Global Journal of Science Frontier Research: F
Mathematics and Decision Sciences
Volume 21 Issue 4 Version 1.0 Year 2021
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Half-Step Implicit Linear Multistep Hybrid Block Third Derivative Methods of order Four for the Solution of Third order Ordinary Differential Equations

By Skwame, Yusuf, Raymond, Dominic, Tumba, Pius & Adiku, Lydia

Adamawa State University

Abstract- This paper proposes a half-step third derivative hybrid block method with two cases of order four for solution of third Order Ordinary Differential Equations. Method of interpolation and collocation of power series approximate solution was used to generate the continuous hybrid linear multistep method, which was then evaluated at non-interpolating points to give a continuous block method. The discrete block method was recovered when the continuous block was evaluated at all step points. The basic properties of the methods were investigated and were found to be zero-stable, consistent and convergent. The develop half-step method is applied to solve some third order initial value problems of ordinary differential equations and from the numerical results obtained, it is observed that our methods gives better approximation than the existing method compared with.

Keywords: *half-step, hybrid block method, third derivative.*

GJSFR-F Classification: *MSC 2010: 33E30*

HALFSTEPIMPLICITLINEARMULTISTEPHYBRIDBLOCKTHIRDDERIVATIVEMETHODSOFORDERFOURFORTHESOLUTIONOFTHIRDORDERORDINARYDIFFERENTIALEQUATIONS

Half-Step Implicit Linear Multistep Hybrid Block Third Derivative Methods of order Four for the Solution of Third order Ordinary Differential Equations

Skwame, Yusuf [α], Raymond, Dominic [σ], Tumba, Pius [ρ] & Adiku, Lydia [ω]

Abstract- This paper proposes a half-step third derivative hybrid block method with two cases of order four for solution of third Order Ordinary Differential Equations. Method of interpolation and collocation of power series approximate solution was used to generate the continuous hybrid linear multistep method, which was then evaluated at non-interpolating points to give a continuous block method. The discrete block method was recovered when the continuous block was evaluated at all step points. The basic properties of the methods were investigated and were found to be zero-stable, consistent and convergent. The develop half-step method is applied to solve some third order initial value problems of ordinary differential equations and from the numerical results obtained, it is observed that our methods gives better approximation than the existing method compared with.

Keywords: half-step, hybrid block method, third derivative.

I. Introduction

This paper consider third order initial value problems of the form

$$y''' = f\left(x, y(x), y'(x)\right),\ y(x_0) = y_0,\ y'(x_0) = y'_0 \tag{1}$$

Where f is continuous within the interval of integration, solving higher order derivatives method by reducing them to a system of first-order approach involves more functions to evaluate which then leads to a computational burden as in Kayode and Obuarhua (2013) and James et al. (2013). Various approaches have been proposed to find the analytic solution of (1) ranging from predictor-corrector method to hybrid methods. Despite the success recorded by the predictor-corrector methods, its major setback is that the predictor are in reducing order of accuracy especially when the value of the step-length is high and moreover the result are at overlapping interval. The hybrid method was established to circumvent the Dahlquist barrier theorem which gives a better approximation to solutions of initial value problems of stiff ordinary differential equations than the k-step method.

Scholars who recently adopted the hybrid method other than the direct method in approximating (1) include among others Adesanya et al. (2013), Adebayo and

Author α: Department of Mathematical Sciences, Faculty of Science, Adamawa State University, Mubi-Nigeria
Author σ ω: Department of Mathematics and Statistics, Faculty of Pure and Applied Sciences, Federal University Wukari Nigeria.
e-mails: domehbazza@yahoo.co.uk, raymond@fuwukari.edu.ng
Author ρ: Department of Mathematics, Faculty of Science, Federal University Gashua, Nigeria.

Adebola (2016), Adoghe and Omole (2019), Kuboye and Omar (2015), Olabode and Momoh (2016), Kayode and Adeyeye (2011) and Mohammed and Adeniyi (2014). We adopted the method of collocation and interpolation of the power series approximation as the basis function to generate continuous linear multistep method as other scholars in solving (1). The derivation of continuous linear multistep methods for direct solution of ordinary differential equations have been discussed over the years in literature and these include, among others collocation, interpolation, integration, interpolating polynomials and basis functions such as, Chebyshev polynomials, trigonometric functions, exponential functions.

Notes

In this paper, we developed half-step third derivative hybrid block method of order four with two cases for direct solution of third order ordinary differential equation which is implemented in block. The method developed evaluates less function per step and circumventing the Dahlquist barrier's by introducing a hybrid points.

The paper is organized as follows: First is a discussion of the new half-step third derivative hybrid block method and the materials for the development of the method. This is followed by a consideration of analysis of the basic properties of the new half-step third derivative hybrid block method, which include convergence, stability region, numerical experiments where the efficiency of the derived method is tested on some stiff numerical examples and discussion of results. Lastly, the study concludes by comparing the results obtained with an existing work of Adeyeye and Omar (2018), Adebayo and Adebola (2016), Adoghe and Omole (2019) and Mohammed and Adeniyi (2014).

II. Derivation of the New Half-Step Hybrid Block Method

In this section we intend to develop a family of half-step third derivative hybrid method with three hybrid points u,v and w, which are all rational numbers $u,v,w\in\left[0,\frac{1}{2}\right]$ of the form

$$y_{n+t}=\sum_{i=0,u,v}\left(\alpha_i(t)y_{n+i}\right)+h^3\left[\sum_{j=0}^{k}\beta_j(t)f_{n+j}+\beta_u(t)f_{n+u}+\beta_v(t)f_{n+v}+\beta_w(t)f_{n+w}\right] \tag{2}$$

$\alpha_0(t),\alpha_u(t),\alpha_v(t),\beta_j(t),\beta_u(t),\beta_v(t),\beta_w(t)$ are polynomials, $y_{n+j}=y\left(x_{n+j}\right)$, $f_{n+j}=f\left(x_{n+j},y_{n+j}\right)$

$t=\dfrac{x-x_n}{h}$ Consider the power series approximate solution of the form

$$y(x)=\sum_{j=0}^{s+r-1}a_i\left(\frac{x-x_n}{h}\right)^j \tag{3}$$

where $r=3$ and $s=5$ are the numbers of interpolation and collocation points respectively, is considered to be a solution to (1).

The third derivative of (3) gives

$$y'''(x)=\sum_{j=3}^{s+r-1}\frac{a_j\,j!}{h^3(j-3)}\left(\frac{x-x_n}{h}\right)^{j-3} \tag{4}$$

Substituting (4) into (1) gives

$$f\left(x,y,y''\right)=\sum_{j=3}^{s+r-1}\frac{a_j j!}{h^3\left(j-3\right)}\left(\frac{x-x_n}{h}\right)^{j-3} \quad (5)$$

Notes

Collocating (5) at all points $x_{n+s}, s=0,u,v,w,\frac{1}{2}$ and Interpolating Equation (3) at $x_{n+r}, r=0,u,v$, gives a system of non linear equation of the form

$$AX=U \quad (6)$$

Where

$$A=\left[a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7\right]^T,$$

$$U=\left[y_n,y_{n+u},y_{n+v},f_n,f_{n+u},f_{n+v},f_{n+w},f_{n+\frac{1}{2}}\right]^T,$$

$$X=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & u & u^2 & u^3 & u^4 & u^5 & u^6 & u^7 \\ 1 & v & v^2 & v^3 & v^4 & v^5 & v^6 & v^7 \\ 0 & 0 & 0 & \frac{3!}{0!h^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3!u}{0!h^3} & \frac{4!u^2}{1!h^3} & \frac{5!u^3}{2!h^3} & \frac{6!u^4}{3!h^3} & \frac{7!u^5}{4!h^3} \\ 0 & 0 & 0 & \frac{3!v}{0!h^3} & \frac{4!v^2}{1!h^3} & \frac{5!v^3}{2!h^3} & \frac{6!v^4}{3!h^3} & \frac{7!v^5}{4!h^3} \\ 0 & 0 & 0 & \frac{3!w}{0!h^3} & \frac{4!w^2}{1!h^3} & \frac{5!w^3}{2!h^3} & \frac{6!w^4}{3!h^3} & \frac{7!w^5}{4!h^3} \\ 0 & 0 & 0 & \frac{3!\frac{1}{2}}{0!h^3} & \frac{4!(\frac{1}{2})^2}{1!h^3} & \frac{5!(\frac{1}{2})^3}{2!h^3} & \frac{6!(\frac{1}{2})^4}{3!h^3} & \frac{7!(\frac{1}{2})^5}{4!h^3} \end{bmatrix}$$

Solving (6) for a_i's using Gaussian elimination method, gives a continuous hybrid linear multistep method of the form

$$y(x)=\sum_{j=0,u,v}\alpha_j y_{n+j}+h^3\left[\sum_{i=0}^{1/2}\beta_j f_{n+j}+\beta_k f_{n+k}\right], k=u,v,w \quad (7)$$

Differentiating (7) twice yields

$$p''(x)=\frac{1}{h^2}\sum_{j=0,u,v}\alpha_j y_{n+j}+h\left[\sum_{j=u,v,w}\beta_j f_{n+j}+\sum_{j=0}^{1/2}\beta_j f_{n+j}\right] \quad (8)$$

We then impose the following conditions on $y(x)$ in (3) for

$$y\left(x_{n+j}\right)=y_{n+j} \text{ and } y'''\left(x_{n+j}\right)=f_{n+j}$$

The coefficient of $y_{n+j}, j=0,u,v$ and $f_{n+j}, j=0,u,v,w,\frac{1}{2}$ gives

$$y_{n+t}=\sum_{i=0,u,v}\left(\alpha_i(t)y_{n+i}\right)+h^3\left[\beta_0(t)f_n+\beta_u(t)f_{n+u}+\beta_v(t)f_{n+v}+\beta_w(t)f_{n+w}+\beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}}\right]$$

Notes

Where

$$\alpha_0 = 1 - \frac{(u+v)\left(-x_n + x\right)}{u\,v\,h} + \frac{\left(-x_n + x\right)^2}{h^2\,u\,v}$$

$$\alpha_u = \frac{\left(-x_n + x\right)^2}{u\,(u-v)\,h^2} - \frac{v\left(-x_n + x\right)}{u\,(u-v)\,h}$$

$$\alpha_v = \frac{u\left(-x_n + x\right)}{v\,(u-v)\,h} - \frac{\left(-x_n + x\right)^2}{v\,(u-v)\,h^2}$$

$$\beta_0 = -\frac{1}{840}\,\frac{1}{w}\Big(\big(6u^4 - 8u^3v - 14u^3w - 8u^2v^2 + 28u^2vw - 8uv^3 + 28uv^2w + 6v^4$$
$$- 14v^3w - 7u^3 + 14u^2v + 21u^2w + 14uv^2 - 84uvw - 7v^3 + 21v^2w\big)\left(-x_n\right.$$
$$\left. + x\right)h^2\Big) + \frac{1}{840}\,\frac{1}{u\,v\,w}\Big(\big(6u^5 - 8u^4v - 14u^4w - 8u^3v^2 + 28u^3vw - 8u^2v^3$$
$$+ 28u^2v^2w - 8uv^4 + 28uv^3w + 6v^5 - 14v^4w - 7u^4 + 14u^3v + 21u^3w + 14u^2v^2$$
$$- 84u^2vw + 14uv^3 - 84uv^2w - 7v^4 + 21v^3w\big)\left(-x_n + x\right)^2h\Big) + \frac{1}{6}\left(-x_n + x\right)^3$$
$$- \frac{1}{24}\,\frac{(2uvw + uv + uw + vw)\left(-x_n + x\right)^4}{u\,v\,w\,h}$$
$$+ \frac{1}{60}\,\frac{(2uv + 2uw + 2vw + u + v + w)\left(-x_n + x\right)^5}{u\,v\,w\,h^2}$$
$$- \frac{1}{120}\,\frac{(2u + 2v + 2w + 1)\left(-x_n + x\right)^6}{u\,v\,w\,h^3} + \frac{1}{105}\,\frac{\left(-x_n + x\right)^7}{u\,v\,w\,h^4}$$

$$\beta_u = \frac{1}{840}\,\frac{1}{(2u-1)(u-v)(u-w)}\Big(v\big(8u^4 - 6u^3v - 14u^3w - 6u^2v^2 + 14u^2vw$$
$$- 6uv^3 + 14uv^2w - 6v^4 + 14v^3w - 7u^3 + 7u^2v + 14u^2w + 7uv^2 - 21uvw + 7v^3$$
$$- 21v^2w\big)\left(-x_n + x\right)h^2\Big) - \frac{1}{840}\,\frac{1}{(2u-1)\,u\,(u-v)(u-w)}\Big(\big(8u^5 - 6u^4v$$
$$- 14u^4w - 6u^3v^2 + 14u^3vw - 6u^2v^3 + 14u^2v^2w - 6uv^4 + 14uv^3w - 6v^5$$
$$+ 14v^4w - 7u^4 + 7u^3v + 14u^3w + 7u^2v^2 - 21u^2vw + 7uv^3 - 21uv^2w + 7v^4$$
$$- 21v^3w\big)\left(-x_n + x\right)^2h\Big) - \frac{1}{24}\,\frac{vw\left(-x_n + x\right)^4}{(2u-1)\,u\,(u-v)(u-w)\,h}$$
$$+ \frac{1}{60}\,\frac{(2vw + v + w)\left(-x_n + x\right)^5}{(2u-1)\,u\,(u-v)(u-w)\,h^2} - \frac{1}{120}\,\frac{(2v + 2w + 1)\left(-x_n + x\right)^6}{(2u-1)\,u\,(u-v)(u-w)\,h^3}$$
$$+ \frac{1}{105}\,\frac{\left(-x_n + x\right)^7}{(2u-1)\,u\,(u-v)(u-w)\,h^4}$$

$$\beta_v = \frac{1}{840}\frac{1}{(2v-1)(u-v)(v-w)}\Big(u\big(6u^4+6u^3v-14u^3w+6u^2v^2-14u^2vw$$

$$+6uv^3-14uv^2w-8v^4+14v^3w-7u^3-7u^2v+21u^2w-7uv^2+21uvw+7v^3$$

$$-14v^2w\big)\big(h^2x-x_n\big)\Big)-\frac{1}{840}\frac{1}{(2v-1)(u-v)v(v-w)}\Big(\big(6u^5+6u^4v$$

$$-14u^4w+6u^3v^2-14u^3vw+6u^2v^3-14u^2v^2w+6uv^4-14uv^3w-8v^5$$

$$+14v^4w-7u^4-7u^3v+21u^3w-7u^2v^2+21u^2vw-7uv^3+21uv^2w+7v^4$$

$$-14v^3w\big)(-x_n+x)^2h\Big)+\frac{1}{24}\frac{uw(-x_n+x)^4}{(2v-1)(u-v)v(v-w)h}$$

$$-\frac{1}{60}\frac{(2uw+u+w)(-x_n+x)^5}{(2v-1)(u-v)v(v-w)h^2}+\frac{1}{120}\frac{(2u+2w+1)(-x_n+x)^6}{(2v-1)(u-v)v(v-w)h^3}$$

$$-\frac{1}{105}\frac{(-x_n+x)^7}{(2v-1)(u-v)v(v-w)h^4}$$

$$\beta_w = -\frac{1}{840}\frac{1}{(2w-1)(v-w)(u-w)w}\Big(uv\big(6u^4-8u^3v-8u^2v^2-8uv^3+6v^4$$

$$-7u^3+14u^2v+14uv^2-7v^3\big)(-x_n+x)h^2\Big)$$

$$+\frac{1}{840}\frac{1}{(2w-1)(v-w)(u-w)w}\Big(\big(6u^5-8u^4v-8u^3v^2-8u^2v^3-8uv^4+6v^5$$

$$-7u^4+14u^3v+14u^2v^2+14uv^3-7v^4\big)(-x_n+x)^2h\Big)$$

$$-\frac{1}{24}\frac{uv(-x_n+x)^4}{(2w-1)(v-w)(u-w)wh}+\frac{1}{60}\frac{(2uv+u+v)(-x_n+x)^5}{(2w-1)(v-w)(u-w)wh^2}$$

$$-\frac{1}{120}\frac{(2u+2v+1)(-x_n+x)^6}{(2w-1)(v-w)(u-w)wh^3}+\frac{1}{105}\frac{(-x_n+x)^7}{(2w-1)(v-w)(u-w)wh^4}$$

$$\beta_{\frac{1}{2}} = \frac{2}{105}\frac{1}{(2w-1)(2v-1)(2u-1)}\Big(uv\big(3u^4-4u^3v-7u^3w-4u^2v^2+14u^2vw$$

$$-4uv^3+14uv^2w+3v^4-7v^3w\big)(-x_n+x)h^2\Big)$$

Notes

$$-\frac{2}{105}\frac{1}{(2w-1)(2v-1)(2u-1)}\Big(\big(3u^5-4u^4v-7u^4w-4u^3v^2+14u^3vw$$

$$-4u^2v^3+14u^2v^2w-4uv^4+14uv^3w+3v^5-7v^4w\big)(-x_n+x)^2h\Big)$$

$$+\frac{2}{3}\frac{uvw(-x_n+x)^4}{(2w-1)(2v-1)(2u-1)h}-\frac{4}{15}\frac{(uv+uw+vw)(-x_n+x)^5}{(2w-1)(2v-1)(2u-1)h^2}$$

$$+\frac{2}{15}\frac{(u+v+w)(-x_n+x)^6}{(2w-1)(2v-1)(2u-1)h^3}-\frac{8}{105}\frac{(-x_n+x)^7}{(2w-1)(2v-1)(2u-1)h^4}$$

a) *Specification of the New Half-step Third Derivative Method*

The family of hybrid half-step method with three off-grid hybrid points u,v and w which are rational numbers. Equation (7) is evaluated at the non-interpolating points $\left\{x_{n+w},x_{n+\frac{1}{2}}\right\}$ and (8) at all points $\left\{x_n,x_{n+u},x_{n+v},x_{n+w},x_{n+\frac{1}{2}}\right\}$, which produces the following general equations in block form

$$AY_L=BR_1+CR_2+DR_3 \tag{9}$$

Case One:

We consider the case where $\left(u=\frac{1}{8},v=\frac{1}{7},w=\frac{1}{6}\right)$ equation (9) becomes

$$A=\begin{bmatrix}
\frac{16}{9} & -\frac{49}{18} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
80 & \frac{147}{2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{64}{h} & \frac{49}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{48}{h} & -\frac{49}{h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{64}{h} & -\frac{63}{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{256}{3h} & -\frac{245}{3h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
\frac{384}{h} & -\frac{343}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
\frac{896}{h^2} & -\frac{784}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{896}{h^2} & -\frac{784}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
\frac{896}{h^2} & -\frac{784}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
\frac{896}{h^2} & -\frac{784}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
\frac{896}{h^2} & -\frac{784}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},\ y_L=\begin{bmatrix}
y_{n+\frac{1}{8}}\\ y_{n+\frac{1}{7}}\\ y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{2}}\\ y'_{n+\frac{1}{8}}\\ y'_{n+\frac{1}{7}}\\ y'_{n+\frac{1}{6}}\\ y'_{n+\frac{1}{2}}\\ y''_{n+\frac{1}{8}}\\ y''_{n+\frac{1}{7}}\\ y''_{n+\frac{1}{6}}\\ y''_{n+\frac{1}{2}}
\end{bmatrix},\ B=\begin{bmatrix}
\frac{1}{18} & 0 & 0\\
\frac{15}{2} & 0 & 0\\
-\frac{15}{h} & 1 & 0\\
-\frac{1}{h} & 0 & 0\\
\frac{1}{h} & 0 & 0\\
\frac{11}{3h} & 0 & 0\\
\frac{41}{h} & 0 & 0\\
\frac{112}{h^2} & 0 & 1\\
\frac{112}{h^2} & 0 & 0\\
\frac{112}{h^2} & 0 & 0\\
\frac{112}{h^2} & 0 & 0\\
\frac{112}{h^2} & 0 & 0
\end{bmatrix},\ R_1=\begin{bmatrix} y_n\\ y'_n\\ y''_n\end{bmatrix},\ C=\begin{bmatrix}
\frac{764401h^3}{669139107840}\\
-\frac{833821h^3}{550731776}\\
\frac{411933h^3}{688414720}\\
-\frac{21251h^3}{1032622080}\\
\frac{297119h^3}{14456709120}\\
\frac{9470009h^3}{125463582720}\\
-\frac{3132947h^3}{147517440}\\
-\frac{32189281h^3}{1032622080}\\
\frac{1191359h^3}{516311040}\\
\frac{158321h^3}{68841472}\\
\frac{64551877h^3}{27880796160}\\
-\frac{38277037h^3}{206524416}
\end{bmatrix},\ R_2=\begin{bmatrix} f_n\end{bmatrix}$$

Notes

Global Journal of Science Frontier Research (F) Volume XXI Issue IV Version I Year 2021

$$D=\begin{bmatrix}
\frac{816523h^3}{11762210880} & -\frac{1827557h^3}{29262643200} & \frac{4340411h^3}{223046369280} & -\frac{310657h^3}{30111259852800}\\
\frac{140041h^3}{1075648} & -\frac{1810517h^3}{8028160} & \frac{59340195h^3}{550731776} & \frac{703301h^3}{2753658880}\\
\frac{61669h^3}{4033680} & -\frac{22313h^3}{1204224} & \frac{774009h^3}{137682944} & -\frac{2369h^3}{688414720}\\
-\frac{39563h^3}{32269440} & \frac{4601h^3}{3763200} & -\frac{239733h^3}{688414720} & \frac{1921h^3}{10326220800}\\
\frac{35261h^3}{28235760} & -\frac{83539h^3}{70246400} & \frac{332721h^3}{963780608} & -\frac{13399h^3}{72283545600}\\
\frac{3356641h^3}{735138180} & -\frac{291271h^3}{73156608} & \frac{6156091h^3}{4646799360} & -\frac{51451h^3}{75278149632}\\
\frac{150989h^3}{96040} & -\frac{5895259h^3}{2150400} & \frac{12481155h^3}{9834496} & \frac{1174009h^3}{245862400}\\
-\frac{2673353h^3}{6050520} & \frac{2792877h^3}{5017600} & -\frac{59549067h^3}{344207360} & \frac{1743611h^3}{15489331200}\\
\frac{100712h^3}{756315} & -\frac{209213h^3}{1505280} & \frac{6755481h^3}{172103680} & -\frac{6473h^3}{309786624}\\
\frac{853303h^3}{6050520} & -\frac{1928953h^3}{15052800} & \frac{13220469h^3}{344207360} & -\frac{320773h^3}{15489331200}\\
\frac{22476493h^3}{163364040} & -\frac{4982251h^3}{45158400} & \frac{9878035h^3}{206524416} & -\frac{8856199h^3}{418211942400}\\
\frac{74773303h^3}{6050520} & -\frac{322920953h^3}{15052800} & \frac{3309380469h^3}{344207360} & \frac{1176351227h^3}{15489331200}
\end{bmatrix},\quad R_3=\begin{bmatrix} f_{n+\frac{1}{8}}\\ f_{n+\frac{1}{7}}\\ f_{n+\frac{1}{6}}\\ f_{n+\frac{1}{2}}\end{bmatrix}$$

The modified form of (9) gives the half-step hybrid block for case one in the form

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix} y_{n+\frac{1}{8}}\\ y_{n+\frac{1}{7}}\\ y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{2}}\\ y'_{n+\frac{1}{8}}\\ y'_{n+\frac{1}{7}}\\ y'_{n+\frac{1}{6}}\\ y'_{n+\frac{1}{2}}\\ y''_{n+\frac{1}{8}}\\ y''_{n+\frac{1}{7}}\\ y''_{n+\frac{1}{6}}\\ y''_{n+\frac{1}{2}}\end{bmatrix}
=\begin{bmatrix}
1 & \frac{h}{8} & \frac{h^2}{128}\\
1 & \frac{h}{7} & \frac{h^2}{98}\\
1 & \frac{h}{6} & \frac{h^2}{72}\\
1 & \frac{h}{2} & \frac{h^2}{8}\\
0 & 1 & \frac{h}{8}\\
0 & 1 & \frac{h}{7}\\
0 & 1 & \frac{h}{6}\\
0 & 1 & \frac{h}{2}\\
0 & 0 & 1\\
0 & 0 & 1\\
0 & 0 & 1\\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} y_n\\ y'_n\\ y''_n\end{bmatrix}
+\begin{bmatrix}
\frac{1327h^3}{7864320}\\
\frac{22987h^3}{98825160}\\
\frac{13h^3}{38880}\\
\frac{h^3}{480}\\
\frac{537h^2}{163840}\\
\frac{977h^2}{252105}\\
\frac{109h^2}{23328}\\
-\frac{h^2}{160}\\
\frac{2057h}{61440}\\
\frac{2411h}{72030}\\
\frac{217h}{6480}\\
-\frac{37h}{240}
\end{bmatrix}
\begin{bmatrix} f_n\end{bmatrix}
+\begin{bmatrix}
\frac{71h^3}{46080} & \frac{79919h^3}{39321600} & \frac{1701h^3}{2621440} & -\frac{53h^3}{117964800}\\
\frac{86144h^3}{37059435} & -\frac{4369h^3}{1440600} & \frac{15849h^3}{16470860} & -\frac{487h^3}{741188700}\\
\frac{8h^3}{2187} & -\frac{343h^3}{72900} & \frac{77h^3}{51840} & -\frac{7h^3}{6998400}\\
\frac{8h^3}{45} & -\frac{343h^3}{1200} & \frac{81h^3}{640} & \frac{7h^3}{28800}\\
\frac{223h^2}{5760} & -\frac{40817h^2}{819200} & \frac{513h^2}{32768} & -\frac{77h^2}{7372800}\\
\frac{7424h^2}{151263} & -\frac{457h^2}{7350} & \frac{3267h^2}{168070} & -\frac{97h^2}{7563150}\\
\frac{688h^2}{10935} & -\frac{45619h^2}{583200} & \frac{53h^2}{2160} & -\frac{7h^2}{437400}\\
\frac{16h^2}{9} & -\frac{2401h^2}{800} & \frac{27h^2}{20} & \frac{17h^2}{3600}\\
\frac{23h}{40} & -\frac{213689h}{307200} & \frac{4347h}{20480} & -\frac{41h}{307200}\\
\frac{20992h}{36015} & -\frac{719h}{1050} & \frac{2538h}{12005} & -\frac{8h}{60025}\\
\frac{704h}{1215} & -\frac{2401h}{3600} & \frac{53h}{240} & -\frac{13h}{97200}\\
\frac{64h}{5} & -\frac{26411h}{1200} & \frac{783h}{80} & \frac{91h}{1200}
\end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{8}}\\ f_{n+\frac{1}{7}}\\ f_{n+\frac{1}{6}}\\ f_{n+\frac{1}{2}}\end{bmatrix}
\quad (10)$$

Case Two:

We consider the case where $\left(u=\frac{1}{5}, v=\frac{1}{4}, w=\frac{1}{3}\right)$ equation (9) becomes

$$A=\begin{bmatrix} \frac{25}{9} & -\frac{32}{9} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{25}{2} & -12 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{25}{h} & \frac{16}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{15}{h} & -\frac{16}{h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{25}{h} & -\frac{24}{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{125}{3h} & -\frac{112}{3h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{75}{h} & -\frac{64}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad y_L=\begin{bmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{1}{2}} \\ y'_{n+\frac{1}{5}} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{1}{2}} \\ y''_{n+\frac{1}{5}} \\ y''_{n+\frac{1}{4}} \\ y''_{n+\frac{1}{3}} \\ y''_{n+\frac{1}{2}} \end{bmatrix}, \quad B=\begin{bmatrix} \frac{2}{9} & 0 & 0 \\ \frac{3}{2} & 0 & 0 \\ -\frac{9}{h} & 1 & 0 \\ -\frac{1}{h} & 0 & 0 \\ \frac{1}{h} & 0 & 0 \\ \frac{13}{3h} & 0 & 0 \\ \frac{11}{h} & 0 & 0 \\ \frac{40}{h^2} & 0 & 1 \\ \frac{40}{h^2} & 0 & 0 \\ \frac{40}{h^2} & 0 & 0 \\ \frac{40}{h^2} & 0 & 0 \\ \frac{40}{h^2} & 0 & 0 \end{bmatrix}, \quad R_1=\begin{bmatrix} y_n \\ y'_n \\ y''_n \end{bmatrix}, \quad C=\begin{bmatrix} \frac{24127h^3}{1166400000} \\ \frac{1693h^3}{12800000} \\ \frac{36887h^3}{22400000} \\ -\frac{31321h^3}{336000000} \\ \frac{3107h^3}{33600000} \\ \frac{1331209h^3}{3265920000} \\ \frac{13723h^3}{16800000} \\ -\frac{247967h^3}{4800000} \\ \frac{18017h^3}{4800000} \\ \frac{2943h^3}{800000} \\ \frac{56099h^3}{14400000} \\ -\frac{2033h^3}{4800000} \end{bmatrix}, \quad R_2=\begin{bmatrix} f_n \end{bmatrix}$$

$$D=\begin{bmatrix} \frac{42181h^3}{559872000} & -\frac{1723h^3}{6075000} & \frac{34679h^3}{259200000} & -\frac{6103h^3}{874800000} \\ \frac{1093h^3}{204800} & -\frac{371h^3}{200000} & \frac{67149h^3}{25600000} & \frac{41h^3}{3200000} \\ \frac{24461h^3}{1075200} & -\frac{22667h^3}{1050000} & \frac{262791h^3}{44800000} & -\frac{5743h^3}{16800000} \\ -\frac{3469h^3}{1075200} & \frac{11447h^3}{5250000} & -\frac{125091h^3}{224000000} & \frac{2603h^3}{84000000} \\ \frac{229h^3}{67200} & -\frac{1347h^3}{700000} & \frac{6021h^3}{11200000} & -\frac{169h^3}{5600000} \\ \frac{2262227h^3}{156764160} & -\frac{201223h^3}{51030000} & \frac{759131h^3}{241920000} & -\frac{353201h^3}{2449440000} \\ \frac{4041h^3}{89600} & -\frac{6281h^3}{262500} & \frac{389313h^3}{11200000} & \frac{2217h^3}{1400000} \\ -\frac{89701h^3}{230400} & \frac{9883h^3}{25000} & -\frac{355077h^3}{3200000} & \frac{23863h^3}{3600000} \\ \frac{5407h^3}{46080} & -\frac{6959h^3}{75000} & \frac{73467h^3}{3200000} & -\frac{4553h^3}{3600000} \\ \frac{16087h^3}{115200} & -\frac{2363h^3}{37500} & \frac{33399h^3}{1600000} & -\frac{2131h^3}{1800000} \\ \frac{778073h^3}{6220800} & \frac{532h^3}{2025000} & \frac{534769h^3}{9600000} & -\frac{155699h^3}{97200000} \\ \frac{60299h^3}{230400} & -\frac{20351h^3}{75000} & \frac{994923h^3}{3200000} & \frac{173863h^3}{3600000} \end{bmatrix}, \quad R_3=\begin{bmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{2}} \end{bmatrix}$$

Notes

Notes

The modified form of (9) gives the half-step hybrid block for case two in the form

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{5}}\\ y_{n+\frac{1}{4}}\\ y_{n+\frac{1}{3}}\\ y_{n+\frac{1}{2}}\\ y'_{n+\frac{1}{5}}\\ y'_{n+\frac{1}{4}}\\ y'_{n+\frac{1}{3}}\\ y'_{n+\frac{1}{2}}\\ y''_{n+\frac{1}{5}}\\ y''_{n+\frac{1}{4}}\\ y''_{n+\frac{1}{3}}\\ y''_{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 1&\frac{h}{5}&\frac{h^2}{50}\\ 1&\frac{h}{4}&\frac{h^2}{32}\\ 1&\frac{h}{3}&\frac{h^2}{18}\\ 1&\frac{h}{2}&\frac{h^2}{8}\\ 0&1&\frac{h}{5}\\ 0&1&\frac{h}{4}\\ 0&1&\frac{h}{3}\\ 0&1&\frac{h}{2}\\ 0&0&1\\ 0&0&1\\ 0&0&1\\ 0&0&1 \end{bmatrix} \begin{bmatrix} y_n\\ y'_n\\ y''_n \end{bmatrix} + \begin{bmatrix} \frac{4619h^3}{6562500}\\ \frac{2069h^3}{1720320}\\ \frac{239h^3}{102060}\\ \frac{31h^3}{5376}\\ \frac{537h^2}{62500}\\ \frac{349h^2}{30720}\\ \frac{233h^2}{14580}\\ -\frac{h^2}{40}\\ \frac{1039h}{18750}\\ \frac{85h}{1536}\\ \frac{h}{18}\\ -\frac{5h}{96} \end{bmatrix} \begin{bmatrix} f_n \end{bmatrix} + \begin{bmatrix} \frac{2039h^3}{630000} & -\frac{5888h^3}{1640625} & \frac{9153h^3}{8750000} & -\frac{316h^3}{4921875}\\ \frac{13375h^3}{2064384} & -\frac{187h^3}{26880} & \frac{459h^3}{229376} & -\frac{157h^3}{1290240}\\ \frac{3625h^3}{244944} & -\frac{128h^3}{8505} & \frac{197h^3}{45360} & -\frac{4h^3}{15309}\\ \frac{1375h^3}{32256} & -\frac{17h^3}{420} & \frac{243h^3}{17920} & -\frac{13h^3}{20160}\\ \frac{467h^2}{9000} & -\frac{864h^2}{15625} & \frac{1971h^2}{125000} & -\frac{134h^2}{140625}\\ \frac{2875h^2}{36864} & -\frac{19h^2}{240} & \frac{459h^2}{20480} & -\frac{31h^2}{23040}\\ \frac{2125h^2}{17496} & -\frac{416h^2}{3645} & \frac{37h^2}{1080} & -\frac{22h^2}{10935}\\ \frac{125h^2}{576} & -\frac{h^2}{5} & \frac{27h^2}{320} & -\frac{h^2}{720}\\ \frac{38h}{75} & -\frac{4576h}{9375} & \frac{837h}{6250} & -\frac{74h}{9375}\\ \frac{1625h}{3072} & -\frac{11h}{24} & \frac{135h}{1024} & -\frac{h}{128}\\ \frac{125h}{243} & -\frac{32h}{81} & \frac{h}{6} & -\frac{2h}{243}\\ \frac{125h}{192} & -\frac{2h}{3} & \frac{27h}{64} & \frac{h}{24} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{5}}\\ f_{n+\frac{1}{4}}\\ f_{n+\frac{1}{3}}\\ f_{n+\frac{1}{2}} \end{bmatrix} \tag{11}$$

III. ANALYSIS OF BASIC PROPERTIES OF THE METHOD

a) *Order of the Block*

According to fatunla (1991) and lambert (1973) the truncation error associated with (9) is defined by

$$L[y(x);h] = \sum_{j=0,u,v}\left(\alpha_j(t)y_{n+j}\right) - h^3\beta_0 y'''(x+jh) - h^3\beta_u y'''(x+uh) - h^3\beta_v(y'''(x+vh) - h^3\beta_w y'''(x+wh) - h^3\beta_{\frac{1}{2}} y'''(x+\tfrac{1}{2}h) \tag{12}$$

Assumed that $y(x)$ can be differentiated. Expanding (12) in Taylor's series and comparing the coefficient of h gives the expression

$$L\{y(x):h\} = C_0 y(x) + C_1 y'(x) + \,...+ C_p h^p y^p(x) + C_{p+1}h^{p+1}y^{p+1}(x) + C_{p+2}h^{p+2}y^{p+2}(x) + \ldots$$

Where the constant coefficients are given below

$$C_0 = \sum_{j=0}^{k}\alpha_j \quad , \quad C_1 = \sum_{j=1}^{k} j\alpha_j$$

$$C_q = \frac{1}{q!}\sum_{j=0}^{k} j^q\alpha_j - q(q-1)(q-2)\left(\sum_{j=0} j^{q-3}\beta_j + u^{q-3}\beta_u + v^{q-3}\beta_v + w^{q-3}\beta_w + \left(\frac{1}{2}\right)^{q-3}\beta_{\frac{1}{2}}\right), \quad q = 2, 3, \ldots$$

Definition: Linear operator L and associated block formula are said to be of order p, *if* $C_0 = C_1 = , . . . = C_p = C_{p+1} = C_{p+2} = 0.$ *and* $C_{p+3} \neq 0.$ C_{p+3} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+3} h^{p+3} y^{p+3}(x) + O\left(h^{p+4}\right)$

For case one of our the new half-step third derivative method,

$$L\{y(x):h\} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{8}} \\ y_{n+\frac{1}{7}} \\ y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{2}} \\ y'_{n+\frac{1}{8}} \\ y'_{n+\frac{1}{7}} \\ y'_{n+\frac{1}{6}} \\ y'_{n+\frac{1}{2}} \\ y''_{n+\frac{1}{8}} \\ y''_{n+\frac{1}{7}} \\ y''_{n+\frac{1}{6}} \\ y''_{n+\frac{1}{2}} \end{bmatrix} - \begin{bmatrix} 1 & \frac{h}{8} & \frac{h^2}{128} \\ 1 & \frac{h}{7} & \frac{h^2}{98} \\ 1 & \frac{h}{6} & \frac{h^2}{72} \\ 1 & \frac{h}{2} & \frac{h^2}{8} \\ 0 & 1 & \frac{h}{8} \\ 0 & 1 & \frac{h}{7} \\ 0 & 1 & \frac{h}{6} \\ 0 & 1 & \frac{h}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_n \\ y'_n \\ y''_n \end{bmatrix} - \begin{bmatrix} \frac{1327h^3}{7864320} \\ \frac{22987h^3}{98825160} \\ \frac{13h^3}{38880} \\ \frac{h^3}{480} \\ \frac{537h^2}{163840} \\ \frac{977h^2}{252105} \\ \frac{109h^2}{23328} \\ -\frac{h^2}{160} \\ \frac{2057h}{61440} \\ \frac{2411h}{72030} \\ \frac{217h}{6480} \\ -\frac{37h}{240} \end{bmatrix} \left[f_n\right] - \begin{bmatrix} \frac{71h^3}{46080} & \frac{79919h^3}{39321600} & \frac{1701h^3}{2621440} & -\frac{53h^3}{117964800} \\ \frac{86144h^3}{37059435} & -\frac{4369h^3}{1440600} & \frac{15849h^3}{16470860} & -\frac{487h^3}{741188700} \\ \frac{8h^3}{2187} & -\frac{343h^3}{72900} & \frac{77h^3}{51840} & -\frac{7h^3}{6998400} \\ \frac{8h^3}{45} & -\frac{343h^3}{1200} & \frac{81h^3}{640} & \frac{7h^3}{28800} \\ \frac{223h^2}{5760} & -\frac{40817h^2}{819200} & \frac{513h^2}{32768} & -\frac{77h^2}{7372800} \\ \frac{7424h^2}{151263} & -\frac{457h^2}{7350} & \frac{3267h^2}{168070} & -\frac{97h^2}{7563150} \\ \frac{688h^2}{10935} & -\frac{45619h^2}{583200} & \frac{53h^2}{2160} & -\frac{7h^2}{437400} \\ \frac{16h^2}{9} & -\frac{2401h^2}{800} & \frac{27h^2}{20} & \frac{17h^2}{3600} \\ \frac{23h}{40} & -\frac{213689h}{307200} & \frac{4347h}{20480} & -\frac{41h}{307200} \\ \frac{20992h}{36015} & -\frac{719h}{1050} & \frac{2538h}{12005} & -\frac{8h}{60025} \\ \frac{704h}{1215} & -\frac{2401h}{3600} & \frac{53h}{240} & -\frac{13h}{97200} \\ \frac{64h}{5} & -\frac{26411h}{1200} & \frac{783h}{80} & \frac{91h}{1200} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{7}} \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{2}} \end{bmatrix} \tag{13}$$

expanding (13) in Taylor series and comparing the coefficient of h gives $C_0 = C_1 = C_2 = C_3 = . . . = C_7 = 0$ and

$$C_8 = \begin{bmatrix} 3.8985e{-11}, 5.6772e{-11}, 8.6066e{-11}, 5.5361e{-09}, 1.5509e{-07}, 1.551e{-07}, \\ 1.5508e{-07}, 9.0422e{-08}, 1.1252e{-08}, 1.124e{-08}, 1.127e{-08}, 8.7839e{-07} \end{bmatrix}^T$$

Hence our method is of order four (4).

Notes

Notes

For case two of the new half-step third derivative method,

$$L\{y(x):h\}=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{5}}\\y_{n+\frac{1}{4}}\\y_{n+\frac{1}{3}}\\y_{n+\frac{1}{2}}\\y'_{n+\frac{1}{5}}\\y'_{n+\frac{1}{4}}\\y'_{n+\frac{1}{3}}\\y'_{n+\frac{1}{2}}\\y''_{n+\frac{1}{5}}\\y''_{n+\frac{1}{4}}\\y''_{n+\frac{1}{3}}\\y''_{n+\frac{1}{2}}\end{bmatrix}-\begin{bmatrix}1&\frac{h}{5}&\frac{h^2}{50}\\1&\frac{h}{4}&\frac{h^2}{32}\\1&\frac{h}{3}&\frac{h^2}{18}\\1&\frac{h}{2}&\frac{h^2}{8}\\0&1&\frac{h}{5}\\0&1&\frac{h}{4}\\0&1&\frac{h}{3}\\0&1&\frac{h}{2}\\0&0&1\\0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_n\\y'_n\\y''_n\end{bmatrix}-\begin{bmatrix}\frac{4619h^3}{6562500}\\\frac{2069h^3}{1720320}\\\frac{239h^3}{102060}\\\frac{31h^3}{5376}\\\frac{537h^2}{62500}\\\frac{349h^2}{30720}\\\frac{233h^2}{14580}\\-\frac{h^2}{40}\\\frac{1039h}{18750}\\\frac{85h}{1536}\\\frac{h}{18}\\-\frac{5h}{96}\end{bmatrix}\left[f_n\right]-\begin{bmatrix}\frac{2039h^3}{630000}&-\frac{5888h^3}{1640625}&\frac{9153h^3}{8750000}&-\frac{316h^3}{4921875}\\\frac{13375h^3}{2064384}&-\frac{187h^3}{26880}&\frac{459h^3}{229376}&-\frac{157h^3}{1290240}\\\frac{3625h^3}{244944}&-\frac{128h^3}{8505}&\frac{197h^3}{45360}&-\frac{4h^3}{15309}\\\frac{1375h^3}{32256}&-\frac{17h^3}{420}&\frac{243h^3}{17920}&-\frac{13h^3}{20160}\\\frac{467h^2}{9000}&-\frac{864h^2}{15625}&\frac{1971h^2}{125000}&-\frac{134h^2}{140625}\\\frac{2875h^2}{36864}&-\frac{19h^2}{240}&\frac{459h^2}{20480}&-\frac{31h^2}{23040}\\\frac{2125h^2}{17496}&-\frac{416h^2}{3645}&\frac{37h^2}{1080}&-\frac{22h^2}{10935}\\\frac{125h^2}{576}&-\frac{h^2}{5}&\frac{27h^2}{320}&-\frac{h^2}{720}\\\frac{38h}{75}&-\frac{4576h}{9375}&\frac{837h}{6250}&-\frac{74h}{9375}\\\frac{1625h}{3072}&-\frac{11h}{24}&\frac{135h}{1024}&-\frac{h}{128}\\\frac{125h}{243}&-\frac{32h}{81}&\frac{h}{6}&-\frac{2h}{243}\\\frac{125h}{192}&-\frac{2h}{3}&\frac{27h}{64}&\frac{h}{24}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{5}}\\f_{n+\frac{1}{4}}\\f_{n+\frac{1}{3}}\\f_{n+\frac{1}{2}}\end{bmatrix} \quad (14)$$

expanding (14) in Taylor series and comparing the coefficient of h gives $C_0=C_1=C_2=C_3=\ldots=C_7=0$ and

$$C_8=\begin{bmatrix}1.4857e-09, 2.7803e-09, 5.897e-09, 1.5501e-08, 2.1577e-08, 3.0195e-08,\\ 4.4681e-08, 6.7171e-08, 1.7296e-07, 1.718e-07, 1.7623e-07, 7.2338e-07\end{bmatrix}^T$$

Hence our method is of order four (4).

b) Consistency

The hybrid block method (13) and (14) is said to be consistent if it has an order more than or equal to one.

Therefore, the new half-step third derivative hybrid block method is consistent.

c) Zero Stability of Our Method

Definition 2: A block method (10) and (11) is said to be zero-stable if as $h \to 0$, the root $z_i, i=1(1)k$ of the first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right]=0$ Satisfies $|z_i| \le 1$ and for those roots with $|z_i|=1$, multiplicity must not exceed two.

Hence, the new half-step third derivative hybrid block method is zero-stable.

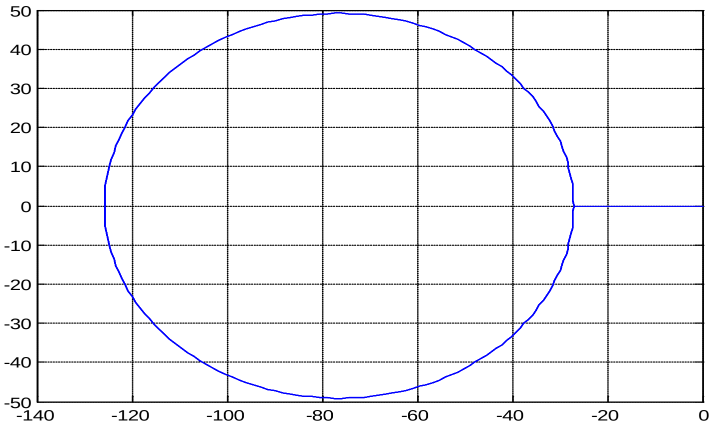
d) Regions of Absolute Stability

For case one: The stability polynomial for half step with three off step point gives

$$\left(-\frac{13}{440484715560960}w^3+\frac{1}{11012117889024000}w^4\right)h^{12}+\left(\frac{31}{7283146752000}w^4\right.$$

$$\left.-\frac{38147}{22658678784000}w^3\right)h^9+\left(\frac{1007}{159318835200}w^4-\frac{16978247}{955913011200}w^3\right)h^6+\left(\right.$$

$$\left.-\frac{235423}{995742720}w^4-\frac{37143145}{1593188352}w^3\right)h^3+w^4-\frac{13}{8}w^3$$



Stability Region for Case Two

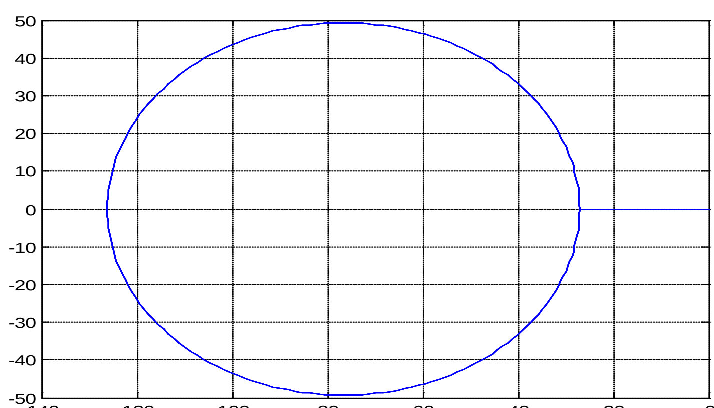
For case two:

The stability polynomial for half step with three offstep point gives

$$\left(-\frac{1}{2239488000000}w^3+\frac{1}{62705664000000}w^4\right)h^{12}+\left(\frac{1}{6220800000}w^4\right.$$

$$\left.-\frac{8983}{1393459200000}w^3\right)h^9+\left(-\frac{525359}{21772800000}w^3+\frac{97}{580608000}w^4\right)h^6$$

$$+\left(\frac{287}{12960000}w^4-\frac{2460731}{103680000}w^3\right)h^3+w^4-\frac{13}{8}w^3$$



Stability Region for Case One

Notes

e) Numerical Example

Problem I We consider a highly stiff problem

$$y'''-y''+y'-y=0\ , y(0)=1, y'(0)=0, y''(0)=-1,$$

Exact Solution: $y(x)=\cos x,\ \ h=\frac{1}{100}$

Notes

Table 1: Comparison of the proposed new half-step method with Adoghe & Omole (2019)

x-values	Error in case one of our method	Error in case two our method	Error in Adoghe & Omole 2019
0.01	6.100e-20	3.0000e-20	0.0000e+00
0.02	1.200e-19	6.0000e-20	1.1102e-16
0.03	1.900e-19	1.2000e-19	4.4409e-16
0.04	2.500e-19	1.6000e-19	5.8842e-15
0.05	3.200e-19	1.9000e-19	2.6201e-14
0.06	3.900e-19	2.5000e-19	8.3822e-14
0.07	4.500e-19	2.8000e-19	2.0750e-13
0.08	5.100e-19	3.2000e-19	4.4142e-13
0.09	5.600e-19	3.5000e-19	8.4743e-13
0.10	6.300e-19	4.0000e-19	1.5086e-12

Problem II We consider a highly stiff problem

$$y'''+5y''+7y'+3y=0, y(0)=1, y'(0)=0, y''(0)=-1,$$

Exact Solution: $y(x)=e^{-x}+xe^{-x},\ \ h=\frac{1}{10}$

Table 2: Comparison of the proposed new half-step method with Mohammed & Adeniyi 2014

x-values	Error in case one of our method	Error in case two our method	Error in Mohammed & Adeniyi 2014
0.1	1.0434e-14	4.1832e-15	1.0000e-10
0.2	9.8731e-14	1.8599e-14	3.0000e-10
0.3	3.1317e-13	4.4245e-14	7.0000e-10
0.4	6.6668e-13	8.0431e-14	7.0000e-10
0.5	1.1507e-12	1.2552e-13	6.0000e-10
0.6	1.7445e-12	1.7743e-13	2.0000e-10
0.7	2.4220e-12	2.3395 e-13	9.0000e-10
0.8	3.1554e-12	2.9298e-13	2.8000e-09
0.9	3.9178e-12	3.5259e-13	5.4000e-09
1.0	4.6852e-12	4.1112e-13	3.5000e-09

Problem III: We consider the third order ODE

$$y'''=-4y'+x,\ \ y(0)=1, y'(0)=0, y''(0)=1$$

Exact Solution: $y(x)=\frac{3}{16}(1-\cos 2x)+\frac{x^2}{8},\ h=\frac{1}{10}$

Table 3: Comparison of the proposed new half-step method with Adebayo & Adebola (2016)

x-values	Error in case one of our method	Error in case two our method	Error in Adebayo &Adebola (2016)
0.1	7.9512e-14	3.1484e-14	2.970e-08
0.2	8.6717e-13	1.5843e-13	1.988e-07
0.3	3.1385e-12	4.2379e-13	6.508e-07
0.4	7.5504e-12	8.5820e-13	1.5480e-06
0.5	1.4585e-11	1.4765e-12	3.062e-06
0.6	2.4504e-11	2.2752e-12	5.3625e-06
0.7	3.7317e-11	3.2313e-12	8.6068e-06
0.8	5.2765e-11	4.3022e-12	1.2926e-05
0.9	7.0321e-11	5.4266e-12	1.8118e-05
1.0	8.9206e-11	6.5277e-12	2.5129e-05

Problem IV: $y'''=\exp(x),\ \ y(0)=3, y'(0)=1, y''(0)=5,\ 0\le x\le 1.$

Exact Solution: $y(x)=2+2x^2+e^x \text{ with } h=\frac{1}{10}$

Notes

Notes

Table 4: Comparison of the proposed new half-step method with Adeyeye & Omar (2018)

x-values	Error in case one of our method	Error in case two our method	Error in Adeyeye & Omar (2018)
0.1	1.70091e-15	4.54792e-15	6.34270e-13
0.2	1.91790e-14	3.06193e-14	2.32882e-12
0.3	7.25153e-14	9.80503e-14	5.44348e-12
0.4	1.83912e-13	2.28761-13	9.85317e-12
0.5	3.77884e-13	4.46995e-13	1.59974e-11
0.6	6.81541e-13	7.79496e-13	2.37223e-11
0.7	1.12480e-12	1.25594e-12	3.35679e-11
0.8	1.74091e-12	1.90890e-12	4.53443e-11
0.9	2.56633e-12	2.77460e-12	5.97084e-11
1.0	3.64152e-12	3.89290e-12	7.64322e-11

IV. Conclusions

It is evident from the above tables that our proposed new half-step third derivative hybrid block methods are converging faster than the existing method and can handle stiff problems.

Comparing the new method with the existing method respectively shows that the new half-step third derivative hybrid block method performs better on stiff and highly stiff problems than the existing method which is of high Order as compared to ours which is of Order four.

References Références Referencias

1. Adeyeye O, Omar Z (2018): New Self Starting Approach for Solving Special Third Order Initial Value Problems. International Journal of Pure and Applied Mathematics Vol 118 No. 3, 511-517.
2. Adesanya AO, Fasansi MK, Odekunle MR (2013): One Step Three Hybrid Block Predictor-Corrector Method for the solution of $y''' = f(x, y, y', y'')$ Applied and computational Mathematics Vol 2 (4)137. doi:10.4172/21689679.1000137.
3. Adebayo OA, Adebola EO (2016): One Step Hybrid Method for the Numerical Solution of General Third Order Ordinary Differential Equations. International Journal of Mathematical Sciences, Vol. 2, 5, 1-12.
4. Adoghe LO, Omole EO (2019): A Fifth-Fourth Continuous Block Implicit Hybrid Method for the Solution of Third Order Initial Value Problems in Ordinary Differential Equations. Applied and Computational Mathematics, Vol. 8(3), 50-57.
5. Kuboye JO, Omar Z (2015): Numerical solution of Third Order Ordinary Differential Equations Using a Seven-Step Block Method. International Journal of Mathematical Analysis Vol.9.No.15,743-754.
6. Olabode BT, Momoh AL (2016): Continuous Hybrid Multistep Methods with Legendre Basis Function for Direct Treatment of Second Order Stiff ODEs American Journal of Computational and Applied Mathematics 2016, 6(2): 38-49 DOI: 10.5923/ j.ajcam.20160602.03.
7. Kayode SJ, Obuarhua FO (2013): Continuous y-function Hybrid Methods for Direct Solution of Differential Equations, International Journal of Differential Equations and Application12, 37-48.

8. James A, Adesanya A, Joshua S (2013): Continuous block method for the solution of second order initial value problems of ordinary differential equation. Int.J.Pure Appl.Math. 83, 405-416.
9. Kayode SJ, Adeyeye O (2011): A 3-step hybrid method for direct solution of second order initial value problems. Australia Journal of Basic Applied Science. 5, 2121-2126.
10. Mohammed U, Adeniyi RB (2014): A Three-step implicit hybrid linear multistep method for the Solution of third order ordinary different equations. General Mathematics Notes, Vol 25, No 1, pp 62-74.
11. Obarhua FO, Kayode SJ, Areo EA, (2018): An Accurate Four-Point Hybrid Block Method for The Solution of General Third Order Differential Equations Journal of the Nigerian Association of Mathematical Physics Vol 48., 47-56.

Notes