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Gauge Symmetries in Physical Fields (Review)

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I. Introduction

Gauge invariance is one of the fundamental symmetries in modern theoretical physics. The gauge invariance was recognized in the 19th century as a mathematical non-uniqueness of potentials that exists despite the uniqueness of observable electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In the 20th century, physical significance of the gauge symmetry was recognized very fundamental and played a role of guiding principle in the study of physical fields such as Electromagnetism, Particle physics and Theory of Gravitation.

It took almost a century to recognize its fundamental physical significance, resulting in, finally, successful formulation of the Gauge Principle. In particular, the *gauge theory* played vital roles in the remarkable development of modern particle physics which was revolutionary (*e.g.* Aitchison & Hey (2013), Utiyama (1956)). In fact, historical development of the gauge theory took gradual and zigzag processes.

In the present paper, firstly, historical developments of gauge theory are reviewed from its initial gauge transformation to later theory of gauge principle taking a zigzag way from one physical field to another, and secondly, possible application of the gauge theory is envisaged to fluid-flow field although the field of fluid-flow is not listed in the literature reviewed.

a) *Historical development of gauge transformations*

What is now generally known as a gauge transformation of the electromagnetic potentials was discovered in 19th century in the process of formulation of classical electrodynamics from mathematical point of view (rather than physics) by its pioneers (Faraday, Neumann, Weber, Kirchhoff, Maxwell, Lorenz, Helmholtz, Lorentz and others: according to Jackson & Okun (2001)). It was, in fact, non-uniqueness of a vector potential $\boldsymbol{A}$ in mathematical representation of electromagnetic field that exists despite the uniqueness of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. This is now referred to as *local gauge invariance* of Maxwell's equations. The law of electromagnetic induction discovered by Faraday (1831) is represented mathematically by the first of the following pair of Maxwell equations:

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Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{1.1}$$

The second is an outcome of the fact that the magnetic field $\boldsymbol{B}$ is generated by electric currents (Jackson (1999, Chap.5)), implying non-existence of magnetic monopoles. In Maxwell's electromagnetic theory (1856), the vector potential played an important role. Introducing a 3-vector potential $\boldsymbol{A} = (A_1, A_2, A_3)$ and a scalar potential $\Phi^{em} = -A_0$, and defining $\boldsymbol{E}$ and $\boldsymbol{B}$ by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -c^{-1}\,\partial_t \boldsymbol{A} - \nabla\Phi^{em}, \tag{1.2}$$

the above pair of equations (1.1) are satisfied identically. This led to a finding that, using an arbitrary differentiable scalar function Ψ^e, the following transformation of the potentials $\boldsymbol{A}$ and Φ^{em},

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla\Psi^e, \qquad \Phi^{em} \to \Phi^{em} - \partial_t\Psi^e, \tag{1.3}$$

revealed a significant property, what is now called the *gauge transformation*, of the electromagnetic field. Maxwell (1873) noticed the invariance of $\boldsymbol{B}$ only by the first of the transformation (1.3), but missed the second one because he relied on the gauge condition $\nabla \cdot \boldsymbol{A} = 0$. The simultaneous two transformations of (1.3) was established by L. V. Lorenz (1867) on the basis of the following gauge condition,

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c}\partial_t \Phi^{em} = 0. \tag{1.4}$$

It is remarkable that the observable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ of (1.2) are invariant in spite of the transformation (1.3). This was the invariance known in the electromagnetic theory of the 19th century. In modern gauge theory, the gauge condition (1.4) is often referred to as *Lorentz* condition, according to Dutch physicist H. A. Lorentz who was one of the key figures in the final formulation of classical electrodynamics (1904) including the condition (1.4), while the former Danish physicist L. V. *Lorenz* (1867) introduced *first* the condition (1.4) (Jackson & Okun, 2001).

In the 19th-century classical electrodynamics, the transformation (1.3) was understood as meaning simply *non-uniqueness* of the vector potential $\boldsymbol{A}$ and scalar potential Φ in a mathematical sense. Its physical significance was not recognized until the 20th-century physics was developed. In the relativity theory of Einstein (1905, 1915), four dimensional (4d) spacetime $x^\nu = (x^0, x^1, x^2, x^3)$ with $x^0 = c\,t$ was introduced under the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) = \eta^{\mu\nu}$.† The structure of electromagnetism is most fitted to the 4d-spacetime. For example, the Lorenz condition (1.4) can be represented compactly as $\partial A^\nu/\partial x^\nu = 0$ in the 4-d spacetime, where $A^\nu = (\Phi, \boldsymbol{A})$. See (1.8) for the difference between the covariant (downstairs) vector A_μ and the contravariant (upstairs) vector A^ν. Scalar product in the Minkowski space is formed like $A_\mu\,\mathrm{d}x^\mu = \eta_{\mu\nu}A^\nu \mathrm{d}x^\mu$ by the pair of a covariant vector A_μ and a contravariant vector $\mathrm{d}x^\mu$ ((see 1.5)). [Concerning the difference of transformation property between the covariant and contravariant vectors, see the footnote to Appendix A.1.]

Stimulated by Einstein's relativity theory, Weyl attempted in 1918 to reinterpret the same transformation (1.3) of electromagnetic 4-potentials A^ν, but turned out to be unsuccessful. The term *gauge* (actually the German term *Eich*) was used to this transformation by Weyl (1918) first. He proposed to unify electromagnetism and gravity geometrically by attaching a scale factor l of the form $l \propto \exp[\int \phi_k(\boldsymbol{x})dx^k]$ where its variation is given by $\delta l = l\phi_k\,\delta x^k$. Although this received unfavorable response from Einstein to be in disagreement with observation, after the advent of the quantum theory, its interpretation was renewed by London (1927) that the Weyl's proposal could be used in quantum theory by changing the *scale* factor to a *phase* factor by attaching it to the wave function $\boldsymbol{\psi}(x^\nu)$ of quantum mechanics in the form,

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\Psi(x^\nu) = \exp\left[i\gamma \int A_\mu(\boldsymbol{x}) dx^\mu\right] \cdot \boldsymbol{\psi}(x^\nu), \tag{1.5}$$

where $\gamma = e/\hbar$ with e a charge, and the function $\boldsymbol{\psi}(x^\nu)$ satisfies the Schrödinger equation:

$$i\hbar\, \partial_t \boldsymbol{\psi} = -(\hbar^2/2m)\, \nabla^2 \boldsymbol{\psi} + eV\, \psi, \tag{1.6}$$

interpreted in section II b) and given by (2.29). Physical significance of the gauge invariance was upheld later by H. Wyle in 1929, who proclaimed this invariance as a *General Principle* and called it *gauge-invariance* (*Eichinvarianz* in German). The gauge invariance is a symmetry rooted at the deepest level of physics, as interpreted next in section I b).

In quantum mechanics, the transformation (1.3) was understood as a phase transformation of the wave function of Schrödinger's equation. In the theory of *gravitation*, on the other hand, the gauge transformation was generalized to such transformations that the vectors or curvature tensors ‡ characterizing the gravitational field as *physical reality* do not change (or satisfy associated transformation laws) in spite of coordinate transformations, where the coordinate frames are taken arbitrarily by the theory (its details are given in section III c) iii. and III d) for weak gravitational field). In *fluid mechanics* too, the convective derivative (following fluid motion) can be shown to satisfy invariance with respect to generalized gauge-transformation, presented in section IV c).

b) A hint of gauge principle with the argument reversed

Historically, the gauge symmetry has been established through zigzag courses. Next formulation may be a typical example. Observing the phase part of the extended wave function $\Psi(x^\nu)$ of (1.5), the phase factor implies existence of the following one-form $\mathcal{A}$ in the spacetime (x^μ), defined by

$$\mathcal{A} = A_\mu dx^\mu = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3, \tag{1.7}$$

$$A_\mu = \eta_{\mu\nu} A^\nu = (-\Phi^{em}, \boldsymbol{A}). \qquad A^\nu = (\Phi^{em}, \boldsymbol{A}). \tag{1.8}$$

The extended wave function $\Psi(x^\nu)$ implies a certain geometrical structure in the spacetime x^μ, furnished with a field A_μ existing in the 4-d spacetime x^ν. The field A_μ possesses an interesting property which is now presented.

The pair of fields $\boldsymbol{E}$ and $\boldsymbol{B}$ of (1.2) are derived from (1.7). In fact, taking exterior differential d of $\mathcal{A}$, we obtain the *field strength* two-form $\mathcal{F}$:

$$\mathcal{F} = \mathsf{d}\mathcal{A} = \sum \frac{1}{2} F_{\nu\lambda}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda, \qquad F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu, \tag{1.9}$$

$$\mathcal{F} \Leftrightarrow (\, F_{\nu\lambda}\,) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{1.10}$$

where $\boldsymbol{E} = (E_k)$ and $\boldsymbol{B} = (B_k)$ are defined by (1.2). The pair of equations (1.1) are also obtained from (1.9) by taking, once more, exterior differential of $\mathcal{F} = d\mathcal{A}$, yielding

† Greek letters such as $\alpha, \beta, \mu, \nu, \lambda, \cdots$ take the quartet $(0, 1, 2, 3)$ to denote 4d-spacetime components, whereas Latin letters such as $i, j, k, \cdots$ take the triplet $(1, 2, 3)$ to denote 3-space components.

‡ In differential geometry, a vector (or a tensor) in an n-dimensional coordinate frame U is not a simple n-tuple array (or a simple $n \times n$ matrix, respectively) of real numbers, but they must follow certain transformation laws when mapped to another n-dimensional coordinate frame V.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$\mathsf{d}\mathcal{F} = \mathsf{d}^2\mathcal{A} \equiv 0$. Its detailed expressions are given in section II a) i. Thus, the definition $\mathcal{A} = A_\mu dx^\mu$ of (1.7) is sufficient for deriving the pair of Maxwell equations (1.1).

Let us consider the *gauge transformation* concerning the one-form $\mathcal{A}$, defined by

$$\mathcal{G}: \; A_\nu \equiv A_\nu^{(old)} \;\to\; A_\nu^{(new)} \equiv A'_\nu = A_\nu^{(old)} - \partial_\nu \Theta, \tag{1.11}$$

equivalent to (1.3), where Θ is an arbitrary differentiable function. Then, we have $\mathcal{A}^{(new)} = A_\nu^{(new)} \mathsf{d}x^\nu = A_\nu^{(old)} \mathsf{d}x^\nu - \partial_\nu \Theta \, \mathsf{d}x^\nu = \mathcal{A}^{(old)} - \mathsf{d}\Theta$. From this, we find the invariance $\mathcal{F}^{(new)} = \mathcal{F}^{(old)}$ since $\mathsf{d}^2\Theta \equiv 0$. Namely, the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant by this local gauge transformation. We will see in section II b) for QED that there is local gauge invariance in quantum electrodynamics (QED) as well (*e.g.* Aitchison & Hey (2013, Chap.2)). It is worth noting that the Maxwell equations are invariant under the local gauge transformations (1.11). The details are given in the section II.

Suppose that we have a particular form of A_μ-field defined by $\tilde{A}_\mu \equiv \partial_\mu \Theta$ with Θ an arbitrary scalar function differentiable two times. Then the one-form $\tilde{A} = \tilde{A}_\mu dx^\mu$ is given by $\mathsf{d}\Theta$, and we have the expression $\Psi = \exp[i\gamma\Theta(x^\nu)] \cdot \boldsymbol{\psi}(x^\nu)$, since $\int \tilde{A}_\mu dx^\mu = \Theta$. In addition, since $\tilde{A} = \mathsf{d}\Theta$, the field strength form $\mathcal{F}$ vanishes identically, because $\mathcal{F} = \mathsf{d}\tilde{A} = \mathsf{d}^2\Theta \equiv 0$. Namely, the observable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish identically, although there exists non-vanishing one-form $\tilde{A}$ in the background spacetime.

Quantum-mechanical probability density is given by $|\Psi|^2 = |\boldsymbol{\psi}|^2$. Namely the probability of a quantum mechanical particle is unchanged formally by the existence of $\tilde{A}_\mu$-field. It is well-known for the wave function $\boldsymbol{\psi} = |\boldsymbol{\psi}| \exp(i\theta)$ that the current conservation law $\partial_\nu j^\nu_{(\mathrm{q})} = 0$† is deduced from the equation (1.6):

$$\partial_\nu j^\nu_{(\mathrm{q})} = 0, \quad \text{with} \quad j^0_{(\mathrm{q})} = \rho_\psi \, c, \quad j^k_{(\mathrm{q})} = (\rho_\psi \lambda) \, \partial_k \theta \quad (k = 1, 2, 3) \tag{1.12}$$

where $j^\nu_{(\mathrm{q})} = (j^0_{(\mathrm{q})},\, j^k_{(\mathrm{q})})$ is a 4-current density with $\rho \equiv |\boldsymbol{\psi}|^2$, $\lambda \equiv \hbar/m$ and $\partial_0 = c^{-1}\partial_t$, In the presence of $\tilde{A}_\mu$-field, the 3-current flux $j^k_{(\mathrm{q})}$ is changed to $\rho_\psi \lambda \, \partial_k(\theta + \gamma\Theta)$. Thus, only effect of the extended phase factor is to change the 3-current $j^k_{(\mathrm{q})}$ from θ to $\theta + \gamma\Theta$.

In the gauge theory, *global* gauge transformation is defined by the following transformation: $\tilde{A}_\mu \to A_\mu = \tilde{A}_\mu + \epsilon_\mu$ for 4 arbitrary constants ϵ_μ. It is trivial to see that the system is invariant with this global transformation, because the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given by derivatives of A_μ. Therefore, the present system is said to be invariant globally. This is the *first* step of the gauge principle, examining whether the system under consideration is equipped with desirable conditions. We will return to see what is the desirable, after having seen the details of the local invariance given in section I c).

Essence of the gauge principle lies in requiring *local* gauge invariance. In the present case, this is defined by $\tilde{A}_\mu \to A_\mu = \tilde{A}_\mu + \alpha_\mu(x^\nu)$ for 4 arbitrary differentiable fields $\alpha_\mu(x^\nu)$ depending on spacetime coordinates x^ν. Since α_μ is assumed to take a general form not limited to the form $\partial_\mu \Theta$, the one-form $\mathcal{A} = A_\mu dx^\mu$ does not necessarily take a form of a total derivative $\mathsf{d}\Theta$. Hence, the field strength two-form $\mathcal{F} = \mathsf{d}\mathcal{A}$ does not vanish in general. This means that we have non-vanishing observable fields of $\boldsymbol{E}$ and $\boldsymbol{B}$, according to (1.9) and (1.10). This changes drastically our battle field of study. Not only the Maxwell equations (1.1) must be satisfied, but also the governing Schrödinger equation should be reformed with partial derivatives ∂'s replaced by covariant derivatives ∇'s, as given by (2.33) below. Thus, the so-called *gauge-potential* A_μ is taken into the equation (2.32) to represent a new interaction force. In this way, a new force is introduced by the local gauge invariance.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

c) Gauge Principle: global invariance and local invariance

From the example just mentioned above, it is seen that there is a crucial difference between global invariance and local invariance of physical fields. Each invariance in its own right composes the significance of the principle.

To understand the distinction between the two is vital to capture the physics of the fields. In a global invariance, the same transformation is carried out at all spacetime points of the field where current conservation (such as the form of (1.12)) is satisfied, while in a local invariance different transformations are carried out at different individual spacetime points. In general, a theory that is globally invariant will not be invariant under locally varying transformations. This is understood to mean that a new field is required in order to satisfy the local invariance. To that end, the system under investigation must have a potential capacity receptive to, *i.e.* able to receive a new field. In fact, the field $\tilde{A}_\mu = \partial_\mu \Theta$ in the previous section played a diagnostic field to test whether the system is receptive to a new field $\alpha_\mu(x^\nu)$. By introducing a new general field $\alpha_\mu(x^\nu)$ in such a receptive system that interacts with the original field and which also transforms the system physically acceptable ways under the local transformations, a *local gauge invariance is established*.

d) Desirable factor for the gauge theory

Reflecting the above analysis of the gauge principle, consider what is the desirable factor playing the role of a game-changer from vanishing-field state to the state of non-vanishing fields of $\boldsymbol{E}$ and $\boldsymbol{B}$ equipped with a new force (electromagnetic, in this case). It is reasonable to identify that most important factor is a geometrical one. Namely, the one-form $\mathcal{A} = A_\mu dx^\mu = \eta_{\mu\nu} A^\nu dx^\mu$ of (1.7) is vested to the spacetime (x^μ) which is a most important geometrical structure. In fact, the present gauge principle sets as a premise the existence of one-form $\mathcal{A}$ in the 4-d spacetime equipped with the metric $\eta_{\mu\nu}$. With this reasoning, one understands that the gauge principle is rooted on the fundamental level of Physics and that the gauge principle works, as proposed by Utiyama (1956), not only in quantum electrodynamics, but also in particle physics and theory of gravitation, because one can define one-form $\mathcal{A} = A_\mu dx^\mu$. Almost needless to say, the field of fluid flows in the 4-d spacetime is not excluded, to be presented in the accompanying paper.

In the gauge theory of particle physics, current conservation law is considered to be a *must*. It is interesting philosophically to investigate how such a current conservation law working in the physics of discrete particles compromises with the physics of continuum, such as in the theory of gravitation (dealing with spacetime continuum) or in the theory of fluid flows (dealing with material continuum with continuous distribution of mass density ρ). The paper accompanying the present paper is concerned with the last problem of fluid-flow fields.

e) Historical reviews

Considering the key role played by the gauge invariance in modern theoretical physics, it would be reasonable and useful to review how it is working in the fundamental fields. On the reviews of historical facts of the initial stage of gauge theory, one can refer two important articles of O'Raifeartaigh (1997) and Jackson and Okun (2001), both of which describe how the modern gauge theory developed in its early days. It took almost a century to formulate the non-uniqueness of potentials in the context of theoretical physics, existing despite the uniqueness of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In regard to the gauge condition (1.4), *Lorenz's* contribution is noted again. In fact, Lorenz (1859) introduced the so-called retarded potentials and showed that those

† This is equivalent to $\partial_t|\psi|^2 + \partial_k(\psi\partial_k\psi^* - \psi^*\partial_k \quad) = 0$, derived from (1.6).

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

satisfied the relation: $\nabla \cdot \boldsymbol{A} + c^{-2}\,\partial_t \Phi \;=\; 0$ (Jackson & Okun, 2001), which is now almost universally known as the *Lorentz* condition, but founded originally by Ludvig V. Lorenz (a Danish physicist) who preceded the Dutch physicist Hendrik A. Lorentz. The English word *gauge*, a translation of German *eichen*, was not used in English until 1929 (Weyl, 1929a) for the transformations such as (1.3).

II. Gauge in Variances in Two Fundamental Physical Fields — A Review

Taking two fundamental physical fields, *Electromagnetism* and *Quantum Electrodynamics*, we review the gauge symmetries and see how the gauge symmetry has been captured historically.

a) *Electromagnetic Field: Gauge Invariance and Charge Conservation*

i. *Maxwell equations*

Electromagnetic fields are represented with a 4-vector potential A^μ in the 4d spacetime $x^\mu = (x^0, x^1, x^2, x^3)$ (where $x^0 \equiv ct$ and $\mu = 0, 1, 2, 3$):

$$A^\mu = (\Phi, \boldsymbol{A}), \qquad \boldsymbol{A} = (A_1, A_2, A_3).$$

Covariant version of A^μ is A_μ defined by

$$A_\mu = \eta_{\mu\nu} A^\nu = (-\Phi, \boldsymbol{A}), \qquad \text{where} \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) = \eta^{\mu\nu}, \tag{2.1}$$

$\eta_{\mu\nu}$ being the Minkowski metric of the *Special Relativity*. To represent electro-magnetic fields, we begin with a frame-independent formulation. To this end, according to the mathematical formalism of differential forms, an electromagnetic one-form $\mathcal{A}$ is defined:

$$\mathcal{A} = A_\nu dx^\nu = -\Phi dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \qquad (x^0 = ct).$$

The pair of electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given by

$$\boldsymbol{E} \;\equiv\; -c^{-1}\,\partial_t \boldsymbol{A} - \nabla\Phi \qquad \boldsymbol{B} \;\equiv\; \nabla \times \boldsymbol{A}\,. \tag{2.2}$$

Taking external differential d of $\mathcal{A}$, we obtain the *field strength* two-form $\mathcal{F}$:

$$\mathcal{F} = \mathsf{d}\mathcal{A} = \sum \frac{1}{2} F_{\nu\lambda}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda, \qquad F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu \tag{2.3}$$

Matrix representation of the tensor $F_{\nu\lambda}$ is given by (1.10). Once again, taking exterior differential of $\mathcal{F} = d\mathcal{A}$, we obtain the following identity equation:

$$\mathsf{d}\mathcal{F} = \mathsf{d}^2\mathcal{A} \equiv 0, \qquad \mathsf{d}\Big(F_{\nu\lambda}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda\Big) = (\partial_\mu F_{\nu\lambda})\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda, \tag{2.4}$$

$$\mathsf{d}\mathcal{F} = \sum F_{[\nu\lambda,\mu]}\; \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda = 0. \qquad F_{\nu\lambda,\mu} \equiv \partial_\mu F_{\nu\lambda}. \tag{2.5}$$

See the footnote for $F_{[\nu\lambda,\mu]}$.† This reduces to the equation expressed compactly:

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{2.6}$$

From this, we obtain a pair of *Maxwell equations* (*cf.* (1.1)):‡

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0, \tag{2.7}$$

By the definitions (2.2) of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, the two equations of (2.7) are satisfied identically. In other words, in stead of using the pair of equations (2.7), it is sufficient that the 4-potential $A^\mu = (\Phi, \boldsymbol{A})$ is used with the understanding that the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given by the definitions (2.2).

The second pair of *Maxwell equations* are given by

$$\nabla \cdot \boldsymbol{E} = 4\pi\, \rho_e, \qquad -\frac{1}{c}\,\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = \frac{4\pi}{c}\,\boldsymbol{j}_e, \tag{2.8}$$

(*cf.* Jackson (1999, §11.9)). This pair of equations are derived from the principle of least action. The action integral $S^{(\mathrm{em})}$ is expressed by a linear combination of two terms with a part $S^{(\mathrm{em})}_{\mathrm{emA}}$ representing an electromagnetic field by the potential A_α and another $S^{(\mathrm{em})}_{\mathrm{int}}$ representing interaction between the field and 4-current $j_e^{\,\nu}$:

$$S^{(\mathrm{em})} = S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}$$

$$S^{(\mathrm{em})}_{\mathrm{emA}} = -\frac{1}{16\pi c}\int F_{\alpha\beta}\, F^{\alpha\beta}\mathrm{d}\Omega, \qquad S^{(\mathrm{em})}_{\mathrm{int}} = \frac{1}{c^2}\int j_e^{\,\alpha}\, A_\alpha\, \mathrm{d}\Omega, \tag{2.9}$$

where $\mathrm{d}\Omega = \mathrm{d}^4 x^\nu$. From the variation δA_α of the field A_α where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, the following equation is deduced in the form of tensor equation (Appendix D: (D.4)):

$$\partial_\beta F^{\alpha\beta} = \frac{4\pi}{c}\, j_e^{\,\alpha}, \tag{2.10}$$

where $j_e^{\,\alpha} = (\rho_e c, \boldsymbol{j}_e)$ with $\boldsymbol{j}_e = \rho_e \boldsymbol{v}$, and $F^{\alpha\beta}$ is given by $F^{\alpha\beta} = \eta^{\alpha\mu}\, F_{\mu\nu}\, \eta^{\nu\beta}$. Practically, the matrix $F^{\alpha\beta}$ is obtained from $F_{\nu\lambda}$ of (1.10) with simply replacing $\boldsymbol{E}$ by $-\boldsymbol{E}$.

ii. *Conservation of electric charge and Gauge invariance*

Conservation law of *electric charge* can be derived from (2.10) by taking 4-divergence of both sides:

$$0 = \partial_\alpha \partial_\beta F^{\alpha\beta} = \frac{4\pi}{c}\,\partial_\alpha j_e^{\,\alpha}. \tag{2.11}$$

The left-hand side vanishes identically because the differential operator $\partial_\alpha\partial_\beta$ is symmetric with respect to α and β, while $F^{\alpha\beta}$ is antisymmetric. Total sum with respect to α and β (taking indices $0, 1, 2, 3$) vanishes identically. Thus, we have the charge conservation equation with $j_e^{\,\beta} = (\rho_e c, \boldsymbol{j}_e)$:

$$\partial_\alpha j_e^{\,\alpha} = \partial_t \rho_e + \nabla \cdot \boldsymbol{j}_e = 0. \tag{2.12}$$

This conservation law is closely related to the gauge symmetry of the electromagnetic field. Let us consider the *gauge transformation* concerning the one-form $\mathcal{A}$, defined by

$$\mathcal{G}:\ A_\nu \equiv A_\nu^{(old)} \ \to\ A_\nu^{(new)} \equiv A'_\nu = A_\nu^{(old)} - \partial_\nu \Theta, \tag{2.13}$$

equivalent to (1.3), where Θ is an arbitrary differentiable function. Then, we have

$$\mathcal{A}^{(new)} = A_\nu^{(new)} \mathsf{d}x^\nu = A_\nu^{(old)} \mathsf{d}x^\nu - \partial_\nu \Theta\, \mathsf{d}x^\nu = \mathcal{A}^{(old)} - \mathsf{d}\Theta.$$

From this, we find the invariance $\mathcal{F}^{(new)} = \mathcal{F}^{(old)}$ as follows:

$$\mathcal{F}^{(new)} = \mathsf{d}\mathcal{A}^{(new)} = \mathsf{d}\mathcal{A}^{(old)} + \mathsf{d}^2\Theta = \mathsf{d}\mathcal{A}^{(old)} = \mathcal{F}^{(old)}, \tag{2.14}$$

† $F_{[\nu\lambda,\mu]} \equiv \frac{1}{3!}\left(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} - \partial_\mu F_{\lambda\nu} - \partial_\nu F_{\mu\lambda} - \partial_\lambda F_{\nu\mu}\right)$ with $F_{\lambda\nu} = -F_{\nu\lambda}$ *etc.* .

‡ The first is obtained with $(\alpha,\, \beta,\, \gamma) = (1, 2, 3)$, while the second is derived when one of α, β and γ takes the suffix number 0.

since $\mathsf{d}^2\Theta = 0$ identically. Thus it is found that the two-form $\mathcal{F}$ defined by (2.3) is invariant with respect to the transformation $\mathcal{G}$, called the *gauge transformation* by the historical reasons explained in the Introduction. Therefore, the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant, said as *gauge-invariant*.

The gauge invariance (2.14) and the charge conservation (2.12) are connected closely. In fact, the connection is *inseparable*, which can be shown as follows. In the expression of $S_{\rm int}$ given in (2.9), we replace the factor A_α by $A_\alpha - \partial_\alpha \Theta$. Then the action $S_{\rm int}$ has an additional term,

$$\int j_e^\alpha \frac{\partial \Theta}{\partial x^\alpha}\, \mathrm{d}\Omega. \tag{2.15}$$

Using (2.12) expressing the charge conservation, one can rewrite the integrand in a form of 4-divergence $\partial(\Theta\, j_e^\alpha)/\partial x^\alpha$. Then the above integral is transformed into vanishing boundary integrals by the conditions of the variational principle.

Thus the gauge transformation has no effect on the equation of motion, so long as the equation of charge conservation (2.12) is valid (*cf.* Landau & Lifshitz (1975) §29). Namely, the charge conservation law ensures the gauge invariance. Conversely, the gauge invariance requires the charge conservation equation $\partial j_e^\alpha/\partial x^\alpha = 0$, because the expression (2.15) is transformed to $-\int \Theta\, \partial_\alpha j_e^\alpha\, \mathrm{d}\Omega$, which is required to vanish to any scalar function Θ by the gauge invariance.

iii. *Electromagnetic wave under Lorenz gauge*

In the previous subsection (i), it is remarked below (2.7) that the 4-potential $A^\alpha = (\Phi, \boldsymbol{A})$ can be used instead of the pair of Maxwell equations (2.7). Now the set of four Maxwell equations are reduced to two equations of (2.8) when the 4-potentials A^α are used as dependent variables and the equation (2.2) for the definition of E and B. The two equations of (2.8) are given by the single tensor equation (2.10): $\partial_\beta F^{\beta\alpha} = -(4\pi/c)\, j_e^\alpha$, where

$$\partial_\beta F^{\beta\alpha} = \partial_\beta(\partial^\beta A^\alpha - \partial^\alpha A^\beta) = \partial_\beta \partial^\beta A^\alpha - \partial^\alpha(\partial_\beta A^\beta), \tag{2.16}$$

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = (\partial_0, \nabla); \qquad \partial^\alpha = \eta^{\alpha\beta}\partial_\beta = (-\partial_0, \nabla), \tag{2.17}$$

and $\partial_0 = \partial/\partial(ct)$ and $\nabla = (\partial_1, \partial_2, \partial_3)$. Therefore, the tensor equation (2.10) becomes

$$\partial_\beta \partial^\beta A^\alpha - \partial^\alpha(\partial_\beta A^\beta) = -\frac{4\pi}{c}\, j_e^\alpha, \tag{2.18}$$

where $\partial_\beta\partial^\beta$ is the differential operator of wave equation and $\partial_\beta A^\beta$ 4-divergence of A^β:

$$\partial_\beta \partial^\beta = -\partial_0^2 + \nabla^2 = \nabla^2 - c^{-2}\partial_t^2, \qquad \partial_\beta A^\beta = c^{-1}\partial_t \Phi + \nabla \cdot \boldsymbol{A}. \tag{2.19}$$

In the last section (ii), it is shown that there is freedom in the potential A^α. This freedom enables choosing a set of potentials $A^\alpha = (\Phi, \boldsymbol{A})$ to satisfy

$$\textit{Lorenz condition:} \qquad \partial_\alpha A^\alpha = c^{-1}\partial_t \Phi + \nabla \cdot \boldsymbol{A} = 0. \tag{2.20}$$

Then, the equation (2.18) reduces to the wave equation with the source term $(4\pi/c)\, j_e^\alpha$:

$$\textit{Wave equation:} \qquad (\nabla^2 - c^{-2}\partial_t^2) A^\alpha = -\frac{4\pi}{c}\, j_e^\alpha. \tag{2.21}$$

Substituting $A^\alpha = (\Phi, \boldsymbol{A})$ and $j_e^\alpha = (\rho_e c, \boldsymbol{j}_e)$, this represents uncoupled wave equations, one for Φ and one for $\boldsymbol{A}$:

$$\nabla^2\Phi - c^{-2}\partial_t^2\Phi \;=\; -\,4\pi\,\rho_e, \tag{2.22}$$

$$\nabla^2\boldsymbol{A} - c^{-2}\partial_t^2\boldsymbol{A} = \;-\,\frac{4\pi}{c}\boldsymbol{j}_e, \tag{2.23}$$

The wave equation (2.21) and the Lorenz condition (2.20) form a set of equations equivalent to the Maxwell equations in vacuum. In a later section, we will see, surprisingly, an analogous set of equations for gravitational waves in generalized form. This implies that a sort of gauge symmetry exists as well in the theory of gravitation.

What is now known as a gauge transformation of the electromagnetic potentials was discovered in the formulation process of classical electrodynamics in the 19th century. However, real recognition of its physical significance required two new fields to be developed: the relativity theory for the structure of 4d-spacetime, like a 4-potential $A^\alpha = (\Phi, \boldsymbol{A})$ and a current 4-vector $j^\nu = (\rho c, \boldsymbol{j})$, and the quantum mechanics (say) for the new dimension of a phase factor $\exp[i\chi(x^\nu)]$ (see next section II b). In fact, the notion of gauge symmetry did not appear in the context of classical electrodynamics, but required the invention of quantum mechanics in particular, according to Jackson & Okun (2001).

As mentioned above, the gauge invariance and charge conservation are connected closely. In fact, the connection is *inseparable*. O'Raifeartaigh L (1997) cites the original paper of Weyl (1918), in which Hermann Weyl commented in the postscript (1955) as

> $\cdots$, *gauge-invariance corresponds to the conservation of electric charge in the same way that coordinate-invariance corresponds to the conservation of energy and momentum. Later the quantum theory introduced the Schrödinger-Dirac potential* (wave function) *of the electron-positron field; it carried with it an experimentally-based principle of gauge-invariance which guaranteed the conservation of charge,* $\cdots\cdots\cdots$. (See O'Raifeartaigh (1997, p.36))

In fact, Noether's theorem shows $\partial_\nu j^\nu = 0$ for 4-current j^ν of relativistic quantum systems such as those governed by Klein-Gordon equation or Dirac equation in Minkowski space (Aitchison & Hey (2013, Chap.3); Frankel (l997, §20.2)).

b) Quantum Electro-Dynamics (QED): Gauge Principle and Covariance

i. *Gauge transformation in QED*

In the context of quantum theory, the attempt of Weyl (1918) is worth mentioning first. He proposed to unify electromagnetism and gravity geometrically by attaching a scale factor of the form $l \propto \exp[\int \phi_k(x^\nu)\,dx^k]$ with its variation given by $\delta l = l\phi_k\,\delta x^k$. This received unfavorable response to be in disagreement with observation.

However, after the advent of the quantum theory, it was revived by London (1927) that Weyl's proposal could be used in quantum theory by changing the scale factor $\exp[\chi]$ (χ : real) to a phase factor $\exp[i\chi]$ and attaching it to the wave function of quantum mechanics. Suppose that ψ_0 describes the zero-field wave function. Then by the transformation from ψ_0 to $\psi = \psi_0\exp[i\gamma\int A_\mu(x^\nu)dx^\mu]$, the wave function describes the state interacting with the electromagnetic potential A_μ (where $\gamma \equiv e/\hbar$).

Earlier than this work, Fock (1926) proposed extension of the freedom of potential A_μ in the classical electrodynamics to the quantum mechanics of a particle with a charge e interacting with the field A_μ. With the transformation of the potential,

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu\chi, \tag{2.24}$$

the wave functions ψ is transformed correspondingly by a phase transformation:

$$\psi \to \psi' = \psi \exp[i\gamma\chi]. \tag{2.25}$$

What Fock discovered for the quantum mechanics was that, for the form of the quantum equation to remain unchanged by these transformations, the wave function is required to undergo the transformation,

$$\psi_0 \to \psi = \psi_0(x^\nu) \exp[i\gamma \int A_\mu(x^\nu)\, dx^\mu], \tag{2.26}$$

whereby ψ is multiplied by a local (space-time dependent) phase factor. Later, the concept was declared a general principle by Hermann Weyl (1928, 1929a,1929b). The invariance of a theory under combined transformations such as (2.24) and (2.25) is known as a gauge symmetry or a gauge invariance and was a touchstone in developing modern gauge theory. (Jackson & Okun)

ii. *Schrödinger's equation and gauge principle in an electromagnetic field*

A wave function ψ of quantum mechanics evolves in time according to the equation $i\hbar\,\partial_t\psi = H\psi$, where $\hbar$ is the Planck constant and H the Hamiltonian operator which is defined, in the absence of the electromagnetic field, by

$$H(\boldsymbol{x}, p) = p^2/2m + eV(x), \tag{2.27}$$

where p is the canonical momentum, V the potential energy and e the charge of the particle. In Schrödinger's equation, the canonical momentum p_k is represented by the differential operator on the wave function ψ expressed as

$$p_k\,\psi = -i\hbar(\partial/\partial x^k)\,\psi, \tag{2.28}$$

while the potential V is a multiplicative operator on ψ. From (2.27), Schrödinger's equation is given by

$$i\hbar\,\partial_t\,\psi = -(\hbar^2/2m)\sum_k(\partial/\partial x^k)^2\,\psi + eV\,\psi. \tag{2.29}$$

When there exists an external electromagnetic field and the particle has a charge e, the Hamiltonian H of (2.27) should be replaced by

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m}(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A})^2 + eV + e\Phi \tag{2.30}$$

where the previous momentum $\boldsymbol{p}$ is replaced by an expression using the new canonical momentum $\boldsymbol{P} = \boldsymbol{p} + (e/c)\boldsymbol{A}$. Replacing P_k with $-i\hbar\partial/\partial x^k$, Schrödinger's equation becomes

$$i\hbar\,\partial_t\,\psi = \frac{1}{2m}\sum_k\Big(-i\hbar\frac{\partial}{\partial x^k} - \frac{e}{c}A_k\Big)^2\psi + eV\,\psi + e\Phi\,\psi. \tag{2.31}$$

This can be rewritten as

$$i\hbar c\,\nabla_0\,\psi = -\frac{\hbar^2}{2m}\sum_k \nabla_k\nabla_k\,\psi + eV\,\psi, \tag{2.32}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

where $\nabla_\alpha = (\nabla_0, \nabla_k)$ are covariant derivatives (with $x^0 = ct$) defined by

$$\nabla_0 = \frac{\partial}{\partial x^0} - \left(\frac{ie}{\hbar c}\right) A_0, \quad \nabla_k = \frac{\partial}{\partial x^k} - \left(\frac{ie}{\hbar c}\right) A_k, \qquad (A_0 = -\Phi). \qquad (2.33)$$

The equation (2.32), equivalent to (2.31), is written compactly by using the covariant derivatives ∇_0 and ∇_k to represent the effect of electromagnetic field A_μ.

Weyl's principle of gauge invariance: If ψ satisfied the Schrödinger's equation (2.32) involving the potential A_μ, then the transformed wave function,

$$\psi' = \exp\left[i\gamma\, \chi(x^\mu) \right] \cdot \psi(x) \qquad (2.34)$$

satisfies Schrödinger's equation when $\mathcal{A} = A_\nu \mathrm{d}x^\nu$ is replaced by $\mathcal{A} + \mathrm{d}\chi$. This is verified if the wave function ψ is represented as

$$\psi(x) = \left(\exp[\, i\gamma \int A_\mu(x)\, \mathrm{d}x^\mu] \right) \cdot \psi_0(x) \qquad (2.35)$$

In fact, with a transformation $\mathcal{A} \to \mathcal{A} + \mathrm{d}\chi$. Then the new function $\psi^{(new)}$ is given by

$$\psi^{(new)}(x) = \exp[\, i\gamma \int (A_\mu(x) + \partial_\mu \chi)\, \mathrm{d}x^\mu] \cdot \psi_0(x) = \exp\left[i\gamma\, \chi(x^\mu) \right] \cdot \psi(x).$$

Thus the form (2.34) is obtained. In the gauge symmetry of QED, the key elements are summarized by the following set of *covariant* transformations (see the item (d) below):

$$A_\mu \to A_\mu + \partial_\mu \chi, \qquad \psi \to \exp\,[\, i\gamma\, \chi\,] \cdot \psi. \qquad (2.36)$$

Here, the transformation of A_μ is equivalent to the pair of transformations $\boldsymbol{A} \to \boldsymbol{A} + \nabla\chi$ and $\Phi \to \Phi - \partial_t \chi$, which keep the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ invariant.‡

Thus, one can uphold the gauge principle to the following general guiding principle.

iii. *Generalized Gauge Principle*

Global gauge invariance:

This is defined by *invariance* under a constant change in the phase of wave function ψ. Writing it explicitly, instead of the added phase factor $\exp[i\gamma\chi(x^\mu)]$ of (2.34) depending on x^μ, the global transformation is given by

$$\psi(x^\mu) \to \psi'(x^\mu) = \exp[\, i\,\alpha\,]\, \psi(x^\mu), \qquad \alpha = const\,, \qquad (2.37)$$

If this transformation does not cause any observable change, it is a *global invariance*.

Local gauge invariance:

This requires invariance with respect to the following local phase transformation:

$$\psi(x^\mu) \to \psi'(x^\mu) = \exp[\, i\,\alpha(x^\mu)\,]\, \psi(x^\mu), \quad \alpha\, :\ \text{dependent on } x^\mu, \qquad (2.38)$$

If our system is not invariant under the local transformation, it is understood to mean that a new field is required in order to satisfy the local invariance. By introducing such a new field interacting with the original field and transforming the system under

‡ The covariant vector-potential (downstairs) is $A_\mu = (-\Phi, A_k)$, while the upstairs vector-potential A^ν is $(A^0, A^k) = \eta^{\nu\mu} A_\mu$ where $A^0 = \Phi$, $(A^k) = \boldsymbol{A}$ and $A_k = A^k$. One-form $\mathcal{A}$ is defined by $\mathcal{A} = \eta_{\mu\nu} A^\nu dx^\mu = A_\mu dx^\mu = -\Phi dt + A_k \mathrm{d}x^k$, where $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. Note that $\nabla_0 = (\partial/\partial x^0) - (ie/\hbar c)\, A_0$.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

investigation according to the local transformation, it is expected that local invariance is established. This is a general scenario to find a new physical field.

In fact, the previous item (*ii*) of Schrödinger's equation is a typical example. For the new field to be received to satisfy the local invariance, the system must be receptive, *i.e.* must have a potential capacity receptive to the new field. Firstly, one can say an elementary aspect of the complex function. Every complex function has a phase factor which absorbs the electromagnetic 4-potenial A_μ within the integral symbol as in (2.35).

Moreover, in the QED case, Schrödinger equation (2.29) represented with partial derivatives ∂'s was reformed and replaced by (2.32) represented with covariant derivatives ∇'s which are defined with (2.33) by taking account of the new field A_μ. Simultaneously the wave function ψ was transformed by (2.34). Thus, local invariance has been established.

In mathematical point of view, the global transformation $\psi \to e^{i\alpha}\psi$ appears to be a trivial transformation. But it is an important step to confirm a *capacity* which is receptive to the (harmless) phase modification. In the context of physics, however, it is understood to express the fact that once phase choice of α has been made at one spacetime point, the same change of phase must be adopted at all other spacetime points. This is unnatural from the view-point of causality.

It would be better if one can find other physically reasonable transformation. In §1.2, for electromagnetic 4-potential A_μ, we saw a particular A_μ-field defined by $\tilde{A}_\mu \equiv \partial_\mu \Theta$ with Θ an arbitrary scalar function. When the A_μ-field is introduced in the field. the wave function is transformed as $\psi \to \exp[i\gamma\Theta(x^\nu)] \cdot \psi$ instead of the uniform phase shift $e^{i\alpha}$. Nevertheless, the observable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish identically, although there exists non-vanishing one-form $\tilde{A}$ in the background spacetime. This signifies that the system is receptive. It has a potential capacity receptive to the new field.

In the flow fields of a perfect fluid to be studied in the last section 5.2, there exists an analogous structure in the fluid-flow field. Hence, the global invariance of the flow field is strengthened by this property.

iv. *Covariance with respect to the gauge transformation*

Next, consider the transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$ from a different angle of mathematical viewpoint. Let us represent this operation as $g\circ$ with the symbol $\circ$ and an element g of a certain continuous differentiable group $\mathcal{G}$ (a Lie g), such that we write it as $A'_\mu = g \circ A_\mu$. Then the new wave function $\psi' \equiv \psi^{(new)}$ is written as $\psi' = g \circ \psi$, where ψ is given by (2.35). The operation g and ψ' are given by (2.36). Namely,

$$\psi' = g \circ \psi(x) = \exp[\, i\gamma\chi] \cdot \psi \tag{2.39}$$

Next, using the covariant derivative ∇_μ defined by (2.33), the covariant derivative of is given by

$$\nabla_\mu \psi = (\partial_\mu \psi_0) \cdot \exp\left[i\gamma \int A_\mu(x) \mathrm{d}x^\mu \right].$$

Its g-transformation is

$$\begin{aligned} g \circ \nabla_\mu \psi = & \; (\partial_\mu \psi_0) \cdot \exp\left[i\gamma \int g \circ A_\mu(x) \mathrm{d}x^\mu \right] = (\partial_\mu \psi_0) \cdot \exp\left[i\gamma \int (A_\mu + \partial_\mu \chi) \mathrm{d}x^\mu \right] \\ = & \; \exp[\, i\gamma\chi] \cdot \nabla_\mu \psi \end{aligned} \tag{2.40}$$

Comparing (2.39) and (2.40), it is seen that the "$g\,\circ$" operations on ψ and $\nabla_\mu \psi$ take the same form, that is, simple multiplication of the same phase factor $\exp[i\gamma\chi]$. In other words, the two functions ψ and $\nabla_\mu \psi$ are transformed covariantly by the operation g, that is by the gauge transformation $\mathcal{A} \to \mathcal{A} + \mathrm{d}\chi$. The *covariance* property of transformation shared by both of ψ and $\nabla_\mu \psi$ can be generalized to other transformations. We will see it later too.

v. *Transformation Group U(1)*

The invariance by the transformation (2.37) or (2.38) is said the gauge symmetry of the type of $U(1)$ group. Multiplication by a phase factor like $\exp[i\alpha]$ corresponds to a kind of rotation of the state vector $\psi = |\psi| \exp[i\theta]$ in the polar representation $(|\psi|, \theta)$ of ψ in the complex plane. The group $U(1)$ is an abelian group corresponding to the circle group, consisting of all complex numbers with absolute value 1 under multiplication.

Imagine doing two successive such transformations: $\psi \to \psi' \to \psi''$, where $\psi'' = \exp[i\beta]\,\psi'$, and the original one was $\psi' = \exp[i\alpha]\,\psi = U_\alpha\,\psi$ with $U_\alpha = \exp[i\alpha]$. So we have $\psi'' = \exp[i(\alpha+\beta)]\,\psi = \exp[i\delta]\,\psi$, where $\delta = \alpha + \beta$. This is a transformation of the same form as the original. The set of all such transformations forms a *group*, in this case called $U(1)$-group, meaning the group of all unitary ($|U_\alpha| = 1$) one-dimensional matrices (ψ, a single complex number). The transformations U_α and a subsequent transformation U_β are commutative. Namely,

$$U_\beta\, U_\alpha = U_{\alpha+\beta} = U_\alpha\, U_\beta.$$

Such a group $U(1)$ is called an *Abelian* group in mathematics where different transformations commute.

The Electro-Weak theory and Quantum Chromodynamics (QCD) are described by non-Abelian gauge symmetries of $SU(2) \times SU(1)$ group and $SU(3)$ group, respectively (see *e.g.* Aitchison & Hey (2013)). All of these theories form what is called today the Standard Model, which is the basis of the theoretical physics except for gravity.

As seen above, the gauge symmetry plays a fundamental role something like a *touchstone* of the theory, testing whether the theory is trustworthy or not. Gauge symmetry exists in other fields too. Geometrical theory of gravitation and Fluid Mechanics are considered below.

III. Geometric Theory of Gravitation

In this section we consider the geometric theory of gravity and the gauge symmetry existing within the theory. Amazingly, there are analogous structures between the quantum electrodynamics (QED) and the theory of gravity. It was known from the initial times of the gravity theory. Most obvious similarity resides in the covariant derivatives of both theories, the former QED including the connection term of the EM potential A_μ and the latter the connection term (Christoffel symbol) associated with the gravity field.

Concerning the theory of gravity at the classic times of Galileo and Newton in the 17th century, a flat Euclidean absolute 3d-space $x^k = (x^1, x^2, x^3)$ and an absolute time t are two distinct physical objects, which are unlinked. A physical object of a point-mass in free motion in an inertial frame in the absence of gravity moves uniformly along a straight line. In the presence of gravitational potential Φ, free motion of a particle takes curved trajectories in flat space. In Einstein's theory of gravitation, world lines of free particles (described by the geodesic equation) are a probe of structure of spacetime.

In Einstein's theory, gravitational field is represented as an object of four-dimensional continuum with curvature (Misner, Thorne & Wheeler (2017, §17.7)). In the equation of gravitation (Einstein, 1915), curvature-tensors are equated to tensors of source-term arising from material motion (mostly motion of fluids or gases), satisfying the conservation laws of energy and momentum of the source material. In this geometro-dynamics, geometry tells matter how to move, such as a free particle taking a curved trajectory, while the matter tells geometry how to curve. Suppose that the source material is a fluid. Being the source of gravity, the fluid tells geometry how to curve

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

in the Einstein's theory. Time t and 3d-space (x^1, x^2, x^3) are two aspects of a single continuum entity, which is an inseparable object of curved *spacetime* $x^\mu = (x^0, x^1, x^2, x^3)$ with $x^0 = ct$. The 4d-spacetime is not flat because of the presence of matter's energy and momentum of the fluid motion.

Squared interval between an event at x^μ and a nearby one at $x^\mu + \mathrm{d}x^\mu$ is given by

$$\mathrm{d}s^2 = g_{\mu\nu}(P)\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu, \quad P = x^\mu = (x^0, x^1, x^2, x^3), \quad \mu, \nu = 0, 1, 2, 3\,. \tag{3.1}$$

where $g_{\mu\nu}$ is the metric tensor. The curved spacetime geometry of physical world is founded by the metric tensor $g_{\mu\nu}$. A special flat space is described by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. This is the space of *Special Relativity* which is a theory invariant under the Lorentz transformation. An important invariant object under the transformation is the proper time τ (the time of comoving frame) defined by

$$\mathrm{d}\tau^2 = -\eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu = (c\,\mathrm{d}t)^2 - |\mathrm{d}\boldsymbol{x}|^2 = c^2\,(1-\beta^2)\,(\mathrm{d}t)^2\,, \quad \beta \equiv |\boldsymbol{v}|/c\,. \tag{3.2}$$

where $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\mathrm{d}t$ with $\boldsymbol{v}$ being a particle velocity. The τ is the time of comoving frame with the particle (where $|\boldsymbol{v}|$ is zero, hence $\mathrm{d}t = c\,\mathrm{d}t$), while the time t is the time observed from other frame, which are related by $\mathrm{d}\tau = c\sqrt{1-\beta^2}\mathrm{d}t$. Appendix C supplemts certain aspects of mathematics of this section III for the Geometric Theory of Gravitation.

a) *An illustrative example: Free motion of a sigle particle and Equivalence Principle*

A free particle of mass m moves along a world line. Its trajectory is determined as an extremal of the action $S^{(\mathrm{m})} = -mc\int \mathrm{d}s$. The action principle is given by

$$\delta S^{(\mathrm{m})} = -mc\,\delta \int_a^b \mathrm{d}s = 0. \tag{3.3}$$

In the flat space of *Special Relativity* (Appendix B), the free motion takes a straight path, while in gravitational field it is curved. Let us consider a free motion taking a curved trajectory according to Newtonian mechanics.

Motion of a free particle in the Earth's gravity potential $\Phi_E(x^k)$ is described by

$$\frac{\mathrm{d}}{\mathrm{d}t} v^k + \frac{\partial \Phi_E}{\partial x^k} = 0, \qquad v^k \equiv \frac{\mathrm{d}x_p^k}{\mathrm{d}t}, \qquad k = 1, 2, 3\,, \tag{3.4}$$

yielding a curved trajectory for the particle path $x_p^k(t)$. In the modern view to take the space and time linked to form a 4d-continuum, the curved trajectory of a free particle is described as a *geodesic* curve in the linked space-time.

Let us take an illustrative example according to Utiyama (1987, §2.3), and consider a free-falling elevator in the Earth's gravitational field $\Phi_E(x^\nu)$. The free-falling elevator provides a particular *inertial* system of spacetime, in which free motion of a particle is described by

$$\mathrm{d}^2 X^\mu / \mathrm{d}\tau^2 = 0, \tag{3.5}$$

where X^μ is the particle coordinates in the frame F_{el} fixed to the free-falling elevator. The gravity effect does not appear apparently because the acceleration owing to the gravity acting on both of the elevator and the particle are the same and cancel out in the free-falling frame F_{el}. Thus, the particle takes a straight path $X^\mu = a^\mu \tau + b^\mu$ with respect to F_{el} with a^μ and b^μ being constants.

Let us observe the same motion from another general frame, and as an example take the frame F_E fixed to the Earth surface, where the coordinates are given by x^μ. The squared interval $\mathrm{d}s^2$ in the frame F_E is given as (3.1). In the particular frame F_{el}, the metric is given by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Suppose that the transformation between the two frames is connected according to $X^\mu = X^\mu(x^\nu)$. Under this transformation from X^μ to x^ν, the equation of free motion $\mathrm{d}^2X^\mu/\mathrm{d}\tau^2 = 0$ in the frame F_{el} is transformed to that of the frame F_E as follows,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\frac{\partial X^\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\Big] = A^\mu_\nu\Big[\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \Gamma^\nu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\Big] = 0$$

Using the inverse A^{-1} of $A^\mu_\nu = X^\mu_\nu$ and multiplying by $(A^{-1})^\lambda_\mu \equiv \partial x^\lambda/\partial X^\mu$, this becomes

$$\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0, \quad \text{where} \quad \Gamma^\lambda_{\alpha\beta} = \frac{\partial x^\lambda}{\partial X^\sigma}\frac{\partial X^\sigma}{\partial x^\alpha\,\partial x^\beta} = \Gamma^\lambda_{\beta\alpha}. \qquad (3.6)$$

This states that the particle trajectory is curved in general when $\Gamma^\lambda_{\alpha\beta} \neq 0$.

The 4-velocity $u^\nu \equiv \mathrm{d}x^\nu/\mathrm{d}\tau$ of the particle is defined by

$$u^\nu = \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \Big(\frac{1}{\sqrt{1-\beta^2}}, \frac{\boldsymbol{v}}{c\sqrt{1-\beta^2}}\Big), \quad x^0 \equiv ct, \quad \boldsymbol{v} = (v^k) = (\mathrm{d}x^k/\mathrm{d}t). \qquad (3.7)$$

In the non-relativistic limit as $\beta \ll 1$ for the particle velocity $|\boldsymbol{v}|$ is much less than the light velocity c, this leads to $u^\nu = \mathrm{d}x^\nu/\mathrm{d}\tau \to (1, \boldsymbol{v}/c)$ in the limit. In this case, the equation (3.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}v^\lambda + c^2\,\Gamma^\lambda_{\alpha\beta}\,v^\alpha\,v^\beta = 0, \quad \text{in particular} \quad \frac{\mathrm{d}}{\mathrm{d}t}v^k + c^2\,\Gamma^k_{00}\cdot 1\cdot 1 = 0, \qquad (3.8)$$

where the second equation is given for $\lambda = k = 1, 2, 3$, $(\alpha, \beta) = (0, 0)$, and the factors $\Gamma^\lambda_{\alpha\beta}$ other than Γ^k_{00} are set to zero. Compare this with (3.4). By assuming the following relation of equality,

$$c^2\,\Gamma^k_{00} = \partial\Phi_E/\partial x^k\,, \qquad (3.9)$$

the second equation of (3.8) becomes equivalent to the equation (3.4). This implies an interesting relation between the gravitational potential Φ_E and the symbol $\Gamma^\lambda_{\alpha\beta}$ (called the Christoffel symbol). The equation (3.11) of the next part b) includes the same symbol Γ and expresses the geodesic equation of a free particle in curved spacetime. By replacing the proper time τ with an equivalent parameter λ, the equation (3.6) reduces to (3.11). We will come back to this point at the item (ii) given below.

In fact, the above simplified example illustrates the conceptual aspects of the geometrical theory of gravitation in three respects. (i) Any curved spacetime has a flat space (the freely-falling elevator in the above case) at any point (locally tangent to it). This is assured by a mathematical theorem, *i.e.* the *local flatness theorem* (Schutz, 1985, §6.2). One can always construct such a local *inertial* frame at any event.

(ii) Gravitational potential Φ_g is related to the metric tensor $g_{\mu\nu}$. In fact, Einstein had a view that there is a similarity between the gravitational field and Riemannian geometry. This is based on the particular feature of the gravity which is distinguished from other forces such as the electromagnetic force (say) and characterized by the fact that all bodies are given same acceleration. The potential Φ_g is related to the tensor $g_{\mu\nu}$, and covariant derivatives depending on $g_{\mu\nu}$ are defined in the curved spacetime.

In the above example of a free particle moving in a weak gravitational field of potential Φ_g, the squared interval $\mathrm{d}s^2$ defined by (3.1) is given by

$$\mathrm{d}s^2 = -(1 + 2\Phi_g/c^2)(c\,\mathrm{d}t)^2 + (1 + 2\Phi_g/c^2)^{-1}\,(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \qquad (3.10)$$

as a leading order representation (Misner *et al.* , 2017, §16.2), where only diagonal elements $g_{\mu\nu}|_{\mu=\nu}$ are non-vanishing. Noting $x^0 = c\,\mathrm{d}t$, the metric tensor g_{00} is given by $-1-2\Phi_g/c^2$. In the theory of weak gravitational field ($\Phi_g/c^2 \ll 1$), the metric tensor $g_{\mu\nu}$ is set as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ by using the Minkowski metric $\eta_{\mu\nu}$ on the assumption $|h_{\mu\nu}| \ll 1$.

In the Earth's gravitational field, the potential Φ_g is replaced by $\Phi_E = -G_0 M/r$ and $h_{00} = -2\Phi_E/c^2$, where M is the Earth's mass and r the radial distance from its center.

Returning the equation (3.9): $\Gamma^k_{00} = c^{-2}\,\partial\Phi_E/\partial x^k$ again, the definition of the Christoffel symbol Γ is given by (3.12) of the next section, leading to $\Gamma^k_{00} = g^{k\mu}\Gamma_{\mu 00} \approx \eta^{k\mu}\Gamma_{\mu 00} = \Gamma^k{}_{00} = -\frac{1}{2}\,\partial_k h_{00} = c^{-2}\,\partial_k \Phi_E$. Thus, the the equation (3.9) was confirmed by the squared interval $\mathrm{d}s^2$ of (3.10).

(*iii*) *Cornerstone* of the Einstein's theory is the *Principle of equivalence* between gravity and acceleration. Consider a uniformly accelerating *rocket* moving in empty space free of gravity (Schutz, 1985, §5.1). Viewed from an observer inside, it appears that there is a gravitational field within the rocket. All objects released from the observer are subjected to uniformly accelerating motion, just as in gravity field. A frame falling freely within the ship is an inertial frame. It can be seen from this that frames accelerating uniformly in empty space are equivalent to uniform gravitational fields. This is a conceptual aspect of the equivalence principle.

Its technical aspect is stated as follows. Transition from the equation (3.5) in flat space-time to the equation (3.6) in a curved spacetime is enabled by the Equivalence Principle. The equation (3.5) can be written as $\mathrm{d}u^\mu/\mathrm{d}\tau = u^\mu{}_{,\tau} = 0$ where $u^\mu \equiv \mathrm{d}X^\mu/\mathrm{d}\tau$, while the equation (3.6) can be written as $\hat{\nabla}_\tau u^\mu \equiv \mathrm{d}u^\mu/\mathrm{d}\tau + \Gamma^\mu_{\alpha\beta}\,u^\alpha\,u^\beta \equiv u^\mu{}_{;\tau} = 0$. Hence, for the transition from *flat* spacetime to *curved* one, the *comma* of $u^\mu{}_{,\tau}$ is replaced by a *semicolon* like $u^\mu{}_{;\tau}$ (§3.2(c). This is the technical aspect of the Equivlence Principle.

The metric $g_{\mu\nu}$ describing the geometry of space-time is a symmetric tensor having ten independent components $g_{\mu\nu}(P)$ in 4-dimesional spacetime, functions of a world point P. Einstein's geometrodynamics is governed by ten tensor equations of the form: $G_{\mu\nu} = 8\pi k\,T_{\mu\nu}$. Among the ten equations, only six are effective. Its detailed account is given in §3.2(e).

The gravitational field considered in this paper is assumed to be weak so that the formulation can be compared with the electromagnetic field presented in the previous section and the fluid-flow field to be considered next in this paper.

b) *Review of Einstein's Theory*

Einstein's theory of gravitation (Einstein 1915) is founded on the Riemannian Geometry. Appendix A describes some of its basics.

i. *Geodesics and Covariant derivative*

In a gravitational field, its 4d-spacetime K_g is curved, and the line element $\mathrm{d}s$ is represented in terms of the metric tensor $g_{\mu\nu}(x^\alpha)$ of (3.1). A free particle in such a space moves along a geodesic line $x^\alpha(\lambda)$, governed by the following *geodesic* equation:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\,\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\,\frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0. \tag{3.11}$$

where λ is an affine parameter defined as $\lambda = a\tau + b$ with τ the particle's proper time and a, b constants. The factors Γ's are the Christoffel symbol, defined by

$$\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma}, \qquad \Gamma_{\mu\beta\gamma} = \tfrac{1}{2}\left(\frac{\partial g_{\mu\beta}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu}\right). \tag{3.12}$$

In such a curved space K_g, a *covariant derivative* of a vector field $v^\alpha(x^\mu)$ along a curve $P(\lambda)$ with its tangent $u^\beta = \mathrm{d}x^\beta/\mathrm{d}\lambda$ is defined by

$$(\hat{\nabla}_{\boldsymbol{u}}\boldsymbol{v})^\alpha \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda}v^\alpha + \Gamma^\alpha{}_{\beta\gamma}\,v^\beta u^\gamma \equiv \hat{\nabla}_\lambda. \tag{3.13}$$

where $\hat{\nabla}$ denotes the nabla-operator in the 4-d spacetime. Using this definition, the geodesic equation (3.11) can be written simply as

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\widehat{\nabla}_{\boldsymbol{u}}\boldsymbol{u}=0, \quad \text{or} \quad \widehat{\nabla}_{\lambda}\boldsymbol{u}=0, \quad \text{where} \quad u^{\alpha}\equiv \mathrm{d}x^{\alpha}(P)/\mathrm{d}\lambda. \tag{3.14}$$

According to the differential geometry (Misner *et al.* 2017, Chap.8), this states that the geodesic is a curve $P(\lambda)$ which parallel-transports its tangent $u^{\alpha}=\mathrm{d}x^{\alpha}(P)/\mathrm{d}\lambda$. In the flat space of special relativity where $g_{\mu\nu}$ is given by the metric $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$, the geodesic takes a straight path $\mathrm{d}^2x^{\alpha}/\mathrm{d}\lambda^2=0$, since $\Gamma_{\mu\beta\gamma}=0$ by (3.12).

ii. *Geodesic deviation and Riemann curvature tensors* $R^{\alpha}_{\beta\gamma\delta}$

Equation of the geodesic deviation, that is now going to be presented, has a special term which represents the gravitation with *curvature tensors* mathematically. Consider a family of geodesics parameterized by λ, so that world points are expressed as $x^{\alpha}(\lambda,n)$, with each geodesic curve discriminated by a second parameter n.

Let us introduce the separation vector η^{α} defined by $\eta^{\alpha}=\partial x^{\alpha}/\partial n$, measuring the separation (deviation) $\Delta x^{\alpha}=\eta^{\alpha}\Delta n$ between the geodesic n and the nearby geodesic $n+\Delta n$ at the same value of λ. In curved spaces, parallel lines when extended do not necessarily remain parallel, which is formulated in terms of the Riemannian tensors.

To that end, we will make mathematical expressions more general than those of the previous section and define a general derivative form D for a general vector field $\boldsymbol{v}=v^{\alpha}\boldsymbol{e}_{\alpha}$ where $\boldsymbol{v}$ is expanded in terms of unit basis vectors $\boldsymbol{e}_{\alpha}$. Then the exterior derivative of the vector $\boldsymbol{v}$ is given one-form expression as

$$\mathrm{D}\,\boldsymbol{v}=(\mathrm{D}v^{\alpha})\,\boldsymbol{e}_{\alpha}+v^{\alpha}\,(\mathrm{D}\boldsymbol{e}_{\alpha}), \tag{3.15}$$

where $\mathrm{D}v^{\alpha}=(\partial_{\beta}v^{\alpha})\,\mathrm{d}x^{\beta}$ is a one-form, and the term $\mathrm{D}\boldsymbol{e}_{\alpha}$ is a vector-valued one-form which is expanded by using the connection coefficient (Christoffel symbol) in the form,

$$\mathrm{D}\,\boldsymbol{e}_{\alpha}=\boldsymbol{e}_{\nu}\,\Gamma^{\nu}_{\ \alpha\mu}\,\mathrm{d}x^{\mu}.$$

Thus, we have the expansion of $\mathrm{D}\,\boldsymbol{v}$ represented as

$$\mathrm{D}\,\boldsymbol{v}=\boldsymbol{e}_{\nu}\left(\frac{\partial v^{\nu}}{\partial x^{\beta}}+\Gamma^{\nu}_{\ \alpha\beta}v^{\alpha}\right)\mathrm{d}x^{\beta}\,,\ =\boldsymbol{e}_{\nu}\left(\frac{\mathrm{d}v^{\nu}}{\mathrm{d}\lambda}+\Gamma^{\nu}_{\ \alpha\beta}\,v^{\alpha}u^{\beta}\right)\mathrm{d}\lambda. \tag{3.16}$$

With these notations, we define

$$\mathrm{D}\eta^{\alpha}\equiv\left(\frac{\partial\eta^{\alpha}}{\partial\lambda}+\Gamma^{\alpha}_{\ \beta\gamma}\eta^{\beta}u^{\gamma}\right)\mathrm{d}\lambda, \qquad \frac{\mathrm{D}}{\mathrm{d}\lambda}\,\eta^{\alpha}\equiv\frac{\partial\eta^{\alpha}}{\partial\lambda}+\Gamma^{\alpha}_{\ \beta\gamma}\,\eta^{\beta}u^{\gamma}.$$

It is seen that the operator D is one-form expression of the covariant derivative ∇. Then, the separation vector η^{α} is governed by the following equation of *geodesic deviation*:

$$\frac{\mathrm{D}}{\mathrm{d}\lambda}\frac{\mathrm{D}}{\mathrm{d}\lambda}\,\eta^{\alpha}=R^{\alpha}_{\ \beta\gamma\delta}\,u^{\beta}\,u^{\gamma}\,\eta^{\delta}, \tag{3.17}$$

where $\eta^{\alpha}=\partial x^{\alpha}(\lambda,n)/\partial n$ is the separation vector and $u^{\beta}=\partial x^{\beta}/\partial\lambda$ the tangent vector.

The covariant derivative of $\boldsymbol{v}$ with respect to the coordinate x^{μ} is given by

$$(\widehat{\nabla}_{\mu}\boldsymbol{v})^{\nu}\left(=\frac{\mathrm{D}v^{\nu}}{\partial x^{\mu}}\right)=\partial_{\mu}v^{\nu}+\Gamma^{\nu}_{\ \alpha\mu}v^{\alpha}\equiv\widehat{\nabla}_{\mu}v^{\nu}, \qquad v^{\nu}_{\ ;\mu}=v^{\nu}_{\ ,\mu}+\Gamma^{\nu}_{\ \alpha\mu}v^{\alpha}. \tag{3.18}$$

(See next (c) for the notations of the second equation). The equation (3.17) serves as a definition of the Riemann curvature tensors $R^{\alpha}_{\ \beta\gamma\delta}$, which are defined by

$$R^{\alpha}_{\ \beta\gamma\delta}=\frac{\partial\Gamma^{\alpha}_{\ \beta\delta}}{\partial x^{\gamma}}-\frac{\partial\Gamma^{\alpha}_{\ \beta\gamma}}{\partial x^{\delta}}+\Gamma^{\alpha}_{\ \nu\gamma}\Gamma^{\nu}_{\ \beta\delta}-\Gamma^{\alpha}_{\ \nu\delta}\Gamma^{\nu}_{\ \beta\gamma}. \tag{3.19}$$

This can be represented in terms of the metric tensors $g_{\alpha\beta}$ and their derivatives (see (C.7)). According to (3.17), geodesics in flat space where $R^{\alpha}_{\ \beta\gamma\delta}=0$ maintain their separation, while those in curved spaces where $R^{\alpha}_{\ \beta\gamma\delta}\neq 0$ do not. This is said in the beginning that *geometry tells matter how to move.*

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

iii. *Equivalence Principle: Transition from flat spacetime to curved one*

How the matter influences the geometry for curving is the subject of subsequent sections. In the present theory of geometro-dynamics, the matter is a perfect fluid. Relativistic expressions of the stress-energy tensor of a perfect fluid are to be given in the section IV, d) by (4.25) and (4.26):

$$T_{\alpha\beta} = (\rho\, c^2 + \rho\, \epsilon(\rho) + p)\, u_\alpha u_\beta + p\, \eta_{\alpha\beta}, \tag{3.20}$$

where u^μ and $\eta^{\mu\nu}$ are defined in (3.7) and (4.21) respectively.§

Conservation law of energy-momentum given by (4.24) is cited here,

$$\partial_\beta T^{\alpha\beta} = T^{\alpha\beta}{}_{,\beta} = 0. \tag{3.21}$$

where the *comma* notation ', β' denotes the *partial* derivative with respect to x^β. This is an expression in global Lorentz (Minkowski) frame of flat spacetime. For the transition (to be considered next) from flat to curved spacetime, the *comma* is replaced by a *semicolon* such as $T^{\alpha\beta}{}_{;\beta}$, *i.e.* the covariant derivative of $T^{\alpha\beta}$.

From the equivalence principle explained in the section III, a), (iii) the same equation as (3.21) is given in local *Lorentz frame* (*Lf* in short) of curved spacetime as well by

$$T^{\alpha\beta}{}_{,\beta} = 0 \quad \text{at origin of local Lorentz frame.} \tag{3.22}$$

In such a frame of local *Lf*, free particles are viewed to move along straight lines at least locally. This means that the term $\Gamma^\alpha_{\beta\gamma}$ of (3.11) must vanish at the origin in the local *Lf*. Namely, all the laws of physics must take their forms known in the special relativity. This is the *Principle of Equivalence.*

Because the Christoffel symbols $\Gamma's$ vanish at the origin of local *Lf*, the equation (3.22) can be rewritten as

$$T^{\alpha\beta}{}_{;\beta} = 0 \quad \text{at origin of local Lorentz frame.}$$

Thus the conservation law given by the form $T^{\alpha\beta}{}_{,\beta} = 0$ at origin of local Lorentz frame is extended to curved spacetime of the form $T^{\alpha\beta}{}_{;\beta} = 0$ in any reference frame owing to the definitive character of tensor. Thus, we have

$$T^{\alpha\beta}{}_{;\beta} = 0 \qquad : \text{extended to any reference frame of curved spacetime.} \tag{3.23}$$

iv. *Einstein field equations*

Equations of the gravitational field are obtained from the principle of least action $\delta(S_g + S_m) = 0$, where S_g and S_m are the actions of the gravitational field and matter field respectively. Accordintg to the variational formulation of Appendix C.2, the variation of S_g with respect to the metric field $g^{\alpha\beta}$ is

$$\delta S_g = -A_g \int \Big(R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta} \hat{R}\Big) \delta g^{\alpha\beta} \sqrt{-g}\, \mathrm{d}\mathit{\Omega}, \quad A_g \equiv \frac{c^3}{16\pi G_0}, \tag{3.24}$$

where $\mathrm{d}\mathit{\Omega} = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$ and $\sqrt{-g}\, \mathrm{d}\mathit{\Omega}$ is the proper volume $[\mathrm{d}\mathit{\Omega}]_{prop}$ in a local Lorentz frame with $g = \det(g_{\mu\nu})$, and $R_{\alpha\beta}$ is the Ricci curvature tensor (C.11), and $\hat{R} \equiv g^{\alpha\nu} R_{\alpha\nu}$ is the scalar curvature, and G_0 is the gravitational constant.

§ The expression of stress-energy tensor $T_{\alpha\beta}$ given here is equivalent to the expression of (a) the equation (133.2) of §133 of "LL (1987)" and that of (b) Box 5.1 of §5.1 of "Gravitation (2017)", under the understanding that $\overline{\rho}\,(m_1 c^2 + \overline{\epsilon}) + \overline{p}$ (where $m_1 = 1$) is equivalent to $w = \rho\, e + p$ of (a) where $e = m_1 c^2 + \epsilon$, and to $\rho + p$ of (b) where ρ is defined by $\overline{\rho}\,(1 + \overline{\epsilon})$ since $m_1 c^2 = 1$ by the assumption $c = 1$ of the text (b). Note that the present Minkowski metric $\eta_{\alpha\beta}$ is equal to $-g_{\alpha\beta}$ of (a). Thus, all the stress-energy tensors $T_{\alpha\beta}$ of the three texts are equivalent under the above understanding.

On the other hand, the variation of the action S_m of the matter field is

$$\delta S_m = \frac{1}{2c}\int T_{\alpha\beta}\,\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}\Omega. \tag{3.25}$$

where $T_{\alpha\beta}$ is the stress-energy tensor of the matter (*i.e.* the fluid in the present case). Note that variation of the coordinates from x^ν to $x'^\nu = x^\nu + \xi^\nu$ results in variation of the metric $\delta g^{\alpha\beta}$ ‖

From the action principle $\delta S_g + \delta S_m = 0$, we find the *Einstein field equation*:

$$G_{\alpha\beta} = 8\pi k\, T_{\alpha\beta}\,, \qquad k = G_0/c^4, \tag{3.26}$$

in view of the arbitrariness of the $\delta g^{\alpha\beta}$. (See Appendix C.2 for its derivation). The tensor $G_{\alpha\beta}$ is defined by

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta}\hat{R}, \tag{3.27}$$

called the Einstein curvature tensor, while $T_{\alpha\beta}$ is the stress-energy tensor.

v. *Degree of freedom of geometro-dynamics*

Einstein's geometro-dynamics is governed by ten tensor equations (3.26): $G_{\alpha\beta} = 8\pi k\, T_{\alpha\beta}$. Among the ten equations, only six are effective. How can the ten equations be in reality only six ? This is because, owing to the four Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$, the equations $G_{\mu\nu} = 8\pi\, T_{\mu\nu}$ place four local conservation laws $T^{\mu\nu}{}_{;\nu} = 0$ of energy and momentum of the source fluid. Instead, four conditions become free, which enable four coordinates chosen arbitrarily. Hence the geometry is constrained by the six independent equations from (3.26).

It is worth emphasizing the ingenious composition of the theory by repeating the concept with other words. The ten equations of $G_{\alpha\beta} = 8\pi k\, T_{\alpha\beta}$ place four constraints on the source motion in the form of the four conservation equations $T^{\mu\nu}{}_{;\nu} = 0$, owing to the four Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$. This is exactly the meaning given in the beginning as "*the geometry tells the matter how to move*". The four conditions, instead, enable four coordinate frames chosen freely. Remaining six constraints from $G_{\mu\nu} = 8\pi\, T_{\mu\nu}$ are those meant by "*the matter tells geometry how to curve*".

The geometro-dynamics in vacuum space requires special attention. Because no matter exists in the vacuum, the six constraints to be imposed by matters mentioned above must be replaced by conditions of vacuum-space own. Here is the place where the Lorentz gauge condition comes into play. This is presented next.

c) Similarity between Gravity Theory and QED

There exist various similarities between the gravity field of the present section and the field of quantum electrodynamics (QED) considered in the section II. Those are reviewed with comparing corresponding mathematical expressions from three aspects here.

i. *Covariant derivatives*

The similarity is clearly seen in the form of the covariant derivatives of both fields. In the gravity, the covariant derivative of $\boldsymbol{v} = v^\nu \boldsymbol{e}_\nu$ with respect to x^μ is given by (3.18):

$$(\widehat{\nabla}_\mu \boldsymbol{v})^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\alpha\mu} v^\alpha. \tag{3.28}$$

In QED, according to (2.33) of the section II, b), (ii), corresponding form of its covariant derivative of wave function is given as

$$\nabla_\mu \boldsymbol{\psi} = \partial_\mu \boldsymbol{\psi} - i\gamma A_\mu\, \psi, \qquad \gamma = e/\hbar c. \tag{3.29}$$

‖ $\delta g^{\alpha\beta} = -\xi^\nu \partial_\nu g^{\alpha\beta} + g^{\alpha\nu}\partial_\nu \xi^\beta + g^{\beta\nu}\partial_\nu \xi^\alpha$. See LL (1975) §94.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

The coefficients of second connection term of each covariant derivative are directly connected to the source field of each case. The former ${\Gamma^\nu}_{\alpha\mu}$ are given by derivatives of metric tensors $g_{\mu\nu}$ including the gravity potential Φ_g (see (3.12) and (3.10)). The latter γA_μ is obvious since A_μ is the electromagnetic (EM) potential.

The covariant derivative $\widehat{\nabla}_\mu \boldsymbol{v}$ denotes the derivative in curved spacetime, leading to curved geodesic lines. Analogously, the latter derivative $\nabla_\mu \psi$ signifies curved motion of microscopic particles because the term $p_k\psi = -i\hbar\partial_k\psi$ of (2.28) denotes rectilinear momentum in the absence of the EM field A_μ.

ii. *Invariant variations*

Equations of the gravitational field are obtained from the principle of least action with total action defined by $S_{total} = S_g + S_m$, where S_g and S_m are the actions of gravitational field and matter field respectively. Variations of both actions δS_g and δS_m are given in Appendix C.2. From the action principle $\delta(S_g + S_m) = 0$, we obtain

$$\delta S_g + \delta S_m = -A_g \int \Big(G_{\alpha\beta} - 8\pi k\, T_{\alpha\beta}\Big)\, \delta g^{\alpha\beta} \sqrt{-g}\, \mathrm{d}\Omega = 0, \tag{3.30}$$

where $G_{\alpha\beta}$ is the Einstein's curvature tensor defined by (C.19), $A_g = c^3/(16\pi G_0)$ and $k = G_0/c^4$ with G_0 the gravitational constant. The action principle requires invariance of $S_g + S_m$, namely *vanishing* of $\delta(\, S_g + S_m\,)$ for arbitrary variations of the metric tensor $\delta g^{\alpha\beta}$. Thus, we obtain the Einstein equation,

$$G_{\alpha\beta} = 8\pi k\, T_{\alpha\beta}\,, \qquad k = G_0/c^4. \tag{3.31}$$

The action principle, *i.e.* the *invariant variation* described above, yields the Einstein field equation (3.31).

On the other hand, corresponding part of EM (electromagnetism) is the second pair of Maxwell equations presented in the section II a) (i). derived from the electromagnetic action composed of two components $S^{(\mathrm{em})}_{\mathrm{emA}}$ and $S^{(\mathrm{em})}_{\mathrm{int}}$ defined in the section II a) (i). Hence, from the action principle $\delta(\, S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}\,) = 0$, we obtain

$$\delta S^{(\mathrm{em})} \equiv \delta\Big(S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}\Big) = \int \Big(\frac{1}{c}\, j_e^{\,\nu} - \frac{1}{4\pi}\,\frac{\partial F^{\nu\lambda}}{\partial x^\lambda}\Big)\delta A_\nu\, \mathrm{d}\Omega = 0. \tag{3.32}$$

The action principle requires invariance of $S^{(\mathrm{em})} \equiv S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}$, namely *vanishing* of $\delta S^{(\mathrm{em})}$ for arbitrary variations of the potential δA_ν. Thus, we obtain

$$\partial_\lambda F^{\nu\lambda} = (4\pi/c)\, j_e^{\,\nu}\,. \tag{3.33}$$

This *invariant variation* yields the second pair of Maxwell equations (2.8).

Similarity between the gravity and the electromagnetism is seen not only in the form of the action principle by comparing (3.30) and (3.32), but also remarkable similarity is observed in the derived equations (3.31) and (3.33). Left-hand side of (3.31), $G_{\alpha\beta}$, denotes the spacetime structure of gravity, while that of (3.33), $\partial_\lambda F^{\nu\lambda}$, denotes the structure of electromagnetic field. Those are generated by the sources on the right-hand side: $T_{\alpha\beta}$ of (3.31) being the stress-energy tensor of the source perfect fluid, and $j_e^{\,\nu}$ of (3.33) being the source current flux.

iii. *Waves in vacuum space and gauge conditions*

In the section II a) (iii), we have seen electromagnetic waves governed by the wave equation (2.21) for the electromagnetic 4-potential A^α. In vacuum space, this reduces to

$$(\nabla^2 - c^{-2}\partial_t^2)A^\nu = 0. \tag{3.34}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

This can be derived from (3.33), which becomes, on substituting $F^{\nu\lambda} = \partial^\nu A^\lambda - \partial^\lambda A^\nu$,

$$-\partial_\lambda \partial^\lambda A^\nu + \partial^\nu (\partial_\lambda A^\lambda) = (4\pi/c)\, j_e^{\ \nu} \tag{3.35}$$

Imposing the Lorenz gauge condition (2.20),

$$\partial_\lambda A^\lambda = 0, \tag{3.36}$$

setting $j_e^{\ \nu} = 0$ in the vacuum space, and noting $-\partial_\lambda \partial^\lambda = c^{-2}\partial_t^2 - \nabla^2$, the equation (3.35) reduces to (3.34).

Similar structure is found in the gravitational waves as well to be presented in the next section d). In weak gravitational field, the metric tensor is represented as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ under the condition $|h_{\alpha\beta}| \ll 1$. Linearizing the Einstein equation (3.31), the wave equation (3.47) is derived under the gauge condition (3.46), both of which are cited here in advance for comparison purpose:

$$(\nabla^2 - c^{-2}\partial_t^2)\, \overline{h}^{\mu\nu} = -16\pi k\, T^{\mu\nu}\,, \tag{3.37}$$

$$\partial_\nu\, \overline{h}^{\mu\nu} = 0\,, \tag{3.38}$$

where $\overline{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}(h^\alpha_{\ \alpha})$. One can recognize similar structures between EM and Gravity, although there is an obvious difference, vectorial fields of the former EM field and tensorial fields of the latter Gravity. Inspite of such difference, their similarity is remarkable.

Consider the EM wave equation (2.21) and apply the divergence operator ∂_ν on it, then we obtain

$$(\nabla^2 - c^{-2}\partial_t^2)(\partial_\nu A^\nu) = -(4\pi/c)\,(\partial_\nu j_e^{\ \nu}).$$

Hence, the gauge condition (3.36) requires the current conservation $\partial_\nu j_e^{\ \nu} = 0$.

Next, consider the gravitational wave equation (3.37) and apply the divergence operator ∂_ν on it, then we obtain

$$(\nabla^2 - c^{-2}\partial_t^2)\,(\partial_\nu \overline{h}^{\mu\nu}) = -16\pi k\,(\partial_\nu T^{\mu\nu})\,,$$

It is consistent with the formulation of the theory that the gauge condition (3.38) requires the conservation of stress-energy of dynamical motion of the source material (fluid) $\partial_\nu T^{\mu\nu} = 0$.

In vacuum space where both of the current flux $j_e^{\ \nu}$ and the stress-energy of material motion are absent. the gauge freedom resulting from the absence of materials is filled up by the gauge conditions $\partial_\nu A^\nu = 0$ or $\partial_\nu \overline{h}^{\mu\nu} = 0$. It is understood that the gauge conditions play the role of filling in the blanks of degrees of freedom.

d) Gravitational waves (weak gravitational field)

The spacetime is flat in the absence of gravity, and presence of a weak gravitational field is one in which spacetime is curved but close to flat. In the spacetime continuum object (*manifold* in mathematics), the metric components are represented as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{3.39}$$

where

$$|h_{\alpha\beta}| \ll 1\,, \qquad \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1), \tag{3.40}$$

assuming small ripples in flat spacetime. Such spacetime is called nearly-Lorentz system and studied by a linearized theory. Merits of linearized theory lie not only in its manageability of analytic handling, but also in the fact that one can apply a *gauge* transformation to the weak gravitational field as well.

In fact, the weak field has a remarkable analogy with the electromagnetic field, as seen in the previous part c, evidenced by the similarity of corresponding wave equations (3.34) and (3.37). However, the difference is clearly recognized in the source terms on the right-hand sides of the two wave equations. In the former field, the source is the current density 4-vector $j_{\rm e}^{\mu}$. while in the latter, it is the stress-energy tensor $T^{\mu\nu}$ of fluid motion. Namely, the vector $j_{\rm e}^{\mu}$ and tensor $T^{\mu\nu}$ symbolize the difference of both fields. However it is more important to have an insight (and recognize) that they share a common physical mechanism for generation of each field despite their difference.

i. *Linearized theory and gravitational gauge transformation*

From the metric form (3.39) under the condition (3.40), one obtains a resulting form of the Christoffel symbol $\Gamma^{\alpha}_{\beta\gamma}$ from the definition (3.12), in which all three terms are linear without approximation: $\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\,(h^{\alpha}{}_{\beta,\gamma} + h^{\alpha}{}_{\gamma,\beta} - h_{\beta\gamma}{}^{,\alpha})$. A linearized form of Riemann tensor is

$$R_{\alpha\mu\beta\nu} = \tfrac{1}{2}\Big(h_{\alpha\nu,\mu\beta} + h_{\mu\beta.\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}\Big), \tag{3.41}$$

and the Ricci tensor is given by $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu} = \partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} - \partial_{\nu}\Gamma^{\alpha}_{\mu\alpha}$ from (C.12). Then, the linearized field equation is derived from the Einstein equation (3.26): $G_{\mu\nu} = 8\pi k\, T_{\mu\nu}$ as

$$-\overline{h}_{\mu\nu,\alpha}{}^{\alpha} - \eta_{\mu\nu}\,\overline{h}_{\alpha\beta,}{}^{\alpha\beta} + \overline{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} + \overline{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = 16\pi k\, T_{\mu\nu}\,, \tag{3.42}$$

(Misner et al. (2017), Chap.18), where

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\,\eta_{\mu\nu}\,h, \qquad h = h^{\alpha}_{\alpha} = \eta^{\alpha\beta}h_{\alpha\beta}. \tag{3.43}$$

We are now in an important stage where one can conceive a *gravitational* gauge transformation, which is quite analogous to the electromagnetic one. Let us consider an infinitesimal transformation of the coordinates of a spacetime point $\mathcal{P}$ from old ones (x^{μ}) to new ones (x'^{μ}), expressed as

$$x'^{\mu}(\mathcal{P}) = x^{\mu}(\mathcal{P}) + \xi^{\mu}(\mathcal{P}), \tag{3.44}$$

where $x^{\mu}(\mathcal{P})$ and $x'^{\mu}(\mathcal{P})$ represent the same spacetime point $\mathcal{P}$, and only their reference frames are changed. Metric perturbations in the new (x'^{μ}) and old (x^{μ}) coordinate frames are related to first order in small quantities by¶

$$h^{\rm new}_{\mu\nu} = h^{\rm old}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}\ . \tag{3.45}$$

This is regarded as a gravitational gauge transformation since the Riemannian tensors are left unchanged by the transformation (3.45). This can be immediately verified by substituting the expression of $h^{\rm new}_{\mu\nu}$ into (3.41), finding $R^{\rm new}_{\alpha\mu\beta\nu} = R^{\rm old}_{\alpha\mu\beta\nu}$. This is reasonable because the change of reference frame only should not influence the physical world. Since the the curvature tensor $R_{\alpha\mu\beta\nu}$ is unchanged, the Ricci tensor $R_{\alpha\beta}$, scalar curvature $\hat{R}$, Einstein tensor $G_{\alpha\beta}$ are all unchanged. This is the gravitational gauge invariance, and the geometrical tensors are essentially the same whether calculated in an orthonormal frame $\eta_{\mu\nu}$, in the old frame $g^{\rm old}_{\mu\nu}$, or in the new frame $g^{\rm new}_{\mu\nu}$.

In general, one can impose the following *gauge condition*:

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0, \tag{3.46}$$

¶ Defining matrix element of transformation by $\Lambda^{\alpha}_{\ \overline{\beta}} \equiv \partial x^{\alpha}/\partial x'^{\beta} = \delta^{\alpha}{}_{\beta} - \xi^{\alpha}{}_{,\beta}$, neglecting higher order terms of smallness, transformation of the metric tensor is given by $g^{\rm new}_{\alpha\beta} = \Lambda^{\mu}{}_{\overline{\alpha}}\Lambda^{\nu}{}_{\overline{\beta}}\, g^{\rm old}_{\mu\nu} = \Lambda^{\mu}{}_{\overline{\alpha}}\Lambda^{\nu}{}_{\overline{\beta}}\,\eta_{\mu\nu} + \Lambda^{\mu}{}_{\overline{\alpha}}\Lambda^{\nu}{}_{\overline{\beta}}\,h_{\mu\nu} = (\eta_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}) + h_{\mu\nu}$.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

called the *Lorentz gauge* for gravitational waves. Under this Lorentz gauge condition, the linearized field equation (3.42) reduces to

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 16\pi k\, T_{\mu\nu}\,, \qquad \text{or equivalently} \quad \partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -16\pi k\, T_{\mu\nu}\,, \quad (3.47)$$

since the second, third and forth terms on the left-hand side of (3.42) vanish, as follows:

$$\bar{h}_{\alpha\beta,}{}^{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}\bar{h}^{\mu\nu,\alpha\beta} = \bar{h}^{\mu\nu}{}_{,\mu\nu} = (\bar{h}^{\mu\nu}{}_{,\nu})_{\mu} = 0, \qquad \text{by (3.46)},$$

$$\bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} = \eta_{\mu\lambda}\bar{h}^{\lambda\beta}{}_{,\beta\nu} = \eta_{\mu\lambda}(\bar{h}^{\lambda\beta}{}_{,\beta})_{\nu} = 0, \qquad \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = \eta_{\nu\lambda}\bar{h}^{\lambda\beta}{}_{,\beta\mu} = 0.$$

The equation (3.47) represents gravitational wave-generation by the source term on the right-hand side, since the operator $\partial_\alpha\partial^\alpha$ is nothing but that of wave equation:

$$\partial_\alpha\partial^\alpha = -\partial_0{}^2 + \nabla^2 = \Box, \qquad \partial_\alpha = (\partial_0, \nabla), \quad \partial^\alpha = \eta^{\alpha\lambda}\partial_\lambda = (-\partial_0, \nabla).$$

Thus, we have found the gauge condition (3.46) and wave equation (3.47) for gravitational waves, which are equivalent to the equations (3.37) and (3.38) presented already in §3.3(c). Note that the indices of $\bar{h}_{\mu\nu}$ and $T_{\mu\nu}$ are raised with the Minkowski metrics $\eta^{\alpha\mu}\eta^{\beta\nu}$ multiplied on both sides of (3.47), obtaining $\bar{h}^{\alpha\beta}$ and $T^{\alpha\beta}$. Since the factors $\eta^{\alpha\mu}\eta^{\beta\nu}$ are constant, they enter through the diffrential operators.

ii. *Justification of Lorentz gauge*

Suppose that the tensors $\bar{h}_{\mu\nu}$ satisfy the equation (3.42), but do not satisfy the condition (3.46). Then, one can apply a gauge transformation (3.45) to obtain $(\bar{h}^{\text{new}})_{\mu\nu}$ from $(\bar{h}^{\text{old}})_{\mu\nu}$, and demand that $(\bar{h}^{\text{new}})_{\mu\nu}$ satisfies the gauge condition:

$$(\bar{h}^{\text{new}})^{\mu\alpha}{}_{,\alpha} = 0 \ = (\bar{h}^{\text{old}})^{\mu\alpha}{}_{,\alpha} - \partial_\alpha\partial^\alpha\,\xi^\mu - \partial^\mu(\partial_\alpha\xi^\alpha). \qquad (3.48)$$

Under the condition $\partial_\alpha\xi^\alpha = 0$ (compatible with the transversality of the waves), one can find the perturbation ξ^μ satisfying the wave equation,

$$\partial_\alpha\partial^\alpha\,\xi^\mu \left[= (-c^{-2}\partial_t{}^2 + \nabla^2)\xi^\mu \right] = (\bar{h}^{\text{old}})^{\mu\alpha}{}_{,\alpha} \quad (\neq 0, \ \text{assumed}).$$

The new field $(\bar{h}^{\text{new}})^{\mu\nu}$ satisfies the Lorentz condition (3.48), $(\bar{h}^{\text{new}})^{\mu\alpha}{}_{,\alpha} = 0$ and the wave equation (3.47).

Even the new metric $(\bar{h}^{\text{new}})^{\mu\nu}$ satisfy the condition (3.48), there is arbitrariness. To fix it, consider a *restricted gauge transformation* $(\bar{h}^{\text{new}})_{\mu\nu} \to (\bar{h}^{\text{new}})'_{\mu\nu}$:

$$(\bar{h}^{\text{new}})'_{\mu\nu} = (\bar{h}^{\text{new}})_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha_{\ \alpha}, \qquad (3.49)$$

derived from the form (3.43) and (3.45). Provided that ξ^μ satisfies the following wave equation,

$$\partial_\alpha\partial^\alpha\,\xi^\mu = (-c^{-2}\partial_t{}^2 + \nabla^2)\,\xi^\mu = 0 \quad [\textit{Restricting condition}], \qquad (3.50)$$

the Lorentz condition $(\bar{h}^{\text{new}})'^{\,\mu\alpha}{}_{,\alpha} = 0$ is satisfied according to an equation equivalent to (3.48). Namely, the restricted gauge transformation preserves the Lorentz gauge condition. Therefore the Lorentz gauge is really a class of gauges.

iii. *Gravitational waves in vacuum*

Just as wavy deformations over sea surface propagate across the ocean, so small ripples of the gravitational metric tensor propagate across spacetime. Propagation of the latter gravitational wave in vacuum space (where $T_{\mu\nu} = 0$) is given by the wave equation (3.47) under the gauge condition (3.46):

$$\partial_\alpha\partial^\alpha\,\bar{h}_{\mu\nu} = (\nabla^2 - c^{-2}\partial_t{}^2)\,\bar{h}_{\mu\nu} = 0, \qquad (3.51)$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\partial_\alpha \overline{h}^{\mu\alpha} = 0, \qquad (\text{where} \quad \partial_\alpha \overline{h}^{\mu\alpha} = \eta^{\mu\nu}\, \partial^\alpha \overline{h}_{\nu\alpha}). \tag{3.52}$$

Plane Wave: For simplicity reason, let us consider a plane wave, described by the following monochromatic wave:

$$\overline{h}_{\mu\nu} = A_{\mu\nu} \exp[i\, k_\alpha x^\alpha], \qquad \Big(k_0 = -\omega/c, \quad \boldsymbol{k} = (k_1, k_2, k_3),\Big) \tag{3.53}$$

where $x^\alpha = (ct, x^1, x^2, x^3)$. Substituting this to the equation (3.51), one obtains

$$i^2\, k_\alpha\, k^\alpha = k_0^2 - |\boldsymbol{k}|^2 = 0, \qquad \therefore |\boldsymbol{k}|^2 = \omega^2/c^2, \tag{3.54}$$

which is referred to as the dispersion relation of the wave and k^α is called the null vector. The equation of gauge condition (3.52) requires the four (orthogonality) conditions:

$$k_\alpha\, A^{\mu\alpha} = 0. \tag{3.55}$$

iv. *Degree of freedom of gravitational waves*

Let us consider the degree of freedom of gravitational waves in vacuum space. Its degree of freedom is found to be *Two*. The reason is as follows. The metric perturbation $\overline{h}_{\mu\nu}$ of a plane wave is given by (3.53), which is a solution to the field equation (3.51) in the form of wave equation. Its wave amplitude $A_{\mu\nu}$ has ten independent components in general. The field equation (originally of the form $G_{\alpha\beta} = 8\pi k\, T_{\alpha\beta}$) is controlled by four constraints due to the four Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$, as mentioned at section III b) (v) The four conditions, instead, enable four frames of coordinate chosen freely. Those are provided by the orthogonality gauge-conditions (3.55): $k_\alpha A^{\mu\alpha} = 0$. Thus, the degree of freedom of $A_{\mu\nu}$ is reduced to six.

Wave propagation in vacuum space requires special attention. Because of absence of matters in the vacuum, the six constraints to be imposed by matters (if they existed) must be replaced by conditions of vacuum-space own. Here is the place where another gauge conditions come into play. However, even when the gauge condition (3.46) is satisfied, there is arbitrariness. Namely without violating the gauge condition (3.55), one can introduce the restricted gauge condition (3.50).

Let us express a solution to the restricted gauge condition (3.50) by another plane wave:

$$\xi_\alpha = B_\alpha \exp[i\, k_\mu x^\mu], \tag{3.56}$$

where B_α is a constant and k_μ is given by (3.54). Consequent change in $\overline{h}_{\alpha\beta}$ is given according to (3.49) as $(\overline{h}^{\text{new}})'_{\alpha\beta} = (\overline{h}^{\text{new}})_{\alpha\beta} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha{}_\alpha$. From (3.56), this gives

$$A'^{(\text{new})}_{\alpha\beta} = A^{(\text{new})}_{\alpha\beta} - iB_\alpha k_\beta - iB_\beta k_\alpha + i\,\eta_{\alpha\beta} B^\mu k_\mu, \tag{3.57}$$

by removing the exponential factor. One can show (Schutz 1985, Chap.9) that B_α can be chosen to impose two further restrictions on $A'^{(\text{new})}_{\alpha\beta}$:

$$A^\alpha{}_\alpha = 0 \qquad (\text{traceless}), \tag{3.58}$$

$$A_{\alpha\beta}\, u^\beta = 0 \qquad (\text{transverse}), \tag{3.59}$$

where u^β is any constant timelike unit vector.

Note that the condition (3.59) gives only three because $k^\alpha A_{\alpha\beta}\, u^\beta = 0$ is valid identically for any B_α. Hence, the constraints (3.55), (3.58) and (3.59) together give the eight conditions, which are called the *transverse-traceless* (TT) gauge conditions. The remaining two must be physically significant. Namely, the degree of freedom of the

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

wave is found to be *Two*. The gravitational wave has two dynamic degrees of freedom, which is analogous to the electromagnetic waves propagating in vacuum space.

The TT-gauge is based on the vector u^β. Let us take the frame of background vacuum Minkowski spacetime (through which the wave is propagating) defined by the time basis set along the vector $u^\beta = \delta^\beta_{\ 0}$. Then, the condition (3.59) implies $A_{\alpha 0} = 0$ for all α. In this frame, we take the spatial x^3-axis parallel to the direction of wave propagation. Then we have $k_\alpha = (-k, 0, 0, k)$ from (3.54), and the equation (3.55) implies $A_{\alpha 0} + A_{\alpha 3} = 0$. Hence we have $A_{\alpha 3} = -A_{\alpha 0} = 0$ for all α.

Thus, using the symmetry of $A_{\alpha\beta}$ and the traceless condition $A_{11} + A_{22} = 0$, we can write the coefficient matrix $A_{\alpha\beta}$ in the TT-gauge (transverse-traceless gauge) as

$$A^{TT}_{\ \ \alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (3.60)$$

IV. Fluid Mechanics: Smooth Sequence of Non-Commutative Diffeomorphisms

In Fluid-Mechanics of a perfect fluid, the fluid *medium* is assumed as a continuum (*i.e.* a continuous distribution of mass) in the spacetime $x^\nu = (t, \boldsymbol{x}) = (ct, x^1, x^2, x^3)$. Flow variables such as the mass density ρ, pressure p or flow velocity $\boldsymbol{v}$ are represented by continuously differentiable functions of x^ν. Dynamical motion of fluid flows is characterized by the presence of *convective derivative* in the equation of motion. It is a derivative following the fluid motion, also called sometimes the advective derivative, Lagrange derivative or material derivative, but most importantly it is gauge-invariant *covariant derivative* under local gauge transformations. A fluid flow is a smooth sequence of diffeomophisms of particle configuration, which is a continuous sequence of transformations from one time to another, and two different sequences are not commutative. This is contrasted with the commutative $U(1)$ gauge transformations of QED, seen in §2.2.

a) Euler's equation of motion

To capture dynamical motion of fluids, we have two distinct kinds of specification: *Eulerian* type and *Lagrangian* type. With respect to each specification, one finds a gauge symmetry associated with the fluid mass in motion. In the first Eulerian type of specification, the mass density, pressure or flow velocity are represented by differentiable field variables of $\rho(t, \boldsymbol{x}), p(t, \boldsymbol{x})$ or $\boldsymbol{v}(t, \boldsymbol{x})$ respectively. Fluid motions are governed by two kinds of equations, the continuity equation and Euler's equation of motion:

$$\partial_t \rho + \nabla \cdot (\rho\, \boldsymbol{v}) = 0, \qquad (4.1)$$

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho} \nabla p, \qquad (4.2)$$

In the second Lagrangian type of specification, as in particle mechanics, flow variables such as mass-density ρ or velocity $\boldsymbol{v}$ are defined by functions of three parameters $\boldsymbol{a} = (a^1, a^2, a^3)$ identifying each fluid particle (a piece of material element of fluid) and time t_a, like $\rho(t_a, a^1, a^2, a^3)$ or $\boldsymbol{v}(t_a, a^1, a^2, a^3)$. In this specification, the quartet members (t_a, a^1, a^2, a^3) play as independent variables replacing the spacetime coordinates (x^0, x^1, x^2, x^3) of the Eulerian type. For example, the spatial position of a fluid particle at a time t_a

specified by the parameter $\boldsymbol{a} = (a^1, a^2, a^3)$ is described by $\boldsymbol{X}_a(t_a, a^1, a^2, a^3) = \boldsymbol{X}_a(t_a, \boldsymbol{a})$. However, to denote a point in Euclidian 3-space, we keep the symbols $(x^k) = (x^1, x^2, x^3)$.

In the case of Lagrangian type of specification, local gauge transformation (LGT) is considered with respect to the specification of position coordinate of a fluid particle identified with the Lagrange-parameter $\boldsymbol{a}$. To describe the particle motion, a *convective* derivative D_t is defined by

$$\mathrm{D}_t \equiv \partial_t + (\boldsymbol{v} \cdot \nabla), \tag{4.3}$$

in addition to partial derivatives such as $\partial_t \equiv \partial/\partial t$ or $\partial_k \equiv \partial/\partial x^k$.

The convective derivative D_t is a generalization of the time derivative ∂_t having a remarkable property of invariance with respect to an LGT transformation defined below. This property is investigated in the section IV c) as another kind of gauge invariance, and also investigated as a *covariant* derivative in curved space-time. In fact, using D_t, the above Euler's equation of motion (4.2) can be rewritten as

$$\mathrm{D}_t \boldsymbol{v} + \rho^{-1} \nabla p = 0. \tag{4.4}$$

This is viewed as a generalization of Newton's equation of motion to a continuous matter of a perfect fluid, because the term $\mathrm{D}_t \boldsymbol{v}$ is regarded as an acceleration of a fluid particle of a unit mass identified with a fixed parameter $\boldsymbol{a}$.

b) Fluid flow: Sequence of non-commutative diffeomorphisms

i. *One-parameter sub-group of diffeomorphisms*

A fluid flow is a smooth sequence of diffeomorphisms of particle configuration on a spacetime manifold M^4 with a point $x = (x^\nu) = (t, \boldsymbol{x}) \in M^4$ ($\boldsymbol{x} = (x^k)$ with $k = 1, 2, 3$). Suppose that a vector field $U(x) = U^\nu e_\nu = \partial_t + U^k \partial_k$ is given at every point $x \in M^4$ (with $U^0 = 1$) as a vector operator U. With such a vector field, one can associate a particular flow, *i.e.* one-parameter sub-group of diffeomorphisms ξ_t with $\xi_0 = I$ (identity). This is a transformation of the particle configuration at the initial moment $\xi_0(x) = Ix = (0, \boldsymbol{X}_0)$ to the particle configuration $\xi_t(x) = (t, \boldsymbol{X}_t)$ at time t. The initial velocity field at $t = 0$ is given by $(\mathrm{d}/\mathrm{d}t)\, \xi_t(x)|_{t=0} = U \xi_t(x)|_{t=0} = U^\nu e_\nu = \partial_t + U^k \partial_k$. where U is an operator on $\xi_0(x)$. In this flow, the initial material point $\boldsymbol{X}_0 = \boldsymbol{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$ in 3-space is transformed to a 3-space point $\boldsymbol{X}_t(\sigma)$ at $t\,(> 0)$. The transformation ξ_t is, as it were, an infinite-dimensional diffeomorphisms from $\boldsymbol{X}_0 = \boldsymbol{\sigma}$ to $\boldsymbol{X}_t(\boldsymbol{\sigma})$. (See, *e.g.* Kambe (2010) Chap.1 and its Appendix C).

On such a group (a Lie group) of diffeomorphisms, one-parameter subgroup with a tangent vector U at the origin I is represented by

$$\xi_t = I + tU + \frac{1}{2!} t^2 U^2 + O(t^3). \tag{4.5}$$

With a second vector field $V(x) = V^k e_k$, a second flow of one-parameter subgroup $\eta_s(x)$ is generated analogously by V with $\eta_0 = I$. Noting that the composition $\eta_s \xi_t(x)$ is understood as $\eta_s(\xi_t(x))$, we have

$$\eta_s\, \xi_t - \xi_t\, \eta_s = st\, [U, V] + O(st^2, s^2 t), \tag{4.6}$$

$$[U, V] \equiv \left(U^k \partial_k V^i - V^k \partial_k U^i \right) \partial_i \tag{4.7}$$

The commutator $[U, V]$ signifies the degree of non-commutativity of the two flows of diffeomorphisms represented by ξ_t and η_s. This non-commutativity signifies the spacetime being curved.

(*b*) *Geodesic equation of a fluid-flow*

With respect to a flow $\xi(t)$, consider a trajectory $X^\mu(t)$ of a fluid particle on a Riemannian manifold M^4 with its tangent vector defined by $T(x^\mu) = \mathrm{d}\xi/\mathrm{d}t$. The curve is said to be a *geodesic* if its tangent is displaced parallel along the curve $\xi(t)$, *i.e.* if

$$\nabla_T T = 0\,. \tag{4.8}$$

See (A.17) of Appendix A, where general interpretation of geodesic equation and covariant derivative are given (*cf.* Kambe (2010) Chap.3, say). In local coordinates, we have $T \equiv \mathrm{d}\xi/\mathrm{d}t = T^\alpha e_\alpha = (\mathrm{d}X^\alpha/\mathrm{d}t)\, e_\alpha$.

Same geodesic equation $\nabla_T T = 0$ is also given by the action principle, *i.e.* by the equation deduced from the extremum of an action integral (*cf.* Appendix A.6). *Relativistic* form of the action integral of a perfect fluid is given as

$$S^{(\mathrm{pf})} = -c \int \rho\, \mathrm{d}\mathcal{V} \int \Big(1 + c^{-2}\,\bar{\epsilon}(\rho)\Big)\, \mathrm{d}\tau \tag{4.9}$$

This is an extended form of the relativistic action integral of a single particle of mass m, $S^{(\mathrm{m})} = -c\,m \int \mathrm{d}\tau$, to the perfect fluid, where the overlined value $\bar{\epsilon}$ denote proper value of the internal energy ϵ of the perfect fluid (the value in the *rest frame*, *i.e.* comoving frame where the fluid is at rest). Comparison of $S^{(\mathrm{pf})}$ with $S^{(\mathrm{m})}$ and considering $\int \rho\, \mathrm{d}\mathcal{V}$ equivalent to m of $S^{(\mathrm{m})}$, one can see that the term $c^{-2}\,\bar{\epsilon}$ is a small correction term to the fluid medium in non-relativistic case.

From the variation analysis, the geodesic equation of a perfect fluid is given as

$$\mathrm{D}_t v^k + \rho^{-1}\, \partial_k p = 0, \tag{4.10}$$

for non-relativistic limit of ordinary fluid flows (Kambe 2020, §2). This coincides with the Euler's equation of motion (4.4) of ordinary fluid mechanics.

c) Convective derivative D_t and its Gauge invariance

The convective derivative $\mathrm{D}_t = \partial_t + (\boldsymbol{v}\cdot\nabla)$ has a special property which is invariant with respect to a group of transformations like the gauge invariance of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Hence the following transformation may be a fluid version of the gauge transformation. The derivative D_t is also regarded as the covariant derivative analogously with the electromagnetic case. The operator D_t is also invariant with respect to the Lorentz transformation, i.e. a relativistic invariant (see Sec. I, d) of Kambe T (2021), Fluid Gauge Theory, GJSFR, vol. 21, iss.4).

i. *Local gauge transformation*

Suppose that we have two coordinate frames F and F' which are overlapping and each fluid particle is identified by the Lagrange-parameter $\boldsymbol{\alpha}$. Let us denote the position of the same particle $\boldsymbol{\alpha}$ with the coordinate $\boldsymbol{X}_a$ in the frame F and $\boldsymbol{X}'_a$ in the frame F'. Relaltive motion of the two frames is not assumed to be time-independent. Hence the frames are not necessarily inertial. We consider the relation between the two coordinates to be a transformation between $\boldsymbol{X}_a(t,\boldsymbol{\alpha})$ and $\boldsymbol{X}'_a(t',\boldsymbol{\alpha})$, which is given by the following *local gauge transformation* (dependent on $\boldsymbol{\alpha}$) at $t' = t$:

$$LGT: \qquad \boldsymbol{X}'_\alpha(t',\boldsymbol{\alpha}) = \boldsymbol{X}_\alpha(t,\boldsymbol{\alpha}) + \xi(t,\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{X}_\alpha}, \qquad t' = t, \tag{4.11}$$

This is rewritten in the form of transformation acted by an element g of the group $\mathcal{G}$ defined by $\mathcal{G} = LGT$:

$$\boldsymbol{X}'_\alpha|_{t'=t} = g(t,\boldsymbol{\alpha}) \circ \boldsymbol{X}_\alpha, \qquad g \in \mathcal{G}. \tag{4.12}$$

This LGT is considered as a local transformation between two coordinates (of the same particle identified by $\boldsymbol{\alpha}$) specified in the two non-inertial reference frames F and F'. In fact, the same particle $\boldsymbol{\alpha}$ has a spatial position coordinate $\boldsymbol{X}_\alpha(t,\boldsymbol{\alpha})$ in the frame F and another one $\boldsymbol{X}'_\alpha = \boldsymbol{X}_\alpha + \xi(t,\boldsymbol{\alpha})$ in the frame F'. Therefore, its velocity at $\boldsymbol{x} \in F$,

$$\boldsymbol{v}(t,\boldsymbol{x})|_{\boldsymbol{\alpha}} = \partial_t \boldsymbol{X}_\alpha, \tag{4.13}$$

is transformed to the velocity at $\boldsymbol{x}' = \boldsymbol{X}'_\alpha \in F'$ and $t' = t$:

$$\boldsymbol{v}'(t',\boldsymbol{x}')|_{\boldsymbol{\alpha}} = \partial_t \boldsymbol{X}'_\alpha(t,\boldsymbol{\alpha}) = \boldsymbol{v}(t,\boldsymbol{X}_\alpha) + (\mathrm{d}/\mathrm{d}t)\boldsymbol{\xi}_\alpha, \tag{4.14}$$

$$\boldsymbol{\xi}_\alpha = \boldsymbol{\xi}(t,\boldsymbol{X}_\alpha), \qquad (\mathrm{d}/\mathrm{d}t)\boldsymbol{\xi}_\alpha = \partial_t\boldsymbol{\xi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi}|_{\boldsymbol{x}=\boldsymbol{X}_\alpha}. \tag{4.15}$$

One may rewrite the equation (4.14) in a form analogous to (4.12) as

$$\boldsymbol{v}'_\alpha(\boldsymbol{X}'_\alpha) = g(t,\boldsymbol{\alpha}) \circ \boldsymbol{v}_\alpha(\boldsymbol{X}_\alpha). \tag{4.16}$$

This is a transformation of motion of the same particle between two different reference frames F and F'. Physically speaking, two vectors $\boldsymbol{X}_\alpha$ and $\boldsymbol{X}'_\alpha$ denote the same material point, represented by the common Lagrange parameter $\boldsymbol{\alpha}$. Namely, we are considering a gauge transformation between two reference frames.

According to the transformation (4.11), the time derivative ∂_t and space derivative $\partial_k = \partial/\partial x^k$ in the frame F are related to the derivatives ∂'_t and $\partial'_k = \partial/\partial x'^k$ of F' as follows:

$$\partial_t = \partial_{t'} + (\partial_t\boldsymbol{\xi})\cdot\nabla', \qquad \nabla' = (\partial'_k), \tag{4.17}$$

$$\partial_k = \partial'_k + (\partial_k\xi_l)\,\partial'_l, \qquad \partial'_k = \partial/\partial x'_k\,. \tag{4.18}$$

ii. *Gauge invariance of the convective derivative D_t*

The convective derivative $\mathrm{D}_t \equiv \partial_t + (\boldsymbol{v}\cdot\nabla)$ is invariant with respect to *LGT*: *i.e.* $\mathrm{D}_t = \mathrm{D}'_t$. In fact from (4.14) and (4.18), we have

$$\boldsymbol{v}\cdot\nabla = \boldsymbol{v}\cdot\nabla' + (\boldsymbol{v}\cdot\nabla\xi)\cdot\nabla' = \boldsymbol{v}'(\boldsymbol{x}')\cdot\nabla' + (\,-(\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t) + \boldsymbol{v}\cdot\nabla\xi)\cdot\nabla',$$

where $\boldsymbol{v} = \boldsymbol{v}' - \mathrm{d}\boldsymbol{\xi}/\mathrm{d}t$ is used. The last term is rewritten as

$$(\,-(\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t) + \boldsymbol{v}\cdot\nabla\xi)\cdot\nabla' \;=\; -\partial_t\boldsymbol{\xi}\cdot\nabla' = \partial_{t'} - \partial_t, \tag{4.19}$$

by using (4.15) and (4.17). Hence, we have

$$\mathrm{D}_t \;=\; \partial_t + \boldsymbol{v}\cdot\nabla = \partial_{t'} + \boldsymbol{v}'\cdot\nabla' \;=\; \mathrm{D}'_t. \tag{4.20}$$

This means that the operator D_t satisfies the invariance with respect to *LGT*.

In addition, it can be shown that the operator D_t is a covariant derivative in the sense of gauge theory. As shown in (*a*), under the transformation by $g \in \mathcal{G}$, the expression (4.12) gives $\boldsymbol{X}_\alpha \to \boldsymbol{X}'_\alpha = g\circ\boldsymbol{X}_\alpha = \boldsymbol{X}_a + \boldsymbol{\xi}$, and its derivative (velocity) $\boldsymbol{v}(\boldsymbol{X}_a) = \mathrm{D}_t\boldsymbol{X}_a$ is transformed as

$$\boldsymbol{v}'(\boldsymbol{X}'_a) = \mathrm{D}'_t\boldsymbol{X}'_a = \mathrm{D}_t(\boldsymbol{X}_a + \boldsymbol{\xi}) = \boldsymbol{v}(\boldsymbol{X}_a) + \mathrm{D}_t\boldsymbol{\xi} = g\boldsymbol{v} = g\circ\mathrm{D}_t\boldsymbol{X}_a,$$

where the equality $\boldsymbol{v}+\mathrm{D}_t\boldsymbol{\xi} \equiv g\boldsymbol{v}$ is consistent with (4.13) and (4.14). The above sequence of equalities states that $\mathrm{D}_t\boldsymbol{X}_a$ is transformed to $g\circ\mathrm{D}_t\boldsymbol{X}_a$ in the same way as $\boldsymbol{X}_a$ is transformed to $g\circ\boldsymbol{X}_\alpha$. Therefore, the operator D_t has the covariance property and is reasonably called *Covariant Derivative*.

One can see that the equation of motion (4.4) of a perfect fluid is expressed in terms of the time derivative D_t. The fact that the covariant derivative D_t plays a role of time derivative in place of the partial time derivative ∂_t implies that the free motion according to (4.4) is like a motion in curved space. Rewriting it as $\mathrm{D}_t\boldsymbol{v} = -\rho^{-1}\nabla p$, the equation has a pressure force $-\rho^{-1}\nabla p$, which is not an external force, but an internal force. In fact, each fluid particle does not take a straight trajectory but a *curved* one, in general, owing to the internal pressure force.

d) Relativistic formulation of a perfect fluid

Let us investigate how the fluid mechanics of a perfect fluid is formulated according to the theory of special relativity, which is based on the Minkowski space equipped with

$$\text{Minkowski metric}: \quad \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1). \tag{4.21}$$

In the space, a world element $\mathrm{d}s$ and an element of proper time $\mathrm{d}\tau/c$ are defined by

$$\mathrm{d}s^2 \equiv -\,\mathrm{d}\tau^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\mathrm{d}x^\mu x^\nu = -(1-\beta^2)\,c^2\mathrm{d}t^2,$$
$$c^{-1}\,\mathrm{d}\tau = \sqrt{1-\beta^2}\,\mathrm{d}t, \qquad \beta \equiv v/c, \quad v = |\boldsymbol{v}|, \tag{4.22}$$

where $\mathrm{d}x^0 = c\,\mathrm{d}t$, and c the light speed, and $\boldsymbol{v} = (v^k)$ is the particle velocity, with its 3-space displacement $\mathrm{d}X^k = v^k \mathrm{d}t$ $(k = 1, 2, 3)$. Relativistic 4-velocity u^ν is defined by

$$u^\nu = \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} = \Big(\frac{1}{\sqrt{1-\beta^2}}, \frac{\boldsymbol{v}}{c\sqrt{1-\beta^2}}\Big), \quad X^0 \equiv ct, \quad \boldsymbol{v} = (v^k) = (\mathrm{d}X^k/\mathrm{d}t). \tag{4.23}$$

Relativistic form of the action integral of a perfect fluid is already given by (4.9). Relativistic equations of conservation of energy-momentum are expressed in the form,

$$\frac{\partial}{\partial x^\nu}\,T^{\mu\nu} = 0 \qquad (\mu,\, \nu = 0, 1, 2, 3), \tag{4.24}$$

where the stress-energy tensor $T^{\mu\nu}$ is given by Kambe (2020) for a *perfect fluid*$^+$ as

$$T^{\mu\nu} \equiv H\,u^\mu u^\nu + p\,\eta^{\mu\nu}. \quad H \equiv \rho\,\varepsilon + p = \rho\,c^2 + \rho\,\epsilon + p, \tag{4.25}$$

$$\varepsilon \equiv c^2 + \epsilon(\rho), \quad H \equiv \rho\,c^2 + \rho\,h, \quad h \equiv \epsilon(\rho) + p/\rho, \tag{4.26}$$

(*cf.* Landau & Lifshitz (1987) calling $T^{\mu\nu}$ as energy-momentum tensor), where $\varepsilon \equiv m_1c^2 + \epsilon = c^2 + \epsilon$ (with $m_1 = 1$) is the relativistic internal energy per unit mass including the mass energy m_1c^2. The thermodynamic variables like $\epsilon(\rho)$ (internal energy) denote the *proper* value (*i.e.* the value in the comoving frame where the fluid is at rest).† The term $\rho\,c^2$ in H denotes the relativistic energy of rest-mass ρ per unit volume.

The above stress-energy tensor $T^{\mu\nu}$ of (4.25) was derived from the Lagrangian density $L \equiv -c\,(\rho\,\mathrm{d}\mathcal{V})\,(1 + c^{-2}\,\bar{\epsilon}(\rho))$ in the action $S^{(\mathrm{pf})}$ of (4.9) under the mass conservation condition $\rho\,\mathrm{d}\mathcal{V} = const$ (see Kambe 2020, §2.2). Present study to be carried out below (and the accompanying paper) does not assume the mass conservation *a priori* (from the outset), but it is deduced from the formulation under a pertinent symmetry. Therefore, the stress-energy tensor $T^{\mu\nu}$ should be derived with taking a different way, which is given in Landau & Lifshitz (1987, §133) and presented here now.

The derivation is as follows. The momentum flux through a surface element $\mathrm{d}\sigma_k$ is just the force acting on the element. Hence $T^{ik}\mathrm{d}\sigma_k$ is the i-th component of the force acting on the surface element $(i, k = 1, 2, 3)$. Let us take a certain volume element within the fluid in which it is at rest (the local rest frame). In this frame, Pascal's law is valid, that is, the pressure force exerts independently of the direction of the surface element $\mathrm{d}\sigma_k$ and is everywhere perpendicular to the surface on which it acts. Therefore, one can write $T^{ik}\mathrm{d}\sigma_k = p\,\delta^{ik}\,\mathrm{d}\sigma_i$, whence $T^{ik} = p\,\delta^{ik}$.

$^+$ Note: There is no energy dissipation in the present case of perfect fluid, hence no entropy change. Assuming the entropy is uniform throughout the fluid, the internal energy ϵ depends only on ρ.

† Some textbooks such as Misner *et al.* (2017), *etc.* use the definition $T^{\mu\nu} \equiv (\rho + p)\,u^\mu u^\nu + p\eta^{\mu\nu}$ where ρ is understood to denote $\rho\varepsilon = \rho(c^2 + \epsilon)$ including the internal energy $\rho\epsilon$ with $c = 1$ in their unit.

In the local rest frame, then, the energy-momentum tensor has the form

$$T^{\mu\nu}_{\rm rest} = \begin{pmatrix} \rho\varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{4.27}$$

where ε is the relativistic internal energy per unit mass including the mass energy $m_1 c^2$, hence $\rho\varepsilon$ denotes the energy per unit volume. In order to find the expression of the tensor $T^{\mu\nu}$ in arbitrary reference system, we introduce the 4-velocity u^ν defined by (4.23) for the motion of the fluid. In the rest frame of the particular fluid particle, we have $v^k = 0$ and $u^\nu = (1, 0, 0, 0)$. The expression to be sought for $T^{\mu\nu}$ must be such a form that it takes the form (4.27) when transformed to this rest frame. Such a second-rank tensor $T^{\mu\nu}$ must be

$$T^{\mu\nu} = (\rho\varepsilon + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}. \tag{4.28}$$

for the 4-velocity u^μ of (4.23) and the metric $\eta^{\mu\nu}$ of (4.21). This can be shown as follows, by using the Appendix B.

In the current unprimed frame x^μ, the particles are in motion with the velocity of (4.23). Lorentz transformation from this unprimed frame x^μ to the primed frame x'^α comoving with the particle P ($i.e.$ $\beta = |\boldsymbol{v}|/c$) is carried out by the transformation matrix $\Lambda^{\alpha'}_{\ \mu}$ defined with (B.6) and (B.7). By this transformation, the second rank tensor $T^{\mu\nu}$ in the unprimed frame x^μ is transformed to that in primed frame as follows:

$$T^{\mu\nu} \quad \Rightarrow \quad T^{\alpha'\beta'}_{\rm rest} = \Lambda^{\alpha'}_{\ \mu}\Lambda^{\beta'}_{\ \nu}\, T^{\mu\nu} = (\rho\varepsilon + p)\,(\Lambda^{\alpha'}_{\ \mu}\, u^\mu)\,(\Lambda^{\beta'}_{\ \nu} u^\nu) + p\,(\Lambda^{\alpha'}_{\ \mu}\Lambda^{\beta'}_{\ \nu})\,\eta^{\mu\nu}$$

$$= {\rm diag}(\rho\varepsilon + p, 0, 0, 0) + {\rm diag}(-p, p, p, p). \tag{4.29}$$

by using the transformation $u'^\alpha = \Lambda^{\alpha'}_{\ \nu}\, u^\nu$ and (B.9) of Appendix B. The last expression (4.29) reduces to the matrix of (4.27).

This is a wonderful derivation of $T^{\mu\nu}$ of (4.28) for a perfect fluid by Landau & Lifshitz (1987). From the point of view of the present study, however, there exists a crucial aspect to be remarked now. In regard to the momentum flux, the isotropic expression $p\,\delta^{ik}$ (Pascal's law) is taken at the rest frame and Lorentz-trandformed to arbitrary inertial systems of reference, $i.e.$ from the rest frame to frames of arbitrary high velocity, even turbulent, or close to the light velocity. If the medium is solid, then it may be one of choices. However, the fluid is receptive of diffeomorphic transformations among constituent fluid particles in infinitely different ways. Its degree of freedom is infinte (say). It is very likely that tensor form of momentum flux may be quite complex. The paper accompanying the present study, *Fluid Gauge Theory*, intends to present one of possible structures of a perfect fluid.

V. Motivations for Fluid Gauge Theory

A symmetry implies a conservation law (Noether's theorem). However it is shown below that, from a single relativistic energy equation of fluid motion, two conservation equations are obtained in the non-relativistic limit according to the current formulation of fluid mechanics: one is the mass conservation and the other is the traditional form of energy equation. This is a *riddle*. We are concerned particularly with the mass conservation equation and investigate what symmetry implies the mass conservation, and conversely what symmetry the mass conservation implies. A key to resolve this *Riddle* is hinted by the general representation of rotational flows of an ideal compressible fluid satisfying the Euler's equation, derived by Kambe (2013). This gives us a hint

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

of existence of a set of gauge fields, suggesting that our physical system should be a combined system consisting of a fluid flow field and a set of background gauge fields. The gauge symmetry of the latter ensures the law of mass conservation. Conversely as far as the mass conservation law is valid, the gauge invariance is ensured for the action representing interaction between the two components of the combined fields.

a) A riddle: By what symmetry the mass conservation law is implied?

It is well-known that the energy conservation is associated with the symmetry of time translation of mechanical systems. Main object of this section is to state motivation by raising a question of what physical symmetry implies the mass conservation law. This query is raised in regard to the relativistic equation of energy conservation of fluid flows when its non-relativistic limit is taken. In the ordinary fluid-mechanics valid in non-relativistic limit, the mass conservation law is given as valid *a priori*. However, let us see what happens in relativistic mechanics. It is reminded that the relativistic energy-momentum tensor has been given in the previous section IV, d).

The equation (4.24) represents four conservation equations. The *space* components of the equation (4.24) are given by $\partial_\nu T^{k\nu} = 0$ with $\mu = k = 1, 2, 3$, representing the momentum conservation of the k-th component.

On the other hand, its *time* component ($\partial_\nu T^{0\nu} = 0$) is the equation of energy conservation. In order to see its explicit representation in terms of flow variables in the non-relativistic limit ($\beta \equiv v/c \to 0$), the stress-energy tensors $T^{\mu\nu}$ are now written by leading-order terms of expansion with respect to small β in a matrix form:

$$T^{\alpha\beta} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}, \quad \left.\begin{array}{rcl} T^{00} & = & \underline{\rho c^2} + \frac{1}{2}\rho v^2 + \rho\epsilon \quad + \cdots, \\ T^{0k} & = & \underline{c\rho v^k} + c^{-1}\rho v^k(\frac{1}{2}v^2 + h) \quad + \cdots, \\ T^{k0} & = & c\rho v^k + c^{-1}\rho v^k(\frac{1}{2}v^2 + h) \quad + \cdots, \\ T^{ik} & = & \rho v^i v^k + p\,\delta_{ik} \quad + \cdots = T^{ki} \\ & & h \equiv \epsilon + p/\rho. \end{array}\right\} \tag{5.1}$$

where matrix elements are given together with flow variables on the right-hand part of the expression (5.1). The term T^{00} is the energy density, while T^{0k} ($k = 1, 2, 3$) is the energy flux density. The underlined terms ρc^2 in T^{00} and $c\rho v^k$ in T^{0k} came from the rest-mass energy part of the tensor $T^{\alpha\beta}$, which do not appear in the ordinary fluid mechanics. There exists the symmetry of $T^{0k} = T^{k0}$ in the relativistic expression of (5.1). This symmetry is lost in the non-relativistic ordinary fluid mechanics when the underlined terms are removed.

The equation $\partial_\nu T^{0\nu} = 0$ of energy conservation can be written down now as,

$$c^{-1}\partial_t\overline{T}^{00} + \partial_k\overline{T}^{0k} = c\left(\partial_t\rho + \partial_k(\rho v^k)\right) + \frac{1}{c}\left(\partial_t(\rho\hat{E}) + \partial_k(\rho v^k\hat{H})\right) + O(\beta^3) = 0, \tag{5.2}$$

$$\hat{E} = \frac{1}{2}v^2 + \epsilon, \quad \hat{H} = \frac{1}{2}v^2 + h. \tag{5.3}$$

(see (2.17) for ∂_ν). In the non-relativistic limit as $\beta \to 0$, we obtain the mass conservation equation from the first term,

$$\partial_t\rho + \partial_k(\rho v^k) = 0. \tag{5.4}$$

Then, deleting it, the remaining expression reduces to the energy equation of ordinary fluid mechanics in the non-relativistic limit. Thus, we obtain the energy conservation equation of fluid flow (Landau & Lifshitz (1987), Eq.(6.1)):

$$\partial_t(\rho\hat{E}) + \partial_k(\rho v^k\hat{H}) = 0. \tag{5.5}$$

Here we have obtained two conservation equations (5.4) and (5.5) from the single energy equation $\partial_\nu T^{0\nu} = 0$. However, the Noether's theorem (Noether 1918) of theoretical physics states 'A symmetry implies a conservation law', as noted in §1 (Introduction). Therefore, we must ask a question whether the above analysis is satisfactory, and we propose a resolution to this query in a separate paper. (Kambe 2021, "Fluid Gauge Theory",GJSFR).

b) *Hint to resolve the riddle: General solution of Euler's equation with helicity*

A hint to resolve the *Riddle* mentioned in the previous section is found in the general representation of rotational flows given by by Kambe (2013) for an ideal compressible flow solution satisfying the Euler's equation. Its expression in details is cited in Kambe (2020, §3). This solution was derived from the action principle with the action

$$S^{(\mathrm{Eul-rot})} = S^{(\mathrm{nR})} + S^{(\mathrm{Ga-inv})} = \int \rho \, \mathrm{d}\mathcal{V} \left[\int \Lambda_{\mathrm{nR}} \, \mathrm{d}t + \int \Lambda_{\mathrm{Gi}} \, \mathrm{d}t \right], \tag{5.6}$$

$$\Lambda_{\mathrm{nR}} = \tfrac{1}{2} v^2 - \epsilon, \qquad \Lambda_{\mathrm{Gi}} = -\mathrm{D}_t \quad - \mathrm{D}_t \langle \boldsymbol{U}, \boldsymbol{Z} \rangle \tag{5.7}$$

$$\nabla \cdot (\rho \boldsymbol{Z}) = 0, \qquad \nabla \cdot \boldsymbol{U} = 0, \tag{5.8}$$

$$\mathcal{L}[\boldsymbol{Z}] \equiv \partial_t \boldsymbol{Z} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{Z} - (\boldsymbol{Z} \cdot \nabla)\boldsymbol{v} = 0, \tag{5.9}$$

for non-relativistic flow fields, where Λ_{nR} is nothing but the ordinary non-relativistic Lagrangian density, while Λ_{Gi} is a gauge-invariant Lagrangian newly introduced in the study of Kambe (2013). Actually, this study had double aims. One was an attempt to obtain general representation of rotational flow field with non-zero helicity (Kambe 2012). Second aim was more fundamental, striving to establish equivalence between two formulations of Eulerian and Lagrangian specifications under the action principle. Each term of the Lagrangian densities Λ_{nR} and Λ_{Gi} satisfies local gauge invariance with respect to translation and rotation, hence it is consistent with the gauge theory.

As discussed in details in Kambe (2020, §1 and 3.1), this new formulation introduced four independent fields. In fact, regarding the 3-vector potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$, each has three components. Those six fields have two invariance conditions of (5.8), *i.e.* divergence-free condition in 3-space. In addition, from (5.9) and the equation, $(\mathcal{L}_t^*[\boldsymbol{U}])_i \equiv \partial_t U_i + v^k \partial_k U_i + U_k \partial_i v^k = 0$ obtained from the variational analysis of Kambe (2013), we have the third invariance condition:

$$\mathrm{D}_t \langle \boldsymbol{U}, \boldsymbol{Z} \rangle (t, \boldsymbol{x}) \equiv \langle \mathcal{L}_*[\boldsymbol{U}], \boldsymbol{Z} \rangle + \langle \boldsymbol{U}, \mathcal{L}[\boldsymbol{Z}] \rangle = 0. \tag{5.10}$$

Hence, the value of scalar product $\langle \boldsymbol{U}, \boldsymbol{Z} \rangle$ is invariant along the particle path $\boldsymbol{x} = \boldsymbol{X}_p(t, \boldsymbol{x})$, keeping its initial value along each trajectory. This is the third invariance imposed on the potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$. Therefore we have only three independent fields remaining among the six components of $\boldsymbol{U}$ and $\boldsymbol{Z}$. Furthermore, if we add the scalar field ψ which is also unconstrained, we have four independent fields in this solution.

Thus, four independent background fields are newly introduced in this solution. Those must be either given externally or determined internally within the framework of theory. In this paper, we take the latter approach, and the general solution given here is understood to predict existence of a new field, which is to be introduced according to the *fluid gauge theory* proposed in Kamle (2021). Hence, the present section describes a partial success, because we are lead to unavoidable circumstances which take us to a new step in two respects. First, owing to the existence of four components of background field, a set of new gauge fields must be introduced in the 4-spacetime according to the gauge-theoretic scenario. Second, it is understood that the newly introduced action $S^{(\mathrm{Ga-inv})} \equiv S^{(\mathrm{int})}$ of (5.12) given below represents interaction of the flow field with unknown background fields. Amazingly this action is analogous to the interaction form

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021
32

$S^{em}_{(\text{int})}$ of (2.9) in the case of Electromagnetism section II, a). This implies a possible approach, by the formulation analogous to that of Electromagnetism.

What is the hint to resolve the riddle mentioned in section V, a)? It is as follows. We rewrite the part of action $S^{(\text{Ga-inv})}$ of gauge-invariant terms of (5.6) as $S^{(\text{int})}$, since this term is considered to describe interaction between the flow-current j^ν and background vector-potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$, and ψ. In addition to $S^{(\text{int})}$, we denote the scalar product $\langle \boldsymbol{U}, \boldsymbol{Z} \rangle$ by W, and define a 4-current j^ν as follows:

$$S^{(\text{int})} = \int \rho \, \mathrm{d}\mathcal{V} \int \Lambda_{\text{Gi}} \mathrm{d}t, \qquad j^\nu \equiv (\rho\, c, \, \rho\, \boldsymbol{v}), \qquad W \equiv \langle \boldsymbol{U}, \, \boldsymbol{Z} \rangle. \tag{5.11}$$

Then the interaction part of action is expressed by

$$S^{(\text{int})} = -\int\!\!\int \Big(\rho\, \mathrm{D}_t \quad + \rho\, \mathrm{D}_t W \Big) \mathrm{d}\mathcal{V}\, \mathrm{d}t = \int\!\!\int \; j^\nu \, \tilde{a}_\nu \; \mathrm{d}\mathcal{V}\, \mathrm{d}t. \tag{5.12}$$

where $\tilde{a}_\nu = -\partial_\nu(\,\psi + W\,)$ and ∂_ν is the same as ∂_α of (2.17).

Note that the field $\tilde{a}_\nu = -\partial_\nu(\,\psi + W\,)$ is analogous to the particular field $\tilde{A}_\nu = \partial_\nu \Theta$ considered in Section I, b) where all the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish identically, In other words, those fields are potentially existing, but vanish in this particular potential form of $\tilde{A}_\nu = \partial_\nu \Theta$. Same can be said that new potential field $\tilde{a}_\nu$ can exist. But with the particular form $\tilde{a}_\nu = -\partial_\nu(\,\psi + W\,)$, the *potentially existing* new field does not show in observable world.

Based on this observation, new *Fluid Gauge Theory* is developed in the accompanying paper (Kambe 2021).

VI. Summary

Gauge invariance is one of the fundamental symmetries in modern theoretical physics. In this paper, the gauge symmetry is reviewed to see how it is working in fundamental physical fields: *Electromagnetism, Quantum ElectroDynamics* and *Geometric Theory of Gravity*. In the 19th century, the gauge invariance was recognized as a mathematicl non-uniqueness of the electromagnetic potentials, existing despite the uniqueness of observable electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In the 20th century, physical significance of the gauge symmetry was recognized but in zigzag ways. Real recognition of its physical significance required two new fields: the relativity theory for recognizing the structure of linked 4d-spacetime $x^\mu = (ct, \boldsymbol{x})$ together with, say, a 4-potential $A^\nu = (\Phi, \boldsymbol{A})$ and a current 4-vector $j^{\,\nu} = (\rho c, \, \boldsymbol{j})$, and the quantum mechanics for the new dimension of a phase factor $\exp{[i\chi(x^\nu)]}$ (§2.2). Finally the gauge symmetry was understood to be very fundamental, and the gauge invariance played a role of guiding principle in the study of physical fields such as Quantum Electrodynamics, Particle Physics and Theory of Gravitation.

There exist similarities in mathematical formulation of physical fields between the quantum electrodynamics (QED, Section II, b) and the gravity theory section III, c) Those are consequences of gauge-invariance property of each field more or less. For example, the covariant derivative of wave function ψ is $\nabla_\mu\, \psi = \partial_\mu \psi - i\gamma A_\mu \psi$, while in the gravity the covariant derivative of a vector $\boldsymbol{v} = v^\nu \boldsymbol{e}_\nu$ is represented as $(\nabla_\mu \boldsymbol{v})^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\alpha\mu} v^\alpha$. Second terms in each expression represent the effects from the electromagnetic potential A_μ in the former and from the gravity through the factor $\Gamma^\nu{}_{\alpha\mu}$ in the latter.

Fundamental governing equations of both fields are derived from the action principle (*i.e.* the action should be invariant for arbitrary variations). A (second) pair of Maxwell equations (3.33) is the one for the electromagnetic field, while the Einstein equation (3.31) is the corresponding equation for the gravitational field, which are, respectively,

$$\partial_\lambda F^{\nu\lambda} = (4\pi/c)\, j_e^{\,\nu}\,. \tag{6.1}$$

$$G^{\alpha\beta} \; = 8\pi k\, T^{\alpha\beta}\,. \tag{6.2}$$

The terms on the right hand side are the sources of each field. Taking 4-divergence ∂_ν of the first equation, the left hand side vanishes identically: $\partial_\nu\partial_\lambda F^{\nu\lambda} \equiv 0$, ensuring the current conservation: $\partial_\nu j_e^\nu = 0$. This is an outcome of the gauge symmetry of the field strength tensor $F^{\nu\lambda}$, which is anti-symmetric: $F^{\nu\lambda} = -F^{\lambda\nu}$. On the other hand, taking 4-divergence ∂_α of the second equation, the right hand side vanishes : $\partial_\alpha T^{\alpha\beta} = 0$ which is the conservation laws of stress-energy deduced as the Noether's theorem from the invariance of the action integral with respect to variations of 4-spacetime coordinates. Corresponding left hand side vanishes by the Bianchi identity of the gravitational field (Misner *et al.* (2017, Chap. 15)).

Waves in vacuum space and gauge conditions (there) are also seen to be similar between the two fields. Electromagnetic waves propagating in vacuum space are governed by the wave equation (3.34) for the potential A^ν under the gauge condition:

$$(\nabla^2 - c^{-2}\partial_t^2)A^\nu = 0. \qquad \partial_\nu A^\nu = 0. \tag{6.3}$$

In weak gravitational field, a linearized theory gives the wave equation (3.37) for the modified metric $\overline{h}^{\mu\nu}$ under the gauge condition (3.38). In vacuum space, we have

$$(\nabla^2 - c^{-2}\partial_t^2)\,\overline{h}^{\mu\nu} = 0\,, \qquad \partial_\nu\,\overline{h}^{\mu\nu} = 0\,, \tag{6.4}$$

In vacuum space where both of the current flux j_e^ν and the stress-energy tensor $T^{\alpha\beta}$ are absent. the gauge freedom resulting from the absence of materials is filled up by the gauge conditions $\partial_\nu A^\nu = 0$ or $\partial_\nu \overline{h}^{\mu\nu} = 0$. Namely, the gauge conditions play the role of filling in the blanks of degrees of freedom.

The section III, d), (iv) describes why the gravitational waves propagating in vacuum space have only two dynamic degrees of freedom, analogous to the electromagnetic waves, although in general, the metric perturbation $\overline{h}^{\mu\nu}$ has ten independent components.

Present review on the *gauge symmetry* is motivated from the previous study of Kambe (2020) having arrived at the conclusion that there exists a new gauge field within flow fields of a perfect fluid, and that the new field ensures the mass conservation. The gauge field is not recognized so far in the framework of mechanics of a perfect fluid.

This was an endeavor to resolve a *riddle*, which is presented in the section V a) as follows. *A symmetry implies a conservation law* (Noether 1918). However it can be shown that, from a single relativistic energy equation of fluid motion, two conservation equations are obtained in the non-relativistic limit according to the current formulation of fluid mechanics: one is the mass conservation and the other is the traditional form of energy equation. We are concerned particularly with the mass conservation equation and investigate what symmetry implies the mass conservation, and conversely what symmetry the mass conservation implies. A key to resolve this Riddle is hinted by the general representation of rotational flows (Kambe 2012, 2013) of an ideal compressible fluid satisfying the Euler's equation, described in the section V b). This gives us a hint of existence of a set of gauge fields, suggesting that our physical system should be a combined system consisting of a fluid flow field and a set of new gauge fields (Kambe 2017). From the gauge symmetry of the latter field, the law of mass conservation is deduced, rather than given *a priori*. As far as the mass conservation law is satisfied conversely, gauge invariance is ensured for the action representing interaction between the two components of the combined field.

Acknowledgments

For writing the present review paper motivated from the previous study (Kambe 2020), the author is grateful to Professor Yasuhide Fukumoto (Kyushu University) who is not

only interested in the subject itself, but encouraged to contemplate on writing a review paper covering the gauge symmetry lying in the background of Physics.

Appendix A. Riemannian Geometry

Gauge theory of physics is formulated on the basis of Riemannian geometry. To help the formulations in the main text, basics of Riemannian geometry are summarized here.

Appendix A.1. Tangent vectors and inner product

We consider the *inner* geometry of a Riemannian manifold M which is not a part of an Euclidean space. If a manifold M under consideration were a part of an Euclidean n-dimensional space E^n, it would inherit a local Euclidean geometry (such as the length) from the enveloping Euclidean space, as is the case of a 2-d surface in E^3. The manifold M^n under consideration is not a part of an Euclidean space, so the existence of a local geometry must be postulated. Let M^n be an n-dimensional manifold. The problem is how to define a tangent vector X when we are constrained to the manifold M^n. Let us introduce a local coordinate frame $(x^1, \cdots, x^n)$, and define a tangent vector $X \in T_x M^n$ at each point x of M^n by

$$X = X^i \frac{\partial}{\partial x^i} = X^i \, \partial_i \, ,$$

where $\partial_i = [\partial/\partial x^1, \cdots, \partial/\partial x^n]$ is a natural frame associated with the coordinate system. Furthermore, we define a vector-valued one-form by $\omega = \partial_i \otimes \mathrm{d}x^i$, where ∂_i and $\mathrm{d}x^i$ are bases of vector and covectors.† From the calculus of differential forms, we have $\omega[X] = \partial_i \otimes \mathrm{d}x^i[X] = X^i \partial_i = X$ where $\mathrm{d}x^i(X) = X^i$. By *eating* a vector X, the 1-form ω *yields* the same vector X, *i.e.* vector-valued one-form.

We consider *intrinsic* geometry of the manifold M^n. It is supposed that an inner product $\langle \cdot , \cdot \rangle$ is given in the tangent space $T_x M^n$. If X and Y are two smooth tangent vector fields of the tangent bundle $T_x M^n$, then $\langle X, \, Y \rangle$ is a smooth real function on M^n.

Appendix A.2. Riemannian metric

On a Riemannian manifold M^n, an inner product $\langle \cdot , \cdot \rangle$ is defined on the tangent space $T_x M^n$ at $x \in M$ and assumed to be differentiable. For two tangent fields $X = X^i(x)\partial_i$, $Y = Y^j(x)\partial_j \in T_x M^n$ (tangent bundle), the *Riemannian metric* is given by‡

$$\langle X, \, Y \rangle (x) \; = \; g_{ij} \, X^i(x) Y^j(x) \, ,$$

where the metric tensor, $g_{ij}(x) = \langle \partial_i, \, \partial_j \rangle = g_{ji}(x)$, is symmetric and differentiable with respect to x^i. This bilinear quadratic form is called the *first fundamental form*. In terms of differential 1-forms $\mathrm{d}x^i$, this is equivalent to $I \equiv g_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j$. *Eating* two vectors $X = X^i(x)\partial_i$ and $Y = Y^j(x)\partial_j$, this yields

$$I\,(X, Y) = g_{ij} \, \mathrm{d}x^i(X) \, \mathrm{d}x^j(Y) = \; g_{ij} \, X^i Y^j \, . \tag{A.1}$$

The inner product is said to be *non-degenerate*,

$$\text{if} \quad \langle X, \, Y \rangle = 0, \quad {}^\forall Y \in TM^n, \quad \text{only when} \quad X = 0 \, . \tag{A.2}$$

† These define symbols independent of local coordinate frames. If $u^1, \cdots, u^n$ is another frame, then we have transformation from ∂_i to $\partial/\partial u^i = (\partial x^l/\partial u^i)(\partial/\partial x^l)$ and from $\mathrm{d}x^i$ to $\mathrm{d}u^i = (\partial u^i/\partial x^k)\mathrm{d}x^k$, Then, their combination is $(\partial/\partial u^i) \otimes \mathrm{d}u^i = (\partial x^l/\partial u^i)(\partial/\partial x^l) \otimes (\partial u^i/\partial x^k)(\mathrm{d}x^k) = \delta^l_k \, (\partial/\partial x^l) \otimes \mathrm{d}x^k = \partial_k \otimes \mathrm{d}x^k$. Also, inner product is independent of frames: $U_i U^i = (\partial x^l/\partial u^i) X_l \, (\partial u^i/\partial x^k) X^k = \delta^l_k \, X_l X^k = X_k X^k$.
‡ If the inner product is only non-degenerate rather than positive definite, the resulting structure on M^n is called a *pseudo*-Riemannian.

As an example, consider a manifold of one-sphere S^1 of continuous interval of real numbers, $S^1 \equiv M^{\infty}_{[0,2\pi]} : [0, 2\pi]$. Its dimension is *infinite*, because the real number $x \in M^{\infty}_{[0,2\pi]}$ distributes continuously within the section $[0, 2\pi]$. Suppose that two fields $X = u(x)\,\partial_x$ and $Y = v(x)\,\partial_x$ are given in the tangent space $T_x M^{\infty}_{[0,2\pi]}$ at a point $x \in S^1$. Their inner product is defined by

$$\langle X,\, Y\rangle \equiv \int_0^{2\pi} u(x)\, v(x)\, \mathrm{d}x\,.$$

This kind of metric is used for electromagnetic fields or flow fields of a fluid.

Appendix A.3. Covariant derivative (Connection)

We introduce an additional structure to the manifold M^n that allows to form a *covariant derivative*. In mathematics, general definition is given to a covariant derivative (called a connection) on a Riemannian *curved* manifold M^n. Let two vector fields X, Y defined in the neighborhood of a point $p \in M^n$ and two vectors U and V defined at p. A *covariant derivative* (or *connection*) is an operator ∇. The operator ∇ assigns a vector $\nabla_U X$ at p to each pair (U, X) and satisfies the following relations:

$$\left.\begin{array}{lrcl} \text{(i)} & \nabla_U\,(aX + bY) &=& a\nabla_U X + b\nabla_U Y\,, \\ \text{(ii)} & \nabla_{aU+bV}\, X &=& a\nabla_U X + b\nabla_V X\,, \\ \text{(iii)} & \nabla_U(f(x)X) &=& (Uf)X + f(x)\nabla_U X\,, \end{array}\right\} \tag{A.3}$$

for a smooth function $f(x)$ and $a,\ b \in \mathcal{R}$, where $U = U^j\partial_j$ and $Uf = U^j\partial_j f \equiv \mathrm{d}f[U]$. Using the representations, $X = X^i\,\partial_i$ and $Y = Y^j\,\partial_j$, and applying the above properties (i)∼(iii), we obtain

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i\partial_i}(Y^j\partial_j) = X^i\nabla_{\partial_i}(Y^j\partial_j) \\ &= (X^i\partial_i Y^k)\partial_k + X^iY^j\,\Gamma^k_{ij}\,\partial_k = (\nabla_X Y)^k\,\partial_k\,, \end{aligned} \tag{A.4}$$

$$\nabla_{\partial_i}\partial_j := \Gamma^k_{ij}\partial_k\,, \tag{A.5}$$

where Γ^k_{ij} is called the *Christoffel symbol*. The i-th component of $\nabla_X Y$ is

$$(\nabla_X Y)^i = X^j\,\frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk}X^jY^k = \mathrm{d}Y^i(X) + (\Gamma^i_{jk}Y^k)\,\mathrm{d}x^j(X) := \nabla Y^i(X), \tag{A.6}$$

$$\nabla Y^i = \mathrm{d}Y^i + \Gamma^i_{jk}\,Y^k\,\mathrm{d}x^j\,, \qquad \nabla_j Y^i = \partial_j Y^i + \Gamma^i_{jk}Y^k, \tag{A.7}$$

where ∇Y^i is called a *connection one-form*. On a manifold M^n, a coordinate frame consists of n vector fields $e_k = \partial_k$ $(k = 1, \cdots, n)$, which are linearly independent and furnish a basis of the tangent space at each point p. Writing (A.5) and (A.6) in the form of vector-valued one-forms, we have $\nabla \boldsymbol{e}_j = \boldsymbol{e}_k\,\Gamma^k_{ij}\mathrm{d}x^i$, and $\nabla Y = (\mathrm{d}Y^k)\,\boldsymbol{e}_k + Y^j\,\Gamma^k_{ij}\mathrm{d}x^i\,\boldsymbol{e}_k$. The operator ∇ is called the *affine connection*, and we have the following representation,

$$\nabla Y\,(X) = \nabla_X Y\,. \tag{A.8}$$

Appendix A.4. Riemannian connection

There is one connection that is of special significance, having the property that *parallel displacement* preserves inner products, and the connection is symmetric.

Definition: There is a unique connection ∇ on a Riemannian manifold M called the *Riemannian connection* or *Levi-Civita* connection that satisfies

$$\text{(i)} \qquad Z\,\langle X,\, Y\rangle \;= \langle \nabla_Z X,\, Y\rangle + \langle X,\, \nabla_Z Y\rangle \tag{A.9}$$

$$\text{(ii)} \quad \nabla_X Y - \nabla_Y X = [X,\, Y] \qquad \text{(torsion free)}, \tag{A.10}$$

for vector fields $X, Y, Z \in TM$, where $Z\langle\cdot,\cdot\rangle = Z^j\partial_j\langle\cdot,\cdot\rangle$. The property (i) is a compatibility condition with the metric. The torsion-free property (ii) requires the following symmetry, $\Gamma^k_{ij} = \Gamma^k_{ji}$, with respect to i and j. In fact, writing as $X = X^i\,\partial_i$ and $Y = Y^j\,\partial_j$, the definitive expression (A.4) leads to

$$(\nabla_X Y - \nabla_Y X)^k = (XY - YX)^k + (\Gamma^k_{ij} - \Gamma^k_{ji})X^iY^j\,. \qquad \text{(A.11)}$$

Christoffel symbol:

The Christoffel symbol Γ^k_{ij} can be represented in terms of the metric tensor $g = (g_{ij})$ by the following formula:

$$\Gamma^k_{ij} = g^{k\alpha}\Gamma_{ij,\alpha}, \qquad \Gamma_{ij,\alpha} = \tfrac{1}{2}\left(\partial_i\,g_{j\alpha} + \partial_j\,g_{\alpha i} - \partial_\alpha\,g_{ij}\right), \qquad \text{(A.12)}$$

where $g^{k\alpha}$ denotes the inverse g^{-1}, $g^{k\alpha} = (g^{-1})^{k\alpha}$, satisfying $g^{k\alpha}g_{\alpha l} = g_{l\alpha}g^{\alpha k} = \delta^k_l$. The symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ follows immediately from (A.12) and $g_{ij} = g_{ji}$.

Appendix A.5. Covariant derivative along a curve

Consider a curve $x(t)$ on M^n passing through a point p whose tangent at p is given by

$$T = T^k\,\partial_k = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = \dot{x}^k\,\partial_k\,,$$

and let Y be a tangent vector field defined along the curve $x(t)$. According to (A.4) or (A.6), the covariant derivative $\nabla_T Y$ is given by

$$\nabla_T Y := \frac{\nabla Y}{\mathrm{d}t} = \left[\mathrm{d}Y^i(T) + \Gamma^i_{kj}\,T^kY^j\right]\partial_i = \left[\frac{\mathrm{d}}{\mathrm{d}t}Y^i + \Gamma^i_{kj}\,\dot{x}^kY^j\right]\partial_i\,, \qquad \text{(A.13)}$$

since $T^k = \dot{x}^k$. When Y^i is a function of $x^k(t)$, then $(\mathrm{d}/\mathrm{d}t)Y^i = \dot{x}^k(\partial Y^i/\partial x^k)$. The expression $\nabla Y/\mathrm{d}t$ emphasizes the derivative along the curve $x(t)$ parameterized with t.

Parallel translation:

On the manifold M^n, one can define *parallel displacement* of a tangent vector $Y = Y^i\partial_i$ along a parameterized curve $x(t)$. Parallel displacement is given by (A.15) below. Mathematically, this is defined by

$$\frac{\nabla Y}{\mathrm{d}t} = \nabla_T Y = 0\,; \quad \text{namely,} \quad \dot{x}^k(\partial Y^i/\partial x^k) + \Gamma^i_{kj}\dot{x}^kY^j = 0\,. \qquad \text{(A.14)}$$

For two vector fields X and Y translated parallel along the curve, we obtain

$$\langle\, X,\, Y\,\rangle = \text{constant} \quad \text{(under parallel translation)}, \qquad \text{(A.15)}$$

because the scalar product is invariant by (A.9) and (A.14):

$$T\,\langle X,\,Y\rangle = \langle\nabla_T X,\,Y\rangle + \langle X,\,\nabla_T Y\rangle = 0\,. \qquad \text{(A.16)}$$

Appendix A.6. Geodesic equation

One curve of special significance in a curved space is the geodesic curve. A curve $\gamma(t)$ on a Riemannian manifold M^n is said to be *geodesic* if its tangent $T = \mathrm{d}\gamma/\mathrm{d}t$ is displaced parallel along the curve $\gamma(t)$, *i.e.* if

$$\nabla_T T = \frac{\nabla}{\mathrm{d}t}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) = 0\,. \qquad \text{(A.17)}$$

In local coordinates $\gamma(t) = (x^i(t))$, we have $\mathrm{d}\gamma/\mathrm{d}t = T = T^i\partial_i = (\mathrm{d}x^i/\mathrm{d}t)\partial_i$. By setting $Y = T$ in (A.13), we obtain

$$\nabla_T T = \left[\frac{\mathrm{d}T^i}{\mathrm{d}t} + \Gamma^i_{jk} T^j T^k \right] \partial_i = 0\,, \quad \text{where} \quad T^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}. \tag{A.18}$$

Thus the *geodesic equation* $\nabla_T T = 0$ is expressed by local coordinates as

$$\frac{\mathrm{d}T^i}{\mathrm{d}t} + \Gamma^i_{jk} T^j T^k = 0, \qquad \text{or} \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{A.19}$$

Parallel translation again: Parallel translation of a tangent vector X along a geodesic $\gamma(s)$ with unit tangent T is defined by (A.14) as $\nabla_T X = 0$. By setting $Y = Z = T$ in the second property (A.9) of the Riemannian connection, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle X,\, T \rangle = T \langle X,\, T \rangle = \langle \nabla_T X,\, T \rangle\,, \tag{A.20}$$

since $\nabla_T T = 0$ by the definition of a geodesic. Hence, the inner product $\langle X,\, T\rangle$ is kept constant by the parallel translation.

Extremum of arc length : A geodesic curve denotes a path of shortest distance connecting two nearby points, or globally of an extremum for all variations with fixed end points. Let $C_0 : \gamma_0(s)$ be a geodesic curve with a length parameter $s \in [0, L]$. A varied curve is denoted by $C_\alpha : \gamma(s, \alpha)$ with $\gamma(s, 0) = \gamma_0(s)$, where $\alpha \in (-\varepsilon, +\varepsilon)$ is a variation parameter and s the arc length for $\gamma_0(s)$. The arc length of the curve C_α is

$$L(\alpha) = \int_0^L \left\| \frac{\partial \gamma(s, \alpha)}{\partial s} \right\| \mathrm{d}s = \int_0^L \langle T(s, \alpha),\, T(s, \alpha) \rangle^{1/2}\, \mathrm{d}s, \qquad T = \frac{\partial \gamma}{\partial s}\,.$$

Its variation is given by $L'(\alpha) = \int_0^L \partial_\alpha < \partial_s \gamma,\, \partial_s \gamma >^{1/2}\, \mathrm{d}s$. In case that the variation vanishes at both ends of $s = 0$ and L, the first variation $L'(0)$ at $\alpha = 0$ is given by

$$L'(0) = -\int_0^L \langle J,\, \nabla_T T \rangle\, \mathrm{d}s, \qquad \langle J,\, \nabla_T T \rangle = 0 \quad \text{for} \quad 0 < s < L\,, \tag{A.21}$$

where $J = \partial_\alpha \gamma((s, 0)$ is the variation vector. Thus, *the geodesic curve* $\nabla_T T = 0$ *takes the extremum of arc length among nearby curves having common endpoints*, in particular characterized by a path of the shortest distance if endpoints are sufficiently near.

Appendix B. Basics of Special Relativity

Suppose that a material particle or fluid particles are moving with high velocities in an inertial frame K: (ct, x^1, x^2, x^3) with c the light velocity. In a time interval $\mathrm{d}t$, the position of a particle changes with time and its displacement is given by a 4-vector:

$$\mathrm{d}x^\mu = (c\,\mathrm{d}t, \mathrm{d}X^1, \mathrm{d}X^2, \mathrm{d}X^3), \qquad \mathrm{d}X^k = v^k\, \mathrm{d}t \quad (k = 1, 2, 3), \tag{B.1}$$

where $\mu = 0, 1, 2, 3$, and the upper-case notation $\mathrm{d}X^k$ denotes material displacement with v^k being components of 3-velocity $\boldsymbol{v}$. In the relativity theory, an infinitesimal interval $\mathrm{d}s$ is defined by its squared form, $\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu$, which is a scalar product of a line-element 4-vector $\mathrm{d}x^\mu$ with its covariant version $\mathrm{d}x_\mu = \eta_{\mu\nu} \mathrm{d}x^\nu = (-c\,\mathrm{d}t, \mathrm{d}X^1, \mathrm{d}X^2, \mathrm{d}X^3)$, where $\eta_{\mu\nu}$ is the Minkowski metric, sometimes called the Lorentz metric, defined by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \tag{B.2}$$

Hence, we have $\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2 \mathrm{d}t^2 + |\mathrm{d}\boldsymbol{X}|^2$.†

† Note that the metric $g_{\mu\nu}$ used by Landau & Lifshitz (1975) is defined by $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = -\eta_{\mu\nu}$. Hence, $\mathrm{d}\tau^2$ [present] $= -\eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}s^2$ [Landau & Lifshitz] $= -\mathrm{d}s^2$ [present].

The interval ds is a relativistic invariant, *i.e.* invariant under the Lorentz transformation now defined. Suppose that the coordinate transformation is expressed by $x^\mu \to x'^\alpha = \Lambda^{\alpha'}_{\ \mu}\, x^\mu$ with $\Lambda^{\alpha'}_{\ \mu}$ a matrix of Lorentz transformation. Then we have

$$ds'^2 = \eta_{\alpha'\beta'}\, dx'^\alpha\, dx'^\beta = \eta_{\alpha'\beta'}\, \Lambda^{\alpha'}_{\ \mu}\Lambda^{\beta'}_{\ \nu}\, dx^\mu\, dx^\nu = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = ds^2\,,$$

where $\Lambda^{\alpha'}_{\ \beta}\Lambda^{\beta}_{\ \gamma'} = \delta^{\alpha'}_{\ \gamma'}$ is required for the Lorentz transformation. The equalities,

$$\eta_{\mu\nu} = \eta_{\alpha'\beta'}\, \Lambda^{\alpha'}_{\ \mu}\Lambda^{\beta'}_{\ \nu} = (\Lambda^{\rm T})_\mu^{\ \alpha'}\, \eta_{\alpha'\beta'}\, \Lambda^{\beta'}_{\ \nu} = (\,\Lambda^{\rm T}\,\eta'\Lambda\,)_{\mu\nu},$$

define the *Lorentz invariance*, or relativistic invariance.

Another relativistic invariant is the *proper time* τ. Its increment dτ is defined by the time increment (multiplied by c) in the instantaneously rest frame where $\boldsymbol{v} = 0$. Squared interval of the proper time is defined by $d\tau^2 = -dx_\nu dx^\nu = -ds^2$. From this, noting $dX^k = v^k\, dt$, we obtain

$$d\tau = c\, dt\, \sqrt{1-\beta^2}\,, \qquad \beta \equiv v/c\,, \quad v = \sqrt{v_k v^k}. \tag{B.3}$$

Using the displacement dX^ν of a fluid particle P, its relativistic 4-velocity is defined by

$$u^\nu = \frac{dX^\nu}{d\tau} = \Big(\frac{1}{\sqrt{1-\beta^2}}, \frac{\boldsymbol{v}}{c\,\sqrt{1-\beta^2}}\Big), \qquad \boldsymbol{v} = (v^k) = (dX^k/dt). \tag{B.4}$$

This fluid particle P is moving with the 4-velocity u^ν relative to the frame x^μ.

Consider the following useful transformation defined by the matrix components $\Lambda^{\alpha'}_{\ \mu}$:

$$v^1/c = \beta\, n^1, \quad v^2/c = \beta\, n^2, \quad v^3/c = \beta\, n^3, \quad \gamma \equiv 1/\sqrt{1-\beta^2}, \tag{B.5}$$

$$\Lambda^{0'}_{\ 0} = \gamma, \qquad \Lambda^{0'}_{\ j} = \Lambda^{j'}_{\ 0} = -\beta\,\gamma\, n^j, \tag{B.6}$$

$$\Lambda^{j'}_{\ k} = \Lambda^{k'}_{\ j} \ = \ (\gamma-1)\, n^j\, n^k + \delta^{jk}, \tag{B.7}$$

where the condition of unit 3-vector $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$ defines $\beta^2 = |\boldsymbol{v}|^2/c^2$.

With the matrix $\Lambda^{\alpha'}_{\ \mu}$ of (B.6) and (B.7), the unprimed frame x^μ is transformed to the primed frame x'^α by the coordinate transformation law: $x'^\alpha = \Lambda^{\alpha'}_{\ \mu}\, x^\mu$ at the instant when the origins of both frames coincide instantaneously. However, the primed frame x'^α is moving with the velocity $v^k/c = \beta\, n^k$ as seen in the unprimed frame x^μ.

It is remarkable that the 4-velocity u^ν is transformed by the same law: $u'^\alpha = \Lambda^{\alpha'}_{\ \nu}\, u^\nu$. Suppose that the particle P is comoving with the unprimed frame, hence its 4-velocity being $u^\nu = (1,0,0,0)$, and that the primed frame x'^α is moving with the velocity $-v^k = -|\boldsymbol{v}|\, n^k$ as seen in the unprimed frame x^μ (*i.e.* $\beta = |\boldsymbol{v}|/c$). It is not difficult to show that the 4-velocity $u'^\alpha = \Lambda^{\alpha'}_{\ \nu}\, u^\nu$ in the primed frame coincides with (B.4). Thus,

$$u^\nu = (1,0,0,0) \quad \Rightarrow \quad u'^\alpha = \gamma\,\Big(1,\, |\beta|\, n^j\Big) = \Big(\frac{1}{\sqrt{1-\beta^2}},\, \frac{\boldsymbol{v}}{c\,\sqrt{1-\beta^2}}\Big), \tag{B.8}$$

Conversely, suppose that the particle P is moving in the unprimed frame with the 4-velocity u^ν of (B.4), and that the primed frame x'^α is comoving with the particle P (*i.e.* $\beta = +|\boldsymbol{v}|/c$), hence moving with the velocity $v^k = |\boldsymbol{v}|\, n^k$ relative to the unprimed frame x^μ. Under the Lorentz transformation of (B.6) and (B.7), the 4-velocity $u'^\alpha = \Lambda^{\alpha'}_{\ \nu}\, u^\nu$ transformed from the u^ν of (B.4) is found as

$$u^\nu = \gamma\left(1,\ \beta\, n^j\right) \quad \Rightarrow \quad u'^{\,\alpha} = \Lambda^{\alpha'}_{\ \ \nu}\, u^\nu = (1,0,0,0), \tag{B.9}$$

where $\gamma \equiv 1/\sqrt{1-\beta^2}$, $\beta \equiv |\boldsymbol{v}|/c$ and $j = 1,2,3$.

Appendix C. Supplements to the Gravity Theory of Main Text

Appendix C.1. Useful formulae of gravity theory

- Covariant derivatives:

$$F\text{: scalar :} \qquad F_{;\gamma} = F_{,\gamma}\ , \tag{C.1}$$

$$V^\alpha\text{: vector :} \qquad V^\alpha_{\ ;\gamma} = V^\alpha_{\ ,\gamma} + \Gamma^\alpha_{\ \mu\gamma} V^\mu, \tag{C.2}$$

$$U_\alpha\text{: 1-form :} \qquad U_{\alpha\,;\gamma} = U_{\alpha\,,\gamma} - \Gamma^\mu_{\ \alpha\gamma} U_\mu, \tag{C.3}$$

$$T^\alpha_\beta\text{: tensor :} \qquad T^\alpha_{\ \beta\,;\gamma} = T^\alpha_{\ \beta\,,\gamma} + \Gamma^\alpha_{\ \mu\gamma} T^\mu_{\ \beta} - \Gamma^\mu_{\ \beta\gamma} T^\alpha_{\ \mu} \tag{C.4}$$

- Curvature tensors and symmetry properties:

$$\text{Riemann tensor :}\ R^\alpha_{\ \beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\nu\gamma}\Gamma^\nu_{\beta\delta} - \Gamma^\alpha_{\nu\delta}\Gamma^\nu_{\beta\gamma}, \tag{C.5}$$

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\nu} R^\nu_{\beta\gamma\delta} \tag{C.6}$$

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \tfrac{1}{2}\left(\partial_\alpha\partial_\delta\, g_{\beta\gamma} + \partial_\beta\partial_\gamma\, g_{\alpha\delta} - \partial_\alpha\partial_\gamma\, g_{\beta\delta} - \partial_\beta\partial_\delta\, g_{\alpha\gamma}\right) \\ &\quad + g_{\mu\nu}(\Gamma^\mu_{\beta\gamma}\Gamma^\nu_{\alpha\delta} - \Gamma^\mu_{\beta\delta}\Gamma^\nu_{\alpha\gamma}), \end{aligned} \tag{C.7}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{C.8}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{C.9}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0. \tag{C.10}$$

$$\text{Ricci tensor :} \qquad R_{\mu\nu} \equiv R^\alpha_{\ \mu\alpha\nu} \tag{C.11}$$

$$= \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{C.12}$$

$$\text{Scalar curvature :}\ \ R_{\rm sc} \equiv g^{\alpha\nu} R_{\alpha\nu} \tag{C.13}$$

Appendix C.2. Variational formulation

Equations of the gravitational field are obtained from the principle of least action $\delta(S_g + S_m) = 0$, where S_g and S_m are the actions of the gravitational field and matter field respectively. The action for the gravitational field is defined by

$$S_g = -A_g \int g^{\alpha\beta} R_{\alpha\beta}\sqrt{-g}\,\mathrm{d}\Omega, \qquad A_g \equiv \frac{c^3}{16\pi G_0}, \qquad \mathrm{d}\Omega = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3, \tag{C.14}$$

where $\sqrt{-g}\,\mathrm{d}\Omega$ is the proper volume $[\mathrm{d}\Omega]_{prop}$ in a local Lorentz frame with $g = \det(g_{\mu\nu})$, and $R_{\alpha\beta}$ is the Ricci curvature tensor (C.11), and $g^{\alpha\beta}R_{\alpha\beta} = R^\alpha_\alpha \equiv R_{\rm sc}$ is the scalar curvature. The variation of S_g with respect to the metric field $g^{\alpha\beta}$ is given by

$$\delta S_g = -A_g \int \left(R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta} R^\nu_\nu\right)\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}\Omega. \tag{C.15}$$

On the other hand, the action S_m of the matter field is

$$S_m = \frac{1}{c}\int \Lambda_m\Big(q, \frac{\partial q}{\partial x^\nu}\Big)\sqrt{-g}\,\mathrm{d}\Omega. \tag{C.16}$$

where the Lagrangian density Λ_m contains only the tensors $q = g_{\alpha\beta}$ and their first derivatives $\partial_\nu q = \partial_\nu g_{\alpha\beta}$. Noting that variation of the coordinate from x^ν to $x^\nu + \xi^\nu$ results in variation of the metric $\delta g^{\alpha\beta}$, we obtain the variation of action S_m given after some analyses as

$$\delta S_m = \frac{1}{2c}\int T_{\alpha\beta}\,\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}\Omega, \tag{C.17}$$

(Landau & Lifshitz (1975) Eq.(94.5)), where $T_{\alpha\beta}$ is the stress-energy tensor defined by

$$\tfrac{1}{2}\sqrt{-g}\,T_{\alpha\beta} = \frac{\partial\sqrt{-g}\,\Lambda}{\partial q} - \frac{\partial}{\partial x^\nu}\frac{\partial\sqrt{-g}\,\Lambda}{\partial(\partial_\nu q)}, \qquad q \equiv g^{\alpha\beta}. \tag{C.18}$$

From the action principle $\delta(S_g + S_m) = 0$, we find

$$-A_g\int\Big(R_{\alpha\beta} - \tfrac{1}{2}\,g_{\alpha\beta}R_{\rm sc} - 8\pi k\,T_{\alpha\beta}\Big)\,\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}\Omega = 0,$$

where $k = G_0/c^4$. In view of the arbitrariness of the $\delta g^{\alpha\beta}$, we obtain the Einstein field equation:

$$G_{\alpha\beta} = 8\pi k\,T_{\alpha\beta}\,, \qquad k = G_0/c^4, \qquad G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2}\,g_{\alpha\beta}R^\nu{}_\nu. \tag{C.19}$$

where $G_{\alpha\beta}$ is the Einstein curvature tensor.

Appendix C.3. Bianchi identity

The Bianchi identity is deeply rooted in geometrical structure of physical fields. But superficially, it is just expressed by a linear combination of three terms, each of which is given by covariant-derivative of a component of Riemann curvature tensor:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0. \tag{C.20}$$

This can be easily verified in the local Lorentz frame by using the representation obtained from (C.7) with all Γ's (but not derivatives) set to 0. The equation thus obtained is the identity like (C.20) but the ";"-operator replaced by ",". Namely, the equation is verified only for the local Lorentz frame. Finally, transition to any frame of curved spacetime can be done just by replacing "comma" by "semicolon".

Physical significance of the Bianchi identity

From the viewpoint of physics, the set of curvature tensors $R_{\alpha\beta\mu\nu}$ has a remarkable geometrical property, and surprisingly shows a striking analogy to the electromagnetic field. First we spotlight the relevant part of *Electromagnetic field*

In terms of the electromagnetic four-potentials A_μ, one-form $\mathcal{A} = A_\mu \mathrm{d}x^\mu$ was defined (see §2.1 (a)). Out of this one-form, a two-form $\mathcal{F} = \mathrm{d}\mathcal{A}$ is derived by taking its exterior differentiation $\mathrm{d}\mathcal{A}$. The two-form field $\mathcal{F}$ satisfies the identity $\mathrm{d}\mathcal{F} \equiv 0$, because $\mathrm{d}^2\mathcal{A} \equiv 0$, *i.e.* $\partial\,\partial = 0$ by the language of differential geometry, in other words by the principle "*boundary of a boundary is zero*". This yields the identity equation (2.5): $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$, giving rise to a pair of Maxwell equations of (2.7). The last can be rewritten as

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$F_{\alpha\beta\,,\gamma} + F_{\beta\gamma\,,\alpha} + F_{\gamma\alpha\,,\beta} = 0, \quad \text{in short} \quad F_{[\alpha\beta,\gamma]} = 0. \tag{C.21}$$

The symbol $[\alpha\beta,\gamma]$ denotes cyclic permutation of the parameters of three anti-symmetric pairs $[\alpha\beta]$, $[\beta\gamma]$ and $[\gamma\alpha]$. It is amazing to find that the equation (C.20) can be written analogously as

$$R_{\alpha\beta[\mu\nu;\lambda]} = 0. \tag{C.22}$$

This is a startling *coincidence*. In fact, there exists a common structure in their backgrounds, which is now highlighted.

Using the exterior derivative D defined by (3.16), the vector-valued one-form $\mathrm{D}\boldsymbol{v}$ is

$$\mathrm{D}\,\boldsymbol{v} = \boldsymbol{e}_\mu \left(\frac{\mathrm{D}v^\mu}{\mathrm{d}x^\beta} + \Gamma^\mu_{\alpha\beta} v^\alpha \right) \mathrm{d}x^\beta\,. \tag{C.23}$$

Now differentiate this once again to get $\mathrm{D}^2\boldsymbol{v}$:

$$\mathrm{D}^2\boldsymbol{v} = \boldsymbol{e}_\mu\, \mathcal{R}^\mu{}_\nu\, v^\nu, \tag{C.24}$$

(Misner *et al.* (2017), §14.5, eq.(14.17)), where $\mathcal{R}^\mu{}_\nu$ is the curvature 2-form defined by

$$\mathcal{R}^\alpha{}_\beta \equiv \mathrm{d}(\Gamma^\alpha{}_{\beta\nu}\mathrm{d}x^\nu) + \Gamma^\alpha{}_{\lambda\mu}\,\Gamma^\lambda{}_{\beta\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{C.25}$$

$$= R^\alpha{}_{\beta\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \quad (\mu < \nu).$$

where the summation of the last line is taken over μ, ν with $\mu < \nu$, and $R^\alpha{}_{\beta\mu\nu}$ is the Riemann curvature tensor of (C.5).

In order to take our last step, we consider the curvature two-form $\mathcal{R}^\alpha{}_\beta$ in the local Lorentz frame where the second term of (C.25) drops as is done in the proof of Bianchi-id. Then we have $\mathcal{R}^\alpha{}_\beta = \mathrm{d}(\Gamma^\alpha{}_{\beta\nu}\mathrm{d}x^\nu)$. Taking exterior differentiation again, we obtain

$$0 = \mathrm{d}\mathcal{R}^\alpha{}_\beta = \mathrm{d}^2(\Gamma^\alpha{}_{\beta\nu}\mathrm{d}x^\nu) = R^\alpha{}_{\beta\mu\nu,\lambda}\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu,$$

because $\mathrm{d}^2 = 0$. From this we find, with cyclic permutation of (λ, μ, ν):

$$R^\alpha{}_{\beta\mu\nu,\lambda} + R^\alpha{}_{\beta\lambda\mu,\nu} + R^\alpha{}_{\beta\nu\lambda,\mu} = 0.$$

in the local Lorentz frame. Final transition to any frame of curved spacetime can be done by replacing "comma" by "semicolon", obtaining the Bianchi identity (C.20).

Appendix D. Second Pair of Maxwell Equations

Second pair of Maxwell equations (2.8) for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ can be derived from the action principle. The total action $S^{(\mathrm{em})}$ is expressed as $S^{(\mathrm{em})} = S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}$, where $S^{(\mathrm{em})}_{\mathrm{emA}}$ is represented with a free-field Lagrangian of Lorentz-invariant quadrutic form of the field strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $S^{(\mathrm{em})}_{\mathrm{int}}$ represents interaction between the field and 4-current $j_e^{\,\nu}$, defined by

$$S^{(\mathrm{em})}_{\mathrm{emA}} = -\frac{1}{16\pi c}\int F_{\mu\nu}\,F^{\mu\nu}\mathrm{d}\Omega, \qquad S^{(\mathrm{em})}_{\mathrm{int}} = \frac{1}{c^2}\int j_e^{\,\nu}\,A_\nu\,\mathrm{d}\Omega, \tag{D.1}$$

with $\mathrm{d}\Omega = c\,\mathrm{d}t\,\mathrm{d}\mathcal{V}$, and $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$. We vary only the A_ν (serving as the coordinates) with the material 4-current $j_e^{\,\nu}$ assumed given (Landau & Lifshitz, 1975).

Thus, we have the action variation caused by the variation of A_ν:

$$\delta S^{(\mathrm{em})} = \frac{1}{c}\int \Big(\frac{1}{c}\, j^\nu\, \delta A_\nu - \frac{1}{8\pi} F^{\nu\lambda}\, \delta F_{\nu\lambda}\Big)\, \mathrm{d}\Omega = 0, \tag{D.2}$$

where we used the equality $F_{\nu\lambda}\, \delta F^{\nu\lambda} = F^{\nu\lambda}\, \delta F_{\nu\lambda}$. In $S^{(\mathrm{em})}_{\mathrm{int}}$, we must not vary j^ν which is a material current, not the field. Substituting $F_{\nu\lambda} = \partial A_\lambda/\partial x^\nu - \partial A_\nu/\partial x^\lambda$, we have

$$\delta S = \frac{1}{c}\int \Big(\frac{1}{c}\, j^\nu\, \delta A_\nu - \frac{1}{8\pi} F^{\nu\lambda}\, \frac{\partial}{\partial x^\nu}\delta A_\lambda + \frac{1}{8\pi} F^{\nu\lambda}\, \frac{\partial}{\partial x^\lambda}\delta A_\nu\Big)\, \mathrm{d}\Omega,$$

We interchange the indices ν and λ in the middle term. Using the antisymmetry of the matrix $F^{\lambda\nu}$, one can replace the factor $F^{\lambda\nu}$ by $-F^{\nu\lambda}$. Then we obtain

$$\delta S = \frac{1}{c}\int \Big(\frac{1}{c}\, j^\nu\, \delta A_\nu + \frac{1}{4\pi} F^{\nu\lambda}\, \frac{\partial}{\partial x^\lambda}\delta A_\nu\Big)\, \mathrm{d}\Omega,$$

To the second term, we perform integration by parts. Since the surface integral thus obtained vanishes by the imposed boundary conditions. Thus, the principle of least action leads to

$$\int \Big(\frac{1}{c}\, j^\nu - \frac{1}{4\pi}\, \frac{\partial F^{\nu\lambda}}{\partial x^\lambda}\Big)\delta A_\nu\, \mathrm{d}\Omega = 0. \tag{D.3}$$

Since the variation δA_ν is arbitrary, the coefficient of δA_ν must vanish:

$$\frac{\partial F^{\nu\lambda}}{\partial x^\lambda} = \frac{4\pi}{c}\, j^\nu, \tag{D.4}$$

where $j^\nu = (\rho_e c, \boldsymbol{j}_e)$ with $\boldsymbol{j}_e = \rho_e \boldsymbol{v}$, The field strength tensor $F_{\nu\lambda}$ is defined by (1.9), and its matrix representation by (1.10), while $F^{\nu\lambda}$ is defined by $g^{\nu\alpha}\, F_{\alpha\beta}\, g^{\beta\lambda}$.

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