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Global Existence and Intrinsic Decay Rates for the Energy of a Kirchhoff Type in a Nonlinear Viscoelastic Equation

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GLOBALEXISTENCEANDINTRINSICDECAYRATESFORTHEENERGYOFAKIRCHHOFFTYPEINANONLINEARVISCOELASTICEQUATION

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Draifia Alaeddine

Abstract- In this work we consider a nonlinear hyperbolic equations of Kirch-hoff type in viscoelasticity. By using the potential well theory we obtain the existence of a global solution. Then, we prove the intrinsic decays for the energy of the nonlinear hyperbolic equations of Kirchhoff type in viscoelasticity of relaxation kernels described by the inequality $h'(t) \leq -H(h(t))$ for all $t \geq 0$, with *H* convex.

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I. Introduction

In this works, we study the global existence and intrinsic decay rates for the energy of a kirchhoff type in a nonlinear viscoelastic equation

$$\begin{cases} u_{tt}(x,t) - \Phi(x)\left[\mu\left(\|\nabla u(t)\|^2_{L^2(\Omega)}\right)\Delta u(x,t) - \int_0^t h(t-s)\,\Delta u(x,s)ds\right] \\ +bu_t(x,t) = 0, \quad x \in \Omega \times \mathbb{R}_+^*, \end{cases} \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $h(.) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions which will be spaced later and $u_0(x)$, $u_1(x)$ are given initial data belonging to appropriate space. All the parameter b are assumed to be positive constants. The function $\Phi(x)$ is the density, $(\rho(x))^{-1} = \Phi(x)$, $\Phi(x) > 0$, for all $x \in \Omega$, and

$$\mu(s) := \xi_0 + \xi_1 s^\gamma,$$

where $s > 0$, $\xi_0 > 0$, $\xi_1 > 0$ and $\gamma \geq 1$. For more information on using Kirchhoff type, see $[4-6]$.

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Also, a result of local existence for problem $(1.1)-(1.3)$ for $\xi_1 = 0$ has been proved in $[1]$, for $\xi_1 \neq 0$, in the same way as [1], we get the same basic results for the local existence of problem $(1.1)-(1.3)$ with a slight change in some calculations that do not affect the basic results.

The motivation of our work is due to some results regarding the following research papers: Boumaza, N and Boulaaras, S. [2] studied the general decay for Kirchhoff type in viscoelasticity with not necessarily decreasing kernel of $(1.1)-(1.3)$. Marcelo M. Irena Lasiecka and Claudete M. Webler. [3] studied the intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka and F. A. Falcao Nascimento. [7] studied the intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects. I. Lasiecka, S. A. Messaoudi and M. I. Mustafa. [8] studied the note on intrinsic decay rates for abstract wave equations with memory. I. Lasiecka and X. Wang. [9] studied the intrinsic decay rate estimates for semilinear abstract second order equations with memory. Cavalcanti M. Filho VND. Cavalcanti JSP. Soriano JA. [10] studied the existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. For more results in this direction, see $[11-15]$.

However, $[1-3]$, $[4-6]$ and $[11-15]$ did not study the intrinsic decay rates for the energy of problem $(1.1)-(1.3)$ of relaxation kernels described by the inequality $h'(t) \leq -H(h(t))$ for all $t \geq 0$, with H convex. Motivated by the above research, we will consider the intrinsic decay rates for the energy of relaxation kernels described by the inequality $h'(t) \leq -H(h(t))$ for all $t \geq 0$ of the model $(1.1)-(1.3)$ in this paper.

The outline of the paper is as follows. In the second section we define the energy $E(t)$ associated to $(1.1)-(1.3)$ and show that it is a non-increasing function of t. In section 3, we prove global existence of solution of $(1.1)-(1.3)$. Finally, in section 4, we prove the intrinsic decay rates for the energy of the posed problem.

II. Assumptions and Main Results

In this section, we define the energy $E(t)$ associated to $(1.1)-(1.3)$ and show that it is a non-increasing function of t. We suppose that the kernel $h(t)$ is a function satisfying

Assumptions 1

The relaxation function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a $C^1 \cap L^1$ decreasing function and satisfies

$$h(0) > 0 \text{ and } \int_0^t h(s)\,ds < \xi_0.$$

Assumptions 2

(**i**) In addition to **Assumption 1**, we require

$$h'(t) \leq -H(h(t)) \text{ for all } t \geq 0,$$

where $H \in C^1(R_+)$ which $H(0) = 0$ is a given strictly increasing and convex function. Moreover,

$$H \in C^2(0,\infty) \text{ and } \lim \inf_{x \to 0^+} \left\{ x^2 H''(x) - xH'(x) + H(x) \right\} \geq 0.$$

(**ii**) With reference to the function H introduced above, let $y(t)$ be the solution of the ODE

$$y'(t) + H(y(t)) = 0, \; y(0) = h(0) > 0.$$

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1. Medjden, M.M., Tatar, N.: Asymptotic behavior for a viscoelastic problem with not necessarily decreasing kernel. Appl. Math. Comput. 67; 1221 - 1235(2005).

(iii) We assume that there exists $\alpha_0 \in [0,1)$ such that $y^{1-\alpha_0} \in L_1(1,\infty)$.
In order to formulate the long-time behavior results, we recall the binary notation

$$\begin{cases} (h * w)(t) := \int_0^t h(t-s)\, w(s)\, ds, \\ \int_\Omega (h \circ w)(t)\, dx := \int_0^t h(t-s) \|w(x,s) - w(x,t)\|_{L^2(\Omega)}^2 ds, \\ (h \diamond w)(t) := \rho(x) \int_0^t h(t-s)(w(t) - w(s))\, ds. \end{cases} \tag{2.1}$$

Notes

We define the corresponding energy functional by

$$\begin{aligned} E(t) \;:&= \frac{1}{2}\|u_t(t)\|_{L^2_\rho(\Omega)}^2 + \frac{1}{2}\left(\xi_0 - \int_0^t h(s)\, ds\right) \|\nabla u(t)\|_{L^2(\Omega)}^2 \\ &+ \frac{\xi_1}{2(\gamma+1)} \|\nabla u(t)\|_{L^2(\Omega)}^{2(\gamma+1)} + \frac{1}{2}\int_\Omega (h \circ \nabla u)(t)\, dx. \end{aligned} \tag{2.2}$$

Note that, in view of (2.1), we have that

$$0 < l := \left(\xi_0 - \int_0^\infty h(s)\, ds\right) \le \xi_0 \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+. \tag{2.3}$$

The energy satisfies the following identity

Lemma 1. We have the identity

$$\begin{aligned} \frac{d}{dt}\{E(t)\} &= \frac{1}{2}\int_\Omega (h' \circ \nabla u)(t)\, dx - \frac{1}{2} h(t) \|\nabla u(t)\|_{L^2(\Omega)}^2 - b\|u_t(t)\|_{L^2_\rho(\Omega)}^2 \\ &\le 0. \end{aligned} \tag{2.4}$$

Proof. Multiplying (1.1) by $\rho(x)\, u_t$ and integration over Ω, we have

$$\begin{aligned} &\int_\Omega \rho(x)\, u_{tt} u_t dx - \int_\Omega \left(\xi_0 + \xi_1 \|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right) \Delta u(x,t) u_t(x,t)\, dx \\ &+ \int_\Omega u_t(x,t) \left[\int_0^t h(t-s)\, \Delta u(x,s)\, ds\right] dx + b\int_\Omega \rho(x)\, u_t^2(x,t) dx \\ &= 0. \end{aligned} \tag{2.5}$$

We have

$$\int_\Omega \rho(x)\, u_{tt} u_t dx = \frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|_{L^2_\rho(\Omega)}^2\right\}. \tag{2.6}$$

And by using integration by parts, we have

$$\begin{aligned}
& -\int_{\Omega}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\Delta u(x,t)u_t\left(x,t\right)dx \\
= & -\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\int_{\Omega}\Delta u(x,t)u_t\left(x,t\right)dx \\
= & \left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\int_{\Omega}\nabla u\left(x,t\right).\nabla u_t\left(x,t\right)dx \\
= & \left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\frac{1}{2}\frac{d}{dt}\left\{\int_{\Omega}\left|\nabla u\left(x,t\right)\right|^2dx\right\} \\
= & \frac{\xi_0}{2}\frac{d}{dt}\left\{\|\nabla u\left(t\right)\|_{L^2(\Omega)}^2\right\}+\frac{\xi_1}{2}\|\nabla u\left(t\right)\|_{L^2(\Omega)}^{2\gamma}\frac{d}{dt}\left\{\|\nabla u\left(t\right)\|_{L^2(\Omega)}^2\right\} \\
= & \frac{\xi_0}{2}\frac{d}{dt}\left\{\|\nabla u\left(t\right)\|_{L^2(\Omega)}^2\right\}+\frac{\xi_1}{2\left(\gamma+1\right)}\frac{d}{dt}\left\{\|\nabla u\left(t\right)\|_{L^2(\Omega)}^{2(\gamma+1)}\right\} \\
= & \frac{d}{dt}\left\{\frac{1}{2}\left(\xi_0+\frac{\xi_1}{\left(\gamma+1\right)}\|\nabla u\left(t\right)\|_{L^2(\Omega)}^{2\gamma}\right)\|\nabla u\left(t\right)\|_{L^2(\Omega)}^2\right\}. \qquad (2.7)
\end{aligned}$$

And by using integration by parts, we have

$$\begin{aligned}
& \int_{\Omega}u_t\left(x,t\right)\left[\int_0^t h\left(t-s\right)\Delta u\left(x,s\right)ds\right]dx \\
= & -\int_0^t h\left(t-s\right)\left[\int_{\Omega}\nabla u\left(x,s\right).\nabla u_t\left(x,t\right)dx\right]ds,
\end{aligned}$$

and using

$$-\nabla u\left(x,s\right).\nabla u_t\left(x,t\right)=\frac{1}{2}\frac{d}{dt}\left\{\left|\nabla u\left(x,s\right)-\nabla u\left(x,t\right)\right|^2\right\}-\frac{1}{2}\frac{d}{dt}\left\{\left|\nabla u\left(x,t\right)\right|^2\right\},$$

then

$$\begin{aligned}
& \int_{\Omega}u_t\left(x,t\right)\left[\int_0^t h\left(t-s\right)\Delta u\left(x,s\right)ds\right]dx \\
= & \int_0^t h\left(t-s\right)\int_{\Omega}\left(\frac{1}{2}\frac{d}{dt}\left\{\left|\nabla u\left(x,s\right)-\nabla u\left(x,t\right)\right|^2\right\}\right)dxds \\
& -\int_0^t h\left(t-s\right)\int_{\Omega}\left(\frac{1}{2}\frac{d}{dt}\left\{\left|\nabla u\left(x,t\right)\right|^2\right\}\right)dxds \\
= & \frac{1}{2}\int_0^t h\left(t-s\right)\left(\frac{d}{dt}\left\{\int_{\Omega}\left|\nabla u\left(x,s\right)-\nabla u\left(x,t\right)\right|^2dx\right\}\right)ds \\
& -\frac{1}{2}\int_0^t h\left(t-s\right)\left(\frac{d}{dt}\left\{\|\nabla u\left(t\right)\|_{L^2(\Omega)}^2\right\}\right)ds, \qquad (2.8)
\end{aligned}$$

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by using (2.1), we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^t h(t-s)\frac{d}{dt}\left\{\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right\}ds \\
=\;& \frac{1}{2}\int_0^t \frac{d}{dt}\left\{h(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)\right\}ds \\
&-\frac{1}{2}\int_0^t h'(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds \\
=\;& \frac{1}{2}\frac{d}{dt}\left\{\int_0^t h(t-s)\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dxds\right\} \\
&-\frac{1}{2}\int_0^t h'(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds \\
=\;& \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx\right\}-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx, \qquad (2.9)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}\int_0^t h(t-s)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\right)ds \\
=\;& -\frac{1}{2}\left(\int_0^t h(t-s)\,ds\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\right) \\
=\;& -\frac{1}{2}\left(\int_0^t h(s)\,ds\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\right) \\
=\;& -\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}. \qquad (2.10)
\end{aligned}
$$

By replacement (2.9) and (2.10) into (2.8), we get

$$
\begin{aligned}
&\int_\Omega u_t(x,t)\left[\int_0^t h(t-s)\Delta u(x,s)\,ds\right]dx \\
=\;& \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx\right\}-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx \\
&-\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)} \\
=\;& \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx-\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\} \\
&-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}. \qquad (2.11)
\end{aligned}
$$

By combining (2.6), (2.7) and (2.11) into (2.5), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2_{L^2_\rho(\Omega)}\right\}\\
&+\frac{d}{dt}\left\{\frac{1}{2}\left(\xi_0+\frac{\xi_1}{(\gamma+1)}\|\nabla u(t)\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\\
&+\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx-\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\\
&-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}+b\|u_t(t)\|^2_{L^2_\rho(\Omega)}\\
=\;&0,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\frac{1}{2}\left(\xi_0-\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right.\\
&\left.+\frac{\xi_1}{2(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)}_{L^2(\Omega)}+\frac{1}{2}\int_\Omega (h\circ\nabla u)(t)\,dx\right\}\\
=\;&\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx-\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}-b\|u_t(t)\|^2_{L^2_\rho(\Omega)}, \qquad (2.12)
\end{aligned}$$

by using (2.2) into (2.12), we get (2.4).
The proof of **Lemma 1** is completes.

III. Global Existence

In this section we show that any solution of (1.1) – (1.3) is bounded and global, provided that $E(0)$ is positive and small enough.

Theorem 1. Assume that (2.3) *holds. Then the solution to problem* (1.1) – (1.3) *is bounded and global.*

Proof. It suffices to show that $\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\|\nabla u(t)\|^2_{L^2(\Omega)}$ is bounded independently of t.
By using (2.3) and (**A1**) into (2.12), we get

$$\omega_1\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\omega_2\|\nabla u(t)\|^2_{L^2(\Omega)}\le E(t)\le E(0),$$

where $\omega_1>0$ and $\omega_2>0$, then

$$\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\|\nabla u(t)\|^2_{L^2(\Omega)}\le\omega_3 E(0),$$

where $\omega_3>0$.
Then the solution to problem (1.1) – (1.3) is bounded and global.
The proof of **Theorem 1** is completes.

IV. Decay of Solutions

Now, we are in a position to state our main result.

Lemma 2. Let us assume that ***Assumption 1*** *and* ***Assumption 2*** *are the place. Then, there exists a positive constant $T_0>0$ such that*

$$E((n+1)T)+\tilde{H}\left(C_9^{-1}E((n+1)T)\right)\le E(nT),\quad n=1,2,3...,$$

for all $T>T_0$ and all $n\in N$, where $\tilde{H}$ is given in (4.49) *and C_9 is given in* (4.52).

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Proof. For this purpose, a by now standard procedure is to multiply (1.1) by the viscoelastic multiplier

$$(h \diamond u)(t) = \rho(x) \int_0^t h(t-s)(u(t)-u(s))\,ds,$$

and integrating over $\Omega \times (nT, (n+1)T)$, we infer that

$$\begin{aligned}
&\int_{nT}^{(n+1)T} (u_{tt}(t), (h \diamond u)(t))_{L^2(\Omega)}\,dt \\
&-\int_{nT}^{(n+1)T} \left(\Phi(x)\left(\xi_0 + \xi_1 \|\nabla u(t)\|_{L^2(\Omega)}^{2\gamma}\right)\Delta u(t), (h \diamond u)(t)\right)_{L^2(\Omega)} dt \\
&+\int_{nT}^{(n+1)T} \left(\Phi(x)\left(\int_0^t h(t-s)\,\Delta u(s)\,ds\right), (h \diamond u)(t)\right)_{L^2(\Omega)} dt \\
&+b\int_{nT}^{(n+1)T} (u_t(t), (h \diamond u)(t))_{L^2(\Omega)}\,dt \\
&= \ 0. \qquad (4.1)
\end{aligned}$$

We shall analyze the above terms separately.
Direct calculation give

$$\begin{aligned}
&(u_{tt}(t), (h \diamond u)(t))_{L^2(\Omega)} \\
= \ & \frac{d}{dt}\left\{\left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\right\} \\
&-\left(u_t(t), \frac{d}{dt}\left(\int_0^t h(t-s)(u(t)-u(s))\,ds\right)\right)_{L^2_\rho(\Omega)} \\
= \ & \frac{d}{dt}\left\{\left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\right\} \\
&-\left(u_t(t), \left(\int_0^t h'(t-s)(u(t)-u(s))\,ds\right)\right)_{L^2_\rho(\Omega)} \\
&-\left(u_t(t), \left(\int_0^t h(t-s)\,u_t(t)\,ds\right)\right)_{L^2_\rho(\Omega)},
\end{aligned}$$

then

$$\begin{aligned}
&\int_{nT}^{(n+1)T} (u_{tt}(t), (h \diamond u)(t))_{L^2(\Omega)}\,dt \\
= \ & \left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\Bigg|_{nT}^{(n+1)T} \\
&-\int_{nT}^{(n+1)T} \left(u_t(t), \int_0^t h'(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)} dt \\
&-\int_{nT}^{(n+1)T} \left(\int_0^t h(s)\,ds\right)\|u_t(x,t)\|_{L^2_\rho(\Omega)}^2\,dt. \qquad (4.2)
\end{aligned}$$

For the second term, by using integration by parts, we have

$$\begin{aligned}
& -\int_{nT}^{(n+1)T}\left(\Phi(x)\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\Delta u(t),(h\diamond u)(t)\right)_{L^2(\Omega)}dt \\
= & -\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\left(\Delta u(t),\int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2(\Omega)}dt \\
= & \int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\left(\nabla u(t),\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt.
\end{aligned} \tag{4.3}$$

For the third term, by using integration by parts, we have

$$\begin{aligned}
& \int_{nT}^{(n+1)T}\left(\Phi(x)\int_0^t h(t-s)\Delta u(s)\,ds,(h\diamond u)(t)\right)_{L^2(\Omega)}dt \\
= & -\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(s)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt \\
= & \int_{nT}^{(n+1)T}\left\|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|_{L^2(\Omega)}^2dt \\
& -\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(t)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt.
\end{aligned} \tag{4.4}$$

Combining $(4.2)-(4.4)$ into (4.1), we arrive at

$$\begin{aligned}
& \left(u_t(t),\int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\Bigg|_{nT}^{(n+1)T} \\
& -\int_{nT}^{(n+1)T}\left(u_t(t),\int_0^t h'(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}dt \\
& -\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|u_t(x,t)\|_{L^2_\rho(\Omega)}^2dt \\
& +\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\left(\nabla u(t),\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt \\
& +\int_{nT}^{(n+1)T}\left\|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|_{L^2(\Omega)}^2dt \\
& -\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(t)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt \\
& +b\int_{nT}^{(n+1)T}(u_t(t),(h\diamond u)(t))_{L^2(\Omega)}dt \\
= & \ 0,
\end{aligned} \tag{4.5}$$

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then (4.5) is equivalent

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
=\ &\left.\left(u_t(t),\int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\right|_{nT}^{(n+1)T} \\
&-\int_{nT}^{(n+1)T}\left(u_t(t),\int_0^t h'(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}dt \\
&+\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\left(\nabla u(t),\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt \\
&+\int_{nT}^{(n+1)T}\left\|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|^2_{L^2(\Omega)}dt \\
&-\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(t)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt \\
&+b\int_{nT}^{(n+1)T}(u_t(t),(h\diamond u)(t))_{L^2(\Omega)}\,dt \\
=\ &J_1+J_2+J_3+J_4+J_5+J_6. \qquad (4.6)
\end{aligned}$$

Estimate for $|J_1|$, where

$$\begin{aligned}
J_1:\ =\ &\left. u_t((n+1)T),\int_0^{(n+1)T} h((n+1)T-s)(u((n+1)T)-u(s))\,ds\right)_{L^2_\rho(\Omega)} \\
-\ &\left. u_t(nT),\int_0^{nT} h(nT-s)(u(nT)-u(s))\,ds\right)_{L^2_\rho(\Omega)}.
\end{aligned}$$

Now, let $m\in N$ be an arbitrary, natural number.
By using Young's inequality (for $\varepsilon=1$), we get

$$\begin{aligned}
&\left. u_t(mT),\int_0^{mT} h(mT-s)(u(mT)-u(s))\,ds\right)_{L^2_\rho(\Omega)} \\
=\ &\int_0^{mT} h(mT-s)(u_t(mT),(u(mT)-u(s)))_{L^2_\rho(\Omega)}\,ds \\
\le\ &\int_0^{mT} h(mT-s)\left[\frac{1}{2}\|u_t(mT)\|^2_{L^2_\rho(\Omega)}+\frac{1}{2}\|u(mT)-u(s)\|^2_{L^2_\rho(\Omega)}\right]ds \\
=\ &\frac{1}{2}\left.\int_0^{mT} h(mT-s)\,ds\right)\|u_t(mT)\|^2_{L^2_\rho(\Omega)} \\
&+\frac{1}{2}\int_0^{mT} h(mT-s)\|u(mT)-u(s)\|^2_{L^2_\rho(\Omega)}\,ds, \qquad (4.7)
\end{aligned}$$

by using

$$\|u(t)\|^2_{L^2_\rho(\Omega)}\le\|\rho\|^2_{L^2(\Omega)}\|\nabla u(t)\|^2_{L^2_\rho(\Omega)}, \qquad (4.8)$$

we get

$$\begin{aligned}
&\frac{1}{2}\int_0^{mT} h\left(mT-s\right)\left\|u(mT)-u\left(s\right)\right\|_{L^2_\rho(\Omega)}^2 ds \\
\leq\ & \frac{1}{2}\left\|\rho\right\|_{L^2(\Omega)}^2 \int_0^{mT} h\left(mT-s\right)\left\|\nabla u(mT)-\nabla u\left(s\right)\right\|_{L^2(\Omega)}^2 ds \\
=\ & \frac{1}{2}\left\|\rho\right\|_{L^2(\Omega)}^2 \int_\Omega \left(h\circ\nabla u\right)\left(mT\right)dx, \qquad (4.9)
\end{aligned}$$

by replacement (4.9) into (4.7) and using $\int_0^{mT} h\left(mT-s\right)ds = \int_0^{mT} h\left(s\right)ds$, we get

$$\begin{aligned}
&\left(u_t\left(mT\right), \int_0^{mT} h\left(mT-s\right)\left(u\left(mT\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(\Omega)} \\
\leq\ & \frac{1}{2}\left(\int_0^{mT} h\left(s\right)ds\right)\left\|u_t(mT)\right\|_{L^2_\rho(\Omega)}^2 \\
&+\frac{1}{2}\left\|\rho\right\|_{L^2(\Omega)}^2 \int_\Omega \left(h\circ\nabla u\right)\left(mT\right)dx, \qquad (4.10)
\end{aligned}$$

by using (2.2), we get

$$\begin{cases}
\frac{1}{2}\left\|u_t(mT)\right\|_{L^2_\rho(\Omega)}^2 \leq E\left(mT\right), \\
\text{and} \\
\frac{1}{2}\int_\Omega \left(h\circ\nabla u\right)\left(mT\right)dx \leq E\left(mT\right),
\end{cases} \qquad (4.11)$$

then, by combining (4.11) into (4.10), we get

$$\begin{aligned}
&\left(u_t\left(mT\right), \int_0^{mT} h\left(mT-s\right)\left(u\left(mT\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(\Omega)} \\
\leq\ & \left(\int_0^{mT} h\left(s\right)ds\right)E\left(mT\right)+\left\|\rho\right\|_{L^2(\Omega)}^2 E\left(mT\right) \\
\leq\ & \left\{\left\|h\right\|_{L^1(0,\infty)}+\left\|\rho\right\|_{L^2(\Omega)}^2\right\}E\left(mT\right),
\end{aligned}$$

then

$$\left|J_1\right| \leq C_1\left[E((n+1)T)+E(nT)\right], \qquad (4.12)$$

where

$$C_1 := \left\|h\right\|_{L^1(0,\infty)}+\left\|\rho\right\|_{L^2(\Omega)}^2 .$$

Estimate for $\left|J_2\right|$, where

$$J_2 := -\int_{nT}^{(n+1)T}\left(u_t\left(t\right), \int_0^t h'\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(\Omega)} dt.$$

Notes

Notes

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_1}{2}\right)$, we get

$$\begin{aligned}
|J_2| \quad &\leq \quad \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\, dt \\
&\quad + \frac{1}{4\varepsilon_1} \int_{nT}^{(n+1)T} \left\| \int_0^t h'(t-s)\,(u(t)-u(s))\, ds \right\|^2_{L^2_\rho(\Omega)} dt. \qquad (4.13)
\end{aligned}$$

By using (4.8), Cauchy-Schwarz inequality and (2.1), we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left\| \int_0^t h'(t-s)\,(u(t)-u(s))\, ds \right\|^2_{L^2_\rho(\Omega)} dt \\
\leq\ & \|\rho\|^2_{L^2(\Omega)} \int_{nT}^{(n+1)T} \left\| \int_0^t h'(t-s)\,(\nabla u(t)-\nabla u(s))\, ds \right\|^2_{L^2(\Omega)} dt \\
\leq\ & -\|\rho\|^2_{L^2(\Omega)}\, h(0) \int_{nT}^{(n+1)T} \int_0^t h'(t-s)\, \|\nabla u(t)-\nabla u(s)\|^2_{L^2(\Omega)}\, ds dt \\
=\ & -\|\rho\|^2_{L^2(\Omega)}\, h(0) \int_{nT}^{(n+1)T} \int_\Omega (h' \circ \nabla u)(t)\, dx dt. \qquad (4.14)
\end{aligned}$$

By replacement (4.14) into (4.13), we get

$$\begin{aligned}
|J_2| \quad &\leq \quad \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\, dt \\
&\quad - \frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)}\, h(0) \int_{nT}^{(n+1)T} \int_\Omega (h' \circ \nabla u)(t)\, dx dt. \qquad (4.15)
\end{aligned}$$

Estimate $|J_3|$, where

$$J_3 := \int_{nT}^{(n+1)T} \left(\xi_0 + \xi_1 \|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right) \left(\nabla u(t), \int_0^t h(t-s)\,(\nabla u(t)-\nabla u(s))\, ds\right)_{L^2(\Omega)} dt.$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_2}{2}\right)$, we get

$$\begin{aligned}
|J_3| \quad &\leq \quad \varepsilon_2 \int_{nT}^{(n+1)T} \left(\xi_0 + \xi_1 \|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)^2 \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt \\
&\quad + \frac{1}{4\varepsilon_2} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)\,(\nabla u(t)-\nabla u(s))\, ds \right\|^2_{L^2(\Omega)} dt, \qquad (4.16)
\end{aligned}$$

by using $\|\nabla u(t)\|^{2\gamma}_{L^2(\Omega)} \leq \left(\frac{2(\gamma+1)}{\xi_1} E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}}$, we get

$$\begin{aligned}
&\varepsilon_2 \int_{nT}^{(n+1)T} \left(\xi_0 + \xi_1 \|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)^2 \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt \\
\leq\ & \varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1} E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}} \Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt, \qquad (4.17)
\end{aligned}$$

by using Cauchy-Schwarz inequality and (2.1), we get

$$\left\| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2$$

$$\leq \left(\int_0^t h(t-s)\, ds \right) \int_0^t h(t-s) \|\nabla u(t) - \nabla u(s)\|_{L^2(\Omega)}^2\, ds$$

$$\leq \|h\|_{L^1(0,\infty)} \int_\Omega (h \circ \nabla u)(t)\, dx, \tag{4.18}$$

by replacement (4.17) and (4.18) into (4.16), we get

$$|J_3| \leq \varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1} E(0) \right)^{\frac{2\gamma}{2(\gamma+1)}} \Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|_{L^2(\Omega)}^2\, dt$$

$$+ \frac{1}{4\varepsilon_2} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dx dt. \tag{4.19}$$

Estimate $|J_4|$, where

$$J_4 := \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2 dt.$$

By using (4.18), we get

$$|J_4| \leq \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dx dt. \tag{4.20}$$

Now, estimate $|J_5|$, where

$$J_5 := -\int_{nT}^{(n+1)T} \left(\int_0^t h(t-s) \nabla u(t)\, ds, \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right)_{L^2(\Omega)} dt.$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_3}{2}\right)$, Cauchy-Schwarz inequality and (4.18), we get

$$|J_5| \leq \varepsilon_3 \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s) \nabla u(t)\, ds \right\|_{L^2(\Omega)}^2 dt$$

$$+ \frac{1}{4\varepsilon_3} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2 dt$$

Notes

$$\leq \varepsilon_3 \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t h(t-s) \|\nabla u(t)\|^2_{L^2(\Omega)} dsdt$$

$$+\frac{1}{4\varepsilon_3} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt$$

$$= \varepsilon_3 \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds\right) \|\nabla u(t)\|^2_{L^2(\Omega)} dt$$

$$+\frac{1}{4\varepsilon_3} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt. \tag{4.21}$$

Now, estimate $|J_6|$, where

$$J_6 \; : \; = b \int_{nT}^{(n+1)T} (u_t(t), (h \diamond u)(t))_{L^2(\Omega)}\, dt$$

$$: \; = b \int_{nT}^{(n+1)T} \left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\, ds\right)_{L^2_\rho(\Omega)} dt.$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_1}{2}\right)$, (4.8) and (4.18), we get

$$|J_6| \leq b^2 \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)} dt$$

$$+\frac{1}{4\varepsilon_1} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(u(t)-u(s))\, ds \right\|^2_{L^2_\rho(\Omega)} dt$$

$$\leq b^2 \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)} dt$$

$$+\frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\, ds \right\|^2_{L^2(\Omega)} dt$$

$$\leq b^2 \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)} dt$$

$$+\frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt. \tag{4.22}$$

Combining (4.12), (4.15) and (4.19)−(4.22) into (4.6), and recalling that $\|h\|_{L^1(0,\infty)} < \xi_0$, we write

$$\int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds\right) \|u_t(x,t)\|^2_{L^2_\rho(\Omega)} dt$$

$$\leq C_1 [E((n+1)T) + E(nT)]$$

Notes

$$+\varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\, dt$$

$$-\frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} h(0) \int_{nT}^{(n+1)T} \int_\Omega (h' \circ \nabla u)(t)\, dxdt$$

$$+\varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1} E(0) \right)^{\frac{2\gamma}{2(\gamma+1)}} \Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt$$

$$+\frac{1}{4\varepsilon_2}\xi_0 \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt$$

$$+\xi_0 \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt$$

$$+\varepsilon_3\xi_0 \int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds \right) \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt$$

$$+\frac{1}{4\varepsilon_3}\xi_0 \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt$$

$$+b^2\varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)}\, dt$$

$$+\frac{1}{4\varepsilon_1}\xi_0 \|\rho\|^2_{L^2(\Omega)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt. \tag{4.23}$$

Since $h(0) > 0$, we can select a points $t_1 < T$ with t_1 close to zero such that for all $t \geq t_1$

$$\int_0^t h(s)\, ds \geq t_1 h(t_1) := c_0.$$

Then (4.23) is equivalent

$$\int_{nT}^{(n+1)T} \left\{ t_1 h(t_1) - \varepsilon_1 \left(1 + b^2\right) \right\} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\, dt$$

$$\leq \quad C_1 \left[E((n+1)T) + E(nT) \right]$$

$$-\frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} h(0) \int_{nT}^{(n+1)T} \int_\Omega (h' \circ \nabla u)(t)\, dxdt$$

$$+\varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1} E(0) \right)^{\frac{2\gamma}{2(\gamma+1)}} \Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt$$

$$+\xi_0 \left\{ \frac{1}{4\varepsilon_2} + 1 + \frac{1}{4\varepsilon_3} + \frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} \right\} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt$$

$$+\varepsilon_3\xi_0 \int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds \right) \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt. \tag{4.24}$$

Notes

Notes

Now, multiplying (1.1) by $\rho(x)\,u(x,t)$ and integrating over $\Omega\times(nT,(n+1)T)$, we infer that

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(u_{tt}(t),u(t)\right)_{L^2_\rho(\Omega)}dt\\
&-\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\Delta u(t),u(t)\right)_{L^2(\Omega)}dt\\
&+\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\,\Delta u(s)\,ds,u(t)\right)_{L^2(\Omega)}dt\\
&+b\int_{nT}^{(n+1)T}\left(u_t(x,t),u(x,t)\right)_{L^2_\rho(\Omega)}dt\\
=&\ 0.
\end{aligned}\tag{4.25}$$

By using

$$u_{tt}(t)\,u(t)=\frac{d}{dt}\left\{u_t(t)\,u(t)\right\}-u_t^2(t),$$

we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(u_{tt}(x,t),u(x,t)\right)_{L^2_\rho(\Omega)}dt\\
=&\ \left(u_t(t),u(t)\right)_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T}-\int_{nT}^{(n+1)T}\|u_t(x,t)\|^2_{L^2_\rho(\Omega)}dt.
\end{aligned}\tag{4.26}$$

By using integration by parts, we get

$$\begin{aligned}
&-\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\Delta u(t),u(t)\right)_{L^2(\Omega)}dt\\
=&\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\left(\nabla u(t),\nabla u(t)\right)_{L^2(\Omega)}dt\\
=&\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}dt.
\end{aligned}\tag{4.27}$$

By using integration by parts, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\,\Delta u(s)\,ds,u(t)\right)_{L^2(\Omega)}dt\\
=&-\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
=&\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t)-\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
&-\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
=&\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t)-\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
&-\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}dt.
\end{aligned}\tag{4.28}$$

And

$$\begin{aligned} & b\int_{nT}^{(n+1)T}\left(u_t(x,t),u\left(x,t\right)\right)_{L^2_\rho(\Omega)}dt \\ = \;& \frac{b}{2}\int_{nT}^{(n+1)T}\frac{d}{dt}\left\{\left\|u\left(x,t\right)\right\|^2_{L^2_\rho(\Omega)}\right\}dt \\ = \;& \frac{b}{2}\left\{\left\|u\left(\left(n+1\right)T\right)\right\|^2_{L^2_\rho(\Omega)}-\left\|u\left(nT\right)\right\|^2_{L^2_\rho(\Omega)}\right\}. \end{aligned} \tag{4.29}$$

By combining $(4.26)-(4.29)$ into (4.25), we get

$$\begin{aligned} & \left(u_t\left(t\right),u\left(t\right)\right)_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T}-\int_{nT}^{(n+1)T}\left\|u_t\left(x,t\right)\right\|^2_{L^2_\rho(\Omega)}dt \\ & +\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\left\|\nabla u\right\|^{2\gamma}_{L^2(\Omega)}\right)\left\|\nabla u\left(t\right)\right\|^2_{L^2(\Omega)}dt \\ & +\int_{nT}^{(n+1)T}\int_0^t h\left(t-s\right)\left(\nabla u\left(t\right)-\nabla u\left(s\right),\nabla u\left(t\right)\right)_{L^2(\Omega)}dsdt \\ & -\int_{nT}^{(n+1)T}\left(\int_0^t h\left(s\right)ds\right)\left\|\nabla u\left(t\right)\right\|^2_{L^2(\Omega)}dt \\ & +\frac{b}{2}\left\{\left\|u\left(\left(n+1\right)T\right)\right\|^2_{L^2_\rho(\Omega)}-\left\|u\left(nT\right)\right\|^2_{L^2_\rho(\Omega)}\right\} \\ = \;& 0. \end{aligned} \tag{4.30}$$

Then (4.30) is equivalent

$$\begin{aligned} & -\int_{nT}^{(n+1)T}\left\|u_t\left(x,t\right)\right\|^2_{L^2_\rho(\Omega)}dt \\ & +\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\left\|\nabla u\right\|^{2\gamma}_{L^2(\Omega)}\right)\left\|\nabla u\left(t\right)\right\|^2_{L^2(\Omega)}dt \\ = \;& -\left(u_t\left(t\right),u\left(t\right)\right)_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T} \\ & -\int_{nT}^{(n+1)T}\int_0^t h\left(t-s\right)\left(\nabla u\left(t\right)-\nabla u\left(s\right),\nabla u\left(t\right)\right)_{L^2(\Omega)}dsdt \\ & +\int_{nT}^{(n+1)T}\left(\int_0^t h\left(s\right)ds\right)\left\|\nabla u\left(t\right)\right\|^2_{L^2(\Omega)}dt \\ & -\frac{b}{2}\left\{\left\|u\left(\left(n+1\right)T\right)\right\|^2_{L^2_\rho(\Omega)}-\left\|u\left(nT\right)\right\|^2_{L^2_\rho(\Omega)}\right\}. \end{aligned} \tag{4.31}$$

To estimate the term

$$\begin{aligned} I_1 \;:\; & =-\left(u_t\left(t\right),u\left(t\right)\right)_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T} \\ : \; & =-\left(u_t\left(\left(n+1\right)T\right),u\left(\left(n+1\right)T\right)\right)_{L^2_\rho(\Omega)}+\left(u_t\left(nT\right),u\left(nT\right)\right)_{L^2_\rho(\Omega)}. \end{aligned}$$

Notes

Notes

By using Young's inequality (for $\varepsilon = 1$), (4.8), $\frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)} \le E(t)$ and $\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} \le l^{-1}E(t)$, we get

$$\begin{aligned}
& (u_t(t), u(t))_{L^2_\rho(\Omega)} \\
\le\ & \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|u(t)\|^2_{L^2_\rho(\Omega)} \\
\le\ & \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|\rho\|^2_{L^2(\Omega)}\|\nabla u(t)\|^2_{L^2(\Omega)} \\
\le\ & E(t) + \|\rho\|^2_{L^2(\Omega)} l^{-1}E(t) \\
=\ & \left\{1 + \|\rho\|^2_{L^2(\Omega)} l^{-1}\right\} E(t),
\end{aligned}$$

then

$$|I_1| \le C_2\{E((n+1)T) + E(nT)\}, \tag{4.32}$$

where

$$C_2 := 1 + \|\rho\|^2_{L^2(\Omega)} l^{-1}.$$

To estimate the term

$$I_2 := -\int_{nT}^{(n+1)T}\int_0^t h(t-s)(\nabla u(t) - \nabla u(s), \nabla u(t))_{L^2(\Omega)}\, dsdt.$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_4}{2}\right)$ and (2.1), we get

$$\begin{aligned}
|I_2| \le\ & \frac{1}{4\varepsilon_4}\int_{nT}^{(n+1)T}\int_0^t h(t-s)\|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)}\, dsdt \\
& +\varepsilon_4\int_{nT}^{(n+1)T}\int_0^t h(t-s)\|\nabla u(t)\|^2_{L^2(\Omega)}\, dsdt \\
=\ & \frac{1}{4\varepsilon_4}\int_{nT}^{(n+1)T}\int_\Omega (h \circ \nabla u)(t)\, dxdt \\
& +\varepsilon_4\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\, ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\, dt.
\end{aligned} \tag{4.33}$$

By using (4.8) and $\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} \le l^{-1}E(t)$, we get

$$\begin{aligned}
& -\frac{b}{2}\left\{\|u((n+1)T)\|^2_{L^2_\rho(\Omega)} - \|u(nT)\|^2_{L^2_\rho(\Omega)}\right\} \\
\le\ & \frac{b}{2}\|\rho\|^2_{L^2(\Omega)}\left\{\|\nabla u((n+1)T)\|^2_{L^2(\Omega)} + \|\nabla u(nT)\|^2_{L^2(\Omega)}\right\} \\
\le\ & b\|\rho\|^2_{L^2(\Omega)} l^{-1}\{E((n+1)T) + E(nT)\} \\
=\ & C_2'\{E((n+1)T) + E(nT)\},
\end{aligned} \tag{4.34}$$

where

$$C_2' := b\|\rho\|^2_{L^2(\Omega)} l^{-1} > 0.$$

By combining (4.32) – (4.34) into (4.31), we can write

$$
\begin{aligned}
&-\int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
&+\int_{nT}^{(n+1)T} \left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
\leq\ & [C_2+C_2']\{E((n+1)T)+E(nT)\} \\
&+\frac{1}{4\varepsilon_4}\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt \\
&+(\varepsilon_4+1)\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt. \qquad (4.35)
\end{aligned}
$$

On multiplied (4.24) by γ_1 and multiplied (4.35) by γ_2 and combining suitably, we get

$$
\begin{aligned}
&\left[\gamma_1\left\{t_1h(t_1)-\varepsilon_1\left(1+b^2\right)\right\}-\gamma_2\right]\int_{nT}^{(n+1)T}\|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
&+\gamma_2\xi_0\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt+\gamma_2\xi_1\int_{nT}^{(n+1)T}\|\nabla u\|^{2(\gamma+1)}_{L^2(\Omega)}\,dt \\
\leq\ & \{\gamma_1C_1+\gamma_2[C_2+C_2']\}[E((n+1)T)+E(nT)] \\
&-\gamma_1\frac{1}{4\varepsilon_1}\|\rho\|^2_{L^2(\Omega)}h(0)\int_{nT}^{(n+1)T}\int_\Omega (h'\circ\nabla u)(t)\,dxdt \\
&+\gamma_1\varepsilon_2\quad \xi_0+\xi_1\left(\left(\frac{2(\gamma+1)}{\xi_1}E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}}\right)^2\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
&+\left\{\gamma_1\xi_0\left\{\frac{1}{4\varepsilon_2}+1+\frac{1}{4\varepsilon_3}+\frac{1}{4\varepsilon_1}\|\rho\|^2_{L^2(\Omega)}\right\}+\gamma_2\frac{1}{4\varepsilon_4}\right\}\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt \\
&+\{\gamma_1\varepsilon_3\xi_0+\gamma_2(\varepsilon_4+1)\}\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt. \qquad (4.36)
\end{aligned}
$$

Let

$$
\left\{
\begin{aligned}
\varepsilon_1 &:= \frac{t_1h(t_1)}{2(1+b^2)}, \\
\varepsilon_2 &:= \frac{3\varepsilon\xi_0}{\gamma_1\left(\xi_0+\xi_1\left(\frac{2(\gamma+1)}{\xi_1}E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}}\right)^2}, \\
\varepsilon_3 &:= \frac{\varepsilon}{\gamma_1\xi_0}, \\
\varepsilon_4 &:= 2\varepsilon,
\end{aligned}
\right. \qquad (4.37)
$$

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and

$$\left\{\begin{array}{l}\gamma_1 := \dfrac{4}{t_1 h(t_1)},\\[2ex] \gamma_2 := 1,\end{array}\right. \tag{4.38}$$

by using (4.37) and (4.38) into (4.36) , we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt\\
&+\xi_0\int_{nT}^{(n+1)T} \|\nabla u(t)\|^2_{L^2(\Omega)}\,dt+\xi_1\int_{nT}^{(n+1)T}\|\nabla u(t)\|^{2(\gamma+1)}_{L^2(\Omega)}\,dt\\
\leq\ & C_3\left[E((n+1)T)+E(nT)\right]\\
&-C_4\int_{nT}^{(n+1)T}\int_\Omega (h'\circ\nabla u)(t)\,dxdt\\
&+3\varepsilon\xi_0\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt\\
&+C_5\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt\\
&+(3\varepsilon+1)\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt,
\end{aligned} \tag{4.39}$$

where

$$C_3 := \gamma_1 C_1 + C_2 + C_2',$$

$$\left\{\begin{array}{l}
C_4 := \gamma_1\dfrac{(1+b^2)}{2t_1 h(t_1)}\|\rho\|^2_{L^2(\Omega)}\,h(0),\\[3ex]
C_5 := \frac{4\xi_0}{t_1 h(t_1)}\left\{\dfrac{\gamma_1\left(\xi_0+\xi_1\left(\frac{2(\gamma+1)E(0)}{\xi_1}\right)^{\frac{2\gamma}{2(\gamma+1)}}\right)^2}{12\varepsilon\xi_0}+1+\frac{\gamma_1\xi_0}{4\varepsilon}+\frac{(1+b^2)}{2t_1h(t_1)}\|\rho\|^2_{L^2(\Omega)}\right\}+\frac{1}{8\varepsilon}.
\end{array}\right.$$

Adding and subtracting in (4.39) the term

$$-\int_{nT}^{(n+1)T}\int_\Omega\left(\int_0^t h(s)\,ds\right)|\nabla u|^2\,dxdt \quad\text{and}\quad \int_{nT}^{(n+1)T}\int_\Omega a(x)(h\circ\nabla u)(t)\,dxdt,$$

in order to recover the energy $E(t)$,we obtain

$$(1-3\varepsilon)\int_{nT}^{(n+1)T}\int_\Omega\left(\xi_0-\int_0^t h(s)\,ds\right)|\nabla u(t)|^2\,dxdt$$

$$+\int_{nT}^{(n+1)T} \|u_t(x,t)\|_{L^2_\rho(\Omega)}^2 \, dt$$

$$+\xi_1 \int_{nT}^{(n+1)T} \|\nabla u(t)\|_{L^2(\Omega)}^{2(\gamma+1)} dt + \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t) \, dxdt$$

$$\leq C_3 \left[E((n+1)T) + E(nT)\right]$$

$$+C_5 \int_{nT}^{(n+1)T} \int_\Omega k_1 \left(-h' \circ \nabla u\right)(t) \, dxdt$$

$$+C_5 \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t) \, dxdt, \tag{4.40}$$

where

$$k_1 := \frac{C_4}{C_5}.$$

From (4·40), choosing ε sufficiently small, $k_1 > 0$ and T large enough and using

$$\alpha_1 \left\{ \|u_t(t)\|_{L^2_\rho(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^{2(\gamma+1)} + \int_\Omega (h \circ \nabla u)(t) \, dx \right\}$$

$$\leq E(t) \leq \alpha_2 \left\{ \|u_t(t)\|_{L^2_\rho(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^2 + \|\nabla u(t)\|_{L^2(\Omega)}^{2(\gamma+1)} + \int_\Omega (h \circ \nabla u)(t) \, dx \right\},$$

where

$$\begin{cases} \alpha_1 := \dfrac{1}{2} \min\left\{1, l, \dfrac{\xi_1}{(\gamma+1)}\right\}, \\ \qquad\qquad \text{and} \\ \alpha_2 := \dfrac{1}{2} \max\left\{1, \xi_0, \dfrac{\xi_1}{(\gamma+1)}\right\}, \end{cases}$$

we get

$$\int_{nT}^{(n+1)T} E(t) \, dt \leq C_6 \left[E((n+1)T) + E(nT)\right]$$

$$+C_7 \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t) \, dxdt$$

$$+C_7 \int_{nT}^{(n+1)T} \int_\Omega k_1 (-h' \circ \nabla u)(t) \, dxdt, \tag{4.41}$$

where

$$\begin{cases} C_6 := \dfrac{\alpha_2 C_3}{\min\left\{(1-3\varepsilon) l, 1, \xi_1\right\}}, \\ \qquad\qquad \text{and} \\ C_7 := \dfrac{\alpha_2 C_5}{\min\left\{(1-3\varepsilon) l, 1, \xi_1\right\}}. \end{cases}$$

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In the last step, we need to relate the viscoelastic energy to the viscoelastic damping. In the case when the relaxation function obeys a linear equation, this relation is straightforward and is expressed by a suitable multiplication. However, in the case of general decays, additional arguments are used. Here, we follow [17]. From the **assumption 2** made on the viscoelastic kernel h and from [17, **Lemma 4**] we obtain

$$(h \circ \nabla u)(t) \leq \hat{H}_{\alpha}^{-1}(-h' \circ \nabla u)(t), \quad t \in [nT, (n+1)T], \tag{4.42}$$

where $\hat{H}_{\alpha}$ is a rescaling of H_{α} with

$$H_{\alpha}(s) = \alpha s^{1-\frac{1}{\alpha}} H\left(s^{\frac{1}{\alpha}}\right),$$

and $\alpha \in (0,1)$ is such that

$$\sup_{t>0} \int_0^t h^{1-\alpha}(t-s) \left\|\nabla u(t) - \nabla u(s)\right\|^2 ds < \infty.$$

From **Assumption 2** it is clear that $\alpha \geq \alpha_0$. The main point, however, is that the argument can be reiterated (based on [16, **Lemma 8**] leading to $\alpha = 1$). This allows us to replace H_{α}, the function in (4.42), by the original function $\hat{H}$ which is a rescaling of $H(s)$. This means that $\hat{H} = cH\left(\frac{C}{s}\right)$ for some c, $C > 0$. Now, from (4.42) and taking (4.41) into account, we deduce that

$$\begin{aligned}\int_{nT}^{(n+1)T} E(t)\,dt \quad &\leq \quad C_6\left[E((n+1)T) + E(nT)\right] \\ &+ C_7 \int_{nT}^{(n+1)T} \int_{\Omega} \left[\hat{H}^{-1} + k_1\right](-h' \circ \nabla u)(t)\,dxdt.\end{aligned} \tag{4.43}$$

Next, we shall employ the following version of **Jensen's inequality** applied to measures and convex functions F. Let F be a convex increasing function on $[\alpha, b]$, let $f : \Omega \longmapsto [\alpha, b]$, and let h be an integrable function such that $h(x) \geq 0$ and

$$\int_{\Omega} h(x)\,dx = h_0 > 0.$$

Then, we have

$$\int_{\Omega} F^{-1}(f(x))\,h(x)\,dx \leq h_0 F^{-1}\left[h_0^{-1} \int_{\Omega} f(x)\,h(x)\,dx\right]. \tag{4.44}$$

We shall use (4.44) in order to bring the functions H in front of the integrals. Let us denote

$$\alpha_0 := meas(\Omega).$$

We note that the function $\hat{H}^{-1}+k_1$ is concave.
Let

$$\begin{cases} F^{-1}=\hat{H}^{-1}+k_1, \\ f(x)=(-h'\circ\nabla u)(t), \\ h(x)=T, \\ h_0=T\alpha_0, \\ h_0^{-1}=\alpha_0^{-1}T^{-1}, \end{cases}$$

thus, we have

$$\begin{aligned} &\int_{nT}^{(n+1)T}\int_\Omega\left[\hat{H}^{-1}+k_1\right](-h'\circ\nabla u)(t)\,dxdt \\ \leq\ &\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_\Omega(-h'\circ\nabla u)(t)\,dxdt\right]. \qquad (4.45) \end{aligned}$$

On the other hand, from the identity (2.4) for the energy, we can write

$$\begin{aligned} &E((n+1)T)-E(nT) \\ =\ &\frac{1}{2}\int_{nT}^{(n+1)T}\left\{\int_\Omega(h'\circ\nabla u)(t)\,dx-h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}-2b\|u_t(t)\|^2_{L^2_\rho(\Omega)}\right\}dt \\ =\ &-\int_{nT}^{(n+1)T}D(t)\,dt, \end{aligned}$$

where

$$D(t):=\frac{1}{2}\left\{\int_\Omega(-h'\circ\nabla u)(t)\,dx+h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}+2b\|u_t(t)\|^2_{L^2_\rho(\Omega)}\right\}. \qquad (4.46)$$

En replacement (4.45) into (4.43) and using

$$E(nT)=E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt, \qquad (4.47)$$

we get

$$\begin{aligned} &\int_{nT}^{(n+1)T}E(t)\,dt \\ \leq\ &C_6\left\{2E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt\right\} \\ &+C_7\alpha_0T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_\Omega(-h'\circ\nabla u)(t)\,dxdt\right], \end{aligned}$$

and using (4.46) we get

$$\int_{nT}^{(n+1)T}\int_\Omega(-h'\circ\nabla u)(t)\,dxdt\leq 2\int_{nT}^{(n+1)T}D(t)\,dt,$$

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thus, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T} E(t)\,dt \\
\leq\ & C_6\left\{2E((n+1)T)+\int_{nT}^{(n+1)T} D(t)\,dt\right\} \\
&+2C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T} D(t)\,dt\right] \\
=\ & 2C_6E((n+1)T)+C_6\int_{nT}^{(n+1)T} D(t)\,dt \\
&+2C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T} D(t)\,dt\right] \\
\leq\ & 2C_6E((n+1)T)+C_8\left[\hat{H}^{-1}+k_2\right]\left[\int_{nT}^{(n+1)T} D(t)\,dt\right] \\
=\ & 2C_6E((n+1)T)+C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right], \qquad (4.48)
\end{aligned}$$

where

$$\begin{cases} C_8 := \max\{2C_7,1\}, \\ \tilde{H} := \left[\hat{H}^{-1}+k_2\right]^{-1}, \\ k_2 := (C_6+2C_7k_1). \end{cases} \qquad (4.49)$$

By integrating t to $(n+1)T$ on both sides of the inequality $\frac{d}{dt}\{E(t)\}\leq 0$ yields

$$E((n+1)T)\leq E(t) \quad \text{for all } (n+1)T\geq t, \qquad (4.50)$$

integrating (4.50) from nT to $(n+1)T$ yields

$$\begin{aligned}
\int_{nT}^{(n+1)T} E(t)\,dt &\geq \int_{nT}^{(n+1)T} E((n+1)T)\,dt \\
&= \int_{nT}^{(n+1)T} dtE((n+1)T) \\
&= TE((n+1)T), \qquad (4.51)
\end{aligned}$$

by replacement (4.51) into (4.48), we get

$$TE((n+1)T)\leq 2C_6E((n+1)T)+C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right],$$

then

$$(T-2C_6)E((n+1)T)\leq C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right].$$

For T large enough, where C_6 is a positive constant, which implies that

$$E\left((n+1)T\right) \leq C_9 \tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D\left(t\right)dt\right],$$

where

$$C_9 := \frac{C_8}{(T-2C_6)}, \tag{4.52}$$

which gives that

$$\tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq \int_{nT}^{(n+1)T} D\left(t\right)dt, \tag{4.53}$$

by using (4.47) into (4.53), we get

$$\tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq E\left(nT\right) - E\left((n+1)T\right),$$

from the above we have

$$E\left((n+1)T\right) + \tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq E\left(nT\right), \quad n = 1, 2, 3....$$

Then the Proof of **Lemma 2** is complete.

Lemma 3. Let p be a positive, increasing function such that $p(0) = 0$. Since p is increasing, we can define an increasing function q, $q(x) \equiv x-(I+p)^{-1}(x)$. Consider a sequence F_n of positive numbers which satisfies

$$F_{m+1} + p\left(F_{m+1}\right) \leq F_m. \tag{4.54}$$

Then $F_m \leq S(m)$ where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}\left\{S\left(t\right)\right\} + q\left(S\left(t\right)\right) = 0, \quad S\left(0\right) = F_0. \tag{4.55}$$

Moreover, if $p(x) > 0$ for $x > 0$ then $\lim_{t\longrightarrow\infty} S(t) = 0$.

Proof. Proof of the Lemma use the proof retraction. Assume $F_m \leq S(m)$ and prove that $F_{m+1} \leq S(m+1)$.
Inequality (4.54) is equivalent to

$$(I+p)\,F_{m+1} \leq F_m,$$

and since $(I+p)^{-1}$ is monotone increasing, $F_{m+1} \leq (I+p)^{-1} F_m$, and using

$$(I+p)^{-1} F_m = (I-q)\,F_m,$$

we get

$$\begin{aligned} F_{m+1} &\leq (I-q)\,F_m \\ &= F_m - q\left(F_m\right). \end{aligned} \tag{4.56}$$

On the other hand, since q is an increasing function, the solution $S(t)$ of equation (4.55) is described by a nonlinear contraction.

In particular integrating $\frac{d}{dt}\{S(t)\} \leq 0$ from m to τ yields

$$S\left(\tau\right) \leq S\left(m\right) \quad \text{for all } t \geq \tau. \tag{4.57}$$

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Integrating equation (4.55) from m to $(m+1)$ yields

$$S(m+1) - S(m) + \int_m^{m+1} q(S(\tau))\, d\tau = 0. \tag{4.58}$$

Since q is increasing, by using (4.57) we obtain for all $m \leq \tau \leq m+1$

$$\begin{aligned} \int_m^{m+1} q(S(\tau))\, d\tau &\leq \int_m^{m+1} q(S(m))\, d\tau \\ &= q(S(m)) \int_m^{m+1} d\tau \\ &= q(S(m)), \end{aligned}$$

then

$$-\int_m^{m+1} q(S(\tau))\, d\tau \geq -q(S(m)), \quad \text{for all } m \leq \tau \leq m+1, \tag{4.59}$$

by replacement (4.59) into (4.58) and using the inductive assumption $F_m \leq S(m)$, we get

$$\begin{aligned} S(m+1) &\geq S(m) - q(S(m)) \\ &= (I-q)\, S(m) \\ &\geq (I-q)\, F_m \\ &= F_m - q(F(m)), \end{aligned} \tag{4.60}$$

comparing (4.60) with (4.56) yields

$$S(m+1) \geq F_{m+1}.$$

Then the Proof of **Lemma 3** is complete.

Theorem 2. Let us assume that ***Assumption 1*** *and* ***Assumption 2*** *ar the place. Then there exist positive constants c_1, c_2 and T_0 such that the solution of problem* $(1.1)-(1.3)$ *satisfies $E(t) \leq s(t)$, where $s(t)$ verifies the ODE*

$$s_t + \hat{H}(s) = 0, \quad s(0) = E(0), \; t \geq T_0 > 0,$$

with $\hat{H}(s) = c_1 H(c_2 s)$.

Proof. Thus, we are in a position to apply the result of **Lemma 2** with

$$F_m \equiv E(mt), \quad F_0 \equiv E(0).$$

This yields

$$E(mT) \leq S(m), \quad m = 0, 1, 2, 3....$$

Setting $t = mT + \tau$ and recalling the evolution property gives

$$E(t) \leq E(mT) \leq S(m) \leq S\left(\frac{t-\tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right),$$

which completes the proof of **Theorem 2.**

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