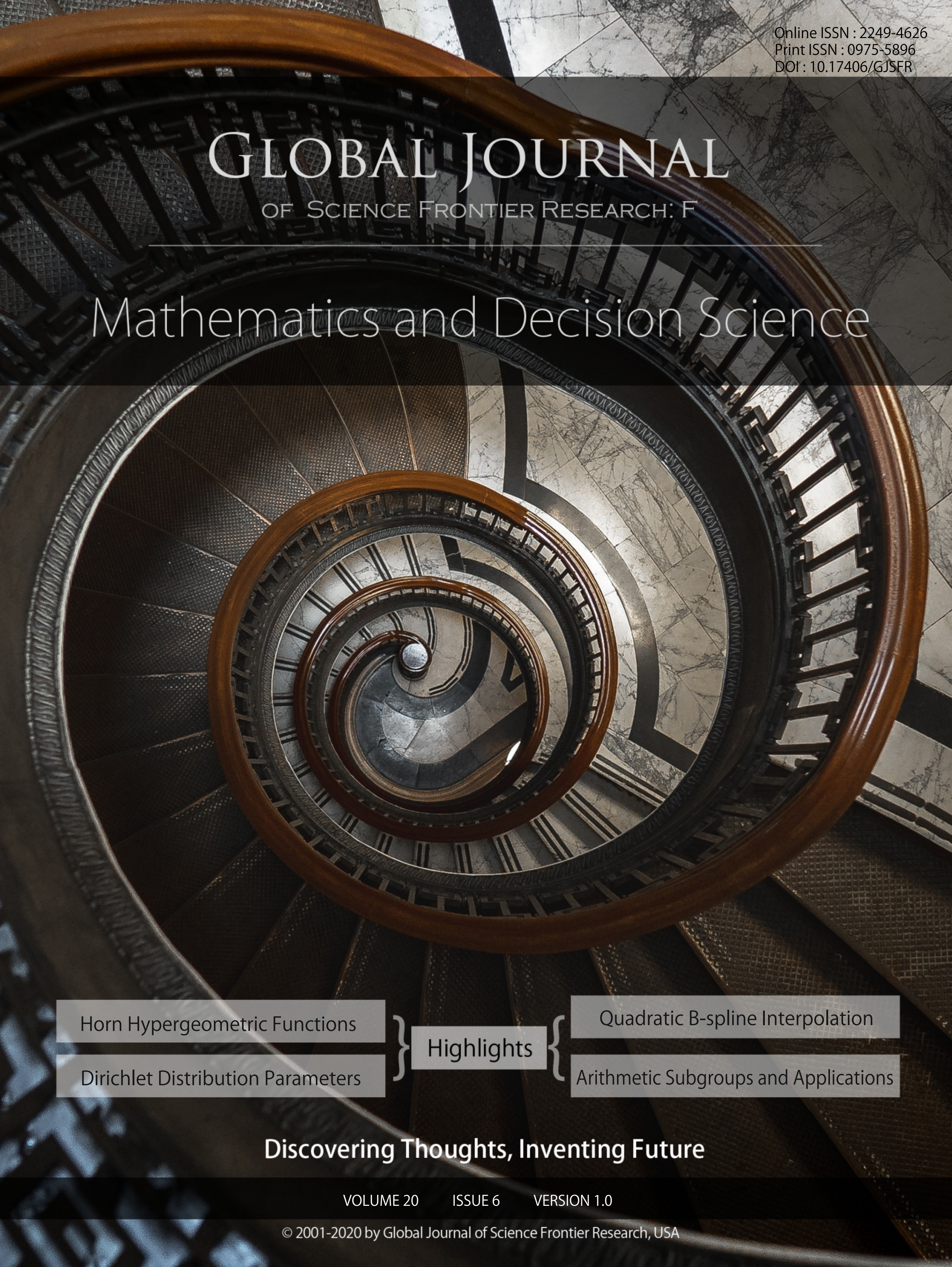

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Horn Hypergeometric Functions

Dirichlet Distribution Parameters

Highlights

Quadratic B-spline Interpolation

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Discovering Thoughts, Inventing Future

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Arithmetic Subgroups and Applications

By Mariam Almahdi Mohammed Mull'a, Amal Mohammed Ahmed Gaweash & Hayat Yousuf Ismail Bakur

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Abstract- Arithmetic subgroups are an important source of discrete groups acting freely on manifolds. We need to know that there exist many torsion-free $\boldsymbol{SL}(\mathbf{2},\ \mathbb{R})$ is an "arithmetic" subgroup of $\boldsymbol{SL}(\mathbf{2},\ \mathbb{R})$. The other arithmetic subgroups are not as obvious, but they can be constructed by using quaternion algebras. Replacing the quaternion algebras with larger division algebras yields many arithmetic subgroups of $\boldsymbol{SL}(\boldsymbol{n},\ \mathbb{R})$, with $\boldsymbol{n}>2$. In fact, a calculation of group cohomology shows that the only other way to construct arithmetic subgroups of $\boldsymbol{S}\mathrm{L}(\boldsymbol{n},\ \mathbb{R})$ is by using arithmetic groups. In this paper justifies Commensurable groups, and some definitions and examples, $\mathbb{R}$-forms of classical simple groups over $\mathbb{C}$, calculating the complexification of each classical group, Applications to manifolds. Let us start with $\boldsymbol{SS}(n,\mathbb{C})$. This is already a complex Lie group, but we can think of it as a real Lie group of twice the dimension. As such, it has a complexification.

Keywords: *lie group, commensurable groups, orthogonal group, symplectic group, subgroups, congruence subgroup, arithmetic subgroups, cohomology, equivalence class, automorphism.*

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2. S. Helgason: Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York, 1978. ISBN 0-12-338460-5, MR 0514561.

Arithmetic Subgroups and Applications

Mariam Almahdi Mohammed Mull'a [α], Amal Mohammed Ahmed Gaweash [σ] & Hayat Yousuf Ismail Bakur [ρ]

Abstract- Arithmetic subgroups are an important source of discrete groups acting freely on manifolds. We need to know that there exist many torsion-free $\boldsymbol{SL}(\mathbf{2},\mathbb{Z})$ is an "arithmetic" subgroup of $\boldsymbol{SL}(\mathbf{2},\mathbb{R})$. The other arithmetic subgroups are not as obvious, but they can be constructed by using quaternion algebras. Replacing the quaternion algebras with larger division algebras yields many arithmetic subgroups of $\boldsymbol{SL}(\boldsymbol{n},\mathbb{R})$, with $\boldsymbol{n} > 2$. In fact, a calculation of group cohomology shows that the only other way to construct arithmetic subgroups of $\boldsymbol{SL}(\boldsymbol{n},\mathbb{R})$ is by using arithmetic groups. In this paper justifies Commensurable groups, and some definitions and examples,$\mathbb{R}$-forms of classical simple groups over $\mathbb{C}$, calculating the complexification of each classical group, Applications to manifolds. Let us start with $\boldsymbol{SL}(n,\mathbb{C})$. This is already a complex Lie group, but we can think of it as a real Lie group of twice the dimension. As such, it has a complexification.

Subject Areas: mathematical analysis, mathematical modern algebra and foundation of mathematics.

Keywords: lie group, commensurable groups, orthogonal group, symplectic group, subgroups, congruence subgroup, arithmetic subgroups, cohomology, equivalence class, automorphism.

I. Introduction

In This paper we will give a quite explicit description of the arithmetic subgroups of almost every classical Lie group G. (Recall that a simple Lie group $G$is "classical" if it is eithera special linear group, an orthogonal group, a unitary group, or a symplectic group The key point is that all the $\mathbb{Q}$-forms of Garealso classical, not exceptional, so they are fairly easy to understand. However, there is an exception to this rule, some 8-dimensional orthogonal groups have $\mathbb{Q}$-forms of so-called triality type, that are not classical and will not bediscussed in any detail here given G, which is a Lie group over $\mathbb{R}$, we would like to know all of its$\mathbb{Q}$-forms (because, by definition, arithmetic groups are made from $\mathbb{Q}$-forms) [1,2,3]. However, we will start with the somewhat simpler problem that replaces the fields $\mathbb{Q}$and $\mathbb{R}$with the fields $\mathbb{R}$and $\mathbb{C}$: finding the $\mathbb{R}$-forms of the classicalLie groups over $\mathbb{C}$.In this paper we construct methods by arithmetic groups. The associated symmetric space $\boldsymbol{SL_2}(\mathbb{R}) = \boldsymbol{SO_2}$ is the hyperbolic plane $\mathbb{H}^{\mathbf{2}}$. There are uncountably many lattices in $\boldsymbol{SL_2}(\mathbb{R})$ (with theassociated locally symmetric spaces being nothing other than Riemann surfaces), but only countably many of them are arithmetic. But in higher rank Lie groups, there is the following truly

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remarkable theorem known as Margulis arithmeticity. Let G be a connected semi simple Lie group with trivial centre and no compact factors, and assume that the real rank of G is at least two. Then every irreducible lattice $\Gamma \subset G$ is arithmetic [4, 5, 6]. These groups play a fundamental role in number theory, and especially in the study of automorphic forms, which can be viewed as complex valued functions on a symmetric domain which are invariant under the action ofan arithmetic group. Appeared that some arithmetic groups are the symmetry groups of several string theories. This is probably why this survey fits into these proceedings[7].

II. Commensurable Groups

Subgroups H_1 and H_2 of a group are said to be commensurable if $H_1 \cap H_2$ is of finiteindex in both H_1 and H_2.The subgroups $a\mathbb{Z}$ and $b\mathbb{Z}$ of $\mathbb{R}$ are commensurable if and only if $\frac{a}{b} \in \mathbb{Q}$. For example, $6\mathbb{Z}$ and $4\mathbb{Z}$are commensurable because they intersect in $12\mathbb{Z}$, but $1\mathbb{Z}$ and $\sqrt{2}\mathbb{Z}$ are not commensurable because they intersect in $\{\mathbf{0}\}$. More generally, lattices $L$and L'in a realvector space $V$are commensurable if and only if they generate the same $\mathbb{Q}$-subspace of V.Commensurability is an equivalence relation, it is reflexive and symmetric, and if H_1, H_2 and H_2, H_3 are commensurable, one shows easily that $H_1 \cap H_2 \cap H_3$ is offinite index in H_1, H_2 and H_3 [8,9].

a) Definition

Let H_1 and H_2 be subgroups of a group $\boldsymbol{G}$. We say that H_1 and H_2 are commensurable if $[H_1{:}\,\boldsymbol{K}], [H_2{:}\,\boldsymbol{K}] < \infty$, where $\boldsymbol{K} = H_1 \cap H_2$, [6,10]

b) Remark

"Being commensurable" is an equivalence relation [11].

c) Examples

$\boldsymbol{G}$ *finite:* any two H_1 and H_2 are commensurable. $\boldsymbol{G} = \boldsymbol{Z}$: H_1 and H_2 are commensurable iff they are isomorphic [13].

d) The geometry topology

Let H_1 and H_2 as fundamental groups.For instance, let $\boldsymbol{G} = \boldsymbol{PSL}(\mathbf{2}, \mathbb{C})$ acting on H_3 and let H_1 and H_2 be lattices which are fundamental groups of hyperbolic manifolds (or, more generally, or bifolds). if $H_1$and $H_2$are commensurable $\boldsymbol{X}/\boldsymbol{H_1}$ and $\frac{\boldsymbol{X}}{\boldsymbol{H_2}}$ have a common finite cover. Since (orbifold) fundamental groups are defined as subgroups of $\boldsymbol{G}$ only up to conjugacy, it is natural to allow subgroups to have a finite index intersection only up to conjugacy[5,4].

e) Definition

Let H_1 and H_2 be subgroups of a group G. We say that $H_1$and H_2 are weakly commensurable if there is a g in $\boldsymbol{G}$ suchthat $[H_1{:}\,\boldsymbol{K}], [H_2{:}\,\boldsymbol{K}] < \infty$, where $\boldsymbol{K} = H_1 \cap \boldsymbol{g} H_2 \boldsymbol{g}^{-1}$[9]

f) Remark

"Weak commensurability" is also an equivalence relation [13].

Ref

9. HELGASON, S. 1978. Differential geometry, Lie groups, and symmetric spaces, volume 80 of Pure and Applied Mathematics. Academic Press Inc., New York.

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g) The Geometry Topology in dimension 2

Let S_g denote the fundamental group of the genus g close orientable surface. of course $S_g \subset S_2$ for all $g \geq 2$. On the other hand one can find discrete surface groups H_1 and H_2 inside $G = PSL(2, R)$ which are not (weakly) commensurable[14].

h) Definition

Let H_1 and H_2 be groups. We say that $H_1$and H_2 are abstractly commensurable if there are subgroups K_i of H_i,$i = 1,2$, such that $[H_1: K_1], [H_2: K_2] < \infty$ and $K_1 \cong K_2$ [5].

III. Definitions

Let G be an algebraic group over $\mathbb{Q}$. Let $\rho: G \to GL_{\boldsymbol{V}}$ be a faithful representation of Gona finite-dimensional vector space V, and let Lbe a lattice in V. Define

$$.G(\mathbb{Q})_L = \{g \in G(\mathbb{Q}) | \rho(g)L = L\}. \quad (1)$$

Anarithmetic subgroup of $\boldsymbol{G}(\mathbb{Q})$ is any subgroup commensurable with $\boldsymbol{G}(\mathbb{Q})_{\boldsymbol{L}}$. For an integer $N > 1$, the principal congruence subgroup of level N is:

$$\Gamma(N)_L = \{g \in G(\mathbb{Q})_L | g \ acts \ as \ 1 \ on \ L/NL\} \quad (2)$$

In other words,.$\Gamma(N)_L$is the kernel of

$$G(\mathbb{Q})_L \to Aut\ (L/NL).$$

In particular, it is normal and of finite index in $G(\mathbb{Q})_{\boldsymbol{L}}$ congruence subgroup of $G(\mathbb{Q})$ is any subgroup containing some $\Gamma(N)$ as a subgroup of finite index, so congruence subgroups are arithmetic subgroups [15].

a) Example

Let $G = GL_n$ with its standard representation on $\mathbb{Q}^{\boldsymbol{n}}$ and its standard lattice $L = Z^n$. Then $G(\mathbb{Q})_L$ consists of the $A = GL_n(\mathbb{Q})$ such that $A\mathbb{Z}^n = \mathbb{Z}^{\boldsymbol{n}}$. On applying A to $e_1, \dots, e_n$, we see that this implies that A has entries in $\mathbb{Z}$. Since $A^{-1}\mathbb{Z}^{\boldsymbol{n}} = \mathbb{Z}^{\boldsymbol{n}}$, the same is true of A^{-1}. Therefore, $G(\mathbb{Q})_L$ is:

$$GL_n(\mathbb{Z}) = \{A \in M_{\boldsymbol{n}}(\mathbb{Z}) | det(A) = \pm 1\} \quad (3)$$

The arithmetic subgroups of $GL_n(\mathbb{Q})$ are those commensurable with $GL_n(\mathbb{Z})$[16]. By definition,

$$\Gamma(N) = \{A \in GL_n(\mathbb{Z}) |\ A \equiv I \ mod\ N\}$$

$$= \{(a_{ij}) \in GL_n(\mathbb{Z}) | N \ divides\ (a_{ij} - \delta_{ij})\}, \quad (4)$$

Which is the kernel of

$$GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/N\mathbb{Z}).$$

b) Example

The group

$$Sp_{2n}(\mathbb{Z}) = \left\{A \in GL_{2n}(\mathbb{Z}) |\ A^t \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right\} \quad (5)$$

Ref

15. N. Bourbaki: Lie Groups and Lie Algebras, Chapters 7. Springer, Berlin, 2005. ISBN 3-540-43405-4, MR 2109105.

is an arithmetic subgroup of $Sp_{2n}(\mathbb{Z})$, and all arithmetic subgroups are commensurable with it [3].

IV. R-Forms of Classical Simple Groups Over ℂ

To set the stage, let us recall the classical result that almost all complex simple groups are classical:

a) Theorem

All but finitely many of the simple Lie groups over ℂare isogenous to either $SL(n,\mathbb{C})$, $SO(n,\mathbb{C})$, or $Sp(2n,\boldsymbol{C})$, for some n, [15, 17]

b) Remark

Up to isogeny, there are exactly five simple Lie groups over ℂthat are not classical. They are the "exceptional" simple groups, and are called E_6, E_7, E_8, F_4, and G_2.

We would like to describe the ℝ-forms of each of the classical groups. For example, finding all the ℝ-forms of $SL(n,\mathbb{C})$ would mean making a list of the (simple) Lie groups G, such that the "complexification" of G is $SL(n,\mathbb{C})$. This is not difficult, but we should perhaps begin by explaining more clearly what it means. It has already been mentioned that, intuitively, the complexification of $G$is the complex Lie group that is obtained from $G$by replacing real numbers with complex numbers. For example, the complexification of $SL(n,\mathbb{R})$ is $\boldsymbol{SL}(n,\mathbb{C})$. In general, $G$is (isogenous to) the set of real solutions of a certain set of equations, and we let $G_{\mathbb{C}}$ be the set of complex solutions of the same set of equations [18]

c) Notation of complex, semisimple

Assume $G \subseteq SL(\ell,\mathbb{R})$, for some ℓ. Since $G$is almost Zariski closed, there is a certain subset $\mathcal{Q}$ofℝ$[x_{1,1},\dots,x_{\ell,\ell}]$, such that $G^{\square} = Var(\mathcal{Q})^{\square}$. Let:

$$G_{\mathbb{C}} = Var_{\mathbb{C}}(\mathcal{Q}) = \{\, g \in SL(\ell,\mathbb{C}) \mid Q(g) = 0, \ \text{for all } Q \in \mathcal{Q}\}. \tag{6}$$

Then $G_{\mathbb{C}}$ is a (complex, semisimple) Lie group.

d) Example

- $SL(n,\mathbb{R})_{\mathbb{C}} = SL(n,\mathbb{C})$.
- $SO(n)_{\mathbb{C}} = SO(n,\mathbb{C})$.
- $SO(m,n)_{\mathbb{C}} \cong SO(m+n,\mathbb{C})$.

e) Definition

If $G_{\mathbb{C}}$ is isomorphic to H, then we say that

- $H$is the complexification of G, and that
- $G$is an ℝ-form of H

The following result lists the complexification of each classical group. It is not difficult to memorize the correspondence. For example, it is obvious from the notation that the complexification of $Sp(m,n)$ should be symplectic.Indeed, the only case that really requires memorization is the complexificationof $SU(m,n)$ [19,2].

Ref

18. J. E. Humphreys: Introduction to Lie Algebras and Representation Theory. Springer, Berlin Heidelberg New York, 1972. MR 0323842.

f) Proposition

Here is the complexification of each classical Lie group.

Real forms of special linear group:

1 $SL(n,\mathbb{R})_{\mathbb{C}} = SL(n,\mathbb{C})$,

2 $SL(n,\mathbb{C})_{\mathbb{C}} \cong SL(n,\mathbb{C}) \times SL(n,\mathbb{C})$,

3 $SL(n,\mathbb{H})_{\mathbb{C}} \cong SL(2n,\mathbb{C})$,

4 $SU(m,n)_{\mathbb{C}} \cong SL(m+n,\mathbb{C})$.

Real forms of orthogonal groups:

1 $SO(m,n)_{\mathbb{C}} \cong SO(m+n,\mathbb{C})$,

2 $SO(n,\mathbb{C})_{\mathbb{C}} \cong SO(n,\mathbb{C}) \times SO(n,\mathbb{C})$,

3 $SO(n,\mathbb{H})_{\mathbb{C}} \cong SO(2n,\mathbb{C})$.

Real forms of symplectic groups:

1 $Sp(n,\mathbb{R})_{\mathbb{C}} = Sp(n,\mathbb{C})$,

2 $Sp(n,\mathbb{C})_{\mathbb{C}} \cong Sp(n,\mathbb{C}) \times Sp(n,\mathbb{C})$,

3 $Sp(m,n)_{\mathbb{C}} \cong Sp(2(m+n),\mathbb{C})$.

V. Calculating the Complexification of Classical G

Here is justifies Proposition 4.6, by calculating the complexification of each classical group. Let us start with $SL(n,\mathbb{C})$. This is already a complex Lie group, but wecan think of it as a real Lie group of twice the dimension. As such, it has a complexification [20,6].

a) Lemma

The tensor product $H \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $Mat_{2\times 2}(\mathbb{C})$.

Proof. Define an $\mathbb{R}$-linear map $\phi\colon \mathbb{H} \to Mat_{2\times 2}(\mathbb{C})$ by

$$\phi(1) = Id,\ \ \phi(i) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},\ \phi(j) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \ \phi(k) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{7}$$

It is straight forward to verify that $\phi$is an injective ring homomorphism. Furthermore, $\phi(\{1,i,j,k\})$ is a $\mathbb{C}-$basis of $Mat_{2\times 2}(\mathbb{C})$. Therefore, the map $\hat{\phi}\colon \mathbb{H} \otimes \mathbb{C} \to Mat_{2\times 2}(\mathbb{C})$ defined by $\hat{\phi}(v \otimes \lambda) = \phi(v)\,\lambda$is a ring isomorphism [1]

b) How to find the real forms of complex groups in $\mathbb{C}$

Now, we will explain how to find all of the possible $\mathbb{R}-$forms of $SL(n,\mathbb{C})$. We take an algebraic approach, based on Galois theory, and we first review the most basic terminology from the theory of (nonabelian) group cohomology [1].

c) Definitions

Suppose a group Xacts (on the left) by automorphismson a group M. (For $x \in X$ and $m \in M$, we write ${}^{x}m$ for the image of munder x.) A function $\alpha\colon X \to M$ is a 1-cocycle (or "crossed homomorphism") if

$$\alpha(xy) = \alpha(x) \cdot {}^{x}\alpha(y) \text{for all} x;\ y\ \in\ X.$$

Two 1-cocycles α and β are equivalent (or "cohomologous") if there is some $m \in M$, such that

Ref

1. Z. I. Borevich and I. R. Shafarevich: Number Theory, Academic Press, New York, 1966. MR 0195803390 18. Arithmetic Subgroups of Classical Groups.

$$\alpha(x) = m^{-1} \cdot \beta(x) \cdot {}^{x}m \; for \; all \; x \in X.$$

$\mathcal{H}^1(X, M)$ is the set of equivalence classes of all $1 - cocycles$. It is calledthe first cohomology of X with coefficients in M. A $1 - cocycle$ is a coboundary if it is cohomologoy us to the trivial $1 - cocycle$ defined by $\tau(x) = e$ for all $x \in X$ [6].

d) Galoiscohomology

For convenience, let $G_{\mathbb{C}} = SL(n, \mathbb{C})$. Suppose $\rho: G_{\mathbb{C}} \to SL(N, \mathbb{C})$ is an embedding, such that $\rho(G_{\mathbb{C}})$is defined over $\mathbb{R}$. We wish to find all the possibilities for the group $\rho(G_{\mathbb{C}})_{\mathbb{R}} = \rho(G_{\mathbb{C}}) \cap SL(N, \mathbb{R})$ that can be obtained by considering all the possible choices ofρ.Let σdenote complex conjugation, the nontrivial Galois automorphism of C over R. Since $\mathbb{R} = \{\mathcal{Z} \in \mathbb{C} | \sigma(\mathcal{Z}) = \mathcal{Z}\}$, we have

$$SL(N, \mathbb{R}) = \{g \in SL(N, \mathbb{C}) | \sigma(g) = g\}, \tag{8}$$

where we apply σ to a matrix by applying it to each of the matrix entries.Therefore

$$\rho(G_{\mathbb{C}})_{\mathbb{R}} = \rho(G_{\mathbb{C}}) \cap SL(N, \mathbb{R}) = \{g \in \rho(G_{\mathbb{C}}) | \sigma(g) = g\} \tag{9}$$

Since $\rho(G_{\mathbb{C}})$ is defined over $\mathbb{R}$, we know that it is invariant under σ, so wehave

$$G_{\mathbb{C}} \xrightarrow{\rho} \rho(G_{\mathbb{C}}) \xrightarrow{\sigma} \rho(G_{\mathbb{C}}) \xrightarrow{\rho^{-1}} G_{\mathbb{C}}.$$

Let $\tilde{\sigma} = \rho^{-1}\sigma\rho: G_{\mathbb{C}} \to G_{\mathbb{C}}$be the composition. Then the real form corresponding to ρ is

$$G_{\mathbb{R}} = \rho^{-1}\big(\rho(G_{\mathbb{C}}) \cap SL(N, \mathbb{R})\big) = \{g \in G_{\mathbb{C}} | \tilde{\sigma}(g) = g\} \tag{10}$$

To summarize, the obvious $\mathbb{R}$-form of $G_{\mathbb{C}}$is the set of fixed points of the usual complex conjugation, and any other $\mathbb{R}$-form is the set of fixed points of some other auto morphism of $G_{\mathbb{C}}$. Now let

$$\alpha(\sigma) = \tilde{\sigma}\sigma^{-1}: G_{\mathbb{C}} \to G_{\mathbb{C}}. \tag{11}$$

It is not difficult to see that

- $\alpha(\sigma)$ is an automorphism of $G_{\mathbb{C}}$(as an abstract group), and
- $\alpha(\sigma)$ is holomorphic (since $\rho^{-1}$and $\sigma\rho\sigma^{-1}$are holomorphic - in fact, they can be represented by polynomials in local coordinates). So $\alpha(\sigma) \in \mathrm{Aut}(G_{\mathbb{C}})$. Thus, by defining $\alpha(1)$ to be the trivial automorphism,we obtain a function $\alpha: \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to \mathrm{Aut}(G_{\mathbb{C}})$. Let $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ act on $\mathrm{Aut}(G_{\mathbb{C}})$, by defining

$${}^{\sigma}\varphi = \sigma\varphi\sigma^{-1} \; \text{for} \varphi \in \mathrm{Aut}(G_{\mathbb{C}}). \tag{12}$$

Then $\alpha(\sigma) = \varphi^{-1}\,{}^{\sigma}\varphi$, so $\alpha(\sigma) \cdot {}^{\sigma}\alpha(\sigma) = \alpha(1)$ (since $\sigma^2 = 1$). This means that$\alpha$is $1 - cocycle$ of group cohomology, and therefore defines an element of the cohomology set $\mathcal{H}^1\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, $\mathrm{Aut}(G_{\mathbb{C}})$. In fact: This construction provides a one-to-one correspondence between $\mathcal{H}^1\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, $\mathrm{Aut}(G_{\mathbb{C}})$. and the set of $\mathbb{R}$ -forms of $G_{\mathbb{C}}$ [21,14,19].

Ref

6. J. Tits: Classification of algebraic semi simple groups, in A. Borel and G. D. Mostow, eds.: Algebraic Groups and Discontinuous Subgroups(Boulder,Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966,pp. 33–62. MR 0224710.

VI. Applications to Manifolds

$\mathbb{Z}^{n^2}$ is a discrete subset of $\mathbb{R}^{n^2}$, i.e., every point of $\mathbb{Z}^{n^2}$ has an open neighbourhood(for the real topology) containing no other point of $\mathbb{Z}^{n^2}$, Therefore, $GL_n(\mathbb{Z})$ is discrete in $GL_n(\mathbb{R})$ and it follows that every arithmetic subgroup Γ of a group $G$is discrete in $G(\mathbb{R})$Let Gbe an algebraic group over $\mathbb{Q}$. Then $G(\mathbb{R})$is a Lie group, and for every compactsubgroup $K$of $G(\mathbb{R}), M = G(\mathbb{R})/K$is a smooth manifold [22].

a) Torsion-free arithmetic groups

$SL_2(\mathbb{Z})$ is not torsion-free. For example, the following elements have finite order:

$$\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^2 = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}^3 \tag{13}$$

b) Theorem

Every arithmetic group contains a torsion-free subgroup of finite index.For this, it suffices to prove the following Lemma [12].

c) Lemma

For any prime $p \in 3$, the subgroup $\Gamma(p)/$ of $GL_n(\mathbb{Z})$ is torsion-free.

Proof. If not, it will contain an element of order a prime ℓ and so we will have an equation:

$$(1 + p^m A)^{\ell} = I \tag{14}$$

with $m \geq 1$and Aa matrix in $M_n(\mathbb{Z})$ not divisible by p. Since I and A commute, we can expand this using the binomial theorem, and obtain an equation:

$$\ell p^m A = -\sum_{i=2}^{\ell} \binom{\ell}{i} p^{mi} A^i \tag{15}$$

In the case that $\ell \neq p$, the exact power of $p$dividing the left hand side is p^m, but p^{2m} divides the right hand side, and so we have a contradiction. In the case that $\ell = p$, the exact power of $p$dividing the left hand side is p^{m+1}, but, for $2 \leq i < p, p^{2m+1} | \binom{p}{i} p^{mi}$ because $p \binom{p}{i}$, and $p^{2m+1} | p^{mi}$ because $p \geq 3$. A gain we have a contradiction [12].

d) Application to quadratic forms

Consider a binary quadratic form:

$$q(x, y) = ax^2 + bxy + cy^2, \quad a, b, c \in \mathbb{R} \tag{16}$$

Assume q is positive definite, so that its discriminant $\Delta = b^2 - 4ac < 0$. There are many questions one can ask about such forms. For example, for which integers N is there a solution to $q(x, y) = N$ with $x, y \in \mathbb{Z}$ For this, and other questions,the answer depends only on the equivalence class of q, where two forms are said to be equivalent if each can be obtained from the other by an integer change of variables. More precisely, $q$and $q'$are equivalent if there is a matrix $A \in SL_2(\mathbb{Z})$ taking q into $q'$by thechange of variables,

Ref

12. LEE, D. H. 2002. The structure of complex Lie groups, volume 429 of Chapman & Hall/CRC Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton, FL.

$$\binom{x'}{y'} = A\binom{x}{y}. \tag{17}$$

In other words, the forms:

$$q(x,y) = (x,y).Q.\binom{x}{y}, \quad q^{'}(x,y) = (x,y).Q^{'}.\binom{x}{y} \tag{18}$$

are equivalent if $Q = A^t.Q^{'}.A$ for $A \in SL_2(\mathbb{Z})$. Every positive-definite binary quadratic form can be written uniquely:

$$q(x,y) = a(x-\omega y)(x-\bar{\omega}\, y), a \in \mathbb{R},\ \ \omega \in \mathcal{H}. \tag{19}$$

If we let Q denote the set of such forms, there are commuting actions of $\mathbb{R}$ and $SL_2(\mathbb{Z})$ on it, and

$$\mathcal{Q}/\mathbb{R} \cong \mathcal{H}$$

as $SL_2(\mathbb{Z})$ sets. We say that q is reduced if

$$|\omega| > 1 \text{ and } -\frac{1}{2} \le \Re(\omega) < \frac{1}{2}, \text{or} |\omega| = 1 \text{ and } -\frac{1}{2} \le \Re(\omega) < 0 \tag{20}$$

More explicitly, $q(x,y) = ax^2 + bxy + cy^2$ is reduced if and only if either [9].

$$-a < b \le a < c \text{ or } 0 \le b \le a = c.$$

e) Applications of the classification of arithmetic groups

Consequences of the classification of F-forms. Suppose Γ is an arithmetic subgroup of $SO(m,n)$, and $m+n \ge 5$ is odd. Then there is a finite extension F of Q, with ring of integers $\mathcal{O}$, such that $\Gamma$is commensurable to $SO(A,\mathcal{O})$, for some invertible, symmetric matrix A in $\mathrm{Mat}_{n\times n}(F)$. So $G = SO(m,n)$. Restriction of scalars implies there isa group $\hat{G}$that is defined over an algebraic number field $F$and has a simple factor that is is ogenous to G, such that Γ is commensurable to $\hat{G}_{\mathcal{O}}$. By inspection, we see that a group of the form $SO(m,n)$ never appears at two places. However, we know that $m+n$ is odd, so the only possibility for $\hat{G}_F$ is $SO(A,F)$. Therefore, Γ is commensurable to $SO(A,\mathcal{O})$ [5, 1, 12].

VII. Conclusion

To state the conclusion in our applications, we can expand binomial and obtain an equation. The coefficient group M is sometimesnon-abelian. In case, $H^1(X,M)$ is a set with no obvious algebraic structure. However, if M is an abelian group (as is often assumed ingroup cohomology), $H^1(\mathrm{X},\mathrm{M})$ is an abelian group.The general principle: if $X$is an algebraic object that is defined over $\mathbb{R}$, then $\mathcal{H}^1\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, $\mathrm{Aut}(X_{\mathbb{C}})$ is in one-to-one correspondence with the set of $\mathbb{R}$-isomorphism classes of $\mathbb{R}$-defined objects whose $\mathbb{C}$-points are isomorphic to $X_{\mathbb{C}}$. We will explain how to find all of the possible $\mathbb{R}$-forms of $SL(n,\mathbb{C})$. The techniques can be used algebraic structure, but additional calculations are needed.

Notes

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We would like to thank Prof Shawgy Hussein Abdallaand Dr. Muhsin Hassan Abdallah who were a great help to us. We would also like to thank Mr. Bashir Alfadol Albdawi.

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Notes

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Analysis and Application of Quadratic B-Spline Interpolation for Boundary Value Problems

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Abstract- B-splines interpolations are very popular tools for interpolating the differential equations under boundary conditions which were pioneered by Maria et.al.[16] allowing us to approximate the ordinary differential equations (ODE). The purpose of this manuscript is to analyze and test the applicability of quadratic B-spline in ODEwithdata interpolation, and the solving of boundary value problems.A numerical example has been given and the error in comparison with the exact value has been shown in tabulated form, and also graphical representations are shown. Maple soft and MATLAB 7.0 are used here to calculate the numerical results and to represent the comparative graphs.

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GJSFR-F Classification: *MSC 2010: 11D09*

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Analysis and Application of Quadratic B-Spline Interpolation for Boundary Value Problems

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Abstract- B-splines interpolations are very popular tools for interpolating the differential equations under boundary conditions which were pioneered by Maria et.al.[16] allowing us to approximate the ordinary differential equations (ODE). The purpose of this manuscript is to analyze and test the applicability of quadratic B-spline in ODEwithdata interpolation, and the solving of boundary value problems.A numerical example has been given and the error in comparison with the exact value has been shown in tabulated form, and also graphical representations are shown. Maple soft and MATLAB 7.0 are used here to calculate the numerical results and to represent the comparative graphs.

Keywords: B-splines interpolation, polynomial, approximate solution, curve approximation.

I. Introduction

B-splines play an important role in many areas such as mathematics, engineering and computer science in recent years. Specially B-splines were used for approximation purposes. I. J. Schoenberg [1]initiate the idea of a spline in 1946 and letter on, in the early 1970s, de Boor [2] defined the splines. At present, these are popular in computer graphing due to their velvetiness, tractability, and precision.

It is well-known that polynomial B-splines, particularly the quadratic and cubic B-splines, have gained widespread application and approximate solutions are obtain using different types of quadratic B-spline methods.S.Kutuay, et.al.[5] demonstrate the numerical solutions of the Burgers' equation by the least-squares quadratic B-spline finite element method. B. Saka et.al.[6] obtained a numerical solution of the Regularised Long Wave (RLW) equation using the quadratic B-spline Galerkin finite element method. A.A. Soliman & K.R. Raslan[7] presented Regularised Long Wave (RLW) equation by Collocation method using quadratic B-splines at mid points as element shape functions. Curve a pproximation with quadratic B -splines can broadly divided in two categories.First, curve approximation with data point interpolation.The use of arc length and curvature characteristics of the given curve to extract the interpolation points was presented in a method for knot placement of the piecewise polynomial approximation of curves was given in [8] and global reparametrization for curve approximation was also been published in [9]. Article on rational parametric curve approximation was published in [10] where the data points, in this case, may not be located on the curve and not much of the work on curve approximation is available using this technique.

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Farago and Horvath [12] (1999) obtained numerical solutions of the heat equation using the finite difference method. Bhatti and Bracken [13] (2006) presented approximate solutions to linear and nonlinear ordinary differential equations using Bernstein polynomials. Bhatta and Bhatti [14] (2006) obtained a numerical solution of the KdV equation using modified Bernstein polynomials via Galerkin method. Munguia*M. and Bhatta. D.* [15] al. (2014) discussed the usage of cubic B-spline functions in interpolation.

The rest of the manuscript is designed as: in section 2, materials and methods are furnished. An illustrative example is shown in section 3 conclusions are prescribed in section 4.

II. Materials and Methods

Theorem-1: If $f(x)$ is defined at $a = x_0 < x_1 < x_2 .... < x_N = b$ then f has a unique natural interpolation on the nodes $x_0, x_1, x_2, ..., x_N$ that is a spline interpolation that satisfies the boundary conditions $s''(0) = 0$ and $s''(b) = 0$

Proof: The boundary conditions $c_0 = 0$ and $c_N = 0$ together with the following equations

$$\frac{c}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}) = h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} \tag{2.1}$$

are produced by a linear system described by the vector equation $\overline{A}\overline{X} = \overline{B}$ where $\overline{A}$ is the $(n+1)$ by $(N+1)$ matrix.

$$\overline{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & .... & 0 \\ h_0 & 2(h_0 + h_1) & 0 & 0 & ...... & 0 \\ .... & ..... & ..... & ..... & .... & ..... \\ .... & ...... & ...... & ..... & ..... & ..... \\ 0 & ...... & ..... & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ 0 & ...... & ...... & 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

$$\overline{B} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ .... \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix} \tag{2.3}$$

And
$$\overline{X} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ ...... \\ c_n \end{bmatrix} \tag{2.4}$$

The matrix is $\overline{A}$ strictly diagonally dominant. So it satisfies the hypothesis. Therefore the linear system has a unique solution for $c_0, c_1, ..., c_n$. The solution to the

Ref

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cubic spline problem with the boundary conditions $s''(x_0)=s''(x_n)=0$ can be obtained the theorem.

Theorem-2: If $f(x)$ is defined at $a=x_0<x_1<x_2<........<x_n=b$ and differentiable at a and b then has a unique clamed interpolation on the nodes $x_0, x_1, x_2, ..., x_n$ that is a spline interpolation that satisfies the boundary conditions $s'(a)=f'(a)$ and $s'(b)=f'(b)$

Notes

Proof: It can be seen using the fact that $s'(a)=s'(x_0)=b_0$

Now we have $f'(a)=\frac{a_1-a_0}{h_0}-\frac{h_0}{3}(2c_0+c_1)$

Consequently, $2h_0c_0+h_0c_1=\frac{h_0}{3}(2c_0+c_1)-3f'(a)$

Similarly $f'(b)=b_n=b_{n-1}+h_{n-1}(c_{n-1}+c_n)$

Now we have the equation $b_j=\frac{1}{h_j}(a_{j+1}-a_j)-\frac{h_j}{3}(2c_j+c_{j+1})$

Putting $j=n-1$ in the above equation, implies that

$$f'(b)=\frac{a_n-a_{n-1}}{h_{n-1}}-\frac{h_{n-1}}{3}(2c_{n-1}+c_n)+h_{n-1}(c_{n-1}+c_n)=\frac{a_n-a_{n-1}}{h_{n-1}}+\frac{h_{n-1}}{3}(c_{n-1}+2c_n)$$

And $h_{n-1}c_{n-1}+2h_{n-1}c_n=3f'(b)-\frac{3}{h_{n-1}}(a_n-a_{n-1})$

Also $2h_0c_0+h_0c_1=\frac{3}{h_0}(a_1-a_0)-3f'(a)$

determine the linear system $\overline{A}\overline{X}=\overline{B}$, where

$$\overline{A}=\begin{bmatrix} 2h_0 & h_0 & 0 & 0 & .... & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & ...... & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & .... & 0 \\ .... & ...... & ...... & ..... & ..... & ..... \\ 0 & ...... & ..... & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & ...... & ...... & 0 & h_{n-1} & 2h_n \end{bmatrix} \quad (2.5)$$

$$\overline{B}=\begin{bmatrix} \frac{3}{h_0}(a_1-a_0)-3f'(a) \\ \frac{3}{h_1}(a_2-a_1)-\frac{3}{h_0}(a_1-a_0) \\ .... \\ \frac{3}{h_{n-1}}(a_n-a_{n-1})-\frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\ 3f'(b)-\frac{3}{h_{n-1}}(a_n-a_{n-1}) \end{bmatrix} \quad (2.6)$$

$$\text{and, } \overline{X} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdots\cdots \\ c_n \end{bmatrix} \tag{2.7}$$

Notes

The matrix is A strictly diagonally dominant. So it satisfies the hypothesis. Therefore the linear system has a unique solution for $c_0, c_1, \ldots\ldots, c_n$. The solution to the cubic spline problem with the boundary conditions $s''(x_0) = s''(x_n) = 0$ can be obtained the theorem

Quadratic B-spline:

$$B_i^2(x) = \begin{cases} \dfrac{(x-x_i)^2}{(x_{i+2}-x_i)(x_{i+1}-x_i)} & \text{if} \quad x_i \le x < x_{i+1}, \\ \dfrac{(x-x_i)(x_{i+2}-x)}{(x_{i+2}-x_i)(x_{i+2}-x_{i+1})} + \dfrac{(x_{i+3}-x)(x-x_{i+1})}{(x_{i+3}-x_{i+1})(x_{i+2}-x_{i+1})} & \text{if} \quad x_{i+1} \le x < x_{i+2}, \\ \dfrac{(x_{i+3}-x)^2}{(x_{i+3}-x_{i+1})(x_{i+3}-x_{i+2})} & \text{if} \quad x_{i+2} \le x < x_{i+3}, \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

The last equation is a quadratic spline with knots $x_i, x_{i+1}, x_{i+2}, x_{i+3}$. Note that the quadratic B-spline is zero except on the interval $[x_i, x_{i+3})$. This is true for all B-splines. In fact, $B_i^k(x) = 0$ if $x \notin [x_i, x_{i+k+1})$, otherwise $B_i^k(x) > 0$ if x $\in (x_i, x_{i+k+1})$.

Since we are only referring to B-splines of degree 2, we write B_i instead of B_i^2. Therefore, after including four additional knots, we assume that

$$\Delta: x_{-2} < x_{-1} < x_0 < x_1 < \cdots < x_{N-1} < x_N < x_{N+1} < x_{N+2}$$

is a uniform grid partition.

Using (4) and letting h = $x_{i+1} - x_i$ for any 0 ≤ i ≤ N, we define the uniform quadratic B-spline $B_i(x)$ as

$$B_i(x) = \frac{1}{2h^2} \begin{cases} (x-x_{i-2})^2 & \text{if} \quad x_{i-2} \le x < x_{i-1} \\ -2(x-x_{i-1})^2 + 2h(x-x_{i-1}) + h^2 & \text{if} \quad x_{i-1} \le x < x_i \\ (x_{i+1}-x)^2 & \text{if} \quad x_i \le x < x_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{2.9}$$

Notes

If we choose $h=1$, then in the interval $[-2,1]$ we have the following

$$B_0(x)=\frac{1}{2}\begin{cases}(x+2)^2 & if\ -2\le x<-1\\ -2x^2-2x+1 & if\ -1\le x<0\\ (1-x)^2 & if\ 0\le x<1\\ 0 & otherwise\end{cases} \tag{2.10}$$

and its graph is shown in Figure 1. We know that B_i lies in the interval $[x_{i,}, x_{i+1})$ this interval has nonzero contributions from B_{i-1}, B_i, B_{i+1} and B_{i+2}. We have a better understanding of this from Figure 2. Next, we derive the quadratic B-spline method for approximating solutions to second-order linear equations.

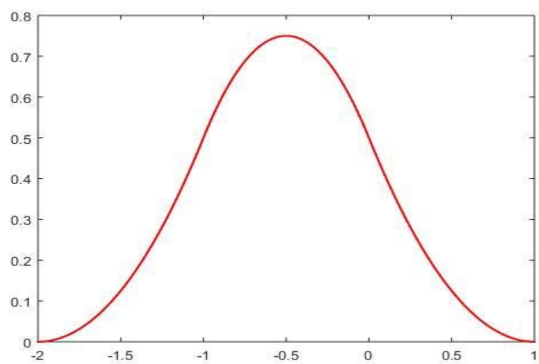

Fig. 1: Quadratic B-spline

Quadratic B-spline Solution Procedure:

To approximate the solution of this BVP using quadratic B-splines, we let $Y(x)$ be a quadratic spline with knots Δ. Then $Y(X)$ can be written as linear combinations of $B_i(x)$

$$Y(x)=\sum_{i=-1}^{N+1} c_i B_i(x) \tag{2.11}$$

Where the constants c_i are to be determined and the $B_i(x)$ are defined in (2.9). It is required that (2.11) satisfies our BVP (3.1-3.2) at x = x_i where x_i is an interior point. That is

$$a_1(x_i)Y''(x_i)+a_2(x_i)Y'(x_i)+a_3(x_i)Y(x_i)=f(x_i) \tag{2.12}$$

and the boundary conditions are

$$Y(x_0)=\alpha \text{ for } x_0=a,$$

$$Y(x_N)=\beta \text{ for } x_N=b.$$

From (2.11), we have

$$Y(x_i)=c_{i-1}B_{i-1}(x_i)+c_iB_i(x_i)+c_{i+1}B_{i+1}(x_i)+c_{i+2}B_{i+2}(x_i),$$

$$Y'(x_i) = c_{i-1}B'_{i-1}(x_i) + c_i B'_i(x_i) + c_{i+1}B'_{i+1}(x_i) + c_{i+2}B'_{i+2}(x_i),$$

$$Y''(x_i) = c_{i-1}B''_{i-1}(x_i) + c_i B''_i(x_i) + c_{i+1}B''_{i+1}(x_i) + c_{i+2}B''_{i+2}(x_i),, \tag{2.13}$$

and these yield

$$\begin{aligned} &c_{i-1}[a_1(x_i)B''_{i-1}(x_i) + a_2(x_i)B'_{i-1}(x_i) + a_3(x_i)B_{i-1}(x_i)] \\ &+ c_i[a_1(x_i)B''_i(x_i) + a_2(x_i)B'_i(x_i) + a_3(x_i)B_i(x_i)] \\ &+ c_{i+1}[a_1(x_i)B''_{i+1}(x_i) + a_2(x_i)B'_{i+1}(x_i) + a_3(x_i)B_{i+1}(x_i)] \\ &+ c_{i+2}[a_1(x_i)B''_{i+2}(x_i) + a_2(x_i)B'_{i+2}(x_i) + a_3(x_i)B_{i+2}(x_i)] = f(x_i), \end{aligned} \tag{2.14}$$

also by the properties of quadratic B-spline functions, we obtain the following

$$\begin{array}{lll} B''_{i-1}(x_i) = 0, & B'_{i-1}(x_i) = 0, & B_{i-1}(x_i) = 0, \\ B''_i(x_i) = \dfrac{1}{h^2}, & B'_i(x_i) = -\dfrac{1}{h}, & B_i(x_i) = \dfrac{1}{2} \\ B''_{i+1}(x_i) = -\dfrac{2}{h^2}, & B'_{i+1}(x_i) = -\dfrac{1}{h}, & B_{i+1}(x_i) = \dfrac{1}{2}, \\ B''_{i+2}(x_i) = 0, & B'_{i+2}(x_i) = 0, & B_{i+2}(x_i) = 0. \end{array} \tag{2.15}$$

If we combine (2.14) and (2.15), we obtain

$$\begin{aligned} &c_i[2a_1(x_i) - 2ha_2(x_i) + a_3(x_i)h^2] \\ &+ c_{i+1}[-4a_1(x_i) - 2ha_2(x_i) + a_3(x_i)h^2] = 2h^2 f(x_i). \end{aligned} \tag{2.16}$$

Now we apply the boundary conditions:

$$\begin{aligned} &Y(x_0) = c_{-1}B_{-1}(x_0) + c_0 B_0(x_0) + c_1 B_1(x_0) + c_2 B_2(x_0) = \alpha, \\ &Y(x_N) = c_{N-1}B_{N-1}(x_N) + c_N B_N(x_N) + c_{N+1}B_{N+1}(x_N) + c_{N+2}B_{N+2} = \beta, \end{aligned} \tag{2.17}$$

where the value of $B_i(x)$ at $x = x_0$ and $x = x_N$ are given below

$$\begin{aligned} &B_{-1}(x_0) = 0 = B_{N-1}(x_N), \\ &B_0(x_0) = \frac{1}{2} = B_N(x_N), \\ &B_1(x_0) = \frac{1}{2} = B_{N+1}(x_N), \\ &B_2(x_0) = 0 = B_{N+2}(x_N). \end{aligned} \tag{2.18}$$

Therefore,

$$c_0 + c_1 = 2\alpha \tag{2.19}$$

$$c_N + c_{N+1} = 2\beta \tag{2.20}$$

Notes

Now that we have found all the constant coefficients in (2.15), (2.19), and (2.20), we can write a system of N + 1 linear equations in N + 1 unknowns. This system is represented in (2.21) where the coefficient matrix is an (N + 1)×(N + 1) matrix.

$$\begin{pmatrix} 0 & 0_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & p_1 & q_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & p_2 & q_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_{N-2} & q_{N-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{N-1} & q_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0_2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{N-2} \\ c_{N-1} \\ c_N \end{pmatrix} = 2 \begin{pmatrix} z_0 \\ h^2 f(x_1) \\ h^2 f(x_2) \\ \vdots \\ h^2 f(x_{N-2}) \\ h^2 f(x_{N-1}) \\ z_N \end{pmatrix} \tag{2.21}$$

Where

$$p_i = 2a_1(x_i) - 2ha_2(x_i) + a_3(x_i)h^2,$$

$$q_i = -4a_1(x_i) - 2ha_2(x_i) + a_3(x_i)h^2,$$

$$0_1 = q_0 - p_0,$$

$$0_2 = p_N - q_N,$$

$$z_0 = h^2 f(x_0) - \alpha p_0,$$

$$z_N = h^2 f(x_N) - \beta q_N.$$

The quadratic B-spline approximation for the BVP (3.1-3.2) is obtained using (2.13), where the constant coefficients c_i satisfy the system defined in (2.21).

III. Numerical Example

Let us consider a linear boundary value problem with constant coefficients

$$y'' + y' - 6y = x \quad for \quad 0 < x < 1 \tag{3.1}$$

With boundary conditions

$$y(0) = 0, \quad y(1) = 1 \tag{3.2}$$

The exact solution to the boundary value problem is

$$y(x) = \frac{(43 - e^2)e^{-3x} - (43 - e^{-3})e^{2x}}{36(e^{-3} - e^2)} - \frac{x}{6} - \frac{1}{36} \tag{3.3}$$

The graph of the exact solution using MATLAB 7.0 is given below

Notes

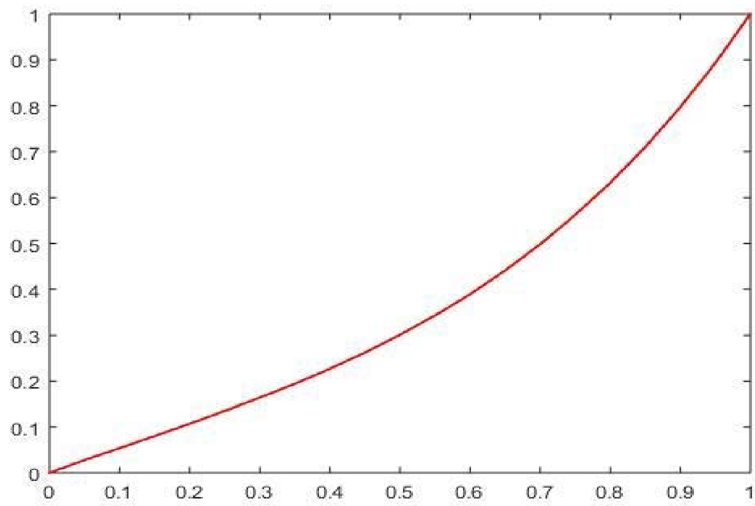

Fig. 2: Exact solution curve

We approximate the solution of (3.1) with the boundary conditions (3.2) using the quadratic B-spline method with N = 20.

In order to use (2.13), we first need to find the constant coefficients c_i for $i = -1, 0, 1, \cdots, 21$ using the system of linear equations (2.21) where the coefficient matrix is an 21×21 matrix and using (2.19) and (2.20) to find c_{-1} and c_{21} respectively. These coefficients are given below

$c_{-1} = 3.37050$	$c_0 = -1.37050$	$c_1 = -0.62963$	$c_2 = -0.28932$	$c_3 = -0.133041$
$c_4 = -0.61303$	$c_5 = -0.28407$	$c_6 = -0.133544$	$c_7 = -0.64997$	$c_8 = -0.34113$
$c_9 = -0.20532$	$c_{10} = -0.00149$	$c_{11} = -0.129210$	$c_{12} = -0.12619$	$c_{13} = -0.13087$
$c_{14} = -0.13910$	$c_{15} = -0.14896$	$c_{16} = -0.159565$	$c_{17} = -0.17051$	$c_{18} = -0.18161$
$c_{19} = -0.19279$	$c_{20} = -0.20400$	$c_{21} = 2.0020400$		

and therefore the quadratic polynomials are as follows

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$$
Y(x)=\begin{cases}
-22.24775x^2+14.81739x-1.00007 & for\ x\in[0.00,0.05)\\
-10.19669x^2+7.82581x-0.82528 & for\ x\in[0.05,0.10)\\
-4.64808x^2+4.05526x-0.57023 & for\ x\in[0.10,0.15)\\
-2.08680x^2+2.06079x-0.35934 & for\ x\in[0.15,0.20)\\
-0.89796x^2+1.01712x-0.21236 & for\ x\in[0.20,0.25)\\
-0.33963x^2+0.47087x-0.11737 & for\ x\in[0.25,0.30)\\
-0.70978x^2+0.17968x-0.57443 & for\ x\in[0.30,0.35)\\
0.64597x^2+0.16550x-0.18661 & for\ x\in[0.35,0.40)\\
0.13903x^2-0.84064x+0.86481 & for\ x\in[0.40,0.45)\\
0.18538x^2-0.15558x+0.30699 & for\ x\in[0.45,0.50)\\
0.21882x^2-0.21486x+0.51335 & for\ x\in[0.50,0.55)\\
0.24634x^2-0.27037x+0.72908 & for\ x\in[0.55,0.60)\\
0.27113x^2-0.32629x+0.90995 & for\ x\in[0.60,0.65)\\
0.29467x^2-0.38472x+0.12422 & for\ x\in[0.65,0.70)\\
0.31764x^2-0.44666x+0.15558 & for\ x\in[0.70,0.75)\\
0.34034x^2-0.51263x+0.19149 & for\ x\in[0.75,0.80)\\
0.36292x^2-0.58286x+0.23237 & for\ x\in[0.80,0.85)\\
0.38544x^2-0.65747x+0.27861 & for\ x\in[0.85,0.90)\\
0.40794x^2-0.73653x+0.33057 & for\ x\in[0.90,0.95)\\
0.43043x^2-0.82006x+0.38861 & for\ x\in[0.95,1.00)
\end{cases}
$$

The figure of quadratic B-spline by these polynomials using MATLAB 7.0 is given below

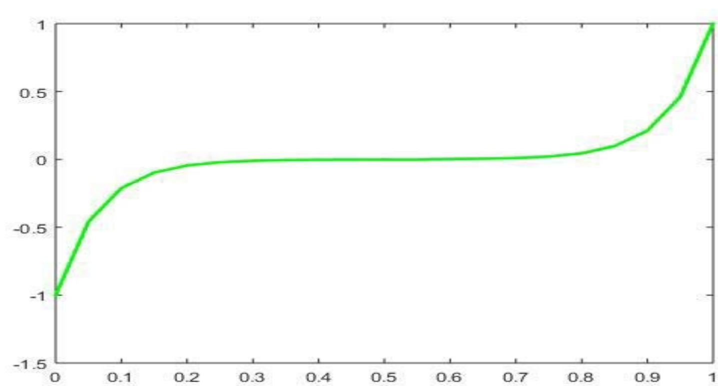

Fig. 3: Quadratic Polynomial

Comparing Exact values with Quadratic B-spline for Example-1

x_i	Quadratic B - spline	Exact	Absolute Error
0·00	-1.00006612	0·0000000000	-1.00006612
0·05	– 0.45947752	0·0275370031	0·487014523
0.10	– 0.211182808	0.0542570003	0·265439808
0·15	– 0.097172806	0·0807133503	0.177886156
0.20	– 0.0448544	0·1074285617	0·152282961
0·25	– 0.020879375	0·1349034523	0·155782827
0·30	– 0.0099242	0·1636255435	0·173549743
0·35	– 0.004955225	0·1940768511	0·199032076
0·40	– 0.0027292	0·2267412146	0·229470414
0·45	– 0.00178155	0·2621112965	0·263892846
0·50	– 0.001385	0·3006953693	0·302080369
0·55	– 0.00127565	0·3430239998	0·344299649
0·60	– 0.0071772	0·3896567348	0·396833934
0·65	– 0.001349925	0·4411888843	0·442538809
0·70	– 0.0014384	0·4982584988	0·499696898
0·75	– 0.001544125	0·5615536314	0·563097756
0·80	– 0.0016492	0·6318199790	0·633469179
0·85	– 0.0017591	0·7098689951	0·711628095
0·90	– 0.0018756	0·7965865702	0·79846217
0·95	– 0.001983925	0·8929423791	0·894926304
1·00	1.0000001	1·0000000000	– 0.0000001

Notes

IV. Conclusion

This paper presented the application of quadratic B-spline function to solve the boundary value problems.At first, we have derived the quadratic B-spline functions and hence derived the methods.Then we used these methods to solve second order linear boundary value problems.After solving these problems, we compare the numerical solution with the exact solution. The comparative graph have shown some error in quadratic B-spline in comparison to the exact solution of the same function.

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New Integrals for Horn Hypergeometric Functions in Two Variables

By Jihad Ahmed Younis
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Abstract- In this paper, we establish several new integral representations of Euler-type involving Horn's functions G1,G2,G3,H1,H2,H5,H6 and H7. Some corollaries have also been obtained as special cases of our main results.

Keywords: *gamma function, beta function, horn double functions, appell functions, eulerian integrals.*

GJSFR-F Classification: *MSC 2010: 05B25*

NEWINTEGRALSFORHORNHYPERGEOMETRICFUNCTIONSINTWOVARIABLES

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New Integrals for Horn Hypergeometric Functions in Two Variables

Jihad Ahmed Younis

Abstract- In this paper, we establish several new integral representations of Euler-type involving Horn's functions $G_1, G_2, G_3, H_1, H_2, H_5, H_6$ and H_7. Some corollaries have also been obtained as special cases of our main results.

Keywords: gamma function, beta function, horn double functions, appell functions, eulerian integrals.

I. Introduction

Integral representations of hypergeometric functions have found applications in divers fields such as mathematics, physics, statistics, and engineering. Hasanov et al. [9] studied some of the properties of the Horn type second-order double hypergeometric function H_2^* involving integral representations, differential equations, and generating functions. Choi et al. [6] introduced certain integral representations for Srivastava's triple hypergeometric functions H_A, H_B and H_C. Younis and Bin-Saad [19, 20] establish several integral representations and operational relations involving quadruple hypergeometric functions $X_i^{(4)}(i = 38, 40, 45, 48, 50)$. Younis and Nisar [21] introduce new integral representations of Euler-type for Exton's hypergeometric functions of four variables D_1, D_2, D_3, D_4 and D_5. Also, in [2-5], authors introduced many integral representations for certain hypergeometric functions in four variables.

Let us recall the Gauss hypergeometric function ${}_2F_1$ is defined as (see, e.g., [14] and [16, Section 1.5])

$$ {}_2F_1\left(a, b; c; x\right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{x^m}{m!}, (|x| < 1), \tag{1.1}$$

where $(a)_m$ is the well known Pochhammer symbol given by (see, e.g., [16, p. 2 and pp. 4-6])

$$ (a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1 & (m = 0), \\ a(a+1)...(a+m-1) & (m \in \mathbb{N} := \{1, 2, ...\}). \end{cases} \tag{1.2}$$

Euler's integral representation of ${}_2F_1$ is defined by (see, e.g., [14, p. 85] and [16, p. 65])

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$$
{}_2F_1(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\alpha^{a-1}(1-\alpha)^{c-a-1}(1-x\alpha)^{-b}\,d\alpha,
$$

$$
(Re(a)>0,Re(c-a)>0).
$$

Appell hypergeometric functions of two variables F_1, F_2 and F_3 are respectively defined by (see [17, p. 53, Eq. (4) - (6)])

$$
F_1(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.3}
$$

$$
F_2(a,b,c;d,e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}\tag{1.4}
$$

and

$$
F_3(a,b,c,d;e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}.\tag{1.5}
$$

Integral representations of Euler type for the functions F_1, F_2, F_3 were already given by Appell [1, Chap. III]. For various integral representations of hypergeometric functions, the interested reader may refer to [8-10, 12, 13, 15, 18].

Other hypergeometric functions of two variables are the following Horn's functions $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_5, H_6$ and H_7 defined by (cf. [7], [8], [11])

$$
G_1(a,b,c;x,y)=\sum_{m,n=0}^{\infty}(a)_{m+n}(b)_{n-m}(c)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.6}
$$

$$
G_2(a,b,c,d;x,y)=\sum_{m,n=0}^{\infty}(a)_m(b)_n(c)_{n-m}(d)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.7}
$$

$$
G_3(a,b;x,y)=\sum_{m,n=0}^{\infty}(a)_{2n-m}(b)_{2m-n}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.8}
$$

$$
H_1(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.9}
$$

$$
H_2(a,b,c,d;e;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_m(c)_n(d)_n}{(e)_m}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.10}
$$

$$
H_3(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.11}
$$

$$
H_4(a,b;c,d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_m(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.12}
$$

$$
H_5(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_{n-m}}{(c)_n}\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.13}
$$

$$
H_6(a,b,c;x,y)=\sum_{m,n=0}^{\infty}(a)_{2m-n}(b)_{n-m}(c)_n\frac{x^m}{m!}\frac{y^n}{n!},\tag{1.14}
$$

$$
H_7(a,b,c;d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m-n}(b)_n(c)_n}{(d)_m}\frac{x^m}{m!}\frac{y^n}{n!}.\tag{1.15}
$$

Ref

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In this paper, we aim to establish the further integral representation of Euler type for Horn double hypergeometric functions $G_1, G_2, G_3, H_1, H_2, H_5, H_6$ and H_7

II. Main Results

Theorem 2.1. The following integral representations hold:

Notes

$$G_1(a,b,c;x,y) = \frac{\Gamma(b+b')}{\Gamma(b)\Gamma(b')} \int_0^\infty \alpha^{b-1}(1+\alpha)^{-(b+b')} H_1\left(c,a,b+b';b';\frac{x}{\alpha},\frac{\alpha y}{(1+\alpha)}\right) d\alpha,$$

$$\left(Re(b)>0, Re(b')>0\right), \quad (1)$$

$$G_1(a,b,c;x,y) = \frac{\Gamma(b+b')\Gamma(c+c')}{2^{b+b'+c+c'-2}\Gamma(b)\Gamma(b')\Gamma(c)\Gamma(c')} \int_{-1}^{1}\int_{-1}^{1} (1+\alpha)^{b'-1}(1-\alpha)^{b-1}(1+\beta)^{c'-1}$$

$$\times(1-\beta)^{c-1} F_2\left(a,c+c',b+b';b',c';\frac{(1+\alpha)(1-\beta)x}{2(1-\alpha)},\frac{(1-\alpha)(1+\beta)y}{2(1-\beta)}\right) d\alpha d\beta,$$

$$\left(Re(b)>0, Re(b')>0, Re(c)>0, Re(c')>0\right), \quad (2)$$

$$G_1(a,b,c;x,y) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)(S-R)^{a+b-1}} \int_R^S (\alpha-R)^{a-1}(S-\alpha)^{b+c-1}\left[(S-\alpha)-(\alpha-R)x\right]^{-c}$$

$$\times {}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};1-c;\frac{-4\left[(\alpha-R)(S-\alpha)-(\alpha-R)^2x\right]y}{(S-R)^2}\right) d\alpha,$$

$$\left(Re(a)>0, Re(b)>0, R<S\right), \quad (3)$$

$$G_1(a,b,c;x,y) = \frac{2M^{a+b}\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \int_0^{\frac{\pi}{2}} (sin^2\alpha)^{b-\frac{1}{2}}(cos^2\alpha)^{c-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{-(b+c)}$$

$$\times\left[M-xcot^2\alpha-M^2ytan^2\alpha\right]^{-a} d\alpha,$$

$$\left(Re(b)>0, Re(c)>0, M>0\right). \quad (4)$$

Proof. To prove the result in equality (1) asserted in Theorem 2.1, let $\mho$ denote the right-hand side of the equality (1). Then from the definition of Horn's function H_1 in (1.9), we obtain

$$\mho = \frac{\Gamma(b+b')}{\Gamma(b)\Gamma(b')}\sum_{m,n=0}^{\infty}\frac{(c)_{m-n}(a)_{m+n}(b+b')_n}{(b')_m}\int_0^\infty \frac{\alpha^{b+n-m-1}}{(1+\alpha)^{b+b'+n}}d\alpha. \quad (5)$$

Employing the integral representation of the Beta function (see, e.g., [7, p. 9, Eq. (2)])

$$B(a,b) = \int_0^\infty \frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}d\alpha, \left(Re(a)>0, Re(b)>0\right),$$

in (5), we have

$$\mho = \frac{\Gamma(b+b')}{\Gamma(b)\Gamma(b')}\sum_{m,n=0}^{\infty}\frac{(c)_{m-n}(a)_{m+n}(b+b')_nB(b+n-m,b'+m)}{(b')_m}. \quad (6)$$

Now applying well known beta function (see, e.g., [16, Section 1.1])

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

in (6), we are led to the desired result. Then, the similar way we can easily get (2)-(4).

The following theorem can be proved, like Theorem 2.1. So the details are omitted.

Theorem 2.2. The following integral representations holds:

$$G_2\left(a,b,c,d;x,y\right)=\frac{\Gamma(a+b)\Gamma(c+c')\Gamma(d+d')}{2^{a+b+c+c'+d+d'-6}\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(c')\Gamma(d)\Gamma(d')}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{a-\frac{1}{2}}$$

$$\times\left[(1-\alpha)^2\right]^{b-\frac{1}{2}}\left(1+\alpha^2\right)^{-(a+b)}\left[(1+\beta)^2\right]^{c'-\frac{1}{2}}\left[(1-\beta)^2\right]^{c-\frac{1}{2}}\left(1+\beta^2\right)^{-(c+c')}\left[(1+\gamma)^2\right]^{d'-\frac{1}{2}}$$

$$\times\left[(1-\gamma)^2\right]^{d-\frac{1}{2}}\left(1+\gamma^2\right)^{-(d+d')}$$

$$\times F_2\left(a+b,d+d',c+c';c',d';\frac{(1+\alpha)^2(1+\beta)^2(1-\gamma)^2x}{4(1+\alpha^2)(1-\beta)^2(1+\gamma^2)},\frac{(1-\alpha)^2(1-\beta)^2(1+\gamma)^2y}{4(1+\alpha^2)(1+\beta^2)(1-\gamma)^2}\right)d\alpha d\beta d\gamma,$$

$$\left(Re(a)>0,Re(b)>0,Re(c)>0,Re(c')>0,Re(d)>0,Re(d')>0\right), \tag{7}$$

$$G_2\left(a,b,c,d;x,y\right)=\frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}\int_0^\infty\int_0^\infty\left(e^{-\alpha}\right)^a\left(1-e^{-\alpha}\right)^{b-1}\left(e^{-\beta}\right)^c\left(1-e^{-\beta}\right)^{a+b+d-1}$$

$$\times\left[\left(1-e^{-\beta}\right)-e^{-\alpha+\beta}\left(1-e^{-\beta}\right)^2x-e^{-\beta}\left(1-e^{-\alpha}\right)y\right]^{-(a+b)}d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(c)>0,Re(d)>0\right), \tag{8}$$

$$G_2\left(a,b,c,d;x,y\right)=\frac{\Gamma(a+c)\Gamma(b+d)\left(S_1-T_1\right)^a\left(R_1-T_1\right)^{b+c+d}\left(S_2-T_2\right)^b\left(R_2-T_2\right)^{a+c+d}}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}$$

$$\times\int_{R_1}^{S_1}\int_{R_2}^{S_2}\left(\alpha-R_1\right)^{a-1}\left(S_1-\alpha\right)^{b+c+d-1}\left(\beta-R_2\right)^{b-1}\left(S_2-\beta\right)^{a+c+d-1}$$

$$\times\left[\left(R_1-T_1\right)\left(S_1-\alpha\right)\left(S_2-R_2\right)\left(\beta-T_2\right)-\left(S_1-T_1\right)\left(\alpha-R_1\right)\left(R_2-T_2\right)\left(S_2-\beta\right)x\right]^{-(b+d)}$$

$$\times\left[\left(S_1-R_1\right)\left(\alpha-T_1\right)\left(R_2-T_2\right)\left(S_2-\beta\right)-\left(R_1-T_1\right)\left(S_1-\alpha\right)\left(S_2-T_2\right)\left(\beta-R_2\right)y\right]^{-(a+c)}d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(c)>0,Re(d)>0\right), \tag{9}$$

$$G_2\left(a,b,c,d;x,y\right)=\frac{2M^{a+c}\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\int_0^\infty cosh\alpha\left(sinh^2\alpha\right)^{a+d-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{-(c+d)}$$

$$\times\left(Msinh^2\alpha-x\right)^{-a}\left(1-Mysinh^2\alpha\right)^{-b}d\alpha,$$

$$\left(Re(c)>0,Re(d)>0,M>0\right). \tag{10}$$

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Theorem 2.3. The following integral representations hold:

$$G_3(a,b;x,y)=\frac{2M^a\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a-\frac{1}{2}}(cos^2\alpha)^{a'-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{-(a+a')}$$

$$\times H_6\left(b,a+a',1-a';\left(1+\frac{1}{M}cot^2\alpha\right)x,-\frac{M^2ysin^2\alpha tan^2\alpha}{cos^2\alpha+Msin^2\alpha}\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0,M>0\right), \quad (11)$$

$$G_3(a,b;x,y)=\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a'-1}\left(\frac{1}{2}-\alpha\right)^{a-1}$$

$$\times H_7\left(b,\frac{a+a'}{2},\frac{a+a'+1}{2};a';\frac{(1+2\alpha)x}{(1-2\alpha)},(1-2\alpha)^2y\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right), \quad (12)$$

$$G_3(a,b;x,y)=\frac{(1+M_1)^a(1+M_2)^b\Gamma(a+a')\Gamma(b+b')}{\Gamma(a)\Gamma(a')\Gamma(b)\Gamma(b')}\int_0^1\int_0^1\alpha^{a-1}(1-\alpha)^{a'-1}(1+M_1\alpha)^{-(a+a')}$$

$$\times\beta^{b-1}(1-\beta)^{b'-1}(1+M_2\beta)^{-(b+b')}{}_2F_1\left(\frac{b+b'}{2},\frac{b+b'+1}{2};a';\frac{4(1+M_2)^2(1-\alpha)\beta^2x}{(1+M_1)\alpha(1+M_2\beta)^2}\right)$$

$$\times{}_2F_1\left(\frac{a+a'}{2},\frac{a+a'+1}{2};b';\frac{4(1+M_1)^2\alpha^2(1-\beta)y}{(1+M_2)(1+M_1\alpha)^2\beta}\right)d\alpha d\beta,$$

$$\left(Re(a)>0,Re(a')>0,Re(b)>0,Re(b')>0,M_1>-1,M_2>-1\right), \quad (13)$$

$$G_3(a,b;x,y)=\frac{2(1+M)^{a+2b}\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{b-\frac{1}{2}}(cos^2\alpha)^{a-\frac{1}{2}}$$

$$\times\left[(1+M)\left(1+Msin^2\alpha\right)-(1+M)^3xsin^2\alpha tan^2\alpha-ycos^2\alpha cot^2\alpha\right]^{-(a+b)}d\alpha,$$

$$\left(Re(a)>0,Re(b)>0,M>-1\right), \quad (14)$$

Theorem 2.4. The following integral representations hold:

$$H_1(a,b,c;d;x,y)=\frac{\Gamma(2d)}{\Gamma(a)\Gamma(2d-a)(S-R)^{2d-1}}\int_R^S(\alpha-R)^{2d-a-1}(S-\alpha)^{a-1}$$

$$\times F_1\left(b,d+\frac{1}{2},c;2d-a;\frac{4(\alpha-R)(S-\alpha)x}{(S-R)^2},\frac{(\alpha-R)y}{(S-\alpha)}\right)d\alpha,$$

$$\left(Re(a)>0,Re(2d-a)>0,R<S\right), \quad (15)$$

$$H_1(a,b,c;d;x,y)=\frac{2\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a-\frac{1}{2}}(cos^2\alpha)^{a'-\frac{1}{2}}$$

$$\times F_2\left(b,a+a',c;d,a';xsin^2\alpha,ycot^2\alpha\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right), \quad (16)$$

$$H_1(a,b,c;d;x,y)=\frac{\Gamma(d)(S-T)^b(R-T)^{d-b}}{\Gamma(b)\Gamma(d-b)(S-R)^{d-a-1}}\int_R^S(\alpha-R)^{b-1}(S-\alpha)^{d-b-1}(\alpha-T)^{a-d}$$

$$\times\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)x\right]^{-a}$$

$$\times{}_2F_1\left(c,1+b-d;1-a;\frac{(S-T)(\alpha-R)\left[(S-R)(\alpha-T)-(S-T)(\alpha-R)x\right]y}{(R-T)(S-R)(S-\alpha)(\alpha-T)}\right)d\alpha,$$

$$(Re(b)>0,Re(d-b)>0,T<R<S), \qquad (17)$$

$$H_1(a,b,c;d;x,y)=\frac{(1+M)^{a+c}\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}(1+M\alpha)^{-(a+b)}$$

$$\times\left[(1+M)\alpha-(1-\alpha)y\right]^{-c}{}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};d;\frac{4(1+M)\alpha(1-\alpha)x}{(1+M\alpha)^2}\right)d\alpha,$$

$$(Re(a)>0,Re(b)>0,M>-1). \qquad (18)$$

Corollary 2.5. Let $y=0$ in (18). Then the following result holds true:

$${}_2F_1(a,b;d;x)=\frac{(1+M)^a\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}(1+M\alpha)^{-(a+b)}$$

$$\times{}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};d;\frac{4(1+M)\alpha(1-\alpha)x}{(1+M\alpha)^2}\right)d\alpha,$$

$$(Re(a)>0,Re(b)>0,M>-1). \qquad (19)$$

Theorem 2.6. The following integral representations hold:

$$H_2(a,b,c,d;e;x,y)=\frac{\Gamma(b+c)\Gamma(2e)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(2e-a)(S_1-R_1)^{b+c-1}(S_2-R_2)^{2e-1}}\int_{R_1}^{S_1}\int_{R_2}^{S_2}$$

$$\times(\alpha-R_1)^{b-1}(S_1-\alpha)^{c-1}(\beta-R_2)^{a-1}(S_2-\beta)^{2e-a-1}$$

$$\times F_1\left(b+c,e+\frac{1}{2},d;2e-a;\frac{4(\alpha-R_1)(\beta-R_2)(S_2-\beta)x}{(S_1-R_1)(S_2-R_2)^2},\frac{(S_1-\alpha)(S_2-\beta)y}{(S_1-R_1)(\beta-R_2)}\right)d\alpha d\beta,$$

$$(Re(a)>0,Re(b)>0,Re(c)>0,Re(2e-a)>0,R_1<S_1,R_2<S_2), \qquad (20)$$

$$H_2(a,b,c,d;e;x,y)=\frac{\Gamma(2e)}{2^{2(e-1)}\Gamma(a)\Gamma(2e-a)}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{2e-a-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a-\frac{1}{2}}$$

$$\times\left(1+\alpha^2\right)^{-e}F_3\left(b,c,e+\frac{1}{2},d;2e-a;\left(\frac{1-\alpha^2}{1+\alpha^2}\right)^2x,\left(\frac{1+\alpha}{1-\alpha}\right)^2y\right)d\alpha,$$

$$(Re(a)>0,Re(2e-a)>0), \qquad (21)$$

$$H_2(a,b,c,d;e;x,y)=\frac{(1+M)^a\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1\alpha^{a-1}(1-\alpha)^{b-1}(1+M\alpha)^{-(a+b)}$$

$$\times H_7\left(a+b,c,d;e;\frac{(1+M)\alpha(1-\alpha)x}{(1+M\alpha)^2},\frac{(1+M\alpha)y}{(1+M)\alpha}\right)d\alpha,$$

$$(Re(a)>0,Re(b)>0,M>-1), \qquad (22)$$

Notes

$$H_2(a,b,c,d;e;x,y)=\frac{2M^{e-b}\Gamma(e)}{\Gamma(b)\Gamma(e-b)}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{e-b-\frac{1}{2}}\left(cos^2\alpha\right)^{b-\frac{1}{2}}\left(cos^2\alpha+Msin^2\alpha\right)^{a-e}$$

$$\times\left[\left(cos^2\alpha+Msin^2\alpha\right)-xcos^2\alpha\right]^{-a}{}_2F_1\left(c,d;1-a;\left[\frac{xcos^2\alpha}{(cos^2\alpha+Msin^2\alpha)}-1\right]y\right)d\alpha,$$

$$(Re(b)>0,Re(e-b)>0,M>0)\,. \tag{23}$$

Notes

Theorem 2.7. The following integral representations hold:

$$H_5(a,b;c;x,y)=\frac{\Gamma(2c)}{\Gamma(b)\Gamma(2c-b)}\int_0^{\infty}\left(e^{-\alpha}\right)^b\left(1-e^{-\alpha}\right)^{2c-b-1}$$

$$\times H_3\left(a,c+\frac{1}{2};2c-b;\left(e^{\alpha}-1\right)x,4e^{-\alpha}\left(1-e^{-\alpha}\right)y\right)d\alpha,$$

$$(Re(b)>0,Re(2c-b)>0)\,, \tag{24}$$

$$H_5(a,b;c;x,y)=\frac{\Gamma(b+b')}{2^{b+b'-1}\Gamma(b)\Gamma(b')}\int_{-1}^{1}(1+\alpha)^{b'-1}(1-\alpha)^{b-1}$$

$$\times H_4\left(a,b+b';b',c;\left(\frac{1+\alpha}{1-\alpha}\right)x,\frac{(1-\alpha)}{2}y\right)d\alpha,$$

$$\left(Re(b)>0,Re(b')>0\right), \tag{25}$$

$$H_5(a,b;c;x,y)=\frac{2M^{a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{a-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{b-c}$$

$$\times\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]^{-b}$$

$$\times{}_2F_1\left(\frac{1+a-c}{2},\frac{a-c}{2}+1;1-b;-\frac{4M^2xsinh^4\alpha\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]}{(1+Msinh^2\alpha)}\right)d\alpha,$$

$$(Re(a)>0,Re(c)>0,M>0)\,, \tag{26}$$

$$H_5(a,b;c;x,y)=\frac{2M^{c-a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{c-a-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{b-c}$$

$$\times\left[\left(1+Msinh^2\alpha\right)-y\right]^{-b}$$

$$\times{}_2F_1\left(\frac{1+a-c}{2},\frac{a-c}{2}+1;1-b;-\frac{4x\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]}{M^2sinh^4\alpha\,(1+Msinh^2\alpha)}\right)d\alpha,$$

$$(Re(a)>0,Re(c)>0,M>0)\,. \tag{27}$$

Corollary 2.8. Let $x=0$ in (26). Then the following result holds true:

$${}_2F_1(a,b;c;y)=\frac{2M^{a}\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{a-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{b-c}$$

$$\times\left[\left(1+Msinh^2\alpha\right)-Mysinh^2\alpha\right]^{-b}d\alpha,$$

$$(Re(a)>0,Re(c)>0,M>0)\,. \tag{28}$$

Theorem 2.9. The following integral representations hold:

$$H_6\left(a,b,c;x,y\right)=\frac{\Gamma(a+c)\Gamma(b+b')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c)}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{c-1}\left(\frac{1}{2}-\alpha\right)^{a-1}$$

$$\times\left(\frac{1}{2}+\beta\right)^{b'-1}\left(\frac{1}{2}-\beta\right)^{b-1}$$

$$F_3\left(\frac{a+c}{2},\frac{b+b'}{2},\frac{a+c+1}{2},\frac{b+b'+1}{2};b';\frac{(1-2\alpha)^2(1+2\beta)\,x}{(1-2\beta)},\frac{(1+2\alpha)\left(1-4\beta^2\right)y}{(1-2\alpha)}\right)d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(b')>0,Re(c)>0\right), \tag{29}$$

$$H_6\left(a,b,c;x,y\right)=\frac{\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\infty}\alpha^{a-1}\left(1+\alpha\right)^{-(a+a')}G_2\left(a+a',c,b,1-a';\frac{-\alpha^2x}{(1+\alpha)},\frac{-y}{\alpha}\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right), \tag{30}$$

$$H_6\left(a,b,c;x,y\right)=\frac{\Gamma(b+b')}{2^{b+b'-2}\Gamma(b)\Gamma(b')}\int_{-1}^{1}\left[(1+\alpha)^2\right]^{b'-\frac{1}{2}}\left[(1-\alpha)^2\right]^{b-\frac{1}{2}}\left(1+\alpha^2\right)^{-(b+b')}$$

$$\times H_6\left(a,1-b',c;-\left(\frac{1+\alpha}{1-\alpha}\right)^2x,-\left(\frac{1-\alpha}{1+\alpha}\right)^2y\right)d\alpha,$$

$$\left(Re(b)>0,Re(b')>0\right), \tag{31}$$

$$H_6\left(a,b,c;x,y\right)=\frac{4(1+M_1)^{a+b+b'}(1+M_2)^{b'}\Gamma(a+c)\Gamma(b+b')}{\Gamma(a)\Gamma(b)\Gamma(b')\Gamma(c)}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(sin^2\alpha\right)^{a-\frac{1}{2}}$$

$$\times\left(cos^2\alpha\right)^{c-\frac{1}{2}}\left(sin^2\beta\right)^{b'-\frac{1}{2}}\left(cos^2\beta\right)^{b-\frac{1}{2}}\left(1+M_1sin^2\alpha\right)^{-(a+c)}$$

$$\times\left[(1+M_1)(1+M_2sin^2\beta)-ycot^2\alpha cos^2\beta\right]^{-(b+b')}$$

$$\times\,{}_2F_1\left(\frac{a+c}{2},\frac{a+c+1}{2};b';\frac{4(1+M_1)^2(1+M_2)xsin^4\alpha tan^2\beta}{(1+M_1sin^2\alpha)^2}\right)d\alpha d\beta,$$

$$\left(Re(a)>0,Re(b)>0,Re(b')>0,Re(c)>0,M_1>-1,M_2>-1\right). \tag{32}$$

Theorem 2.10. The following integral representations hold:

$$H_7\left(a,b,c;d;x,y\right)=\frac{\Gamma(a+a')(S-T)^a(R-T)^{a'}}{\Gamma(a)\Gamma(a')(S-R)^{a+a'-1}}\int_R^S(\alpha-R)^{a-1}(S-\alpha)^{a'-1}(\alpha-T)^{-(a+a')}$$

$$\times{}_2F_1\left(\frac{a+a'}{2},\frac{a+a'+1}{2};d;\left(\frac{2(S-T)(\alpha-R)}{(S-R)(\alpha-T)}\right)^2x\right){}_2F_1\left(b,c;a';\frac{(R-T)(S-\alpha)y}{(S-T)(\alpha-R)}\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0,T<R<S\right), \tag{33}$$

Notes

Notes

$$H_7(a,b,c;d;x,y)=\frac{\Gamma(d)}{\Gamma(b)\Gamma(d-b)}\int_0^{\infty}\left(e^{-\alpha}\right)^{b}\left(1-e^{-\alpha}\right)^{d-b-1}$$

$$\times H_6\left(a,1+b-d,c;\left(e^{-\alpha}-1\right)x,\frac{y}{(1-e^{\alpha})}\right)d\alpha,$$

$$\left(Re(b)>0,Re(d-b)>0\right), \qquad (34)$$

$$H_7(a,b,c;d;x,y)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)(S-R)^{a+b-1}}\int_R^{S}(\alpha-R)^{a+c-1}(S-\alpha)^{b-1}$$

$$\times\left[(\alpha-R)-(S-\alpha)y\right]^{-c}{}_2F_1\left(\frac{a+b}{2},\frac{a+b+1}{2};d;4\left(\frac{\alpha-R}{S-R}\right)^2x\right)d\alpha,$$

$$\left(Re(a)>0,Re(b)>0,R<S\right), \qquad (35)$$

$$H_7(a,b,c;d;x,y)=\frac{2\Gamma(a+a')}{\Gamma(a)\Gamma(a')}\int_0^{\frac{\pi}{2}}(sin^2\alpha)^{a'-\frac{1}{2}}(cos^2\alpha)^{a-\frac{1}{2}}$$

$$\times H_2\left(1-a',a+a',b,c;d;-xcot^2\alpha cos^2\alpha,-ytan^2\alpha\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0\right). \qquad (36)$$

Corollary 2.11. Let $y=0$ in (32). Then the following result holds true:

$${}_2F_1\left(\frac{a}{2},\frac{a+1}{2};d;4x\right)=\frac{\Gamma(a+a')(S-T)^{a}(R-T)^{a'}}{\Gamma(a)\Gamma(a')(S-R)^{a+a'-1}}\int_R^{S}(\alpha-R)^{a-1}(S-\alpha)^{a'-1}$$

$$\times(\alpha-T)^{-(a+a')}{}_2F_1\left(\frac{a+a'}{2},\frac{a+a'+1}{2};d;4\left(\frac{(S-T)(\alpha-R)}{(S-R)(\alpha-T)}\right)^2x\right)d\alpha,$$

$$\left(Re(a)>0,Re(a')>0,T<R<S\right). \qquad (37)$$

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Extension of Comparative Analysis of Estimation Methods for Dirichlet Distribution Parameters

By Halid, M. A., Akomolafe, A. A., Oyegoke, O. A. & Oladimeji O. A.
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Abstract- The Dirichlet distribution is a generalization of the Beta distribution. This research deals with the estimation of scale parameter for Dirichlet distribution with known shapes. We examined three methods to estimate the parameters of Dirichlet distribution which are Maximum Likelihood Estimator, Method of Moment Estimator and Quasi-Likelihood Estimator. The performance of these methods were compared at different sample sizes using Bias, Mean Square Error, Mean Absolute Error and Variance criteria, an extensive simulation study was carried out on the basis of the selected criterion using statistical software packages as well as the application of the criterion to real life data, all these were done to obtain the most efficient method. The simulation study and analysis revealed that Quasi- Likelihood Estimator perform better in terms of bias while Method of Moment Estimator is better than the other two methods in terms of variance; the Maximum Likelihood Estimation was the best estimation method in terms of the Mean square Error and Mean Absolute Error; while Quasi- Likelihood Estimation method was the best estimation method with real life data.

Keywords: dirichlet distribution, parameter estimation, maximum likelihood estimator, method of moment estimator and quasi- likelihood estimator.

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Abstract- The Dirichlet distribution is a generalization of the Beta distribution. This research deals with the estimation of scale parameter for Dirichlet distribution with known shapes. We examined three methods to estimate the parameters of Dirichlet distribution which are Maximum Likelihood Estimator, Method of Moment Estimator and Quasi-Likelihood Estimator. The performance of these methods were compared at different sample sizes using Bias, Mean Square Error, Mean Absolute Error and Variance criteria, an extensive simulation study was carried out on the basis of the selected criterion using statistical software packages as well as the application of the criterion to real life data, all these were done to obtain the most efficient method. The simulation study and analysis revealed that Quasi- Likelihood Estimator perform better in terms of bias while Method of Moment Estimator is better than the other two methods in terms of variance; the Maximum Likelihood Estimation was the best estimation method in terms of the Mean square Error and Mean Absolute Error; while Quasi- Likelihood Estimation method was the best estimation method with real life data.

Keywords: dirichlet distribution, parameter estimation, maximum likelihood estimator, method of moment estimator and quasi- likelihood estimator.

I. Introduction

In Bayesian Statistics, the Dirichlet distribution is a popular conjugate prior for multinomial distribution. The Dirichlet distribution has a number of applications in various fields. Samuel S. Wilk (1962), gave an example, where he applied the Dirichlet distribution in deriving the distribution of order statistics. Again Kenneth Lange (1995), also used the Dirichlet distribution in biology to demonstrate and to compute forensic match probabilities from several distinct populations. In addition, Brad N (2009), used the Dirichlet distribution to model a player's abilities in Major League Baseball. It is shown that the Dirichlet distribution can be used to model consumer behavior Gerald et al (1984). Dirichlet Distribution can be extended to various fields of study such as biology, astronomy, text mining and so on. The Dirichlet Distribution (DD) is usually employed as a conjugate prior for the multinomial modeling and Bayesian analysis of complete contingency tables (Agresti (2002)). Gupta and Richards (1987, 1991, and 1992) extended the Dirichlet Distribution to the Liouville distribution. Fang, Kotz and Ng (1990) gave an extensive exposition of the Liouville family and its ramifications.

The problem of estimating the parameters which determine a mixture has been the subject of diverse studies (Redner and Walker 1984). During the last two decades,

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the method of maximum likelihood (ML) (Bishop. C.M .1995) and (Rao. P. 1987) has become the most common approach to this problem. Of the variety of the iterative methods which has been subjected as an alternative to optimize the parameters of a mixture, the one most likely used is the expectation maximization (EM). EM was originally proposed by Dempster et al 1977 for estimating the maximum likelihood estimator (MLE) of stochastic models. This algorithm gives an iterative procedure and the practical form is usually simple and easy to implement .The EM algorithm can be viewed as an approximation of the Fisher scoring method (Ikeda. S. (2000). In this research we showed that the Dirichlet distribution can be a very good choice for modelling data, MLE was used to estimate the parameters of the Dirichlet Mixture Model alongside with EM algorithm. This mixture decomposition algorithm incorporates a penalty term in the objective function to find the number of components required to model the data. This algorithm suffers some set back: the need to specify the number of components each time, which will be determine by selected criterion functions such as AIC, BIC, MDL which has been in existence to validate the model and justify the more efficient one.

This research centered on studying how the different estimators of the unknown parameters of a Dirichlet distribution can behave for different sample sizes. Here, we are mainly comparingthe Maximum Likelihood Estimator, Method of Moment Estimator and Quasi-Likelihood Estimator with respect to efficiency, bias, mean absolute error and variance using extensive simulation techniques as well as application of the estimation methods to real life data set.

Notes

II. Literature Review

The Dirichlet model describes patterns of repeat purchases of brands within a product category. It models simultaneously the counts of the number of purchases of each brand over a period of time, so that it describes purchase frequency and brand choice at the same time. It assumes that consumers have an experience of the product category, so that they are not influenced by previous purchase and marketing strategies; for this reason, consumer characteristics and marketing-mix instruments are not included in the model. As the market is assumed to be stationary, these effects are already incorporated in each brand market share which influences other brand performance indexes calculated by the model. The market is also assumed to be un segmented. The theory and development of the model is fully described in Ehrenberg (1972).Good hardt, Ehrenberg and Chatfield (1984), summarise the situation by stating that the Dirichlet model makes explicit that there are simple, general and rather precise regularities in a substantial area of human behaviour where this has not always been expected. In setting the context for this particular approach to the modeling of consumer behaviour viz. the largely explanatory models of consumer behaviour, Ehrenberg (1988) claims that it describes how consumers behave, rather than why, and takes into account only those factors necessary for an adequate description.

Many aspects of buyer behaviour can be predicted simply from the penetration and the average purchase frequency of the item, and even these two variables are interrelated (Ehrenberg, 1988, pg. ii). The Dirichlet model integrates the reported regularities, and predicts many aggregate brand performance measures. These measures are the distribution of purchases for a brand, the proportion of a brand's buyers buying that brand only, and the proportion of people purchasing a brand, given that they have previously purchased that brand. When these predictions are compared with observed figures, Ehrenberg claims that it is not unreasonable to expect to obtain correlations in the order of 0.9 and sometimes much higher, (Ehrenberg 1975, Ehrenberg and Bound 1993).

Applications and theory can be used to provide norms for examining brand performance, or diagnostic information for the "health" of a brand. In addition, the Dirichlet model can provide interpretative norms for evaluating situations where some trend in sales has occurred, say after a promotion or advertising scheme. Ehrenberg also claims that the Dirichlet model provides valuable insights into the nature and implications of brand-loyalty (e.g., Ehrenberg and Uncles 1995; Ehrenberg and Uncles 1999). The use of likelihood theory to estimate the parameters of the Dirichlet model, providing an alternative to the standard procedure based on the method of zeros and ones and on marginal moments (Rungie 2003b). In order to write the likelihood function, the data should be in the form of joint frequencies, like those contained in a contingency table with n-rows, representing the number of consumers, and g columns, for the number of brands. Alternatively, the iterative procedures based on the approach that computations are easy to use, and require only aggregated data as input, as access to original panel data is not necessary as proposed by Goodhardt, Ehrenberg and Chatfield (1984). Raw panel data cannot always be used since panel operators who measure sales and household consumption provide information only in some aggregate format such as market share, penetration, and average purchase rate with reference to the various brands (Wright et al. 2002). In these situations, the only way to estimate the Dirichlet model is to use the traditional method. Dirichlet modeling continues to be a successful and influential approach, and is increasingly being used to provide norms against which brand performance can be interpreted (Uncles et al. 1995; Bhattacharya 1997; Ehrenberg et al. 2000). Dirichlet model is useful for the provision of norms for stationary markets, to supply baselines for interpreting change (i.e., non-stationary situations) without having to match the results against a control sample, to help strategic decision-making, and to understand the nature of markets.

There are diverse ways of applying the distribution, where the Dirichlet has proved to be particularly useful is in modeling the distribution of words in text documents [9]. If we have a dictionary containing k possible words, then a particular document can be represented by a probability mass function [pmf] of length k-produced by normalizing the empirical frequency of its words. A group of documents produces a collection of pmfs, and we can fit a Dirichlet distribution to capture the variability of these pmfs.

Notes

III. Methodology

a) *Deriving the Dirichlet Distribution*

Let X_i be a random variable from the Gamma distribution $G(\alpha_i, 1), i = 1, \ldots, k$, and let $X_1, \ldots, X_k$ be independent. The joint pdf of $X_1, \ldots, X_k$ is

$$f(x_1, \ldots, x_k) = \begin{cases} \prod_{i=1}^{k} \frac{1}{\Gamma(\alpha_i)} x_i^{\alpha_i - 1} e^{-x_i}, & \text{if} 0 < x_i < \infty \\ 0, & \text{otherwise} \end{cases}$$

Let

$$Y_i = \frac{X_i}{X_1 + X_2 + \cdots + X_k}, \quad i = 1,2, \ldots, k-1$$

and

$$Z_k = X_1 + X_2 + \cdots + X_k \; .$$

By using the change of variables technique, this transformation maps $M = \{(x_i, \dots, x_k): 0 < x_i < \infty, i = 1, \dots, k\}$ onto $N = \{(y_i, \dots, y_{k-1}, z_k): y_i > 0, i = 1, \dots, k-1, 0 < z_k < \infty, y_1 + \cdots + y_{k-1} < 1\}$.The inverse functions are$x_1 = y_1 z_k, x_2 = y_2 z_k, \dots, x_{k-1} = y_{k-1} z_k, x_k = z_k(1 - y_1 - \cdots - y_{k-1})$. Hence, the Jacobian is

$$J = \begin{vmatrix} z_k & 0 & \dots & & 0 y_1 \\ 0 & z_k & \dots & & 0 y_2 \\ \vdots & \vdots & \vdots & & \vdots \vdots \\ 0 & 0 & \dots & Z_k & y_{k-1} \\ -z_k & -z_k & \dots & -z_k & (1-y_1) - \cdots - y_{k-1} \end{vmatrix} = z_k^{k-1}$$

Then, the joint pdf of $Y_1, \dots, Y_{k-1}, Z_k$ is

$$f(y_1, \dots, y_{k-1}, z_k) = \frac{y_1^{\alpha_1 - 1} \dots y_{k-1}^{\alpha_{k-1}-1} (1 - y_1 - \cdots - y_{k-1})^{\alpha_k - 1}}{\Gamma(\alpha_1) \dots \Gamma(\alpha_k)} e^{-z_k} z_k^{\alpha_1 + \cdots + \alpha_{k-1}}$$

By integrating out z_k, the joint pdf of $Y_1, \dots, Y_{k-1}$ is

$$f(y_1, \dots, y_{k-1}) = \frac{\alpha_1 + \cdots + \alpha_k}{\Gamma(\alpha_1) \dots \Gamma(\alpha_k)} y_1^{\alpha_1 - 1} \dots y_{k-1}^{\alpha_{k-1}-1} (1 - y_1 - \cdots - y_{k-1})^{\alpha_k - 1} ,$$

where $y_i > 0, y_1 + \cdots + y_{k-1} < 1, i = 1, \dots, k-1$. The joint pdf of the random variables $Y_1, \dots, Y_{k-1}$is known as the pdf of the Dirichlet distribution with parameters $\alpha_1, \dots, \alpha_k$. Furthermore, it is clear that Z_k has a Gamma distribution $G(\sum_{i=1}^{k} \alpha_i , 1)$ and Z_k is independent of $Y_1, \dots, Y_{k-1}$. Robert V Hogg and Allen T Craig.1970.

b) Moment generating function

The moment generating function of $Y^k = [Y_1, \dots, Y_k]$. Let $t = (t_1, \dots, t_k)^T \in \Re^k$.

The moment generating function of Y^k at t is

$$E\left(e^{t^T Y^k}\right) = \int \dots \int e^{t^T y} f(y^k) dy_1 \dots dy_k$$

$$= \int \dots \int \sum_{m=0}^{\infty} \frac{(t^T y^k)^m}{m!} f(y^k) dy_1 \dots dy_k \quad (1)$$

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \int \dots \int (t^T y^k)^m f(y^k) dy_1 \dots dy_k \quad (2)$$

Step (a)

$$= \sum_{m=0}^{\infty} \frac{1}{m!} \left[\int \dots \int \sum_{n_1 + n_2 + \cdots + n_k = m} \frac{m!}{n_1! n_2! \dots n_k!} \times \prod_{i=1}^{k} (t_i y_i)^{n_i} f(y^k) dy_1 \dots dy_k \right]$$

Notes

$$= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{m=0}^{\infty}\frac{1}{m!}\left[\sum_{n_1+n_2+\cdots+n_k=m}\frac{m!}{n_1!\,n_2!\ldots n_k!}\prod_{i=1}^{k}(t_i)^{n_i}\,E\left(\prod_{i=1}^{k}Y_i^{n_i}\right)\right]$$

$$= \sum_{m=0}^{\infty}\frac{1}{m!}\left[\sum_{n_1+n_2+\cdots+n_k=m}\frac{m!}{n_1!n_2!\ldots n_k!}\prod_{i=1}^{k}(t_i)^{n_i}\times\left[\frac{\Gamma(\alpha_1+\cdots+\alpha_k)}{\Gamma(\alpha_1+n_1+\cdots\alpha_k+n_k)}\prod_{i=1}^{k}\frac{\Gamma(\alpha_i+n_i)}{\Gamma(\alpha_i)}\right]\right]. \quad (3)$$

Notes

In step (a), we apply the multinomial theorem

$$(x_1+x_2+\cdots+x_k)^m = \sum_{n_1+n_2+\cdots+n_k=m}\frac{m!}{n_1!n_2!\ldots n_k!}\prod_{i=1}^{k}x_i^{n_i} \quad (4)$$

for any positive integer k and any non-negative integer m.

c) Maximum Likelihood Estimation

The ML estimation method concerns choosing parameters to maximize the joint density function of the sample (likelihood function). Therefore, we consider

$$\max_{\Theta}\, p(x^k|\Theta) \quad (5)$$

with constraints $\sum_{j=1}^{m}p(j)=1$ and $p(j)>0$ for $j=1,2,\ldots,m$. We can consider $p(j)$ as prior probabilities under these constraints. Now suppose we have a sample that contains n random vectors X_i^k, which are i.i.d., $i=1,\ldots,n$. We maximize the following function with respect to Θ and Λ

$$\phi(\mathrm{x}^{\mathrm{k}},\Theta,\Lambda) = \sum_{i=1}^{n} ln\left(\sum_{j=1}^{m}p\left(x^k|\Theta_j\right)p(j)\right) + \Lambda\left(1-\sum_{j=1}^{m}p(j)\right) + \mu\left(\sum_{j=1}^{m}p(j)\ln\left(p(j)\right)\right) \quad (6)$$

The first term of equation 8 is the log-likelihood function. $\Lambda$is the Lagrange multiplier in the second term. In the last term of eq. 8, we use an entropy-based criterion. Also, μ is the ratio of the first term to the last term in of each iteration t byNizar Bouguila, Djemel Ziou, and Jean Vaillancourt (2004)

$$\mu(\mathrm{t}) = \frac{\sum_{j=1}^{m}\ln\left(\sum_{j=1}^{m}p^{t-1}\left(x_i^k|\Theta_j\right)p^{t-1}(j)\right)}{\left(\sum_{j=1}^{m}p^{t-1}(j)\ln(p^{t-1}(j)\right)}, \quad (7)$$

In order to optimize (8), we need to solve the following equations:

$$\frac{\partial}{\partial\Theta}\phi(x^k,\Theta,\Lambda)=0$$

$$\frac{\partial}{\partial\Lambda}\phi(x^k,\Theta,\Lambda)=0$$

It is shown that the estimator of the prior probability p(j) is

$$\mathrm{p(j)}^{\mathrm{new}} = \frac{\sum_{i=1}^{n}p^{old}(j|x^k,\Theta_j)+\mu\left[p(j)^{old}\left(1+ln\ p(j)^{old}\right)\right]}{n+\mu\sum_{j=1}^{m}p(j)^{old}\left(1+\ln p(j)^{old}\right)}, \quad j=1,2,\ldots,m. \quad (8)$$

Note that μ is defined by (4.3) and $p(j|x_i^k,\Theta_j)$ is the posterior probability where

$$p\left(j\middle|x_i^k,\Theta_j\right)=\frac{p\left(x_i^k,\Theta_j\right)p(j)}{p(x_i^k,\Theta)}, i=1,\dots,n,\ \ j=1,2,\dots,m. \qquad (9)$$

Now we want to estimate the parameters α_j^k, $j=1,2,\dots,m$. The Fisher scoring method is used to find these estimates. Denote α_{jl} as one element of the parameter vector α_j^k for each component $j=1,2,\dots,m$.The derivative of $\phi(x^k,\Theta,\Lambda)$ with respect to α_{jl} is

$$\frac{\partial}{\partial\alpha_{jl}}\phi(x^k,\Theta,\Lambda)=\sum_{i=1}^{n}p(j|x_i^k,\alpha_j^k)(\ln x_{il})+\left[\psi\left(\alpha_{0_j}\right)-\psi(\alpha_{jl})0\right]\sum_{i=1}^{n}p\left(j\middle|x_i^k,\alpha_j^k\right), \qquad (10)$$

$$l=1,\dots,k,\quad j=1,\dots,m,$$

where ψ(.) is the Digamma function. However, α_{jl} can become negative during iterations. In order to keep α_{jl} strictly positive, set $\alpha_{jl}=e^{\beta jl}$. βjl is any real number. Then, the derivative of $\phi(x^k,\Theta,\Lambda)$ with respect to βjl is

$$\frac{\partial}{\partial\beta jl}\phi(x^k,\Theta,\Lambda)=\alpha_{jl}\left[\sum_{i=1}^{n}p\left(j\middle|x_i^k,\alpha_j^k\right)(\ln x_{il})+\ [\psi(\alpha_{0j})-\psi(\alpha_{jl})]\sum_{i=1}^{n}p(j|x_i^k,\alpha_j^k)\right], \qquad (11)$$

$$l=1,\dots,k,\quad j=1,\dots,m.$$

By using the iterative scheme of the Fisher scoring method, we obtain

$$\begin{pmatrix}\hat\beta jl\\ \vdots\\ \hat\beta jk\end{pmatrix}^{new}=\begin{pmatrix}\hat\beta jl\\ \vdots\\ \hat\beta jk\end{pmatrix}^{old}+\begin{pmatrix}VAR(\hat\beta jl) & \dots & COV(\hat\beta jl,\hat\beta jk)\\ \vdots & \ddots & \vdots\\ COV(\hat\beta jk,\hat\beta jl) & \dots & VAR(\hat\beta jk)\end{pmatrix}^{old}\times\begin{pmatrix}\frac{\partial}{\partial\hat\beta_{jk}}\phi(x^k,\Theta,\Lambda)\\ \vdots\\ \frac{\partial}{\partial\hat\beta_{jl}}\phi(x^k,\Theta,\Lambda)\end{pmatrix}^{old} \qquad (12)$$

$$j=1,\dots,m.$$

Note that the variance-covariance matrix.is obtained by the inverse of the Fisher information matrix I and

$$\mathrm{I}=-E\left[\frac{\partial^2}{\partial\beta_{jl_1}\partial\beta_{jl_2}}\phi(x^k,\Theta,\Lambda)\right]. \qquad (13)$$

IV. Analysis and Results

In this chapter, the results of the simulation study on the basis of the entire criterion at different sample sizes are presented and Performance of parameter estimation method in terms of Bias as the sample size and parameter dimension varies was discussed.

Alpha	N	QLE	MLE	MOM
$\alpha`1 = 0.15$	10	0.2328	0.3125	**0.0994**
	20	0.1013	0.1848	**0.1**
	30	0.067	0.151	0.1
$\alpha`2 = 0.30$	40	0.0524	0.1371	0.0999
	50	0.0401	0.1247	0.1
	75	0.0286	0.1136	0.1
$\alpha`3 = 0.45$	100	0.0219	0.1074	0.1
	250	0.01	0.0964	0.1001

Notes

Table 1: Results of the Bias at different alpha level as sample size varies are presented below

Alpha	N	QLE	MLE	MOM
$\alpha^{\cdot}1 = 0.45$	10	0.4683	0.4771	1.1
	20	0.2223	0.2324	1.1
	30	0.1521	0.1638	1.1
$\alpha^{\cdot}2 = 0.75$	40	0.1072	0.1195	1.1
	50	0.0853	0.0977	1.1
	75	0.0615	0.0743	1.1
$\alpha^{\cdot}3 = 0.90$	100	0.0442	0.0567	1.1
	250	0.0235	0.0362	1.0999

Alpha	N	QLE	MLE	MOM
$\alpha^{\cdot}1 = 0.90$	10	0.6564	0.6576	1.889
	20	0.3141	0.3152	1.889
	30	0.185	0.1862	1.888
$\alpha^{\cdot}2 = 0.99$	40	0.1462	0.1476	1.8891
	50	0.1264	0.1277	1.8891
	75	0.0862	0.0876	1.8891
$\alpha^{\cdot}3 = 0.999$	100	0.0638	0.0652	1.889
	250	0.0324	0.0339	1.889

Notes

Based on the above table, it was observed that the level of alpha the Quasi-likelihood estimator (QLE) has the least value of Biasas the sample size increases and performs better compared with the Method of Moment and Maximum likelihood estimator.

Table 2: Results of the Variance at different alpha level as sample size varies are presented

Alpha	N	QLE	MLE	MOM
$\alpha^{\cdot}1 = 0.15$	10	0.108	0.1167	0.0325
	20	0.0355	0.0379	0.0158
	30	0.0187	0.02	0.0103
$\alpha^{\cdot}2 = 0.30$	40	0.0139	0.0148	0.0082
	50	0.0108	0.0116	0.0065
	75	0.0063	0.0067	0.0041
$\alpha^{\cdot}3 = 0.45$	100	0.0048	0.0051	0.0033
	250	0.0018	0.0019	0.0013

Alpha	n	QLE	MLE	MOM
$\alpha^{\cdot}1 = 0.45$	10	0.4132	0.4128	0.0207
	20	0.144	0.1444	0.0101
	30	0.0863	0.0858	0.0071
$\alpha^{\cdot}2 = 0.75$	40	0.0565	0.0563	0.0051
	50	0.043	0.0426	0.004
	75	0.0279	0.0277	0.0028
$\alpha^{\cdot}3 = 0.90$	100	0.0201	0.0199	0.0021
	250	0.0076	0.0075	0.0008

Alpha	N	QLE	MLE	MOM
$\alpha^{\cdot}1 = 0.90$	10	0.6887	0.6885	0.0173
	20	0.2398	0.2396	0.0086
	30	0.1286	0.1285	0.0056
$\alpha^{\cdot}2 = 0.99$	40	0.0926	0.0925	0.0043
	50	0.0772	0.0772	0.0034
	75	0.05	0.0498	0.0023
$\alpha^{\cdot}3 = 0.999$	100	0.0353	0.0353	0.0018
	250	0.0134	0.0134	0.0006

Based on the above table and appendix 2, it was observed that the performance of parameter estimation method in terms of Variance, as the sample size increases Method of Moment performs better compared with the Quasi-likelihood estimator and Maximum likelihood estimator.

Table 3: Results of the Mean Absolute Error (MAE) at different alpha level as sample sizes varies are presented below

Alpha	N	QLE	MLE	MOM
$\alpha^{\cdot}1 = 0.15$	10	0.3652	0.3982	0.2668
	20	0.2171	0.2484	0.1915
	30	0.1671	0.2003	0.1633
$\alpha^{\cdot}2 = 0.30$	40	0.1438	0.1792	0.1512
	50	0.1262	0.1626	0.1396
	75	0.0995	0.1396	0.1233
$\alpha^{\cdot}3 = 0.45$	100	0.0873	0.1292	0.1181
	250	0.052	0.104	0.1043

Notes

Notes

Alpha	n	QLE	MLE	MOM
α˙1 = 0.90	10	1.0665	1.0665	1.8917
	20	0.6602	0.6601	1.889
	30	0.4866	0.4865	1.889
α˙2 = 0.99	40	0.4212	0.4211	1.8891
	50	0.3827	0.3825	1.8891
	75	0.3059	0.3057	1.8891
α_3 = 0.999	100	0.2622	0.2622	1.889
	250	0.1588	0.1588	1.889

Alpha	n	QLE	MLE	MOM
α˙1 = 0.45	10	0.7774	0.7778	1.1001
	20	0.486	0.4859	1.1
	30	0.3818	0.3815	1.1
α˙2 = 0.75	40	0.3138	0.3147	1.1
	50	0.2748	0.2748	1.1
	75	0.2216	0.2225	1.1
α˙3 = 0.90	100	0.186	0.1863	1.1
	250	0.1157	0.1168	1.0999

Performance of parameter estimation method in terms of Mean Absolute Error (MAE) as the sample size increases, Maximum likelihood estimator performs better than Quasi-likelihood estimator and Method of moment as shown in table 3.

Table 4: Results of the *Mean Square Error (MSE)* at different alpha level as sample sizes varies are presented below

Alpha	N	QLE	MLE	MOM
α˙1 = 0.15	10	0.1335	0.1569	0.0375
	20	0.0401	0.0503	0.0199
	30	0.0207	0.0281	0.0144
α˙2 = 0.30	40	0.0152	0.0214	0.0122
	50	0.0115	0.017	0.0105
	75	0.0066	0.0112	0.0081
α_3 = 0.45	100	0.0051	0.0091	0.0071
	250	0.0018	0.0051	0.0052

Alpha	N	QLE	MLE	MOM
α_1 = 0.45	10	0.4945	0.4969	0.4532
	20	0.1624	0.1633	0.4421
	30	0.095	0.0957	0.439
α˙2 = 0.75	40	0.0608	0.0615	0.4371
	50	0.0457	0.0461	0.4363
	75	0.0293	0.0298	0.435
α˙3 = 0.90	100	0.0207	0.021	0.445
	250	0.0078	0.008	0.4328

Alpha	n	QLE	MLE	MOM
α˙1 = 0.90	10	0.8341	0.8342	1.2088
	20	0.273	0.273	1.2005
	30	0.1401	0.1401	1.1976
α˙2 = 0.99	40	0.0999	0.0999	1.1961
	50	0.0827	0.0827	1.1953
	75	0.0525	0.0525	1.1944
α˙3 = 0.999	100	0.0367	0.0367	1.1938
	250	0.0137	0.0137	1.1927

Performance of parameter estimation method in terms of Mean Square Error (MSE) as the sample size increases the Quasi- likelihood estimator and Maximum likelihood estimator performs better as compare to Method of Moment. as shown in table 4.

Conclusively, the best method for each criterion was based on the modal class for the entire criterion as summarized in table 5 below

Table 5: Shows the Count of Quasi- likelihood estimator, Maximum likelihood estimator and Method of Moment using Bias, Variance, Mean absolute error and Mean square error

		Count		Best Method
Criterion	**QLE**	**MLE**	**MOM**	
Bias	77	0	2	QLE
MSE	57	35	11	QLE
MAE	34	48	6	MLE
VAR	4	11	66	MOM

Graphical Reresentation of the Criterion

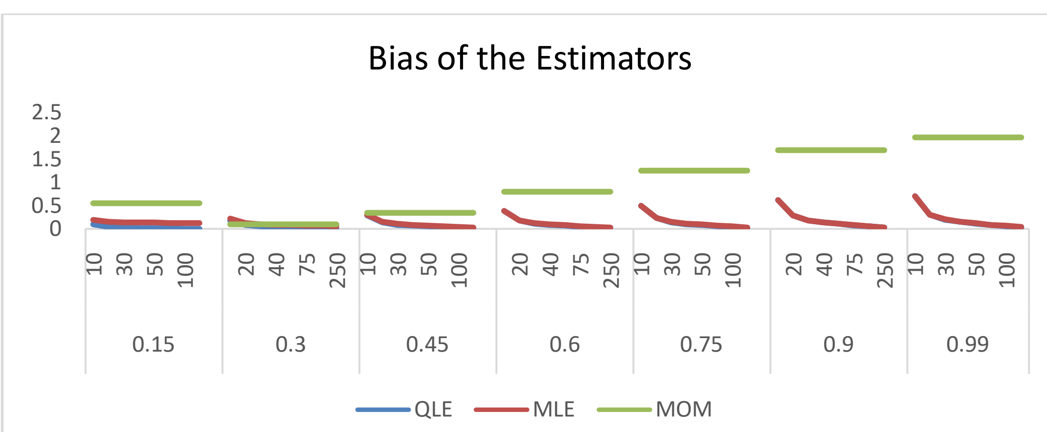

Fig. 1: The graph above shows the Bias of the estimator of QLE, MLE, MOM at different sample sizes

Notes

The graph shows the *Bias* of the estimator of QLE, MLE, MOM at different sample sizes. Judging by the bias criterion, the Quasi Likelihood method (QLE) was the best for the lower and medium level of alpha, but for the higher level of alpha, Method of moment performs better.

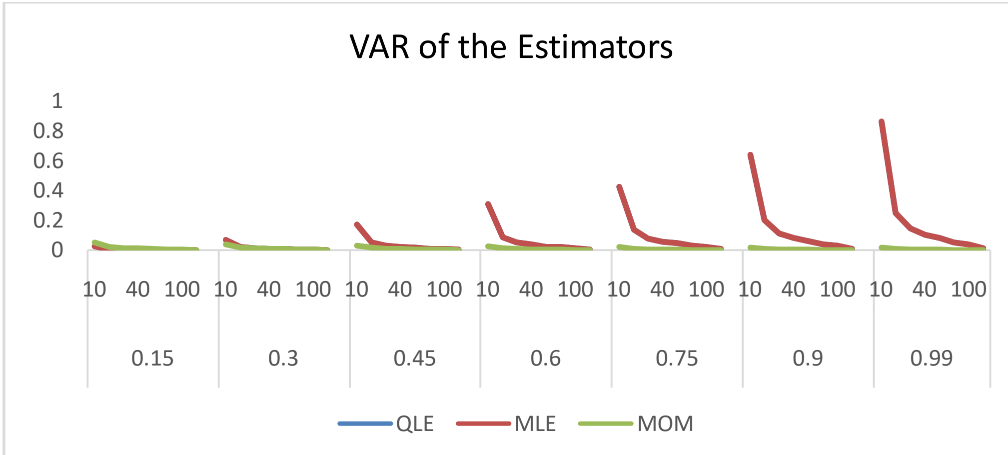

Fig. 2: The graph above shows the Variance of the estimator of QLE, MLE, MOM at different sample sizes

The graph shows the *Variance* of the estimator of QLE, MLE, MOM at different sample sizes. From the graph, Method of moment consistently performed better across the alpha(parameter) level which implies that the method of Moment is the best method.

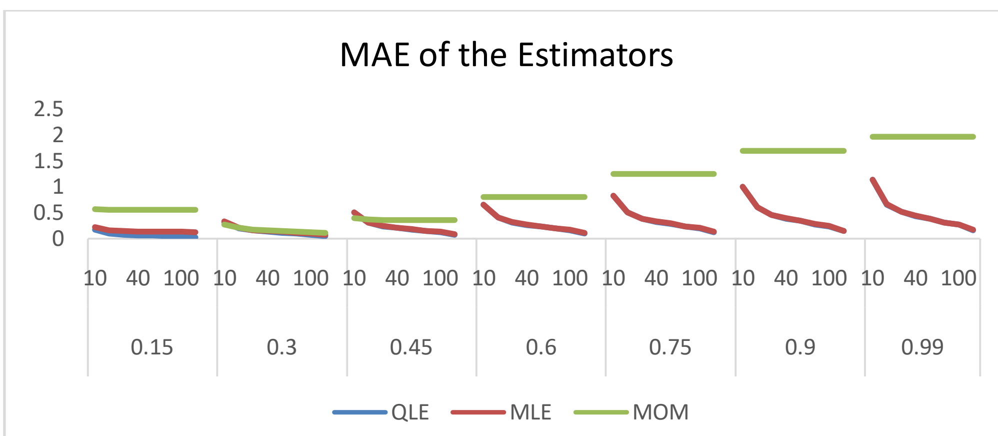

Fig. 3: The graph above shows the Mean Absolute Error (MAE) of the estimator of QLE, MLE, MOM at different sample sizes

The graph shows the *Mean Absolute Error (MAE)* of the estimator of QLE, MLE, MOM at different sample sizes. The Quasi Likelihood method (QLE) outperformed the other methods for lower level of alpha but as the alpha level increases (medium level and above) the Maximum Likelihood method (MLE) and the QLE has just a slight difference in their estimates. Out of the three methods considered, the method of Moment consistently gives the higher estimate of Mean Absolute Error.

Notes

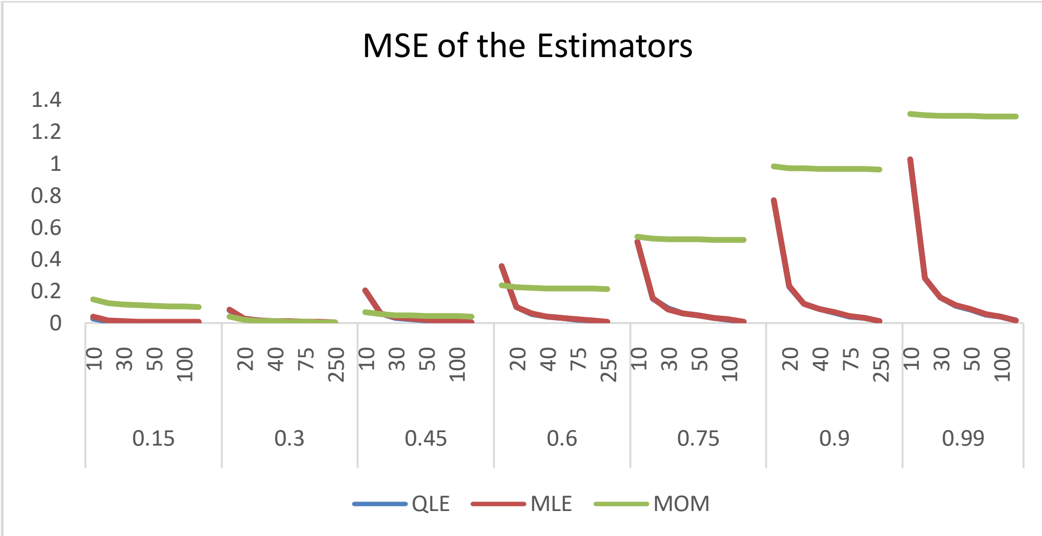

Fig. 4: The graph above shows the Mean Square Error (MSE) of the estimator of QLE, MLE, MOM at different sample sizes

The graph shows the Mean Square Error (MSE) of the estimator of QLE, MLE, MOM at different sample sizes. QLE method was the best for lower level of alpha, but as for the medium and higher level of alpha the QLE and the MLE does not give a significant different estimate. While on the other hand, the MOM gives a consistently higher estimate of Mean square error.

Real Life Data Results

Fitting Dirichlet Model to Data Containing Selected Agricultural Products in Nigeria (2008-2017).

Table 6: Parameter Estimation (QLE)

Coefficients	Estimate	Std. Error
α_1	0.2260442	0.1438496
α_2	0.1729296	0.1390305
α_3	0.2410169	0.1453349

From the table above, we obtained the estimates of the mean and standard error of Millet, millet rice, Sorghum. The parameter estimate of millet rice is more efficient due to its lowest standard error as compared to others.

Table 7: Parameter Estimation (MLE)

Coefficients	Estimate	Std. Error
α_1	0.03325620	0.455928
α_2	0.02179287	0.401919
α_3	0.03987597	0.471691

From the table above, we obtained the estimates of the mean and standard error of Millet, millet rice, Sorghum. The parameter estimate of millet rice is more efficient due to its lowest standard error as compared to others.

Notes

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Table 8: Parameter Estimation (MOM)

Coefficients	Estimate	Std. Error
α_1	0.04000534	0.4591688
α_2	0.01302276	0.309861
α_3	0.05461017	0.4750334

Notes

From the table above, we obtained the estimates of the mean and standard error of Millet, millet rice, Sorghum. The parameter estimate of millet rice is more efficient due to its lowest standard error as compared to others.

Conclusively, the quasi-likelihood estimator performs the best as compared to others.

V. Conclusion

The Dirichlet distribution is a multivariate generalization of the Beta distribution. In this research, we introduced three methods of estimation for Dirichlet distribution which are maximum likelihood estimator (MLE), Method of Moment (MOM) and Quasi-likelihood estimator. This was done in other to obtain the most efficient method. An extensive simulation study was carried out on the basis of selected criterion (Bias, Variance, Mean absolute error and Mean square error) considering various sample sizes, also the methods were subjected to real life data. The performance of these methods were compared at different sample sizes it shows that the Quasi-likelihood estimator performs better in terms of Bias, than the other methods, while Method of Moment performs better in terms of Variance, than the other methods. Maximum likelihood estimator performs better in terms of Mean Absolute Error (MAE) and (MSE) than the other methods. The real life result shows that Quasi-likelihood estimator performs better as compared to Method of moment and Maximum likelihood estimator, also the Bayes factor of Dirichlet distribution gives 57.95215, which implies a very strong evidence of the Goodness of fits. Hence, The Dirichlet distribution is efficient based on what we have done with higher precision and more adequacies in the estimate of the model, also the estimate of the model should be used in taking any prospective decision and can be reliable if large samples is involved.

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Univariate and Vector Autocorrelation Time Series Models for Some Sectors in Nigeria

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GJSFR-F Classification: *MSC 2010: 37M10*

UNIVARIATEANDVECTORAUTOCORRELATIONTIMESERIESMODELSFORSOMESECTORSINNIGERIA

Strictly as per the compliance and regulations of:

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AMADI, Godpower Dike [α], BIU, Oyinebifun Emmanuel [σ] & ARIMIE, Christopher Onyema [ρ]

Abstract- This work on univariate and Vector Autocorrelation (VAR) time series model for the sectors in Nigeria, aims at providing an in-depth quantitative analysis of the variables (Agriculture, Industry, Building & Construction, Wholesale & Retail trade and Services). The study made use of secondary data, of all the variables investigated in the model, collected from the National Bureau of Statistics' Statistical Bulletin (2018). The sample covers quarterly data from 1981 to 2018. Univariate and Multivariate time series estimation techniques – Autoregressive Integrated Moving Average (ARIMA) and Vector Autoregressive (VAR) were employed. Plots of the five sectors indicate that they all have Quadratic trend with appreciation and depreciation. Correlation analysis of the data set show that there exists a strong relationship among each variable. Each of the economic variables ARIMA model was built using Minitab 18 statistical software. Vector autoregressive (VAR) model was also obtained using Gretl statistical software. Two model selection criteria (AIC and BIC) were used to identify and select the suitable models. The identified ARIMA and VAR (2) models were used to make forecasts for the next 6 years for each of the variables. Furthermore, forecast accuracy measure and coefficient of variation (CV) were used to compare and identify the best model to forecast each of the variable. The result confirm that the best model to forecast the Agriculture and Building/Construction variables is the VAR(2) model; while Industry, Wholesale/Retail and Services variables was the ARIMA model [ARIMA(2,1,1)$(1,1,1)_4$, ARIMA(2,1,1)$(1,1,1)_4$and ARIMA(1,1,1)$(1,1,1)_4$].

Keywords: gross domestic product (GDP), vector autoregressive (VAR) model, ARIMA, forecast accuracy measure, model selection criteria.

I. Introduction

The analysis of time dependent variables is one of the methods designed for prediction of future events. Most variables are economical in nature and the economy of any nation partly depends on the interplay of these variables with respect to time. Indeed, time series plays a vital role in planning and predicting of future economy of any nation(Sani and Abdullahi, 2012). Nigeria Economy is not stable and for decades now, the country has been facing some economic crises, challenges or shocks both internally and externally. Due to the present unstable state of the economy resulting from fall in the oil price which Nigeria so much depend on for generating its internal Revenue (provides 75% of Nigeria's IGR) there is need to go back to Agriculture. However, focusing attention only to Agricultural sector may not solve the problem. There is need for Government to diversify by considering other sectors ofthe economy involved, in order to capture these variables along with Agricultural sector with the sole

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aim of studying their inter-relationship with respect to time and possibly develop a model for predicting the future of the various sectors under consideration.

The aim of this work is to build a suitable multivariate time series model (VAR Model) and univariate (ARIMA) Model for the sectors in Nigeria's Gross Domestic products (GDP). This would be achieved via the following specific objectives:(1) Obtain the correlation between the variables and stationarity of each variables, (2) Fit suitable ARIMA model of each variables and their forecasts, (3) Fit suitable VAR Model by using each variable as the dependent variable that can be used to predict future time series of the sectors in Nigeria's Gross Domestic products (GDP) and (4) To compare the forecasts of the two models for each variable. This work would help to improve macroeconomic policy formulation in Nigeria especially, by predicting future trend of output from major sectors(Agriculture, Manufacturing industries, Oil and Gas, Solid minerals, Transportation and General services).

Data for the five sectors under consideration in Nigeria's Gross Domestic products (GDP)were obtained from the National Bureau of Statistics. The data were quarterly data running from 1986-2018, making a total of 32 years.This work is limited to the sectors in Nigeria's Gross Domestic Product such as Agriculture, Industry, Building & Construction, and wholesale & Retail and Services. It did not consider other Nigeria's Economic variable such as Consumer Price Index, producer price index, crude oil production, etc.

II. Literature Review

Time series find application in Statistics, Signal Processing, Econometrics, Mathematical finance, Pattern recognition, Weather forecasting etc. Time Series analysis comprises methods for analyzing time series data in order to extract meaningful statistics and other characteristics of the data while time series forecasting involves use of model to predict future values based on previous observed values. In forecasting events that will occur in the future, a forecaster must rely on information concerning events that have occurred in the past. This is carried out by identifying a pattern that can be used to describe it. Believing that the identified pattern will continue to repeat itself in the future, this pattern is extended into the future in order to prepare a forecast. Methods of analysis of time series may be divided into Frequency Domain and Time domain. Frequency domain includes spectral analysis and Wavelet analysis. The latter include auto-correlation and cross correlation analysis.

Works on univariate and multivariate time series have been reported in the literatures.

Abdul and Marwan (2013), studied the effect of interest rate, inflation rate, GDP, on real economic growth rate in Jordan. Unit root test was employed to check the integration order of the variables. The results showed that inflation causes interest rate while all the other variables are independent of each other. Regression was also conducted to test the growth rate and interest rate which suggested that current interest rate has an influence on growth rate. Finally, it was shown that current GDP and one lag GDP have influence on growth rate.

Omoke (2010) investigated Inflation and Economic growth in Nigeria. Co-integration and Granger Causality was used to carry out the test. Consumer price index was used as proxy for inflation and GDP as perfect proxy for Economic growth. The results showed that there was a co- integration relationship between inflation and economic growth for the Nigerian data used. Also, Fadli (2011) noted that from

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empirical findings, Causality that runs from inflation to Economic growth indicates that inflation has an impact on growth.

Okororie (2012) used Buy-Balloutto model Nigeria Domestic Crude Oil production applying inverse square root transformation to stabilize the Variance.Quadratic trend was fitted and the error component was discovered to be random and normally distributed with mean zero and some constant Variance. Aminu *et al*., (2013) examined the effect of unemployment and inflation on economic growth in Nigeria. Augmented Dickey-fuller technique was used in testing the unit root property of the series. Also, Granger Casualty test of causation between GDP, Unemployment and inflation implied that all the variables in the model are stationary. The results revealed that unemployment and inflation impacted positively on economic growth. However, Nasiru and Solomon (2012) showed that unemployment does not affect economic growth.

Amos (2010) applied Time Series Modeling to South Africa Inflation data. Autoregressive integrated moving average (ARIMA) and Conditional heteroscedastic (ARCH) models were fitted to financial time series data. Box and Jenkins (1976) strategies were employed and the best fitted model for each family of model was selected. Onwukwe, and Nwafor (2014)used Multivariate Time Series model to study Major Economic Indicators in Nigeria. They obtained a stable Vector Autoregressive Model for the six economic variables. The result of the Granger causality analysis is unique enough, and there exist causality between variables. The Exchange rate as at the period of study reveals weak correlation which signifies the weak and devaluation of the Nigeria currency. Gross Domestic Product was seen as a good predictor to other economic indicators and the External Reverse. The relationship between these economic indicators is however significant and positive in either direction. This implies that the connection among these economic indicators and economic activities in Nigeria over some period of time is not automatic and the study also provides forecast value for the next two years from the last period of investigation.Other works reviewed which discussed the relationship among economic variables under consideration using univariate and multivariate times modeling techniques include Mphumuzi (2013), Francis and Charles (2012), Basutor (2014), Ruey (2013), Rokas (2012) and Abdulrasheed (2005). Whereas these works discussed co-integration, causality effects and relationships among the economic variables, our study highlights the most suitable model to forecast future performance of the five economic sectors with respect to the GDP.

III. Methods

This section explains the methods used in conducting this study and the reason(s)for choosing each method. Statistical tools used include: Autoregressive Integrated Moving Average (ARIMA), Stationary and Non-Stationary Time Series, Pre-whitening, Akaike Information Criterion (AIC), Schwartz-Bayesian Information Criteria (BIC) and Vector Auto regression (VAR).

a) *Vector Autoregressive (VAR) Model*

The objectives of univariate time series analysis is to find the dynamic dependence of Y_t that is the dependence of $Y_t$on its past values$Y_{t-1,}Y_{t-2,}Y_{t-3\ldots}$.A linear model implies that Y_t depends on its past values. The Vector Autoregressive (VAR) model is an approach in modeling dynamics among a set of variables. The approach usually focuses on the dynamic of multiple time series. Vector Autoregressive(VAR)

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model is also an independent reduced form dynamic model which involves constructing an equation that makes each endogenous variable a function of their own past values and past values of all other endogenous variables. The basic p-lag Vector autoregressive VAR(p) model has the form.

$$Y_t = C + \Pi_1 Y_{t-1} + \Pi_2 Y_{t-2} + \ldots + \Pi_p Y_{t-p} + \varepsilon_t \quad t = 1, \ldots, T. \tag{3.1}$$

where

$Y_t = (y_{1t}, y_{2t}, \ldots y_{nt})$ is an $(n \times n)$ vector of time series variable

$\Pi = (n \times n)$ coefficient matrices

ε_t is an $(n \times 1)$ unobserved zero mean with white noise vector process(serially uncorrelated and independent) with invariant covariance matrix Σ

The model can be written in the matrix form as

$$\begin{pmatrix} y_{1t} \\ y_{2t} \\ . \\ . \\ y_{nt} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix} + \begin{pmatrix} \pi_{11}^1 \, \pi_{12}^1 \cdots \pi_{1n}^1 \\ \pi_{21}^1 \, \pi_{22}^1 \cdots \pi_{2n}^1 \\ . \quad . \quad \cdots \quad . \\ . \quad . \quad \cdots \quad . \\ \pi_{n1}^1 \pi_{n2}^1 \cdots \pi_{nn}^1 \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \\ . \\ . \\ y_{nt-1} \end{pmatrix} + \begin{pmatrix} \pi_{11}^2 \, \pi_{12}^2 \cdots \pi_{1n}^2 \\ \pi_{21}^2 \, \pi_{22}^2 \cdots \pi_{2n}^2 \\ . \quad . \quad \cdots \quad . \\ . \quad . \quad \cdots \quad . \\ \pi_{n1}^2 \pi_{n2}^2 \cdots \pi_{nn}^2 \end{pmatrix} \begin{pmatrix} y_{1t-2} \\ y_{2t-2} \\ . \\ . \\ y_{nt-2} \end{pmatrix} + \cdots +$$

$$\begin{pmatrix} \pi_{11}^p \, \pi_{12}^p \cdots \pi_{1n}^p \\ \pi_{21}^p \, \pi_{22}^p \cdots \pi_{2n}^p \\ . \quad . \quad \cdots \quad . \\ . \quad . \quad \cdots \quad . \\ \pi_{n1}^p \pi_{n2}^p \cdots \pi_{nn}^p \end{pmatrix} \begin{pmatrix} y_{1t-1} \\ y_{2t-1} \\ . \\ . \\ y_{nt-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ . \\ . \\ \varepsilon_{nt} \end{pmatrix} \tag{3.2}$$

The data is a secondary source obtained from National Bureau of Statistics for the five sectors under consideration in Nigeria's Gross Domestic products (GDP). The data is quarterly data running from 1986-2018 making a total of 32 years.

b) Stationary and non-stationary time series

A time series is said to be stationary if the statistical property e.g. the mean and variance are constant through time. If we have n values of observations $x_1, x_2, x_3, \ldots x_n$ of a time series, we can plot this values against time to help us determine if the time series is stationary. If the n-values fluctuate with constant variation around a constant mean μ, then we can say that the time series is stationary and all processes which do not possess, this property are called "non-stationary". A non-stationary time series can be made stationary by transforming the time series into series of stationary time series value (differencing).

The general linear process allows us to represent X_t as the weighted sum of present and paste values of the white noise ε_t. ε_t can be represented as

$$X_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \ldots$$

$$= \varepsilon_t + \sum_{j=1}^{\infty} \psi_j \varepsilon_{t-j}$$

$$= \sum_{j=o}^{\infty} \psi_j \varepsilon_{t-j} \quad where\ \psi_0 = 1 \tag{3.3}$$

Thus, white noise ε_t consist of a sequence of uncorrelated random variable with zero mean and constant variance. That is

$$E[\varepsilon_t] = 0 \qquad Var[\varepsilon_t] = \delta^2$$

c) Mixed Autoregressive moving average (ARMA) model

Notes

According to Box and Jenkins (1970), mixed autoregressive moving average model is the combination of MA(q) and AR(p). Let $X_t$be the deviation from the mean μ, the ARMA(p, q) model can be written as

$$x_t - \phi x_{t-1} - \phi_2 x_{t-2} - \cdots - \phi_p x_{t-p} = \varepsilon_t - \theta\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \cdots - \theta_q\varepsilon_{t-q}.$$

Thus,

$$\phi(B)x_t = \theta(B)\varepsilon_t \tag{3.4}$$

The equation (2.9) can be written as

$$x_t = \phi^{-1}(B)\theta(B)\varepsilon_t$$

$$= \frac{\theta(B)}{\phi(B)}\varepsilon_t = \frac{1-\theta_1 B - \cdots - \theta_q B^q}{1-\phi_1 B - \cdots \phi_p B^P}\varepsilon_t \tag{3.5}$$

ARIMA model are resultant time series obtained if a non stationary time series which has variation in mean is differenced to get rid of the variation. The resultant series becomes stationary after differencing. The word “integrated” is used whenever differencing is applied to achieve stationarity. The ARIMA model is based on prior values in autoregressive term and the error made by previous prediction. The order of ARIMA model is given by p, d, q where,p represents the autoregressive component, $d$stands for the differencing to achieve stationarity and$q$ is the order of the moving average.

d) Seasonal Autoregressive integrated moving average.

Seasonal autoregressive integrated moving average (SARIMA) model is used for time series with seasonal and non-seasonal behavior. For ARIMA model, the autoregressive part deals with past observations while the moving average is concern with the errors associated with the series. The order of ARIMA model is $p, d, and\ q$ where,$p, d, and\ q$ are whole numbers ≥ 0.A process y_t is said to be an ARIMA model with parameter $p, d, and\ q$ if $\nabla^d y_t$ is shown by a stationary ARMA(p, q)model where ∇ stands for the difference operator. Thus, we write

$$\phi(B)\nabla^d z_t = \theta(B)\varepsilon_t \tag{3.6}$$

SARIMA model has multiplicative and additive part. The multiplicative is so applied because of the assumption that there exist a significant parameter resulting from the multiplication between non seasonal parameters. By the use of ∇ and B notation, ARIMA(p, d, q) model can be written as

$$\phi(B)w_t = \theta(B)\varepsilon_t \tag{3.7}$$

where the polynomial in B is given as

$$\phi(B) = 1 - \phi_1(B) - \cdots - \phi_p B^p$$

$$\theta(B) = 1 - \theta_1(B) - \cdots - \theta_q B^p$$

For further purpose of this study, we shall be focusing on multiplication model because of the assumptions that there is a major parameter because of the multiplication between the non-seasonal and seasonal model. This is denoted by $ARIMA\ (p, d, q) \times (P, D, Q)$ written as

$$\phi_p(B)\phi_P(B^s)\nabla^d\nabla_s^D z_t = \theta_q(B)\theta_Q(B^s)\varepsilon_t \quad (3.8)$$

Notes

where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots \phi_p B^p$$

$$\Phi(B) = 1 - \Phi_1, sB^s - \Phi_2, sB^{2s} - \cdots - \Phi_p, sB^p$$

$$\nabla^d = 1 - B - B^2 - \cdots - B^d$$

$$\nabla_s^D = 1 - B^s - B^{2s} - \cdots - B^{2D}$$

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q$$

$$\Theta(B^s) = 1 - \Theta_1, sB^s - \Theta_2, sB^{2s} - \cdots - \Theta_Q, sB^{QS}$$

where z_t is the time series at period t, ε_t stands for the white noise, B represents the backshift operator, S is the duration of the seasonal model which could be weekly, quarterly or yearly, p=Autoregressive parameter, P=Seasonal autoregressive parameter, d=order of the monthly difference(quarterly difference), D=order of seasonal difference, q=moving average parameter and Q=seasonal moving average parameter.

e) *Pre-whitening*

This refers to fitting an ARIMA model for the input series sufficient to reduce the residual to white noise, then fitter the input series with this model to get white noise residual series, then filter the response series with the same model and cross-correlate the filtered response with filtered input series.

f) *Model selection criteria*

Before an ARMA(p, q) may be estimated for a time series $Y_{t,}$ the AR and MA order p and q have to be determined by examining the SACF and SPACF for $Y_{t..}$ The idea is to fit all ARMA(p q) models with order $p \leq p_{max}$and $q \leq q_{max}$ and choose the value of p and q which minimizes some model selection criteria. For ARMA(p, q), the model selection criteria (MSC) is given by

$$MSC(p, q) = Ln\left(\sigma^2(p, q)\right) + c_T.\varphi(p, q) \quad (3.9)$$

where $\sigma^2(p, q)$ is the MLE of $\mathrm{var}(\varepsilon_t) = \sigma^2$ without a degree of freedom correction from the ARMA(p q) model, c_T is a sequence indexed by the sample size T, and $\varphi(p, q)$ is a penalty function that penalizes large ARMA(p, q) models.

g) *Information Criteria*

The two most common information criteria are the AkaikeInformation Criteria (AIC) and the Schwarz-Bayesian(BIC).

Notes

i. *Akaike Information Criteria*

This was developed by Hirotugu Akaike under the name "Information criteria". The AIC is a measure of the relative goodness of fit of a statistical model. It was first published by Akaike in 1974.It tends to describe the exchange between bias and variance in model specification for a data set. Individual model could be ranked based on their AIC value. Thus

$$AIC = -2InL(\theta) + 2K \tag{3.10}$$

where, $L(\theta)$ is the value of the likelihood function and$K$ is the number of parameters in the model.

ii. *Schwartz-Bayesian Information Criteria (SBIC or BIC)*

This is model selection criteria that involves selections among finite set of models. Whenever models are fitted, it is necessary sometimes to increase the likelihood by adding parameters but sometimes adding parameters may result to over fitting which can be resolved by introducing a penalty term for the number of parameter in the model, BIC is given by

$$BIC = -2InL(\theta) + KIn(N) \tag{3.11}$$

where, $L(\theta)$ is the value of the likelihood function evaluated,

K is the sum number of parameters estimated and N is the number of usable observation

Procedures for developing linear time series by Box and Jenkins (1976) proposed four steps in developing a linear time series which are Model identification, Estimation of parameters, Diagnostic Checking and Forecasting.

h) Accuracy Measures of the Estimated Values

To gauge the accuracy of our estimates, the estimated errors will be used to compare the two methods forecasts enumerated in the objective. This is done by subtracting the estimated forecast values (EFV) from the original values or [actual values (AV)] to obtain the estimate errors. The estimate error is denoted by

$$e_i = \mathrm{AV}_i - EFV_i \text{ , } i = 1 \text{ , } 2, \ldots v \tag{3.12}$$

where, v is the number of forecast values.

In addition, the accuracy measures are: Mean Error (ME), Mean Absolute Error (MAE), Mean Squared Error (MSE), Root Mean Squared Error (RMSE) and Mean Absolute Percentage Error (MAPE). However, this study will only consider three out of the five accuracy measures which are: Mean Error (ME), Mean Squared Error (MSE) and Mean Absolute Percentage Error (MAPE).

i. *Mean Error (ME)*

The first descriptive Statistics of Error used is called the Mean Error. It indicates the deviation between the actual values and estimates, Mean Error is given as

$$\mathrm{ME} = \left[\frac{1}{v}\sum_{i=1}^{v} e_i\right] \tag{3.13}$$

ii. *Mean Squared Error (MSE)*

MSE also indicates the fluctuations of the deviations and it can be calculated as

$$\text{MSE} = \left[\frac{1}{v}\sum_{i=1}^{v} e_i^2\right] \quad (3.14)$$

iii. *Mean Absolute Percentage Error (MAPE)*

This accounts for the percentage of deviation between the actual values and estimates. This can be obtained as

$$\text{MAPE} = 100 \times \left[\frac{1}{v}\sum_{i=1}^{v}\left|\frac{e_i}{AV_i}\right|\right] \quad (AV_i \neq 0) \quad (3.15)$$

Notes

i) Coefficient of Variation

To be able to identify the suitable model to use to forecast each of the variables, we will introduce the coefficient of variation. The coefficient of variation (CV) is a statistical measure of the dispersion of data points in a data series around the mean. The coefficient of variation represents the ratio of the standard deviation to the mean, and it is a useful statistic for comparing the degree of variation from one data series to another, even if the means are drastically different from one another. The coefficient of variation shows the extent of variability of data in a sample in relation to the mean of the population. In finance, the coefficient of variation allows investors to determine how much volatility, or risk, is assumed in comparison to the amount of return expected from investments. Ideally, the coefficient of variation formula should result in a lower ratio of the standard deviation to mean return, meaning a better risk-return trade-off. Note that if the expected return in the denominator is negative or zero, the coefficient of variation could be misleading.

This can be obtained by;

$$\text{CV} = \frac{\sigma}{\mu} \quad (3.16)$$

where, σ is the standard deviation and ` is the mean.

IV. Results Presentation and Analysis

The data used in this research were quarterly Gross Domestic Product (GDP) and some Nigeria Sectors Current Basic Prices [Quarterly (N' Billion)] collected from the 2018 Statistical Bulletin Real Sector Statistics (SBRS) for the period of 1981-2018 making a total of 37 years (Appendix A). Results of analysis of the data are presented in the following subsections. Minitab statistical software version 17 was used for the analysis.

a) Time Series Plots and Correlation Analysis

The five variables were plotted in the Figures 4.1 to 4.5 below. The correlation matrix in Table 4.1 shows that there is strong relationship among all the variables including the GDP.

Notes

Table 4.1: Correlation Matrix between the variables

	Agriculture	Industry	Building Construction	Wholesale Retail Trade	Services	GDP
Agriculture	1					
Industry	0.953431	1				
Building Construction	0.926086	0.896342	1			
Wholesale Retail Trade	0.960794	0.921519	0.988916	1		
Services	0.942882	0.897907	0.993628	0.993051	1	
GDP	0.979138	0.951794	0.980361	0.994021	0.987319	1

Figures 4.1 to 4.5 show an upward trend in all the sectors examined with respect to the GDP series and the trend appear to be quadratic since they all appreciate from the first year to the last year (or curve shaped).

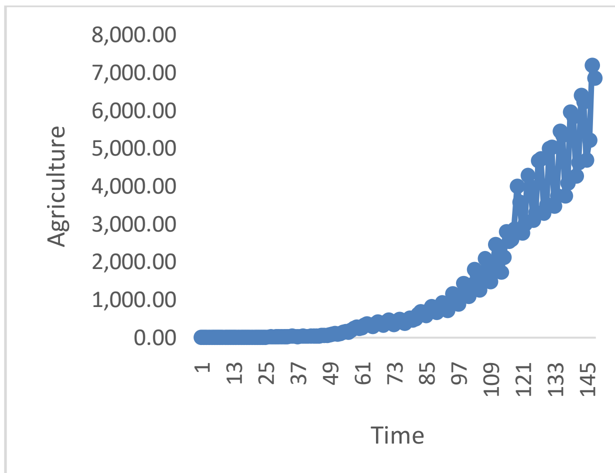

Figure 4.1: Agriculture Series Plot

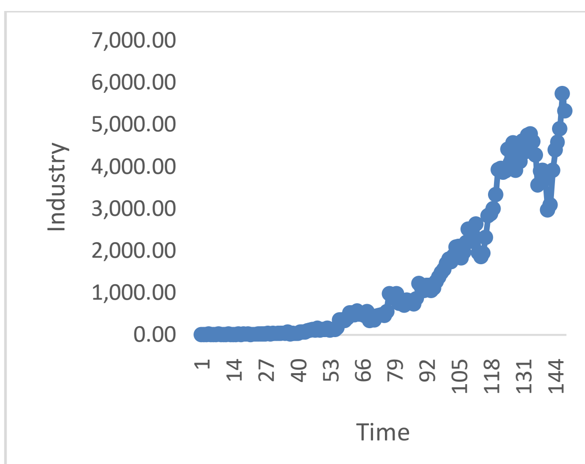

Figure 4.2: Industry Series Plot

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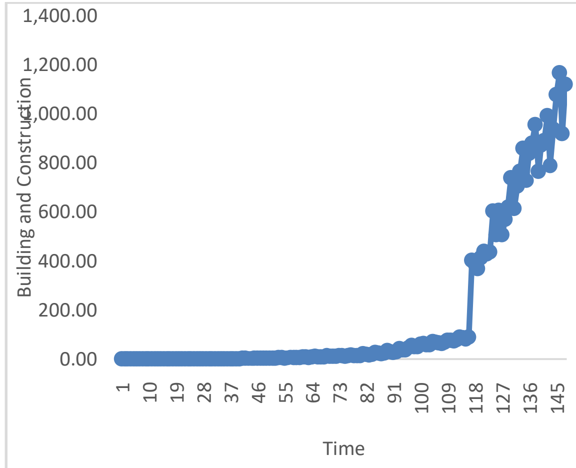


Figure 4.3: Building & Construction Series Plot

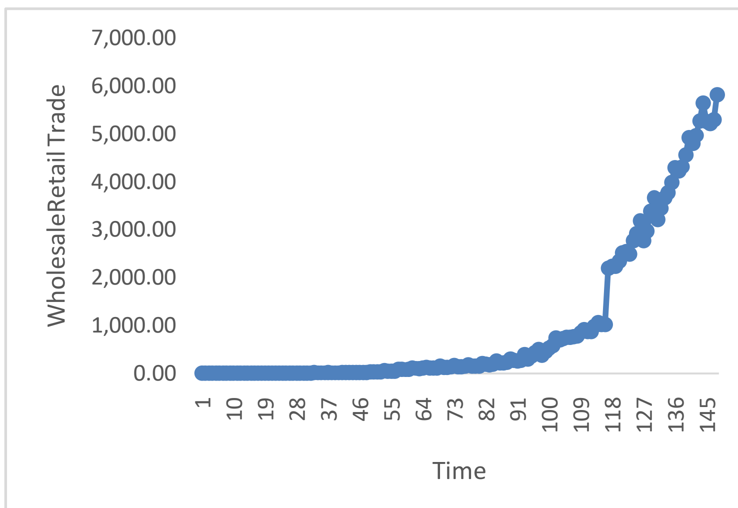


Figure 4.4: Wholesale Retail TradeSeries Plot

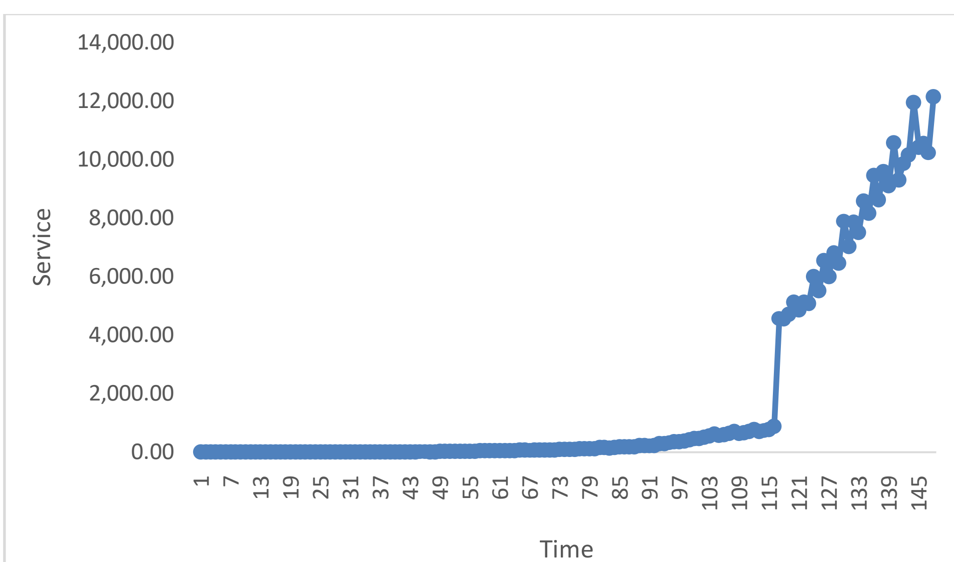


Figure 4.5: Service Series Plot

Notes

Notes

b) *Univariate Time Series Analysis*

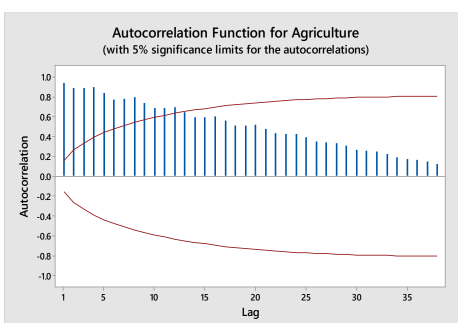
The ACF plot for Agriculture suggest an Autoregressive (AR) process while the PACF plot shows that the data is AR(5) with little sign of seasonality as shown in fig 4.8 and fig 4.9 respectively. The ACF plot for Industry suggest an Autoregressive (AR) process while the PACF plot shows that the data is AR(3) with little sign of seasonality as shown in fig 4.10 and fig 4.11 respectively. The ACF plot for Building & Construction suggest an Autoregressive (AR) process while the PACF plot shows that the data is AR(4) with little sign of seasonality in fig 4.12 and fig 4.13 respectively. The ACF plot for Wholesale and Retail suggest an Autoregressive (AR) process while the PACF plot shows that the data is AR(1) as shown in fig 4.15 and 4.16 respectively. The ACF plot for Service suggest an Autoregressive (AR) process while the PACF plot shows that the data is AR(2) as shown in fig 4.17 and 4.18 respectively. The ACF plot for Gross Domestic Product (GDP) suggest an Autoregressive (AR) process while the PACF plot shows that the data is AR(1) as shown in fig 4.19 and 4.20 respectively.

i. *Univariate Time Series Plots for the Differenced Data*

The plots of the ACF, PACF in figure 4.8 to 4.20 of the six variables shows that the series are not stationary and hence, need to be differenced to attain stationarity before fitting the ARIMA models.Figure 4.21 shows that the movement of the plot is sinusoidal (sine wave pattern) after first differencing indicating that the series is now stationary with mean zero and constant variance. The ACF and PACF plots in Figures4.22 and 4.23 respectively indicate the presence of seasonarity of order 2 and also have both AR(2) and MA(2) process in it.The difference series, ACF and PACF Plots show that the variables are now stationary with a seasonality of order 2.

ii. *ARIMA Model Parameter Estimate and Time Series Modelling*

In this section we show several fitted ARIMA models with the results of ACF and PACF correlogram. Also shown are the computed model residual sum of square, AIC and BIC as discussed in chapter three of this work. The parameter estimates and p-value of univariate time series model (ARIMA) are summarized for each variable as follows (Tables 4.2 to 4.6).



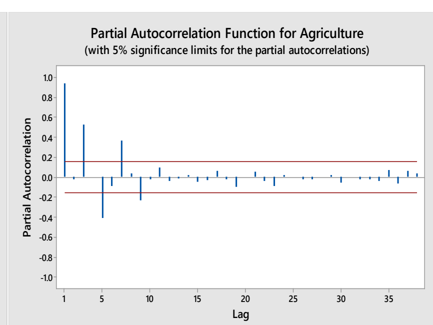
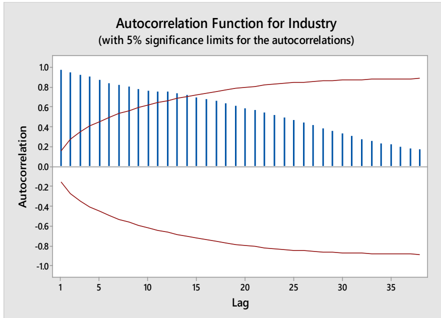


Figure 4.8: ACF Plot for Agriculture *Figure 4.9:* PACF Plot for Agriculture

Notes



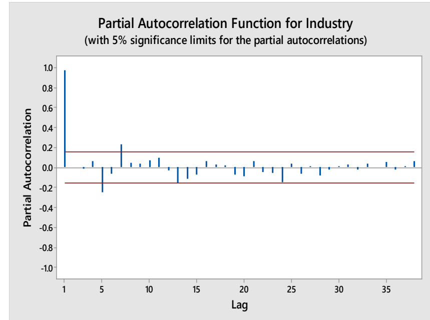
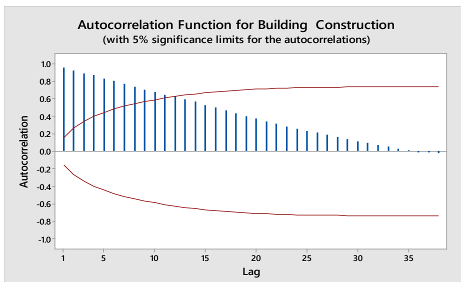


Figure 4.10: ACF Series Plot for Industry *Figure 4.11:* PACF Plot for Industry



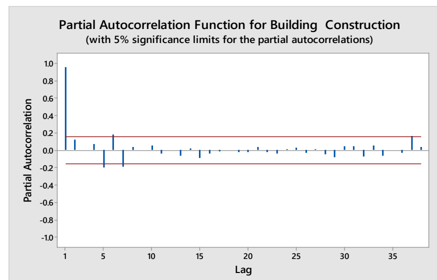
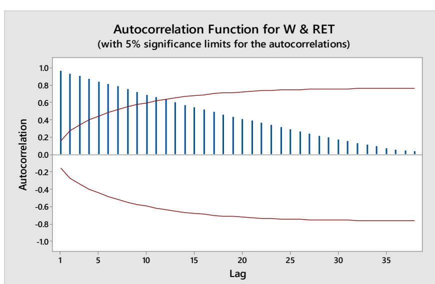


Figure 4.12: ACF Plot for Building & Construction

Figure 4.13: PACF Plot for Building & Construction



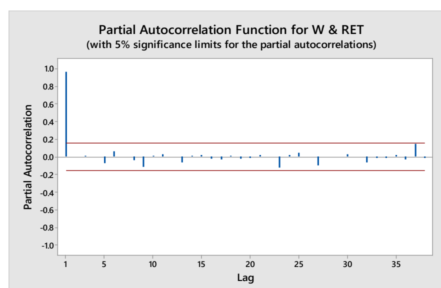
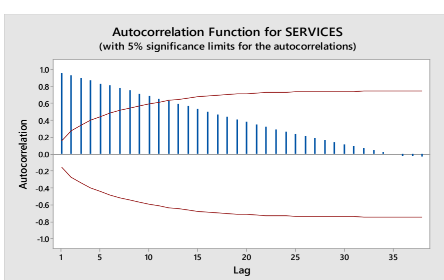


Figure 4.15: ACF Plot for Wholesale & Retail

Figure 4.16: PACF Plot for Wholesale & Retail



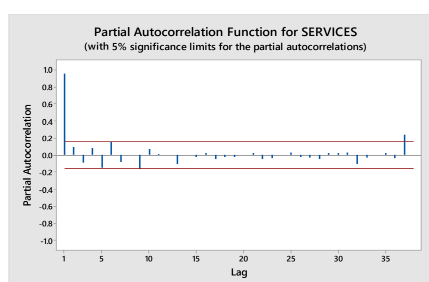


Figure 4.17: ACF Plot for Services *Figure 4.18:* PACF Plot for Services

Notes

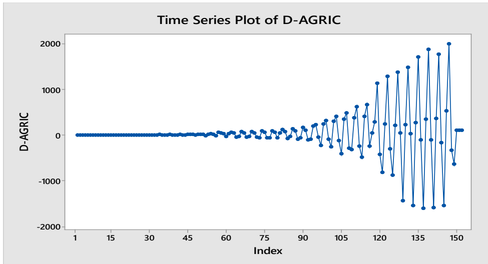


Figure 4.21: Series Plot for Differenced Agriculture

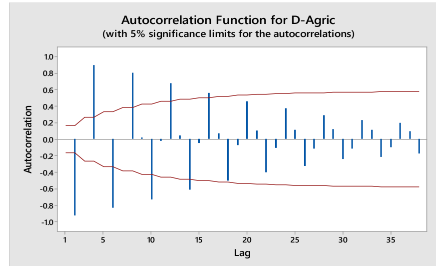


Figure 4.22: ACF for Differenced Agric Plot

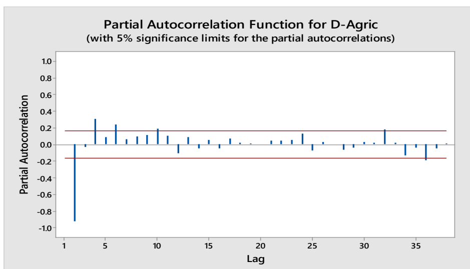


Figure 4.23: PACF for Differenced Agric Plot

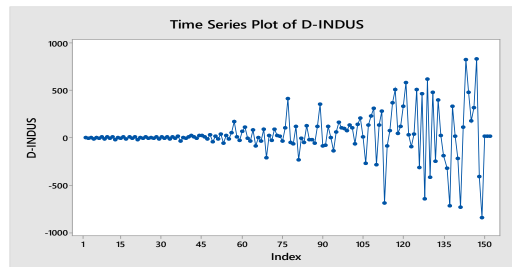


Figure 4.24: Series Plot for Differenced Industry

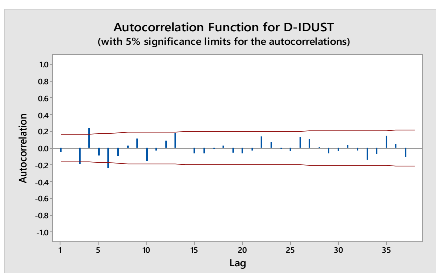


Figure 4.25: ACF for Differenced Industry Plot

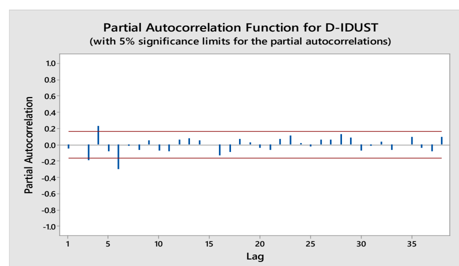


Figure 4.26: PACF for Differenced Industry Plot

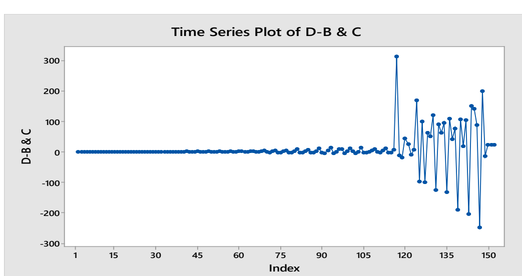


Figure 4.27: Series Plot for Differenced Building & Construction

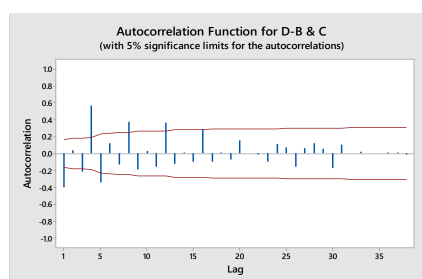


Figure 4.28: ACF for Differenced B & Con Plot

Notes

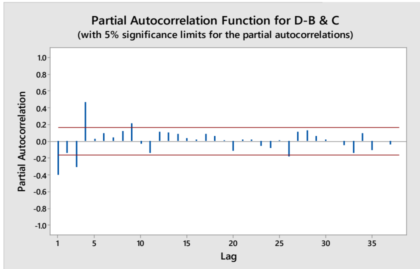


Figure 4.29: PACF for Differenced B & Con Plot

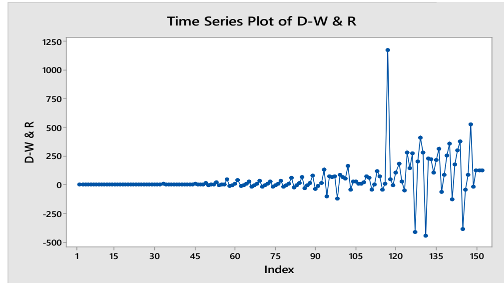


Figure 4.30: Series Plot for Differenced Wholesale & Retail

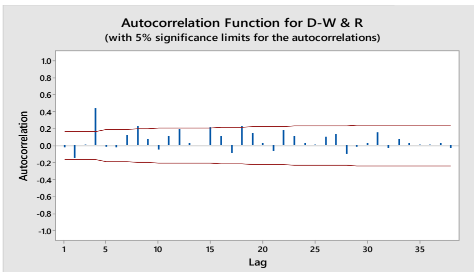


Figure 4.31: ACF for Differenced W & Retail Plot

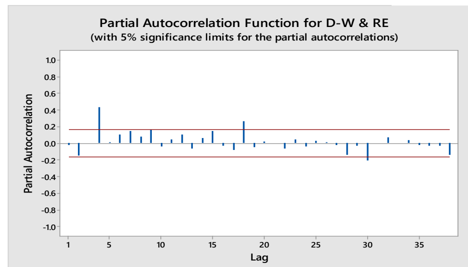


Figure 4.32: PACF for Differenced W & Retail Plot

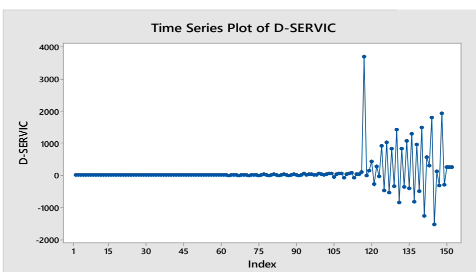


Figure 4.33: Series Plot for Differenced Services

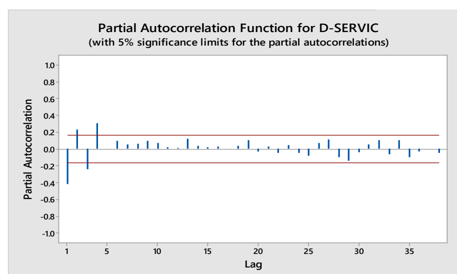


Figure 4.34: ACF for Differenced Services Plot

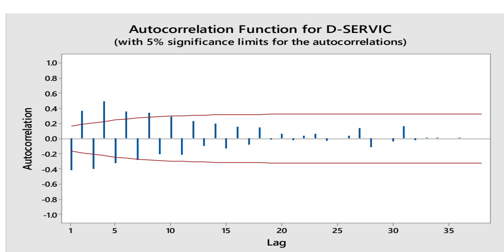


Figure 4.35: PACF for Differenced Services Plot

Notes

Table 4.2: Univarite Model for Agricultural variable

Model	Regular AR(p) Estimates		Regular MA(q) Estimates		Seasonal AR(p) Estimates		Seasonal MA(q) Estimates		K	RSS	AIC	Rank	BIC	Rank
	ϕ_1	ϕ_2	θ_1	θ_2	$\phi_{1,4}$	$\phi_{2,4}$	$\theta_{1,4}$	$\theta_{2,4}$						
ARIMA $(1,1,1)(1,1,1)_4$	0.4886 (0.000***)		0.9269 (0.000***)		-0.8870 (0.000***)		-0.5997 (0.014**)		4	5262373	1549.39	1	1561.35	1
ARIMA $(0,1,1)(1,1,?\ 1)_4$			-0.4479 (0.000***)		-0.8812 (0.000***)		-0.4844 (0.029**)		3	6122286	1569.64	2	1578.61	2

*Footnote: ***-sig. at 1%, **-sig. at 5%,*-sig. at 10%,*

Table 4.3: Univariate Model for industry Variable

Model	Regular AR(p) Estimates		Regular MA(q) Estimates		Seasonal AR(p) Estimates		Seasonal MA(q) Estimates		k	RSS	AIC	Rank	BIC	Rank
	ϕ_1	ϕ_2	θ_1	θ_2	$\phi_{1,4}$	$\phi_{2,4}$	$\theta_{1,4}$	$\theta_{2,4}$						
ARIMA $(2,1,1)(0,1,1)_4$	-0.5247 (0.234)	0.1340 (0.205)	-0.5129 (0.233)				0.7943 (0.000***)		4	8305152	1616.47	1	1628.43	1
ARIMA $(2,1,1)(2,1,1)_4$	-0.4365 (0.255)	0.1875 (0.066*)	-0.4195 (0.268)		-0.3633 (0.082*)	-0.3981 (0.003**)	0.1999 (0.386)		6	8215884	1618.88	2	1636.82	2

*Footnote: ***-sig. at 1%, **-sig. at 5%,*-sig. at 10%,*

Table 4.4: Univariate Model for Building & Construction Variable

Model	Regular AR(p) Estimates		Regular MA(q) Estimates		Seasonal AR(p) Estimates		Seasonal MA(q) Estimates		k	RSS	AIC	Rank	BIC	Rank
	ϕ_1	ϕ_2	θ_1	θ_2	$\phi_{1,4}$	$\phi_{2,4}$	$\theta_{1,4}$	$\theta_{2,4}$						
ARIMA $(1,1,1)(2,1,1)_4$	0.3879 (0.003**)		0.8084 (0.000***)		-1.2593 (0.000***)	-0.4402 (0.000***)	-0.9285 (0.000***)		5	294377	1127.52	1	1142.47	1
ARIMA $(2,1,1)(2,1,1)_4$	-0.3794 (0.000***)	-0.4269 (0.000***)	-0.9870 (0.000***)		-1.3543 (0.000***)	-0.5217 (0.000***)	-0.9233 (0.000***)		6	297477	1131.06	2	1149.00	2

*Footnote: ***-sig. at 1%, **-sig. at 5%,*-sig. at 10%,*

Notes

Table 4.5: Univariate Model for Wholesale & Retail Variable

Model	Regular AR(p) Estimates		Regular MA(q) Estimates		Seasonal AR(p) Estimates		Seasonal MA(q) Estimates		K	RSS	AIC	Rank	BIC	Rank
	ϕ_1	ϕ_2	θ_1	θ_2	$\phi_{1,4}$	$\phi_{2,4}$	$\theta_{1,4}$	$\theta_{2,4}$						
ARIMA $(2,1,1)(1,1,1)_4$	0.4280 (0.229)	-0.1569 (0.146)	0.5335 (0.135)		0.2934 (0.042*)		0.8336 (0.000***)		5	2633258	1449.62	1	1464.57	1
ARIMA $(1,1,1)(2,1,1)_4$	0.6748 (0.000***)		0.9064 (0.000***)		-1.2722 (0.000***)	-0.2853 (0.007**)	-0.9506 (0.000***)		5	2990481	1468.32	2	1483.27	2

*Footnote: ***-sig. at 1%, **-sig. at 5%,*-sig. at 10%,*

Table 4.6: Univariate Model for Service Variable

Model	Regular AR(p) Estimates		Regular MA(q) Estimates		Seasonal AR(p) Estimates		Seasonal MA(q) Estimates		k	RSS	AIC	Rank	BIC	Rank
	ϕ_1	ϕ_2	θ_1	θ_2	$\phi_{1,4}$	$\phi_{2,4}$	$\theta_{1,4}$	$\theta_{2,4}$						
ARIMA $(1,1,0)(2,1,1)_4$	-0.2170 (0.013*)				0.3164 (0.056*)	0.1865 (0.153)	0.9099 (0.000***)		4	27398974	1791.93	2	1803.89	2
ARIMA $(1,1,1)(1,1,1)_4$	0.5163 (0.033*)		0.2458 (0.000***)		0.7332 (0.083*)		0.8101 (0.000***)		4	27214710	1790.94	1	1802.90	1

*Footnote: ***-sig. at 1%, **-sig. at 5%,*-sig. at 10%,*

From Tables 4.2 to 4.6, the identified model for the sectors are ARIMA$(1,1,1)(1,1,1)_4$ for Agricultural sector, ARIMA$(2,1,1)(0,1,1)_4$ for Industrial sector ARIMA$(1,1,1)(2,1,1)_4$ for Building and Construction sector ARIMA$(2,1,1)(1,1,1)_4$ for Wholesale and Retail sector and ARIMA$(1,1,1)(1,1,1)_4$ for Service sector.
Mathematically, the models can be expression as

$$\nabla\nabla_s^4(1-\phi_1 B)(1-\phi_{1,4}B^4)X_t = (1+\theta_1 B)(1+\theta_{1,4}B^4)e_t$$ for Agricultural sector

$$\nabla\nabla_s^4(1-\phi_1 B-\phi_{1,2}B^2)(1-\phi_{1,4}B^4)X_t = (1+\theta_1 B)(1+\theta_{1,4}B^4)e_t$$ for Industrial sector

$$\nabla\nabla_s^4(1-\phi_1 B)(1-\phi_{1,4}B^4-\phi_{2,4}B^8)X_t = (1+\theta_1 B)(1+\theta_{1,4}B^4)e_t$$ for Building and Construction sector

$$\nabla\nabla_s^4(1-\phi_1 B-\phi_{1,2}B^2)(1-\phi_{1,4}B^4)X_t = (1+\theta_1 B)(1+\theta_{1,4}B^4)e_t$$ for Wholesale and Retail sector

$$\nabla\nabla_s^4(1-\phi_1 B)(1-\phi_{1,4}B^4)X_t = (1+\theta_1 B)(1+\theta_{1,4}B^4)e_t$$ for Services sector.

Hence, substituting the values of the parameters becomes,

$$\nabla\nabla_s^4(1-0.4886B)(1+0.8870B^4)X_t = (1+0.9269B)(1-0.5997B^4)e_t$$ for Agricultural sector

$$\nabla\nabla_s^4(1+0.5247B-0.1340B^2)X_t = (1-0.5129B)(1+0.7943B^4)e_t$$ for Industrial sector

$$\nabla\nabla_s^4(1-0.3879B)(1+1.2593B^4+0.4402B^8)X_t = (1+0.8084B)(1-0.9285B^4)e_t$$ for Building and Construction sector

$$\nabla\nabla_s^4(1-0.4280B+0.1569B^2)(1-0.2934B^4)X_t = (1+0.5335B)(1+0.8336B^4)e_t$$ for Wholesale and Retail sector

$$\nabla\nabla_s^4(1-0.5163B)(1-0.7332B^4)X_t = (1+0.2458B)(1+0.8101B^4)e_t$$ for Services sector

c) Vector Autocorrelation (VAR) Model for each Variable

Multivariate analyses of the different sectors were done, with each variable placed as dependent against the others and VAR(2) model fitted to each of the variables. The results of the parameter estimate and p-values, in parenthesis, are summarized in Table 4.8

Table 4.8: VAR(2) Model and Its Parameter Estimates

VRA(2)	Agriculture	Industry	Building & Construction	Wholesale & Retail	Services
Constant	16.5611	22.2805	-3.9284	-10.7831	-46.7275
	(0.501)	(0.408)	(0.352)	(0.404)	(0.263)
Π_{1Agric}	0.3652	0.2035	0.0147	0.1339	0.2291
	(0.000)***	(0.011)**	(0.234)	(0.001)***	(0.062)*
Π_{2Agric}	-0.7765	-0.1662	0.0481	0.1695	0.1291
	(0.000)***	(0.036)**	(0.000)***	(0.000)***	(0.289)
$\Pi_{1Industry}$	0.2533	0.9674	0.0199	0.0800	0.3748
	(0.000)***	(0.000)***	(0.134)	(0.049)**	(0.005)***
$\Pi_{2Industry}$	0.3019	-0.0429	-0.0359	-0.1749	-0.4716
	(0.000)***	(0.628)	(0.010)**	(0.000)***	(0.001)***
$\Pi_{1B\ \&\ Con}$	2.9369	3.3954	0.1377	0.6971	-0.4844
	(0.000)***	(0.000)***	(0.304)	(0.091)*	(0.714)
$\Pi_{2B\ \&\ Con}$	0.0592	-2.5361	0.0579	2.0639	4.2928
	(0.943)	(0.006)***	(0.683)	(0.000)***	(0.003)***
$\Pi_{1W\ \&\ Retail}$	1.1688	0.2073	0.0710	0.8270	1.5163
	(0.000)***	(0.425)	(0.082)*	(0.000)***	(0.000)***
$\Pi_{2W\ \&\ Retail}$	1.1226	0.0169	-0.1302	-0.2959	-1.8169
	(0.000)***	(0.948)	(0.002)***	(0.019)**	(0.000)***
$\Pi_{1Services}$	-0.5402	-0.3347	0.0014	-0.1061	0.1506
	(0.000)***	(0.002)***	(0.932)	(0.041)**	(0.364)
$\Pi_{2Services}$	-0.3189	0.1586	0.0782	-0.0274	0.5247
	(0.000)***	(0.182)	(0.000)***	(0.630)	(0.005)***
No. of Significance	9	6	5	9	7
R^2	0.9898	0.9827	0.98998	0.9965	0.9917
AIC	13.5756	13.7553	10.0476	12.2868	14.6286
BIC	13.7964	13.9761	10.2683	12.5076	14.8494

*Footnote: ***-sig. at 1%, **-sig. at 5%,*-sig. at 10%,*

From Table 4.8, it is clear that VAR(2) models are very adequate, since the R^2 values are very close to one, the AIC and BIC have the least values when compared to VAR(1) and VAR(3).

d) Graphical Comparison of Univariate and Multivariate Fitted Value for each Sector

The fitted values for both models (ARIMA and VAR model) were obtained, so as to compare them graphically and also, using measurement of accuracy techniques.

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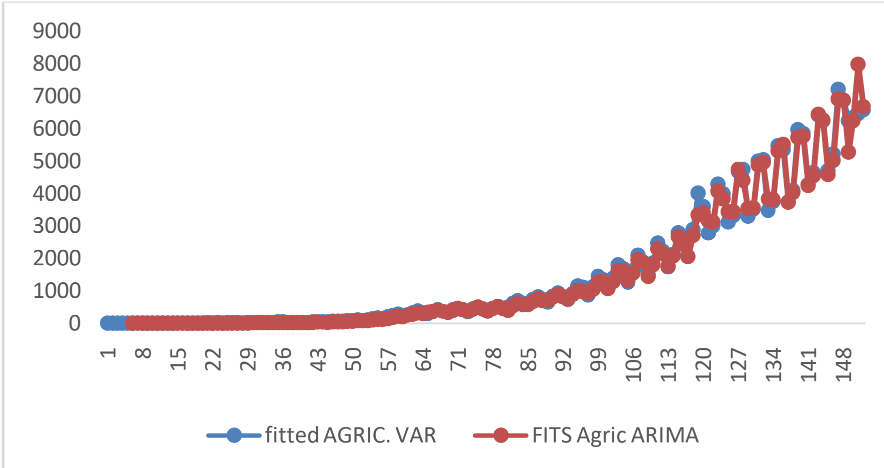

Figure 4.39: Plot of Fitted values Comparison for Agriculture Sector

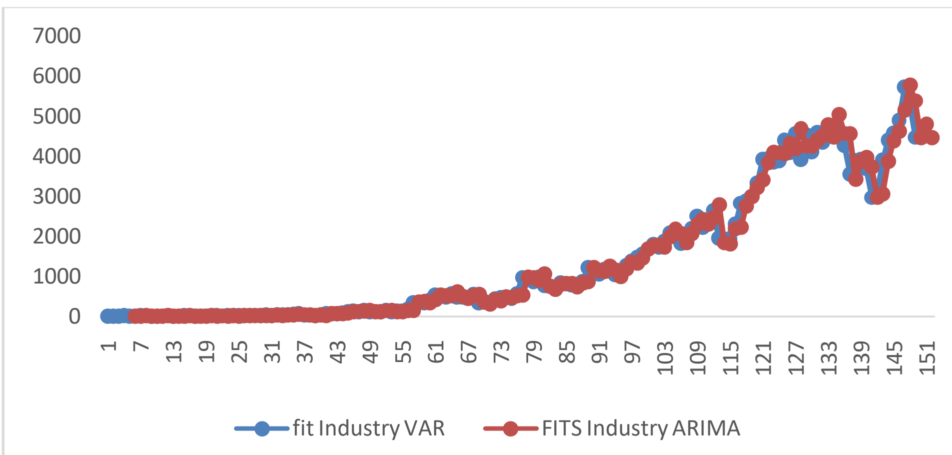

Figure 4.40: Plot of Fitted Values Comparison for Industry Sector

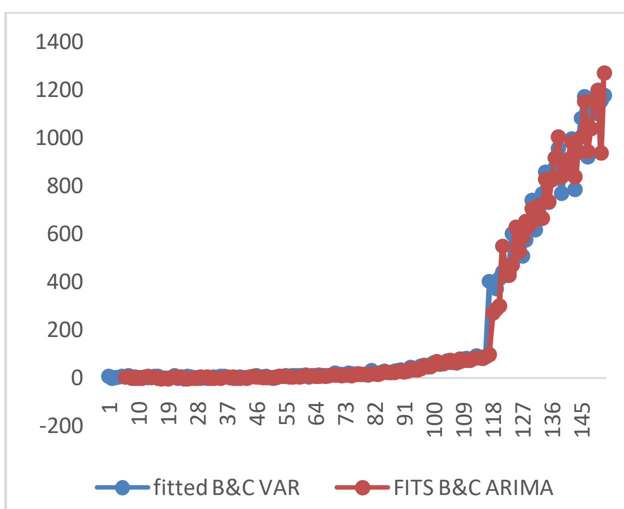

Figure 4.41: Plot of Fitted Values Comparison for Building and Construction Sector

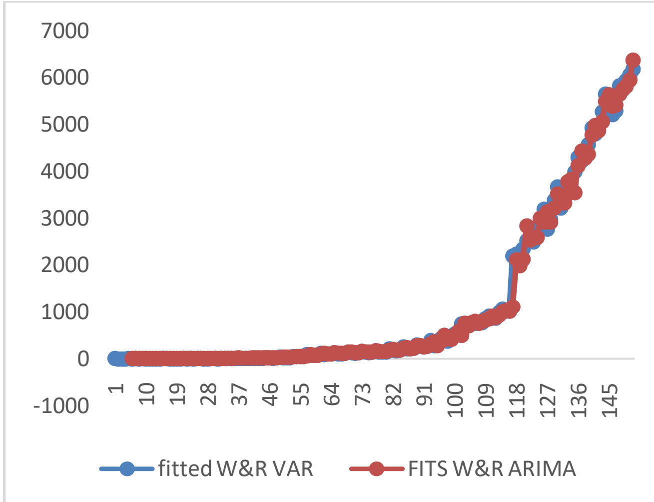

Figure 4.42: Plot of Forecast Comparison for Wholesale and Retail Sector

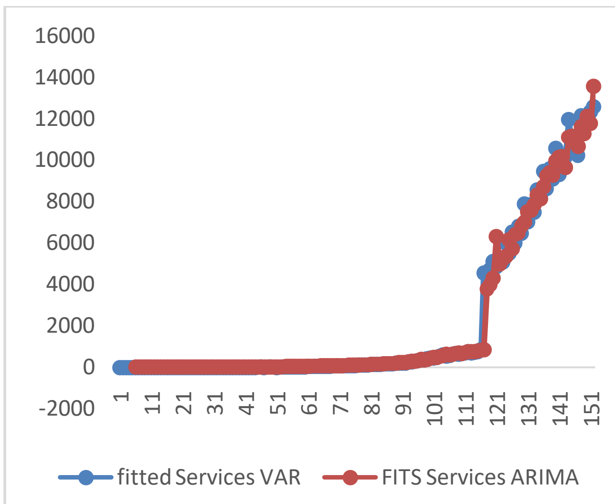

Figure 4.43: Plot of Fitted Values Comparison for Service Sector

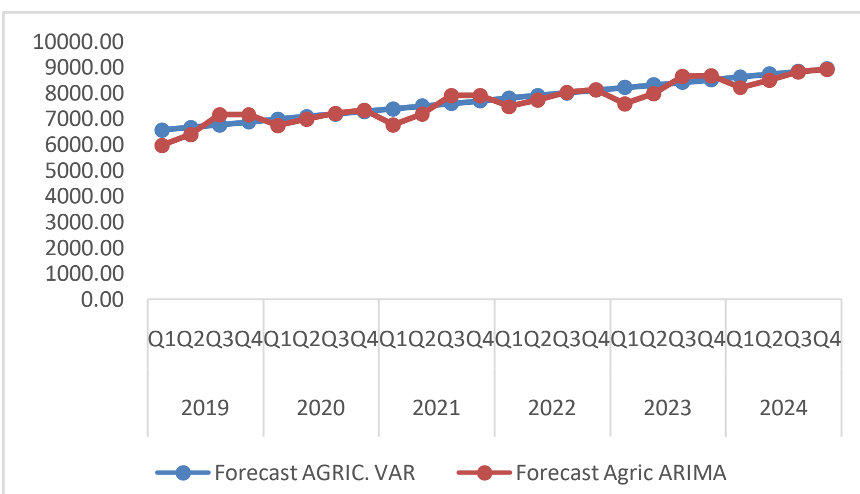

Figure 4.44: Plot of Forecast Comparison for Agriculture Sector

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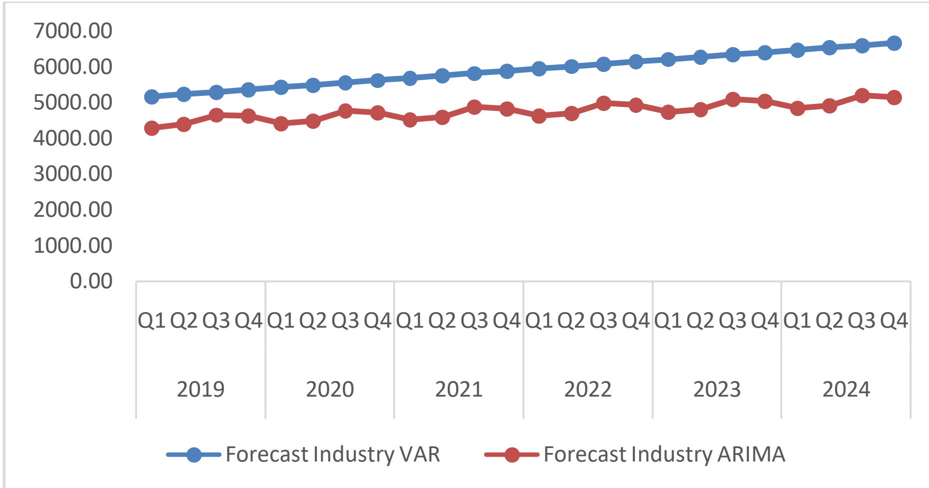

Figure 4.45: Plot of Forecast Comparison for Industry Sector

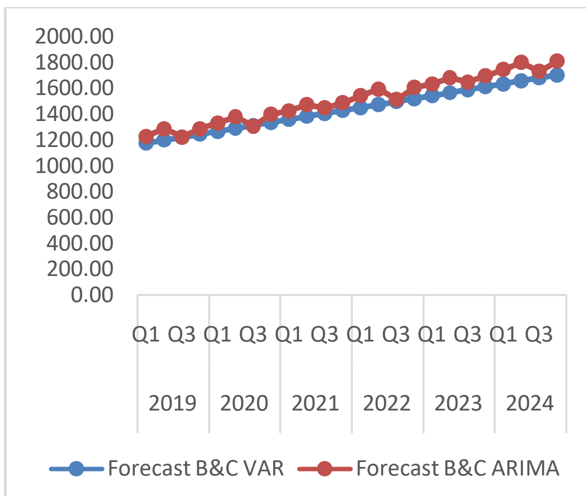

Figure 4.46: Plot of Forecast Comparison for Building and Construction Sector

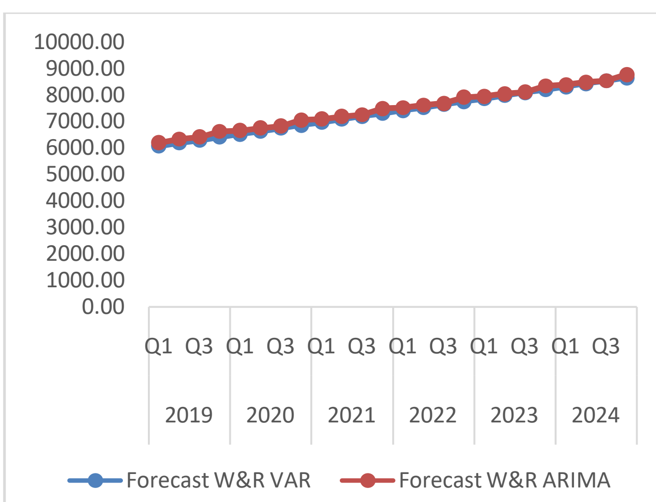

Figure 4.47: Plot of Forecast Comparison for Service Sector

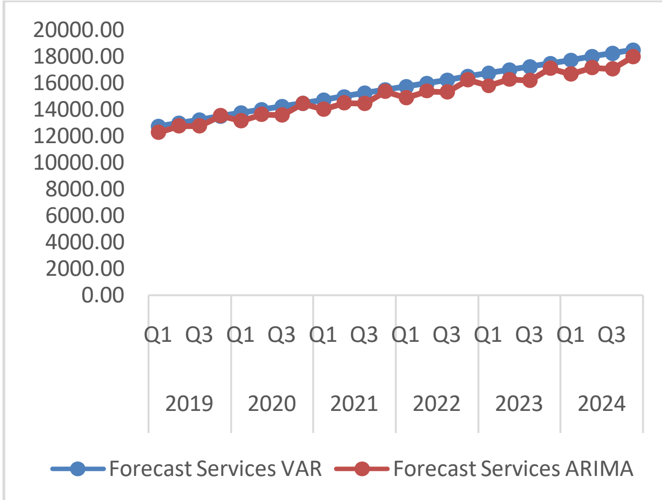

Figure 4.48: Plot of Forecast Comparison for Building and Construction Sector

Figures4.39 to 4.43 showed the comparison of the fitted values using adequate ARIMA and VAR models for each variable. The graphical representation shows no difference between both models in all the fitted graphs.

Table 4.9: Measure Accuracy of Fitted values for ARIMA and VAR Model

Variables	Models	ME	MSE	MAPE
Agriculture	ARIMA	1.9085×10^{-6}	2527.87	9.55%
	VAR	20.43415	10524747	**7.41%**
Industry	ARIMA	2.24×10^{-6}	2594.53	**4.92%**
	VAR	2.8877	16610305	11.80%
Building & Con	ARIMA	-2.032×10^{-6}	61530.14	78.89%
	VAR	2.9942	588754.3	**67.52%**
Wholesale & Ret	ARIMA	-2.359×10^{-6}	2525.96	25.01%
	VAR	18.39138	5266516	**10.91%**
Service	ARIMA	4.27×10^{-6}	2517.29	17.15%
	VAR	41.1486	54429421	**8.16%**

Footnote: The bold MAPE are best model for each variable.

Table 4.9 shows that for the variables (Agriculture, Building and Construction, Wholesale and Retail and Services) the VAR model have a better Mean Absolute Percentage Error (MAPE) values when compared to the ARIMA model. For Industry variable, the ARIMA model have better MAPE than the VAR model.

e) Graphical Comparison of Univariate and Multivariate Forecasts for Each Sector

The identified ARIMA and VAR models were used to obtained forecasts, so as to compare them using Coefficient of Variation (CV) and graphically. Figures 4.44 to 4.48 showed the forecasts comparison between the adequacy ARIMA and VAR models for each variable. The graphical representation shows no difference for thevariables, except the Industry variable.

f) FORECAST Comparison

Notes

Notes

Table 4.10: Variables Forecasts Comparison using Coefficient of Variation (CV)

Years	Quarterly	Forecast AGRIC. VAR	Forecast Agric ARIMA	Forecast Industry VAR	Forecast Industry ARIMA	Forecast B&C VAR	Forecast B&C ARIMA	Forecast W&R VAR	Forecast W&R ARIMA	Forecast Services VAR	Forecast Services ARIMA
2019	Q1	6560.13	5963.30	5159.96	4283.53	1174.59	1229.17	6068.29	6185.7	12744.20	12278.7
	Q2	6662.83	6377.50	5225.20	4384.21	1197.63	1289.50	6180.47	6320.6	12994.94	12759.9
	Q3	6765.53	7154.20	5290.44	4649.02	1220.66	1223.79	6292.65	6413.1	13245.68	12783.4
	Q4	6868.23	7150.20	5355.67	4615.60	1243.69	1288.37	6404.83	6616.4	13496.43	13559
2020	Q1	6970.93	6730.70	5420.91	4405.14	1266.73	1333.91	6517.01	6642.4	13747.17	13152.7
	Q2	7073.63	6975.40	5486.14	4480.42	1289.76	1380.19	6629.19	6752.3	13997.91	13640.7
	Q3	7176.33	7209.30	5551.38	4763.33	1312.79	1308.95	6741.36	6822.4	14248.65	13583
	Q4	7279.03	7330.40	5616.61	4717.01	1335.83	1400.12	6853.54	7047.6	14499.40	14475.1
2021	Q1	7381.73	6767.20	5681.85	4515.74	1358.86	1426.21	6965.72	7079.5	14750.14	14037.5
	Q2	7484.43	7168.20	5747.09	4584.46	1381.89	1474.13	7077.90	7182.8	15000.88	14523.5
	Q3	7587.13	7886.80	5812.32	4872.04	1404.93	1449.23	7190.08	7246.5	15251.62	14444
	Q4	7689.83	7898.20	5877.56	4822.39	1427.96	1488.92	7302.26	7478.1	15502.37	15363.8
2022	Q1	7792.53	7463.30	5942.79	4623.50	1450.99	1543.02	7414.44	7511.7	15753.11	14918
	Q2	7895.23	7725.80	6008.03	4690.53	1474.03	1595.06	7526.61	7613.1	16003.85	15403.3
	Q3	7997.93	8014.20	6073.27	4979.32	1497.06	1514.24	7638.79	7674.8	16254.60	15318.3
	Q4	8100.63	8123.10	6138.50	4928.81	1520.09	1607.06	7750.97	7908.3	16505.34	16244.8
2023	Q1	8203.33	7574.40	6203.74	4730.52	1543.13	1634.44	7863.15	7942.4	16756.08	15797
	Q2	8306.03	7959.80	6268.97	4797.12	1566.16	1680.57	7975.33	8043.2	17006.82	16282.1
	Q3	8408.73	8630.20	6334.21	5086.22	1589.19	1649.77	8087.51	8104.4	17257.57	16195.8
	Q4	8511.43	8652.50	6399.44	5035.49	1612.23	1698.35	8199.69	8338.4	17508.31	17123.9
2024	Q1	8614.13	8204.80	6464.68	4837.36	1635.26	1747.05	8311.86	8372.7	17759.05	16675.6
	Q2	8716.83	8481.20	6529.92	4903.85	1658.29	1798.81	8424.04	8473.3	18009.79	17160.6
	Q3	8819.53	8812.40	6595.15	5193.03	1681.33	1729.64	8536.22	8534.4	18260.54	17074
	Q4	8922.24	8911.60	6660.39	5142.24	1704.36	1810.53	8648.40	8768.5	18511.28	18002.5
Coefficient of Variation (CV)		9.38	10.25	7.8	5.11	11.31	12.07	10.78	10.27	11.35	10.69

Footnote: The bold CV are best forecasts for each variable.

The results in Table 4.10, confirm that the best model to forecasts Agriculture and, Building and Construction variable is the VAR model. It also, confirm that the best model to forecasts Industry variable is ARIMA model. However, there is contradiction for both Wholesale/Retail and Services variables, where their VAR models have a better Mean Absolute Percentage Error (MAPE), but their forecast values show the ARIMA model as being better.

Notes

V. Summary and Conclusion

This paper studied univariate and multivariate time series analysis for the sectors in Nigeria's Gross Domestic Product (GDP) which are Agriculture, Industry, Building & Construction, Wholesale & Retail, and Services. First, the plot for the five sectors was done to determine the trend component that exist among them. It was discovered that the trend component was Quadratic with appreciation and depreciation over time. The Correlation matrix was obtained to determine the degree of relationship among the variables. It was shown that a strong positive relationship, above 90% exists among all the variables. Modeling of the sectors performance was done using ARIMA and VAR models. For each of the economic variables ARIMA and VAR models were built and the best model selected for further analysis. VAR 2 of lag 4 was identified as the suitable model for the data set using the AIC and BIC selection criteria. Furthermore, the VAR (2) model was used to forecast for the next 6 years to come. Finally, coefficient of variation, CV was used to compare and identify the best model forecast for each of the variable which confirm that the best model to forecast Agriculture and Building/Construction variables is the VAR model while for Industry, Wholesale/Retail and Services variables the best model to forecast is the ARIMA model.

This study was able to identify

1. The correlation among the multivariate time series data and was considered as positive correlation.
2. Suitable model using the univariate and multivariate time series method.
3. Forecast for the next six years for all the sectors in Nigeria's GDP.
4. The best model for forecasting of each of the variable.

In conclusion, this study identified VAR model to be more suitable for forecasting Agriculture and Building/Construction variables and ARIMA model for forecasting Industry, Wholesale/Retail and Services variables.

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- Computer programs
- Electronic material
- Any other original work

Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

Copyright

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Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- o You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- o Give details of all of your remarks as much as possible, focusing on mechanisms.
- o Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- o One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- o Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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