(\lambda,\mu)}$$

It is easy to verify $\forall x_{\lambda}, y_{\mu}, z_{\alpha}\in\tilde{X}$, the following conditions hold:
BCI-1'. $((x_{\lambda}*y_{\mu})*(x_{\lambda}*z_{\alpha}))*(z_{\alpha}*y_{\mu})=0_{\min(\lambda,\mu,\alpha)}$;

BCI-2'. $(x_{\lambda}*(x_{\lambda}*y_{\mu}))*y_{\mu}=0_{\min(\lambda,\mu)}$;

BCI-3'. $x_{\lambda}*x_{\mu}=0_{\min(\lambda,\mu)}$;

BCK-5'. $0_{\mu}*x_{\lambda}=0_{\min(\lambda,\mu)}$;

Remark 2.3 (C. Lele [6]). The condition BCI-4 is not true $(\tilde{X},*)$. So the partial order $\leq(X,*)$ can not be extended to $(\tilde{X},*)$.
We can also establish the following conditions $\forall x_{\lambda}, y_{\mu}, z_{\alpha}\in\tilde{X}$:

1'. $x_{\lambda}*0_{\mu}=x_{\min(\lambda,\mu)}$;

2'. $x_{\lambda}*y_{\mu}=0_{\min(\lambda,\mu)}$ and $y_{\mu}*z_{\alpha}=0_{\min(\mu,\alpha)}\Rightarrow x_{\lambda}*z_{\alpha}=0_{\min(\lambda,\alpha)}$;

3'. $x_{\lambda}*y_{\mu}=0_{\min(\lambda,\mu)}\Rightarrow(x_{\lambda}*z_{\alpha})*(y_{\mu}*z_{\alpha})=0_{\min(\lambda,\mu,\alpha)}$ and

$$(z_{\alpha}*y_{\mu})*(z_{\alpha}*x_{\lambda})=0_{\min(\lambda,\mu,\alpha)};$$

4'. $(x_{\lambda}*y_{\mu})*z_{\alpha}=(x_{\lambda}*z_{\alpha})*y_{\mu}$;

5'. $(x_{\lambda}*y_{\mu})*x_{\lambda}=0_{(\lambda,\mu)}$;

6'. $x_{\lambda}*(x_{\lambda}*(x_{\lambda}*y_{\mu}))=x_{\lambda}*y_{\mu}$;

We recall that if A is a fuzzy subset of a BCI-algebra X, then we have the following:

$$\tilde{A} = \{x_\lambda \in \tilde{X} : A(x) \geq \lambda \ , \ \lambda \in (0,1]\}. \tag{i}$$

$$\forall \lambda \in (0,1], \tilde{X}_\lambda = \{x_\lambda : x \in X\}, \text{ and } \tilde{A}_\lambda = \{x_\lambda \in \tilde{X}_\lambda : A(x) \geq \lambda\} \tag{ii}$$

also have $\tilde{X}_\lambda \subseteq \tilde{X}, \tilde{A} \subseteq \tilde{X}, \tilde{A}_\lambda \subseteq \tilde{A}, \tilde{A}_\lambda \subseteq \tilde{X}_\lambda$ and one can easily check that $(\tilde{X}_\lambda ; *, 0_\lambda)$ is a BCK-algebra.

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Definition 2.4(Isèki [2]). A nonempty subset of BCI-algebra X is called an ideal of X if it satisfies

1. $0 \in I$;

2. $\forall x, y \in X, (x * y \in I \text{ and } y \in I) \Rightarrow x \in I$

Definition 2.5 (Liu and Meng [7]). A nonempty subset I of BCI-algebra X is BCI-commutative ideal if it satisfies:

1. $0 \in I$;

2. $\forall x, y, z \in X$

$$((x * y) * z) \in I \text{ and } z \in I) \Rightarrow \left(x * \left(\left(y * (y * x)\right)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right) \in I$$

Definition 2.6Xi [11]). A fuzzy subset A of a BCI-algebra X is a fuzzy ideal if

1. $\forall x \in X, A(0) \geq A(x)$;

2. $\forall x, y \in X, A(x) \geq \min(A(x * y), A(y))$.

Definition 2.7 (Xi [11]). A fuzzy subset A of a BCI-algebra X is called a fuzzy BCI- commutative ideal of X if.

1. $\forall x \in X, A(0) \geq A(x)$;

2. $\forall x, y, z \in X$

$$A\left(x * \left(\left(y * (y * x)\right)\right) * \left(0 * \left(0 * (x * y)\right)\right)\right) \geq \left(A\left((x * y) * z\right), A(z)\right)$$

Definition 2.8(C . Lele, [6]). $\tilde{A}$ is a weak ideal of $\tilde{X}$ if

1. $\forall \nu \in \mathrm{Im}(A)$; $0_\nu \in \tilde{A}$;

2. $\forall x_\lambda, y_\mu \in X$. Such that $x_\lambda * y_\mu \in \tilde{A}$ and $y_\mu \in \tilde{A}$, we have $x_{\min(\lambda,\mu)} \in \tilde{A}$.

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Theorem 2.9 (Lele, [6]). Suppose that A is a fuzzy subset of a BCK - algebra X , then the following conditions are equivalent:

1. A is a fuzzy ideal ;

2. $\forall x_\lambda, y_\mu \in \tilde{A}$, $(z_\alpha * y_\mu) * x_\lambda = 0_{\min(\lambda,\mu,\alpha)} \Rightarrow z_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$;

3. $\forall t \in (0,1]$, the t-level subset $A^t = \{x \in X : A(x) \geq t\}$ is an ideal when $A^t \neq \phi$;

4. $\tilde{A}$ is a weak ideal.

III. FUZZY N-FOLD BCI-COMMUTATIVE IDEALS IN BCI - ALGEBRAS

Throughout this paper $\tilde{X}$ is the set of fuzzy points on BCI-algebra X and $n \in \mathrm{N}$ (where N the set of all the natural numbers).

Let us denote $(\cdots((x * y) * y) * \cdots) * y$ by $x * y^n$

and $(\cdots((x_{\min(\lambda,\mu)} * 0_\mu) * 0_\mu) * \cdots) * 0_\mu$ by $x_\lambda * y_\mu^n$ (where y and y_μ occurs respectively n times) with $x, y \in X$, $x_\lambda, y_\lambda \in \tilde{X}$.

Definition 3.1. A nonempty subset I of a BCI -algebra X is an n-fold BCI-commutative ideal of X if it satisfies :

1. $0 \in I$;

2. $\forall x, y, z \in X$;

$$((x * y) * z) \in I \text{ and } z \in I) \Rightarrow \left(x * \left(\left(y * (y * x)\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right) \in I$$

Definition 3.2 A fuzzy subset A of X is called a fuzzy n-fold BCI-commutative ideal of X if it satisfies :

1. $\forall x \in X \ , A(0) \geq A(x)$;

2. $\forall x, y, z \in X$,

$$A\left(\left(x*\left(y*(y*x)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)\geq \min\left(A\left((x*y)*z\right), A(z)\right).$$

Definition 3.3. $\tilde{A}$ is BCI-commutative weak ideal of $\tilde{X}$ if

1. $\forall v \in \mathrm{Im}(A)$, $0_v \in \tilde{A}$;

2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

$$\left(\left(x_\lambda * y_\mu\right) * z_\alpha\right) \in I \text{ and } z_\alpha \in I) \Rightarrow \left(\left(x_\lambda * \left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right.\right.$$
$$\left.\left. * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu\right)\right)\right)\right)\right) \in I$$

Definition 3.4. $\tilde{A}$ is n-fold BCI- commutative weak ideal of $\tilde{X}$ if

1. $\forall v \in \mathrm{Im}(A)$, $0_v \in \tilde{A}$;

2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

$$\left(\left(x_\lambda * y_\mu\right) * z_\alpha\right) \in I \text{ and } z_\alpha \in I) \Rightarrow \left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right)\right.$$
$$\left. * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu^{\ n}\right)\right)\right)\right) \in I$$

Example 3.5. Let $X = \{0, a, b, c, d\}$ with $*$ defined by the following table

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	c	c	0	0
d	d	d	d	d	0

By simple computations, one can prove that $(X, *, 0)$ is BCI – algebra. let $t_1, t_2 \in (0,1]$ and define a fuzzy subset $t_1 = A(0) = A(a) = A(b) = A(c) \geq A(d) = t_2$.

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One can easily check that for any $n \geq 3$

$$\tilde{A}=\{0_\lambda:\lambda\in(0,t_1]\}\cup\{a_\lambda:\lambda\in(0,t_1]\}\cup\{b_\lambda:\lambda\in(0,t_1]\}\cup\{c_\lambda:\lambda\in[0,t_1)\}\cup\{d_\lambda:\lambda\in(0,t_2]\}$$

It is an n-fold BCI- commutative weak ideal.

Remark 3.6. $\tilde{A}$ Is a 1-fold BCI- commutative weak ideal of a BCK-algebra $\tilde{X}$ if $\tilde{A}$ is BCI- commutative weak ideal of $\tilde{X}$.

Proposition 3.7 An ideal I of BCI -algebra X Is an n-fold BCI - commutative ideal if

$$\forall x,y\in X, x*y\in I\Rightarrow\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^n\right)\right)\right)\right)\in I$$

Proof. If an ideal I is an n-fold BCI - commutative and $x*y\in I$ then $(x*y)*0\in I$ and $0\in I$, then we have

$$\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^n\right)\right)\right)\right)\in I ,$$

thus this means that the condition satisfies.

Conversely, let an I an ideal satisfies the condition. If $(x*y)*z\in I$ and $z\in I$, then by the definition of ideas we have $x*y\in I$. It follows from the given condition that $\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^n\right)\right)\right)\right)\in I$; this means that I is an n-fold BCI - commutative ideal .this finishes the proof.

Proposition 3.8. An n-fold BCI - commutative weak ideal is a weak ideal.

Proof. $\forall x_\lambda, y_\mu\in\tilde{X}$ let $x_\lambda*y_\mu=(x_\lambda*0_\lambda)*y_\mu\in\tilde{A}$ and $y_\mu\in\tilde{A}$, since $\tilde{A}$ n-fold BCI - commutative ideal we have

$$x_{\min(\lambda,\mu)}=\left(\left(x_\lambda*\left(0_\mu*\left(0_\mu*x_\lambda\right)\right)*\left(0_\mu*\left(0_\mu*\left(x_\lambda*0^n{}_\mu\right)\right)\right)\right)\right)=x_{\min(\lambda,\mu)}\in\tilde{A} ,$$

Thus $\tilde{A}$ is a weak ideal.

Proposition 3.9. Any fuzzy n-fold BCI - commutative ideal of BCI – algebras X is the fuzzy ideal of X .

Proof. let A be a fuzzy n-fold BCI - commutative ideal of X and let $x, z \in X$.

Then
$$\min\left(A(x*z),A(z)\right)$$
$$=\min\left(A\left((x*0)*z\right),A(z)\right)$$
$$\leq A\left(\left(x*\left(0*\left(0*x\right)\right)*\left(0*\left(0*\left(x*0^{n}\right)\right)\right)\right)\right)$$
$$=A\left(\left(x*\left(0*\left(0*x\right)\right)*\left(0*\left(0*\left(x*0\right)\right)\right)\right)\right)$$
$$=A(x*0)$$
$$=A(x)$$

Thus A is a fuzzy ideal of X.

Theorem 3.10. If A is a fuzzy subset of X, then A is a fuzzy n-fold BCI -commutative ideal if $\tilde{A}$ is an n-fold BCI -commutative weak ideal.

Proof. $\Rightarrow$ - Let $\lambda \in \mathrm{Im}(A)$, it is easy to prove that $0_{\lambda} \in \tilde{A}$;

- Let $(x_{\lambda} * y_{\mu}) * z_{\alpha} \in \tilde{A}$ and $z_{\alpha} \in \tilde{A}$

$A((x*y)*z) \geq \min(\lambda,\mu,\alpha)$ and $A(z) \geq \alpha$.

Since A is a fuzzy n-fold BCI -commutative ideal, we have

$$A\left(\left(x*\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)\right)\geq\min\left(A\left((x*y)*z\right),A(z)\right)$$

$$\geq\min\left(\min(\lambda,\mu,\alpha),\alpha\right)=\min(\lambda,\mu,\alpha).$$

Therefore $\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)} =$

$$\left(x_{\lambda}*\left(\left(y_{\mu}*\left(y_{\mu}*x_{\lambda}\right)\right)\right)*\left(0_{\alpha}*\left(0_{\alpha}*\left(x_{\lambda}*y_{\mu}^{\;n}\right)\right)\right)\right)\in\tilde{A}$$

$\Leftarrow$ - Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$;

- Let $x, y, z \in X$ and let $A((x*y)*z)=\beta$ and $A(z)=\alpha$, then

$$((x*y)*z)_{\min(\beta,\alpha)} = (x_\beta * y_\alpha)*z_\alpha \in \tilde{A} \text{ and } z_\alpha \in \tilde{A}.$$

Since $\tilde{A}$ is n-fold BCI -commutative weak ideal, we have

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu{}^n\right)\right)\right)\right) =$$

$$\left(x * \left(\left(y * \left(y * x\right)\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$$

$$\text{Thus } \left(x * \left(\left(y * \left(y * x\right)\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right) \geq \min\left(\beta, \alpha\right)$$

$$= \min\left(A\left((x*y)*z\right), A(z)\right)$$

Theorem 3.11. Suppose that $\tilde{A}$ is a weak ideal (namely A is a fuzzy ideal by Theorem 2.12), then the following conditions are equivalent:

1. A is a fuzzy n-fold BCI - commutative ideal ;

2. $\forall x_\lambda, y_\mu \in \tilde{X}$ such that $x_\lambda * y_\mu \in \tilde{A}$, we have

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu{}^n\right)\right)\right)\right) \in \tilde{A} \;;$$

3. $\forall t \in (0,1]$, the t-level subset $A^t = \{x \in X : A(x) \geq t\}$ is an n-fold BCI – commutative ideal when $A^t \neq \phi$;

4. $A\left(\left(x * \left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right)\right) \geq A\left(x * y\right)$

5. $\tilde{A}$ is an n-fold BCI – commutative weak ideal.

Proof. $1 \Rightarrow 2$ Let $x_\lambda, y_\mu \in \tilde{A}$ such that $x_\lambda * y_\mu \in \tilde{A}$. Since A is a fuzzy n-fold BCI - commutative ideal, we have

$$A\left(\left(x * \left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right)\right) \geq \min\left(A\left((x*y)*(x*y)\right), A\left(x*y\right)\right)$$

$$= \min(A(0), A(x*y)) = A(x*y) \geq \min(\lambda,\mu).$$

Therefore $\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)} =$

$$\left(x_{\lambda}*\left(\left(y_{\mu}*\left(y_{\mu}*x_{\lambda}\right)\right)\right)*\left(0_{\alpha}*\left(0_{\alpha}*\left(x_{\lambda}*y_{\mu}{}^{n}\right)\right)\right)\right) \in \tilde{A}$$

$2 \Rightarrow 3$ $\forall t \in (0,1], 0 \in A^{t}$.

Let $(x*y)*z \in A^{t}$ and $z \in A_{t}$ then we have $((x*y)*z)_{t} = (x_{t}*y_{t})*z_{t} \in \tilde{A}$ and $z_{t} \in \tilde{A}$.

Since $\tilde{A}$ it is a weak ideal, we have $x_{t}*y_{t} = (x*y)_{t} \in \tilde{A}$.

Using the hypothesis, we obtain

$$\left(x_{t}*\left(\left(y_{t}*\left(y_{t}*x_{t}\right)\right)\right)*\left(0_{t}*\left(0_{t}*\left(x_{t}*y_{t}{}^{n}\right)\right)\right)\right) =$$

$$\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)_{t} \in \tilde{A} \text{ hence}$$

$\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right) \in A^{t}$. By Proposition 3.7, we obtain that $A^{t} = \{x \in X : A(x) \geq t\}$ is an n-fold BCI - commutative ideal.

$3 \Rightarrow 4$ Let $x, y \in X$ and $t = A(x*y)$, then $x*y \in A^{t}$.

Since A^{t} is an n-fold BCI – commutative ideal, we have

$\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right) \in A^{t}$. Hence

$A\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right) \geq t = A(x*y)$ 4. $\Rightarrow$ 5. Let $\lambda \in \mathrm{Im}(A)$. Obviously $0_{\lambda} \in \tilde{A}$.

- Let $(x_{\lambda}*y_{\mu})*z_{\alpha} \in \tilde{A}$ and $z_{\alpha} \in \tilde{A}$. Since $\tilde{A}$ is a weak ideal, we obtain $(x*y)_{\min(\lambda,\mu,\alpha)} \in \tilde{A}$. According to the hypothesis, we obtain

Notes

$$A\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)\geq A\left(x*y\right)\geq \min\left(\lambda,\mu,\alpha\right)$$, hence

$$\left(x*\left(\left(y*\left(y*x\right)\right)\right)*\left(0*\left(0*\left(x*y^{n}\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)}=$$

Notes

$$\left(x_{\lambda}*\left(\left(y_{\mu}*\left(y_{\mu}*x_{\lambda}\right)\right)\right)*\left(0_{\alpha}*\left(0_{\alpha}*\left(x_{\lambda}*y_{\mu}{}^{n}\right)\right)\right)\right)\in\tilde{A}\ .$$

5. $\Rightarrow$ 1. Follows from Theorem 3.10

Theorem 3.12 Let $\{\tilde{A}_{i\in I}\}$ be a family of *n*-fold BCI – commutative weak ideals and $\{A_{i\in I}\}$ be a family of fuzzy *n*-fold BCI – commutative ideals. then:(1) $\bigcap_{i\in I}\tilde{A}_i$ is an *n*-fold BCI – commutative weak ideal.

(2))$\bigcup_{i\in I}\tilde{A}_i$ is an *n*-fold BCI – commutative weak ideal.

(3))$\bigcap_{i\in I}A_i$ is a fuzzy *n*-fold BCI – commutative ideal.

(4))$\bigcup_{i\in I}A_i$ is a fuzzy *n*-fold BCI – commutative ideal.

Proof. (1) $\forall\lambda\in \mathrm{Im}\left(\bigcap_{i\in I}\tilde{A}_i\right)$, then $\lambda\in\mathrm{Im}(\tilde{A}_i),\forall i$, so, $0_\lambda\in\tilde{A}_i,\forall i$, i.e. $0_\lambda\in\bigcap_{i\in I}\tilde{A}_i$. For every $x_\mu,y_\lambda,z_\alpha\in\tilde{X}$, if $((x_\lambda*y_\mu)*z_\alpha)\in\bigcap_{i\in I}\tilde{A}_i$ *and* $z_\alpha\in\bigcap_{i\in I}\tilde{A}_i$, then $((x_\lambda*y_\mu)*z_\alpha)\in\tilde{A}_i$ *and* $z_\alpha\in\tilde{A}_i\ \forall i$, thus

$$\left(x_{\lambda}*\left(\left(y_{\mu}*\left(y_{\mu}*x_{\lambda}\right)\right)\right)*\left(0_{\alpha}*\left(0_{\alpha}*\left(x_{\lambda}*y_{\mu}{}^{n}\right)\right)\right)\right)\in\tilde{A}_i$$

So $\left(x_{\lambda}*\left(\left(y_{\mu}*\left(y_{\mu}*x_{\lambda}\right)\right)\right)*\left(0_{\alpha}*\left(0_{\alpha}*\left(x_{\lambda}*y_{\mu}{}^{n}\right)\right)\right)\right)\in\bigcap_{i\in I}\tilde{A}_i$. Thus $\bigcap_{i\in I}\tilde{A}_i$ is an *n*-fold BCI - commutative weak ideals.

(2) $\forall\lambda\in\mathrm{Im}\left(\bigcup_{i\in I}\tilde{A}_i\right)$, then $\exists i_0\in I$, *such,that* $\lambda\in\tilde{A}_{i_0}$, so, $0_\lambda\in\tilde{A}_{i_0}$, i.e. $0_\lambda\in\bigcup_{i\in I}\tilde{A}_i$. For every $x_\mu,y_\lambda,z_\alpha\in\tilde{X}$, if

$((x_\lambda * y_\mu) * z_\alpha) \in \bigcup_{i\in I} \tilde{A}_i$ *and* $z_\alpha \in \bigcup_{i\in I} \tilde{A}_i$, then $\exists i_0 \in I$ such that

$$((x_\lambda * y_\mu) * z_\alpha) \in \tilde{A}_{i_0} \text{ and } z_\alpha \in \tilde{A}_{i_0}\ \forall i \text{ , thus}$$

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu{}^n\right)\right)\right)\right) \in \tilde{A}_{i_0}$$

So $\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(x_\lambda * y_\mu{}^n\right)\right)\right)\right) \in \bigcup_{i\in I} \tilde{A}_i$. Thus $\bigcup_{i\in I} \tilde{A}_i$ is an

n-fold BCI – commutative weak ideals.

(3) Follows from (1) and Theorem 3.10.

(4) Follows from (2) and Theorem 3.10.

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IV. Fuzzy N - Fold Weak BCI – Commutative Ideals in BCI - Algebras

In this section, we define and give some characterizations of (fuzzy) n-fold weak BCI - commutative(weak) ideals in BCK-algebras.

Definition 4.1. A nonempty subset I of X is called an n-fold weak BCI – a commutative ideal of X if it satisfies

1. $0 \in I$;

2. $\forall x, y, z \in X, \left(x * y^n\right) * z \in I, and, z \in I$

$$\Rightarrow x * \left(\left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right) \in I$$

Lemma 4.2. An ideal I of X is called an n-fold weak BCI - commutative ideal if

$$\forall x, y, z \in X, \left(x * y^n\right) * z \in I \Rightarrow x * \left(\left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right) \in I$$

Definition 4.3. A fuzzy subset A of X is called a fuzzy n-fold weak BCI – commutative ideal of X if it satisfies

1. $\forall x \in X \ , A(0) \geq A(x)$;

2. $\forall x, y, z, A\left(x * \left(\left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right)\right) \geq \min\left(A\left(\left(x * y^n\right) * z\right), A\left(z\right)\right)$

Notes

Definition 4.4. $\tilde{A}$ is a weak BCI – commutative weak ideal of $\tilde{X}$ if

1. $\forall \nu \in \mathrm{Im}(A)$, $0_\nu \in \tilde{A}$;

2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$

$$(x_\lambda * y_\mu) * z_\alpha \in I, z_\alpha \in I \Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right.$$
$$\left. * \left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right) \in I$$

Definition 4.5. $\tilde{A}$ is an n-fold a weak BCI – commutative weak ideal of $\tilde{X}$ if

1. $\forall \nu \in \mathrm{Im}(A)$, $0_\nu \in \tilde{A}$;

2. $\forall x_\lambda, y_\mu, z_\alpha \in \tilde{X}$;

$$(x_\lambda * y_\mu{}^n) * z_\alpha \in I,\ z_\alpha \in I \Rightarrow x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\right.$$
$$\left. * \left(0_\alpha * \left(0_\lambda * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right) \in I$$

Example 4.6 Let $X = \{0,1,2,3\}$ in which $*$ is given by the following table

*	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	c	0

Then $(X; *, 0)$ it is a BCI-algebra. Let $t_1, t_2 \in (0,1]$ and let us define a fuzzy subset $A : X \to [0,1]$ by

$$t_1 = A(0) = A(a) = A(b) > A(c) = t_2$$

It is easy to check that for any $n > 2$

$$\tilde{A} = \{0_\lambda : \lambda \in (0,t_1]\} \cup \{a_\lambda : \lambda \in (0,t_2]\} \cup \{b_\lambda : \lambda \in (0,t_1]\} \cup \{c_\lambda : \lambda \in (0,t_2]\}$$

It is an n-fold weak BCI – commutative weak ideal.

Remark 4.7 $\tilde{A}$ is a 1-fold weak BCI – commutative weak ideal of a BCK-algebra X if $\tilde{A}$ is a weak BCI – commutative weak ideal.

Theorem 4.8 If A it is a fuzzy subset of X , then A is a fuzzy n-fold weak BCI – commutative ideal if $\tilde{A}$ is an n-fold weak BCI – commutative weak ideal.

Proof. $\Rightarrow$ - Let $\lambda \in \operatorname{Im}(A)$ obviously $0_\lambda \in \tilde{A}$;

- Let $\left(x_\lambda * y_\mu{}^n\right) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$, then

$$A\left(\left(x * y^n\right) * z\right) \geq \min(\lambda, \mu, \alpha) \text{ and } A(z) \geq \alpha .$$

Since A is a fuzzy n-fold weak BCI - commutative ideal, we have

$$\forall x, y, z, A\left(x * \left(\left(y * (y * x)\right) * \left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right)\right) \geq \min\left(A\left(\left(x * y^n\right) * z\right),\right.$$

$$\left.A(z)\right) \geq \min(\min(\lambda, \mu, \alpha), \alpha) = \min(\lambda, \mu, \alpha) .$$

$$\text{Therefore } \left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \tilde{A} =$$

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \tilde{A} .$$

$\Leftarrow$ - Let $x \in X$, it is easy to prove that $A(0) \geq A(x)$;

- Let $x, y, z \in X$,$A\left(\left(x * y^n\right) * z\right) = \beta$ and $A(z) = \alpha$.

Then $\left(\left(x * y^n\right) * z\right)_{\min(\beta, \alpha)} = \left(\left(x_\beta * y_\beta{}^n\right) * z_\alpha\right) \in \tilde{A}$ and $z_\alpha \in \tilde{A}$

Since $\tilde{A}$ is n-fold weak BCI - commutative weak ideal, we have

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) =$$

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \tilde{A}$$

Hence

Notes

$$A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(\left(x*y\right)*y\right)\right)\right)\right)\right)\geq \min\left(\beta,\alpha\right)$$

$$=\min\left(A\left(\left(x*y^{n}\right)*z\right),A\left(z\right)\right)$$

Notes

Proposition 4.9. Any fuzzy n-fold weak BCI – commutative ideal of X is the fuzzy ideal of X.

Proof. Let A be an n-fold weak BCI – commutative ideal of X and let

$x,z\in X$, then $\min\{A(x*z),A(z)\}$

$$\min\{A((x*0)*z),A(z)\}$$

$$\leq A\left(x*\left(\left(0*\left(0*x\right)\right)*\left(0*\left(0*\left(\left(x*0\right)*0\right)\right)\right)\right)\right)$$

$$=A\left(x*\left(\left(0*\left(0*x\right)\right)*\left(0*\left(0*x\right)\right)\right)\right)$$

$$=A\left(x*0\right)$$

$$=A\left(x\right).$$

Thus A is a fuzzy ideal of X .

Corollary 4.10. An n-fold weak BCI – commutative weak ideal is a weak ideal.

Theorem 4.11. Suppose that $\tilde{A}$ is a weak ideal (namely A is a fuzzy ideal by Theorem 2.9), then the following conditions are equivalent:

1. A is a fuzzy n-fold weak BCI – commutative ideal ;

2. $\forall x_{\lambda},y_{\mu}\in\tilde{X}$ such that $(x_{\lambda}*y^{n}_{\min(\lambda,\mu)})\in\tilde{A}$, we have

$$\left(x_{\lambda}*\left(\left(y_{\mu}*\left(y_{\mu}*x_{\lambda}\right)\right)*\left(0_{\alpha}*\left(0_{\alpha}*\left(\left(x_{\lambda}*y_{\mu}\right)*y_{\mu}\right)\right)\right)\right)\right)\in\tilde{A}.$$

3. $\forall t\in(0,1]$, the t-level subset $A^{t}=\{x\in X:A(x)\geq t\}$,

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is an n-fold weak IBCI – commutative ideal when $A^t \neq \phi$;

4. $,\forall x,y \in X, A\left(x * \left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(\left(x*y\right)*y\right)\right)\right)\right)\right) \geq A\left(x*y^n\right)$;

5. $\tilde{A}$ is an n-fold weak BCI – commutative weak ideal

Proof. $1 \Rightarrow 2$ - Let $(x_\lambda * y^n_{\min(\lambda,\mu)}) \in \tilde{A}$. Since A is a fuzzy n-fold weak BCI – commutative ideal, we have

$$A\left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(\left(x*y\right)*y\right)\right)\right)\right)\right) \geq \min\left(A\left(\left(x*y^n\right)*0\right), A\left(0\right)\right)$$

$$= A(((x*y^n))) \geq \min(\lambda,\mu) \geq \min\left(\lambda,\mu,\alpha\right).$$

$$\text{Therefore } \left(x*\left(\left(y*\left(y*x\right)\right)*\left(0*\left(0*\left(\left(x*y\right)*y\right)\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)}$$

$$= \left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \tilde{A}$$

$2 \Rightarrow 3$ – Obviously, $\forall t \in (0,1]$, $0 \in A^t$.

Let $(x*y^n) \in A^t$, we have

$$(x*y^n)_t = (x_t * y^n_t) \in \tilde{A}.$$

By the hypothesis, one obtains,

$$\left(x_t * \left(\left(y_t * \left(y_t * x_t\right)\right)\left(0_t * \left(0_t * \left(\left(x_t * y_t\right) * y_t\right)\right)\right)\right)\right) \in \tilde{A}$$

therefore $\left(x * \left(\left(y*\left(y*x\right)\right)\left(0*\left(0*\left(\left(x*y\right)*y\right)\right)\right)\right)\right) \in A^t$. Using Lemma 4.2. , we can conclude that

$A^t = \{x \in X : A(x) \geq t\}$ it is an n-fold weak BCI – commutative ideal.

$3 \Rightarrow 4$- Let $x, y \in X$ and $t = A(x*y^n)$, then $(x*y^n) \in A^t$.

Since A^t is an n-fold weak is BCI – commutative ideal, we have

$$\left(x * \left(\left(y * \left(y * x\right)\right)\left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right)\right) \in A^t \text{, therefore}$$

$$A\left(x * \left(\left(y * \left(y * x\right)\right)\left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right)\right) \geq t = A\left(x * y^n\right).$$

Notes

$4 \Rightarrow 5$ – Let $\lambda \in \mathrm{Im}(A)$, it is clear that $0_\lambda \in \tilde{A}$.

- Let $(x_\lambda * y_\mu^n) * z_\alpha \in \tilde{A}$ and $z_\alpha \in \tilde{A}$. Since $\tilde{A}$ it is a weak ideal, $(x * y^n)_{\min(\lambda,\mu)} \in \tilde{A}$. Using the hypothesis, we obtain

$$A\left(x * \left(\left(y * \left(y * x\right)\right)\left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right)\right) \geq A\left(x * y^n\right) \geq \min\left(\lambda, \mu, \alpha\right).$$

From this, one can deduce that

$$\left(x * \left(\left(y * \left(y * x\right)\right) * \left(0 * \left(0 * \left(\left(x * y\right) * y\right)\right)\right)\right)\right)_{\min(\lambda,\mu,\alpha)}$$

$$= \left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right) * \left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \tilde{A}$$

$5 \Rightarrow 1$ Follows from Theorem 4.8

Theorem 4.12. Let $\{\tilde{A}_{i\in I}\}$ be a family of n-fold weak BCI – commutative weak ideals and $\{A_{i\in I}\}$ be a family of fuzzy n-fold weak BCI – commutative ideals. then (1) $\bigcap_{i\in I} \tilde{A}_i$ is an n-fold weak BCI – commutative weak ideal.

(2))$\bigcup_{i\in I} \tilde{A}_i$ is an n-fold weak BCI – commutative weak ideal.

(3)) $\bigcap_{i\in I} A_i$ is a fuzzy n-fold weak BCI – commutative ideal.

(4))$\bigcup_{i\in I} A_i$ is a fuzzy n-fold weak BCI – commutative ideal.

Proof. (1) $\forall \lambda \in \mathrm{Im}\left(\bigcap_{i\in I} \tilde{A}_i\right)$, then $\lambda \in \mathrm{Im}(\tilde{A}_i), \forall i$, so, $0_\lambda \in \tilde{A}_i, \forall i$, i.e. $0_\lambda \in \bigcap_{i\in I} \tilde{A}_i$. For every $x_\mu, y_\lambda, z_\alpha \in \tilde{X}$, if $\left(x_\lambda * y^n{}_\mu\right) * z_\alpha \in \bigcap_{i\in I} \tilde{A}_i$ *and* $, z_\alpha \in \bigcap_{i\in I} \tilde{A}_i$, then $\left(x_\lambda * y^n{}_\mu\right) * \in \tilde{A}_i$ *and* $, z_\alpha \in \tilde{A}\ \forall i$, thus

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \tilde{A}_i \forall i$$

So $\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \bigcap_{i \in I} \tilde{A}_i$. Thus $\bigcap\limits_{i \in I} \tilde{A}_i$ is an *n*-fold weak BCI – commutative weak ideals

(2) $\forall \lambda \in \operatorname{Im}\left(\bigcup_{i \in I} \tilde{A}_i\right)$, then $\exists i_0 \in I$, *such that* $\lambda \in \dot{\tilde{A}}_{i_0}$, so, $0_\lambda \in \tilde{A}_{i_0}$, i.e.

$0_\lambda \in \bigcup_{i \in I} \tilde{A}_i$. For every $x_\mu, y_\lambda, z_\alpha \in \tilde{X}$, if

$\left(x_\lambda * y^n{}_\mu\right) * z_\alpha \in \bigcup_{i \in I} \tilde{A}_i$ *and* $, z_\alpha \in \bigcup_{i \in I} \tilde{A}_i$, then $\exists i_0 \in I$ such that

$$\left(x_\lambda * y^n{}_\mu\right) * z_\alpha \in \tilde{A}_{i_0} \text{ and } , z_\alpha \in \tilde{A}_{i_0} \ \forall i \text{ , thus}$$

$$\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \tilde{A}_{i_0}$$

So $\left(x_\lambda * \left(\left(y_\mu * \left(y_\mu * x_\lambda\right)\right)\left(0_\alpha * \left(0_\alpha * \left(\left(x_\lambda * y_\mu\right) * y_\mu\right)\right)\right)\right)\right) \in \bigcup_{i \in I} \tilde{A}_i$. Thus $\bigcup\limits_{i \in I} \tilde{A}_i$ is an *n*-fold weak BCI – commutative weak ideals.

(3) Follows from (1) and Theorem 4.8.

(4) Follows from (2) and Theorem4.8.

V. Algorithms

Here We Give Some Algorithms For Studding The Structure Of The Foldness Of (Fuzzy BCI- commutative Ideals In BCI-Algebras)

Notes

Notes

Algorithm for ABCI- Commutative Ideals of BCI-Algebra

Input (X :*BCI-algebra*, $*$: binary operation, I : subset of X) ;

Output(" I is a BCI- commutative ideal of X or not");

Begin

If $I = \phi$ then

go to (1.);

End If

If $0 \notin I$ then

go to (1.);

End If

Stop:=false;

$i := 1$;

While $i \le |X|$ and not (*Stop*) do

$j := 1$;

While $j \le |X|$ and not (*Stop*) do

$k := 1$;

While $k \le |X|$ and not (*Stop*) do

If $(x_i * y_j) * z_k \in I$ and $z_k \in I$ then

If $\left(x * \left(\left(y * \left(y * x\right)\right)\right) * \left(0 * \left(0 * \left(x * y\right)\right)\right)\right) \notin I$

Stop:=true;

EndIf

End If

Endwhile

Endwhile

Endwhile

If *Stop* then

Output (" I is a BCI- commutative ideal of X ")

Else

(1.) Output (" I is not a BCI- commutative ideal of X ")

End If

End

Algorithm for N-Fold BCI- Commutative Ideals of BCI - Algebra

Input (X :*BCI -algebra*, $*$: binary operation, I : subset of X) ;

Output(" I is n-fold BCI - commutative ideal of X or not");

Begin

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If $I = \phi$ then
  go to (1.);
End If
If $0 \notin I$ then
  go to (1.);
End If
Stop:=false;
$i := 1$;
While $i \leq |X|$ and not (*Stop*) do
$j := 1$;
  While $j \leq |X|$ and not (*Stop*) do
$k := 1$;
    While $k \leq |X|$ and not (*Stop*) do
     If $(x_i * y_j) * z_k \in I$ and $z_k \in I$ then
      If $\left(x * \left(\left(y * \left(y * x\right)\right)\right) * \left(0 * \left(0 * \left(x * y^n\right)\right)\right)\right) \notin I$
       Stop:=true;
      EndIf
     EndIf
    Endwhile
  Endwhile
 Endwhile
 If *Stop* then
  Output ("I is an n-fold BCI - commutative ideal of X")
 Else
 (1.) Output ("I is not an n-fold BCI - commutative ideal of X")
 End If
End

Algorithm for Fuzzy BCI- Commutative Ideals of Bcialgebra

Input (X : *BCI-algebra*, $*$: binary operation, A : the fuzzy subset of X);
Output("A is a fuzzy BCI- commutative ideal of X or not");
Begin
 Stop:=false;
$i := 1$;
 While $i \leq |X|$ and not (*Stop*) do
  If $A(0) < A(x_i)$ then
   Stop:=true;
  End If

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$j := 1$;
While $j \leq |X|$ and not (*Stop*) do
$k := 1$;
While $k \leq |X|$ and not (*Stop*) do
If $A\left(x * \left(\left(y * \left(y * x\right)\right)\right) * \left(0 * \left(0 * \left(x * y\right)\right)\right)\right) < \left(A\left(\left(x * y\right) * z\right), A\left(z\right)\right)$ then
Stop=true;
End If
Endwhile
Endwhile
Endwhile
If *Stop* then
Output (" A is not a fuzzy BCI- commutative ideal of X ")
Else
Output (" A is a fuzzy BCI- commutative ideal of X ")
End If
End

Algorithm for Fuzzy N–Fold BCI- Commutative Ideals of BCI-Algebra

Input (X : *BCI-algebra*, $*$: binary operation, A : the fuzzy subset of X) ;
Output(" A is a fuzzy n- fold BCI- commutative ideal of X or not");
Begin
Stop:=false;
$i := 1$;
While $i \leq |X|$ and not (*Stop*) do
If $A(0) < A(x_i)$ then
Stop:=true;
End If
$j := 1$;
While $j \leq |X|$ and not (*Stop*) do
$k := 1$;
While $k \leq |X|$ and not (*Stop*) do

$$A\left(x * \left(\left(y * \left(y * x\right)\right)\right) * \left(0 * \left(0 * \left(x * y^{n}\right)\right)\right)\right) < \left(A\left(\left(x * y\right) * z\right), A\left(z\right)\right)$$

Stop=true;
End If
Endwhile
Endwhile
Endwhile

Notes

```
  If Stop then
    Output (" A is not a fuzzy n- fold BCI- commutative ideal of X ")
  Else
    Output (" A is a fuzzy n- fold BCI- commutative ideal of X ")
  End If
End
```

Algorithm for N–Fold Weak BCI- Commutative Ideals of BCI-Algebra

```
Input( X :BCI-algebra, I : subset of X , n ∈ N );
Output(" I is an n-fold weak BCI - commutative ideal of X or not");
Begin
 If I = ϕ then
   go to (1.);
 End If
 If 0 ∉ I then
   go to (1.);
 End If
 Stop:=false;
 i := 1;
 While i ≤ |X| and not (Stop) do
 j := 1;
   While j ≤ |X| and not (Stop) do
 k := 1;
     While k ≤ |X| and not (Stop) do
       If (x * y^n) * z ∈ I ,and ,z ∈ I then
         If x * ((y * (y * x)) * (0 * (0 * ((x * y) * y)))) ∉ I
           Stop:=true;
         EndIf
       EndIf
     Endwhile
   Endwhile
  Endwhile
  If Stop then
    Output (" I is an n-fold weak BCI - commutative ideal of X ")
  Else
   (1.) Output (" I is not an n-fold weak BCI - commutative ideal of X ")
  End If
End
```

Notes

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Notes

Algorithm for Fuzzy N-Fold Weak BCI- Commutative Ideals of BCI-Algebra

Input(X : *BCI-algebra*, $*$: binary operation, A fuzzy subset of X);
Output(" A is a fuzzy n-fold weak BCI -commutative ideal of X or not");
Begin
Stop:=false;
$i := 1$;
While $i \le |X|$ and not (*Stop*) do
If $A(0) < A(x_i)$ then
Stop:=true;
End If
$j := 1$;
While $j \le |X|$ and not (*Stop*) do
$k := 1$;
While $k \le |X|$ and not (*Stop*) do
If $\Big(x * \big((y * (y * x))\big(0 * \big(0 * ((x * y) * y)\big)\big)\big)\Big) < \min\Big(A\big((x * y^n) * z\big), A(z)\Big)$
then
Stop=true;
End If
Endwhile
Endwhile
Endwhile
If *Stop* then
Output (" A is not a fuzzy n-fold weak BC I–a commutative ideal of X ")
Else
Output (" A is a fuzzy n-fold weak BCI –commutative ideal of X ")
End If
End

VI. Conclusion and Future Research

In this paper we introduce new notions of (fuzzy) n-fold BCI-commutative ideals, and (fuzzy) n-fold weak BCI - commutative ideals in BCI - algebras ., Then we studied relationships between different type of n- fold BCI – commutative ideals and investigate several properties of foldness theory of BCI – commutative ideals in BCI -algebras. Finally, we construct some algorithms for studying foldness theory of BCI – commutative ideals in BCI -algebras.

In our future study of foldness ideals in BCK/BCI algebras, maybe the following topics should be considered :

(1) developing the properties of foldness of positive implicative ideals of BCK/BCI algebras.

(2) finding useful results on other structures of foldness theory of ideals of BCK/BCI algebras.

(3) constructing the related logical properties of such structures.

(4) one may also apply this concept to study some applications in many fields like decision making knowledge base systems, medical diagnosis, data analysis, and graph theory.

Notes

References Références Referencias

1. Y. Huang and Z. Chen, On ideals in BCK-algebra, Math. Japonica, 50 (1999), pp. 211-226.
2. K. Iséki, On ideals in BCK-algebras, Math. Seminar Notes (presently, Kobe J. Math.) 3(1975), 1-12.
3. K. Iséki and S. Tanaka, An Introduction to the theory of BCK-algebra, Math. Japonica, 23(1978), pp. 935-942.
4. Y. B. Jun, On n-fold fuzzy implicative/commutative ideals of BCK-algebras IJMMS, (2001), pp. 419-427.
5. Y. B. Jun, (2017) Neutrosophic subalgebras of several types In BCK /BCI-algebras, Ann. Fuzzy Math Inform 14 (1), PP 75-86.
6. C. Lele, C. Wu, P. Weke, T. Mamadou and C.E. Njock, Fuzzy ideals and weak ideals in BCK-algebras, Scientiae Mathematicae Japonicae, 54 (2001), pp. 323-336.
7. Y. Liu and J. Meng, Fuzzy Ideals in BCI-algebras, Fuzzy sets, and systems 123(2001) 227-237.
8. Muhiuddin, G., Kim S. J . and Jun Y. B. (2019) Implicative N – ideals of BCK – algebras based on Neutrosophic N – structures, Discrete mathematics, algorithms, and Applications 11 (1), 1950011.
9. G. Muhiuddin, Y. B. Jun, Further Results of Neutrosophic subalgebras in BCK/BCI -algebras based on Neutrosophic points, TWMS J. App. Eng. Math. V. 10 N. 2, 2020, PP 232-240.
10. J. Meng, Y. B Jun and H. S. KIM, Fuzzy implicative ideals of BCK-algebras, Fuzzy sets, and systems 89 (1997) pp. 243-248.
11. O. G. XI, Fuzzy BCK-algebras, Math. Japonica, 36 (1991), pp. 935-942.
12. L. A. Zadeh, Fuzzy sets, Inform. and Control 8(1965), pp. 338-353.

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Effects of Correlation between the Error Term and Autocorrelation on Some Estimators in a System of Regression Equations

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Abstract- Seemingly unrelated regression model developed to handle the problem of correlation among the error terms of a system of the regression equations is still not without a challenge, where each regression equation must satisfy the assumptions of the standard regression model. When dealing with time-series data, some of these assumptions, especially that of independence of the regressors and error terms leading to multicollinearity and autocorrelation respectively, are often violated. This study examined the effects of correlation between the error terms and autocorrelation on the performance of seven estimators and identify the estimator that yields the most preferred estimates under the separate or joint influence of the two correlation effects considered by the researcher. A two-equation model was considered, in which the first equation had multicollinearity and autocorrelation problems while the second one had no correlation problem. The error terms of the two equations were also correlated. The levels of correlation between the error terms and autocorrelation were specified between -1 and +1 at interval of 0.2 except when it approached unity.

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EFFECTSOFCORRELATIONBETWEENTHEERRORTERMANDAUTOCORRELATIONONSOMEESTIMATORSINASYSTEMOFREGRESSIONEQUATIONS

Notes

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Abstract- Seemingly unrelated regression model developed to handle the problem of correlation among the error terms of a system of the regression equations is still not without a challenge, where each regression equation must satisfy the assumptions of the standard regression model. When dealing with time-series data, some of these assumptions, especially that of independence of the regressors and error terms leading to multicollinearity and autocorrelation respectively, are often violated. This study examined the effects of correlation between the error terms and autocorrelation on the performance of seven estimators and identify the estimator that yields the most preferred estimates under the separate or joint influence of the two correlation effects considered by the researcher. A two-equation model was considered, in which the first equation had multicollinearity and autocorrelation problems while the second one had no correlation problem. The error terms of the two equations were also correlated. The levels of correlation between the error terms and autocorrelation were specified between -1 and +1 at interval of 0.2 except when it approached unity. A Monte Carlo experiment of 1000 trials was carried out at five levels of sample sizes 20, 30, 50, 100, and 250 at two runs. The seven estimation methods namely; Ordinary Least Squares (OLS), Cochran – Orcutt (CORC), Maximum Likelihood Estimator (MLE), Multivariate Regression, Full Information Maximum Likelihood (FIML), Seemingly Unrelated Regression Model (SUR), and Three-Stage Least Squares (3SLS). Their performances were examined by subjecting the results obtained from each finite property of the estimators into a multi-factor analysis of variance model. The significant factors were further checked using their estimated marginal means and the Least Significant Difference (LSD) methodology to determine the best estimator. The findings generally show that the estimator of MLE is preferred to estimate all the parameters of the model in the presence of correlation between the error terms and autocorrelation at all the sample sizes. This study has applications in areas such as Economics, Econometrics, Social Sciences, Agricultural Economics, and some other fields where the correlation between the error terms and autocorrelation problems can be encountered.

I. Introduction

The seemingly unrelated regression (SUR) model is common in the Econometric literature (Zellner, 1962; Srivastava and Giles, 1987; Greene, 2003) but is less known elsewhere, its benefits have been explored by several authors, and more recently the SUR model is being applied in Agricultural Economics (O' Dorell et al. 1999), Wilde et al. (1999). Its application in the natural and medical sciences is likely to increase once scientists in the disciplines are exposing to its potential.

The SUR estimation procedures which enable an efficient joint estimation of all the regression parameters were first reported by Zellner (1962), which involves the

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application of Aitken's Generalised Least Squares (AGLS) (Aitken 1935, Powell 1965) to the whole system of equations. Zellner (1962 & 1963), submitted that the joint estimation procedure of SUR is more efficient than the equation-by-equation estimation procedure of the Ordinary Least Square (OLS). The gain in efficiency would be magnified if the contemporaneous correlation between each pair of the disturbances in the SUR system of equations is very high and explanatory variables (covariates) in different equations are uncorrelated. In other words, the efficiency in the SUR formulation increases, the more the correlation between error vectors differs from zero, and the closer the explanatory variables for each response are to being uncorrelated.

David (1999), in his work on test for auto correlated errors which are generalized to cover systems of equations and the properties of 18 versions of the test are studied using Monte Carlo methods. However, the size and power properties of all tests deteriorate sharply as the number of equations increases, the system becomes more dynamic, the exogenous variables become more auto correlated, and the sample size decreases. This performance has, in general, an unknown degree since the interaction amongst these factors does not permit a predictive summary, as might be hoped for by response surface-type approaches.

Unger et al. (2009), in their work, developed a regression model for use with ensemble forecasts. Ensemble members are assumed to represent a set of equally likely solutions, one of which will best fit the observation. If standard linear regression assumptions apply to the best member, then a regression relationship can be derived between the full ensemble and the observation without explicitly identifying the best member for each case. The ensemble regression equation is equivalent to linear regression between the ensemble mean and the observed data, but is applied to each member of the ensemble. The "best member" error variance is defined in terms of the correlation between the ensemble mean and the observations, their respective variances, and the ensemble spread.

a) Methods of Parameter Estimation of the Linear Model with Auto correlated Errors

The GLS and the OLS methods are the two methods that can be used to estimate the parameters of the linear model in the presence of auto correlated error. Since the later suffers efficiency, the former is used to improve this efficiency. However, Chipman (1979), Kramer (1980), Kleiber (2001), Olanrewaju S.O. (2017), among many others, have observed that the efficiency of the OLS estimator in a linear regression containing an auto correlated error term depends largely on the structure of X used. The GLS method requires thatΩ, and in particular, ρ is known before the parameters can be estimated. Thus, in a linear model with an auto correlated error term

$$\widetilde{\widetilde{\beta}}_{(\mathrm{GLS})} = (X^1\Omega^{-1}X)^{-1}X^1\Omega^{-1}Y \tag{2.4}$$

$$\mathrm{V}(\widetilde{\widetilde{\beta}}_{(\mathrm{GLS})}) = \sigma^2(X^1\Omega^{-1}X)^{-1} \tag{2.5}$$

Where

$$E(UU^{'}) = \sigma^2\Omega = \sigma^2\begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-2} & \rho^{n-1} \\ \rho & 1 & \rho & \cdots & \rho^{n-3} & \rho^{n-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{n-4} & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^{n-2} & \rho^{n-3} & \rho^{n-4} & \cdots & 1 & \rho \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & \rho & 1 \end{bmatrix}$$

Notes

Notes

And $\sigma^2 = \sigma_u^2 = \frac{\sigma_\varepsilon^2}{(1-\rho^2)}$,

And the inverse of Ω is

$$\Omega^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$$

We now search for a suitable transformation matrix P^*, as discussed in section 2.1. If we consider an (n – 1) x n matrix P^* defined by

$$P^* = \begin{bmatrix} -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ 0 & 0 & -\rho & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}_{(n-1) \times n} \tag{2.6}$$

Multiplying then shows that $P^{*1}P^*$ gives an n x n matrix which apart from a proportional constant is identical with Ω^{-1} except for the first elements in the leading diagonal, which is ρ^2rather than unity.

Now if we consider an n x n matrix P obtained from $P^*$by adding a new row to the first row with $\sqrt{1-\rho^2}$ in the first position and zero elsewhere, that is

$$P = \begin{bmatrix} (1-\rho^2)^{\frac{1}{2}} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}_{(n \times n)} \tag{2.7}$$

Multiplying shows that $P^1P = (1-\rho^2)\Omega^{-1}$. The difference between P^* and P lies only in the treatment of the first sample observation, P^* is much easier to use, provided we are prepared to lose information on the first observation. However, when n is large, the difference is negligible, but in a small samples such as in this study, the difference can be large.

If Ω or more precisely, ρ is known, the GLS estimation can be achieved by applying the OLS via the transformation matrix P^* and P above. However, this is not often the case; we resort to estimating Ω by Ω to have feasible Generalized Least Squares Estimator. This estimator becomes feasible when P is replaced by a consistent estimator ρ (Formby et al. 1988).

b) Notations:

* : Computed F value is significant at $\alpha = 0.01$

** : Computed F value is significant at $\alpha = 0.05$

CR-: Correlation between the error terms
RE-: Autocorrelation
BB-: Bias
AB-: Absolute Bias
MB-: Mean Square Error
VB-: Variance
OLS-: Ordinary Least Squares
COCR-: Cochrane – Orcutt (Generalized Least Squares)
MLE-: Maximum Likelihood Estimator
MULTIREG-: Multivariate Regression
FIML-: Full Information Maximum Likelihood
SUR -: Seemingly Unrelated Regression
3SLS -: Three Stage Least Squares
M-: Method

Notes

II. The Monte - Carlo Approach

Monte-Carlos is a mathematical technique based on experiment for evaluation and estimation of problems which are intractable by probabilistic or deterministic approach. By probabilistic Monte-Carlo experiment, random numbers are observed and chosen in such a way that they directly simulate the physical random process of the original problem. The desired solutions from the behavior of these random numbers are then inferred. The idea of a Monte-Carlo approach to deterministic problems is to exploit the strength of theoretical Mathematics, which cannot be solved by theoretical means but now being solved by a numerical approach.

The Monte-Carlo approach has been found useful to investigate the small (finite) sample properties of these estimators. The use of this approach is because real-life observation on economic variables is in most cases, plagued by one or all of the problems of nonspherical disturbances and measurement and misspecification errors. By this approach, data sets and stochastic terms are generated, which are free from all the problems listed above and, therefore, it can be regarded as data obtained from a controlled laboratory experiments.

In a Monte-Carlo experiment, the experimenter artificially sets up a system (model) and specifies the distribution of the independent variables alongside with the values of the model parameters. Those values are then generated for the error term and the independent variables as specified for a specified sample size. By using those generated values and the parameter values, the value of the dependent variable is thus determined. Next is to treat the generated data as if they are real-life data by estimating the parameters of the model via the estimation methods (estimators). This process of generating values for the disturbance term, independent variables, and estimating the parameters of the model is then replicated a large number of times. The experimenter then builds up empirical distributions of the parameter estimates, which are then used to evaluate the performance of the estimators in estimating the parameter values.

The Monte – Carlo studies can be designed generally by using the following summarized five steps as given below:

(a) The researcher specifies a model and assigns specific numeric values as in parameters. The assigned values are assumed to be the real values of the parameter.
(b) The distribution of error terms is also specified by the researcher.

Notes

(c) He uses the distribution of U's with the random drawings from it to obtain different values for the error terms.
(d) The experimenter now selects or generates values for the regressors (X's) depending on the specifications of the model.
(e) The researcher obtains or generates values for the dependent variable using the real values of the regressors and the error terms. (Olanrewaju et al. 2017)

The five steps mentioned above are repeated several times, say R, to have R replications.

Thus, the experimenter obtains an estimate of the model parameters for each replication, treating the generated data as real-life data.
(i) Seven estimation methods under consideration
(ii) Different number of replication (replication of 1000 in this context)
(iii) Different sample sizes of 20, 30, 50, 100, and 250 as used in this study. (Olanrewaju et al. 2017)

III. The Model Formulation

The System of regression equations used in this research work as proposed by Olanrewaju S.O. (2013) is given as:

$$y_{1t} = \beta_{01} + \beta_{11}x_{1t} + \beta_{12}x_{2t} + u_{1t} \tag{3.1}$$

where,

$$u_{1t} = \rho u_{1(t-1)} + e_{1t} \ , \ e_{1t} \approx (0, \sigma^2) \ .$$

$$y_{2t} = \beta_{02} + \beta_{21}x_{1t} + \beta_{22}x_{3t} + u_{2t} \ , \ u_{2t} \approx N(0, \sigma^2) \tag{3.2}$$

Note: (1) Multicollinearity exists between X_1 and X_2 in equation (3.1)
(2) Autocorrelation exists in equation (3.1)
(3) There is a correlation between U_1 and U_2 of the two equations
(4) There is no correlation between X_1 and X_3 in equation (3.2). Thus, equation (3.2) appears as a control equation.

a) *The Equation used for generating values in the simulation*

The equation used for generating values of the variables in the simulation study as proposed by Ayinde K. (2007) is given below:

Suppose, $W_i \sim N(\mu, \sigma_i^2) \quad i = 1,2$. If these variables are correlated, then, W_1 and W_2 can be generated with the following equations:

$$\begin{aligned} W_1 &= \mu_1 + \sigma_1 z_1 \\ W_2 &= \mu_2 + \rho\sigma_2 z_1 + \sigma_2 z_2 \sqrt{1-\rho^2} \end{aligned} \tag{3.3}$$

Where $Z_i \tilde{} N(0,1)$ i = 1,2 and $|\rho| < 1$ is the value of the correlation between the two variables.

b) *Other Specifications*

1. Sample Size(n) of 20, 30, 50, 100 and 250 were used in the simulation
2. The following levels were used for the correlations studied:
a. Autocorrelation(RE): -0.99, -0.9,-0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8,0.9, 0.99

b. Correlation between error term (CR) : -0.99, -0.9,-0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.99
c. Replication (RR) : we make use of 1000 replications
d. Two RUNS were done for the simulations which were averaged at analysis stage.

c) Criteria for comparison

The evaluation and comparison of the seven (7) estimators considered in this study were examined using the finite sampling properties of estimators, which include Bias (BB), Absolute Bias (AB), Variance (VB), and the Mean Square Error (MB) criteria.

Notes

Mathematically, for any estimator $\hat{\beta}_i$ *of* β_i of the models (3.1) & (3.2)

$$(i)\ \bar{\hat{\beta}}_i = \frac{1}{R}\sum_{j=1}^{R}\hat{\beta}_{ij} \qquad (ii)\ Bias\left(\hat{\beta}_i\right) = \frac{1}{R}\sum_{j=1}^{R}\left(\hat{\beta}_{ij} - \beta_j\right) = \bar{\hat{\beta}}_i - \beta_i$$

$$(iii)\ AB\left(\hat{\beta}_i\right) = \frac{1}{R}\sum_{j=1}^{R}\left|\hat{\beta}_{ij} - \beta_j\right| \qquad (iv)\ Var\left(\hat{\beta}_i\right) = \frac{1}{R}\sum_{j=1}^{R}\left(\hat{\beta}_{ij} - \bar{\hat{\beta}}_i\right)^2$$

$$(v)\ MSE\left(\hat{\beta}_i\right) = \frac{1}{R}\sum_{j=1}^{R}\left(\hat{\beta}_{ij} - \beta_i\right)^2, \quad \text{for } i = 0, 1, 2 \text{ and } j = 1,2,\ldots, R.$$

Using a computer program which was written with TSP software package to estimate all the model parameters and the criteria, the performances of seven estimation methods; Ordinary Least Squares (OLS), Cochran – Orcutt (COCR), Maximum Likelihood Estimator (MLE), Multivariate Regression, Full Information Maximum Likelihood (FIML), Seemingly Unrelated Regression (SUR) and Three-Stage Least Squares (3SLS) were examined by subjecting the results obtained from each finite properties of the estimators into a multi-factor analysis of variance model. Consequently, the highest order significant interaction effect, which has a "method" as a factor, is further examined using the Least Significance Difference (LSD) test. The estimated marginal mean of the factor was investigated out at a particular combination of levels of the correlations in which estimators were preferred if the marginal mean is the smallest.

IV. Analysis and Results

The summary of results from the Analysis of variance tables of the criteria showing the effect of the estimators, the correlation between the error term and autocorrelation on β_i are presented in Table 1 below.

Notes

Table 1: ANOVA for a sample size of 20

s.no.	SOV	EQN	βi	df	TYPE III SUM OF SQUARES: Bias	Absolute Bias	Variance	Mean Square
20	RE	1	β01	12	892.446***	115926.509***	2445822.237***	3951716.298***
			β11	12	.029***	32.515***	95.927***	96.084***
			β21	12	.012	24.373***	23.509***	87.017***
		2	β02	12	.112	103.206***	122116.658***	128548.527***
			β12	12	.063	.628***	.093***	.091***
			β22	12	.132***	.605***	.113***	.125***
	CR	1	β01	12	.670	.003	.003	.005
			β11	12	.001	6.016E-5	8.532E-5	8.897E-5
			β21	12	7.468***	3.807***	3.176***	5.004***
		2	β02	12	3.519	45.130***	113879.706***	119769.347***
			β12	12	.513***	.032	.224***	.032***
			β22	12	3.006***	.404***	.139***	.011***
	M	1	β01	6	315.786***	83483.317***	4080093.223***	6466311.896***
			β11	6	.000	4.612***	5.977***	5.990***
			β21	6	.007	5.564***	2.320***	9.779***
		2	β02	6	.042	45.091***	232859.705***	243905.100***
			β12	6	.476***	.141***	.002***	.006***
			β22	6	.086***	2.096***	.361***	.391***
	RE*CR		β01	144	.458	.026	.021	.037
			β11	144	.001	.000	.001	.001
			β21	144	5.046	1.759	7.126***	19.761**
			β02	144	5.506	195.745***	360069.375***	378405.077***
			β12	144	.048	.423***	.054***	.052***
			β22	144	.019	.256***	.053***	.052***
	RE*M	1	β01	72	5540.631***	454326.369***	1.038E7***	1.557E7***
			β11	72	.011	15.816***	40.207***	40.276***
			β21	72	.014	15.506***	13.080***	56.497***
		2	β02	72	.675	199.404***	716966.573***	755012.778***
			β12	72	.201	.078	.007**	.007
			β22	72	.116***	.529***	.131***	.134***
	CR*M	1	β01	72	.515	.002	.002	.004
			β11	72	.001	4.520E-5	6.406E-5	6.680E-5
			β21	72	5.943	2.889***	2.473**	3.902
		2	β02	72	3.940	196.384***	683436.471***	721549.650***
			β12	72	.243	.407***	.004	.022***
			β22	72	.148***	1.340***	.289***	.246***
	RE*CR*M	1	β01	864	.348	.020	.017	.030
			β11	864	.001	.000	.000	.000
			β21	864	3.917	1.358	5.731	15.895
		2	β02	864	33.150	884.547***	2141981.317***	2251208.104***
			β12	864	.072	.059	.006	.005
			β22	864	.082	.433	.102	.103
	ERROR	1	β01	1183	3595.810	8759.488	8834975.252	8871627.167
			β11	1183	.245	11.841	50.548	50.564
			β21	1183	84.299	23.465	28.185	128.394
		2	β02	1183	214.134	135.089	1384793.426	1438377.807
			β12	1183	16.754	2.534	.089	.150
			β22	1183	.659	1.140	.197	.256
	TOTAL	1	β01	2365	10346.814	662542.071	2.575E7	3.487E7
			β11	2365	.288	64.783	192.657	192.913
			β21	2365	106.707	78.742	85.610	326.275
		2	β02	2365	261.126	1804.890	5756781.519	6037516.906
			β12	2365	18.374	4.302	.480	.365
			β22	2365	4.259	6.818	1.387	1.320

Effect on B_0

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators except GLS2 are preferred to estimate β_0 at all the levels of autocorrelation.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria except in the bias criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators except GLS2 are preferred for β_0 at all levels of autocorrelation and correlation between the error terms.

Effect on B_1

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation under all criteria except for the bias. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred for β_1 at all the levels of autocorrelation.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria except in the bias criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators are preferred to get β_1 at all levels of autocorrelation and correlation between the error terms.

Effect on B_2

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to estimate β_2 at all the levels of autocorrelation.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria. The results of the LSD further test visa-vice their estimated marginal means revealed that SUR and 3SLS estimators are preferred to get β_2 at all levels of autocorrelation and correlation between the error terms EXCEPT for -0.9 and -0.8 correlation levels between the error terms under the bias that is significantly different.

Summarily, GLS2, MLE, SUR, and 3SLS are preferred to estimate the model at the sample size of 20.

Notes

Notes

Table 2: ANOVA for the sample size of 30

s. no.	SOV	EQN	β_i	df	TYPE III SUM OF SQUARES			
					Bias	Absolute Bias	Variance	Mean Square
30	RE	1	β01	12	1368.073***	165272.612***	1.008E12***	1.009E12***
			β11	12	.029***	37.228***	125.437***	127.031***
			β21	12	.075	29.385***	69.897***	83.604***
		2	β02	12	.095	51.854***	6251392.175***	6258276.131***
			β12	12	.005	.334***	.025***	.025***
			β22	12	.011	.175***	.008***	.043***
	CR	1	β01	12	.147	10.954	1.102E8	1.102E8
			β11	12	.271***	.043	.144	.137
			β21	12	1.980***	5.721***	5.706***	5.296***
		2	β02	12	.200	13.613***	5187309.671***	5192414.391***
			β12	12	2.338***	.012	.096***	.001
			β22	12	.695	15.467***	.018***	2.358***
	M	1	β01	6	187.891***	82932.248***	6.302E11	6.312E11***
			β11	6	.013**	5.296***	7.289***	7.403***
			β21	6	.009	4.316***	8.386***	8.620***
		2	β02	6	.007	9.955***	5467535.629***	5474905.799***
			β12	6	.029	.034	.004***	.004
			β22	6	.001	.114***	.052***	.040***
	RE*CR		β01	144	1.846	131.775	1.362E9	1.363E9
			β11	144	.143**	.163	1.140	1.113
			β21	144	1.132	2.945***	20.161***	18.791***
			β02	144	1.196	70.924***	3.657E7***	3.661E7***
			β12	144	.024	.222	.016***	.016
			β22	144	.034	.108	.005***	.016
	RE*M	1	β01	72	7396.149***	696530.165***	6.041E12***	6.050E12***
			β11	72	.012	17.083***	51.754***	52.348***
			β21	72	.045	15.567***	46.940***	48.988***
		2	β02	72	583	43.075***	3.735E7***	3.739E7***
			β12	72	.002	.034	.003**	.004
			β22	72	.028	.105	.016***	.026
	CR*M	1	β01	72	.988	45.595	6.622E8	6.623E8
			β11	72	.203***	.032	.108	.103
			β21	72	1.491***	4.279***	4.281	3.968
		2	β02	72	.863	40.424***	3.189E7***	3.193E7***
			β12	72	.012	.085	.004***	.005
			β22	72	.009	.414***	.035***	.065
	RE*CR*M	1	β01	864	10.321	546.890	8.173E9	8.174E9
			β11	864	.107	.121	.855	.835
			β21	864	.853	2.202	15.134	14.102
		2	β02	864	7.135	249.877***	2.190E8***	2.192E8***
			β12	864	.001	.041	.003	.003
			β22	864	.014	.174	.013***	.030
	ERROR	1	β01	1183	3150.131	4943.259	4.933E10	5.003E10
			β11	1183	.916	19.633	110.203	112.032
			β21	1183	13.579	13.356	69.548	66.923
		2	β02	1183	44.274	32.901	2.213E7	2.212E7
			β12	1183	7.545	6.245	.041	.357
			β22	1183	123.484	4.942	.011	1.476
	TOTAL	1	β01	2365	12115.647	950459.247	7.739E12	7.751E12
			β11	2365	1.693	79.598	296.927	301.002
			β21	2365	19.164	77.788	240.059	250.302
		2	β02	2365	54.357	512.744	3.639E8	3.642E8
			β12	2365	9.965	7.003	.192	.413
			β22	2365	124.277	21.562	.158	4.059

Effect on β_0

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators are preferred to get β_0 at all the levels of autocorrelation except for GLS2, which differed significantly at 0.8, 0.9 and 0.99 autocorrelation levels.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria except in the bias criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators are preferred to get values for β_0 at all levels of autocorrelation and correlation between the error terms except for GLS2, which differed significantly at autocorrelation level of 0.9 and a correlation between the error terms of 0.99 under the bias criterion.

Effect on β_1

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to calculate β_1 at all the levels of autocorrelation and correlation between the error terms.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under variance criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to get β_1 at all levels of autocorrelation and correlation between the error terms.

Effect on B_2

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to estimate β_1 at all the levels of autocorrelation and correlation between the error terms except that we have to be cautious when using them at some levels of autocorrelation.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under variance criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators except OLS, GLS2, and MLE estimators are preferred to calculate β_2 at all levels of autocorrelation and correlation between the error terms.

Summarily, GLS2 and MLE estimators are preferred to estimate the model at the sample size of 30

Notes

Notes

Table 3: ANOVA for the sample size of 50

s.no.	SOV	β_i	df	TYPE III SUM OF SQUARES			
				Bias	Abs.Bias	Var	MSE
50	RE	β01	12	452.571***	74575.669***	1.764E11***	1.770E11***
		β11	12	.050***	18.709***	24.791***	24.976***
		β21	12	1.014***	6.985***	1.255***	2.662***
		β02	12	.515***	35.964***	251158.322***	252591.912***
		β12	12	.417***	.167***	.004***	.007***
		β22	12	.129**	.174***	.001***	.003***
	CR	β01	12	1.992	234.178	1.404E9	1.406E9
		β11	12	.022***	.972***	1.780***	1.786***
		β21	12	5.131***	.177**	.052	.030
		β02	12	1.353***	6.205***	161713.711***	162539.579***
		β12	12	3.505***	1.093***	.026***	.024***
		β22	12	.221***	.373***	.003***	.003***
	M	β01	6	227.569***	24107.884***	8.971E10***	9.003E10***
		β11	6	.001	3.527***	2.193***	2.209***
		β21	6	.085	2.311***	.619***	.709***
		β02	6	.105***	1.285***	178487.825***	179459.300***
		β12	6	.021***	.003	.002***	8.218E-5
		β22	6	.023	.307***	.010***	.012***
	RE*CR	β01	144	23.036	2764.733	1.713E10	1.714E10
		β11	144	.019	4.251***	12.667***	12.699***
		β21	144	1.698	1.158	.538***	1.049***
		β02	144	2.520***	28.684***	1365792.064***	1373566.001***
		β12	144	.136***	.165***	.005***	.009
		β22	144	.058	.055	.001	.001
	RE*M	β01	72	3285.331***	280727.544***	1.056E12***	1.060E12***
		β11	72	.021	8.107***	10.064***	10.139***
		β21	72	.847	3.449***	1.724***	1.696***
		β02	72	2.363***	10.635***	1480512.907***	1488990.865***
		β12	72	.132***	.218***	.003***	.008***
		β22	72	.009	.064	.002***	.003
	CR*M	β01	72	11.459	1223.011	8.561E9	8.571E9
		β11	72	.016	.241	.671	.673
		β21	72	3.879***	.469	.307***	.207
		β02	72	1.161***	6.135***	982226.051***	987452.884***
		β12	72	.012	.092***	.002**	.007***
		β22	72	.010	.276***	.006***	.007***
	RE*CR*M	β01	864	137.160	14657.361	1.026E11	1.027E11
		β11	864	.013	1.423	4.996	5.009
		β21	864	1.263	.490	1.123	.968
		β02	864	13.881***	59.010***	8150369.132***	8196863.794***
		β12	864	.048	.067	.002	.003
		β22	864	.008	.050	.002	.002
	ERROR	β01	1183	6296.390	82375.378	8.427E11	8.460E11
		β11	1183	.278	8.667	23.218	23.233
		β21	1183	12.234	8.004	2.939	4.219
		β02	1183	4.175	9.073	1571334.088	1580748.512
		β12	1183	.661	.922	.026	.070
		β22	1183	6.136	1.510	.012	.043
	TOTAL	β01	2365	10435.662	480676.871	2.295E12	2.303E12
		β11	2365	.420	45.888	80.372	80.717
		β21	2365	26.153	23.042	8.557	11.540
		β02	2365	26.079	157.021	1.414E7	1.422E7
		β12	2365	4.946	2.729	.070	.128
		β22	2365	6.593	2.811	.038	.073

Effect on β_0

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators are preferred to estimate β_0 at all the levels of autocorrelation except for GLS2, which differed significantly at 0.99 autocorrelation level.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria. The results of the LSD further test visa-vice their estimated marginal means revealed that all estimators are preferred to compute β_0 at all levels of autocorrelation and correlation between the error terms except for GLS2 which differed significantly at autocorrelation levels of 0.9 & 0.99 and correlation between the error terms of 0.99 under all criteria.

Effect on β_1

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to get β_1 at all the levels of autocorrelation and correlation between the error terms.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under variance criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to compute β_1 at all levels of autocorrelation and correlation between the error terms.

Effect on β_2

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to estimate β_2 at all the levels of autocorrelation and correlation between the error terms.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria. The results of the LSD further test visa-vice their estimated marginal means revealed that all estimators except OLS, GLS2, and MLE estimators are preferred to get β_2 at all levels of autocorrelation and correlation between the error terms.

Summarily, GLS2 and MLE estimators are preferred to estimate the model at a sample size of 50

Notes

Notes

Table 4: ANOVA for the sample size of 100

s.no.	SOV	β_i	df	TYPE III SUM OF SQUARES			
				Bias	Abs.Bias	Var	MSE
100	RE	β01	12	47.699	48743.394***	7.898E9***	7.945E9***
		β11	12	.022***	21.108***	33.739***	33.776***
		β21	12	.014	7.287***	3.435***	4.171***
		β02	12	.031	23.608***	27762.719***	27802.782***
		β12	12	.004	.122***	.002***	.002***
		β22	12	.007	.019***	.001***	.001***
	CR	β01	12	.001	.001	.000	.000
		β11	12	.011***	.002	.006	.006
		β21	12	1.366***	1.539***	.352***	.320***
		β02	12	.486	.139***	12228.857***	12185.714***
		β12	12	.047***	.692***	.018***	.016***
		β22	12	.057	1.315***	.002***	.019***
	M	β01	6	12.616	13036.510***	3.909E9***	3.932E9***
		β11	6	.005***	3.739***	2.145***	2.151***
		β21	6	.000	1.601***	.568***	.585***
		β02	6	.044	.218***	13231.759***	13252.284***
		β12	6	.003	.058***	.002***	.002***
		β22	6	.002	.095***	.004***	.005***
	RE*CR	β01	144	.001	.008	.002	.002
		β11	144	.022***	.026	.077	.077
		β21	144	.810***	.784***	1.083***	.978***
		β02	144	.204	15.466***	147798.362***	148004.003***
		β12	144	.011	.053***	.002***	.002***
		β22	144	.002	.021	.000***	.000
	RE*M	β01	72	151.558	156345.485***	4.691E10***	4.719E10***
		β11	72	.015***	9.522***	13.766***	13.781***
		β21	72	.005	4.081***	2.357***	2.457***
		β02	72	.167	2.498***	158513.192***	158738.340***
		β12	72	.011	.044***	.001***	.001***
		β22	72	.009	.029***	.001***	.001***
	CR*M	β01	72	.001	7.294E-6	5.759E-5	6.569E-5
		β11	72	.009***	.001	.004	.004
		β21	72	1.024***	1.158***	.264***	.240***
		β02	72	.063	1.170***	73197.801***	73298.063***
		β12	72	.001	.052***	.001***	.001***
		β22	72	.002	.123***	.002***	.002***
	RE*CR*M	β01	864	.001	.000	.001	.001
		β11	864	.018	.016	.058	.057
		β21	864	.610	.587	.813	.734
		β02	864	.899	13.854***	877548.891***	878767.713***
		β12	864	.011	.029	.001	.001
		β22	864	.005	.038	.001***	.001
	ERROR	β01	1183	3458.358	1982.236	5.755E10	5.739E10
		β11	1183	.034	8.157	27.089	27.121
		β21	1183	1.548	1.645	1.959	2.328
		β02	1183	28.879	4.729	142202.156	142616.134
		β12	1183	.422	.148	.005	.005
		β22	1183	13.178	.257	.000	.009
	TOTAL	β01	2365	3670.242	220115.863	1.163E11	1.165E11
		β11	2365	.137	42.568	76.883	76.972
		β21	2365	5.378	18.685	10.833	11.814
		β02	2365	30.774	61.678	1452558.902	1454739.728
		β12	2365	.510	1.199	.031	.030
		β22	2365	13.260	1.905	.012	.039

Effect on β_0

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation under absolute bias, variance, and mean square error criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators are good to estimate β_0 at all the levels of autocorrelation except for GLS2 which differed significantly at 0.99 autocorrelation levels.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria. The results of the LSD further test visa-vice their estimated marginal means show that all estimators are good for the computation of β_0 at all levels of autocorrelation and correlation between the error terms except for GLS2, which differed significantly at autocorrelation level of 0.99 and correlation between the error terms of -0.99 and +0.99 under all the criteria considered.

Effect on β_1

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to get β_1 at all the levels of autocorrelation and correlation between the error terms.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under all the criteria. The results of the LSD further test visa-vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred for the computation of β_1 at all levels of autocorrelation and correlation between the error terms.

Effect on β_2

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to compute β_2 at all the levels of autocorrelation and correlation between the error terms. However, they too are significantly different at some limited levels of autocorrelation.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under variance criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators except OLS, GLS2, and MLE estimators are preferred to estimate β_2 at all levels of autocorrelation and correlation between the error terms.

Summarily, GLS2, SUR and MLE estimators are preferred to estimate the model at the sample size of 100

Notes

Notes

Table 5: ANOVA for the sample size of 250

s.no.	SOV	β_i	df	TYPE III SUM OF SQUARES			
				Bias	Abs. Bias	Var	MSE
250	RE	β01	12	1.478***	4632.931***	1.059E8***	1.059E8***
		β11	12	.030***	6.158***	2.703***	2.730***
		β21	12	.001	2.812***	.540***	.658***
		β02	12	.003	6.709***	83.297***	83.466***
		β12	12	.319	.035***	17.412	.001**
		β22	12	.002	.021***	8.986E-5***	.000***
	CR	β01	12	.008	6.104E-5	8.877E-5	8.761E-5
		β11	12	.001	5.698E-5	3.105E-5	3.798E-5
		β21	12	.205***	.627***	.062***	.058***
		β02	12	.338***	.295***	30.040	29.626
		β12	12	.778**	.356***	17.229	.006***
		β22	12	.036	.303***	.000***	.001***
	M	β01	6	.190***	873.346***	5.133E7***	5.133E7***
		β11	6	7.732E-5	1.168***	.229***	.230***
		β21	6	.001	.583***	.102***	.105***
		β02	6	.001	.021	28.674	28.775
		β12	6	.182	.007	8.763	.000
		β22	6	.001	.073***	.001***	.001***
	RE*CR	β01	144	.005	.000	.001	.001
		β11	144	.012	9.808E-5	.000	.000
		β21	144	.176	.311**	.155***	.148***
		β02	144	.027	2.815***	352.222	353.150
		β12	144	3.609	.015	209.071	.000
		β22	144	.005	.012	6.516E-5***	.000
	RE*M	β01	72	2.434	10508.921***	6.160E8***	6.161E8***
		β11	72	.011***	2.858***	1.109***	1.120***
		β21	72	.001***	1.528***	.355***	.376***
		β02	72	.003	.195	344.710***	345.580***
		β12	72	1.811	.071	104.517	.001
		β22	72	.002	.042***	.000***	.000***
	CR*M	β01	72	.006	4.581E-5	6.662E-5	6.574E-5
		β11	72	.001	3.947E-5	2.313E-5	2.817E-5
		β21	72	.152**	.469***	.046	.044
		β02	72	.005	.090	162.122	162.533
		β12	72	1.763	.013	104.616	.000
		β22	72	.006	.075***	.001***	.001***
	RE*CR*M	β01	864	.004	.000	.001	.001
		β11	864	.009	8.591E-5	.000	.000
		β21	864	.131	.233	.116	.111
		β02	864	.055	1.121	1945.383	1950.633
		β12	864	21.396	.030	1254.431	.001
		β22	864	.004	.021	.000***	.000
	ERROR	β01	1183	3.854	16061.289	7.821E8	7.822E8
		β11	1183	.082	3.122	2.240	2.268
		β21	1183	1.781	2.054	.749	.792
		β02	1183	3.320	7.305	2945.654	2953.427
		β12	1183	37.560	1.475	1717.611	.034
		β22	1183	3.873	.314	6.234E-5	.003
	TOTAL	β01	2365	7.981	32077.122	1.555E9	1.556E9
		β11	2365	.146	13.305	6.281	6.348
		β21	2365	2.449	8.618	2.125	2.292
		β02	2365	3.756	18.552	5892.407	5907.490
		β12	2365	67.419	2.003	3433.660	.042
		β22	2365	3.929	.863	.003	.007

Effect on β_0

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators are preferred to estimate β_0 at all the levels of autocorrelation except for GLS2, which differed significantly at 0.99 autocorrelation levels.

In equation 2, the estimators are affected by autocorrelation under variance and mean square error criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators are preferred to compute β_0 at all levels of autocorrelation except for GLS2, which differed significantly at autocorrelation level of 0.99 in both criteria considered.

Summarily, we can infer that all the estimators are preferred to estimate β_0 except GLS2 at all the five sample sizes under consideration.

Effect on β_1

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation under all criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to estimate β_1 at all the levels of autocorrelation.

In equation 2, the estimators are neither affected by autocorrelation nor correlation between the error terms under all criteria.

Summarily, we can infer that GLS2 and MLE estimators are preferred to estimate β_1 at all five sample sizes under consideration and at all levels of autocorrelation and correlation between the error terms.

Effect on β_2

Consequently, in equation 1, it can be inferred that the performances of the estimators are affected by autocorrelation and correlation between the error terms under all the criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that GLS2 and MLE estimators are preferred to get values for β_2 at all the levels, except at -0.99 and +0.99 levels for correlation between the error terms under absolute bias.

In equation 2, the estimators are affected by autocorrelation and correlation between the error terms under variance criterion. The results of the LSD further test visa- vice their estimated marginal means revealed that all estimators except OLS, GLS2, and MLE estimators are preferred to estimate β_2 at all the levels of autocorrelation and correlation between the error terms. We can now infer that GLS2 and MLE estimators are preferred to estimate β_2.

Summarily, MLE estimator is the most preferred for the model at the sample size of 250

Conclusively, MLE is the most preferred to estimate all the parameters of the model in the presence of correlation between the error terms and autocorrelation at the entire five different sample sizes.

Notes

Notes

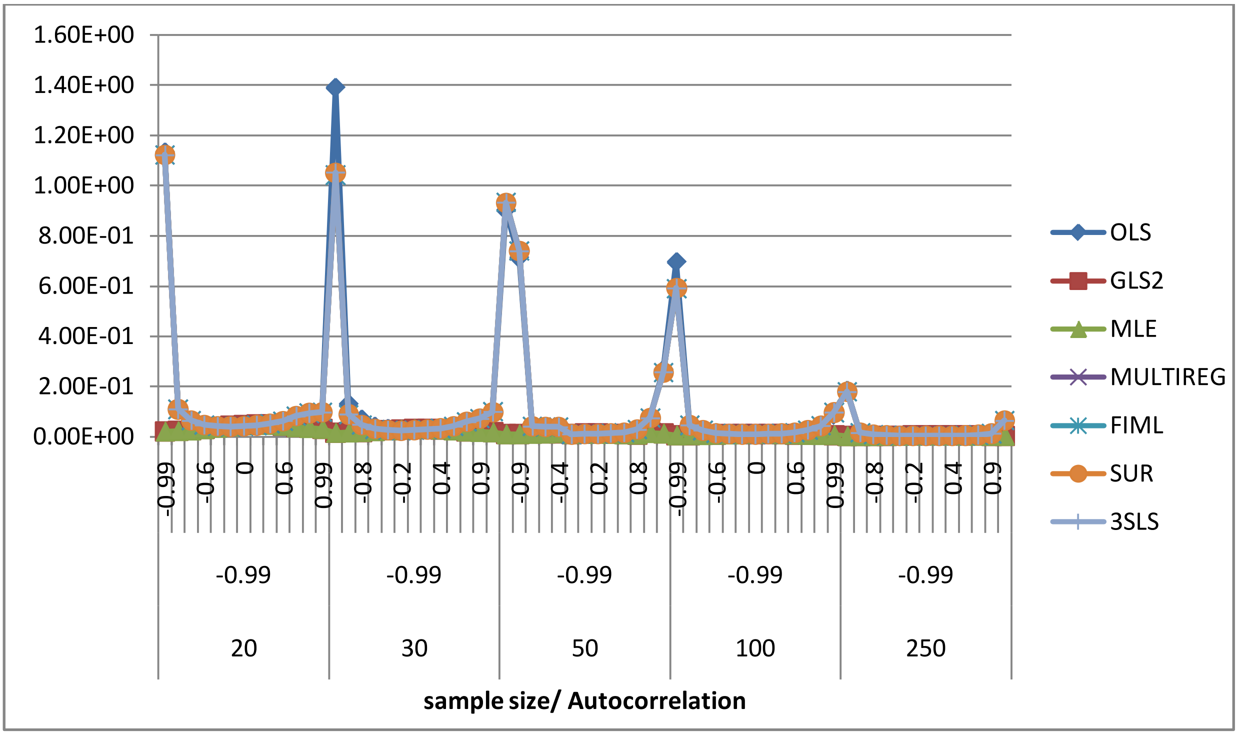

Figure 1: Performaces of the estimators using MSE(B_{11}) at different levels of sample size, correlation bet the error term and autocorrelation at CR = -0.99

In figure 1, the plot of MSE values against different sample sizes for all the estimators revealed an appreciable increase in efficiency (lower MSE) of the estimators as sample size increases with MLE estimator showing a better performance over GLS2.

V. Summary of the Findings

a) When there is a correlation between the error terms and Autocorrelation

The summary of results from the analysis of variance tables of the criteria showing the performances of the estimators and sample sizes on parameters of the two equations model when there is the presence of correlation between the error terms and autocorrelation are given in Table 6 below:

Table 6: Summary of results when there is a correlation between the error terms and in the presence of autocorrelation

s. no.	Eq. no.	Parameters	Preferred	Overall Assessment	Most Preferred
20	1	β_{01}	All except CORC	CORC, MLE	MLE
		β_{11}	CORC, MLE		
		β_{21}	CORC, MLE		
	2	β_{02}	All except CORC	All except CORC	
		β_{12}	All		
		β_{22}	SUR, 3SLS		
30	1	β_{01}	All except CORC	CORC, MLE	MLE
		β_{11}	CORC, MLE		
		β_{21}	CORC, MLE		
	2	β_{02}	All except CORC	MLE.SUR,3SLS	
		β_{12}	CORC, MLE		
		β_{22}	MulReg,FIML,SUR,3SLS		
50	1	β_{01}	All	CORC, MLE	CORC, MLE
		β_{11}	CORC, MLE		
		β_{21}	CORC, MLE		
	2	β_{02}	All	CORC,MLE.SUR,3SLS	
		β_{12}	CORC, MLE		
		β_{22}	MulReg,FIML,SUR,3SLS		
100	1	β_{01}	All except CORC	CORC, MLE	MLE
		β_{11}	CORC, MLE		
		β_{21}	CORC, MLE		
	2	β_{02}	All except CORC	MLE,MulReg,FIML,SUR ,3SLS	
		β_{12}	CORC, MLE		
		β_{22}	MulReg,FIML,SUR,3SLS		
250	1	β_{01}	All except CORC	CORC, MLE	MLE
		β_{11}	CORC, MLE		
		β_{21}	CORC, MLE		
	2	β_{02}	All except CORC	All except CORC	
		β_{12}	All		
		β_{22}	MulReg, FIML,SUR, 3SLS		

From table 6 when there is the presence of correlation between the error terms and autocorrelation in the model under the equation 1 in all the five sample sizes, all the estimating methods except CORC are equally good in estimating the parameter β_{01}, meanwhile, for parameters β_{11} and β_{21} CORC and MLE estimators are preferred, thus, it can be concluded that MLE estimating method can be used in estimating all the model parameters in equation 1.

Under equation 2, it was observed that all estimation methods except CORC can be used in estimating all the parameters of the model at all levels of the sample sizes. However, observing the two equations together, we can conclude that MLE is the most preferred in estimating all the parameters of the two equations among all the estimation methods used.

VI. Recommendation

The research work has revealed that the MLE method of estimation is the most preferred estimator in estimating all the parameters of the model based on the four criteria used, namely, Bias, Absolute Bias, Variance, and Mean Square Error under the five-level of sample sizes considered. It can, therefore, be recommended that when the validity of correlation assumptions considered cannot be authenticated in a system of regression equation, the most preferred estimator to use is MLE.

Notes

References Références Referencias

Notes

1. Aitken, A.C. (1935). On Least Squares and Linear Combinations of observations. Proceedings of the Royal Statistical Society. Edinburgh, 55, 42-48.
2. Ayinde K. and Oyejola B.A. (2007): A comparative Study of the Performances of the OLS and Some GLS Estimators when Stochastic Regressors are correlated with the error terms. *Research Journal of Applied Sciences. 2 (3):215-220.* Published by Medwell Journals, Faisalabad, Pakistan.
3. Chipman, J.S. (1979). Efficiency of Least Squares Estimation of linear Trend when residuals are auto correlated. ECTRA, 47,115-127.
4. David Edgerton & Ghazi Shukur (1999) Testing autocorrelation in a system perspective testing autocorrelation, Econometric Reviews, 18:4, 343-386, DOI: 10.1080/07474939908800351
5. Fomby, B.T.; Carter, R.H. and Johnson, R.S. (1988): Advanced Econometric Methods. Springer-Verlag, New York.
6. GreenW.(2003). Econometric Analysis (5th Edition). New York Macmillan. Pearson Education Inc. USA.
7. Kleiber, C. (2001). Finite Sample Efficiency of OLS in Linear Regression Models with Long – Memory Disturbances. Economics Letters, 72, 131-136.
8. Kramer W. (1980). Finite Sample Efficiency of Ordinary Least Squares in the Linear Regression Model with Auto correlated Errors. JASA, 81,150-154.
9. Nwabueze, J.C. (2000). Estimation of Parameters of Linear Regression Models with Auto correlated Errors terms. Unpublished Ph.D Thesis. University of Ibadan, Nigeria.
10. O'Donnell, C.J.; Shumway, C.R. & Ball, V.E. (1999). Input demands and inefficiency in US agriculture. *American Journal of Agricultural Economics.* 81:865-880.
11. Olanrewaju S.O. (2013). Effects of Multicollinearity, Autocorrelation, and Correlation between the error terms on some Estimators in a System of Regression Equation. Unpublished PhD Thesis submitted to the Department of Statistics, University of Ilorin, Ilorin. Kwara State Nigeria.
12. Olanrewaju S.O.; Yahaya H.Uand Nasiru M.O. (2017) -Effects of Multicollinearity and Autocorrelation on Some Estimators in a System of Regression Equation. *European Journal of Statistics and Probability.* Vol.5(3), Pg 1-15.
13. Powell, A. A., (1965), Aitken Estimators as a tool in Allocating Predetermined Aggregates. *Journal of the American Statistical Association.* 64: 913 – 922.
14. Srivasta, V. K. & Giles D. E. A.(1987). SURE equations model: Estimation and inference, New York: Marcel Dekker.
15. Unger, D.A., H. van den Dool, E. O'Lenic, and D. Collins, (2009): Ensemble Regression. *Mon. Wea. Rev.,* 137, 2365–2379, https://doi.org/10.1175/2008 MWR2605.1
16. Wilde, P.E.; McNamara P.E.& Ranney, C.K. (1999). The effect of income and food programs on dietry quality: A seemingly unrelated regression analysis with error components. *Amer. J. of Agric. Economics* 81(4):959-971.
17. Zellner, A. (1962). An efficient method of estimating seemingly unrelated regression equations and test for aggregation bias. *J. Amer. Statist. Asso.*57:348-368.
18. Zellner, A. (1963). Estimators for seemingly unrelated regression equations: Some exact finite sample results *J. Amer. Statist. Asso.*58:977-992.

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Effects of Multicollinearity and Correlation between the Error Terms on Some Estimators in a System of Regression Equations

By Olanrewaju, Samuel Olayemi
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Abstract- One of the assumptions of a single equation model is that there is one -way causation between the dependent variable Y and the independent variables X. When the assumption is not valid, as, in many econometric models, of lack of correlation between the independent variables and the error terms (U) is further violated, Ordinary Least Square estimator would no longer efficient, that was why this study examined the effects of multicollinearity and a correlation between the error terms on the performance of seven estimators and identified the estimator that yields the most preferred estimates under the separate or joint influence of the two correlation effects under consideration. A two-equation model in which the two correlation problems were introduced was used in this study. The error terms of the two equations were also correlated. The levels of correlation between the error terms and multicollinearity were specified between -1 and +1 at an interval of 0.2 except when the correlation value approached unity. A Monte Carlo experiment of 1000 trials was carried out at five levels of sample sizes 20, 30, 50, 100, and 250 at two runs.

GJSFR-F Classification: MSC 2010: 62M10

EFFECTSOFMULTICOLLINEARITYANDCORRELATIONBETWEENTHEERRORTERMSONSOMEESTIMATORSINASYSTEMOFREGRESSIONEQUATIONS

Strictly as per the compliance and regulations of:

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Effects of Multicollinearity and Correlation between the Error Terms on Some Estimators in a System of Regression Equations

Olanrewaju, Samuel Olayemi

Abstract- One of the assumptions of a single equation model is that there is one -way causation between the dependent variable Y and the independent variables X. When the assumption is not valid, as, in many econometric models, of lack of correlation between the independent variables and the error terms (U) is further violated, Ordinary Least Square estimator would no longer efficient, that was why this study examined the effects of multicollinearity and a correlation between the error terms on the performance of seven estimators and identified the estimator that yields the most preferred estimates under the separate or joint influence of the two correlation effects under consideration. A two-equation model in which the two correlation problems were introduced was used in this study. The error terms of the two equations were also correlated. The levels of correlation between the error terms and multicollinearity were specified between -1 and +1 at an interval of 0.2 except when the correlation value approached unity. A Monte Carlo experiment of 1000 trials was carried out at five levels of sample sizes 20, 30, 50, 100, and 250 at two runs. The seven estimation methods namely; Ordinary Least Squares (OLS), Cochran – Orcutt (CORC), Maximum Likelihood Estimator (MLE), Multivariate Regression (MR), Full Information Maximum Likelihood (FIML), Seemingly Unrelated Regression Model (SUR) and Three-Stage Least Squares (3SLS) and their performances were thoroughly checked by subjecting the results obtained from each finite properties of the estimators into a multi-factor ANOVA model. The significant factors of the results were further examined using their estimated marginal means and the Least Significant Difference (LSD) methodology to determine the best estimator. The results when there is no correlation show that the OLS, CORC, and MLE estimators are generally preferred. Furthermore, the estimators of MR, FIML, SUR, and 3SLS are preferred for computing all the parameters of the model in the presence of multicollinearity and correlation between the error terms at all the sample sizes chosen.

I. General Introduction

Sometimes we may want to estimate more than one equations, which are closely related, The OLS and GLS estimation methods can be used to estimate the equations simultaneously, which has some advantages over estimation done one by one (Philip et al., 1990). As we will see later, estimating the system of equations is closely related to estimating models based on panel data (data from the same people/firms/countries for two or more periods).

One of the assumptions of a single equation model is that, there is one-way causation between the dependent variable Y and the independent variables Xs. When this assumption is not valid, as we have it in many econometric models, that is, the assumption of lack of correlation between the independent variables and the error terms

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(U) is further violated (i.e. E(XU) ≠ 0). Thus, the function no longer belongs to a one-way causation model but rather a wider system of regression equations (multi-equation model), which describe the relationship among all the relevant variables. In a multi-equation model, the dependent variable Y and independent variables now appear as well as explanatory variables in other equation(s) of the model.

Moreover, in a multi-equation model, there are problems of Autocorrelation and Multicollinearity, together with the presence of correlation between the error terms, which may eventually lead to a seemingly unrelated regression model (Lang et al., 2003; Olanrewaju et al., 2017). Consequently, some degree of Autocorrelation and Multicollinearity may have to be allowed in the system of regression equations. Therefore, this study examined and compared the effect of correlation between the error terms (ƛ) and Multicollinearity (δ) on the performances of seven methods of parameter estimation of a multi-equation model using the Monte Carlo approach.

a) *Aim and Objectives of the Study*

Consequently, the study examines the performances of some estimators of a single-equation and that of a system of Regression equation in the presence of correlation between the error terms, multicollinearity, and autocorrelation, study their effects on those estimators, and then, identify the preferred estimator(s) of the model parameters.

Very specifically, the study aims at the following:

(i) Examine the effect of sample size on the performance of the estimators

(ii) Examine the effect of multicollinearity (ƛ) and the correlation between the error terms (δ) jointly on the performance of seven estimators.

(iii) Identify the estimator that yields the most preferred estimates under the joint influence of the two correlation effects under consideration.

II. Literature Review and Theoretical Frame Work

a) *Estimation Methods under Multicollinearity in Single Equation*

Olanrewaju et al. (2017) stated that, one of the major assumptions of the explanatory variables in the classical linear regression model is that they are independent (orthogonal). Orthogonal variables can be set up in experimental designs, but such variables are not often in business and economic data. Thus when the explanatory variables are strongly interrelated, we have the problem of multicollinearity. When multicollinearity is not exact (i.e., the linear relationship between two explanatory variables is not perfect) but strong, the regression analysis is not affected; however, its results become ambiguous. Consequently, interpreting a regression coefficient as measuring the change in the response variable when the corresponding independent variable is increased by one unit, while other predictor variables are held constant is incorrect. This is because the OLS estimator of β given as;

$$\hat{\beta}_{(OLS)} = (X'X)^{-1}X'Y \tag{2.1}$$

and

$$V(\hat{\beta}_{(OLS)}) = \sigma^2(X'X)^{-1} \tag{2.2}$$

are affected by the sample value of the explanatory variables. Precisely, in this case

Notes

Notes

$$|X'X| \to 0$$

When multicollinearity is exact (perfect), the assumption that X has a full column rank break down and therefore $|X'X| = 0$. Consequently, the OLS estimate of equations (2.1) and (2.2) cannot be obtained. The concept of estimable function in which equations (2.1) and (2.2) now have an infinite solution of vectors is used. (Olanrewaju et al., 2017)

As long as multicollinearity is not perfect, the OLS estimates are still unbiased and BLUE (Johnson, 1984). Multicollinearity is associated with unstable estimated regression coefficients from the presence of strong linear relationships among the predictors. It is not a problem of model misspecification but rather that of data: and therefore, empirical study of this problem should only begin after the model has been satisfactorily specified (Charterjee, 2000). However, there may be some indications of the problem resulting from the process of adding, deleting, and transformation of variables or data points in search of a good model. Indications of multicollinearity that appear as a result of instability in regression coefficients are as follows.

i. Large changes in estimated OLS coefficients when a variable is dropped or added.
ii. Large changes in the estimated OLS coefficients when a data point is dropped or altered.

Once the residual plot indicates that the regression model has been satisfactorily specified, multicollinearity may be present if:

i. The algebraic signs of the estimated coefficients do not conform to prior expectations. This may be because greater covariance between the explanatory variable produces greater sampling covariance for the OLS coefficients. Comparing the off–diagonal terms in $X'X$ and $(X'X)^{-1}$ show that a positive covariance for the X's gives a negative covariance for the $\hat{\beta}$'s, and vice versa. In a specific application, if $\hat{\beta}_2$ is below $\hat{\beta}_2, \hat{\beta}_3$ is most likely to exceed $\hat{\beta}_3$ and vice versa (provided that X's are positively correlated).
ii. Coefficients of variables X's that are expected to be important have large standard error (small t – value).

A thorough investigation of the presence of multicollinearity in a system of regression equations can be accomplished by several methods which include:

i. The use of variance inflation factor (VIF): Charterjee (2000).
ii. Principal component analysis approach: Seber (1984), Johnson and Wichern (1992), Charterjee (2000).
iii. The use of two-step procedure: Besley, et al. (1980).
iv. The Farrar – Glauber test: Farrar and Glauber (1967).

The assumption that the regressors X are treated as fixed variables in repeated samples is often violated by economists and other social scientists. The reason for this violation is because their X is often being generated by stochastic processes beyond the scientists' control. For instance, consider regressing daily bathing suit sales by a department store on the mean daily temperature. Surely, the department store cannot control daily temperature, so it would not be meaningful to think of repeated sampling when temperature levels are the same from sample to sample. Fomby et al. (1988) demonstrated that under general conditions, the essential results of the classical linear regression model remain intact even with stochastic regressors. Neter and Wasserman (1974) pointed out that all results on estimations, testing, and prediction obtained using the classical linear regression model still applies if the following conditions hold:

- The conditioned distribution of the dependent variable given the regressors are normal and independent, with means $X\beta$ and conditional variance σ^2
- The regressors are independent random variables, whose probability distribution does not involve the parameter of the classical linear regression model and the conditional variance σ^2.

Notes

However, they pointed out that modification would occur in the area of confidence interval calculated for each sample and the power of the test. This problem was also discussed and supported by Chartterjee et al. (2000).

The assumption that the values of X variables in a regression model are measured without error is hardly ever satisfied. Measurement errors may enter the value observed for the independent variable, 'for instance, when it is temperature, pressure, production line speed, or person's age. Consequently, the independent variable becomes correlated with the error terms (Neter and Wasserman, 1974). These measurement errors affect the residual variance, the multiple correlation coefficients, and the estimated regression coefficients. Its effects increase the residual variance and reduce the magnitude of the observed multiple correlations. The effects of these errors on the estimated regression coefficients are more difficult to assess (Chartterjee et al. 2000). A more extensive discussion of the aforementioned problem can be found in Cochran (1970), Fuller (1987), Chartterjee and Hadi (1988), and Chi – Lu and Van Ness (1999).

Dhrymes and Schwarz (1987) stated that "the heart of the problem is that the conditions on the parameters force the singularity of the covariance matrix-and to a certain degree the converse is true, i.e. the singularity of the covariance matrix implies certain restrictions." It is important to note that, as stated by Bewley (1986), "a necessary and sufficient condition for the OLS estimates to satisfy the adding-up criterion is that some linear combination of the regressors must be identically equal to the sum of regressands if the model is to be logically consistent."

Since the constraints in (2) depend on the values of the regressors, we postulate that the constraints are identically valid in the regressors, which induces restrictions on the parameters that are independent from the regressors. Thus, let Z be a T x P matrix of T-vectors $z_1,\ldots,z_p$, which constitute a base of the vector space containing the $\sum_i k_i$ regressors of all n equations. The obvious consequence is the existence of n matrices c_i of order P x k_i with $X_i = z.c_i$, for all i.

III. Research Methodology

a) *The Monte - Carlo Approach*

In a Monte-Carlo experiment, the experimenter artificially sets up a system (model) and specifies the distribution of the independent variables alongside with the values of the model parameters. More so, the values are then generated for the error terms and the independent variables as specified for a specified sample size. We then, make use of the generated and the parameter values to formulate data for the dependent variable. Next is to treat the generated data as if they are real-life data by estimating the parameters of the model via the estimation methods (estimators). This process of generating values for the disturbance term, independent variables, and estimating the parameters of the model which is then, replicated a large number of times. The experimenter then builds up empirical distributions of the parameter estimates, which are then used to evaluate the performance of those estimators in relation to the parameter values. (Olanrewaju et al. 2017)

The Monte – Carlo studies as stated in Olanrewaju et al. 2017, can be designed generally by using the following summarized five steps as given below:

Notes

(a) The researcher specifies a model and assigns specific numeric values as in parameters. The assigned values are assumed to be the real values of the parameters.
(b) There is need to specify the distribution of error terms.
(c) He also uses the distribution of the error terms (U's) with the random drawings to get new different values for it.
(d) The experimenter now selects or generates values for the regressors (X's) depending on the specifications of the model.
(e) The researcher obtains or generates values for the dependent variable using the computed values of the regressors and the error terms.

The five steps mentioned above are repeated several times, say R, to have R replications.

Thus, the experimenter obtains an estimate of the model parameters for each replication, treating the generated data as real-life data.

b) The Model Formulation

The system of regression equation used in this research work as stated in Olanrewaju, 2013, is given as

$$y_{1t} = \beta_{01} + \beta_{11}x_{1t} + \beta_{12}x_{2t} + u_{1t} \tag{3.1}$$

where- $u_{1t} = \rho u_{1(t-1)} + e_{1t}$, $e_{1t} \approx (0, \sigma^2)$.

$$y_{2t} = \beta_{02} + \beta_{21}x_{1t} + \beta_{22}x_{3t} + u_{2t} \quad , \quad u_{2t} \approx N(0, \sigma^2) \tag{3.2}$$

Note: (1) Multicollinearity exists between X_1 and X_2 in equation (3.1)
(2) Autocorrelation exists in equation (3.1)
(3) There is a correlation between u_1 and u_2 of the two equations
(4) There is no correlation between x_1 and x_3 in equation (3.2), thus, equation (3.2) appears as a control equation.

The models (3.1) and (3.2) were studied under two sub-divisions as given below:

1. There is no any form of correlation in the model i.e. δ=0, ρ =0 and λ=0
2. There is correlation between the error term and presence of multicollinearity in the model i.e. $\delta \neq 0$, $\rho = 0$ and $\lambda \neq 0$.

c) Specifications and Choice of Parameters for Simulation Study

For the simulation study in this research work, the parameters of the model in equations 3.1 and 3.2 are fixed as β_{01} = 0.4; $ß_{11}$ = 1.8; $ß_{21}$ =2.5; $ß_{02}$ = 2.0; $ß_{12}$ = 4.5; $ß_{22}$ = -1.2. The Multicollinearity (δ) levels are -0.99, -0.9,-0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.99 and that of Correlation between error terms (λ) levels are -0.99, -0.9,-0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8, 0.9, 0.99. The sample sizes (n) 20, 30, 50, 100 and 250 were used in the simulation. At a particular choice of sample size, multicollinearity level and correlation between the error terms, a Monte-Carlo experiment is performed 1000 times at two runs which were averaged at analysis stage.

d) The Data Generation for the Simulation Study

The generation of the data used in this simulation study is in three stages, which are: Generation of the

(i) independent variables

(ii) error terms

(iii) dependent variables

e) Computation of Data for the Independent Variables

The independent variables used in this study are fixed numbers at each trial of the simulation. They were computed using the equation provided by Ayinde (2007) to create normally distributed random variables with specified intercorrelations, i.e.

$$X_1 \sim N(\mu_1, \sigma_1^2), \qquad X_2 \sim N(\mu_2, \sigma_2^2), \qquad X_3 \sim N(\mu_3, \sigma_3^2)$$

$$Z_1 = \frac{X_1 - \mu_1}{\sigma_1}, \qquad Z_2 = \frac{X_2 - \mu_2}{\sigma_2}, \qquad Z_3 = \frac{X_3 - \mu_3}{\sigma_3}$$

$$Cor(X_1, X_2) = \rho_{12}, \qquad Cor(X_1, X_3) = 0, \qquad Cor(X_2, X_3) = 0$$

For the three normally distributed random variables given above, the newly derived equation is given as:

$$\begin{aligned} X_1 &= \mu_1 + \sigma_1 Z_1 \\ X_2 &= \mu_2 + \rho_{12}\sigma_2 Z_1 + \sqrt{g_{22} Z_2} \\ X_3 &= \mu_3 + \rho_{13}\sigma_3 Z_1 + \frac{g_{23}}{\sqrt{g_{22}}} Z_2 + \sqrt{h_{33} Z_3} \end{aligned} \tag{3.3}$$

Since $\rho_{13} = 0$ and $\rho_{23} = 0$, then X becomes

$$X_3 = \mu_3 + \frac{g_{23}}{\sqrt{g_{22}}} Z_2 + \sqrt{h_{33} Z_3}$$

where-, $g_{22} = \sigma_2^2[1 - \rho_{12}^2]$, $\quad g_{23} = 0$, $\quad g_{33} = \sigma_3^2$,

$$h_{33} = g_{33} - \frac{g_{23}^2}{g_{22}} \quad and\ Z_i \sim N(0,1), \quad for \quad i = 1,2,3.$$

f) Generation of the Error Terms

The two error terms, u1 and u2, assumed to be well behaved with a multivariate normal distribution u ~ NID (0, Σ) as expressed in equations 3.1 and 3.2 were also generated to exhibit correlation ƛ using the technique as provided by Ayinde (2007). Here is the equation in which the error terms values were generated

Suppose, $u_i \sim N(\mu, \sigma_i^2)$ $\quad i = 1,2$. If these variables are correlated, then, u_1 and u_2 can be gotten by equations

$$\begin{aligned} u_1 &= \mu_1 + \sigma_1 z_1 \\ u_2 &= \mu_2 + \rho\sigma_2 z_1 + \sigma_2 z_2 \sqrt{1 - \rho^2} \end{aligned} \tag{3.4}$$

Notes

Where $Z_i \sim N(0,1)$ for $i = 1,2$ and $|\rho| < 1$ is the value of the correlation between the two variables.

g) Generation of Data for the Dependent Variables

Considering the system of regression equation in 3.1 and 3.2, we have

Notes

$$y_{1t} - \beta_{01} - \beta_{11}x_{1t} - \beta_{12}x_{2t} - 0x_{3t} = u_{1t}$$

$$y_{2t} - \beta_{02} - \beta_{21}x_{1t} - 0x_{2t} - \beta_{22}x_{3t} = u_{2t} \qquad (3.5)$$

We can write 3.4 in matrix form as:

$$Ay_t + \gamma x_t = u_t \qquad (3.6)$$

Where

$$A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \gamma = \begin{bmatrix} -\beta_{01} & -\beta_{11} & -\beta_{12} & 0 \\ -\beta_{02} & -\beta_{21} & 0 & -\beta_{22} \end{bmatrix}$$

$$x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \end{bmatrix} \quad \text{and} \quad u_t = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

Now, equation (3.6) becomes

$$y_t + A^{-1}\gamma x_t = A^{-1}u_t$$

$$y_t = -A^{-1}\gamma x_t + A^{-1}u_t$$

Then, equation 3.1 and 3.2 become,

$$y_{1t} = \beta_{01} + \beta_{11}x_{1t} + \beta_{12}x_{2t} + \rho u_{1(t-1)} + e_{1t} \quad \text{since,} \quad u_{1t} = \rho u_{1(t-1)} + e_{1t}$$

$$y_{2t} = \beta_{02} + \beta_{21}x_{1t} + \beta_{22}x_{3t} + u_{2t} \qquad (3.7)$$

Therefore, equation (3.7) above was used to generate dependent variables y_1 and y_2 by substituting the values of model parameters, independent variables, and that of error terms as specified in the previous sections above.

h) The Evaluation, Comparative Analysis and Preference of Estimators

The evaluation and comparative analysis of the seven (7) estimation methods namely, Ordinary Least Squares (OLS), Cochran – Orcutt (CORC), Maximum Likelihood Estimator (MLE), Multivariate Regression (MR), Full Information Maximum Likelihood (FIML), Seemingly Unrelated Regression Model (SUR) and Three-Stage Least Squares (3SLS), were examined using the finite sampling properties of estimators, which include Bias (BB), Absolute Bias (AB), Variance (VAR), and the Mean Square Error (MSE) criteria.

Mathematically, for any estimator $\hat{\beta}_{ij}$ of Model (3.1) & (3.2)

(i) $$\bar{\hat{\beta}}_{ij} = \frac{1}{R}\sum_{l=1}^{R}\hat{\beta}_{ijl}$$

$$(ii)\quad Bias\left(\hat{\beta}_{ij}\right)=\frac{1}{R}\sum_{l=1}^{R}\left(\hat{\beta}_{ijl}-\beta_{ij}\right)=\bar{\hat{\beta}}_{ij}-\beta_{ij}$$

$$(iii)\quad AB\left(\hat{\beta}_{ij}\right)=\frac{1}{R}\sum_{l=1}^{R}\left|\hat{\beta}_{ijl}-\beta_{ij}\right|$$

$$(iv)\quad VAR\left(\hat{\beta}_{ij}\right)=\frac{1}{R}\sum_{l=1}^{R}\left(\hat{\beta}_{ijl}-\bar{\hat{\beta}}_{ij}\right)^{2}$$

$$(v)\quad MSE\left(\hat{\beta}_{ij}\right)=\frac{1}{R}\sum_{l=1}^{R}\left(\hat{\beta}_{ijl}-\beta_{ij}\right)^{2}$$, for i = 0, 1, 2 ; j = 1,2 and l=1,2,…,R.

Using a computer program which was written with TSP software package to estimate all the model parameters and the criteria, the performances of the seven estimation methods considered in this work as mentioned above were examined by subjecting the results obtained from each finite properties of the estimators into a multi-factor analysis of variance model. Consequently, the highest order significant interaction effect, which has a "method" as a factor, is further examined using Duncan Multiple Range Test and the Least Significance Difference (LSD) test. The estimated marginal mean of the factor was investigated at a particular combination of levels of the correlations in which preferred estimators were chosen. An estimator is most preferred at a specific combination of levels of the correlation, if the marginal means is the smallest. Also, all estimators whose estimated marginal means are not significantly different from the most preferred are also preferred.

IV. Analysis of Results and Discussions

a) Results when we do not have any Form of Correlation in the Model

The performances of the estimators under various sample sizes base on finite sampling properties of estimators using the Analysis of Variance technique are presented and discussed in the table below.

Table 1: ANOVA Table showing the effect of estimators when there is no any form of correlation in the model

N	β_i	df	Value of F - Statistic							
			Equation One				Equation Two			
			BB	A B	VAR	MSE	BB	A B	VAR	MSE
20	β0	6, 7	0.0036	0.8051	1.3441	1.3538	7.38E-4	0.0020	2.4196	0.0039
	β1	6, 7	6.42E-5	0.0291	0.0283	0.0294	0.0777	0.3225	55.4948***	0.3018
	β2	6, 7	0.0851	0.1616	0.1830	0.1839	0.3852	0.2830	11.5274***	0.2496
30	β0	6, 7	0.0525	0.6623	0.3889	0.3818	0.0052	1.7846	2.9613	1.6736
	β1	6, 7	0.2897	0.0098	0.0129	0.0127	0.0080	0.0080	18.0066***	0.0033
	β2	6, 7	0.4588	0.0189	0.0164	0.0164	1.15E-4	0.0223	72.7679***	0.0293
50	β0	6, 7	0.2474	9.62E-5	2.86E-4	2.88E-4	0.0080	0.0080	0.4714	0.0289
	β1	6, 7	1.7866	3.33E-4	7.53E-4	7.57E-4	0.1259	0.1257	0.7172	0.1841
	β2	6, 7	0.0724	0.0010	0.0013	0.0013	0.0044	0.0160	0.6077	0.0326
100	β0	6, 7	2.56E-4	0.0297	0.0147	0.0140	3.26E-4	2.45E-5	0.7457	3.38E-4
	β1	6, 7	0.0017	0.0090	0.0048	0.0051	0.0041	0.0390	5.958**	0.0260
	β2	6, 7	0.0099	0.1326	0.1816	0.1780	5.09E-5	0.0257	1.267E3***	0.0237
250	β0	6, 7	0.0025	0.0012	3.06E-5	1.58E-5	2.88E-4	1.62E-4	0.3148	8.72E-4
	β1	6, 7	0.1215	0.0035	0.0078	0.0079	0.0023	0.0012	0.2603	1.72E-4
	β2	6, 7	2.15E-4	0.0058	0.0051	0.0053	0.0024	0.0067	61.5188***	0.0169

Notes

Notes

It was observed from Table1 above, that, all the estimators do not perform differently (P-value > 0.05) under all the criteria except under the variance criterion in some parameters of equation two. Thus, we concluded that all the estimators do not exhibit a significant difference in their performances under all the criteria in equation one. The results of the further test to identify those estimators that perform equivalently in equation2 are presented in Table 2. From the table, we observed that the performances of the OLS, CORC, MLE, MR, SUR, FIML, and 3SLS estimators are not significantly different. Meanwhile, the OLS, CORC, and MLE estimators are generally preferred.

Table 2: Results of a further test to identify Means that are not significantly different

n	β_i	Criterion	Equation	Estimated Marginal Means of the Estimators						
				OLS	CORC	MLE	MR	FIML	SUR	3SLS
20	β_1	VAR	Two	1.1020E-7[a]	4.1333E-7[a]	1.4404E-6[a]	0.000247[b]	0.000494[c]	0.000247[b]	0.000495[c]
	β_2	VAR	Two	8.9795E-5[a]	6.3520E-7[a]	0.00001[a]	0.005447[b]	0.010333[c]	0.005447[b]	0.010344[c]
30	β_1	VAR	Two	0.00001[a]	0.00001[a]	6.8997E-7[a]	0.000002[b]	2.7419E-6[bc]	0.000002b	3.2419E-6[c]
	β_2	VAR	Two	8.4995E-7[a]	0.00001[a]	0.00001[a]	0.001206[b]	0.001548[c]	0.001206[b]	0.001548[c]
100	β_1	VAR	Two	4.2036E-8[a]	0.00001[a]	0.00001[a]	9.0873E-7[b]	9.2553E-7[b]	9.0873E-7[b]	9.2553E-7[b]
	β_2	VAR	Two	2.3192E-8[a]	1.1680E-7[a]	3.0251E-7[a]	0.00009[b]	0.000096[c]	0.00009[b]	0.000097[c]
250	β_2	VAR	Two	1.1307E-8[a]	2.4719E-8[a]	3.4559E-8[a]	0.000016[b]	0.000017[b]	0.000016[b]	0.000017[b]

Note: Means that have the same letter on top (superscript) are not different significantly.

b) Results when there is a Correlation between the Error Terms and Multicollinearity in the Model

The performances of the estimators under the influence of multicollinearity and a correlation between the error terms at various levels of sample sizes based on finite sampling properties of estimators using the Analysis of Variance technique are presented and discussed below.

i. *Effect on β_0*

The effect of estimators, multicollinearity and a correlation between the error terms on estimating β_0 based on the sampling properties of the estimators as revealed by the Analysis of Variance technique are shown in Table 3 below:

Table 3: ANOVA Table showing the effect of estimators, multicollinearity, and a correlation between the error terms on parameter β_0 in the model.

n	Factor	df	Value of F – Statistic							
			Equation One				Equation Two			
			BB	A B	VAR	MS	BB	A B	VAR	MS
20	E	6,1183	.522	126.179***	240.379***	211.902***	.112	.474	103.035***	1.116
	δ	12, 1183	8.084E-5	0.0001	0.0001	0.0001	.002	.001	.728	.001
	λ	12, 1183	102.30***	.335	.558	.530	7.270***	1.011	9.985E3***	3.950***
	E* δ	72.1183	.00001	0.0001	0.0001	0.001	7.229E-7	4.4007E-6	.001	6.9295E-6
	E* λ	72,1183	13.175***	.044	.071	0.068	.003	.011	5.035***	.035
	$\delta * \lambda$	144,1183	9.214E-5	0.001	0.0001	0.001	.002	.001	.728	.001
	E* $\delta * \lambda$	864,1183	.0001	0.001	0.0001	0.001	7.223E-7	4.4008E-6	.001	6.9295E-6
30	E	6,1183	2.938***	208.454***	156.891***	138.158***	.615	198.853***	1.837	180.426***
	δ	12, 1183	.003	.425	.018	.017	1.102	.766	62,427***	5.507***
	λ	12, 1183	50.267***	7.641***	6.468***	5.684***	.053	43.910***	1.502E3***	223.773***
	E* δ	72.1183	3.2778E-4	.378	.002	.002	.005	.583	.005	.539
	E* λ	72,1183	6.316***	1.114	.810	.711	.010	6.519***	.071	4.454***
	$\delta * \lambda$	144,1183	.003	.172	0.0001	0.0001	.030	.041	2.404***	.205
	E* $\delta * \lambda$	864,1183	3.385E-4	.170	0.0001	0.001	1.8595E-4	.027	.002	.022
50	E	6,1183	.104	.299	.488	.459	1.542	.944	.390	2.016
	δ	12, 1183	71.543***	72.758***	63.348***	63.232***	1.323	36.402***	40.609***	42.555***
	λ	12, 1183	57.528***	2.567***	2.802***	2.800***	2.234***	86.860***	283.525***	151.046***
	E* δ	72.1183	.018	.008	.021	.019	.061	.120	.015	.066
	E* λ	72,1183	6.529***	.001	.002	.001	.033	.020	.021	.010
	$\delta * \lambda$	144,1183	1.188	.287	.328	.328	.056	.683	2.553***	2.111***
	E* $\delta * \lambda$	864,1183	.077	7.833E-5	1.8373E-4	1.8E-4	.001	.009	.001	.001
100	E	6,1183	.050	1.754	1.10	1.08	.042	.002	.353	.034
	δ	12, 1183	.106	67.521***	50.945***	48.759***	1.028E-6	.019	23.051***	.041
	λ	12, 1183	.517	.339	.374	.35	1.984**	43.924***	5.958E3***	26.839***
	E* δ	72.1183	1.928E-4	.061	.062	.059	1.3929E-4	.002	.090	.002
	E* λ	72,1183	.056	.050	.050	.048	.002	.005	.358	.005
	$\delta * \lambda$	144,1183	.002	.322	.346	.333	.006	.025	2.564***	.020
	E* $\delta * \lambda$	864,1183	.001	.050	.051	.049	.001	.004	.245	.003
250	E	6,1183	2.499**	.656	1.281	.786	.038	.040	0.0001	.106
	δ	12, 1183	.253	6.806***	6.623***	6.545***	.737	.402	58.915***	.251
	λ	12, 1183	27.426***	1.84**	2.341***	1.844**	9.754***	.359	1.873E3***	.433
	E* δ	72.1183	.076	.002	0.0001	0.001	1.9645E-5	7.3543E-5	0.001	.0001
	E* λ	72,1183	3.318***	.016	0.001	0.0001	.001	.001	0.0001	.003
	$\delta * \lambda$	144,1183	.114	1.813***	1.852***	1.773***	.062	.1041.96E-5	17.846***	.074
	E* $\delta * \lambda$	864,1183	.022	0.0001	0.0001	0.0001	1.1272E-5	.00001	.0001	.0001

From Table 3, the following points are observed:

- The effect of multicollinearity is generally significant under all the criteria when the sample sizes are moderate and high in equations one and two, but occasionally significant under variance and mean square error in equation two.
- The effect of correlation between the error terms is generally significant under all criteria in equations one and two.
- The effect of estimators is generally significant under all the criteria in both equations when the sample sizes are small (i.e., when n = 20 and 30).

Notes

Notes

- The interaction effect of estimators and multicollinearity is not significant under all the criteria in both equations.
- The interaction effect of estimators and the correlation between the error terms is occasionally significant under all the criteria in both equations.
- The interaction effect of estimators, correlation between the error terms and Multicollinearity is not significant under all the criteria in equations one and two.

More so, we can summarize that the performances of the estimators are affected by Multicollinearity and the correlation between the error terms under all criteria. The results of the LSD further test visa- vice their estimated marginal means, as shown in Tables 4 revealed that OLS and MLE estimators are preferred for the estimation of β_0.

Table 4: Results of a further test on β_0 to identify Means that are not significantly different

n	Criterion	Equation	Means of the Estimators						
			OLS	CORC	MLE	MR	FIML	SUR	3SLS
20	AB	1	.4165^{a}	12.3996^{b}	.3985^{a}	.4165^{a}	.4165^{a}	.4162^{a}	.4162^{a}
	VAR	1	0.0514^{a}	0.0605^{b}	0.0525^{a}	0.0508^{a}	0.0508^{a}	0.0509^{a}	0.0509^{a}
	MS	1	0.0261^{a}	0.0362^{b}	0.0258^{a}	0.0262^{a}	0.0262^{a}	0.0262^{a}	0.0262^{a}
30	AB	1	.1537^{b}	.1540^{b}	.1578^{c}	.1526^{a}	.1526^{a}	.1526^{a}	.1528^{a}
	VAR	1	.0369^{b}	.0388^{c}	.0368^{b}	.0363^{a}	.0363^{a}	.0364^{a}	.0364^{a}
	MS	1	.03695^{b}	.03886^{c}	.03704^{b}	.0364^{a}	.0364^{a}	.0364^{a}	.0364^{a}

ii. *Effect on β_1*

The effects of estimators, multicollinearity and the correlation between the error terms on estimating β_1 based on the sampling properties of the estimators as revealed by Analysis of Variance technique are shown in Table 5 below:

Table 5: ANOVA Table showing the effect of estimators, multicollinearity and the correlation between the error terms on β_1 in the model

n	Factor	df	Value of F- Statistic							
			Equation One				Equation Two			
			BB	A B	VAR	MS	BB	A B	VAR	MS
20	E	6,1183	2.0628E-4	8.463***	60.534***	6.530***	5.773***	30.306***	19.129***	44.083***
	δ	12, 1183	.161	754.436***	748.809***	424.704***	.002	.010	.056	.028
	λ	12, 1183	3.4276E-4	10.665***	16.158***	5.522***	3.366***	33.571***	589.634***	135.683***
	E* δ	72.1183	.002	1.232	20.038***	2.124***	1.183E-4	1.1157E-4	0.0001	1.0709E-4
	E* λ	72,1183	4.3797E-5	1.352**	2.091***	.702	.282	1.330**	.810	.377
	$\delta * \lambda$	144,1183	1.269**	2.037***	5.463***	1.867***	.002	.010	.056	.028
	E* $\delta * \lambda$	864,1183	.169	.258	.707	.237	9.688E-5	1.116E-4	.0001	1.0712E-4
30	E	6,1183	.003	158.541***	147.575***	93.709***	.757	.546	1.706	.514
	δ	12, 1183	.010	2.808E3***	1.92E3***	1.641E3***	.095	1.403	21.173***	1.631
	λ	12, 1183	.360	65.244***	42.752***	31.059***	29.425***	2.290***	754.727***	6.762***
	E* δ	72.1183	.003	26.096***	48.948***	31.051***	.003	.001	.007	4.349E-4
	E* λ	72,1183	.045	8.167***	5.353***	3.885***	.020	.025	.110	.01
	$\delta * \lambda$	144,1183	1.810***	12.310***	14.444***	10.497***	.004	.067	.811	.061
	E* $\delta * \lambda$	864,1183	.228	1.541***	1.809***	1.313***	8.3324E-5	1.3466E-4	.001	1.2112E-4
	E	6,1183	.011	11.380***	53.193***	10.178***	1.055	11.985***	2.700**	15.502***

50	δ λ E* δ E* λ $\delta * \lambda$ E* $\delta * \lambda$	12, 1183 12, 1183 72.1183 72,1183 144,1183 864,1183	.061 5.712*** .016 .718 5.041*** .631	2.28E3*** 37.493*** 2.398*** 3.747*** 7.846*** .775	757.932** * 18.736*** 18.731*** 1.958*** 6.925*** .704	1.059E3*** 17.315*** 4.323*** 1.602*** 6.403*** .582	24.371*** 6.055*** .004 .053 .547 .001	39.266*** 9.745*** .058 .504 .267** .006	29.383*** 482.516*** .082 .137 1.810*** .008	50.551*** 1.525 .036 .509 .825 .007
100	E δ λ E* δ E* λ $\delta * \lambda$ E* $\delta * \lambda$	6,1183 12, 1183 12, 1183 72.1183 72,1183 144,1183 864,1183	.011 1.402 .413 .045 .039 11.172** * 1.385***	290.024*** 5.11E3*** 121.913*** 60.79*** 14.127*** 25.653*** 3.242***	2.31E3*** 2.52E4*** 616.392** * 821.734** * 73.903*** 209.668** * 26.401***	488.385*** 7.752E3*** 153.927*** 172.583*** 18.48*** 52.298*** 6.583***	.974 .053 8.505*** .002 .034 .031 .004	1.935 .131 1.795E3*** .015 .147 .266 .044	.874 .312 2.296E3*** .023 .199 .451 .074	1.205 .282 1.818E3*** .022 .132 .342 .056
250	E δ λ E* δ E* λ $\delta * \lambda$ E* $\delta * \lambda$	6,1183 12, 1183 12, 1183 72.1183 72,1183 144,1183 864,1183	.031 .689 .045 .061 .007 1.606*** .203	294.712*** 5.22E3*** 119.336*** 58.504*** 14.923*** 26.177*** 3.372***	357.23*** 3.59E3*** 90.141*** 120.792** * 11.270*** 30.723*** 3.841***	228.023*** 3.651E3*** 72.591*** 77.204*** 9.076*** 24.824*** 3.103***	.182 .115 1.848** .001 .006 .097 .000	.050 .017 14.942*** 1.6106E-4 .006 .323 1.8084E-4	.127 33.861*** 1.402E3*** .0001 .0001 12.951 .0001	.016 .086 10.687*** 0.0001 .001 .295 2.6288E-4

From Table 5, the following are noticed:

- The effect of multicollinearity is generally significant under all criteria except under bias in equation one and occasionally significant under some criteria in equation two.
- The influence of correlation between the error terms is generally significant under all criteria in equations one and two but not significant under bias criterion in equation two.
- The effect of estimators is generally significant under all the criteria except under bias in equation one and occasionally significant in equation two. A further test as shown in Table 6 revealed that MR, FIML, SUR, and 3SLS are preferred to estimate β_1
- The interaction effect of estimators and multicollinearity is generally significant under all criteria except under bias in equation one, but not significant at all in two.
- The interaction effect of estimators and the correlation between the error terms is generally significant under all criteria except under bias in equation one, but not significant at all in equation two.
- The interaction effect of the correlation between the error terms and multicollinearity is generally significant under all criteria in equation one only.
- The interaction effect of estimators, the correlation between the error terms, and Multicollinearity is significant under all criteria in equation one except when the sample sizes are 20 and 50.

Meanwhile, we can now infer that the performances of the estimators are affected by Multicollinearity under all the criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that MR, FIML, SUR, and 3SLS estimators are preferred to estimate β_1.

Notes

Table 6: The Results of a further test on β_1 to identify Means that are not significantly different

n	Criterion	Equation	Means of the Estimators						
			OLS	CORC	MLE	MR	FIML	SUR	3SLS
20	AB	2	-4.5023E-2[a]	-1.624E-2[b]	-1.554E-2[b]	-4.8626E-2[a]	-4.8625E-2[a]	-4.7201E-2[a]	-4.7201E-2[a]
	VAR	2	.0211[b]	.0243[c]	.023587[c]	.0194[a]	.0194[a]	.01937[a]	.01937[a]
	MS	2	.0289[a]	.0361[b]	.0355[b]	.0272[a]	.0272[a]	.0272[a]	.0272[a]

iii. *Effect on* β_2

The effects of estimators, multicollinearity, and the correlation between the error terms on estimating β_2 based on the sampling properties of the estimators as revealed by Analysis of Variance technique are shown in Table 7.

Table 7: ANOVA Table showing the effect of estimators, multicollinearity, and the correlation between the error terms on β_2 in the model

n	Factor	df	Value of F-Statistic							
			Equation One				Equation Two			
			BB	A B	VAR	MS	BB	A B	VAR	MS
20	E	6,1183	.016	12.631***	72.973***	7.606***	21.763***	826.562***	835.104***	696.964***
	δ	12, 1183	.076	658.856***	740.685***	413.919***	.008	.001	.007	.002
	λ	12, 1183	8.333***	20.211***	19.898***	6.698***	437.321***	10.453***	190.904***	51.064***
	E* δ	72.1183	.001	.873	19.804***	2.064***	.001	.001	.002	.001
	E* λ	72,1183	1.112	2.559***	2.575***	.852	4.52	47.414***	56.989***	40.609***
	$\delta * \lambda$	144,1183	.576	1.398***	5.403***	1.819***	.008	.001	.007	.002
	E* $\delta * \lambda$	864,1183	.077	.177	.699	.231	.001	.001	.002	.001
30	E	6,1183	.001	294.82***	187.7***	118.147***	.020	9.803***	1.161E3***	10.327
	δ	12, 1183	.005	2.497E3***	1.99E3***	1.68E3***	1.461	22.377***	8.295***	12.489***
	λ	12, 1183	12.32***	124.811***	54.313***	39.098***	.350	232.749***	249.605***	123.808***
	E* δ	72.1183	.002	19.011***	50.449***	31.715***	1.731E-4	.220	2.372	.063
	E* λ	72,1183	1.549	15.63***	6.801***	4.891***	.001	2.561	55.147***	1.306
	$\delta * \lambda$	144,1183	.813	9.306	14.939***	10.759***	.037	.637	1.105	.448
	E* $\delta * \lambda$	864,1183	.102	1.165***	1.871***	1.346***	4.2151	.006	.751	.012
50	E	6,1183	.112	16.819***	63.519***	12.201***	2.35**	149.153***	93.822***	79.377***
	δ	12, 1183	.040	2.085E3***	748.809***	1.05E3***	45.294***	65.760***	21.965***	39.80***
	λ	12, 1183	29.57***	70.712***	22.333***	20.874***	19.278***	19.495***	20.868***	3.224***
	E* δ	72.1183	.014	1.967***	18.590***	4.496***	.021	.427	2.668***	1.846***
	E* λ	72,1183	3.698***	7.29***	2.335***	1.932***	.087	9.106***	5.732***	4.744***
	$\delta * \lambda$	144,1183	3.07***	5.862***	6.876***	6.422***	0.997	.473	.906	.465
	E* $\delta * \lambda$	864,1183	.384	.559	.692	.575	.002	.093	.249	.214
100	E	6,1183	.128	479.091***	2.48E3***	586.269***	.042	162.452***	2.263E3***	234.664***
	δ	12, 1183	.576	3.90E3***	2.18E4***	7.508***	2.2067E-6	.003	.072	.006
	λ	12, 1183	70.852***	220.649***	662.388***	185.141***	.404	374.329***	547.88***	127.768***
	E* δ	72.1183	.035	36.496***	711.661***	167.242***	1.4205E-4	.594	7.862***	.740
	E* λ	72,1183	7.768***	25.850***	79.451***	22.232***	.001	16.767***	116.846***	9.684***
	$\delta * \lambda$	144,1183	5.393***	15.603***	181.451***	50.67***	.002	.121	0.0001	.028
		864,1183	.752	2.015***		6.389***	4.4135E-4	.070	.408	.035

	E* $\delta * \lambda$				22.888***					
250	E	6,1183	.026	454.998***	414.92***	267.172***	.158	76.360***	2.259E3***	128.267***
	δ	12, 1183	.825	3.744E3***	3.37E3***	3.455E3***	.427	2.495***	8.747***	1.711
	λ	12, 1183	11.413***	191.494***	103.404***	83,962***	.916	54.825***	435.096***	18.869***
	E* δ	72.1183	.035	32.665***	113.669***	73.194***	.001	.052	1.976***	.074
	E* λ	72,1183	1.427	23.939***	12.927***	10.496***	.010	5.122***	98.201***	4.05***
	$\delta * \lambda$	144,1183	.705	16.084***	29.035***	23.643***	.036	.394	4.352***	.134
	E* $\delta * \lambda$	864,1183	.088	2.010***	3.629***	2.955***	0.003298	.069	.985	.047

Notes

From Table 7, the following points are observed:

- The influence or effect of multicollinearity is generally significant under all criteria, but not under bias in equation one and occasionally significant under some of the criteria in equation two.
- The effect of the correlation between the error terms is generally significant under all criteria in both equations, but occasionally significant under bias criterion in equation two.
- The effect of estimators is generally significant under all the criteria except under bias in equation one and occasionally significant under bias criterion again in equation two.
- The interaction effect of estimators and multicollinearity is generally significant under all criteria except under bias in equation 1but occasionally in equation two.
- The interaction effect of estimators and the correlation between the error terms is generally significant under all criteria except under bias in equation two.
- The interaction effect of estimators, Multicollinearity, and the correlation between the error terms is generally significant under all criteria except under bias in equation two.

In summary, it can be inferred that the performances of the estimators are affected by Multicollinearity under all the criteria. The results of the LSD further test visa- vice their estimated marginal means revealed that MR, FIML, SUR, and 3SLS estimators are preferred to estimate β_2.

Conclusively, the estimator of MR, FIML, SUR, and 3SLS is preferred to estimate all the parameters of the regression model in the presence of multicollinearity and the correlation between the error terms at all the levels of sample sizes.

V. Summary of the Findings and Conclusions

a) *When there is no any form of correlation*

The summary of the results from the Analysis of variance tables of the criteria showing the performances of the estimators and sample sizes on parameters of the two-equation model when there is no form of correlation are presented in Table 8 below:

Notes

Table 8: Summary of results when there is no form of correlation

N	Eqn	Parameters	Preferred	Overall Assessment	Most Preferred
20	1	β_{01}	All	All	OLS
		β_{11}	All		
		β_{21}	All		
	2	β_{02}	OLS, CORC	OLS,CORC,MLE	
		β_{12}	OLS, CORC,MLE		
		β_{22}	OLS, CORC,MLE		
30	1	β_{01}	All	All	OLS
		β_{11}	All		
		β_{21}	All		
	2	β_{02}	All	OLS, CORC,MLE	
		β_{12}	OLS, CORC,MLE		
		β_{22}	OLS, CORC,MLE		
50	1	β_{01}	All	All	OLS
		β_{11}	All		
		β_{21}	All		
	2	β_{02}	All	All	
		β_{12}	All		
		β_{22}	All		
100	1	β_{01}	All	All	OLS
		β_{11}	All		
		β_{21}	All		
	2	β_{02}	OLS, CORC	OLS,CORC,MLE	
		β_{12}	OLS, CORC,MLE		
		β_{22}	OLS, CORC,MLE		
250	1	β_{01}	All	l Al	OLS
		β_{11}	All		
		β_{21}	All		
	2	β_{02}	OLS, CORC	OLS,CORC,MLE	
		β_{12}	OLS, CORC,MLE		
		β_{22}	OLS, CORC,MLE		

From table 8, when there is no correlation in the model under the equation one in all the five sample sizes, all the methods are equally good in estimating all the parameters β_{01}, β_{11} and β_{21}, thus it can be concluded that all the estimation methods are preferred in estimating all the model parameters in equation one.

Under the second equation, it was observed that OLS, CORC AND MLE estimation methods can estimate all the parameters of the model in all the sample sizes except when the sample sizes are 30 and 50. However, observing the two equations together, we can deduce that OLS is most preferred in estimating all the parameters of the two equations among all the estimation methods used due to its simplicity and efficiency over others.

b) When there are Multicollinearity and correlation between the error terms

The summary of results from the Analysis of variance tables of the criteria showing the performances of the estimators and sample sizes on parameters of the two-equation model when there is the presence of correlation between the error terms and multicollinearity are presented in Table 9 below:

Table 9: Summary of results of the model in the presence of multicollinearity and the correlation between the error terms

n	Eqn	Parameters	Preferred	Overall Assessment	Most Preferred
20	1	β_{01}	OLS,CORC,MLE	MR,FIML,SUR,3SLS	SUR, 3SLS
		β_{11}	MR,FIML,SUR,3SLS		
		β_{21}	MR,FIML,SUR,3SLS		
	2	β_{02}	All Except CORC	MR,FIML,SUR,3SLS	
		β_{12}	All Except CORC,MLE		
		β_{22}	MR,FIML,SUR,3SLS		
30	1	β_{01}	OLS,CORC,MLE	All	SUR,3SLS,FIML
		β_{11}	All		
		β_{21}	MR,FIML,SUR,3SLS		
	2	β_{02}	All Except CORC	All Except CORC	
		β_{12}	All		
		β_{22}	All		
50	1	β_{01}	OLS,CORC,MLE	MR,FIML,SUR,3SLS	FIML,SUR,3SLS
		β_{11}	MR,FIML,SUR,3SLS		
		β_{21}	MR,FIML,SUR,3SLS		
	2	β_{02}	All	MR,FIML,SUR,3SLS	
		β_{12}	All		
		β_{22}	MR,FIML,SUR,3SLS		
100	1	β_{01}	All	MR,FIML,SUR,3SLS	FIML,SUR,3SLS
		β_{11}	MR,FIML,SUR,3SLS		
		β_{21}	MR,FIML,SUR,3SLS		
	2	β_{02}	All	All	
		β_{12}	All		
		β_{22}	All		
250	1	β_{01}	All	MR,FIML,SUR,3SLS	FIML,SUR,3SLS
		β_{11}	MR,FIML,SUR,3SLS		
		β_{21}	MR,FIML,SUR,3SLS		
	2	β_{02}	All	All	
		β_{12}	All		
		β_{22}	All		

Table 9 summarized the case when there is the presence of correlation between the error terms and multicollinearity in the model under the equation one in all the five sample sizes; we observed that all the estimating methods are equally good in estimating the parameters β_{01} when the sample sizes are 100 and 250, but when the sample sizes are 20, 30 and 50 OLS, CORC and MLE estimation methods are also okay. Meanwhile, for parameters β_{11} and β_{21}, MulReg, FIML, SUR, and 3SLS estimators are preferred for their estimation; thus, it can be concluded that MulReg, FIML, SUR, and 3SLS estimating method are preferred in estimating all the model parameters in equation one.

Under equation two, it was observed that all estimation methods except CORC are good in estimating all the parameters of the model at all level of the sample sizes.

However, critically looking at the two equations considered in this study together, we can conclude that FIML, SUR, and 3SLS are preferred in computing all the parameters of the two equations among all the estimation methods used.

Notes

Notes

Recommendation

The research work has revealed that FIML, SUR, and 3SLS methods of estimation are the most preferred estimator in estimating all the parameters of the model based on the four criteria used, namely, Bias, Absolute Bias, Variance and Mean Square Error under the five-level of sample sizes considered. It can, therefore, be recommended that when the validity of other correlation assumptions cannot be authenticated in a system of the regression model, the most preferred estimators to use are FIML, SUR, and 3SLS. Meanwhile, for any model without a form of correlation, the OLS, CORC, and MLE estimation methods are most preferred.

References Références Referencias

1. Ayinde K. & Oyejola B.A. (2007): A comparative Study of the Performances of the OLS and Some GLS Estimators when Stochastic Regressors are correlated with the error terms. *Research Journal of Applied Sciences. 2(3):215 -220.*Published by Medwell Journals, Faisalabad, Pakistan.
2. Besley, D.A., Kuh, E., and Welsch, R.E (1980).Regression Diagnostics. Identifying Influential Data and sources of Collinearity. New York. John Wiley and Sons.
3. Bewley, R. (1986). Allocation Models: Specification, Estimation and Applications. Cambridge, Mass: Ballinger.
4. Chatterjee, S. and Hadi, A.S. (1988). Sensitivity Analysis in linear Regression, New York, John Wiley and Sons.
5. Chatterjee, S., Hadi, A.S. and Price, B. (2000). Regression Analysis by Example. Third Edition. A Wiley – Interscience publication. John Wiley and Sons.
6. Chi –Lu, C. and Van Ness, J.W (1999). Statistical Regression with measurement error. London, Arnold.
7. Cochran, W. G. (1970). Some effects of errors of measurement on multiple correlation. JASA, 65, 22-34.
8. Dhrymes, P.J. & Schwarz, S. (1987). On the existence of generalized inverse estimators in a singular system of equations. *Journal of furcating* 6:181-192.
9. Farrar, D. and Glauber, R. (1967). Multicollinearity in regression analysis: The problem revisited. *Review of Economics and Statistics*, 49, 92-107.
10. Fomby, B.T.; Carter, R.H. and Johnson, R.S. (1988): Advanced Econometric Methods. Springer-Verlag, New York.
11. Fuller, W.A. (1987). Measurement Error Models. New York, John Wiley and Sons.
12. Johnson, R.A and Wichern, D.W. (1992). Applied Multivariate Statistical Analysis, 3rd Edition. Englewood Cliffs, N.J.: Prentice –Hall.
13. Johnson, J. (1984). Econometric Methods, Third Edition, New York, McGraw Hill. *Journal of Computational and Applied mathematics.*
14. Lang, S., Adebayo, S. B., Fahrmeir, L. and Steiner, W. J., (2003), Bayesian Geoadditive Seemingly Unrelated Regression. *Computational Statistics, 18(2): 263 – 292.*
15. Neter,J. and Wasserman, W. (1974). Applied Linear Model. Richard D. Irwin, Inc.
16. Olanrewaju S.O. (2013). Effects of Multicollinearity, Autocorrelation, and Correlation between the error terms on some Estimators in a System of Regression Equation. Unpublished PhD Thesis submitted to the Department of Statistics, University of Ilorin, Ilorin. Kwara State Nigeria.

17. Olanrewaju S.O.; Yahaya H.U and Nasiru M.O. (2017) - Effects of Multicollinearity and Autocorrelation on Some Estimators in a System of Regression Equation. *European Journal of Statistics and Probability.* Vol.5(3), pg 1-15.
18. Phillips, P.C.B. and Hansen, B.E.(1990). Statistical inference in instrumental variables regression with I(1) processes. *Review of Economic studies.*57:99-125.
19. Seber, G.A.F. (1984). Multivariate Observations. New York, John Wiley and Sons.
20. TSP (2005). Users Guide and Reference Manual. Time Series Processor. New York.

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Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- o You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- o Give details of all of your remarks as much as possible, focusing on mechanisms.
- o Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- o One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- o Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	C-D	E-F
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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