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Difference Sequence Spaces of Second order Defined by a Sequence of Moduli

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Abstract- In this article we introduce the sequence spaces $c_0(u, \triangle^2, F, p), c(u, \triangle^2, F, p)$ and $\ell_\infty(u, \triangle^2, F, p)$ for $F = (f_k)$ a sequence of moduli, $p = (p_k)$ sequence of positive reals and $u \in U$ the set of all sequences and establish some inclusion relations.

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6. Kizmaz, H. On Certain sequence spaces, *Canad.Math.Bull.*24(1981) 169-176.

Difference Sequence Spaces of Second order Defined by a Sequence of Moduli

Khalid Ebadullah α & Kibreab Gebreselassie σ

Abstract- In this article we introduce the sequence spaces $c_0(u,\triangle^2,F,p)$, $c(u,\triangle^2,F,p)$ and $\ell_\infty(u,\triangle^2,F,p)$ for $F=(f_k)$ a sequence of moduli, $p=(p_k)$ sequence of positive reals and $u\in U$ the set of all sequences and establish some inclusion relations.

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I. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively. We write

$$\omega=\{x=(x_k): x_k\in R \text{ or } C\},$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively.

The following subspaces of ω were first introduced and discussed by Maddox [10-12].

$l(p)=\{x\in\omega:\sum_k|x_k|^{p_k}<\infty\}$

$\ell_\infty(p)=\{x\in\omega:\sup_k|x_k|^{p_k}<\infty\}$

$c(p)=\{x\in\omega:\lim_k|x_k-l|^{p_k}=0,\quad \text{for some } l\in C\}$

$c_0(p)=\{x\in\omega:\lim_k|x_k|^{p_k}=0\}$

where $p=(p_k)$ is a sequence of strictly positive real numbers.

The idea of difference sequence sets

$$X_\triangle=\{x=(x_k)\in\omega:\triangle x=(x_k-x_{k+1})\in X\}$$

where $X=\ell_\infty,\ c$ or c_0 was introduced by Kizmaz [6].

Kizmaz [6] defined the following sequence spaces,

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$$\ell_\infty(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in \ell_\infty\}$$

$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\}$$

$$c_0(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c_0\}$$

where $\triangle x = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

Mikail [14] defined the sequence spaces

$$\ell_\infty(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in \ell_\infty\}$$

$$c(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in c\}$$

$$c_0(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in c_0\}$$

Where $(\triangle^2 x) = (\triangle^2 x_k) = (\triangle x_k - \triangle x_{k+1})$.

The sequence spaces $\ell_\infty(\triangle^2), c(\triangle^2)$ and $c_0(\triangle^2)$ are Banach spaces with the norm

$$||x||_\triangle = |x_1| + |x_2| + ||\triangle^2 x||_\infty.$$

Mikail and Colak [15] defined the sequence spaces

$$\ell_\infty(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in \ell_\infty\}$$

$$c(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in c\}$$

$$c_0(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in c_0\}$$

where $m \in N$,

$$\triangle^0 x = (x_k),$$

$$\triangle x = (x_k - x_{k+1}),$$

$$\triangle^m x = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}),$$

and so that

$$\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} \quad x_{k+i}.$$

Ref

14. Mikail. On some difference sequence spaces, *Dogra-Tr.J.Math.*,17,(1993)18-24.

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and showed that these are Banach spaces with the norm

$$||x||_{\triangle} = \sum_{i=1}^{m} |x_i| + ||\triangle^m x||_{\infty}.$$

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ $(k = 1, 2, 3....)$. Malkowsky[13] defined the following sequence spaces

$$\ell_\infty(u, \triangle) = \{x = (x_k) \in \omega : (u_k \triangle x_k) \in \ell_\infty\}$$

$$c(u, \triangle) = \{x = (x_k) \in \omega : (u_k \triangle x_k) \in c\}$$

$$c_0(u, \triangle) = \{x = (x_k) \in \omega : (u_k \triangle x_k) \in c_0\}$$

where $u \in U$.

The concept of paranorm (see[12]) is closely related to linear metric spaces. It is a generalization of that of absolute value.

Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y \in X$,

(PI) $g(x) = 0 \; if \; x = 0$,

(P2) $g(-x) = g(x)$,

(P3) $g(x + y) \leq g(x) + g(y)$,

(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$, in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space.

The idea of modulus was structured by Nakano[16].

A function f : $[0,\infty) \longrightarrow [0,\infty)$ is called a modulus if

(P1) $f(\mathrm{t}) = 0$ if and only if t = 0,

(P2) f(t+u)$\leq$ f(t)+ f(u) for all t,u$\geq$0,

(P3) f is increasing, and

(P4) f is continuous from the right at zero.

Ruckle [17-19] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}$$

Ref

12. Maddox, I.J. Some properties of paranormed sequence spaces, *J. London. Math.* Soc.1 (1969), 316-322.

This space is an FK space. Ruckle[17-19] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[17-19] proved that, for any modulus f.

$$X(f) \subset l_1 \text{ and } X(f)^\alpha = \ell_\infty$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in[4]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by Garling[2-3], Köthe[9] and Ruckle[17-19].

Kolk [7-8] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$$

Khan and Lohani [5] defined the following sequence spaces

$$\ell_\infty(u, \triangle, F) = \{x = (x_k) \in \omega : \sup_{k \geq 0} f_k(|u_k \triangle x_k|) < \infty\}$$

$$c(u, \triangle, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k \triangle x_k - l|) = 0, l \in C\}$$

$$c_0(u, \triangle, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k \triangle x_k|) = 0\}$$

where $u \in U$.

If we take x_k instead of $\triangle x$,then we have the following sequence spaces

$$\ell_\infty(u, F) = \{x = (x_k) \in \omega : \sup_{k \geq 0} f_k(|u_k x_k|) < \infty\}$$

$$c(u, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k x_k - l|) = 0, l \in C\}$$

$$c_0(u, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k x_k|) = 0\}$$

Ref

9. Köthe, G. *Topological Vector spaces*.1.(Springer,Berlin,1970.)

where $u \in U$.

Asma and Colak[1] defined the following sequence spaces

$$\ell_\infty(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|)^{p_k} \in \ell_\infty(p)\}$$

$$c(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|)^{p_k} \in c(p)\}$$

$$c_0(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|)^{p_k} \in c_0(p)\}$$

where $u \in U$, $p = (p_k)$ be any sequence of positive reals.

Khan and Lohani [5] defined the following sequence spaces

$$\ell_\infty(u, \triangle, F, p) = \{x = (x_k) \in \omega : \sup_{k \geq 0}(f_k(|u_k \triangle x_k|))^{p_k} < \infty\}$$

$$c(u, \triangle, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty}(f_k(|u_k \triangle x_k - l|))^{p_k} = 0, l \in C\}$$

$$c_0(u, \triangle, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty}(f_k(|u_k \triangle x_k|))^{p_k} = 0\}$$

which are paranormed spaces paranormed with

$$Q(x) = \sup_{k \geq 0}(f_k(|u_k \triangle x_k|))^{p_k})^{\frac{1}{H}} \leq a$$

where $H = max(1, \sup_{k \geq 0} p_k)$ and $a = f_k(l)$, $l = \sup_{k \geq 0}(|u_k \triangle x_k|)$.

II. Main Results

In this article we introduce the following class of sequence spaces.

$$\ell_\infty(u, \triangle^2, F, p) = \{x = (x_k) \in \omega : \sup_{k \geq 0}(f_k(|u_k \triangle^2 x_k|))^{p_k} < \infty\}$$

$$c(u, \triangle^2, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty}(f_k(|u_k \triangle^2 x_k - l|))^{p_k} = 0, l \in C\}$$

$$c_0(u, \triangle^2, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty}(f_k(|u_k \triangle^2 x_k|))^{p_k} = 0\}$$

Theorem 2.1. $\ell_\infty(u, \triangle^2, F)$ is a Banach space with norm

$$||x||_{\triangle^2} = \sup_{k \geq 0}(f_k(|u_k \triangle^2 x_k|)) \leq \alpha,$$

where $\alpha = f_k(l)$ and $l = \sup_{k \geq 0}(|u_k \triangle^2 x_k|)$.

Ref

1. Asma, C., Colak, R. On the Köthe-Toeplitz duals of generalised sets of difference sequences. *Demonstratio Math;* 33(4), (2000)797-803.

Proof. Let(x^i) be a cauchy sequence in $\ell_\infty(u, \triangle^2, F)$ for each $i \in N$.
Let r, x_0 be fixed.Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer N such that

$$||x^i - x^j||_{\triangle^2} < \frac{\epsilon}{rx_0} \quad \text{for all i, j} \geq N$$

Using the definition of norm, we get

$$\sup_{k\geq 0} f_k(\frac{|u_k(\triangle^2 x^i_k - \triangle^2 x^j_k)|}{||x^i - x^j||_{\triangle^2}}) \leq \alpha, \quad \text{for all i, j} \geq N$$

ie,

$$f_k(\frac{|u_k(\triangle^2 x^i_k - \triangle^2 x^j_k)|}{||x^i - x^j||_{\triangle^2}}) \leq \alpha, \quad \text{for all i, j} \geq N$$

Hence we can find $r > 0$ with $f_k(\frac{rx_0}{2}) \geq \alpha$ such that

$$f_k(\frac{|u_k(\triangle^2 x^i_k - \triangle^2 x^j_k)|}{||x^i - x^j||_{\triangle^2}}) \leq f_k(\frac{rx_0}{2})$$

$$\frac{|u_k(\triangle^2 x^i_k - \triangle^2 x^j_k)|}{||x^i - x^j||_{\triangle^2}} \leq \frac{rx_0}{2}$$

This implies that

$$|u_k(\triangle^2 x^i_k - \triangle^2 x^j_k)| \leq \frac{rx_0}{2}\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$$

Since $u_k \neq 0$ for all k, we have

$$|\triangle^2 x^i_k - \triangle^2 x^j_k| \leq \frac{\epsilon}{2} \quad \text{for all i, j} \geq N$$

Hence $(\triangle^2 x^i_k)$ is a cauchy sequence in R.
For each $\epsilon > 0$ there exists a positive integer N such that $|\triangle^2 x^i_k - \triangle^2 x_k| < \epsilon$ for all $i > N$.
Using the continuity of $F = (f_k)$ we can show that

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x^i_k - \lim_{j\to\infty} \triangle^2 x^j_k)|) \leq \alpha,$$

Thus

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x^i_k - \triangle^2 x_k)|) \leq \alpha,$$

since $(x^i) \in \ell_\infty(u, \triangle^2, F)$ and $F = (f_k)$ is continuous it follows that $x \in \ell_\infty(u, \triangle^2, F)$
Thus $\ell_\infty(u, \triangle^2, F)$ is complete.

Notes

Theorem 2.2. $\ell_\infty(u, \triangle^2, F, p)$ is a complete paranormed space with

$$Q_u(x) = \sup_{k\geq 0}(f_k(|u_k\triangle^2 x_k|)^{p_k})^{\frac{1}{H}} \leq \alpha$$

where $H = max(1, \sup_{k\geq 0} p_k)$ and $\alpha = f_k(l)$, $l = \sup_{k\geq 0}(|u_k\triangle^2 x_k|)$.

Notes

Proof. Let (x^i) be a cauchy sequence in $\ell_\infty(u, \triangle^2, F, p)$ for each $i \in N$.
Let $r > 0, x_0$ be fixed.Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer N such that

$$Q_u(x^i - x^j)_{\triangle^2} < \frac{\epsilon}{rx_0} \quad \text{for all i, j} \geq N$$

Using the definition of paranorm, we get

$$\sup_{k\geq 0} f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}})^{\frac{p_k}{H}} \leq \alpha, \quad \text{for all i, j} \geq N$$

ie,

$$f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}})^{p_k} \leq \alpha, \quad \text{for all i, j} \geq N$$

Hence we can find $r > 0$ with $f_k(\frac{rx_0}{2}) \geq \alpha$ such that

$$f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}}) \leq f_k(\frac{rx_0}{2})$$

$$\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}} \leq \frac{rx_0}{2}$$

This implies that

$$|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)| \leq \frac{rx_0}{2}\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$$

Since $u_k \neq 0$ for all k, we have

$$|\triangle^2 x_k^i - \triangle^2 x_k^j| \leq \frac{\epsilon}{2} \quad \text{for all i, j} \geq N$$

Hence $(\triangle^2 x_k^i)$ is a cauchy sequence in R.
For each $\epsilon > 0$ there exists a positive integer N such that $|\triangle^2 x_k^i - \triangle^2 x_k| < \epsilon$ for all $i > N$.
Using the continuity of $F = (f_k)$ we can show that

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x_k^i - \lim_{j\to\infty} \triangle^2 x_k^j)|)^{\frac{p_k}{H}} \leq \alpha,$$

Thus

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x_k^i - \triangle^2 x_k)|)^{\frac{p_k}{H}} \leq \alpha,$$

since $(x^i) \in \ell_\infty(u, \triangle^2, F, p)$ and $F = (f_k)$ is continuous it follows that $x \in \ell_\infty(u, \triangle^2, F, p)$ Thus $\ell_\infty(u, \triangle^2, F, p)$ is complete.

Notes

Theorem 2.3. Let $0 < p_k \leq q_k < \infty$ for each k. Then we have

$$c_0(u, \triangle^2, F, p) \subseteq c_0(u, \triangle^2, F, q)$$

Proof. Let $x \in c_0(u, \triangle^2, F, p)$ that is

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

This implies that

$$f_k(|u_k(\triangle^2 x_k)|) \leq 1$$

for sufficiently large k, since modulus function is non decreasing.
Hence we get

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{q_k} \leq \lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

Therefore $x \in c_0(u, \triangle^2, F, q)$.

Theorem 2.4. (a) Let $0 < \inf p_k \leq p_k \leq 1$. Then we have

$$c_0(u, \triangle^2, F, p) \subseteq c_0(u, \triangle^2, F).$$

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then we have

$$c_0(u, \triangle^2, F) \subseteq c_0(u, \triangle^2, F, p).$$

Proof. (a) Let $x \in c_0(u, \triangle^2, F, p)$, that is

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|)) \leq \lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

Notes

Hence $x \in c_0(u, \triangle^2, F)$.

(b) Let $p_k \geq 1$ for each k and $\sup_k p_k < \infty$.

Suppose that $x \in c_0(u, \triangle^2, F)$.

Then for each $\epsilon > 0$ there exists a positive integer N such that

$$f_k(|u_k(\triangle^2 x_k)|) \leq \epsilon \quad \text{for all } k \geq N$$

Since $1 \leq p_k \leq \sup_k p_k < \infty$, we have

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} \leq \lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|)) \leq \epsilon < 1$$

Therefore $x \in c_0(u, \triangle^2, F, p)$.

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