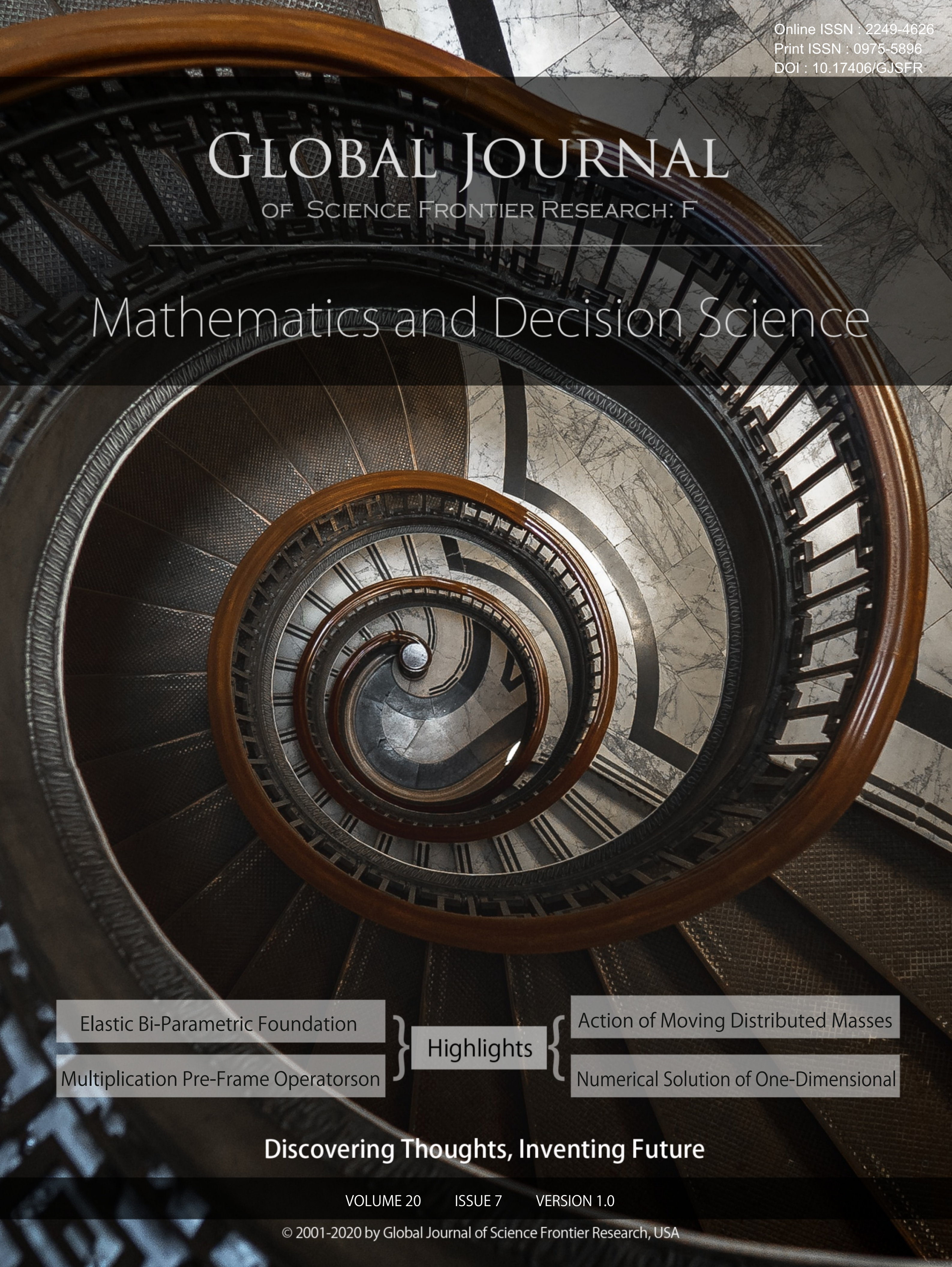

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Highlights

- Elastic Bi-Parametric Foundation
- Multiplication Pre-Frame Operatorson
- Action of Moving Distributed Masses
- Numerical Solution of One-Dimensional

Discovering Thoughts, Inventing Future

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Composite Multiplication Pre-Frame Operatorson the Space of Vector-Valued Weakly Measurable Functions

By S. Senthil, M. Nithya & D. C. Kumar
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Abstract- In this paper,we first characterize the boundedness of the condition under which composite multiplication pre-frame operators on L^2 (μ) -space, namely $M_{u,T,f}$ and its adjoint. Then, we identify the relation between the adjoint of $M_{u,T,f}$ and the composite multiplication frame operators which is denoted by $S_{u,T,f}$ all the results have been obtained in terms of radonnikodymderivative h_T.

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12. Duffin RJ & Shaeffer AC, A class of non-harmonic Fourier series, Trans. Amer. Math. Soc, vol.72, pp.341-366 (1952).

Composite Multiplication Pre-Frame Operatorson the Space of Vector-Valued Weakly Measurable Functions

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Abstract- In this paper,we first characterize the boundedness of the condition under which composite multiplication pre-frame operators on L^2 **(μ)** -space, namely $M_{u,T,f}$ and its adjoint. Then, we identify the relation between the adjoint of $M_{u,T,f}$ and the composite multiplication frame operators which is denoted by $S_{u,T,f}$ all the results have been obtained in terms of radonnikodymderivative h_T.

Keywords: composite multiplication operator, expectation, composite multiplication preframe operator.

I. Introduction

Frames were developed as a powerful tool in signal processing. the frame in a Hilbert space was defined by Duffin and Schaeffer [12] for investigating non-harmonic Fourier series. A discrete frame is a countable family of elements in a separable Hilbert space, which allows stable and not necessarily unique decomposition of arbitrary elements in an expansion of frame elements. In this paper, H refers to a Hilbert space over C and the closed unit ball of H is denoted by H_1.

Let (X, Σ, μ) be a σ-finite measure space. Then a mapping T from X into X is said to be a measurable transformation if $T^{-1}(E)\in\Sigma$ forevery $E\in\Sigma$. A measurable transformation T is said to be non-singular if $\mu(T^{-1}(E))=0$ whenever $\mu(E)=0$. If T is non-singular then the measure μT^{-1} defined as $\mu T^{-1}(E)=\mu(T^{-1}(E))$ for every E in Σ , is an absolutely continuous measure on Σ with respect to μ.Since μ is a σ-finite measure, then by the Radon-Nikodym theorem, there exists a non-negative function h_T in $L^1(\mu)$ such that $\mu T^{-1}(E)=\int_E h_T\, d\mu$ for every $E\in\Sigma$. The function h_T is called the Radon-Nikodym derivative of μT^{-1} with respect to μ.

Every non- singular measurable transformation T from X into itself induces a linear transformation C_T on $L^p(\mu)$ defined as $C_T f = f\circ T$ for every f in $L^p(\mu)$. In case C_T is continuous from $L^p(\mu)$ into itself, then it is called a composition operator on $L^p(\mu)$ induced by T. We restrict our study of the composition operators on $L^2(\mu)$ which has Hilbert space structure. If u is an essentially bounded complex-valued measurable function on X, then the

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mapping M_u on $L^2(\mu)$ defined by $M_u f = u \cdot f$, is a continuous operator with range in $L^2(\mu)$. The operator M_u is known as the multiplication operator induced by u.

A composite multiplication operator is linear transformation acting on a set of complex valued Σ measurable functions f of the form

$$M_{u,T}(f) = C_T M_u(f) = (u \circ T)\ (f \circ T)$$

where u is a complex valued, Σ measurable function. In case $u = 1$ almost everywhere, $M_{u,T}$ becomes a composition operator, denoted by C_T.

In the study considered is the using conditional expectation of composite multiplication operator on L^2-spaces. For each $f \in L^p(X, \Sigma, \mu)$, $1 \le p \le \infty$, there exists an unique $T^{-1}(\Sigma)$-measurable function $E(f)$ such that

$$\int_A g f \, d\mu = \int_A g E(f) \, d\mu$$

for every $T^{-1}(\Sigma)$-measurable function g, for which the left integral exists. The function $E(f)$ is called the conditional expectation of f with respect to the subalgebra $T^{-1}(\Sigma)$. As an operator of $L^p(\mu)$, E is the projection onto the closure of range of T and E is the identity on $L^p(\mu)$, $p \ge 1$ if and only if $T^{-1}(\Sigma) = \Sigma$. Detailed discussion of E is found in [1, 2, 3, 4].

The study of weighted composition operators on L^2 spaces was initiated by R.K.Singh and D.C.Kumar [5]. During the last thirty years, several authors have studied the properties of various classes of weighted composition operator. Boundedness of the composition operators in $L^p(\Sigma)$, $(1 \le p < \infty)$ spaces, where the measure spaces are σ-finite, appeared already in [6]. Also boundedness of weighted operators on $C(X, E)$ has been studied in [7]. Recently S.Senthil, P.Thangaraju and D.C.Kumar have proved several theorems on n-normal, n-quasi-normal, k-paranormal, and (n,k) paranormal of composite multiplication operators on L^2 spaces [8, 9, 10, 11,17].

The theory of weighted translation pre-frame operators is the generalizations of the theory of c-frames and c-Bessel mappings. The properties of c-frames and c-Bessel mappings have been studied in [13]. The change of variable formula will be frequently used throughout this paper and we remind it here as follows:

$$\int_{T^{-1}(B)} f \circ T \, d\mu = \int_{T^{-1}(B)} f \, d\mu \circ T^{-1} = \int_{T^{-1}(B)} f \frac{d\mu \circ T^{-1}}{d\mu} d\mu = \int_B h_T f \, d\mu, \quad B \in \Sigma, \ f \in L^1(\Sigma).$$

In this paper we investigate composite multiplication pre-frame operators on $L^2(\mu)$-spaces.

1.1 Let $L^2(X, H)$ be the class of all measurable mappings $f : X \to H$ such that

$$\| f \|_2^2 = \int_X \| f(x) \|^2 \, d\mu < \infty$$

Ref

7. Takagi, H & Yokouchi, K, Multiplication and Composition operators between two pL - spaces, Contem. Math., vol.232, pp.321-338 (1999).

For any $f, g \in L^2(X,H)$, based on the polar identity, we may conclude that the mapping $x \to \langle f(x), g(x) \rangle$ of X to C, is measurable and it can be seen that $L^2(X,H)$ is a Hilbert space with the inner product defined by

$$\langle f, g \rangle_{L^2} = \int_X \langle f(x), g(x) \rangle d\mu.$$

Notes

We shall write $L^2(X)$ when $H = C$

II. Composite Multiplication Pre-Frame Operator

2.1 Let $f : X \to H$ be a mapping. We say that f is weakly measurable if for each $h \in H$, the mapping $x \to \langle h, f(x) \rangle$ of X to C is measurable.

2.2 Let $f : X \to H$ be weakly measurable. We say that f is a c-frame for H, if there exist $0 < A \le B < \infty$ such that

$$A\|h\|^2 \le \int_X \left|\langle h, f(x) \rangle\right|^2 d\mu \le B\|h\|^2, \quad h \in H.$$

If only the right hand inequality is satisfied, then we say that f is a c-Bessel mapping for H. Let $f : X \to H$ be a c-Bessel for H. Let $M_{u,T,f} : L^2(X) \to H$ be defined by

$$\langle M_{u,T,f}(g), h \rangle = \int_X (u \circ T)(x)(g \circ T)(x)\langle f(x), h \rangle d\mu(x), \quad h \in H, \; g \in L^2(X).$$

It is obvious that $M_{u,T,f}$ is well-defined and linear. For each $g \in L^2(x)$ and $h \in H$, we have

$$\left\| M_{u,T,f}(g) \right\| = \sup_{h \in H} \left| \langle M_{u,T,f}(g), h \rangle \right|$$

$$= \sup_{h \in H} \left| \int_X (u \circ T)(x)(g \circ T)(x) \langle f(x), h \rangle d\mu \right|$$

$$= \sup_{h \in H} \left| \int_X ((ug) \circ T)(x) \langle f(x), h \rangle d\mu \right|$$

$$\le \left(\int_X \left| ((ug) \circ T) \right|^2 d\mu \right)^{\frac{1}{2}} \sup_{h \in H_1} \left(\int_X \left| \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$= \left(\int_X E\left| ((ug) \circ T) \right|^2 d\mu \right)^{\frac{1}{2}} \sup_{h \in H_1} \left(\int_X \left| \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$= \left(\int_X h_T |u|^2 |g|^2 d\mu \right)^{\frac{1}{2}} \sup_{h \in H_1} \left(\int_X \left| \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\le B^{\frac{1}{2}} \|g\|_2 \|J\|_\infty^{\frac{1}{2}}$$

Consequently, $M_{u,T,f}$ is bounded. We shall denoted

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$M_{u,T,f} : L^2(X) \to H$, by $M_{u,T,f}(g) = \int_X (u \circ T)(g \circ T) f \, d\mu, \quad g \in L^2(X)$ is called the composite multiplication pre-frame operator of f.

For each $g \in L^2(X)$ and $h \in H$ by an application of the conditional expectation properties and the change of variable formula,

$$\left\langle g,\ M^*_{u,T,f}(h) \right\rangle = \left\langle M_{u,T,f}(g), h \right\rangle$$

$$= \int_X (u \circ T)(x)(g \circ T)(x) \left\langle f(x), h \right\rangle d\mu$$

$$= \int_X ((ug) \circ T)(x) \left\langle f(x), h \right\rangle d\mu$$

$$= \int_X E\left(((ug) \circ T)(x) \left\langle f(x), h \right\rangle \right) d\mu$$

$$= \int_X h_T\, u(x) g(x) E\left(\left\langle f(x), h \right\rangle \right) \circ T^{-1}\, d\mu$$

$$= \left\langle g,\ h_T\, u E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1} \right\rangle$$

Thus, $M^*_{u,T,f}(h) = h_T\, u E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1}$

Also, for each $h \in H$, we have

$$\left\| M^*_{u,T,f}(h) \right\|^2 = \left\langle M^*_{u,T,f}(h),\ M^*_{u,T,f}(h) \right\rangle$$

$$= \int_X \left\langle M_{u,T,f}\, M^*_{u,T,f}(h),\ h \right\rangle d\mu$$

$$= \int_X \left| u\, h_T\, E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1} \right|^2 d\mu$$

The mapping $M^*_{u,T,f} : H \to L^2(X)$ is called the composite multiplication analysis operator of f.

We define, $S_{u,T,f} : H \to H$ by $S_{u,T,f}(h) = M_{u,T,f}\, M^*_{u,T,f}(h)$

$$= M_{u,T,f}\left(h_T\, u E\left(\overline{\left\langle f, h \right\rangle} \right) \circ T^{-1} \right)$$

$$= \int_X u^2 \circ T\ h_T \circ T\ E\left(\overline{\left\langle f, h \right\rangle} \right) f\, d\mu$$

and it is called the composite multiplication frame operator of f.

Theorem 2.1. Let $S_{u,T,f}$ is composite multiplication frame operator of f. The mapping $S_{u,T,f} : H \to H$ and For each c-Bessel mapping $f : X \to H$, Then $S_{u,T,f}$ is invertible if and only if $M_{u,T,f}$ is surjective.

Notes

Proof. Since $S_{u,T,f}$ is a self-adjoint operator on H then by [14, Theorem 9.2.1], we have

$\inf_{h\in H_1}\left\langle S_{u,T,f}\, h,\, h\right\rangle = \inf_{h\in H_1}\left\| M^*_{u,T,f}\,(h)\right\|^2 \in \mathrm{Sepc}\, S_{u,T,f}$, the spectrum of $S_{u,T,f}$.

By hypothesis $0 \notin \mathrm{Spec}\, S_{u,T,f}$, Hence, $\inf_{h\in H_1}\left\| M^*_{u,T,f}\right\| > 0$. It follows that

$$\inf_{h\in H_1}\left\| M^*_{u,T,f}\right\|\left\| h\right\| \le \left\| M^*_{u,T,f}\right\| \text{and so } M_{u,T,f} \text{ is surjective.}$$

Conversely, Let $M_{u,T,f}$ is surjective. Then there exists $K > 0$ such that for each $h \in H$

$\left\| M^*_{u,T,f}\right\|^2 \ge K\left\| h\right\|^2$

$$\text{So, } \left\langle S_{u,T,f}\,(h),\, h\right\rangle = \left\langle M_{u,T,f}\, M^*_{u,T,f}\,(h),\, h\right\rangle = \left\| M^*_{u,T,f}\right\|^2 \ge K\left\| h\right\|^2$$

For each $h \in H$, we have

$$\begin{aligned}
\left\langle S_{u,T,f}\,(h),\, h\right\rangle &= \left\langle M_{u,T,f}\, M^*_{u,T,f}\,(h),\, h\right\rangle \\
&= \int_X \left\langle f, h\right\rangle \left(u\, M^*_{u,T,f}\right)\circ T\, d\mu \\
&= \int_X u^2 \circ T\; h_T \circ T\; E\left(\overline{\left\langle f, h\right\rangle}\right)\left\langle f, h\right\rangle d\mu \\
&= \int_X E\left(u^2 \circ T\; h_T \circ T\; E\left(\overline{\left\langle f, h\right\rangle}\right)\left\langle f, h\right\rangle\right) d\mu \\
&= \int_X u^2 \circ T\; h_T \circ T\; E\left(\overline{\left\langle f, h\right\rangle}\right) E\left(\left\langle f, h\right\rangle\right) d\mu \\
&\le \int_X u^2 \circ T\; h_T \circ T\; E\left(\left|\left\langle f, h\right\rangle\right|^2\right) d\mu \\
&= \int_X (u^2 h_T)\circ T\; \left|\left\langle f, h\right\rangle\right|^2 d\mu \le 1 \\
&\le \left\|(u^2 h_T)\circ T\right\|_\infty B\left\| h\right\|^2 \text{ for some } B > 0
\end{aligned}$$

Therefore $K \le S_{u,T,f} \le \left\|(u^2 h_T)\circ T\right\|_\infty B$, $S_{u,T,f}$ is invertible.

Theorem 2.2. Let $M_{u,T,f}$ is composite multiplication pre-frame operator of f. For each $x \in X$, the map $x \to \left\langle f(x), h\right\rangle$ is $T^{-1}(\Sigma)$ measurable. Then $f : X \to H$, is a c-frame for H if and only if the operator $M_{u,T,f}$ is a bounded and onto operator.

Proof. Let f be c-frame by definition 2.2, it is clear that $M_{u,T,f}$ is bounded. We have to prove only that $M_{u,T,f}$ is onto.

Since $(u^2 h_T)\circ T > 0$ almost everywhere, Now we assume that $(u^2 h_T)\circ T > \delta$ for some $\delta > 0$. Then, by using the change of variable formula, we get

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$$\left\| M^*_{u,T,f}(h) \right\|^2 = \int_X \left| u\, h_T\, E\left(\overline{\langle f, h \rangle}\right) \circ T^{-1} \right|^2 d\mu$$

$$= \int_X |u|^2 |h_T|^2 \left| E\left(\overline{\langle f, h \rangle}\right) \circ T^{-1} \right|^2 d\mu$$

$$= \int_X |u|^2 |h_T| \left| E\left(\overline{\langle f, h \rangle}\right) \circ T^{-1} \right|^2 d\mu \circ T^{-1}$$

$$= \int_X |u|^2 \circ T\, |h_T| \circ T \left| E\left(\overline{\langle f, h \rangle}\right) \right|^2 d\mu$$

$$= \int_X \left| u^2 h_T \right| \circ T \left| E\left(\overline{\langle f, h \rangle}\right) \right|^2 d\mu$$

$$\geq \delta \int_X \left| E\left(\overline{\langle f, h \rangle}\right) \right|^2 d\mu = \delta \int_X \left| \overline{\langle f, h \rangle} \right|^2 d\mu = \delta \int_X \left| \langle h, f \rangle \right|^2 d\mu$$

$$\geq \delta A \| h \|^2$$

Therefore, by [15, lemma 2.4.1], $M_{u,T,f}$ is onto.

Conversely, let $M_{u,T,f}$ is bounded and onto operator, by [15,Lemma 2.4.1], there exists a constant $c > 0$ such that for each $h \in H$, $c\| h \|^2 \leq \left\| M^*_{u,T,f}(h) \right\|^2$.

On the other hand, by the change of variable formula, we get

$$c\| h \|^2 \leq \left\| M^*_{u,T,f}(h) \right\|^2 = \int_X \left| u\, h_T\, E\left(\overline{\langle f, h \rangle}\right) \circ T^{-1} \right|^2 d\mu$$

$$= \int_X \left| u^2 h_T \right| \circ T \left| E\left(\overline{\langle f, h \rangle}\right) \right|^2 d\mu$$

$$\leq \left\| (u^2 h_T) \circ T \right\|_\infty \int_X \left| \langle h, f \rangle \right|^2 d\mu$$

Since $\left\| (u^2 h_T) \circ T \right\|_\infty > 0$, we get $A \| h \|^2 \leq \int_X \left| \langle h, f \rangle \right|^2 d\mu$ for some constant $A > 0$.

To proved is that f is c-Bessel, For this the change of variable formula and the properties of the conditional expectation are essentially used to obtain by

$$\delta \int_X \left| \langle h, f \rangle \right|^2 d\mu \leq \delta \int_X E \left| \langle h, f \rangle \right|^2 d\mu$$

$$\leq \int_X (u^2 h_T) \circ T\; E \left| \langle h, f \rangle \right|^2 d\mu$$

$$= \int_X (u^2 h_T)\; E \left| \langle h, f \rangle \circ T^{-1} \right|^2 d\mu \circ T^{-1}$$

$$= \int_X (u^2 h_T)\; E \left| \langle h, f \rangle \circ T^{-1} \right|^2 h_T\, d\mu$$

Notes

$$= \int_X \left| u\, h_T \; E\langle h, f \rangle \circ T^{-1} \right|^2 d\mu$$

$$= \left\| M^*_{u,T,f}(h) \right\|^2 \le \left\| M^*_{u,T,f} \right\|^2 \|h\|^2$$

Hence $\int_X |\langle h, f \rangle|^2 d\mu \le B\|h\|^2$ for some $B > 0$

Notes

Theorem 2.3. Let K be a Hilbert space, $f: X \to H$ be a c-Bessel mapping for H and $v: H \to K$ be a bounded linear mapping. Then

(i) The mapping $vf: X \to K$ is a c-Bessel mapping for K and $vM_{u,T,f} = M_{u,T,vf}$

(ii) For each $x \in X$ the map, $x \to \langle h, f(x) \rangle$ is $T^{-1}(\Sigma\ )$ – measurable. Let f be a c-frame for H. Then vf is

Proof. (i).Since $\sup_{h \in H_1} \int_X |\langle h, v(f(x)) \rangle|^2 d\mu \le \|v\|^2 \sup_{h \in H_1} \int_X |\langle h, f(x) \rangle|^2 d\mu$, vf is a c-Bessel mapping for K.

For each $g \in L^2(X)$, we have $\langle M_{u,T,vf}(g), k \rangle = \int_X u \circ T(x)\; g \circ T(x) \langle v(f(x)), k \rangle d\mu$

$$= \int_X (ug) \circ T(x) \left\langle f(x), v^*(k) \right\rangle d\mu$$

$$= \left\langle M_{u,T,f}(g),\; v^*(k) \right\rangle = \left\langle v M_{u,T,f}(g),\; k \right\rangle$$

Hence $M_{u,T,vf} = vM_{u,T,f}$.

(ii). Suppose that v is surjective, by (i) it is clear that $M_{u,T,vf}$ is also surjective.

Hence by Theorem 2.2, vf is a c-frame for K.

Conversely, suppose that vf is a c-frame for K, then by Theorem 2.2, $M_{u,T,vf}$ is surjective and again by (i) v is clearly surjective.

III. Dual of C-Bessel Mapping

3.1 Let f, g be c-Bessel mappings for $h \in H$ we say that f equals weakly to g whenever $M^*_{u,T,f} = M^*_{u,T,g}$, which is equivalent with $\langle h, f \rangle = \langle h, g \rangle$ almost everywhere, for all $h \in H$.

Theorem 3.1. Let f, g be c-Bessel mappings for H. Then the following assertions are equivalent,

(1). For each $h \in H$, $h = M_{u,T,f}\left(\left\langle h,\; g \circ T^{-1} \right\rangle\right)$

(2). For each $k \in H$, $k = M_{\overline{u \circ T},T,g}\left(\left\langle k,\; f \circ T^{-1} \right\rangle\right)$

(3). For each $h, k \in H$, $\langle h, k \rangle = \int_X u \circ T(x) \langle h, g(x) \rangle \langle f(x), k \rangle d\mu$

(4). For each $h \in H$, $\|h\|^2 = \int_X u \circ T(x) \langle h, g(x) \rangle \langle f(x), h \rangle d\mu$

(5). For each orthonormal bases $\{ e_i \}_{i\in I}$ for H

$$\langle e_i,\gamma_j \rangle = \int_X u\circ T(x)\langle e_i, g(x)\rangle \langle f(x), \gamma_j \rangle d\mu, \quad i\in I, \ j\in J$$

(6). For each orthonormal bases $\{ \gamma_j \}_{j\in J}$ and $\{ e_i \}_{i\in I}$ for H

$$\langle e_i,e_j \rangle = \int_X u\circ T(x)\langle e_i, g(x)\rangle \langle f(x), e_j \rangle d\mu, \quad i\in I, \ j\in J$$

Notes

Proof. $(1)\to(2)$, choose $h, k\in H$ arbitrarily then

$$\langle h,k \rangle = \left\langle M_{u,T,f}\left(\left\langle h, g\circ T^{-1} \right\rangle \right), k \right\rangle$$

$$= \int_X u\circ T(x) \left(\left\langle h, g\circ T^{-1} \right\rangle \right)\circ T(x) \ \langle f(x), k \rangle d\mu$$

$$= \int_X u\circ T(x) \langle h, g(x) \rangle \ \langle f(x), k \rangle d\mu$$

$$= \overline{\int_X \overline{u\circ T(x)} \ \langle k, f(x) \rangle \ \langle g(x), h \rangle d\mu}$$

$$= \overline{\left\langle M_{\overline{u\circ T},T,g}\left(\left\langle k, f\circ T^{-1} \right\rangle \right), \ h \right\rangle}$$

$$= \left\langle h, \ M_{\overline{u\circ T},T,g}\left(\left\langle k, f\circ T^{-1} \right\rangle \right) \right\rangle$$

Hence $k = M_{\overline{u\circ T},T,g}\left(\left\langle k, f\circ T^{-1} \right\rangle \right)$

$(2)\to(3)$ is proved in a similar way and proof of the other implications refer[16, Theorem 3.4].

3.2 Let f, g be c-Bessel mappings for H. we say that f, g is a dual pair if one of the assertions of Theorem 3.1 is satisfied.

Note that:

$$\| h \|^2 = \int_X u\circ T(x) \langle h, \ g(x) \rangle \langle f(x), h \rangle d\mu$$

$$\le \int_X \left| u\circ T(x) \langle h, \ g(x) \rangle \langle f(x), h \rangle \right| \ d\mu$$

$$\le \left(\int_X \left| \langle h, \ g(x) \rangle \right|^2 \ d\mu \right)^{\frac{1}{2}} \left(\int_X \left| u\circ T(x) \langle f(x), h \rangle \right|^2 d\mu \right)^{\frac{1}{2}}$$

$$\le \left(\int_X \left| \langle h, \ g(x) \rangle \right|^2 \ d\mu \right)^{\frac{1}{2}} \| u\circ T \|_\infty \ B^{\frac{1}{2}} \| h \|$$

Notes

Hence g is a c-frame for H.

Theorem 3.2. For each $x \in X$ and $h \in H$, the map $x \to \langle h, f(x) \rangle$ is $T^{-1}(\Sigma\ )$ – measurable. Let f be a c-frame for H. Then the following arguments hold.

(1). For each $h \in H$, we find the following formulas $h = M_{u,T,S^{-1}_{u,T,f}}\left(u\,h_T\,E(\langle h, f \rangle)\circ T^{-1}\right)$ and $h = M_{u,T,f}\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),\ f\right\rangle\right)\circ T^{-1}\right)$

(2). In the formula $h = M_{u,T,f}\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),\ f\right\rangle\right)\circ T^{-1}\right)$,

$h = M_{u,T,f}\left(u\,h_T\,E\left(\left\langle h, S^{-1}_{u,T,f}(f)\right\rangle\right)\circ T^{-1}\right)$ has the least norm among all of the retrieval formulas.

(3). For each $h \in H$, $h = M_{u,T,f}\left\langle h, g\circ T^{-1}\right\rangle$ if and only if there exists a c-Bessel mapping $l \in H$ Such that $g\circ T^{-1} = S^{-1}_{u,T,f}\,f + l$, where for each $k \in H, \langle k, l\rangle \in Ker(M_{u,T,f})$.

(4). The map f has just one dual if and only if $R(M^{*}_{u,T,f}) = L^2(X)$.

Proof.(1).Since f is c-frame, then by Theorem 2.2 ,$M_{u,T,f}$ is onto and hence $S_{u,T,f}$ is an invertible operator. Consequently, for each $h \in H$, we obtain that

$$h = S^{-1}_{u,T,f}\,S_{u,T,f}(h) = S^{-1}_{u,T,f}\,M_{u,T,f}\,M^{*}_{u,T,f}(h)$$

$$= M_{u,T,S^{-1}_{u,T,f}}\left(u\,h_T\,E(\langle h, f \rangle)\circ T^{-1}\right)$$

Now, we have $h = S^{-1}_{u,T,f}\,S_{u,T,f}(h) = M_{u,T,f}\,M^{*}_{u,T,f}(h)\left(S^{-1}_{u,T,f}\right)$

$$= M_{u,T,f}\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),\ f\right\rangle\right)\circ T^{-1}\right).$$

(2). Choose $\phi \in L^2(X)$ and $h = M_{u,T,f}(\phi)$. Then for each $g \in H$, we have

$$\langle h, g \rangle = \left\langle M_{u,T,f}\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h), f\right\rangle\right)\circ T^{-1}\right),\ g\right\rangle$$

$$= \int_X u\circ T(x)\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h), f\right\rangle\right)\circ T^{-1}\right)\circ T(x)\,\langle f,\ g\rangle d\mu$$

$$= \int_X u^2\circ T(x)\,h_T\circ T(x)\,E\left(\left\langle S^{-1}_{u,T,f}(h), f\right\rangle\right)\langle f(x),\ g\rangle d\mu$$

Similarly, we have

$$\langle h, g \rangle = \langle M_{u,T,f}(\phi),\ g \rangle = \int_X u\circ T(x)\,\phi\circ T(x)\,\langle f(x), g\rangle\,d\mu$$

Therefore $\langle h,g\rangle-\langle h,g\rangle=\left\langle M_{u,T,f}\left[\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\right],\,g\right\rangle$

$$=\int_X u\circ T(x)\left(\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi(x)\right)\circ T(x)\langle f(x),\,g\rangle d\mu=0$$

$$M_{u,T,f}\left(\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\right)=0$$

Implies that $\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\in Ker(M_{u,T,f})$

Since f is a c-Bessel mapping for H, we obtain that $\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),f\right\rangle\right)\circ T^{-1}\right)\in R(M^*_{u,T,f})$

But $L^2(X)=\ker(M_{u,T,f})\oplus R(M^*_{u,T,f})$

Consequently,

$$\|\phi\|^2=\left\|\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),f\right\rangle\right)\circ T^{-1}\right)-\phi\right\|^2+\left\|\left(u\,h_T\,E\left(\left\langle S^{-1}_{u,T,f}(h),f\right\rangle\right)\circ T^{-1}\right)\right\|^2$$

and (2) is proved.

(3). Let g be a c-Bessel mapping for H. For each $h\in H$, assume that $h=M_{u,T,f}\left\langle h,\,g\circ T^{-1}\right\rangle$

Let $g\circ T^{-1}-S^{-1}_{u,T,f}f=l$ by Theorem 3.1, for each $h,k\in H$ we have

$$\left\langle M_{u,T,f}\langle k,l\rangle,h\right\rangle=\left\langle M_{u,T,f}\left\langle k,g\circ T^{-1}\right\rangle,h\right\rangle-\left\langle M_{u,T,f}\left\langle k,\,S^{-1}_{u,T,f}f\right\rangle,\,h\right\rangle$$

$$=\int_X u\circ T\left\langle k,g\circ T^{-1}\right\rangle\circ T\langle f,h\rangle d\mu-\int_X u\circ T\left\langle k,\,S^{-1}_{u,T,f}f\right\rangle\circ T\langle f,h\rangle d\mu$$

$$=\int_X u\circ T\langle k,g\rangle\langle f,h\rangle d\mu-\int_X u\circ T\left\langle k,\,S^{-1}_{u,T,f}f\circ T\right\rangle\langle f,h\rangle d\mu$$

$$=\langle k,h\rangle-\langle k,h\rangle=0$$

Hence, for each $k\in H$, $\langle k,l\rangle\in R\left(M^*_{u,T,f}\right)^{\perp}=ker\left(M_{u,T,f}\right)$.

Now, let $g\circ T^{-1}=S^{-1}_{u,T,f}f+l$, Then for each $h\in H$,we have

$$\int_X u\circ T\langle f,h\rangle\langle k,g\rangle d\mu=\int_X u\circ T\langle f,h\rangle\left\langle k,\,\left(S^{-1}_{u,T,f}f+l\right)\circ T\right\rangle d\mu$$

$$=\int_X u\circ T(x)\,\langle f(x),h\rangle\left\langle k,\,\left(S^{-1}_{u,T,f}f\right)\circ T\right\rangle d\mu+\int_X u\circ T(x)\,\langle f(x),h\rangle\langle k,\,l\circ T\rangle d\mu$$

$$=\langle k,h\rangle+\left\langle M_{u,T,f}\langle k,l\rangle,\,h\right\rangle=\langle k,h\rangle.$$

By Theorem 3.1, $h=M_{u,T,f}\left\langle h,\,g\circ T^{-1}\right\rangle$.

(4). Let $R\left(M^*_{u,T,f}\right)\neq L^2(X)$ and Let $l\in R\left(M^*_{u,T,f}\right)^{\perp}$ with $\|l\|=1$

Notes

Notes

Consider the map $k:X\to L^2(X)$ defined by $k(x)=l\circ T(x)l$. For each $t\in L^2(X)$, the map $X\to C$, defined by $x\to\langle t,k(x)\rangle$ is Σ -measurable and $\int_X |\langle t,k(x)\rangle|^2 d\mu=\int_X |\langle t,l\circ T(x)l\rangle|^2 d\mu$

$$=\int_X |\langle t,l\rangle|^2 |l\circ T(x)|^2 d\mu=|\langle t,l\rangle|^2\le\|t\|^2$$

Thus, k is a c-Bessel mapping for $L^2(X)$. let $v:L^2(X)\to H$ be a mapping such that $v(l)\neq 0$. Then vk is c-Bessel mapping for H and $S^{-1}_{u,T,f}f+vk$ is a c-Bessel mapping for H.

Let $h\in H$, $\int_X u\circ T(x)\left\langle h,S^{-1}_{u,T,f}f(x)+vk(x)\right\rangle\langle f(x),h\rangle d\mu$

$$=\int_X u\circ T(x)\left\langle h,S^{-1}_{u,T,f}f(x)\right\rangle\langle f(x),h\rangle d\mu+\int_X u\circ T(x)\langle h,vk(x)\rangle\langle f(x),h\rangle d\mu$$

$$=\|h\|^2+\left\langle v^*(h),l\right\rangle\int_X u\circ T(x)\overline{l\circ T(x)}\langle f(x),h\rangle d\mu$$

$$=\|h\|^2+\left\langle v^*(h),l\right\rangle\left\langle M_{u,T,f}(l),h\right\rangle=\|h\|^2$$

Therefore $S^{-1}_{u,T,f}f+vk$ is the dual of f.

The equation $\langle v(l),vk(x)\rangle=\langle v(l),l\circ T(x)v(l)\rangle=\overline{l\circ T(x)}\langle v(l),v(l)\rangle$

This implies that, $S^{-1}_{u,T,f}f+vk$ is not weakly equal to $S^{-1}_{u,T,f}f$

Conversely, Assume that $L^2(X)=R\left(M^*_{u,T,f}\right)$, Now, $g\circ T^{-1}=S^{-1}_{u,T,f}f+l$ where for each $k\in H$ $\langle k,l\rangle\in ker\left(M_{u,T,f}\right)=R\left(M^*_{u,T,f}\right)^{\perp}=\{0\}$, $l=0$ weakly, so f has a dual.

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Response to Simply Supported Orthotropic Rectangular Plate Resting on a Variable Elastic Bi-Parametric Foundation under the Action of Moving Distributed Masses

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Abstract- This work investigates the response to simply supported orthotropic rectangular plate resting on a variable elastic bi-parametric foundation under the action of moving distributed masses. The governing equation is a fourth order partial differential equation with variable and singular co-efficients. The solutions to the problem are obtained by transforming the fourth order partial differential equation for the problem to a set of coupled second order ordinary differential equations using the technique of Shadnam et al[12] which are then simplified using modified asymptotic method of Struble. The closed form solution is analyzed, resonance conditions are obtained and the results are presented in plotted curves for both cases of moving distributed mass and moving distributed force.

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GJSFR-F Classification: *MSC 2010: 35A17*

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I. Introduction

The problems connected with the analysis of thin structural bodies (rods, beams, plates,and shells) with other bodies have widespread application in various fields of science and technology. The physical phenomena involved in the impact event include structural responses, contact effects and wave propagation. These problems are topical issues of research in the field of applied mechanics. Since these problems belong to the problems of dynamic contact interaction, their solution is connected with severe mathematical and calculation difficulties. To this end , several researchers had worked and some are still working on the dynamic behavior of orthotropic rectangular plates. Analytical investigation of the low-velocity impact response of circular orthotropic and transversely isotropic plates possessing curvilinear anisotropy under compressive preloading has been carried out recently by Rossikhin and Shitikova in [1] and [2], respectively. The equations of plate motion take the rotatory inertia and transverse shear deformations into account. In the case of the orthotropic target [1], the changes in the geometrical dimensions of the contact domain have been ignored and the contact interaction is modeled by a linear spring, and a force arising in it is the linear approximation of Herts?z contact force. Ambartsumian [3] examined the five fundamental differential equations describing the equilibrium of an orthotropic plate with a cylindrical anisotropy for the case when all radial planes crossing

Ref

1. Yu. A. Rossikhin and Shitikova M. V. (2006): Dynamic stability of a circular pre-stressed elastic orthotropic plate subjected to shock excitation, Shock Vibr. 13, pp. 197?214.

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the axis of anisotropy are the planes of elastic symmetry. Sveklo [4] suggested the contact theory for two anisotropic bodies under compression according to which the contact pressure is distributed over an elliptical contact region. The same structural effects are also true of the concrete slab in a composite girder bridge, but the steel orthotropic deck is considerably lighter, and therefore allows longer span bridges to be more efficiently designed. Awodola [5] studied the effect of plate parameters on the vibrations under moving masses of elastically supported plate resting on bi-parametric foundation with stiffness variation.Szekrenyes [6] investigated the interface fracture in orthotropic composite plates using second order shear deformation theory. Kadari [7] analyzed buckling in orthotropic nanoscale plates resting on elastic foundations. Yan [8] proposed elastic orthotropic models and used these in the nonlinear analysis of concrete structures subjected to monotonic or pseudo dynamic loading. Since these models can appropriately describe the strain softening behavior of concrete beyond the peak stress and show good agreement with the strength envelope obtained from experimental results Hu and Yao [9] studied the vibration solutions of rectangular orthotropic plates by symplectic geometry method. In the same vien, Alshaya, Hunt and Rowlands [10] investigated stresses and strains in thick perforated orthotropic plates. Gbadeyan and Dada [11] found the natural frequency of rectangular plates traversed by moving concentrated masses. Awodola and Adeoye [13] investigated the behavior of simply supported orthotropic rectangular plate by applying the technique of variable separable. Adeoye and Awodola [14] studied the dynamic behavior of orthotropic rectangular plate with clamped-clamped boundary conditions by making use of the technique of Shadnam Due to inability of researchers to solve orthotropic plates problems by analytical methods, this work aims at solving the governing equation by analytical solution and also considers the effect of the flexural rigidities in both x and y directions.

II. Governing Equation

The dynamic transverse displacement $W(x,y,t)$ of orthotropic rectangular plates when it is resting on a bi-parametric elastic foundation and traversed by distributed mass M_r moving with constant velocity c_r along a straight line parallel to the x-axis issuing from point y=s on the y-axis with flexural rigidities D_x and D_y is governed by the fourth order partial differential equation given as

$$D_x\frac{\partial^4}{\partial x^4}W(x,y,t)+2B\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)+D_y\frac{\partial^4}{\partial y^4}W(x,y,t)+\mu\frac{\partial^2}{\partial t^2}W(x,y,t)-\rho hR_0$$

$$\left[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)\right]+K_0(4x-3x^2+x^3)W(x,y,t)+S_0(-13+$$

$$12x+3x^2)\frac{\partial}{\partial x}W(x,y,t)-S_0(12-13x+6x^2+x^3)\left[\frac{\partial^2}{\partial x^2}W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)\right] \quad (2.1)$$

$$-\sum_{r=1}^{N}\left[M_rgH(x-ct)H(y-s)-M_r\left(\frac{\partial^2}{\partial t^2}W(x,y,t)+2c_r\frac{\partial^2}{\partial x\partial t}W(x,y,t)+c_r^2\right.\right.$$

Ref

10. Alshaya, A., John H., Rowlands R. (2016): Stresses and Strains in Thick Perforated Orthotropic Plates. Journal of Engineering Mechanics. vol. 142(11), 10p.

$$\frac{\partial^2}{\partial x^2}W(x,y,t)\Bigg)H(x-c_rt)H(y-s)\Bigg]=0$$

where D_x and D_y are the flexural rigidities of the plate along x and y axes respectively.

Notes

$$D_x=\frac{E_xh^3}{12(1-\nu_x\nu_y)},\quad D_y=\frac{E_yh^3}{12(1-\nu_x\nu_y)},\quad B=D_xD_y+\frac{G_oh^3}{6} \tag{2.2}$$

E_x and E_y are the Young's moduli along x and y axes respectively, G_o is the rigidity modulus, ν_x and ν_y are Poisson's ratios for the material such that $E_x\nu_y=E_y\nu_x$, ρ is the mass density per unit volume of the plate, h is the plate thickness, t is the time, x and y are the spatial coordinates in x and y directions respectively, R_o is the rotatory inertia correction factor, K_o is the foundation constant, S_o shear modulus and g is the acceleration due to gravity, H(.) is the Heaviside function.

Rewriting equation (2.1), one obtains

$$\mu\frac{\partial^2}{\partial t^2}W(x,y,t)+\mu\omega_n^2W(x,y,t)=\rho hR_0\left[\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)\right]-2B$$

$$\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)-D_y\frac{\partial^4}{\partial y^4}W(x,y,t)-D_x\frac{\partial^4}{\partial x^4}W(x,y,t)+\mu\omega_n^2W(x,y,t)-K_0(4x-$$

$$3x^2+x^3)W(x,y,t)+G_o(-13+12x+3x^2)\frac{\partial}{\partial x}W(x,y,t)-G_0(12-13x+6x^2+x^3)\Bigg[\tag{2.3}$$

$$\frac{\partial^2}{\partial x^2}W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)\Bigg]+\sum_{r=1}^{N}\Bigg[M_rgH(x-c_rt)H(y-s)-M_r\Bigg(\frac{\partial^2}{\partial t^2}W(x,y,t)$$

$$+2c_r\frac{\partial^2}{\partial x\partial t}W(x,y,t)+c_r^2\frac{\partial^2}{\partial x^2}W(x,y,t)\Bigg)H(x-c_rt)H(y-s)\Bigg]$$

Simplifying equation (2.3) further, one obtains

$$\frac{\partial^2}{\partial t^2}W(x,y,t)+\omega_n^2W(x,y,t)=\sum_{r=1}^{N}\Bigg[R_0\Bigg(\frac{\partial^4}{\partial x^2\partial t^2}W(x,y,t)+\frac{\partial^4}{\partial y^2\partial t^2}W(x,y,t)\Bigg)-\frac{2B}{\mu}$$

$$\frac{\partial^4}{\partial x^2\partial y^2}W(x,y,t)-\frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}W(x,y,t)-\frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}W(x,y,t)+\omega_n^2W(x,y,t)-\frac{K_0}{\mu}(4x-3x^2$$

$$+x^3)W(x,y,t)+\frac{G_o}{\mu}(-13+12x+3x^2)\frac{\partial}{\partial x}W(x,y,t)-\frac{G_0}{\mu}(12-13x+6x^2+x^3)\Bigg(\frac{\partial^2}{\partial x^2}$$

$$W(x,y,t)+\frac{\partial^2}{\partial y^2}W(x,y,t)\Bigg)+\sum_{r=1}^{N}\frac{M_r}{\mu}gH(x-c_rt)H(y-s)-\frac{M_r}{\mu}\Bigg(\frac{\partial^2}{\partial t^2}W(x,y,t)+2c_r$$

$$\frac{\partial^2}{\partial x\partial t}W(x,y,t)+c_r^2\frac{\partial^2}{\partial x^2}W(x,y,t)\Bigg)H(x-c_rt)H(y-s)\Bigg)\Bigg] \tag{2.4}$$

where ω_n^2 is the natural frequencies, $n=1,2,3,\ldots$

The initial conditions, without any loss of generality, is taken as

$$W(x,y,t)=0=\frac{\partial}{\partial t}W(x,y,t) \tag{2.5}$$

III. Analytical Approximate Solution

In order to solve equation (2.4), one applies technique of Shadnam et al which requires that the deflection of the plates be in series form as

$$W(x,y,t)=\sum_{n=1}^{N}\Psi_n(x,y)Q_n(t) \tag{3.1}$$

where

$$\Psi_n(x,y)=\Psi_{jm}(x)\Psi_{hm}(y)$$

$$\Psi_{jm}x=\sin\zeta_{jm}x+A_{jm}\cos\zeta_{jm}x+B_{jm}\sinh\zeta_{jm}x+C_{jm}\cosh\zeta_{jm}x$$

$$\Psi_{hm}y=\sin\varphi_{hm}y+A_{hm}\cos\varphi_{hm}y+B_{hm}\sinh\varphi_{hm}y+C_{hm}\cosh\varphi_{hm}y$$

$$\zeta_{jm}=\frac{\phi_{jm}}{L_x},\qquad \varphi_{hm}=\frac{\phi_{hm}}{L_y}$$

The right hand side of equation (2.4), taken into account equation (3.1), written in the form of series takes the form

$$\sum_{n=1}^{\infty}\Bigg[R_0\Bigg(\frac{\partial^2}{\partial x^2}\Psi_n(x,y)\ddot{Q}_n(t)+\frac{\partial^4}{\partial y^2}\Psi_n(x,y)\ddot{Q}_n(t)\Bigg)-\frac{2B}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}\Psi_n(x,y)Q_n(t)-\frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}$$

$$\Psi_n(x,y)Q_n(t)-\frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}\Psi_n(x,y)Q_n(t)+\omega_n^2\Psi_n(x,y)Q_n(t)-\frac{K_0}{\mu}(4x-3x^2+x^3)\Psi_n(x,y)$$

$$Q_n(t)+\frac{G_o}{\mu}(-13+12x+3x^2)\frac{\partial}{\partial x}\Psi_n(x,y)Q_n(t)-\frac{G_0}{\mu}(12-13x+6x^2+x^3)\Bigg(\frac{\partial^2}{\partial x^2}\Psi_n(x,y)$$

Notes

$$Q_n(t) + \frac{\partial^2}{\partial y^2}\Psi_n(x,y)Q_n(t)\Bigg) + \sum_{r=1}^{N}\Bigg(\frac{M_r}{\mu}gH(x-c_rt)H(y-s) - \frac{M_r}{\mu}\Bigg(\Psi_n(x,y)\ddot{Q}_n(t)+$$

$$2c\frac{\partial}{\partial x}\Psi_n(x,y)\dot{Q}_n(t) + c_r^2\frac{\partial^2}{\partial x^2}\Psi_n(x,y)Q_n(t)\Bigg)H(x-c_rt)H(y-s)\Bigg)\Bigg] = \sum_{n=1}^{N}\Psi_n(x,y)\Theta_n(t) \quad (3.2)$$

Notes

Multiplying both sides of equation (3.2) by $\Psi_m(x,y)$ and integrating on area A of the plate and considering the orthogonality of $\Psi_n(x,y)$, one obtains

$$\Theta_n(t) = \frac{1}{\theta^*}\sum_{n=1}^{\infty}\int_A\Bigg[R_0\Bigg(\frac{\partial^2}{\partial x^2}\Psi_n(x,y)\ddot{Q}_n(t) + \frac{\partial^4}{\partial y^2}\Psi_n(x,y)\ddot{Q}_n(t)\Bigg) - \frac{2B}{\mu}\frac{\partial^4}{\partial x^2\partial y^2}\Psi_n(x,y)$$

$$Q_n(t) - \frac{D_y}{\mu}\frac{\partial^4}{\partial y^4}\Psi_n(x,y)Q_n(t) - \frac{D_x}{\mu}\frac{\partial^4}{\partial x^4}\Psi_n(x,y)Q_n(t) + \omega_n^2\Psi_n(x,y)Q_n(t) - \frac{K_0}{\mu}(4x-$$

$$3x^2+x^3)\Psi_n(x,y)Q_n(t) + \frac{G_o}{\mu}(-13+12x+3x^2)\frac{\partial}{\partial x}\Psi_n(x,y)Q_n(t) - \frac{G_0}{\mu}(12-13x$$

$$+6x^2+x^3)\Bigg(\frac{\partial^2}{\partial x^2}\Psi_n(x,y)Q_n(t) + \frac{\partial^2}{\partial y^2}\Psi_n(x,y)Q_n(t)\Bigg) + \sum_{r=1}^{N}\Bigg(\frac{M_r}{\mu}gH(x-c_rt)H(y-s)$$

$$-\frac{M_r}{\mu}\Bigg(\Psi_n(x,y)\ddot{Q}_n(t) + 2c\frac{\partial}{\partial x}\Psi_n(x,y)\dot{Q}_n(t) + c_r^2\frac{\partial^2}{\partial x^2}\Psi_n(x,y)Q_n(t)\Bigg)H(x-c_rt)H(y-s)$$

$$\Bigg)\Bigg]\Psi_m(x,y)dA \quad (3.3)$$

and zero when $n \neq m$
where

$$\theta^* = \int_A \Psi_n^2(x,y)dA \quad (3.4)$$

Making use of equation (3.3) and taking into account equation (3.2), equation (2.4) can be written as

$$\Psi_n(x,y)\Big[\omega_n^2Q_n(t) + \ddot{Q}_n(t)\Big] = \frac{\Psi_n(x,y)}{\theta^*}\sum_{q=1}^{\infty}\int_A\Bigg[R_0 \quad \frac{\partial^2\Psi_q(x,y)}{\partial x^2}\Psi_m(x,y)\ddot{Q}_q(t) + \frac{\partial^2\Psi_q(x,y)}{\partial y^2}$$

$$\Psi_m(x,y)\ddot{Q}_q(t)\Bigg) - \frac{2B}{\mu}\frac{\partial^2\Psi_q(x,y)}{\partial x^2\partial y^2}\Psi_m(x,y)Q_q(t) - \frac{D_y}{\mu}\frac{\partial^4\Psi_q(x,y)}{\partial y^4}\Psi_m(x,y)Q_q(t) + \frac{D_x}{\mu}\frac{\partial^4\Psi_q(x,y)}{\partial x^4}$$

$$\Psi_m(x,y)Q_q(t) - \frac{K_0}{\mu}(4x-3x^2+x^3)\Psi_q(x,y)\Psi_m(x,y)Q_q(t) + \frac{G_o}{\mu}(-13+12x+3x^2)\frac{\partial \Psi_q(x,y)}{\partial x}$$

$$\Psi_m(x,y)Q_q(t) - \frac{G_o}{\mu}(12-13x+6x^2+x^3)\Big(\frac{\partial^2\Psi_q(x,y)}{\partial x^2}\Psi_m(x,y)Q_q(t) + \frac{\partial^2\Psi_q(x,y)}{\partial y^2}\Psi_m(x,y)$$

$$Q_q(t)\Bigg) + \sum_{r=1}^{N}\Bigg(\frac{M_r}{\mu}g\Psi_m(x,y)H(x-c_rt)H(y-s) - \frac{M_r}{\mu}\Bigg(\Psi_q(x,y)\Psi_m(x,y)\ddot{Q}_q(t) + 2c_r$$

$$\frac{\partial \Phi_q(x,y)}{\partial x}\Phi_m(x,y)\dot{Q}_q(t) + c_r^2\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)Q_q(t)\Bigg)H(x-c_rt)H(y-s)\Bigg)\Bigg]dA \qquad (3.5)$$

On further simplification of equation (3.5), one obtains

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) = \frac{1}{\theta^*}\sum_{q=1}^{\infty}\int_A\Bigg[R_0\Big(\frac{\partial^2\Psi_q(x,y)}{\partial x^2}\Psi_m(x,y)\ddot{Q}_q(t) + \frac{\partial^2\Psi_q(x,y)}{\partial y^2}\Psi_m(x,y)\ddot{Q}_q(t)\Big)$$

$$-\frac{2B}{\mu}\frac{\partial^2\Psi_q(x,y)}{\partial x^2\partial y^2}\Psi_m(x,y)Q_q(t) - \frac{D_y}{\mu}\frac{\partial^4\Psi_q(x,y)}{\partial y^4}\Psi_m(x,y)Q_q(t) - \frac{D_x}{\mu}\frac{\partial^4\Psi_q(x,y)}{\partial x^4}\Psi_m(x,y)$$

$$Q_q(t) + \omega_q^2\Psi_q(x,y)\Psi_m(x,y)Q_n(t) - \frac{K_0}{\mu}(4x-3x^2+x^3)\Psi_q(x,y)\Psi_m(x,y)Q_q(t) + \frac{G_o}{\mu}(-13$$

$$+12x+3x^2)\frac{\partial\Psi_q(x,y)}{\partial x}\Psi_m(x,y)Q_q(t) - \frac{G_o}{\mu}(12-13x+6x^2+x^3)\Big(\frac{\partial^2\Psi_q(x,y)}{\partial x^2}\Psi_m(x,y)Q_q(t)$$

$$+\frac{\partial^2\Psi_q(x,y)}{\partial y^2}\Psi_m(x,y)Q_q(t)\Bigg) + \sum_{r=1}^{N}\Bigg(\frac{M_r}{\mu}g\Psi_m(x,y)H(x-c_rt)H(y-s) - \frac{M_r}{\mu}\Bigg(\Psi_q(x,y)\Psi_m(x,y)$$

$$\ddot{Q}_q(t) + 2c_r\frac{\partial\Phi_q(x,y)}{\partial x}\Phi_m(x,y)\dot{Q}_q(t) + c_r^2\frac{\partial^2\Phi_q(x,y)}{\partial x^2}\Phi_m(x,y)Q_q(t)\Bigg)H(x-c_rt)H(y-s)\Bigg)\Bigg]dA$$

$$(3.6)$$

The system of equations in equation (3.6) is a set of coupled ordinary differential equations where $H(x-c_rt)$ and $H(y-s)$ are the Heaviside functions which are defined as

$$H\,(x-c_rt) = \begin{cases} 1, for\ \ x \geq c_rt \\ 0, for\ \ x < c_rt \end{cases}, \qquad H\,(y-s) = \begin{cases} 1, for\ \ y \geq s \\ 0, for\ \ y < s \end{cases} \qquad (3.7)$$

With the properties

$$(i)\frac{d}{dx}[H(x-c_rt)] = \delta(x-c_rt), \qquad \frac{d}{dy}[H(y-s)] = \delta(y-s) \qquad (3.8)$$

Notes

$$(ii) f(x)H\left(x-c_r t\right)=\begin{cases} f(x), for \ x\geq c_r t \\ 0, for \ x<c_r t \end{cases}, \qquad f(y)H\left(y-s\right)=\begin{cases} f(y), for \ y\geq s \\ 0, for \ y<s \end{cases} \tag{3.9}$$

Using the Fourier series representation, the Heaviside functions take the form

Notes

$$H(x-c_r t)=\frac{1}{4}+\frac{1}{\pi}\sum_{r=1}^{N}\frac{\sin(2n+1)\pi(x-c_r t)}{2n+1}, 0<x<1 \tag{3.10}$$

$$H(y-s)=\frac{1}{4}+\frac{1}{\pi}\sum_{r=1}^{N}\frac{\sin(2n+1)\pi(y-s)}{2n+1}, 0<y<1 \tag{3.11}$$

On putting equations (3.7) to (3.11) into equation (3.6) and simplifying, one obtains

$$\ddot{Q}_n(t)+\omega_n^2 Q_n(t)-\frac{1}{\theta^*}\sum_{q=1}^{\infty}\Bigg[R_0 T_1\ddot{Q}_q(t)-\frac{2B}{\mu}T_2 Q_q(t)-\frac{D_y}{\mu}T_3 Q_q(t)-\frac{D_x}{\mu}T_4 Q_q(t)+(\omega_q^2 F_4^*-$$

$$\frac{K_0}{\mu}F_5^*)Q_q(t)+\frac{G_0}{\mu}(T_6+T_7)Q_q(t)-\sum_{r=1}^{N}\frac{M_r}{\mu}\Bigg(\Bigg(T_8+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_1^*\frac{\cos(2j+1)\pi c_r t}{2j+1}-\sum_{j=1}^{\infty}E_2^*$$

$$\frac{\sin(2j+1)\pi c_r t}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_3^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_4^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_5^*\frac{\cos(2j+1)\pi c_r t}{2j+1}$$

$$-\sum_{j=1}^{\infty}E_6^*\frac{\sin(2j+1)\pi c_r t}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty}E_7^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_8^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\ddot{Q}_q(t)+$$

$$2c_r\Bigg(T_9+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_9^*\frac{\cos(2j+1)\pi c_r t}{2j+1}-\sum_{j=1}^{\infty}E_{10}^*\frac{\sin(2j+1)\pi c_r t}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_{11}^*\frac{\cos(2k+1)\pi s}{2k+1}$$

$$-\sum_{k=1}^{\infty}E_{12}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_{13}^*\frac{\cos(2j+1)\pi c_r t}{2j+1}-\sum_{j=1}^{\infty}E_{14}^*\frac{\sin(2j+1)\pi c_r t}{2j+1}\Bigg)$$

$$+\frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty}E_{15}^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{16}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\dot{Q}_q(t)+c_r^2\Bigg(T_{10}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_{17}^*$$

$$\frac{\cos(2j+1)\pi c_r t}{2j+1}-\sum_{j=1}^{\infty}E_{18}^*\frac{\sin(2j+1)\pi c_r t}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_{19}^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{20}^*\frac{\sin(2k+1)\pi s}{2k+1}$$

$$\Bigg) + \frac{1}{4\pi}\left(\sum_{j=1}^{\infty} E_{21}^{*}\frac{\cos(2j+1)\pi c_r t}{2j+1} - \sum_{j=1}^{\infty} E_{22}^{*}\frac{\sin(2j+1)\pi c_r t}{2j+1}\right) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^{*}\frac{\cos(2k+1)\pi s}{2k+1}$$

$$-\sum_{k=1}^{\infty} E_{24}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)Q_q(t)\Bigg)\Bigg] = \sum_{q=1}^{\infty}\sum_{r=1}^{N}\frac{M_r g}{\mu\theta^{*}}\Psi_m(ct)\Psi_m(s) \tag{3.12}$$

Notes

which is the transformed equation governing the problem of an orthotropic rectangular plate resting on bi-parametric elastic foundation.
where

$$T_1 = \int_A \left[\frac{\partial^2}{\partial x^2}\Psi_q(x,y)\Psi_m(x,y) + \frac{\partial^2}{\partial y^2}\Psi_q(x,y)\Psi_m(x,y)\right] dA \tag{3.13}$$

$$T_2 = \int_A \frac{\partial^2}{\partial x^2}\left[\frac{\partial^2}{\partial x^2}\Psi_q(x,y)\right]\Psi_m(x,y)dA \tag{3.14}$$

$$T_3 = \int_A \frac{\partial^4}{\partial y^4}\left[\Psi_q(x,y)\right]\Psi_m(x,y)dA \tag{3.15}$$

$$T_4 = \int_A \frac{\partial^4}{\partial x^4}\left[\Psi_q(x,y)\right]\Psi_m(x,y)dA \tag{3.16}$$

$$F_4^{*} = \int_A \Psi_q(x,y)\Psi_m(x,y)dA \tag{3.17}$$

$$T_5 = 4U_1 - 3U_2 + U_3, \qquad T_6 = -13A_1 + 12A_2 + 3A_3 \tag{3.18}$$

$$T_7 = 12f_1 - 13f_2 + 6f_3 + f_4 + 12f_5 - 13f_6 + 6f_7 + f_8 \tag{3.19}$$

$$T_8 = \frac{1}{16}\int_A \Psi_q(x,y)\Psi_m(x,y)dA \tag{3.20}$$

$$E_1^{*} = \int_A \Psi_q(x,y)\Psi_m(x,y)\sin(2j+1)\pi x dA \tag{3.21}$$

$$E_2^{*} = \int_A \Psi_q(x,y)\Psi_m(x,y)\cos(2j+1)\pi x dA \tag{3.22}$$

$$E_3^{*} = \int_A \Psi_q(x,y)\Psi_m(x,y)\sin(2k+1)\pi y dA \tag{3.23}$$

$$E_4^{*} = \int_A \Psi_q(x,y)\Psi_m(x,y)\cos(2k+1)\pi y dA \tag{3.24}$$

$$E_5^{*} = E_1^{*}, \quad E_6^{*} = E_2^{*}, \quad E_7^{*} = E_3^{*}, \quad E_8^{*} = E_4^{*} \tag{3.25}$$

$$T_9 = \frac{1}{16}\int_A \frac{\partial}{\partial x}\Psi_q(x,y)\Psi_m(x,y)dA \tag{3.26}$$

$$E_9^* = \int_A \frac{\partial}{\partial x}\left(\Psi_q(x,y)\right)\Psi_m(x,y)\sin(2j+1)\pi x dA \tag{3.27}$$

Notes

$$E_{10}^* = \int_A \frac{\partial}{\partial x}\left(\Psi_q(x,y)\right)\Psi_m(x,y)\cos(2j+1)\pi x dA \tag{3.28}$$

$$E_{11}^* = \int_A \frac{\partial}{\partial x}\left(\Psi_q(x,y)\right)\Psi_m(x,y)\sin(2k+1)\pi y dA \tag{3.29}$$

$$E_{12}^* = \int_A \frac{\partial}{\partial x}\Psi_q(x,y)\Psi_m(x,y)\cos(2k+1)\pi y dA \tag{3.30}$$

$$E_{13}^* = E_9^*, \quad E_{14}^* = E_{10}^*, \quad E_{15}^* = E_{11}^*, \quad E_{16}^* = E_{12}^* \tag{3.31}$$

$$T_{10} = \frac{1}{16}\int_A \frac{\partial^2}{\partial x^2}\left(\Psi_q(x,y)\right)\Psi_m(x,y)dA \tag{3.32}$$

$$E_{17}^* = \int_A \frac{\partial^2}{\partial x^2}\left(\Psi_q(x,y)\right)\Psi_m(x,y)\sin(2j+1)\pi x dA \tag{3.33}$$

$$E_{18}^* = \int_A \frac{\partial^2}{\partial x^2}\left(\Psi_q(x,y)\right)\Psi_m(x,y)\cos(2j+1)\pi x dA \tag{3.34}$$

$$E_{19}^* = \int_A \Psi_q(x,y)\Psi_m(x,y)\sin(2k+1)\pi y dA \tag{3.35}$$

$$E_{20}^* = \int_A \frac{\partial^2}{\partial x^2}\left(\Psi_q(x,y)\right)\Psi_m(x,y)\cos(2k+1)\pi y dA \tag{3.36}$$

$$E_{21}^* = E_{17}^*, \quad E_{22}^* = E_{18}^*, \quad E_{23}^* = E_{19}^*, \quad E_{24}^* = E_{20}^* \tag{3.37}$$

$\Psi_m(x,y)$ is assumed to be the products of functions $\Psi_{pm}(x)\Psi_{bm}(y)$ which are the beam functions in the directions of x and y axes respectively. That is

$$\Psi_m(x,y) = \Psi_{pm}(x)\Psi_{bm}(y) \tag{3.38}$$

where

$$\Phi_m(x) = \sin\frac{\Gamma_m x}{L_x} + A_m\cos\frac{\Gamma_m x}{L_x} + B_m\sinh\frac{\Gamma_m x}{L_x} + C_m\cosh\frac{\Gamma_m x}{L_x} \tag{3.39}$$

$$\Phi_m(y) = \sin\frac{\Gamma_m y}{L_y} + A_m\cos\frac{\Gamma_m y}{L_y} + B_m\sinh\frac{\Gamma_m y}{L_y} + C_m\cosh\frac{\Gamma_m y}{L_y} \tag{3.40}$$

Notes

where A_{pm}, B_{pm}, C_{pm}, A_{bm}, B_{bm} and C_{bm} are constants determined by the boundary conditions. And Ψ_{pm} and Ψ_{bm} are called the mode frequencies
where

$$\lambda_{pm} = \frac{\xi_{pm}}{L_x}, \qquad \lambda_{bm} = \frac{\xi_{bm}}{L_y} \tag{3.41}$$

Considering a unit mass, equation (3.12) can be re-written as

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) - \frac{1}{\theta^*}\sum_{q=1}^{\infty}\Bigg[R_0T_1\ddot{Q}_q(t) - \frac{2B}{\mu}T_2Q_q(t) - \frac{D_y}{\mu}T_3Q_q(t) - \frac{D_x}{\mu}T_4Q_q(t) +$$

$$(\omega_q^2F_4^* - \frac{K_0}{\mu}T_5)Q_q(t) + \frac{G_0}{\mu}(T_6+T_7)Q_q(t) - \alpha\varrho\Bigg(\Bigg(T_8 + \frac{1}{\pi^2}\quad\sum_{j=1}^{\infty}E_1^*\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$-\sum_{j=1}^{\infty}E_2^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_3^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty}E_4^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg($$

$$\sum_{j=1}^{\infty}E_5^*\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty}E_6^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty}E_7^*\frac{\cos(2k+1)\pi s}{2k+1} -$$

$$\sum_{k=1}^{\infty}E_8^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\ddot{Q}_q(t) + 2c\Bigg(T_9 + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_9^*\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty}E_{10}^*$$

$$\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_{11}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty}E_{12}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\quad\sum_{j=1}^{\infty}E_{13}^*$$

$$\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty}E_{14}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty}E_{15}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty}E_{16}^*$$

$$\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\dot{Q}_q(t) + c^2\Bigg(T_{10} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_{17}^*\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty}E_{18}^*\frac{\sin(2j+1)\pi ct}{2j+1}$$

Notes

$$\Bigg)\Bigg(\sum_{k=1}^{\infty} E_{19}^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} B_{20}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_{21}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$-\sum_{j=1}^{\infty} E_{22}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)$$

$$\Bigg)Q_q(t)\Bigg)\Bigg] = \sum_{q=1}^{\infty}\sum_{r=1}^{N}\frac{Mg}{\mu\theta^{*}}\Psi_m(ct)\Psi_m(s) \tag{3.42}$$

equation (3.42) is the fundamental equation of the problem. where

$$\alpha = \frac{M}{\mu\varrho}, \quad \varrho = L_x L_y \tag{3.43}$$

$$\Psi_m(ct) = \sin\chi_m t + A_m\cos\chi_m t + B_m\sinh\chi_m t + C_m\cosh\chi_m t \tag{3.44}$$

$$\Psi_m(s) = \sin\nu_m + A_m\cos\nu_m + B_m\sinh\nu_m + C_m\cosh\nu_m \tag{3.45}$$

$$\chi_m = \frac{\phi_m c}{L_x}, \ \nu_m = \frac{\phi_m s}{L_y} \tag{3.46}$$

We shall consider the situation where the orthotropic rectangular plate is simply supported at all its edges. The boundary conditions for an orthotropic rectangular plate having simple supports at all its edges are given by

$$W(0,y,t) = 0 = W(L_x,y,t) = 0 \tag{3.47}$$

$$W(x,0,t) = 0 = W(x,L_y,t) \tag{3.48}$$

$$\frac{\partial^2}{\partial x^2}W(0,y,t) = 0 = \frac{\partial^2}{\partial x^2}W(L_x,y,t) = 0 \tag{3.49}$$

$$\frac{\partial^2}{\partial y^2}W(0,y,t) = 0 = \frac{\partial^2}{\partial y^2}W(x,L_y,t) = 0 \tag{3.50}$$

$$\Psi_m(0) = \Psi_m(L_x) \tag{3.51}$$

$$\Psi_m(0) = \Psi_m(L_y) \tag{3.52}$$

$$\frac{\partial^2}{\partial x^2}\Psi_m(0) = \frac{\partial^2}{\partial x^2}\Psi_m(L_x) \tag{3.53}$$

$$\frac{\partial^2}{\partial y^2}\Psi_m(0) = \frac{\partial^2}{\partial y^2}\Psi_m(L_y) \tag{3.54}$$

$$\Psi_m(x) = \sin\frac{\Gamma_m x}{L_x} + A_m \cos\frac{\Gamma_m x}{L_x} + B_m \sinh\frac{\Gamma_m x}{L_x} + C_m \cosh\frac{\Gamma_m x}{L_x} \tag{3.55}$$

$$\Psi_m(y) = \sin\frac{\Gamma_m y}{L_y} + A_m \cos\frac{\Gamma_m y}{L_y} + B_m \sinh\frac{\Gamma_m y}{L_y} + C_m \cosh\frac{\Gamma_m y}{L_y} \tag{3.56}$$

Notes

On solving equations (3.51) and (3.53) simultaneously, one obtains

$$A_m = 0,\ B_m = 0,\ C_m = 0 \tag{3.57}$$

$$\Gamma_m = m\pi \tag{3.58}$$

On putting equations (3.55) to (3.58) into equations (3.13) to (3.37), the integrals become

$$T_1 = -\left[\frac{\pi^2 q^2}{L_x^2} + \frac{\pi^2 q^2}{L_y^2}\right]\int_0^{L_x} \sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.59}$$

$$T_2 = \frac{\pi^4 q^4}{L_x^2 L_y^2}\int_0^{L_x} \sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.60}$$

$$T_3 = \frac{\pi^4 q^4}{L_y^4}\int_0^{L_x} \sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.61}$$

$$T_4 = \frac{\pi^4 q^4}{L_x^4}\int_0^{L_y} \sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.62}$$

$$T_5 = 4\int_0^{L_x} x\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy - 3\int_0^{L_x} x^2\sin\frac{q\pi x}{L_x}$$

$$\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy + \int_0^{L_x} x^3\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y} \tag{3.63}$$

$$\sin\frac{m\pi y}{L_y}dy$$

$$T_6 = \frac{-13\pi q}{L_x}\int_0^{L_x} \cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx \int_0^{L_y} \sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy + \frac{12\pi q}{L_x}\int_0^{L_x} x\cos\frac{q\pi x}{L_x}$$

$$\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy+\frac{3\pi q}{L_x}\int_0^{L_x}x^2\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.64}$$

Notes

$$T_7=\frac{12\pi^2q^2}{L_x^2}\int_0^{L_x}\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy-\frac{13\pi^2q^2}{L_x^2}\int_0^{L_x}x\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy+\frac{6\pi^2q^2}{L_x^2}\int_0^{L_x}x^2\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy+\frac{\pi^2q^2}{L_x^2}\int_0^{L_x}x^3\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy+\frac{12\pi^2q^2}{L_y^2}\int_0^{L_x}\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy-\frac{13\pi^2q^2}{L_y^2}\int_0^{L_x}x\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy+\frac{6\pi^2q^2}{L_y^2}\int_0^{L_x}x^2\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy+\frac{\pi^2q^2}{L_y^2}\int_0^{L_x}x^3\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.65}$$

$$T_8=\frac{1}{16}\int_0^{L_x}\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.66}$$

$$E_1^*=\int_0^{L_x}\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}\sin(2j+1)\pi x dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.67}$$

$$E_2^*=\int_0^{L_x}\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}\cos(2j+1)\pi x dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.68}$$

$$E_3^*=\int_0^{L_x}\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}\sin(2k+1)\pi y dy \tag{3.69}$$

$$E_4^*=\int_0^{L_x}\sin\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}\cos(2k+1)\pi y dy \tag{3.70}$$

$$E_5^*=E_1^*,\quad E_6^*=E_2^*,\quad E_7^*=E_3^*,\quad E_8^*=E_4^* \tag{3.71}$$

Notes

$$T_9 = \frac{\pi q}{16L_x}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.72}$$

$$E_9^* = \frac{q\pi}{L_x}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}\sin(2j+1)\pi x dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.73}$$

$$E_{10}^* = \frac{q\pi}{L_x}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}\cos(2j+1)\pi x dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.74}$$

$$E_{11}^* = \frac{q\pi}{L_x}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}\sin(2k+1)\pi y dy \tag{3.75}$$

$$E_{12}^* = \frac{q\pi}{L_x}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}\cos(2k+1)\pi y dy \tag{3.76}$$

$$E_{13}^* = E_9^*, \quad E_{14}^* = E_{10}^*, \quad E_{15}^* = E_{11}^*, \quad E_{16}^* = E_{12}^* \tag{3.77}$$

$$T_{10} = -\frac{\pi^2 q^2}{16L_x^2}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.78}$$

$$E_{17}^* = -\frac{q^2\pi^2}{L_x^2}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}\sin(2j+1)\pi x dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.79}$$

$$E_{18}^* = -\frac{q^2\pi^2}{L_x^2}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}\cos(2j+1)\pi x dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}dy \tag{3.80}$$

$$E_{19}^* = \frac{q^2\pi^2}{L_x^2}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}\sin(2k+1)\pi y dy \tag{3.81}$$

$$E_{20}^* = -\frac{q^2\pi^2}{L_x^2}\int_0^{L_x}\cos\frac{q\pi x}{L_x}\sin\frac{m\pi x}{L_x}dx\int_0^{L_y}\sin\frac{q\pi y}{L_y}\sin\frac{m\pi y}{L_y}\cos(2k+1)\pi y dy \tag{3.82}$$

$$E_{21}^* = E_{17}^* \quad E_{22}^* = E_{18}^* \quad E_{23}^* = E_{19}^* \quad E_{24}^* = E_{20}^* \tag{3.83}$$

On solving equations (3.59) to (3.66),(3.72) and (3.78), and substituting into equation (47), one obtains

$$\ddot{Q}_n(t) + \omega_n^2 Q_n(t) - \frac{1}{\theta^*}\sum_{q=1}^{\infty}\Bigg[-\frac{R_0 L_x L_y}{4}\quad\frac{\pi^2 q^2}{L_x^2}+\frac{\pi^2 q^2}{L_y^2}\Bigg)\ddot{Q}_q(t) - \frac{2B\pi^4 q^4}{4\mu L_x L_y}Q_q(t) - \frac{D_y\pi^4 q^4 L_x}{4\mu L_y^3}$$

$$Q_q(t) - \frac{D_x\pi^4 q^4 L_y}{4\mu L_x^3}Q_q(t) + \left(\frac{L_x L_y}{4}\omega_q^2 - \frac{K_0}{\mu}T_5\right)Q_q(t) + \frac{G_0}{\mu}(T_6+T_7)Q_q(t) - \varpi\varrho^*\Bigg(\Bigg($$

Notes

$$\frac{L_xL_y}{64}+\frac{1}{\pi^2}\left(\sum_{j=1}^{\infty}E_1^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_2^*\frac{\sin(2j+1)\pi ct}{2j+1}\right)\left(\sum_{k=1}^{\infty}E_3^*\frac{\cos(2k+1)\pi s}{2k+1}\right.$$

$$\left.-\sum_{k=1}^{\infty}E_4^*\frac{\sin(2k+1)\pi s}{2k+1}\right)+\frac{1}{4\pi}\left(\sum_{j=1}^{\infty}E_5^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_6^*\frac{\sin(2j+1)\pi ct}{2j+1}\right)+\frac{1}{4\pi}$$

$$\left(\sum_{k=1}^{\infty}E_7^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_8^*\frac{\sin(2k+1)\pi s}{2k+1}\right)\Bigg)\ddot{Q}_q(t)+2c\left(\frac{-4mL_x}{64(q^2-m^2)\pi}+\frac{1}{\pi^2}\left(\sum_{j=1}^{\infty}E_9^*\right.\right.$$

$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{10}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\left(\sum_{k=1}^{\infty}E_{11}^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{12}^*\frac{\sin(2k+1)\pi s}{2k+1}\right.$$

$$\Bigg)+\frac{1}{4\pi}\left(\sum_{j=1}^{\infty}E_{13}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{14}^*\frac{\sin(2j+1)\pi ct}{2j+1}\right)+\frac{1}{4\pi}\left(\sum_{k=1}^{\infty}E_{15}^*\frac{\cos(2k+1)\pi s}{2k+1}\right.$$

$$\left.\left.-\sum_{k=1}^{\infty}E_{16}^*\frac{\sin(2k+1)\pi s}{2k+1}\right)\right)\dot{Q}_q(t)+c^2\left(\frac{-\pi^2q^2L_y}{64L_x}+\frac{1}{\pi^2}\left(\sum_{j=1}^{\infty}E_{17}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}\right.\right.$$

$$E_{18}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\left(\sum_{k=1}^{\infty}E_{19}^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{20}^*\frac{\sin(2k+1)\pi s}{2k+1}\right)+\frac{1}{4\pi}\left(\sum_{j=1}^{\infty}E_{21}^*\right.$$

$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{22}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\left(\sum_{k=1}^{\infty}E_{23}^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{24}^*\right.$$

$$\left.\left.\left.\frac{\sin(2k+1)\pi s}{2k+1}\right)\right)Q_q(t)\right)\Bigg]=\sum_{q=1}^{\infty}\frac{Mg}{\mu\theta^*}\Phi_m(ct)\Phi_m(s) \qquad (3.84)$$

The solutions to equation (3.84) shall be obtained by considering two cases:

a) Simply Supported Orthotropic Rectangular Plate Tranversed by Moving Force

For moving force problem, one sets $\varpi=0$ in equation (3.84) which becomes

$$\ddot{Q}_n(t)+\left(1-\frac{L_xL_y}{4\theta^*}\right)\omega_n^2Q_n(t)-\frac{1}{\mu\theta^*}\left(-\frac{\mu R_0L_xL_y}{4}\left(\frac{\pi^2n^2}{L_x^2}+\frac{\pi^2n^2}{L_y^2}\right)\ddot{Q}_n(t)-\frac{2B\pi^4n^4}{4L_xL_y}\right.$$

$$\left.Q_n(t)-\frac{D_y\pi^4n^4L_x}{4L_y^3}Q_n(t)-\frac{D_x\pi^4n^4L_y}{4L_x^3}Q_n(t)-K_0T_5Q_n(t)+\frac{G_0}{\mu}(T_6+T_7)Q_n(t)\right)$$

$$-\frac{1}{\mu\theta^*}\sum_{q=1,q\neq=n}^{\infty}\Bigg(-\frac{\mu R_0 L_x L_y}{4}\quad\frac{\pi^2 q^2}{L_x^2}+\frac{\pi^2 q^2}{L_y^2}\Bigg)\ddot{Q}_q(t)-\frac{2B\pi^4 q^4}{4L_xL_y}Z_q(t)-\frac{D_y\pi^4q^4L_x}{4L_y^3}$$

$$Z_q(t)-\frac{D_x\pi^4q^4L_y}{4L_x^3}Q_q(t)+\Bigg(\frac{L_xL_y}{4}\mu\omega_q^2-K_0T_5\Bigg)Q_q(t)-G_0(T_6+T_7)Q_q(t)\Bigg)$$

$$=\sum_{q=1}^{\infty}\frac{Mg}{\mu\theta^*}\sin\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x} \tag{3.85}$$

On further simplification and re-arrangement, one obtains

$$\Bigg[1+\lambda\frac{\mu R_0 L_x L_y}{4}\quad\frac{\pi^2 n^2}{L_x^2}+\frac{\pi^2 n^2}{L_y^2}\Bigg)\Bigg]\ddot{Q}_n(t)+\Bigg(\epsilon_n^2-\lambda\Bigg(-\frac{2B\pi^4n^4}{4L_xL_y}-\frac{D_y\pi^4n^4L_x}{4L_y^3}-\frac{D_x\pi^4q^4L_y}{4L_x^3}$$

$$-K_0T_5Q_n(t)+G_0(T_6+T_7)Q_n(t)\Bigg)\Bigg)Q_n(t)-\lambda\sum_{q=1,q\neq=n}^{\infty}\Bigg(-\frac{\mu R_0L_xL_y}{4}\Bigg(\frac{\pi^2q^2}{L_x^2}+\frac{\pi^2q^2}{L_y^2}\Bigg)$$

$$\ddot{Q}_q(t)-\frac{2B\pi^4q^4}{4L_xL_y}Q_q(t)-\frac{D_y\pi^4q^4L_x}{4L_y^3}Q_q(t)-\frac{D_x\pi^4q^4L_y}{4L_x^3}Q_q(t)+\Bigg(\frac{L_xL_y}{4}\mu\omega_q^2-K_0T_5\Bigg)$$

$$Q_q(t)+G_0(T_6+T_7)Q_q(t)\Bigg)=\sum_{q=1}^{\infty}Mg\lambda\sin\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x} \tag{3.86}$$

where

$$\epsilon_n^2=\Bigg(1-\frac{L_xL_y}{4\theta^*}\Bigg)\omega_n^2\qquad\lambda=\frac{1}{\mu\theta^*} \tag{3.87}$$

Consider a parameter $\lambda* < 1$ for any arbitrary mass ratio λ, defined as

$$\lambda=\frac{\lambda^*}{1+\lambda^*} \tag{3.88}$$

It can be shown that

$$\lambda=\lambda^*-o(\lambda^{*2}) \tag{3.89}$$

Retaining only $o(\lambda^*)$, one obtains

$$\lambda=\lambda^* \tag{3.90}$$

On putting equation (3.89) into equation (3.86),rewriting and simplifying further , one obtains

Notes

Notes

$$\ddot{Q}_n(t) + \Bigg[\omega_n^2\left(1 - \lambda^*\frac{\mu R_0 L_x L_y}{4}\quad\frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\right) + o(\lambda^{*2}) + \ldots\Bigg) - \lambda^* \quad \frac{2B\pi^4 n^4}{4L_xL_y} - \frac{D_x\pi^4 n^4 L_y}{4L_x^3}$$

$$- \frac{D_y\pi^4 n^4 L_x}{4L_y^3} + \left(\mu\omega_n^2 - \frac{K_0 L_y}{4}\varphi_1(x)\right) + \frac{G_0 L_y}{4}\quad\frac{\pi n}{L_x}\varphi_2(x) + \left(\frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\right)\varphi_3(x)\Bigg)\Bigg(1-$$

$$\lambda^*\frac{\mu R_0 L_x L_y}{4}\left(\frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\right) + o(\lambda^{*2}) + \ldots\Bigg)\Bigg)\Bigg]Q_n(t) - \eta^*\Bigg(1 - \lambda^*\frac{\mu R_0 L_x L_y}{4}\Bigg(\frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}$$

$$\Bigg) + o(\lambda^{*2}) + \ldots\Bigg)\Bigg) \sum_{q=1, q\neq=n}^{\infty}\Bigg(-\frac{\mu R_0 L_x L_y}{4}\quad\frac{\pi^2 q^2}{L_x^2} + \frac{\pi^2 q^2}{L_y^2}\Bigg)\ddot{Q}_q(t) - \Bigg(\frac{2B\pi^4 q^4}{4L_xL_y} + \frac{D_x\pi^4 q^4 L_y}{4L_x^3}$$

$$+ \frac{D_y\pi^4 q^4 L_x}{4L_y^3} + \frac{K_0 L_y}{4}\varphi_1(x) - \frac{G_0 L_y}{4}\Bigg(\frac{\pi q}{L_x}\varphi_2(x) + \quad\frac{\pi^2 q^2}{L_x^2} + \frac{\pi^2 q^2}{L_y^2}\Bigg)\varphi_3(x)\Bigg)\Bigg)\Bigg)Q_q(t) =$$

$$Mg\lambda^* \sin\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x} \tag{3.91}$$

Expanding equation (3.91), and retaining only $o(\lambda^*)$, one obtains

$$\ddot{Q}_n(t) + \Bigg[\omega_n^2\Bigg(1 - \eta^*\frac{\mu R_0 L_x L_y}{4}\quad\frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\Bigg)\Bigg) - \eta^*\Bigg(\frac{2B\pi^4 n^4}{4L_xL_y} - \frac{D_x\pi^4 n^4 L_y}{4L_x^3} - \frac{D_y\pi^4 n^4 L_x}{4L_y^3}$$

$$+ \left(\mu\omega_n^2 - \frac{K_0 L_y}{4}\varphi_1(x)\right) + \frac{G_0 L_y}{4}\Bigg(\frac{\pi n}{L_x}\varphi_2(x) + \left(\frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\right)\varphi_3(x)\Bigg)\Bigg)\Bigg]Q_n(t) - \eta^*\Bigg(1$$

$$- \eta^*\frac{\mu R_0 L_x L_y}{4}\left(\frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\right)\Bigg) \sum_{q=1, q\neq=n}^{\infty}\Bigg(-\frac{\mu R_0 L_x L_y}{4}\left(\frac{\pi^2 q^2}{L_x^2} + \frac{\pi^2 q^2}{L_y^2}\right)\ddot{Q}_q(t) - \Bigg(\frac{2B\pi^4 q^4}{4L_xL_y}$$

$$+ \frac{D_x\pi^4 q^4 L_y}{4L_x^3} + \frac{D_y\pi^4 q^4 L_x}{4L_y^3} + \frac{K_0 L_y}{4}\varphi_1(x) + \frac{G_0 L_y}{4}\Bigg(\frac{\pi q}{L_x}\varphi_2(x) + \left(\frac{\pi^2 q^2}{L_x^2} + \frac{\pi^2 q^2}{L_y^2}\right)\varphi_3(x)\Bigg)\Bigg)$$

$$Q_q(t) = Mg\eta^* \sin\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x} \tag{3.92}$$

Using Struble's technique, equation (3.92) can be rewritten as

$$\ddot{Q}_n(t) + \nu_n^2 Q_n(t) = 0 \tag{3.93}$$

Hence, the entire equation (3.92) becomes

$$\ddot{Q}_n(t) + \nu_n^2 Q_n(t) = Mg\lambda^* \sin\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x} \tag{3.94}$$

where

$$\nu_n = \epsilon_n - \frac{\lambda^*}{2\epsilon_n}\left[\frac{\epsilon_n^2 R_0 L_x L_y}{4} \quad \frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\right) + \frac{2B\pi^4 n^4}{4L_x L_y} - \frac{D_y \pi^4 n^4 L_x}{4L_y^3} - \frac{D_x \pi^4 n^4 L_y}{4L_x^3} - \tag{3.95}$$

$$\frac{K_0 L_x L_y}{4} + \frac{G_0 L_x L_y}{4} \quad \frac{\pi^2 n^2}{L_x^2} + \frac{\pi^2 n^2}{L_y^2}\Bigg)\Bigg]$$

is the modified frequency for simply supported orthotropic rectangular plate traversed by moving force.

On solving equation (3.94) by Laplace transformation techniques one obtains

$$Q_n(t) = \frac{Mg\lambda^*}{\nu_n} \sin\frac{m\pi s}{L_y} \times \left[\frac{\nu_n \sin\frac{m\pi c}{L_x}t - (\frac{m\pi c}{L_x})\sin\nu_n t}{(\frac{m\pi c}{L_X})^2 - \nu_n^2}\right] \tag{3.96}$$

which on inversion becomes

$$W(x,y,t) = \frac{Mg\lambda^*}{\nu_n} \sin\frac{m\pi s}{L_y} \times \left[\frac{\nu_n \sin\frac{m\pi c}{L_x}t - (\frac{m\pi c}{L_x})\sin\nu_n t}{(\frac{m\pi c}{L_X})^2 - \nu_n^2}\right] \times \sin\frac{m\pi x}{L_x}\sin\frac{m\pi y}{L_y} \tag{3.97}$$

is the transverse displacement response to a moving force of a simply supported orthotropic rectangular plate.

b) Simply Supported Orthotropic Rectangular Plate Transversed by Moving Mass

Here, one seeks solution to the entire equation (3.84). To solve this problem, one makes use of the modified asymptotic method of Struble. The equation becomes,

$$\ddot{Q}_n(t) + \nu_n^2 Q_n(t) + \alpha\varrho^* \sum_{q=1}^{\infty}\Bigg[\Bigg(\frac{L_x L_y}{64} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_1^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_2^*$$

$$\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty} E_3^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^* \frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_5^*$$

$$\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_6^* \frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_7^* \frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^*$$

$$\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\ddot{Q}_q(t) + 2c\Bigg(\frac{-4mL_y}{64(q^2-m^2)\pi} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_9^* \frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^*$$

Notes

$$\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_{11}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{12}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_{13}^{*}$$

$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{14}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty}E_{15}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{16}^{*}$$

Notes

$$\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\dot{Q}_q(t)+c^2\Bigg(\frac{-\pi^2q^2L_y}{64L_x}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_{17}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{18}^{*} \tag{3.98}$$

$$\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_{19}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{20}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_{21}^{*}$$

$$-\sum_{j=1}^{\infty}E_{22}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty}E_{23}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{24}^{*}\frac{\sin(2k+1)\pi s}{2k+1}$$

$$\Bigg)\Bigg)Q_q(t)\Bigg]=\frac{g\alpha}{\theta^*}\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x}$$

On further simplifications and rearrangements of equation (3.98), one obtains

$$\ddot{Q}_n(t)+\frac{2c\alpha\varrho^*}{1+\alpha\varrho^*\delta(i,j)\Bigg)}\Bigg(\frac{-4mL_y}{64(q^2-m^2)\pi}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_{9}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{10}^{*}$$

$$\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_{11}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{12}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_{13}^{*}$$

$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{14}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty}E_{15}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{16}^{*}$$

$$\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\dot{Q}_n(t)+\frac{1}{1+\alpha\varrho^*\delta(i,j)\Bigg)}\Bigg[\nu_n^2+c^2\alpha\varrho^*\quad\frac{-\pi^2q^2L_y}{64L_x}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_{17}^{*}$$

$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{18}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_{19}^{*}\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_{20}^{*}$$

$$\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_{21}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{22}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}$$

$$\sum_{k=1}^{\infty} E_{23}^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\Bigg]Q_n(t) + \frac{\alpha\varrho^{*}}{\Big(1+\alpha\varrho^{*}\delta(i,j)\Big)}$$

$$\sum_{q=1,q\neq n}^{\infty}\Bigg[\Bigg(\frac{L_xL_y}{64} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_1^{*}\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_2^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty} E_3^{*}$$

$$\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_5^{*}\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_6^{*}$$

$$\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_7^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\ddot{Q}_q(t)$$

$$+2c\Bigg(\frac{-4mL_y}{64(q^2-m^2)\pi} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_9^{*}\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)$$

$$\Bigg(\sum_{k=1}^{\infty} E_{11}^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{12}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_{13}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$-\sum_{j=1}^{\infty} E_{14}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{15}^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^{*}\frac{\sin(2k+1)\pi s}{2k+1}$$

$$\Bigg)\Bigg)\dot{Q}_q(t) + c^2\Bigg(\frac{-\pi^2q^2L_y}{64L_x} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_{17}^{*}\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)$$

$$\Bigg(\sum_{k=1}^{\infty} E_{19}^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^{*}\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_{21}^{*}\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$-\sum_{j=1}^{\infty} E_{22}^{*}\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^{*}\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^{*}\frac{\sin(2k+1)\pi s}{2k+1} \qquad (3.99)$$

$$\Bigg)\Bigg)Q_q(t)\Bigg] = \frac{g\alpha}{\Big(\theta^{*}\quad 1+\alpha\varrho^{*}\delta(i,j)\Big)}\sin\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x}$$

We shall consider a parameter $\alpha^{*} < 1$ for any arbitrary mass ratio defined by

$$\alpha = \frac{\alpha^{*}}{1+\alpha^{*}} \qquad (3.100)$$

Notes

By using binomial theorem and truncating after second terms, one obtains

$$\alpha = \alpha^* - o(\alpha^{*2}) \tag{3.101}$$

Considering only $o(\alpha^*)$, equation(3.101) becomes

$$\alpha = \alpha^* \tag{3.102}$$

Notes

Applying binomial expansion, one obtains

$$\frac{1}{(1+\alpha^*\varrho^*\delta(i,j))} = 1 - \alpha^*\varrho^*\delta(i,j) + (\alpha^*)^2 + ... \tag{3.103}$$

On putting equation (3.101) into equation (3.99), expanding and retaining only $o(\alpha^{*2})$, one obtains

$$\ddot{Q}_n(t) + 2c\alpha^*\varrho^*\Bigg(\frac{-4mL_y}{64(n^2-m^2)\pi} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_9^*\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{10}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)$$

$$\Bigg(\sum_{k=1}^{\infty} E_{11}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{12}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_{13}^*\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$- \sum_{j=1}^{\infty} E_{14}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{15}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{16}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)$$

$$\dot{Q}_n(t) + \Bigg[\nu_n^2\Bigg(1 - \alpha^*\varrho^*\Bigg(\frac{L_xL_y}{64} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_1^*\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_2^*\frac{\sin(2j+1)\pi ct}{2j+1}$$

$$\Bigg)\Bigg(\sum_{k=1}^{\infty} E_3^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_4^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_5^*\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$- \sum_{j=1}^{\infty} E_6^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_7^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)$$

$$+ c^2\alpha^*\varrho^*\Bigg(\frac{-\pi^2q^2L_y}{64L_x} + \frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty} E_{17}^*\frac{\cos(2j+1)\pi ct}{2j+1} - \sum_{j=1}^{\infty} E_{18}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)$$

$$\Bigg(\sum_{k=1}^{\infty} E_{19}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{20}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty} E_{21}^*\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$- \sum_{j=1}^{\infty} E_{22}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg) + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)$$

$$\Bigg)\Bigg)\Bigg]Q_n(t)+\alpha^*\varrho^*\Bigg[1-\alpha^*\varrho^*\Bigg(\frac{L_xL_y}{64}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_1^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_2^*$$

$$\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_3^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_4^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_5^*$$

$$\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_6^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_7^*\frac{\cos(2j+1)\pi s}{2j+1}-\sum_{j=1}^{\infty}E_8^*$$

$$\frac{\sin(2j+1)\pi s}{2j+1}\Bigg)\Bigg)+o(\alpha^*)^2+\ldots\Bigg]\sum_{q=1,q\neq n}^{\infty}\Bigg[\Bigg(\frac{L_xL_y}{64}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_1^*\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$-\sum_{j=1}^{\infty}E_2^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{k=1}^{\infty}E_3^*\frac{\cos(2k+1)\pi s}{2k+1}-\sum_{k=1}^{\infty}E_4^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)$$

$$+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_5^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_6^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_7^*\frac{\cos(2j+1)\pi s}{2j+1}$$

$$-\sum_{j=1}^{\infty}E_8^*\frac{\sin(2j+1)\pi s}{2j+1}\Bigg)\Bigg)\ddot{Q}_q(t)+2c\Bigg(\frac{-4mL_y}{64(q^2-m^2)\pi}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_9^*\frac{\cos(2j+1)\pi ct}{2j+1}$$

$$-\sum_{j=1}^{\infty}E_{10}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{j=1}^{\infty}E_{11}^*\frac{\cos(2j+1)\pi s}{2j+1}-\sum_{j=1}^{\infty}E_{12}^*\frac{\sin(2j+1)\pi s}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg($$

$$\sum_{j=1}^{\infty}E_{13}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{14}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_{15}^*\frac{\cos(2j+1)\pi s}{2j+1}-$$

$$\sum_{j=1}^{\infty}E_{16}^*\frac{\sin(2j+1)\pi s}{2j+1}\Bigg)\Bigg)\dot{Q}_q(t)+c^2\Bigg(\frac{-\pi^2q^2L_y}{64L_x}+\frac{1}{\pi^2}\Bigg(\sum_{j=1}^{\infty}E_{17}^*\frac{\cos(2j+1)\pi ct}{2j+1}-$$

$$\sum_{j=1}^{\infty}E_{18}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)\Bigg(\sum_{j=1}^{\infty}E_{19}^*\frac{\cos(2j+1)\pi s}{2j+1}-\sum_{j=1}^{\infty}E_{20}^*\frac{\sin(2j+1)\pi s}{2j+1}\Bigg)+\frac{1}{4\pi}$$

$$\Bigg(\sum_{j=1}^{\infty}E_{21}^*\frac{\cos(2j+1)\pi ct}{2j+1}-\sum_{j=1}^{\infty}E_{22}^*\frac{\sin(2j+1)\pi ct}{2j+1}\Bigg)+\frac{1}{4\pi}\Bigg(\sum_{j=1}^{\infty}E_{23}^*\frac{\cos(2j+1)\pi s}{2j+1}-$$

$$\sum_{j=1}^{\infty}E_{24}^*\frac{\sin(2j+1)\pi s}{2j+1}\Bigg)\Bigg)Q_q(t)\Bigg]=\frac{g\alpha^*}{\theta^*}\sin\frac{m\pi s}{L_y}\sin\frac{m\pi ct}{L_x} \qquad (3.104)$$

Notes

Applying the method of Struble technique to equation (3.104),its homogeneous part becomes

$$\ddot{Q}_n(t) + \delta_n^2 Q_n(t) = 0 \tag{3.105}$$

Hence, entire equation becomes

Notes

$$\ddot{Q}_n(t) + \delta_n^2 Q_n(t) = \frac{g\alpha^*}{\theta^*} \sin\frac{m\pi s}{L_y} \sin\frac{m\pi ct}{L_x} \tag{3.106}$$

where

$$\delta_n = \Bigg[\nu_n - \frac{1}{2\nu_n}\Bigg(\nu_n^2\alpha^*\varrho^*\Bigg(\frac{L_xL_y}{64} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_7^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg) -$$

$$c^2\alpha^*\varrho^*\Bigg(\frac{-\pi^2q^2L_y}{64L_x} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\Bigg)\Bigg] \tag{3.107}$$

is the modified frequency for simply supported orthotropic rectangular plate.
On solving equation (3.106) by Laplace transformation techniques, one obtains

$$Q_n(t) = \frac{g\alpha^*}{\theta^*\delta_n}\sin\frac{m\pi s}{L_y} \times \left[\frac{\delta_n \sin\frac{m\pi c}{L_x}t - (\frac{m\pi c}{L_x})\sin\delta_n t}{(\frac{m\pi c}{L_X})^2 - \delta_n^2}\right] \tag{3.108}$$

which on inversion becomes

$$W(x,y,t) = \frac{g\alpha^*}{\theta^*\delta_n}\sin\frac{m\pi s}{L_y} \times \left[\frac{\sigma_n \sin\frac{m\pi c}{L_x}t - (\frac{m\pi c}{L_x})\sin\sigma_n t}{(\frac{m\pi c}{L_X})^2 - \delta_n^2}\right] \times \sin\frac{m\pi x}{L_x}\sin\frac{m\pi y}{L_y} \tag{3.109}$$

is the transverse displacement response to a moving mass of a simply supported orthotropic rectangular plate.

IV. Disscusion of the Analytical Solutions

For this undamped system, it is desirable to examine the phenomenon of resonance. From equation (3.96), it is clearly shown that the simply supported orthotropic rectangular plate on constant elastic foundation and traverse by moving distributed force with uniform speed reaches a state of resonance whenever

$$\nu_n = \frac{m\pi c}{L_x} \tag{4.1}$$

while equation (3.109) shows that the same simply supported orthotropic rectangular plate under the action of a moving mass experiences resonance when

$$\delta_n = \frac{m\pi c}{L_x} \tag{4.2}$$

where

$$\delta_n = \Bigg[\nu_n - \frac{1}{2\nu_n}\Bigg(\nu_n^2\alpha^*\varrho^*\Bigg(\frac{L_xL_y}{64} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_7^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^*\frac{\sin(2k+1)\pi s}{2k+1}$$

$$\Bigg)\Bigg) - c^2\alpha^*\varrho^*\Bigg(\frac{-\pi^2q^2L_y}{64L_x} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\Bigg)\Bigg] \tag{4.3}$$

Comparing equations (4.1) and (4.2), one obtains

$$\delta_n = \nu_n\Bigg[1 - \frac{1}{2\nu_n^2}\Bigg(\nu_n^2\alpha^*\varrho^*\Bigg(\frac{L_xL_y}{64} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_7^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^*\frac{\sin(2k+1)\pi s}{2k+1}$$

$$\Bigg)\Bigg) - c^2\alpha^*\varrho^*\Bigg(\frac{-\pi^2q^2L_y}{64L_x} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\Bigg)\Bigg] \tag{4.4}$$

$$= \frac{m\pi c}{L_x}$$

Obviously

$$1 - \frac{1}{2\nu_n^2}\Bigg(\nu_n^2\alpha^*\varrho^*\Bigg(\frac{L_xL_y}{64} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_7^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_8^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)$$

$$- c^2\alpha^*\varrho^*\Bigg(\frac{-\pi^2q^2L_y}{64L_x} + \frac{1}{4\pi}\Bigg(\sum_{k=1}^{\infty} E_{23}^*\frac{\cos(2k+1)\pi s}{2k+1} - \sum_{k=1}^{\infty} E_{24}^*\frac{\sin(2k+1)\pi s}{2k+1}\Bigg)\Bigg)\Bigg) < 1 \tag{4.5}$$

That is, $\delta_n < \nu_n$ implies that moving mass simply supported system researches the state of resonance earlier than the moving force system.

V. Graphs of the Numerical Solutions

To illustrate the analysis presented in this work, orthotropic rectangular plate is taken to be of length $L_y = 0.923m$, breadth $L_x = 0.432m$ the load velocity c=0.8123 m/s and $s = 0.4m$.The results are presented on the various graphs below for the simply supported boundary conditions.

a) Graphs for Simply Supported Boundary Conditions

Figures 5.1 and 5.2 display the effect of rotatory inertia R_o on the deflection profile of simply supported orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of rotatory inertia R_o increases.

Figures 5.3 and 5.4 display the effect of foundation modulus K_o on the deflection profile of

Notes

Notes

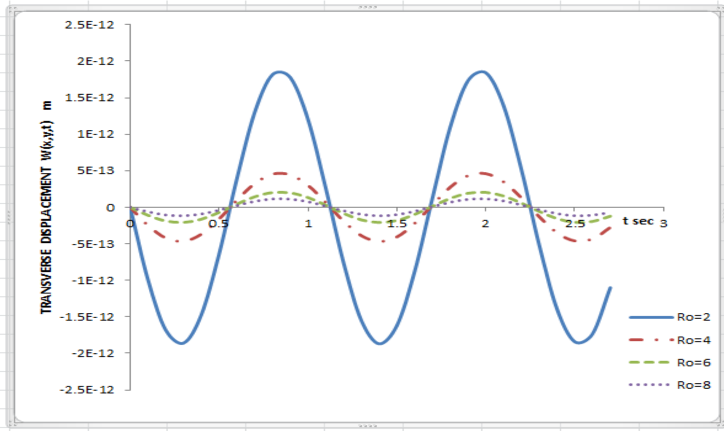

Figure 5.1: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying R_o and Traversed by Moving Force

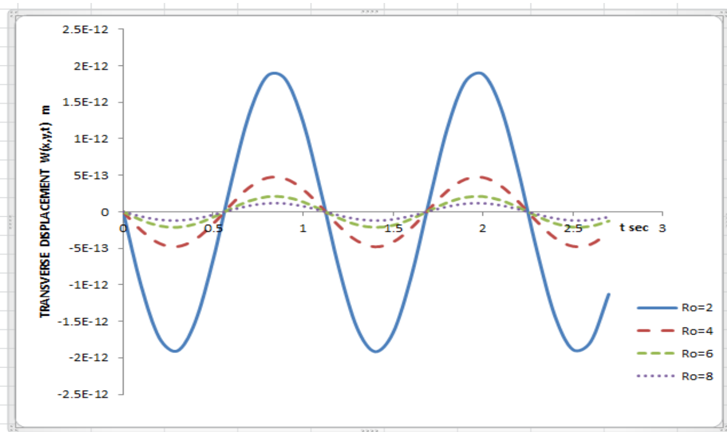

Figure 5.2: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying R_o and Traversed by Moving Mass

simply supported orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of foundation modulus K_o increases.

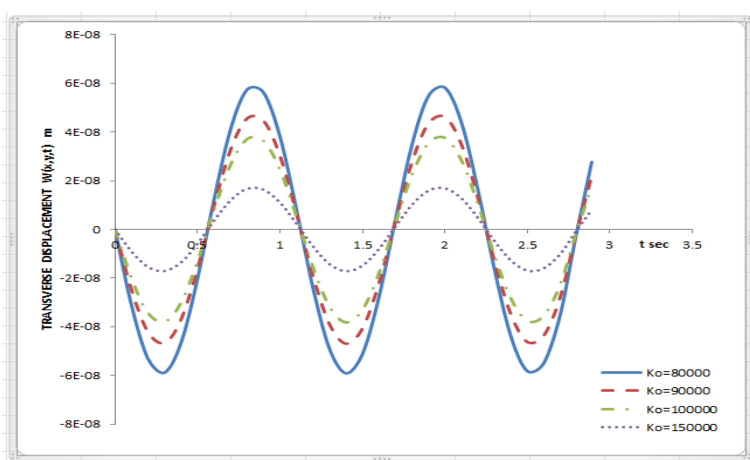

Figure 5.3: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying K_o and Traversed by Moving Force

Notes

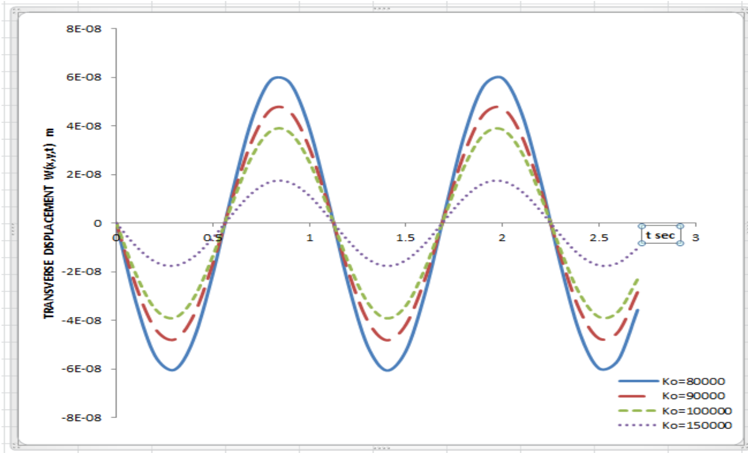

Figure 5.4: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying K_o and Traversed by Moving Mass

Figures 5.5 and 5.6 display the effect of shear modulus G_o on the deflection profile of simply supported orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of shear modulus G_o increases.

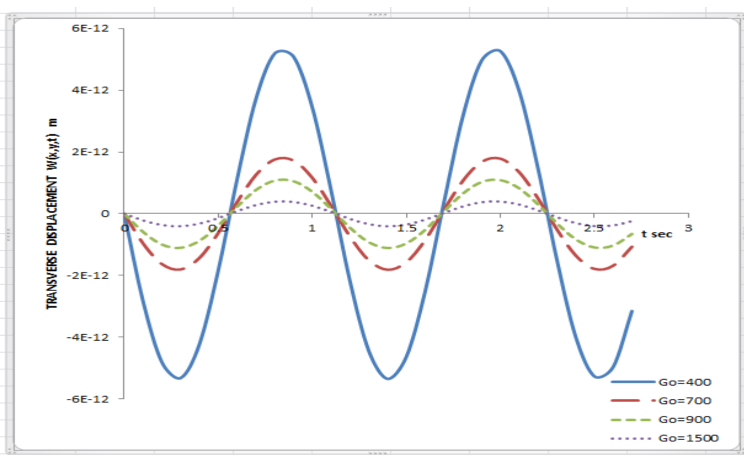

Figure 5.5: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying G_o and Traversed by Moving Force

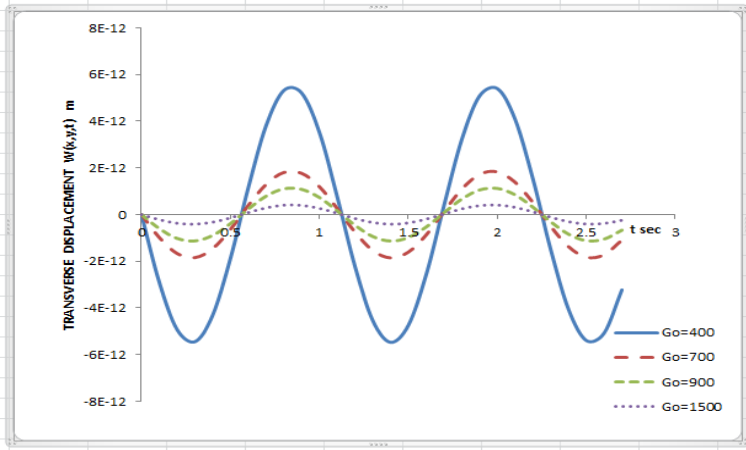

Figure 5.6: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying G_o and Traversed by Moving Mass

Figures 5.7 and 5.8 display the effect of flexural rigidity of the plate along x-axis D_x on the deflection profile of simply supported orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity D_x increases.

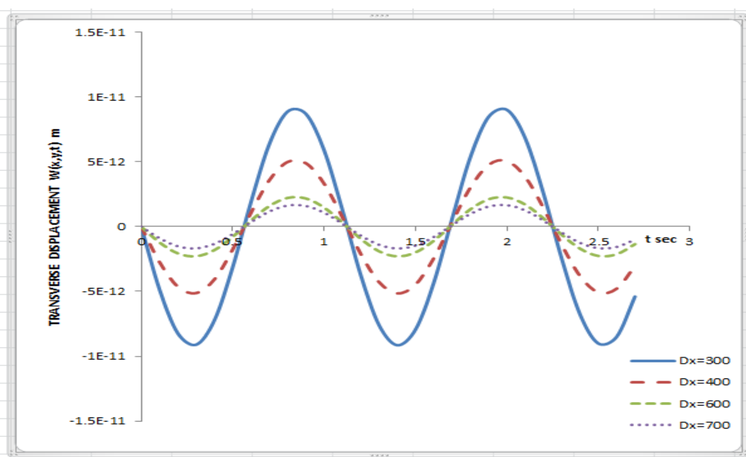

Figure 5.7: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying D_x and Traversed by Moving Force

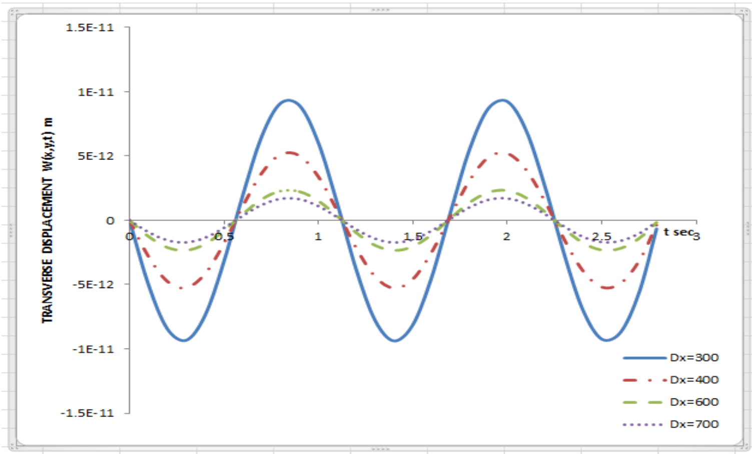

Figure 5.8: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying D_x and Traversed by Moving Mass

Figures 5.9 and 5.10 display the effect of flexural rigidity of the plate along y-axis D_y on the deflection profile of simply supported orthotropic rectangular plate under the action of load moving at constant velocity in both cases of moving distributed forces and moving distributed masses respectively. The graphs show that the response amplitude decreases as the value of flexural rigidity D_y increases.

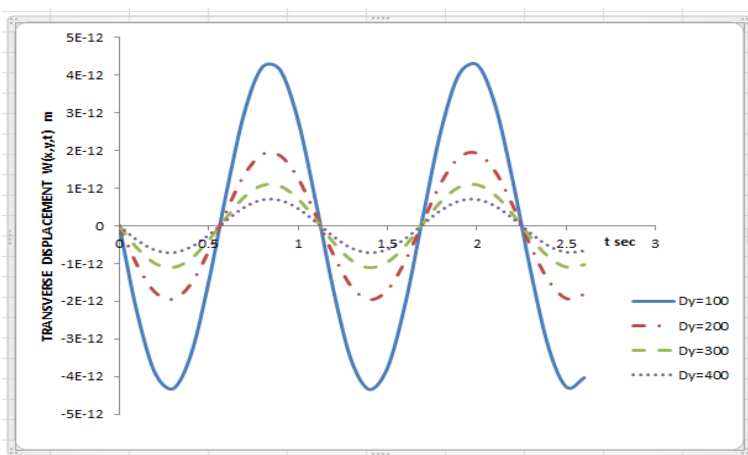

Figure 5.9: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying D_y and Traversed by Moving Force

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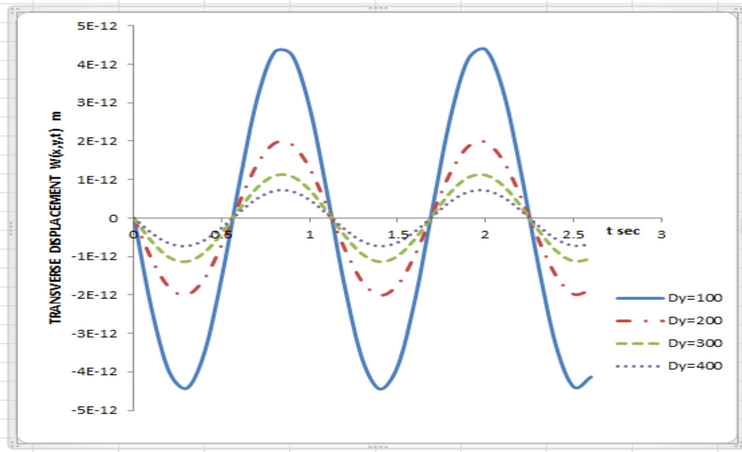

Figure 5.10: Displacement Profile of Simply Supported Orthotropic Rectangular Plate with Varying D_y and Traversed by Moving Mass

Figure 5.11 displays the comparison between moving force and moving mass for fixed values of R_o, G_o, K_o, D_x and D_y.

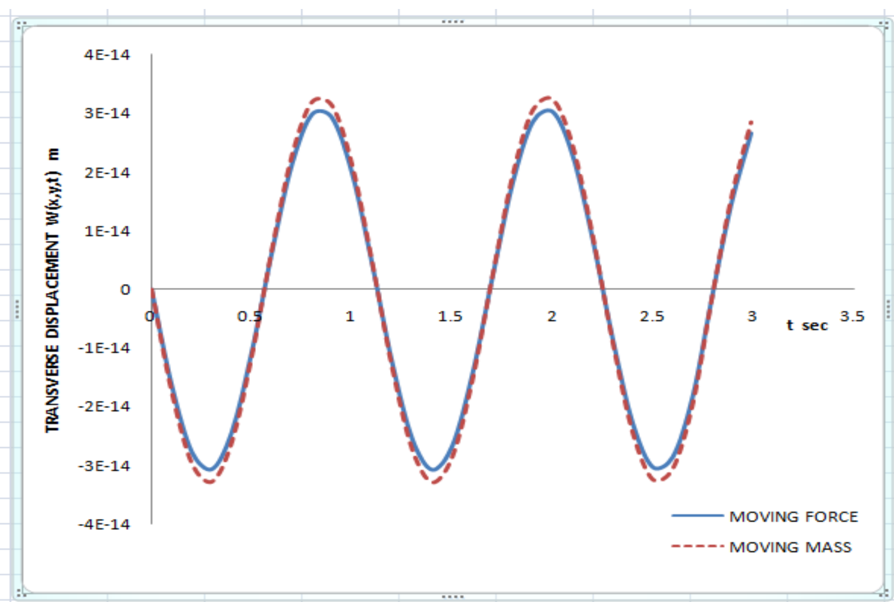

Figure 5.11: Displacement Profile of Comparison between Moving Force and Moving Mass

VI. Conclusion

In this work, the problem of response to simply supported orthotropic rectangular plate resting on a variable elastic bi-parametric foundation under the action of moving distributed masses has been studied. The closed form solutions of the fourth order partial differential equations with variable and singular coefficients governing the orthotropic rectangular plates is obtained for both cases of moving force and moving mass using a solution technique that is based on the separation of variables which was used to remove the singularity in the governing fourth order partial differential equation and thereby reducing it to a sequence of coupled second order differential equations. The modified Struble's asymptotic technique and Laplace transformation techniques are then employed to obtain the analytical solution to the two-dimensional dynamical problem.

The solutions are then analyzed. The analyses show that, for the same natural frequency and the critical speed for the moving mass problem is smaller than that of the moving force problem. Resonance is reached earlier in the moving mass system than in the moving force problem. That is to say the moving force solution is not an upper bound for the accurate solution of the moving mass problem.

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The results in plotted curves show that as the rotatory inertia correction factor R_o increases, the amplitudes of plates decrease for both cases of moving force and moving mass problems. The flexural rigidities along both the x-axis D_x and y-axis D_y increase, the amplitudes of plates decrease for both cases of moving force and moving mass problems. As the shear modulus G_o and foundation modulus K_o increase, the amplitudes of plates decrease for both cases of moving force and moving mass problems.

It is shown further from the results that for fixed values of rotatory inertia correction factor, flexural rigidities along both x-axis and y-axis, shear modulus and foundation modulus, the amplitude for the moving mass problem is greater than that of the moving force problem which implies that resonance is reached earlier in moving mass problem than in moving force problem of simply supported orthotropic rectangular plates resting on bi-parametric foundation.

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13. Awodola, T.O and Adeoye A.S (2020): Behavior under Moving Distributed Masses of Simply Supported Orthotropic Rectangular Plate Resting on a Constant Elastic Bi-Parametric Foundation. Asian Research Journal of Mathematics, 16(8): 64 - 92, 2020, Article no. ARJOM. 57775, ISSN: 2456 - 477X.
14. Adeoye A.S and Awodola, T.O (2020): Dynamic Behavior of Moving Distributed Masses of Orthotropic Rectangular Plate with Clamped-clamped Boundary Conditions Resting on a Constant Elastic Bi-Parametric Foundation. International Journal of Chemistry, Mathematics and Physics, Vol. 4, Jul-Aug.

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Affect of Spatial and Temporal Discretization in the Numerical Solution of One-Dimensional Variably Saturated Flow Equation

By M. S. Islam & R. Ahamad

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Keywords: *richards' equation; finite element; variably saturated flow; spatial discretization; temporal discretization.*

GJSFR-F Classification: *MSC 2010: 37M15*

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M. S. Islam [α] & R. Ahamad [σ]

Abstract- Numerical simulation of the Richards' equation in dynamically saturated soils keeps on being a difficult assignment because of its highly non-linear course of action. This is especially evident as soils approach saturation and the conduct of the principal partial differential equation changes from elliptic to parabolic. In this study, we developed a numerical model for solving Richards' equation with regards to finite element approach in which pressure head-based scheme is proposed to approximate the governing equation, and mass-lumping techniques are used to maintain stability of the numerical simulation. Dynamic adaptive time stepping procedure is implemented in the Picard and Newton linearization schemes. The robustness and accuracy of the numerical model were demonstrated through simulation of two difficult tests, including sharp moisture front that infiltrates into the soil column with time dependent boundary condition and flow into a layered soil with variable initial conditions. The two cases introduced feature various parts of the presentation of the two iterative strategies and the various components that can influence their convergence and efficiency, spatial and temporal discretization, convergence error norm, time weighting, conductivity and moisture content attributes and the degree of completely saturated regions in the soil. Numerical accuracy, mass balance nature and iteration efficiency of Picard and Newton techniquesare compared using different step sizes and spatial resolutions. Results demonstrated that the presented algorithm is vigorous and exact in simulating variably saturated flows and outcomes of some hydrologic process simulations are affected significantly by the spatial and temporal grid scales. Hence it is proposed that the strategy can be adequately actualized and used in numerical models of Richards' equation.

Keywords: richards' equation; finite element; variably saturated flow; spatial discretization; temporal discretization.

I. Introduction

Ground water flow issues are moderately hard to solve because of their nonlinear and parabolic nature, dependent on space and time dependent boundary conditions, nonhomogeneous parameters, etc. Analytical solution can once in a while be acquired for such genuine frameworks. In this way much of the time, flow equations must be illuminated by numerical approximations. However, numerically solving the flow problem is regularly tested by numerical scattering and motions, and as often as possible winds up with misleading outcomes. Inexact results of numerical approximations might be a significant reason for much disarray in the quantifiable analysis of flow problems.

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Existing numerical methodologies to deal with explain Richards' equation vary by the detailing of this equation, for example, grid discretization, time step and resolution strategies. These decisions impact computational time, numerical strength and result exactness. Numerical strategies for Richards' equation have pulled in extensive examination consideration and are generally utilized in reasonable simulations of subsurface procedures. In any case, numerous examinations have been indicated that standard numerical process cannot overcome difficulties for certain flow problems satisfactorily, particularly for the saturation of at first dry soils with non-uniform pore size appropriation [1]. This examination researches the upsides of noniterative adaptive time stepping approximations for Richards' equation and built up a simple cost-effective approximation that takes care of these troublesome issues precisely. The proposed formulation is firmly identified with in backward Euler techniques and henceforth can be utilized to progress existing programming for pragmatic subsurface simulations.

Standard numerical strategies for Richards' equation is principally restricted to straightforward time stepping approximations combined with finite element or finite difference spatial approximations [2]. The time stepping approximations included backward Euler and related schemes [e.g., 3, 4]. A basic advancement in the numerical examination of Richards' equation is the presentation of adaptive time stepping algorithms, which acclimate to the conduct of the solution and are commonly more solid and productive than uncontrolled procedures. Adaptive spatial approximations for Richards' equation incorporate a hierarchic finite element technique [5] and a front-tracking scheme [6].

Variable-order variable-step size differential algebraic equation solvers (DASPK) [1, 7, 8], lower-order backward Euler and similar techniques [9, 10] are depicted and successfully applied in the pressure head form of Richards' equation. Modern high-order techniques gave significant upgrades over existing low-order uniform step-size procedures when a small tolerance is used. In any case, for practical framework, many ordinary differential equation algorithms have certain constraints in the modeling variably saturated flows. By the controlling of formal truncation error, impressive improvements in solution accuracy and efficiency are achievable using fixed step and heuristic time stepping approximations, as well as, enhances the mass balance of models dependent on pressure head form of Richards' equation.

A significant issue in taking care of the flow problem is the mass balance error relating to its nonlinear nature when flow includes physical and chemical responses, for example, degradation, adsorption, evapotranspiration, and production. Mass preservation is an important obligation for accurate numerical solution, while, numerical accuracy is not ensured with a small mass balance. Iterative solution techniques with small step size can reduce the mass balance error, which thus makes the solution procedures very expensive. Numerical encounters for certain cases, contingent upon the nature and level of the nonlinearity, shows that mass balance errors may not be adequately wiped out in any event, when small steps are utilized. Thus, in flow demonstrating, most consideration has been paid to overcoming nonlinearity and eliminating the numerical scattering and false motions of the flow problems.

The governing equation for flow in saturated porous media i.e., Richards' equation, contains nonlinearities arising from pressure head dependencies on soil moisture and hydraulic conductivity. For steadiness reasons an implicit time discretization requiring assessment of the nonlinear coefficients at the current time level, is typically used to tackle the equation numerically. To linearize the subsequent discrete

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1. Miller, C. T., Williams, G. A., Kelly, C. T., and Tocci, M. D.: *Robust solution of Richards' equation for nonuniform porous media.* Water Resour. Res., 1998, 34:2599-2610.

system of equations, Newton or Picard method is ordinarily utilized numerical techniques for solving the nonlinearity of the coupled system [3, 11]. Newton-Krylov method, combined Picard-Newton method, initial slope Newton methods are also used to solve Richards' equation [12, 13, 14]. Basically, Picard scheme is the most famous because of its straightforwardness and normally adequate performance [15], and, is computationally more affordable on a for each iteration premise, and preserves symmetry of the discrete system of equations. Yet, the technique may diverge under specific conditions, as has been watched experimentally [3]. Furthermore, the non-perfection of constitutive relationships depicting a few soils causes poor convergence or complete divergence of Picard and Newton solvers for uncontrolled time stepping algorithms. To enhance the convergence efficiencies for such difficult simulations, improved sophisticated variable-order variable-step size strategies along chord slope iteration integrator and Newton techniques with global line search method can be employed [1, 7]. The Newton technique, yields nonsymmetric system matrices and is more unpredictable and costly than Picard linearization, however it accomplishes a higher rate of convergence and can be more strong than Picard for particular sorts of issues. Utilization of the Newton method has been restricted to one-and two-dimensional saturated-unsaturated flow models. Detail comparison of Picard and Newton strategies has been directed for the transient one-dimensional Richards' equation is found in the study [3], where it was demonstrated that, regarding CPU time expected to accomplish a given degree of solution exactness, Newton scheme can be as or more effective than Picard.

The number of iterations are expected to converge is a deciding component in the linearization schemes such as the Picard and Newton for the accurate, robust and efficient simulations. Therefore to meet this rationale, convergence rate is often enhanced by providing the solver with an initial solution estimate that is closer to the final solution for the current time step. This can be obtained by taking the initial guess from the previous step and by choosing a sufficiently small time step [13]. Hence, empirical dynamic adaptive time step criterion is required for a numerical model [3, 13, 16, 17].

Possible efficiency advantages can be obtained by use of noniterative schemes where formation of a single matrix with inversion per time step is required. For instance, the study [3] demonstrated that the noniterative implicit factored scheme with Newton solver can display equivalent or higher convergence efficiency than Crank-Nicolson method. However, it is not comfortable to handle the Richards' equation, as well as, much complexities are occurred at the saturated-unsaturated interface. Besides, these simpler algorithms, noniterative linearizations are limited for the temporal accuracy to first order. Regardless of these complexities, noniterative linearization techniques are an alluring option in contrast to customary iterative techniques for solving Richards' equation and other nonlinear partial differential equations.

The goal of this study, a general head-based mass conservative numerical procedure with regards to finite element scheme is developed to approximate the governing equation in which mass-lumping strategies are utilized to keep the stability of numerical simulation. To investigate the applicability and accuracy of the mathematical model and solution technique that offers a stable solution without requiring the resizing of the finite element mesh structure. To analyze complete flow behavior, realistic initial and Dirichlet boundary conditions are imposed in the numerical simulator to the head-based form of Richards' equation. Adaptive time-stepping approach is employed to minimize the computational time and maintain small truncation error. The performance

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3. Paniconi, C., Aldama, A. A., and Wood, E. F.: *Numerical evaluation of iterative and noniterative methods for the solution of the nonlinear Richards' equation.* Water Resour. Res., 1991, 27:1147-1163.

of the algorithm is shown to be superior to the conventional pressure head-based form and can easily be used in layered soil.

II. Governing Equations

Move through fluidly saturated permeable media is portrayed by the classical Richards' equation, which is joined by coupling an announcement of mass preservation with the Darcy's equation. Richards' equation contains nonlinearities emerging from pressure head conditions in the soil moisture and hydraulic conductivity. For settling Richards' equation utilizing regular numerical techniques can prompt a progression of numerical troubles including loss of mass protection, inadequately settled sharp fronts, and disappointment for nonlinear solver or iterative linear solvers. Also, precise and effective simulation of ground water flow in the saturated-unsaturated zone is computationally pricey, particularly for issues those are described by sharp fronts in both realities. Normal calculations that utilize homogeneous spatial and transient discretizations for the numerical solution of these issues lead to off base, wasteful, some time shaky and costly simulations. To evade these numerical challenges, mass-preservation plan of flow condition, fine discretization in reality can be utilized, and need to usage of proficient solid nonlinear and linear calculations. While bringing about solutions of adequate precision, these methodologies can be computationally costly, particularly when simulating conditions that include sharp fronts in space and timey, time varying boundary conditions, vertical redistribution, just as various soil materials in flow system.

Richards' equation might be written in three standard structures, with either pressure head or moisture content as dependent variables. The constitutive connection between fluid substance and pressure head takes into account transformation of one type of the condition to another. Three standard types of the saturated-unsaturated flow condition might be distinguished by the 'ψ-based', 'θ-baesd', and the 'mixed (ψ-θ)' form. For one-dimensional vertical flow, these conditions can be composed as follows:

(i) The 'ψ-based' form, where the primary variable is the pressure head,

$$C(\psi)\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial\psi}{\partial z} + 1\right)\right] \tag{1}$$

where, $C(\psi)$ is the specific fluid capacity $[L^{-1}]$ and is defined by $C(\psi) = \frac{d\theta}{d\psi}$, ψ is the pressure head $[L]$, t is time $[T]$, z denotes the vertical distance from reference elevation, assumed positive upward $[L]$, $K(\psi)$ is the hydraulic conductivity $[LT^{-1}]$, and θ is the moisture content.

The 'ψ-based' form permits for both unsaturated and saturated conditions. However, in highly non-linear problems, such as infiltration into very dry heterogeneous soils, these methods can suffer from mass-balance error, convergence problems and poor CPU efficiency. The reason for poor mass balance resides in the time derivative term. While $\frac{d\theta}{dt}$ and $C(\psi)\left(\frac{d\psi}{dt}\right)$ are mathematically equivalent in the continuous partial differential equation, their discrete analogues are not. The inequality in the discrete forms is exacerbated by the highly nonlinear nature of the specific capacity term $C(\psi)$. This leads to significant mass-balance errors in the ψ-based formulations because the change in mass in the system is calculated using discrete values of $\frac{d\theta}{dt}$ while the

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approximating equations use the expansion $C(\psi)\left(\frac{d\psi}{dt}\right)$. Using standard time-integration techniques, mass-balance errors grow with the time-step size. Various approaches have been developed to overcome this problem. A mass-conserving solution that modifies the capacity term to force global mass balance scheme is proposed [18]. A mass distributed algorithm [19] that satisfied mass balance and was free from oscillation. Implementation of method of lines is shown the property of good mass balance through time-step truncation error [7]. Moreover, very fine spatial and temporal discretizations with mass lumping are needed to maintain mass balance property for these scenarios.

(ii) The θ-based form, where the primary variable is the moisture content,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] + \frac{\partial K}{\partial z} \tag{2}$$

where $D(\theta) = \frac{K}{C(\psi)} = K\frac{d\psi}{d\theta}$ is the soil water unsaturated diffusivity $[L^2T^{-1}]$. One of the advantages of the θ-based formulation is that perfectly mass conservative discrete approximations can be applied. However, this form degenerates under fully saturated conditions as heterogeneous material produces discontinuous θ profiles and a pressure-saturation relationship no longer exist [20]. Thus, this form may be useful only for homogeneous porous media.

(iii) The mixed form, where both θ and ψ are the dependent variables,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left[K(\psi)\left(\frac{\partial\psi}{\partial z} + 1\right)\right] \tag{3}$$

It is also expressed in terms of mass conservative formulation. This form can be used to solve for both saturated-unsaturated flow cases. It is commonly viewed as better than the other two structures as a result of vigor as for mass balance. However, conservation of mass alone does not guarantee satisfactory numerical solutions [4, 21]. Numerical strategies that utilize both θ and ψ in the solution system have been developed to reduce the mass balance errors and improve computational efficiency. A primary variable switching technique, which is unconditionally mass conservative [22]. This method involves assembling and solving a nonsymmetric equation system at each time and iteration level which increases CPU time but reported faster convergence behavior. Modified Picard iteration approach guarantees mass balance by assessing the moisture content change in a period step legitimately from the adjustment in the water pressure head [4]. It has been shown to provide excellent mass balance when modelling unsaturated problems with sharp wetting fronts [23]. This method is easy to implement into ψ-based codes, requiring only an additional source term.

More efficient convergence scheme has been proposed for the modified Picard iteration method dependent on utilizing the pressure head as the primary variable [24]. However, problems have been reported when employing the mixed form for free drainage problems [25]. If relatively large values are encountered, mass-balance errors can accumulate with longer simulation times and larger domains. The ψ-based form can achieve good mass balance if the change in ψ is small enough during a time step whereas the mixed form improves mass balance with a sharp wetting front. Therefore, combining these, makes a more efficient procedure for long time simulations of water flow in soils with frequent infiltration and deep drainage processes. The method switches to the ψ-based form when the change in ψ is less than some prescribed value, otherwise the mixed form is applied. Developing robust and efficient algorithms for

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18. Milly, P. C. D.: *A mass-conservative procedures for time-stepping in models of unsaturated flow.* Adv. Water Resour., 1985, 8:32-36.

certain flow problems, such as those that give rise to sharp wetting fronts, has provided a computational challenge to the simulation community. For this class of problem, small time-step sizes and a fine mesh is often required in order to maintain stability when steep wetting fronts develop, making large-scale multi-dimensional infiltration problems impractical to simulate.

a) Constitutive Relationships

For solving Richards' equation numerically, we must define the characteristic functions to illustrate the relationship among fluid pressures, saturations and relative permeabilities. Various mathematical formulations are used in modeling for the soil water moisture curves. The most regularly utilized connections are the Brooks–Corey [26] and the van Genuchten [27] models. These two models are described as follows:

i. *Brooks–Corey Model*

The soil water pressure-moisture mathematical models proposed by Brooks and Corey [26] are given by:

$$\theta(\psi) = \theta_r + (\theta_s - \theta_r)\left(\frac{\psi_d}{\psi}\right)^n \text{ if } \psi \leq \psi_d$$

$$\theta(\psi) = \theta_s \text{if} \psi > \psi_d$$

$$K(\psi) = K_s\left[\frac{\theta(\psi) - \theta_r}{\theta_s - \theta_r}\right]^{3+2/n} \text{if} \psi \leq \psi_d$$

$$K(\psi) = K_s \text{if} \psi > \psi_d$$

$$C(\psi) = n\frac{\theta_s - \theta_r}{|\psi_d|}\left(\frac{\psi_d}{\psi}\right)^{n+1} \text{if} \psi \leq \psi_d$$

$$C(\psi) = 0 \text{ if } \psi > \psi_d$$

where θ_s is the saturated moisture content $[L^3L^{-3}]$, θ_r is the residual moisture content $[L^3L^{-3}]$, $\psi_d = -\frac{1}{\alpha}$ is the air entry pressure head $[L]$ and $m = 1 - \frac{1}{n}$ is a pore-size distribution index.

ii. *Van Genuchten Model*

Van Genuchten model [27]is the most used characteristic function for moisture content and hydraulic conductivity and presented as follows:

$$\theta(\psi) = \theta_r + \frac{\theta_s - \theta_r}{[1 + |\alpha\psi|^n]^m} \text{if} \psi \leq 0$$

$$\theta(\psi) = \theta_s \text{if} \psi > 0$$

$$K(\psi) = K_s\left[\frac{\theta - \theta_r}{\theta_s - \theta_r}\right]^{0.5}\left\{1 - \left[1 - \left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{\frac{1}{m}}\right]^m\right\}^2 \text{ if} \psi \leq 0$$

$$K(\psi) = K_s \text{if} \psi > 0$$

Ref

27. Van Genuchten, M.T.:*A Closed-form Equation for Predicting the Hydraulic Conductivity of Unsaturated Soils.* Soil Sci. Soc. Am. J. 1980, 44:892–898.

$$C(\psi) = \alpha m n \frac{\theta_s - \theta_r}{[1 + |\alpha\psi|^n]^{m+1}} |\alpha\psi|^{n-1} \text{if} \psi \leq 0$$

$$C(\psi) = 0 \text{ if} \psi > 0$$

b) Spatial Discretization

An appropriate technique to divided the boundary-value spatial component of Richards' equation from its initial-value temporal variation is the finite element technique and this approach is very simple and practical to use. To build up the finite element algorithm of the pressure head-based Richards' equation, the weak model of the dependent variable and the constitutive relations were approximated utilizing introducing polynomials [28, 29]. It was expected that the pressure driven conductivity just as capacitance differs linearly inside every component [30].

For solving Richards' equation (1) numerically, finite element Galerkin's approach is applied to discretize spatial domain and finite difference method is used for time derivative term. To build up the finite element model, there are M-1 discretized components for M global nodes in the problem domain.
The approximating function is

$$\psi(z,t) \approx \hat{\psi}(z,t) = \sum_{J=1}^{M} N_J(z)\psi_J(t) \tag{16}$$

where $N_J(z)$ and $\psi_J(t)$ are linear Lagrange basis functions and nodal values of ψ at time t, respectively. The method of weighted residuals is used to set the criteria to solve for the unknown coefficients. In local coordinate space $-1 \leq \xi \leq 1$, the approximating function for each element (e) is $\hat{\psi}^{(e)} = \sum_{i=1}^{2} N_i^{(e)}(\xi)\,\psi_i^{(e)}(t) = \frac{1}{2}(1-\xi)\psi_1^{(e)}(t) + \frac{1}{2}(1+\xi)\psi_2^{(e)}(t)$, which we can write in vector form as $\hat{\psi}^{(e)} = \left(N^{(e)}(\xi)\right)^T \Psi^{(e)}(t)$. The global function (16) becomes:

$$\hat{\psi} = \sum_{e=1}^{M-1} (\boldsymbol{N}^{(e)})^T \boldsymbol{\Psi}^{(e)} = \sum_{e=1}^{M-1} \hat{\psi}^{(e)}$$

The symmetric weak formulation of Galerkin's method applied to (1) yields the system of ordinary differential equations [14]:

$$A(\Psi)\Psi + \mathrm{F}(\Psi)\frac{\mathrm{d}\Psi}{\mathrm{dt}} = \mathrm{q(t)} - \mathrm{b}(\Psi)$$

where Ψ is the vector of undetermined coefficients corresponding to the values of pressure head at each node, A is the stiffness matrix, F is the storage or mass matrix, q contains the specified Darcy flux boundary conditions and b contains the gravitational gradient component. Over local sub domain element $\Omega^{(e)}$, we have:

$$A^{(e)} = \int_{\Omega^{(e)}} K_s^{(e)} K_r(\hat{\psi}^{(e)}) \frac{dN^{(e)}}{dz} \left(\frac{dN^{(e)}}{dz}\right)^T dz$$

Ref

30. Zadeh, K. S.: *Parameter estimation in flow through partially saturated porous materials.* J. Comput. Phys., 2008, 227(24):10243–10262.

$$b^{(e)} = \int_{\Omega^{(e)}} K_s^{(e)} K_r(\hat{\psi}^{(e)}) \frac{dN^{(e)}}{dz} dz$$

$$F^{(e)} = \int_{\Omega^{(e)}} C(\hat{\psi}^{(e)}) N^{(e)} (N^{(e)})^T dz$$

Here, N^T denotes the transpose of N.

c) Time Differencing

Equation (18) can be integrated by the weighted finite difference scheme. We obtain:

$$A(\Psi^{k+\lambda})\Psi^{k+\lambda} + F(\Psi^{k+\lambda}) \frac{\Psi^{k+1} - \Psi^k}{\Delta t^{k+1}} = q(t^{k+\lambda}) - b(\Psi^{k+\lambda})$$

where $\Psi^{k+\lambda} = \lambda\Psi^{k+1} + (1-\lambda)\Psi^k$, with $0 \le \lambda \le 1$ (λ is a weighting parameter)and $k+1$ denotes current time level.

The time step size to ensure a stable solution will be dependent on the spatial discretization, and for nonlinear equations, there will in general also be a dependency on the form of the solution itself at any given time. Equation (22) is $O(\Delta t)$ accurate, except for $\lambda = \frac{1}{2}$. When $\lambda = \frac{1}{2}$, the discretized scheme (22) corresponds to the Crank–Nicolson scheme.

The system of equations (22) is nonlinear in ψ^{k+1}, except when $\lambda = 0$, which corresponds to an explicit Euler scheme. When $\lambda > 0$, the scheme becomes implicit. Some iteration or linearization strategy is thus needed to solve the system of nonlinear equations for the implicit case. For $\lambda = 1$, the scheme corresponds to the backward Euler scheme.

III. Iterative Methods

The system of equations (22) is highly nonlinear because of the nonlinear dependency of hydraulic conductivity K and specific moisture capacity$C$ on ψ. Picard and Newton are the two classical iterative approaches can be applied in the nonlinear system (22) for linearization. Picard method is simpler than Newton and preserves symmetry in the system matrix. Then again, the Newton strategy requires the computation of Jacobian matrix at each iteration and yields a nonsymmetric system. Along these matters, Picard technique is less computational, on a for every cycle premise, than the Newton strategy. The Picard strategy is convergent linearly, whereas, Newton meets quadratically.

a) Newton Scheme

Let us Consider

$$\begin{aligned} & f(\Psi^{k+1}) \\ & = A(\Psi^{k+\lambda})\Psi^{k+\lambda} + F(\Psi^{k+\lambda}) \frac{\Psi^{k+1} - \Psi^k}{\Delta t^{k+1}} \\ & - q(t^{k+\lambda}) + b(\Psi^{k+\lambda}) = 0 \end{aligned}$$

Notes

Notes

The Newton scheme [3] can be written as:

$$f'\left(\psi^{k+1,(m)}\right)\left(\psi^{k+1,(m+1)} - \psi^{k+1,(m)}\right) = -f\left(\psi^{k+1,(m)}\right)$$

where the superscripts m and $m+1$ denote the previous and current iteration levels respectively.

The Jacobian for the system is:

$$f'_{ij} = \lambda A_{ij} + \frac{1}{\Delta t^{k+1}} F_{ij} + \sum_s \frac{\partial A_{is}}{\partial \psi_j^{k+1}} \psi_s^{k+\lambda}$$

$$+ \frac{1}{\Delta t^{k+1}} \sum_s \frac{\partial F_{is}}{\partial \psi_j^{k+1}} (\psi_s^{k+1} - \psi_s^k) + \frac{\partial b_i}{\partial \psi_j^{k+1}}$$

expressed here in terms of ij-th component of the Jacobian matrix $f'(\Psi^{k+1})$.

b) Picard Scheme

Straightforward and simple mathematical expression of Picard iterative method can be derived from (22) by iterating with all linear events of ψ^{k+1}taken at the current iteration level $m+1$ and all nonlinear events at the previous level m [3]. We get:

$$\left[\lambda A^{k+\lambda,(m)} + \frac{1}{\Delta t^{k+1}} F^{k+\lambda,(m)}\right]\left(\psi^{k+1,(m+1)} - \psi^{k+1,(m)}\right)$$

$$= -f\left(\psi^{k+1,(m)}\right)$$

By the comparison of the equations (22) and (26), it is observed that Picard technique is an approximation of Newton technique. To assess the overall efficiency of the two linearization techniques, it is very important to know the structural differences of Picard and Newton techniques, such as, Picard linearization produces symmetric and Newton produces a nonsymmetric system matrix. Three derivative terms are needed to calculate in the Newton procedure, as a result, the Newton strategy is more expensive and arithmetically complex than Picard.

IV. Methodology

The principal objective of this research is to generalize pressure head-based finite element algorithm to handle the nonlinearity, minimize the mass balance errors locally and globally of the flow equation and application of one-dimensional saturated flow conditions for investigating the spatial and temporal discretization affect. This is practiced by linearizing a head-based flow equation with the Picard and Newton iteration techniques. Anusual Galerkin finite element technique is then used to comprehend the linearized definition to acquire the solution of flow problems.

Mass balance errors and computational efficiency are the key factors for the measure of the solution quality. Numerical trials will be introduced to delineate the promising solution execution of the iteration techniques as contrasted and the reference solution which will be made by fine grid resolutions maintain with a tight nonlinear tolerance for the test problems to evaluate the efficiency and robustness and also compare the computed result with other published footprints. Note that the input tolerance level will affect the accuracy of the numerical solution, within limits imposed by spatial and temporal truncation error.

The exhibition of the calculation is contrasted and two illustrative arrangements of distributed exploratory information, every one of which speaks to an alternate physical situation and is frequently used to approve calculations. In the test examples, the accuracy, mass balance character and iteration efficiency of the pressure head-based model is evaluated with the Picard and the Newton iteration schemes using three different spatial and time-step sizes, and applicability of the resultant solutions, and draw methods to assess the computational work required to achieve the results. Numerical experiments are performed with mass lumping, to appraise the robustness of the approach and investigate the advantages of the methods for improving the efficiency of solutions to Richards' equation.

To enhance the convergence the of the nonlinear iterative approaches, dynamic time stepping technique is incorporated in this study. During whenever step, nonlinear convergence tolerance $Tol\,(=10^{-4})$ is assigned for both the test examples, alongside a most extreme number of nonlinear iterations denoted by *maxit* and it is 15. Simulation start with time step size is $\Delta t_0$and proceeds until we arrive at the end of the simulation time T_{max}. Present step size will increase with a predetermined amplification factor $\Delta t_{mag}\,(=1.20)$ if the number of iterations is less than another pre-assigned limit of iterations $maxit_1(=8)$ and this process is repetitive until reach the maximum time step size Δt_{max}. Current step size is constant if number of nonlinear iterations are lies between $maxit_1$ and $maxit_2(=5)$ iterations. If the number of nonlinear iterations is less than $maxit_2$, then the simulation step size will reduce by a reduction factor $\Delta t_{red}\,(=0.5)$ to assigned minimum step size Δt_{min}. Solution will start recalculate if the convergence is not attaining within the specified maximum number of nonlinear iterations, which is called back-stepping. For both the iterative schemes, the infinity norm [11], $\left\|\psi^{k+1,(m+1)} - \psi^{k+1,(m)}\right\| \le Tol$ is used as the stopping criterion.

A correlation of the overall precision of the numerical outcomes got from various plans is not easy [15]. It is depending upon the objectives such as, global or local comparisons of water pressure or water content, minimum or maximum value of the compared variable, etc. One proportion of a numerical test system is its capacity to preserve global mass over the area of intrigue. Small mass balance error is necessary yet not totally satisfactory essential for a correct solution [4, 15, 31]. To quantify the capacity of the test system to conserve mass, one of the most broadly utilized models for assessing the accuracy of a numerical strategy is the mass balance error (MBE) given by [4]:

$$\text{Mass Balance Error} = \left| 1 - \frac{\text{Total additional mass in the domain}}{\text{Total net flux into the domain}} \right|$$

where the complete extra mass in the space is the distinction between the mass estimated at any moment t and the underlying mass in the area, and the total net flux into the region is the flux balance coordinated in time up to t. In this study, this is determined by the accompanying equation [4]:

Ref

4. Celia, M. A., Bouloutas, E. T., and Zarba, R. L.: *A General mass-conservative numerical solution for the unsaturated flow equation*, Water Resour. Res., 1990, 26(7):1483-1496.

$$\mathrm{MB(t)}=\frac{\sum_{i=1}^{E-1}\left(\theta_i^{k+1}-\theta_i^0\right)(\Delta z)+\left(\theta_0^{k+1}-\theta_0^0\right)\left(\frac{\Delta z}{2}\right)+\left(\theta_E^{k+1}-\theta_E^0\right)\left(\frac{\Delta z}{2}\right)}{\sum_{j=1}^{k+1}\left\{\left(q_o^j-q_N^j\right)(\Delta t)\right\}} \tag{27}$$

Notes

with $N = E + 1$ nodes $\{z_0, z_1, z_2, \dots\dots, z_E\}$, and constant nodal spacing Δz is considered and q_0 and q_N being boundary fluxes evaluated from the finite element equations related with the boundary modes z_0 and z_N.

To solve the linearized system of equations, a main drawback of the Newton scheme is insufficiency of linear solvers for large, sparse non symmetric systems. This is not true anymore, as at present accessible form conjugate gradient-type algorithms for handling non symmetric systems have gotten progressively steady and effective. In this work, bi-conjugate gradient stabilized algorithm (BICGSTAB) is used to solve the linear systems. For the symmetric system produced by Picard linearization, incomplete Cholesky conjugate gradient strategy (ICCG) is joined. For all experiments, where ICCG, BICGSTAB, iterative solver, the linear solver boundaries *tolcg* (convergence tolerance 10^{-10}) and *maxitcg* (maximum number of linear iterations is1000) was assigned. Soil moisture properties are evaluated by analytical differentiation.

Hydrological model CATHY (CATchmentHYdrology) [11, 32], where the surface module settles the one-dimensional diffusion wave condition and the subsurface module solves the three-dimensional Richards' equation, is used for all runs. All simulations were executed on a Dell Inspiron 2.56-GHz laptop computer.

V. Results and Discussions

Two challenging one-dimensional test examples are considered to validate the algorithm and to compare the accuracy of the numerical solution of Richards' equation by the CATHY model. Time dependent boundary conditions with a sharp moisture front that infiltrates into the soil column [10, 16, 33] is the first test problem and the second test case involves flow into a layered soil with variable initial conditions [33, 34, 35].

a) *Test problem 1*

This problem considers a soil column of $2\,m$ deep with the initial pressure head distribution is $\psi(z,0) = z - 2$. At the bottom of thecolumn, a water table boundary condition (i.e.,$\psi(0,t) = 0$) is imposed, while a time-dependent Dirichlet condition

$$\psi(2,t)=\begin{cases}-0.05+0.03\sin\left(\dfrac{2\pi t}{100000}\right) & \text{if } 0<t\le 100000\\ 0.1 & \text{if} 100000<t\le 180000\\ -0.05+2952.45\,e^{-\frac{t}{18204.8}} & \text{if } 180000<t\le 300000\end{cases}$$

is applied at the top boundary which is presented in Figure 1. The soil hydraulic properties are described by the van Genuchten model. The soil parameters are $\theta_s = 0.410$, $\theta_r = 0.095$, $\alpha = 1.9/\mathrm{m}$, $\mathrm{n} = 1.31$ and $\mathrm{K_s} = 0.062\ \mathrm{m/day}$.

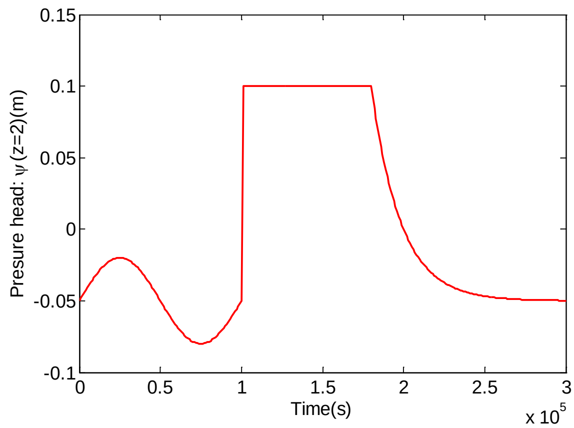


Figure 1: Dirichlet boundary condition imposed at the top of the soil column for Test problem 1

The Dirichlet boundary condition leads to significant ponding between $100000\,s$and $200000\,s$, and as will be found in the outcomes, this kind of boundary condition, leading in coupled groundwater water representation, is a wellspring of huge trouble in the iterative techniques.

Attributes of such soil compare to an unconsolidated clay loam with a nonuniform grain size circulation [36]. Antecedent experiment [37] completed a comparative correlation utilizing a moisture-based type of Richards' equation and an alternate experiment that doesnot include time-differing boundary conditions with surface ponding.

Due to the positive value of pressure head in the second period of simulation time $(100000 < t \leq 180000\,s)$, to achieve the numerical convergence is very challenging for any algorithm. In light of unexpected increment of the upper Dirichlet boundary condition to a positive estimation of $0.1\,m$, it makes a sharp moisture front that infiltrates into the soil section. Toward the start of the third time frame $(t > 180000\,s)$ ponding diminishes exponentially, arriving at asymptotically a last worth $-0.05\,m$, and before the finish of the simulation the whole section is near tofull saturation.

The moisture retention curve is monotonic with a point of inflection that gives the moisture capacity function its typical shape. The soil moisture retention curves for this test problem using the van Genuchten model are represented in Figure 2.

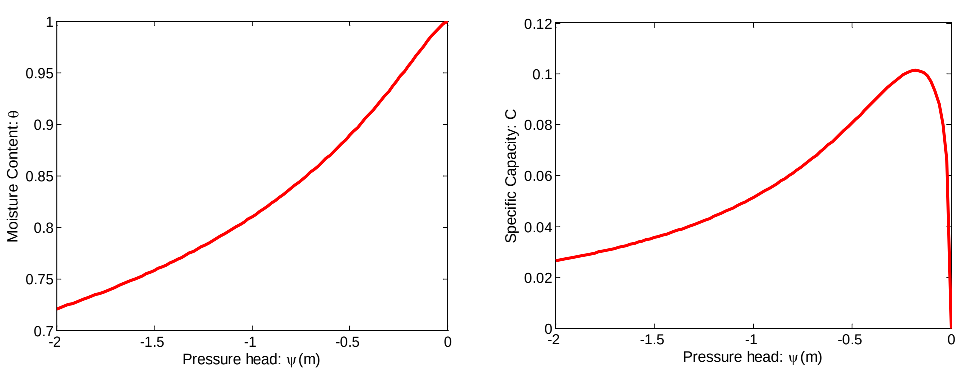


Figure 2: Soil moisture characteristic curves for Test problem 1

Notes

In order to assess the robustness and efficiency of the method, we used three set of grid sizes, i.e., $\Delta z = 0.004\,m, 0.008\,m$ and $0.04\,m$ and each grid discretization is simulated with three temporal sizes$\Delta t_{max} = 1000\,s, 100s$ and $10s$.

The computed pressure head profiles at various meshing obtained with a small tolerance ($10^{-4}\,m$) are displayed in Figure 3. These solutions are very similar to those reported in the literature [10, 16, 33]. Figure 3 shows the initial conditions and pressure head solution profiles at three different times (e.g., $0\,s, 35000\,s, 155000s$ and $300000\,s$). It is evident that the solution profiles are affected by the spatial resolutions. The red profiles, which falls inside the ponding time frame, shows the abundance water that structures at the soil surface and the fairly sharp moisture front that is produced.

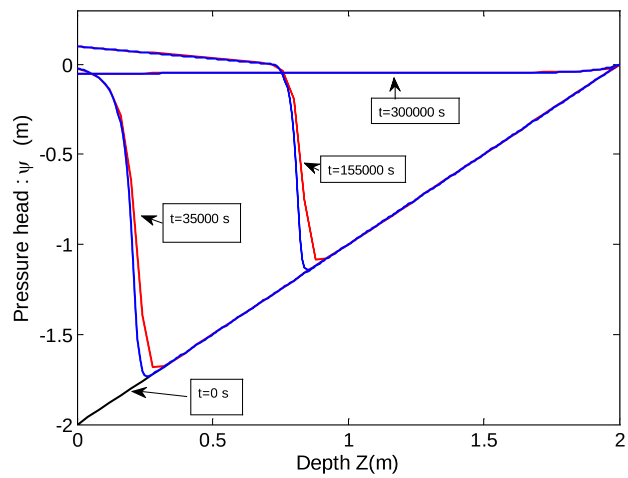


Figure 3: Pressure head profiles at different times for $\Delta z = 0.04\,m$ (red) and $\Delta z = 0.008\,m$ (blue) of Test problem 1

Adaptive time stepping algorithm is applied to the iteration techniques Newton and Picard for all vertical discretizations and three different time stepping scales for investigating the step size behavior. We discovered generally striking here the altogether different conduct between the Newton and Picard methodologies during the ponding time frame. Though the Newton model is compelled to make extremely little step sizes just at the absolute starting point and end of the ponding time frame, the Picard plot needs to arrange a wide scope of step sizes all through the ponding time range, and surely for the $\Delta t_{max} = 1000\,s$ case, it never accomplishes this most extreme incentive during ponding, for any of the vertical grid resolutions. Small time step size is observed from $100000\,s$ to $200000\,s$, as ponding progressively diminishes to zero. Step size is quickly increasing and reaches maximum allowable time step size $1000\,s$ in the simulation period $200000\,s$ to $300000\,s$ for both iteration schemes. This demonstrates simpler nonlinear solver conditions because of smoother infiltration fronts and surface conditions that are no longer fully saturated. Compelling an iteration scheme to take extremely small time steps for prolonged periods during a simulation can represent a massive computational trouble for subsurface solvers. The time stepping behavior of Picard and Newton can be found in the Figure 4 for $\Delta z = 0.008\,m$ and $0.04\,m$ cases with various time step sizes.

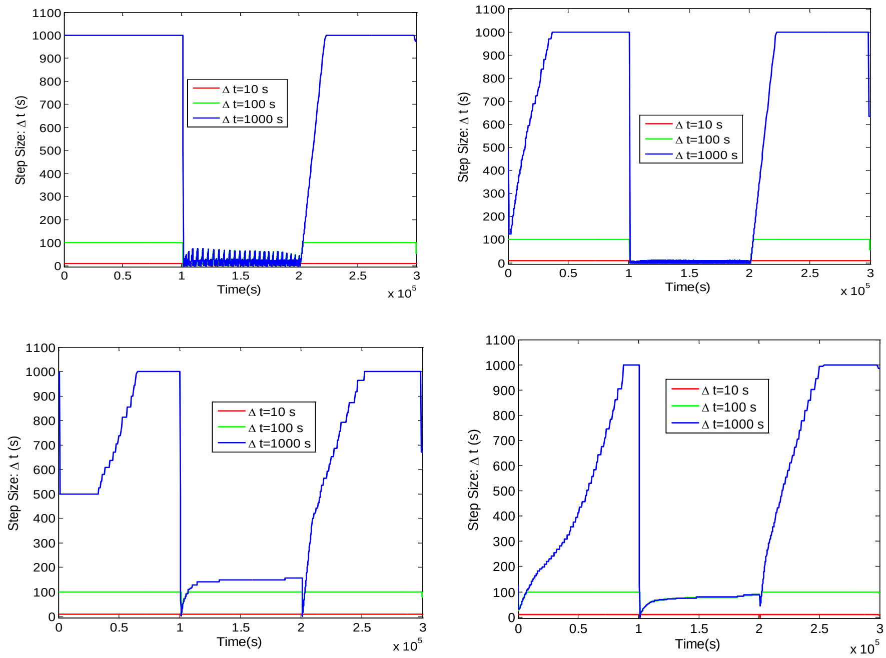


Figure 4: Dynamic time stepping behavior of Picard (top row)and Newton (bottom row) schemes for $\Delta z = 0.04\, m$ (left) and $\Delta z = 0.008\, m$ (right)of Test problem 1

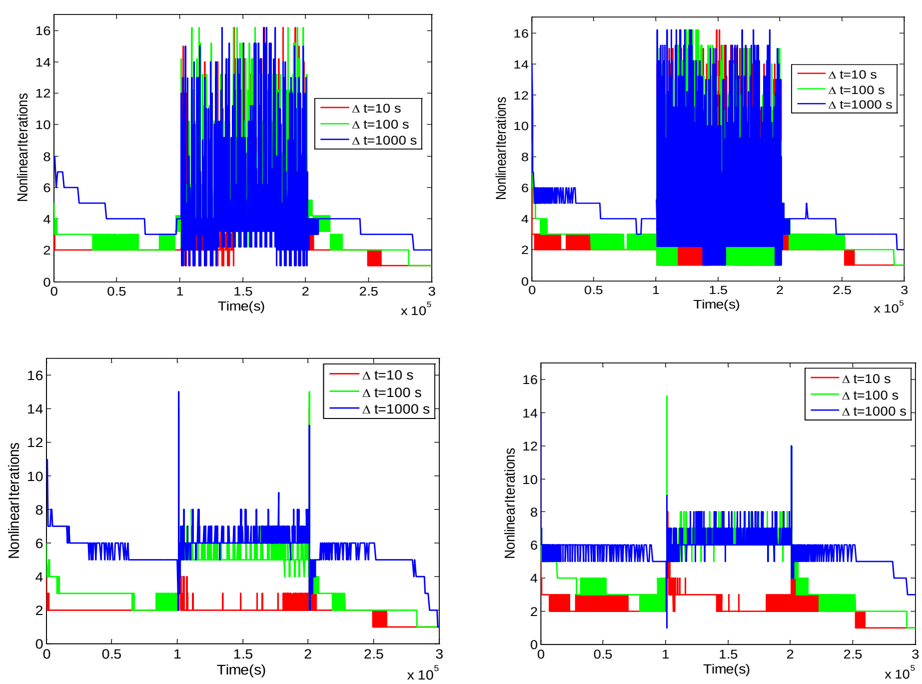


Figure 5: Nonlinear convergence behavior of Picard (top row) and Newton (bottom row) schemes for $\Delta z = 0.04\, m$ (left) and $\Delta z = 0.008\, m$ (right) of Test problem 1

Graphical representation of convergence nature on the basis of number of nonlinear iterations required at each step of Picard and Newton iteration schemes are shown in the Figure 5. Here, we observed that a smoother transition into and out of the ponding period, and without the need for time step adaptation. Solver needs to negotiate a wide range of iteration to achieve converge.

Notes

Using Picard and Newton techniques, cumulative mass balance error (CMBE) plot is presented in the Figure 6 for all temporal discretizations. The mass balance error almost closes to zero with the exception of a couple of cases around $100000\,s$ implied that the accurate solution is ensured. Nonlinear iteration, time stepping and CMBE behavior of $\Delta z = 0.004\,m$ case is not presented graphically as they are almost same as for $\Delta z = 0.008\,m$. Note that Newton method cannot converge for $\Delta t = 100\,s$ and $1000\,s$ for grid spacing $\Delta z = 0.004\,m$.

The computational statistics of the methods under all the cases are summarized in Table 1 and Table 2. The performance indicators are the total number of iterations, cumulative mass balance error, the average number of Picard and Newton iterations taken at each time step, the number of back stepping occurrences i.e., failure of Picard or Newton to converge within the assigned maximum number of iterations, the number of linear solver failures and the computational time (CPU). Examining more closely both iterative results, we note that Newton scheme resulted in significantly fewer back-stepping occurrences. Graphical results and statistics of the simulation clearly indicate that the technique is adequate.

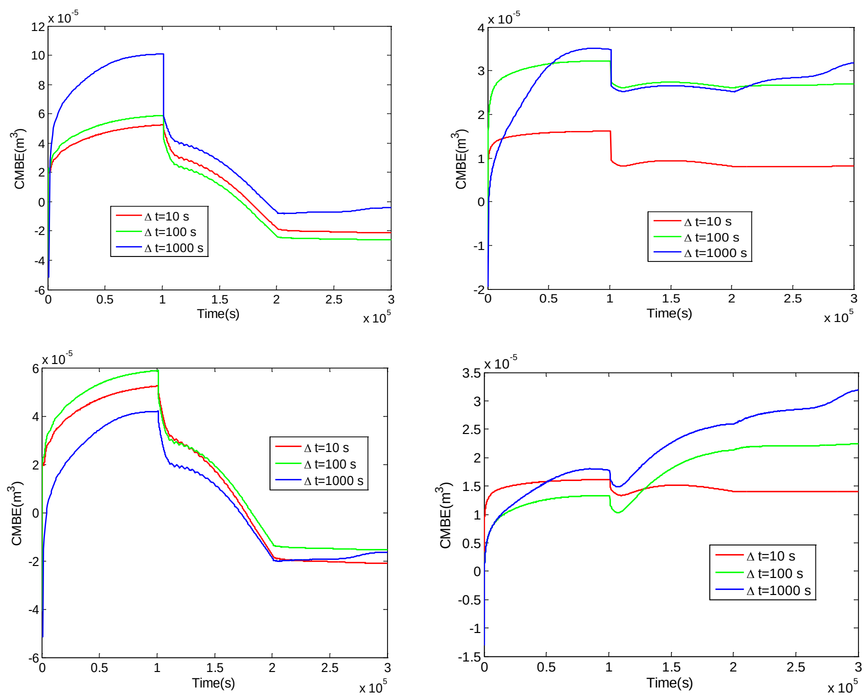


Figure 6: Cumulative mass balance error behavior of Picard (top row) and Newton (bottom row) schemes for $\Delta z = 0.04\,m$ (left) and $\Delta z = 0.008\,m$ (right) of Test problem 1

Another precision of simulation is assessed by the root mean squared error (RMSE) as for the reference solution which is made utilizing very fine grid with very small nonlinear tolerance. Errors are measured at three different times, explicitly, at $35000\,s$, $155000\,s$, and $300000\,s$ for all the temporal discretizations of Picard and Newton techniques (Table 3). We have appeared (Figure 7) that the normal errors are most noteworthy at the coarsest spatial and temporal discretizations, and the pinnacle errors spread with the moisture front that is moving downwards into the soil. Furthermore, Picard scheme gives little higher errors than Newton scheme and sharp increment in absolute error in the range of ponding time is observed. Choice of the

Notes

discretization method of spatial and temporal domain has a great impact on handling soil properties, as a result, numerical accuracy can be affected significantly including the stability and rate of convergence of the numerical scheme.

Table 1: Computational statistics of Picard scheme for Test problem 1

$\Delta t_{max}(s) \rightarrow$	10			100			1000		
$\Delta z\ (m) \rightarrow$	0.04	0.008	0.004	0.04	0.008	0.004	0.04	0.008	0.004
MBE (m^3)	-2.11e-5	8.12e-6	8.44e-6	-2.13e-5	9.85e-6	4.95e-6	-2.11e-5	8.12e-6	8.44e-6
No. of time step	42583	136622	186242	303006	364601	403462	42583	136622	186242
NL Ite/Step	2.22	2.44	2.37	1.43	1.65	1.70	2.22	2.44	2.37
Back step	1084	7911	11419	367	6290	9564	1084	7911	11419
Solver failures	0	0	0	0	0	0	0	0	0
CPU (s)	5546	80729	202349	3679	69892	74496	3055	7790	183895

** NL Ite=Nonlinear Iteration*

Table 2: Computational statistics of Newton scheme for Test problem 1

$\Delta t_{max}(s) \rightarrow$	10			100			1000		
$\Delta z\ (m) \rightarrow$	0.04	0.008	0.004	0.04	0.008	0.004	0.04	0.008	0.004
MBE (m^3)	-2.09e-5	1.40e-5	5.90e-6	-1.52e-5	2.24e-5	Div	-1.65e-5	3.19e-5	Div
No. of time step	30149	30247	30266	3187	3737	Div	1281	2237	Div
NL Ite/Step	1.86	2.05	2.08	3.38	4.26	Div	5.58	5.93	Div
No. of back step	6	16	14	7	12	Div	15	22	Div
Solver failures	0	2	1	0	1	Div	1	3	Div
CPU (s)	6949	52378	70347	1385	11303	Div	911	7790	Div

** Div=Divergent*

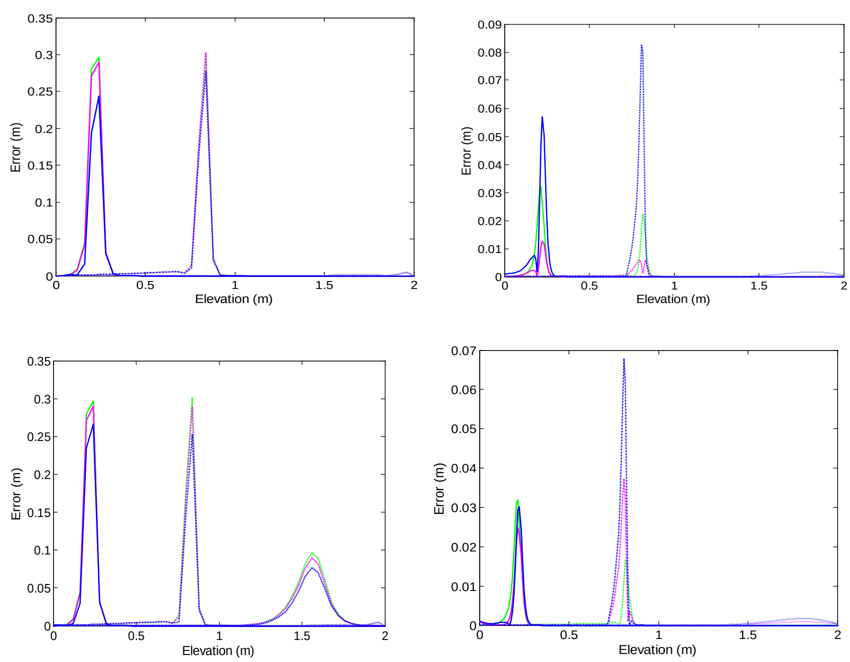


Figure 7: RMSE behavior of Picard (top row) and Newton (bottom row) schemes for $\Delta z = 0.04\ m$ (left) and $\Delta z = 0.008\ m$ (right)of Test problem 1.Errors at 35000s, 155000s and 300000s are marked by solid, dash-dotted, dashed lines with green, magenta, blue colors respectively

Table 3: RMSE of Picard and Newton schemes for Test problem 1

$\Delta z\ (m)$	$\Delta t_{max}\ (s)$	Time (s)→ Method↓	35000	155000	300000
0.04	10	Picard	5.76e-2	4.93e-2	9.33e-4
		Newton	5.76e-2	4.94e-2	9.33e-4
	100	Picard	5.61e-2	4.98e-2	9.37e-4
		Newton	5.62e-2	4.73e-2	9.19e-4
	1000	Picard	4.38e-2	4.54e-2	9.15e-4
		Newton	5.01e-2	4.12e-2	9.05e-4
0.008	10	Picard	4.20e-3	2.50e-3	6.62e-5
		Newton	4.20e-3	1.90e-3	6.89e-5
	100	Picard	1.50e-3	9.80e-3	9.69e-5
		Newton	3.20e-3	4.50e-3	3.32e-4
	1000	Picard	6.60e-3	1.00e-3	5.98e-4
		Newton	3.37e-3	8.10e-3	6.36e-4

Notes

b) Test problem 2

The simulations of this test case with different layer thicknesses with the heterogeneity in the soil moisture retention curves, represented with the Brooks–Corey model. This case involves vertical drainage from initially saturated conditions. At time $t = 0\ s$, the pressure head at the base of the column is reduced from $2\ m$ to $0\ m$. During the subsequent drainage, a no-flow boundary condition is applied to the top of the soil column. These forcing conditions lead to the development of a sharp discontinuity in the moisture content occurs at the interface between two material layers [33, 34, 35]. This type of problem provides a rigorous test case for a numerical algorithm and is well suited for the analysis of numerical convergence and efficiency.

During downward draining, the middle coarse soil tends to restrict drainage from the upper fine soil, and high saturation levels are maintained in the upper fine soil for a considerable period of time. The hydraulic properties of the soils are given in Table 4. The soil profile is Soil 1 for $0 < z < 60\ cm$ and $120\ cm < z < 200\ cm$ and Soil 2 for $60\ cm < z < 120\ cm$.

Table 4: Soil hydraulic properties used in Test problem 2

Parameters	Soil 1	Soil 2
θ_s	0.35	0.35
θ_r	0.07	0.035
$\alpha\ (\text{cm}^{-1})$	0.0286	0.0667
n	1.5	3.0
K_s (cm/s)	9.81×10^{-5}	9.81×10^{-3}

The soil moisture curves of the moisture content (θ) and specific moisture capacity (C) are evaluated by the Brooks–Corey model (Figure 8). The shape of the soil moisture capacity is very sharp near the saturation implies the rigorous complexities are encountered when the analytical differentiation of fluid content is used. As a consequence, numerical accuracy can be affected significantly. To handle such difficulties efficiently, proper choice of grid resolution and temporal discretization is required for heterogeneous porous media.

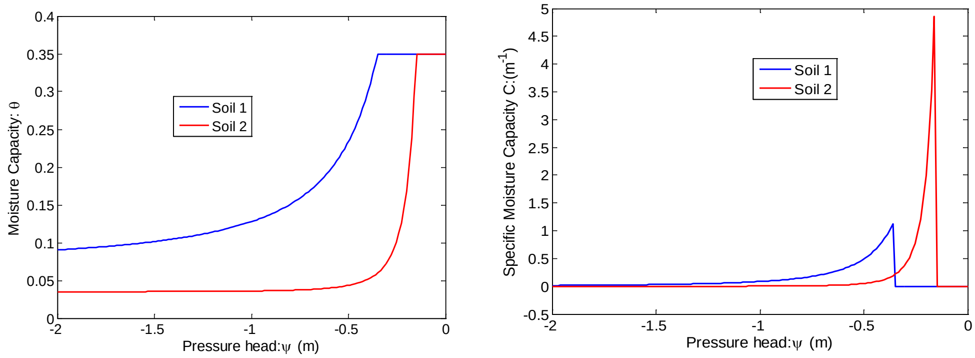


Figure 8: Soil moisture characteristic curves for Test problem 2

To compare the performance of the algorithm, simulations are performed on two fine mesh of 300 and 150 elements and a coarser mesh of 50 elements with three time step sizes ($\Delta t_{max} = 10\,s, 100\,s$ and $1000\,s$) via dynamic time stepping control for nonlinear iterations with mass lumping. The algorithm is used to simulate the experiment and the comparison of water saturation prediction after $1050000\,s$is depicted in Figure 9, which is similar to those presented in the published result [33, 34, 35]. Some oscillations are produced in the middle coarse soil in the solution profile, as our expectation. These oscillations have been attributed to insufficient spatial resolution.

The simulations conducted with small grid spacing produce more acceptable results in that the overall shape of the soil hydraulic characteristic. However, that use of even smaller grid spacing may not significantly improve the simulation results. It is recommended that the computed saturation is sensitive about grid spacing.

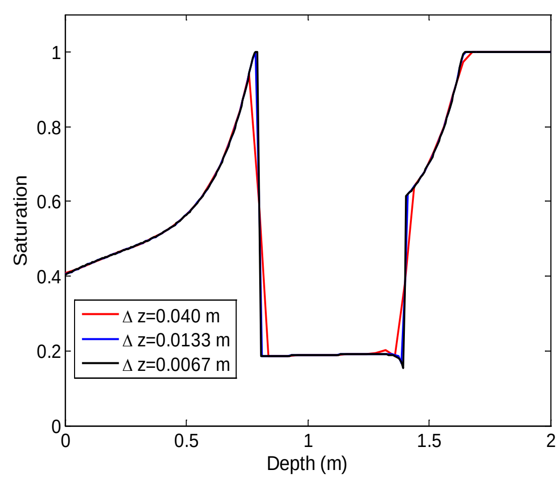


Figure 9: Saturation predictions after 1050000 s for different spatial discretizations of Test problem 2

Adaptive time stepping behavior (Figure 10), nonlinear iterations per time step (Figure 11) and cumulative mass balance error (Figure 12) are presented graphically for the case $\Delta z = 0.04\,m$ and $\Delta z = 0.0133\,m$. Almost similar results are recorded for 150 and 300 elements. So, in the figure analysis on the basis of the mentioned factors are excluded for 300 elements. Time stepping plots shows that Picard scheme has to face very little trouble at $2 \times 10^5\,s$, whereas Newton scheme is highly affected during the

Notes

Notes

simulation for $\Delta t_{max} = 100\ s$ and $1000\ s$. But note that, the large time step size speedup to complete the simulation. Convergence plots demonstrated that, Picard and Newton techniques need only one iteration during entire simulation for $\Delta t_{max} = 10\ s$. There are some differences are observed for other time scales as well grid spacing. Cumulative mass balance errors are almost approaching to zero. This implies that the numerical results are strictly maintained accuracy. Table 5 and 6 summarized the simulation statistics for Picard and Newton iteration methods respectively. The mass balance error at any given time step is calculated as the absolute difference between the changes in water storage during that time step. In this test case, Newton technique needs many back-stepping to achieve the convergence for all spatial and temporal discretizations. RMSE evaluated with respect to the surrogate exact solution with very small nonlinear tolerance. Errors are measured at the three different times, specifically, at $250000\ s$, $550000\ s$, and $1050000\ s$ for all the temporal discretizations of Picard and Newton techniques (Table 7).

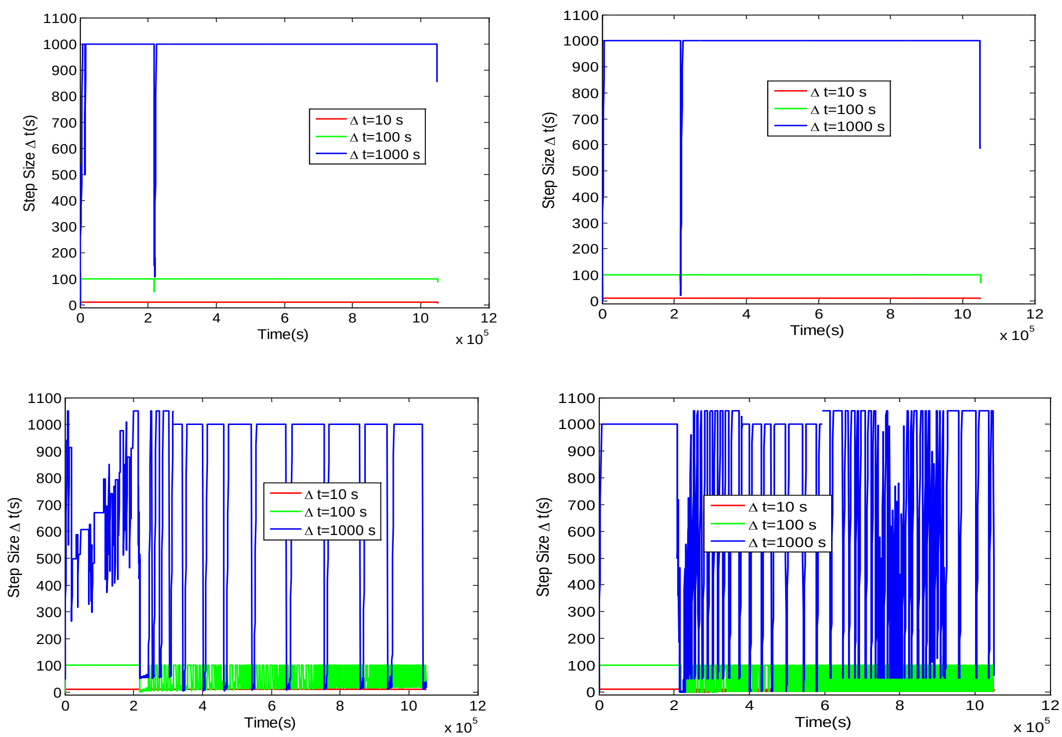
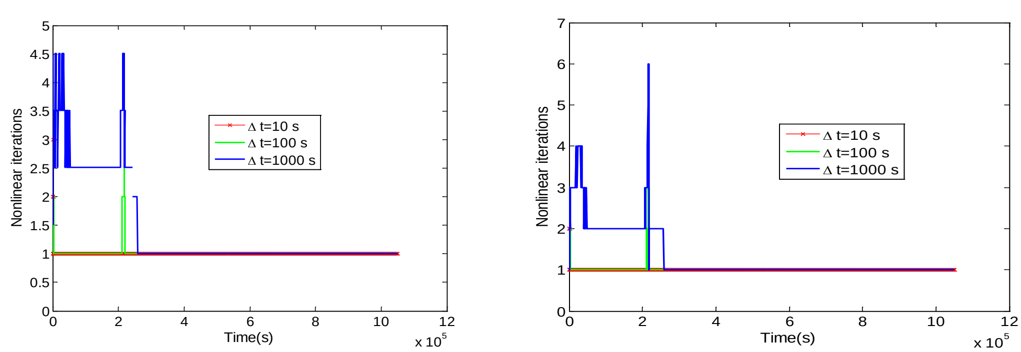


Figure 10: Dynamic time stepping behavior of Picard (top row) and Newton (bottom row) schemes for $\Delta z = 0.04\ m$ (left) and $\Delta z = 0.0133\ m$ (right)of Test problem 2



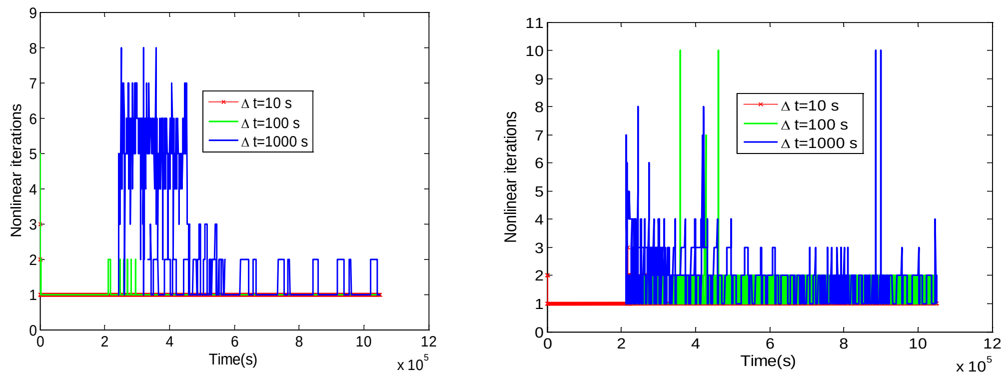


Figure 11: Nonlinear convergence behavior of Picard (top row) and Newton (bottom row) schemes for $\Delta z = 0.04\ m$ (left) and $\Delta z = 0.0133\ m$ (right)of Test problem 2

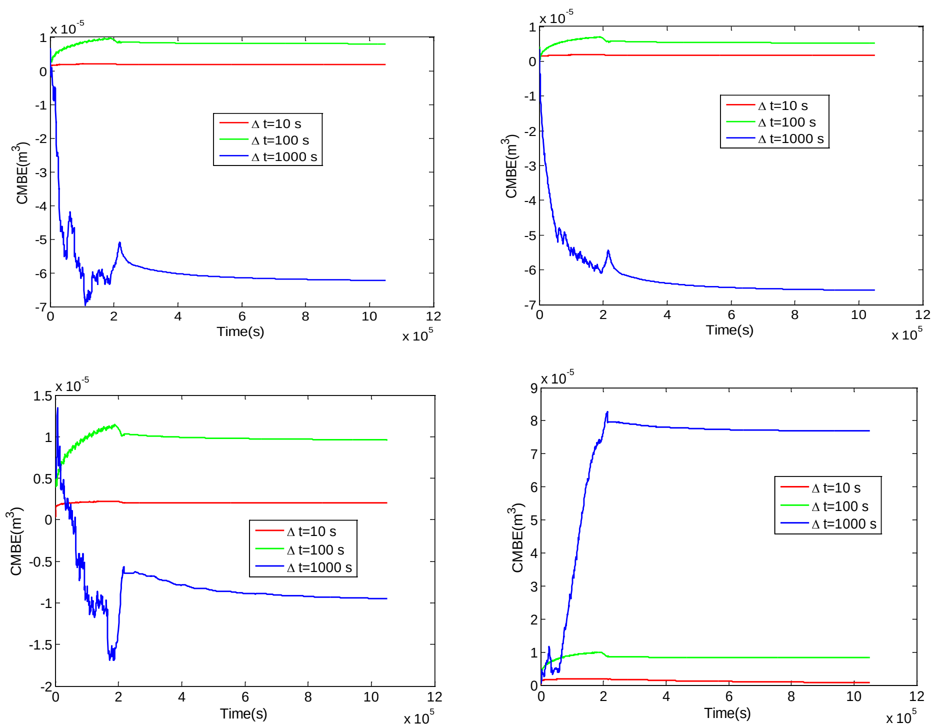


Figure 12: Cumulative mass balance error behavior of Picard (top row) and Newton (bottom row) schemes for $\Delta z = 0.04\ m$ (left) and $\Delta z = 0.0133\ m$ (right) of Test problem 2

Table 5: Computational statistics of Picard scheme for Test problem 2

$\Delta t_{max}(s) \rightarrow$	10			100			1000		
$\Delta z\ (m) \rightarrow$	0.04	0.0133	0.0067	0.04	0.0133	0.0067	0.04	0.0133	0.0067
MBE (m^3)	1.88e-6	1.72e-6	1.43e-6	8.03e-6	0.0067	3.11e-6	-6.23e-5	6.59e	-7.44e-5
No. of time step	105051	105051	105051	10566	5.28e-6	10556	1146	43602	1136
NL Ite/Step	1.00	1.00	1.00	1.04	1.03	7.56	1.62	3.62	1.58
No. of back step	15	15	15	20	1.03	16	29	28	25
Solver failures	0	0	0	0	19	0	0	0	0
CPU (s)	6449	14034	47528	767	2506	4050	134	631	966

Table 6: Computational statistics of Newton scheme for Test problem 2

$\Delta t_{max}(s) \rightarrow$	10			100			1000		
$\Delta z\ (m) \rightarrow$	0.04	0.0133	0.0067	0.04	0.0133	0.0067	0.04	0.0133	0.0067
MBE (m^3)	2.20e-6	7.15e-7	7.32e-7	9.59e-6	8.26e-6	7.63e-6	-9.45e-6	7.69e-5	8.59e-5
No. of time step	106217	240041	371587	29024	79014	143872	11137	43602	86856
NL Ite/Step	1.06	3.07	3.10	3.05	3.38	3.53	3.55	3.62	3.65
No. of back step	663	49640	77980	5972	18698	36050	2682	11287	22697
Solver failures	0	0	0	0	19	0	0	0	0
CPU (s)	14945	416403	1340	31101	145679	168846	11400	74944	367914

Table 7: RMSE of Picard and Newton schemes for Test problem 2

$\Delta z\ (m)$	$\Delta t_{max}\ (s)$	Time (s)→ Method↓	250000	550000	1050000
0.04	10	Picard	4.70e-3	3.70e-3	3.60e-3
		Newton	5.30e-3	3.70e-3	3.60e-3
	100	Picard	4.70e-3	3.70e-3	3.60e-3
		Newton	5.20e-3	3.90e-3	3.80e-3
	1000	Picard	4.60e-3	3.70e-3	3.60e-3
		Newton	5.50e-3	3.30e-3	3.70e-3
0.0133	10	Picard	1.10e-3	1.00e-3	8.21e-4
		Newton	1.10e-3	1.30e-3	8.95e-4
	100	Picard	1.10e-3	1.00e-3	8.21e-4
		Newton	1.30e-3	1.00e-3	7.72e-4
	1000	Picard	4.60e-3	3.70e-3	3.60e-3
		Newton	1.10e-3	9.83e-4	8.08e-4

VI. Conclusions

A finite element algorithm is introduced to solve the Richards' equation for one-dimensional flow problems in variably saturated soils. Specifically, the problem of mass-balance errors is handled, which is in reality a pressing problem for the simulation of such highly nonlinear phenomena as the infiltration into soil column and drainage through layered soil from initially saturated condition. The effectiveness of the algorithm is demonstrated by compare with published results. The conduct of various techniques for solution estimates and adaptive time stepping were experimented for Richards' equation model. Time step adaptation is essential to accomplish sensible figuring execution in reasonable uses of Richards' equation. Head based Picard and

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Newton iteration schemes are compared, where three step-time sizes are implemented for each of three different spatial discretizations. It is demonstrated that both iterative schemes are mass conservative and efficient in terms of nonlinear iteration. For the most part, large time-step size requires modest number of iterations to converge the solution, however, Newton scheme is diverge for drainage problem, as well as significantly many back-stepping occurred. So, the size of the time step can be constrained by the convergence of the iterative scheme for simulating strong nonlinearities. Coarse grid spacing is caused for numerical oscillations for the both test experiments. Therefore, time step size and/ or grid size are the influential factors for the numerical simulation of variably saturated flows. The model presents tremendous mass balance property over whole spatial and temporal mesh for the problems of infiltration fronts and drainage problems. The accomplishment of the finite element algorithm in simulating an assortment of problems leads to confidence in its applicability to many dynamically saturated flow problems for its advantageous flexibility. Further research is needed in the development of multidimensional finite element model for solving problem in saturated–unsaturated regions without special treatment of fluid content discontinuities in heterogeneous porous media.

Notes

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Conjecturing with Some Conjectures

By Balasubramani Prema Rangasamy

Abstract- This world has been seeing so many conjectures from the formation of this world. Particularly, mathematics world has been seeing so many conjectures from the civilization of this world, also venturing through on it. So many Indians, Chinese and Arabian scholars provided their participation in mathematics world. Since before Euclid so many mathematicians ventured on numbers, geometry, astronomy, etc… in number theory, there are so many conjectures like applications of GCD, Fermat Last theorem, Euler's totient function, Gold Bach conjecture, ABC conjecture, etc… some of this has been proved but so many of that still has not been proved. In this paper, I try to prove Gold Bach conjecture, stating abc conjecture for composite numbers, and try to deliver some conjectures.

Keywords: *GCD, euler's totient function, gold bach conjecture, abc conjecture.*

GJSFR-F Classification: *MSC 2010: 05C60*

CONJECTURINGWITHSOMECONJECTURES

Notes

Conjecturing with Some Conjectures

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Abstract- This world has been seeing so many conjectures from the formation of this world. Particularly, mathematics world has been seeing so many conjectures from the civilization of this world, also venturing through on it. So many Indians, Chinese and Arabian scholars provided their participation in mathematics world. Since before Euclid so many mathematicians ventured on numbers, geometry, astronomy, etc… in number theory, there are so many conjectures like applications of GCD, Fermat Last theorem, Euler's totient function, Gold Bach conjecture, ABC conjecture, etc… some of this has been proved but so many of that still has not been proved. In this paper, I try to prove Gold Bach conjecture, stating abc conjecture for composite numbers, and try to deliver some conjectures.

Keywords: GCD, euler's totient function, gold bach conjecture, abc conjecture.

I. Introduction

In mathematics, the **greatest common divisor** (**gcd**) of two or more integers, which are not all zero, is the largest positive integer that divides each of the integers. For twoin tegers x, y, the greatest common divisor of x and y is denoted. For example, the gcd of 8 and 12 is 4, that is, In the name "greatest common divisor", the adjective "greatest" may be replaced by "highest", and the word "divisor" may be replaced by "factor", so that other names include **greatest common factor** (**gcf**), etc. Historically, other names for the same concept have included **greatest common measure**.

In number theory , **Euler's totient function** counts the positive integers up to a given integer n that are relatively prime to n. It is written using the Greek letter phi as $\varphi(n)$ or $\phi(n)$, and may also be called **Euler's phi function**. In other words, it is the number of integers k in the range $1 \leq k \leq n$ for which the greatest common divisor $\gcd(n, k)$ is equal to 1. The integers k of this form are sometimes referred to as totatives of n.

The ***abc*** **conjecture** (also known as the **Oesterlé–Masser conjecture**) is a conjecture in number theory, first proposed by Joseph Oesterle (1988) and David Masser (1985). It is stated in terms of three positive integers, a, b and c (hence the name) that are relatively prime and satisfy $a + b = c$. If d denotes the product of the distinct prime factors of abc, the conjecture essentially states that d is usually not much smaller than c. In other words: if a and b are composed from large powers of primes, then c is usually not divisible by large powers of primes. A number of famous conjectures and theorems in number theory would follow immediately from the *abc* conjecture or its versions. Goldfeld (1996) described the *abc* conjecture as "the most important unsolved problem in Diophantine analysis".

Various attempts to prove the abc conjecture have been made, but none are currently accepted by the mainstream mathematical community and as of 2020, the conjecture is still largely regarded as unproven.

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On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII), in which he proposed the following conjecture:

Every integer that can be written as the sum of two primes can also be written as the sum of as many primes as one wishes, until all terms are units.

A modern version of the marginal conjecture is:

Every even integer greater than 2 can be written as the sum of two primes.

[1]. **Wikipedia** gave some basic ideas about GCD, Euler's totient function, Gold Bach conjecture, ABC conjecture.[2]. **Balasubramani Prema Rangasamy** - Some Extensions on Numbers – Advances in Pure Mathematics – 2019. p. 944-958, gave some conjecturing ideas in Euler's totient function,

In this paper, I try to prove Gold Bach conjecture, stating abc conjecture for composite numbers, and try to deliver some conjectures.

Facts 1:

1. For any n > 2, we get $\varphi[x]$ is always even number.
2. $\varphi[E] \leq \frac{E}{2}$ is always even number, where E is even number.
3. We never find a number n ϵ I, which gives 2+12x|x ϵ I numbers when we find φ[n].
4. We cannot find a number n ϵ I, which gives $a^{\varphi(n)} \equiv 1$ mod n, where φ[n] = 2 + 12x|xϵI.

14	134	254	374	494	614	734	854	974	1094
26	146	266	386	506	626	746	866	986	1106
38	158	278	398	518	638	758	878	998	1118
50	170	290	410	530	650	770	890	1010	1130
62	182	302	422	542	662	782	902	1022	1142
74	194	314	434	554	674	794	914	1034	1154
86	206	326	446	566	686	806	926	1046	1166
98	218	338	458	578	698	818	938	1058	1178
110	230	350	470	590	710	830	950	1070	1190
122	242	362	482	602	722	842	962	1082	1202

Above 100 numbers which are not the value of φ[n], where n is any positive integer.

Following numbers are also having the same character like above numbers i.e. 3(1+4x)-1| x ϵ I.

68	152	188	194	308	428	548	668	788	872
908	1028	1148	…						

All the above numbers having a common relation, that is, digit sum of above numbers would be 2 or 5 or 8. Digit sum of 12x would be 3, 6, and 9. These 3, 6 and 9 are rotational identities. They are stumping their existence in all ways.

38+30x | x ϵ I also not comes for φ[n].

i.e. 38	68	98	158	188	248	278	308	338	398	428
	458	488	518	548	578	608	638	668	698	728
	758	788	818	…						

Ref

2. Balasubramani Prema Rangasamy - Some Extensions on Numbers – Advances in Pure Mathematics – 2019. p. 944-958.

In above numbers some numbers like 38 + 30 x 3 = 38 +90 = 128, 368 … would be value of φ[n], even though their digit sum would be 2 and 8.

114,318, 298, would not be a value of φ[n], contrarily their digit sum values are 3, 1 …
If digit sum 2,5 and 8 numbers mostly would not be a value of φ[n].

Notes

From the above we concluded that, there are so many numbers greater than equal to fourteen exist which would not be the value of φ[n].

Theorem 1: Let x ≥ 2 be the integer then $\varphi[\varphi[\varphi[\ldots\ldots \varphi[x]]]] = 1$. i.e. $\varphi^{n}[x] = 1$. where n is the totient order of x.

Proof: Let x be any number then $x > \varphi[x]$.

Let we take $\varphi[x] = x_1$ then $x_1 > \varphi[x_1]$

By this way we can obtain the totatives, $\varphi[x] > \varphi[x_1] > \varphi[x_2] > \cdots > \varphi[2] = 1$ ■

Ex:

1. Let x = 693 then

$693 > \varphi[693] = 360$

$360 > \varphi[360] = 96 = \varphi^{2}[693]$

$96 > \varphi[96] = 32 = \varphi^{3}[693]$

$32 > \varphi[32] = 16 = \varphi^{4}[693]$

$16 > \varphi[16] = 8 = \varphi^{5}[693]$

$8 > \varphi[8] = 4 = \varphi^{6}[693]$

$4 > \varphi[4] = 2 = \varphi^{7}[693]$

$2 > \varphi[2] = 1 = \varphi^{8}[693]$ Totient order of 693 is 8.■

Facts 2:

1. If GCD (a, b) = k then GCD (a^n, b^n) = k^n, where n ϵ Z.
2. If GCD $(a, b) = k$ and $\frac{a}{k} = c; \frac{b}{k} = d$ then GCD (a^n, b^n) = k^n and $\frac{a^n}{k^n} = c^n; \frac{b^n}{k^n} = d^n$, where n ϵ Z.
3. We can generalize above as If GCD $(a_1, a_2, a_3 \cdots a_i) = k$ and

   $\frac{a_1}{k} = b_1, \frac{a_2}{k} = b_2, \frac{a_3}{k} = b_3 \cdots \frac{a_i}{k} = b_i$ then GCD $(a_1^n, a_2^n, a_3^n \cdots a_i^n) = k^n$ and

   $\frac{a_1^n}{k^n} = b_1^n, \frac{a_2^n}{k^n} = b_2^n, \frac{a_3^n}{k^n} = b_3^n \cdots \frac{a_i^n}{k^n} = b_i^n$, where n ϵ Z.
4. If $p_1 p_2 p_3 \cdots p_i$ are distinct primes then $p_1^a \neq p_2^b \neq p_3^c \cdots \neq p_i^\alpha$
5. We can write any composite number as the product of prime numbers. i.e.

   $c = p_1^a p_2^b p_3^c \cdots$

Theorem 2: Let a, b, c are composite positive integers and a + b =c, also k = GCD (a, b, c) then x + y = z is relatively prime with each other, where $x = \frac{a}{k}; y = \frac{b}{k}; z = \frac{c}{k}$.

Proof:

Let a, b, c are composite positive integers and a + b =c, also k = GCD (a, b, c) then we can write a + b =c as $xk + yk = zk$

If GCD$\left(a_1, a_2, a_3 \cdots a_i\right) = 1$, we multiply each element of GCD $\left(a_1, a_2, a_3 \cdots a_i\right) = 1$ with k, we get GCD $\left(ka_1, ka_2, ka_3 \cdots ka_i\right) = k$. Which means all the relatively prime numbers changed into composite number of k.

So we divide, $xk + yk = zk$ by k

Hence we get $x + y = z$ ■

Theorem 3: Let a, b, a+b are composite positive integers with GCD k, then we can find

$a^n + b^n = (a+b)^n - \sum_{i=1}^{n-1}\binom{n}{i}a^{n-i}b^i$ with GCD k^n.

If we divide $a^n + b^n = (a+b)^n - \sum_{i=1}^{n-1}\binom{n}{i}a^{n-i}b^i$ by k^n, we get x + y = z with GCD 1,

where $x = \frac{a^n}{k^n}; y = \frac{b^n}{k^n}$ and $z = \frac{c^n - \sum_{i=1}^{n-1}\binom{n}{i}a^{n-i}b^i}{k^n} = \frac{\left(a^n + b^n\right)}{k^n}$.

Facts 3:

1. Except two, all prime numbers are odd number.
2. Two only the number stated as even prime.
3. Except 2, $p \pm o \subset E$ Are composite numbers and $p \pm e \subset O$ may be prime or composite.
4. Two and above digits Prime numbers ended with one, three, seven and nine.
5. If even integer ended with zero, we can express 0 = 1 + 9 = 3 + 7 = 5 + 5. These 1, 3, 5, 7 and 9 are ended digit in certain number. But other than 10, we cannot express zero ended number as sum of two five ended prime numbers.
6. If even integer ended with two, we can express 2 = 1 + 1 = 2 + 9 = 5 + 7.
7. If even integer ended with four, we can express 4 = 1 + 3 = 5 + 9 = 7 + 7.
8. If even integer ended with six, we can express 6 = 1 + 5 = 3 + 3 = 7 + 9.
9. If even integer ended with eight, we can express 8 = 1 + 7= 3 + 5 = 9 + 9.
10. Four is the only even number, expressed as sum of two even prime. i.e. 4 = 2 + 2.
11. Two, three, five and seven are base prime. Nine is not base prime but numbers which are ended with nine may be prime number. Here we considered single digit prime numbers are base prime numbers.
12. If digit sum of odd number is either three or six or nine, it would be a composite number.
13. All odd prime numbers having even integer relationship with each other.

Notes

Definition: Residue factors

Let A be a dividend, its factors are abcd and B be a divisor, its factors are abc then residue factor of A ÷ B is d.

Ex

1. Let A =48 and B =16 then factors of A = 2^4 x 3 and factors of B = 2^4 then residue factor $= \frac{2^4 \times 3}{2^4} = 3$. Residue factor is odd.
2. Let A =210 and B =14 then

   Factors of A = 2 x 3 x 5 x 7 and Factors of B = 2 x 7 then residue factor

   $= \frac{2 \times 3 \times 5 \times 7}{2 \times 7} = 3 \times 5 = 15$. Residue factor is odd.
3. Let A = 48 and A = 24 then

   Factors of A = 2^4 x 3 and Factors of B = 2^3 x 3 then residue factor $= \frac{2^4 \times 3}{2^3 \times 3} = 2$.

   Residue factor is even.

Arithmetic operations of odd and even integers

Addition

O + O = E

O + E = O

E + O = O

E + E = E

Subtraction

|O – O | = E

|O – E | = O

|E – O | = O

|E – E | = E

Multiplication

O x O = O

O x E = E

E x O = E

E x E = E

Division

O ÷ O = O

E ÷ E = O if residue factor is odd

E ÷ E = E if residue factor is even ■

Notes

Summations of prime numbers and composite numbers

$$\sum_{O} O_i = O \;; \sum_{E} O_i = E \;; \sum_{O} E_i = E \;; \sum_{E} E_i = E$$

$$\sum_{O} p_i = O \text{ and } \sum_{E} p_i = E \;;$$

Gold Bach conjecture

Every even integer greater than two can be expressed as the sum of two prime numbers.

Proof:

We know 4 = 2 + 2, 6 = 3 + 3, 8 = 3 + 5, 10 = 5 + 5 = 7 + 3, 12 = 7 + 5, 14 = 7 + 7, 16 = 5 + 11 …but is it true for all even numbers? So, we try to prove every even integer greater than two can be expressed as the sum of two prime numbers by some ideological concepts. We know the fact 4 only expressed by sum of two even prime. In other words, No even integer greater than four can be expressed by sum of two even prime. But it can be expressed by two odd prime.

Let E be an even number and O_1 and O_2 are odd numbers.

We can express E as sum of two odd numbers. i. e. $E = O_1 + O_2$.

Using above facts, we can say all odd prime numbers are the members of odd numbers. i.e. $p \subset O$

Here we recall one thing, every even integer can be expressed as the difference of two primes. i.e. $E = p_1 - p_2$

Every prime number can be expressed as the sum of odd number and even number. Also we can express prime number as the difference of odd number and even number.

i.e. $o_1 + e_1 = p_1 \in O$ and $o_2 - e_2 = p_2 \in O$.

Hence, we can express $E = (o_1 + o_2) = ((o_1 + e_1) + (o_2 - e_2)) = p_1 + p_2$.

More precisely we can express above as,

$$E = (o_1 + o_2) = ((o_1 + e) + (o_2 - e)) = p_1 + p_2 = ((o_1 - e) + (o_2 + e)) \blacksquare$$

Ex 1:

Let 94, we can express 94 = 63 + 41 = 25 + 69 = 37 + 57 = … for instance,

1. Let 63 and 41, its sum is 94. 94 is even number. Four ended number. Possibility of summation is 1+3, 5+9, and 7+7. 63 and 41 are odd numbers but 63 is composite number and 41 is prime number. We need one prime number instead of 63. We know prime number 61 is near to 63. But 61+2 is 63 so we subtract 2 from 63. Now we get 61. To balance equality of sum, we should add the same 2 with 41. Now we get 43 and 43 is a prime. Also we get 1 and 3 combination. So we can express 94 = (63-2) + (41+2) = 61 + 43.
2. Let 25 and 69, its sum is 94. 94 is even number. Four ended number. Possibility of summation is 1+3, 5+9, and 7+7. 25 and 69 are odd numbers but both are composite

Notes

Notes

numbers, we need two prime numbers instead of 25 and 69. We know prime number {…17, 19, 23, 29, 31, 37…} is near to 25. Also {…57, 61, 67, 71, 73 …} near to 69. Select

If e = 2 then

$$94 = (25 + 69) = ((25 - 2) + (69 + 2)) = 23 + 71$$

e = 2 is opted for this way of summation.

But $94 = (25 + 69) = ((25 + 2) + (69 - 2)) = 27 + 67$,

e=2 is not suit for this way of summation.

If e = 6 then $94 = (25 + 69) = ((25 - 6) + (69 + 6)) = 19 + 75$, e = 6 is not suit for this.

If e = 6 then $94 = (25 + 69) = ((25 + 6) + (69 - 6)) = 31 + 64$, e = 6 is not suit for this.

If e = 8 then $94 = (25 + 69) = ((25 + 8) + (69 - 8)) = 33 + 61$, e = 8 is not suit for this.

If e = 8 then $94 = (25 + 69) = ((25 - 8) + (69 + 8)) = 17 + 77$, e = 8 is not suit for this.

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If e = 16 then

$$94 = (25 + 69) = ((25 - 16) + (69 + 16)) = 9 + 85 ,$$

e = 16 is not suit for this way of summation.

But, $94 = (25 + 69) = ((25 + 16) + (69 - 16)) = 41 + 53$,

e=16 is opted for this way of summation.

From above we concluded that, until we expressed sum of two prime numbers equal to an even number, we do repeatedly the above.

References Références Referencias

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19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- Single section and succinct.
- An outline of the job done is always written in past tense.
- Concentrate on shortening results—limit background information to a verdict or two.
- Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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