Online ISSN : 2249-4626
Print ISSN : 0975-5896
DOI : 10.17406/GJSFR

Global Journal of Science Frontier Research: A

Physics and Space Science

Fluid Gauge Theory

Coupling of Spin Angular Momenta

Highlights

Models of Financial Markets

Gauge Symmetries in Physical Fields

Discovering Thoughts, Inventing Future

VOLUME 21 ISSUE 4 VERSION 1.0

Global Journal of Science Frontier Research: A
Physics & Space Science

Global Journal of Science Frontier Research: A
Physics & Space Science

Volume 21 Issue 4 (Ver. 1.0)

Open Association of Research Society

Incorporation No.: 0423089
License No.: 42125/022010/1186
Registration No.: 430374
Import-Export Code: 1109007027
Employer Identification Number (EIN):
USA Tax ID: 98-0673427

Global Journals Inc.

*(A Delaware USA Incorporation with "Good Standing"; **Reg. Number: 0423089**)*

Sponsors: *Open Association of Research Society*
Open Scientific Standards

Publisher's Headquarters office

Global Journals® Headquarters
945th Concord Streets,
Framingham Massachusetts Pin: 01701,
United States of America
USA Toll Free: +001-888-839-7392
USA Toll Free Fax: +001-888-839-7392

Offset Typesetting

Global Journals Incorporated
2nd, Lansdowne, Lansdowne Rd., Croydon-Surrey,
Pin: CR9 2ER, United Kingdom

Packaging & Continental Dispatching

Global Journals Pvt Ltd
E-3130 Sudama Nagar, Near Gopur Square,
Indore, M.P., Pin:452009, India

Find a correspondence nodal officer near you

To find nodal officer of your country, please email us at *local@globaljournals.org*

eContacts

Press Inquiries: *press@globaljournals.org*
Investor Inquiries: *investors@globaljournals.org*
Technical Support: *technology@globaljournals.org*
Media & Releases: *media@globaljournals.org*

Pricing (Excluding Air Parcel Charges):

Yearly Subscription (Personal & Institutional)
250 USD (B/W) & 350 USD (Color)

Contents of the Issue

Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 21 Issue 4 Version 1.0 Year 2021
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Gauge Symmetries in Physical Fields (Review)

By Tsutomu Kambe

University of Tokyo

Abstract- Gauge invariance is one of the fundamental symmetries in theoretical physics. In this paper, the gauge symmetry is reviewed to see how it is working in fundamental physical fields: *Electromagnetism, Quantum Electro Dynamics* and *Geometric Theory of Gravity*. In the 19th century, the gauge invariance was recognized as a mathematical non-uniqueness of the electromagnetic potentials. Real recognition of the gauge symmetry and its physical significance required two new fields developed in the 20th century: the relativity theory for physics of the world structure of linked 4d-spacetime and the quantum mechanics for the new dimension of a phase factor in complex representation of wave function. Finally the gauge theory was formulated on the basis of the gauge principle which played a role of guiding principle in the study of physical fields such as Quantum Electrodynamics, Particle Physics and Theory of Gravitation. Fluid mechanics of a perfect fluid can join in this circles, which is another motivation of the present review. There is a hint of fluid gauge theory in the general representation of rotational flows of an ideal compressible fluid satisfying the Euler's equation, found in 2013 by the author. In fact, law of mass conservation can be deduced from the gauge symmetry equipped in the new system of fluid-flow field combined with a gauge field, rather than given *a priori*.

Keywords: *gauge principle, covariant derivative, current conservation, maxwell equations, theory of gravitation.*

GJSFR-A Classification: *FOR Code: 029999p*

GAUGESYMMETRIESINPHYSICALFIELDSREVIEW

Strictly as per the compliance and regulations of:

Gauge Symmetries in Physical Fields (Review)

Tsutomu Kambe

Abstract- Gauge invariance is one of the fundamental symmetries in theoretical physics. In this paper, the gauge symmetry is reviewed to see how it is working in fundamental physical fields: *Electromagnetism, Quantum ElectroDynamics* and *Geometric Theory of Gravity*. In the 19th century, the gauge invariance was recognized as a mathematical non-uniqueness of the electromagnetic potentials. Real recognition of the gauge symmetry and its physical significance required two new fields developed in the 20th century: the relativity theory for physics of the world structure of linked 4d-spacetime and the quantum mechanics for the new dimension of a phase factor in complex representation of wave function. Finally the gauge theory was formulated on the basis of the gauge principle which played a role of guiding principle in the study of physical fields such as Quantum Electrodynamics, Particle Physics and Theory of Gravitation. Fluid mechanics of a perfect fluid can join in this circles, which is another motivation of the present review. There is a hint of fluid gauge theory in the general representation of rotational flows of an ideal compressible fluid satisfying the Euler's equation, found in 2013 by the author. In fact, law of mass conservation can be deduced from the gauge symmetry equipped in the new system of fluid-flow field combined with a gauge field, rather than given *a priori*.

Keywords: gauge principle, covariant derivative, current conservation, maxwell equations, theory of gravitation.

I. Introduction

Gauge invariance is one of the fundamental symmetries in modern theoretical physics. The gauge invariance was recognized in the 19th century as a mathematical non-uniqueness of potentials that exists despite the uniqueness of observable electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In the 20th century, physical significance of the gauge symmetry was recognized very fundamental and played a role of guiding principle in the study of physical fields such as Electromagnetism, Particle physics and Theory of Gravitation.

It took almost a century to recognize its fundamental physical significance, resulting in, finally, successful formulation of the Gauge Principle. In particular, the *gauge theory* played vital roles in the remarkable development of modern particle physics which was revolutionary (*e.g.* Aitchison & Hey (2013), Utiyama (1956)). In fact, historical development of the gauge theory took gradual and zigzag processes.

In the present paper, firstly, historical developments of gauge theory are reviewed from its initial gauge transformation to later theory of gauge principle taking a zigzag way from one physical field to another, and secondly, possible application of the gauge theory is envisaged to fluid-flow field although the field of fluid-flow is not listed in the literature reviewed.

a) *Historical development of gauge transformations*

What is now generally known as a gauge transformation of the electromagnetic potentials was discovered in 19th century in the process of formulation of classical electrodynamics from mathematical point of view (rather than physics) by its pioneers (Faraday, Neumann, Weber, Kirchhoff, Maxwell, Lorenz, Helmholtz, Lorentz and others: according to Jackson & Okun (2001)). It was, in fact, non-uniqueness of a vector potential $\boldsymbol{A}$ in mathematical representation of electromagnetic field that exists despite the uniqueness of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$. This is now referred to as *local gauge invariance* of Maxwell's equations. The law of electromagnetic induction discovered by Faraday (1831) is represented mathematically by the first of the following pair of Maxwell equations:

Author: (Former Professor at) Department of Physics, University of Tokyo, Tokyo, Japan, (Home) Higashi-yama 2-11-3, Meguro-ku, Tokyo, Japan. e-mail: kambe@ruby.dti.ne.jp

$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0. \tag{1.1}$$

The second is an outcome of the fact that the magnetic field $\boldsymbol{B}$ is generated by electric currents (Jackson (1999, Chap.5)), implying non-existence of magnetic monopoles. In Maxwell's electromagnetic theory (1856), the vector potential played an important role. Introducing a 3-vector potential $\boldsymbol{A} = (A_1, A_2, A_3)$ and a scalar potential $\Phi^{em} = -A_0$, and defining $\boldsymbol{E}$ and $\boldsymbol{B}$ by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -c^{-1}\,\partial_t \boldsymbol{A} - \nabla \Phi^{em}, \tag{1.2}$$

the above pair of equations (1.1) are satisfied identically. This led to a finding that, using an arbitrary differentiable scalar function Ψ^e, the following transformation of the potentials $\boldsymbol{A}$ and Φ^{em},

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla \Psi^e, \qquad \Phi^{em} \to \Phi^{em} - \partial_t \Psi^e, \tag{1.3}$$

revealed a significant property, what is now called the *gauge transformation*, of the electromagnetic field. Maxwell (1873) noticed the invariance of $\boldsymbol{B}$ only by the first of the transformation (1.3), but missed the second one because he relied on the gauge condition $\nabla \cdot \boldsymbol{A} = 0$. The simultaneous two transformations of (1.3) was established by L. V. Lorenz (1867) on the basis of the following gauge condition,

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c}\partial_t \Phi^{em} = 0. \tag{1.4}$$

It is remarkable that the observable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ of (1.2) are invariant in spite of the transformation (1.3). This was the invariance known in the electromagnetic theory of the 19th century. In modern gauge theory, the gauge condition (1.4) is often referred to as *Lorentz* condition, according to Dutch physicist H. A. Lorentz who was one of the key figures in the final formulation of classical electrodynamics (1904) including the condition (1.4), while the former Danish physicist L. V. *Lorenz* (1867) introduced *first* the condition (1.4) (Jackson & Okun, 2001).

In the 19th-century classical electrodynamics, the transformation (1.3) was understood as meaning simply *non-uniqueness* of the vector potential $\boldsymbol{A}$ and scalar potential Φ in a mathematical sense. Its physical significance was not recognized until the 20th-century physics was developed. In the relativity theory of Einstein (1905, 1915), four dimensional (4d) spacetime $x^\nu = (x^0, x^1, x^2, x^3)$ with $x^0 = c\,t$ was introduced under the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) = \eta^{\mu\nu}$.† The structure of electromagnetism is most fitted to the 4d-spacetime. For example, the Lorenz condition (1.4) can be represented compactly as $\partial A^\nu / \partial x^\nu = 0$ in the 4-d spacetime, where $A^\nu = (\Phi, \boldsymbol{A})$. See (1.8) for the difference between the covariant (downstairs) vector A_μ and the contravariant (upstairs) vector A^ν. Scalar product in the Minkowski space is formed like $A_\mu\,\mathrm{d}x^\mu = \eta_{\mu\nu} A^\nu \mathrm{d}x^\mu$ by the pair of a covariant vector A_μ and a contravariant vector $\mathrm{d}x^\mu$ ((see 1.5)). [Concerning the difference of transformation property between the covariant and contravariant vectors, see the footnote to Appendix A.1.]

Stimulated by Einstein's relativity theory, Weyl attempted in 1918 to reinterpret the same transformation (1.3) of electromagnetic 4-potentials A^ν, but turned out to be unsuccessful. The term *gauge* (actually the German term *Eich*) was used to this transformation by Weyl (1918) first. He proposed to unify electromagnetism and gravity geometrically by attaching a scale factor l of the form $l \propto \exp[\int \phi_k(\boldsymbol{x}) dx^k]$ where its variation is given by $\delta l = l \phi_k\, \delta x^k$. Although this received unfavorable response from Einstein to be in disagreement with observation, after the advent of the quantum theory, its interpretation was renewed by London (1927) that the Weyl's proposal could be used in quantum theory by changing the *scale* factor to a *phase* factor by attaching it to the wave function $\boldsymbol{\psi}(x^\nu)$ of quantum mechanics in the form,

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I 2 Year 2021

$$\Psi(x^\nu) = \exp\left[i\gamma \int A_\mu(\boldsymbol{x}) dx^\mu\right] \cdot \boldsymbol{\psi}(x^\nu), \tag{1.5}$$

where $\gamma = e/\hbar$ with e a charge, and the function $\boldsymbol{\psi}(x^\nu)$ satisfies the Schrödinger equation:

$$i\hbar\, \partial_t \boldsymbol{\psi} = -(\hbar^2/2m)\, \nabla^2 \boldsymbol{\psi} + eV\, \psi, \tag{1.6}$$

interpreted in section II b) and given by (2.29). Physical significance of the gauge invariance was upheld later by H. Wyle in 1929, who proclaimed this invariance as a *General Principle* and called it *gauge-invariance* (*Eichinvarianz* in German). The gauge invariance is a symmetry rooted at the deepest level of physics, as interpreted next in section I b).

In quantum mechanics, the transformation (1.3) was understood as a phase transformation of the wave function of Schrödinger's equation. In the theory of *gravitation*, on the other hand, the gauge transformation was generalized to such transformations that the vectors or curvature tensors ‡ characterizing the gravitational field as *physical reality* do not change (or satisfy associated transformation laws) in spite of coordinate transformations, where the coordinate frames are taken arbitrarily by the theory (its details are given in section III c) iii. and III d) for weak gravitational field). In *fluid mechanics* too, the convective derivative (following fluid motion) can be shown to satisfy invariance with respect to generalized gauge-transformation, presented in section IV c).

b) A hint of gauge principle with the argument reversed

Historically, the gauge symmetry has been established through zigzag courses. Next formulation may be a typical example. Observing the phase part of the extended wave function $\Psi(x^\nu)$ of (1.5), the phase factor implies existence of the following one-form $\mathcal{A}$ in the spacetime (x^μ), defined by

$$\mathcal{A} = A_\mu dx^\mu = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3, \tag{1.7}$$

$$A_\mu = \eta_{\mu\nu} A^\nu = (-\Phi^{em}, \boldsymbol{A}). \qquad A^\nu = (\Phi^{em}, \boldsymbol{A}). \tag{1.8}$$

The extended wave function $\Psi(x^\nu)$ implies a certain geometrical structure in the spacetime x^μ, furnished with a field A_μ existing in the 4-d spacetime x^ν. The field A_μ possesses an interesting property which is now presented.

The pair of fields $\boldsymbol{E}$ and $\boldsymbol{B}$ of (1.2) are derived from (1.7). In fact, taking exterior differential d of $\mathcal{A}$, we obtain the *field strength* two-form $\mathcal{F}$:

$$\mathcal{F} = \mathsf{d}\mathcal{A} = \sum \frac{1}{2} F_{\nu\lambda}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda, \qquad F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu, \tag{1.9}$$

$$\mathcal{F} \Leftrightarrow (\, F_{\nu\lambda}\,) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \tag{1.10}$$

where $\boldsymbol{E} = (E_k)$ and $\boldsymbol{B} = (B_k)$ are defined by (1.2). The pair of equations (1.1) are also obtained from (1.9) by taking, once more, exterior differential of $\mathcal{F} = d\mathcal{A}$, yielding

† Greek letters such as $\alpha, \beta, \mu, \nu, \lambda, \cdots$ take the quartet $(0,1,2,3)$ to denote 4d-spacetime components, whereas Latin letters such as $i, j, k, \cdots$ take the triplet $(1,2,3)$ to denote 3-space components.

‡ In differential geometry, a vector (or a tensor) in an n-dimensional coordinate frame U is not a simple n-tuple array (or a simple $n \times n$ matrix, respectively) of real numbers, but they must follow certain transformation laws when mapped to another n-dimensional coordinate frame V.

$\mathsf{d}\mathcal{F} = \mathsf{d}^2\mathcal{A} \equiv 0$. Its detailed expressions are given in section II a) i. Thus, the definition $\mathcal{A} = A_\mu dx^\mu$ of (1.7) is sufficient for deriving the pair of Maxwell equations (1.1).

Let us consider the *gauge transformation* concerning the one-form $\mathcal{A}$, defined by

$$\mathcal{G}: \; A_\nu \equiv A_\nu^{(old)} \;\to\; A_\nu^{(new)} \equiv A'_\nu = A_\nu^{(old)} - \partial_\nu \Theta, \tag{1.11}$$

equivalent to (1.3), where Θ is an arbitrary differentiable function. Then, we have $\mathcal{A}^{(new)} = A_\nu^{(new)}\mathsf{d}x^\nu = A_\nu^{(old)}\mathsf{d}x^\nu - \partial_\nu\Theta\,\mathsf{d}x^\nu = \mathcal{A}^{(old)} - \mathsf{d}\Theta$. From this, we find the invariance $\mathcal{F}^{(new)} = \mathcal{F}^{(old)}$ since $\mathsf{d}^2\Theta \equiv 0$. Namely, the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant by this local gauge transformation. We will see in section II b) for QED that there is local gauge invariance in quantum electrodynamics (QED) as well (*e.g.* Aitchison & Hey (2013, Chap.2)). It is worth noting that the Maxwell equations are invariant under the local gauge transformations (1.11). The details are given in the section II.

Suppose that we have a particular form of A_μ-field defined by $\tilde{A}_\mu \equiv \partial_\mu\Theta$ with Θ an arbitrary scalar function differentiable two times. Then the one-form $\tilde{A} = \tilde{A}_\mu dx^\mu$ is given by $\mathsf{d}\Theta$, and we have the expression $\Psi = \exp[i\gamma\Theta(x^\nu)]\cdot\boldsymbol{\psi}(x^\nu)$, since $\int \tilde{A}_\mu dx^\mu = \Theta$. In addition, since $\tilde{A} = \mathsf{d}\Theta$, the field strength form $\mathcal{F}$ vanishes identically, because $\mathcal{F} = \mathsf{d}\tilde{A} = \mathsf{d}^2\Theta \equiv 0$. Namely, the observable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish identically, although there exists non-vanishing one-form $\tilde{A}$ in the background spacetime.

Quantum-mechanical probability density is given by $|\Psi|^2 = |\boldsymbol{\psi}|^2$. Namely the probability of a quantum mechanical particle is unchanged formally by the existence of $\tilde{A}_\mu$-field. It is well-known for the wave function $\boldsymbol{\psi} = |\boldsymbol{\psi}|\exp(i\theta)$ that the current conservation law $\partial_\nu j^\nu_{(\mathrm{q})} = 0$† is deduced from the equation (1.6):

$$\partial_\nu j^\nu_{(\mathrm{q})} = 0, \quad \text{with} \quad j^0_{(\mathrm{q})} = \rho_\psi\, c, \quad j^k_{(\mathrm{q})} = (\rho_\psi\lambda)\,\partial_k\theta \quad (k = 1, 2, 3) \tag{1.12}$$

where $j^\nu_{(\mathrm{q})} = (j^0_{(\mathrm{q})},\ j^k_{(\mathrm{q})})$ is a 4-current density with $\rho\ \equiv |\boldsymbol{\psi}|^2$, $\lambda \equiv \hbar/m$ and $\partial_0 = c^{-1}\partial_t$, In the presence of $\tilde{A}_\mu$-field, the 3-current flux $j^k_{(\mathrm{q})}$ is changed to $\rho_\psi\lambda\,\partial_k(\theta + \gamma\Theta)$. Thus, only effect of the extended phase factor is to change the 3-current $j^k_{(\mathrm{q})}$ from θ to $\theta + \gamma\Theta$.

In the gauge theory, *global* gauge transformation is defined by the following transformation: $\tilde{A}_\mu \to A_\mu = \tilde{A}_\mu + \epsilon_\mu$ for 4 arbitrary constants ϵ_μ. It is trivial to see that the system is invariant with this global transformation, because the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given by derivatives of A_μ. Therefore, the present system is said to be invariant globally. This is the *first* step of the gauge principle, examining whether the system under consideration is equipped with desirable conditions. We will return to see what is the desirable, after having seen the details of the local invariance given in section I c).

Essence of the gauge principle lies in requiring *local* gauge invariance. In the present case, this is defined by $\tilde{A}_\mu \to A_\mu = \tilde{A}_\mu + \alpha_\mu(x^\nu)$ for 4 arbitrary differentiable fields $\alpha_\mu(x^\nu)$ depending on spacetime coordinates x^ν. Since α_μ is assumed to take a general form not limited to the form $\partial_\mu\Theta$, the one-form $\mathcal{A} = A_\mu dx^\mu$ does not necessarily take a form of a total derivative $\mathsf{d}\Theta$. Hence, the field strength two-form $\mathcal{F} = \mathsf{d}\mathcal{A}$ does not vanish in general. This means that we have non-vanishing observable fields of $\boldsymbol{E}$ and $\boldsymbol{B}$, according to (1.9) and (1.10). This changes drastically our battle field of study. Not only the Maxwell equations (1.1) must be satisfied, but also the governing Schrödinger equation should be reformed with partial derivatives ∂'s replaced by covariant derivatives ∇'s, as given by (2.33) below. Thus, the so-called *gauge-potential* A_μ is taken into the equation (2.32) to represent a new interaction force. In this way, a new force is introduced by the local gauge invariance.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

c) Gauge Principle: global invariance and local invariance

From the example just mentioned above, it is seen that there is a crucial difference between global invariance and local invariance of physical fields. Each invariance in its own right composes the significance of the principle.

To understand the distinction between the two is vital to capture the physics of the fields. In a global invariance, the same transformation is carried out at all spacetime points of the field where current conservation (such as the form of (1.12)) is satisfied, while in a local invariance different transformations are carried out at different individual spacetime points. In general, a theory that is globally invariant will not be invariant under locally varying transformations. This is understood to mean that a new field is required in order to satisfy the local invariance. To that end, the system under investigation must have a potential capacity receptive to, *i.e.* able to receive a new field. In fact, the field $\tilde{A}_\mu = \partial_\mu \Theta$ in the previous section played a diagnostic field to test whether the system is receptive to a new field $\alpha_\mu(x^\nu)$. By introducing a new general field $\alpha_\mu(x^\nu)$ in such a receptive system that interacts with the original field and which also transforms the system physically acceptable ways under the local transformations, a *local gauge invariance is established*.

d) Desirable factor for the gauge theory

Reflecting the above analysis of the gauge principle, consider what is the desirable factor playing the role of a game-changer from vanishing-field state to the state of non-vanishing fields of $\boldsymbol{E}$ and $\boldsymbol{B}$ equipped with a new force (electromagnetic, in this case). It is reasonable to identify that most important factor is a geometrical one. Namely, the one-form $\mathcal{A} = A_\mu dx^\mu = \eta_{\mu\nu} A^\nu dx^\mu$ of (1.7) is vested to the spacetime (x^μ) which is a most important geometrical structure. In fact, the present gauge principle sets as a premise the existence of one-form $\mathcal{A}$ in the 4-d spacetime equipped with the metric $\eta_{\mu\nu}$. With this reasoning, one understands that the gauge principle is rooted on the fundamental level of Physics and that the gauge principle works, as proposed by Utiyama (1956), not only in quantum electrodynamics, but also in particle physics and theory of gravitation, because one can define one-form $\mathcal{A} = A_\mu dx^\mu$. Almost needless to say, the field of fluid flows in the 4-d spacetime is not excluded, to be presented in the accompanying paper.

In the gauge theory of particle physics, current conservation law is considered to be a *must*. It is interesting philosophically to investigate how such a current conservation law working in the physics of discrete particles compromises with the physics of continuum, such as in the theory of gravitation (dealing with spacetime continuum) or in the theory of fluid flows (dealing with material continuum with continuous distribution of mass density ρ). The paper accompanying the present paper is concerned with the last problem of fluid-flow fields.

e) Historical reviews

Considering the key role played by the gauge invariance in modern theoretical physics, it would be reasonable and useful to review how it is working in the fundamental fields. On the reviews of historical facts of the initial stage of gauge theory, one can refer two important articles of O'Raifeartaigh (1997) and Jackson and Okun (2001), both of which describe how the modern gauge theory developed in its early days. It took almost a century to formulate the non-uniqueness of potentials in the context of theoretical physics, existing despite the uniqueness of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In regard to the gauge condition (1.4), *Lorenz's* contribution is noted again. In fact, Lorenz (1859) introduced the so-called retarded potentials and showed that those

† This is equivalent to $\partial_t |\psi|^2 + \partial_k(\psi \partial_k \psi^* - \psi^* \partial_k \quad) = 0$, derived from (1.6).

satisfied the relation: $\nabla \cdot \boldsymbol{A} + c^{-2}\,\partial_t \Phi = 0$ (Jackson & Okun, 2001), which is now almost universally known as the *Lorentz* condition, but founded originally by Ludvig V. Lorenz (a Danish physicist) who preceded the Dutch physicist Hendrik A. Lorentz. The English word *gauge*, a translation of German *eichen*, was not used in English until 1929 (Weyl, 1929a) for the transformations such as (1.3).

II. Gauge in Variances in Two Fundamental Physical Fields — A Review

Taking two fundamental physical fields, *Electromagnetism* and *Quantum Electrodynamics*, we review the gauge symmetries and see how the gauge symmetry has been captured historically.

a) Electromagnetic Field: Gauge Invariance and Charge Conservation

i. *Maxwell equations*

Electromagnetic fields are represented with a 4-vector potential A^μ in the 4d spacetime $x^\mu = (x^0, x^1, x^2, x^3)$ (where $x^0 \equiv ct$ and $\mu = 0, 1, 2, 3$):

$$A^\mu = (\Phi, \boldsymbol{A}), \qquad \boldsymbol{A} = (A_1, A_2, A_3).$$

Covariant version of A^μ is A_μ defined by

$$A_\mu = \eta_{\mu\nu} A^\nu = (-\Phi, \boldsymbol{A}), \qquad \text{where} \quad \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) = \eta^{\mu\nu}, \tag{2.1}$$

$\eta_{\mu\nu}$ being the Minkowski metric of the *Special Relativity*. To represent electro-magnetic fields, we begin with a frame-independent formulation. To this end, according to the mathematical formalism of differential forms, an electromagnetic one-form $\mathcal{A}$ is defined:

$$\mathcal{A} = A_\nu dx^\nu = -\Phi dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \qquad (x^0 = ct).$$

The pair of electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given by

$$\boldsymbol{E} \equiv -c^{-1}\,\partial_t \boldsymbol{A} - \nabla\Phi \qquad \boldsymbol{B} \equiv \nabla \times \boldsymbol{A}\,. \tag{2.2}$$

Taking external differential d of $\mathcal{A}$, we obtain the *field strength* two-form $\mathcal{F}$:

$$\mathcal{F} = \mathsf{d}\mathcal{A} = \sum \frac{1}{2} F_{\nu\lambda}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda, \qquad F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu \tag{2.3}$$

Matrix representation of the tensor $F_{\nu\lambda}$ is given by (1.10). Once again, taking exterior differential of $\mathcal{F} = d\mathcal{A}$, we obtain the following identity equation:

$$\mathsf{d}\mathcal{F} = \mathsf{d}^2\mathcal{A} \equiv 0, \qquad \mathsf{d}\Big(F_{\nu\lambda}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda\Big) = (\partial_\mu F_{\nu\lambda})\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda, \tag{2.4}$$

$$\mathsf{d}\mathcal{F} = \sum F_{[\nu\lambda,\mu]}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda = 0. \qquad F_{\nu\lambda,\mu} \equiv \partial_\mu F_{\nu\lambda}. \tag{2.5}$$

See the footnote for $F_{[\nu\lambda,\mu]}$.† This reduces to the equation expressed compactly:

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{2.6}$$

From this, we obtain a pair of *Maxwell equations* (*cf.* (1.1)):‡

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0, \tag{2.7}$$

By the definitions (2.2) of the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, the two equations of (2.7) are satisfied identically. In other words, in stead of using the pair of equations (2.7), it is sufficient that the 4-potential $A^\mu = (\Phi, \boldsymbol{A})$ is used with the understanding that the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given by the definitions (2.2).

The second pair of *Maxwell equations* are given by

$$\nabla\cdot\boldsymbol{E} = 4\pi\,\rho_e, \qquad -\frac{1}{c}\,\partial_t\boldsymbol{E} + \nabla\times\boldsymbol{B} = \frac{4\pi}{c}\,\boldsymbol{j}_e, \tag{2.8}$$

(*cf.* Jackson (1999, §11.9)). This pair of equations are derived from the principle of least action. The action integral $S^{(\mathrm{em})}$ is expressed by a linear combination of two terms with a part $S^{(\mathrm{em})}_{\mathrm{emA}}$ representing an electromagnetic field by the potential A_α and another $S^{(\mathrm{em})}_{\mathrm{int}}$ representing interaction between the field and 4-current $j_e^{\,\nu}$:

$$S^{(\mathrm{em})} = S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}$$

$$S^{(\mathrm{em})}_{\mathrm{emA}} = -\frac{1}{16\pi c}\int F_{\alpha\beta}\,F^{\alpha\beta}\mathrm{d}\varOmega, \qquad S^{(\mathrm{em})}_{\mathrm{int}} = \frac{1}{c^2}\int j_e^{\,\alpha}\,A_\alpha\,\mathrm{d}\varOmega, \tag{2.9}$$

where $\mathrm{d}\varOmega = \mathrm{d}^4x^\nu$. From the variation δA_α of the field A_α where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, the following equation is deduced in the form of tensor equation (Appendix D: (D.4)):

$$\partial_\beta F^{\alpha\beta} = \frac{4\pi}{c}\,j_e^{\,\alpha}, \tag{2.10}$$

where $j_e^{\,\alpha} = (\rho_e c, \boldsymbol{j}_e)$ with $\boldsymbol{j}_e = \rho_e\boldsymbol{v}$, and $F^{\alpha\beta}$ is given by $F^{\alpha\beta} = \eta^{\alpha\mu}\,F_{\mu\nu}\,\eta^{\nu\beta}$. Practically, the matrix $F^{\alpha\beta}$ is obtained from $F_{\nu\lambda}$ of (1.10) with simply replacing $\boldsymbol{E}$ by $-\boldsymbol{E}$.

ii. *Conservation of electric charge and Gauge invariance*

Conservation law of *electric charge* can be derived from (2.10) by taking 4-divergence of both sides:

$$0 = \partial_\alpha\partial_\beta F^{\alpha\beta} = \frac{4\pi}{c}\,\partial_\alpha j_e^{\,\alpha}. \tag{2.11}$$

The left-hand side vanishes identically because the differential operator $\partial_\alpha\partial_\beta$ is symmetric with respect to α and β, while $F^{\alpha\beta}$ is antisymmetric. Total sum with respect to α and β (taking indices $0,1,2,3$) vanishes identically. Thus, we have the charge conservation equation with $j_e^{\,\beta} = (\rho_e c, \boldsymbol{j}_e)$:

$$\partial_\alpha j_e^{\,\alpha} = \partial_t\rho_e + \nabla\cdot\boldsymbol{j}_e = 0. \tag{2.12}$$

This conservation law is closely related to the gauge symmetry of the electromagnetic field. Let us consider the *gauge transformation* concerning the one-form $\mathcal{A}$, defined by

$$\mathcal{G}:\; A_\nu \equiv A_\nu^{(old)} \;\to\; A_\nu^{(new)} \equiv A'_\nu = A_\nu^{(old)} - \partial_\nu\varTheta, \tag{2.13}$$

equivalent to (1.3), where $\varTheta$ is an arbitrary differentiable function. Then, we have

$$\mathcal{A}^{(new)} = A_\nu^{(new)}\mathsf{d}x^\nu = A_\nu^{(old)}\mathsf{d}x^\nu - \partial_\nu\varTheta\,\mathsf{d}x^\nu = \mathcal{A}^{(old)} - \mathsf{d}\varTheta.$$

From this, we find the invariance $\mathcal{F}^{(new)} = \mathcal{F}^{(old)}$ as follows:

$$\mathcal{F}^{(new)} = \mathsf{d}\mathcal{A}^{(new)} = \mathsf{d}\mathcal{A}^{(old)} + \mathsf{d}^2\varTheta = \mathsf{d}\mathcal{A}^{(old)} = \mathcal{F}^{(old)}, \tag{2.14}$$

† $F_{[\nu\lambda,\mu]} \equiv \frac{1}{3!}\left(\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} - \partial_\mu F_{\lambda\nu} - \partial_\nu F_{\mu\lambda} - \partial_\lambda F_{\nu\mu}\right)$ with $F_{\lambda\nu} = -F_{\nu\lambda}$ *etc.* .

‡ The first is obtained with $(\alpha,\,\beta,\,\gamma) = (1,2,3)$, while the second is derived when one of α, β and γ takes the suffix number 0.

since $\mathsf{d}^2\Theta = 0$ identically. Thus it is found that the two-form $\mathcal{F}$ defined by (2.3) is invariant with respect to the transformation $\mathcal{G}$, called the *gauge transformation* by the historical reasons explained in the Introduction. Therefore, the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant, said as *gauge-invariant*.

The gauge invariance (2.14) and the charge conservation (2.12) are connected closely. In fact, the connection is *inseparable*, which can be shown as follows. In the expression of S_{int} given in (2.9), we replace the factor A_α by $A_\alpha - \partial_\alpha \Theta$. Then the action S_{int} has an additional term,

$$\int j_e^\alpha \frac{\partial \Theta}{\partial x^\alpha} \, \mathrm{d}\Omega. \tag{2.15}$$

Using (2.12) expressing the charge conservation, one can rewrite the integrand in a form of 4-divergence $\partial(\Theta\, j_e^\alpha)/\partial x^\alpha$. Then the above integral is transformed into vanishing boundary integrals by the conditions of the variational principle.

Thus the gauge transformation has no effect on the equation of motion, so long as the equation of charge conservation (2.12) is valid (*cf.* Landau & Lifshitz (1975) §29). Namely, the charge conservation law ensures the gauge invariance. Conversely, the gauge invariance requires the charge conservation equation $\partial j_e^\alpha/\partial x^\alpha = 0$, because the expression (2.15) is transformed to $-\int \Theta\, \partial_\alpha j_e^\alpha \, \mathrm{d}\Omega$, which is required to vanish to any scalar function Θ by the gauge invariance.

iii. *Electromagnetic wave under Lorenz gauge*

In the previous subsection (i), it is remarked below (2.7) that the 4-potential $A^\alpha = (\Phi, \boldsymbol{A})$ can be used instead of the pair of Maxwell equations (2.7). Now the set of four Maxwell equations are reduced to two equations of (2.8) when the 4-potentials A^α are used as dependent variables and the equation (2.2) for the definition of E and B. The two equations of (2.8) are given by the single tensor equation (2.10): $\partial_\beta F^{\beta\alpha} = -(4\pi/c)\, j_e^\alpha$, where

$$\partial_\beta F^{\beta\alpha} = \partial_\beta(\partial^\beta A^\alpha - \partial^\alpha A^\beta) = \partial_\beta \partial^\beta A^\alpha - \partial^\alpha(\partial_\beta A^\beta), \tag{2.16}$$

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = (\partial_0, \nabla); \qquad \partial^\alpha = \eta^{\alpha\beta}\partial_\beta = (-\partial_0, \nabla), \tag{2.17}$$

and $\partial_0 = \partial/\partial(ct)$ and $\nabla = (\partial_1, \partial_2, \partial_3)$. Therefore, the tensor equation (2.10) becomes

$$\partial_\beta \partial^\beta A^\alpha - \partial^\alpha(\partial_\beta A^\beta) = -\frac{4\pi}{c}\, j_e^\alpha, \tag{2.18}$$

where $\partial_\beta \partial^\beta$ is the differential operator of wave equation and $\partial_\beta A^\beta$ 4-divergence of A^β:

$$\partial_\beta \partial^\beta = -\partial_0^2 + \nabla^2 = \nabla^2 - c^{-2}\partial_t^2, \qquad \partial_\beta A^\beta = c^{-1}\partial_t \Phi + \nabla \cdot \boldsymbol{A}. \tag{2.19}$$

In the last section (ii), it is shown that there is freedom in the potential A^α. This freedom enables choosing a set of potentials $A^\alpha = (\Phi, \boldsymbol{A})$ to satisfy

$$\textit{Lorenz condition:} \qquad \partial_\alpha A^\alpha = c^{-1}\partial_t \Phi + \nabla \cdot \boldsymbol{A} = 0. \tag{2.20}$$

Then, the equation (2.18) reduces to the wave equation with the source term $(4\pi/c)\, j_e^\alpha$:

$$\textit{Wave equation:} \qquad (\nabla^2 - c^{-2}\partial_t^2) A^\alpha = -\frac{4\pi}{c}\, j_e^\alpha. \tag{2.21}$$

Substituting $A^{\alpha} = (\Phi, \boldsymbol{A})$ and $j_e^{\alpha} = (\rho_e c, \boldsymbol{j}_e)$, this represents uncoupled wave equations, one for Φ and one for $\boldsymbol{A}$:

$$\nabla^2 \Phi - c^{-2} \partial_t^2 \Phi \;=\; -\, 4\pi\, \rho_e, \tag{2.22}$$

$$\nabla^2 \boldsymbol{A} - c^{-2} \partial_t^2 \boldsymbol{A} = \;-\, \frac{4\pi}{c} \boldsymbol{j}_e, \tag{2.23}$$

The wave equation (2.21) and the Lorenz condition (2.20) form a set of equations equivalent to the Maxwell equations in vacuum. In a later section, we will see, surprisingly, an analogous set of equations for gravitational waves in generalized form. This implies that a sort of gauge symmetry exists as well in the theory of gravitation.

What is now known as a gauge transformation of the electromagnetic potentials was discovered in the formulation process of classical electrodynamics in the 19th century. However, real recognition of its physical significance required two new fields to be developed: the relativity theory for the structure of 4d-spacetime, like a 4-potential $A^{\alpha} = (\Phi, \boldsymbol{A})$ and a current 4-vector $j^{\nu} = (\rho c, \, \boldsymbol{j})$, and the quantum mechanics (say) for the new dimension of a phase factor $\exp[i\chi(x^{\nu})]$ (see next section II b). In fact, the notion of gauge symmetry did not appear in the context of classical electrodynamics, but required the invention of quantum mechanics in particular, according to Jackson & Okun (2001).

As mentioned above, the gauge invariance and charge conservation are connected closely. In fact, the connection is *inseparable*. O'Raifeartaigh L (1997) cites the original paper of Weyl (1918), in which Hermann Weyl commented in the postscript (1955) as

> $\cdots$, *gauge-invariance corresponds to the conservation of electric charge in the same way that coordinate-invariance corresponds to the conservation of energy and momentum. Later the quantum theory introduced the Schrödinger-Dirac potential* (wave function) *of the electron-positron field; it carried with it an experimentally-based principle of gauge-invariance which guaranteed the conservation of charge,* $\cdots\cdots\cdots$. (See O'Raifeartaigh (1997, p.36))

In fact, Noether's theorem shows $\partial_{\nu} j^{\nu} = 0$ for 4-current j^{ν} of relativistic quantum systems such as those governed by Klein-Gordon equation or Dirac equation in Minkowski space (Aitchison & Hey (2013, Chap.3); Frankel (l997, §20.2)).

b) Quantum Electro-Dynamics (QED): Gauge Principle and Covariance

i. *Gauge transformation in QED*

In the context of quantum theory, the attempt of Weyl (1918) is worth mentioning first. He proposed to unify electromagnetism and gravity geometrically by attaching a scale factor of the form $l \propto \exp[\int \phi_k(x^{\nu})\, dx^k]$ with its variation given by $\delta l = l\phi_k\, \delta x^k$. This received unfavorable response to be in disagreement with observation.

However, after the advent of the quantum theory, it was revived by London (1927) that Weyl's proposal could be used in quantum theory by changing the scale factor $\exp[\chi]$ (χ : real) to a phase factor $\exp[i\chi]$ and attaching it to the wave function of quantum mechanics. Suppose that ψ_0 describes the zero-field wave function. Then by the transformation from ψ_0 to $\psi = \psi_0 \exp[i\gamma \int A_{\mu}(x^{\nu}) dx^{\mu}]$, the wave function describes the state interacting with the electromagnetic potential A_{μ} (where $\gamma \equiv e/\hbar$).

Earlier than this work, Fock (1926) proposed extension of the freedom of potential A_{μ} in the classical electrodynamics to the quantum mechanics of a particle with a charge e interacting with the field A_{μ}. With the transformation of the potential,

$$A_{\mu} \to A'_{\mu} = A_{\mu} + \partial_{\mu}\chi, \tag{2.24}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

the wave functions ψ is transformed correspondingly by a phase transformation:

$$\psi \to \psi' = \psi \exp[i\gamma\chi]. \tag{2.25}$$

What Fock discovered for the quantum mechanics was that, for the form of the quantum equation to remain unchanged by these transformations, the wave function is required to undergo the transformation,

$$\psi_0 \to \psi = \psi_0(x^\nu) \exp[i\gamma \int A_\mu(x^\nu)\, dx^\mu], \tag{2.26}$$

whereby ψ is multiplied by a local (space-time dependent) phase factor. Later, the concept was declared a general principle by Hermann Weyl (1928, 1929a,1929b). The invariance of a theory under combined transformations such as (2.24) and (2.25) is known as a gauge symmetry or a gauge invariance and was a touchstone in developing modern gauge theory. (Jackson & Okun)

ii. *Schrödinger's equation and gauge principle in an electromagnetic field*

A wave function ψ of quantum mechanics evolves in time according to the equation $i\hbar\,\partial_t\psi = H\psi$, where $\hbar$ is the Planck constant and H the Hamiltonian operator which is defined, in the absence of the electromagnetic field, by

$$H(\boldsymbol{x}, p) = p^2/2m + eV(x), \tag{2.27}$$

where p is the canonical momentum, V the potential energy and e the charge of the particle. In Schrödinger's equation, the canonical momentum p_k is represented by the differential operator on the wave function ψ expressed as

$$p_k\,\psi = -i\hbar(\partial/\partial x^k)\,\psi, \tag{2.28}$$

while the potential V is a multiplicative operator on ψ. From (2.27), Schrödinger's equation is given by

$$i\hbar\,\partial_t\,\psi = -(\hbar^2/2m)\sum_k(\partial/\partial x^k)^2\,\psi + eV\,\psi. \tag{2.29}$$

When there exists an external electromagnetic field and the particle has a charge e, the Hamiltonian H of (2.27) should be replaced by

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m}(\boldsymbol{P} - \frac{e}{c}\boldsymbol{A})^2 + eV + e\Phi \tag{2.30}$$

where the previous momentum $\boldsymbol{p}$ is replaced by an expression using the new canonical momentum $\boldsymbol{P} = \boldsymbol{p} + (e/c)\boldsymbol{A}$. Replacing P_k with $-i\hbar\partial/\partial x^k$, Schrödinger's equation becomes

$$i\hbar\,\partial_t\,\psi = \frac{1}{2m}\sum_k\Big(-i\hbar\frac{\partial}{\partial x^k} - \frac{e}{c}A_k\Big)^2\psi + eV\,\psi + e\Phi\,\psi. \tag{2.31}$$

This can be rewritten as

$$i\hbar c\,\nabla_0\,\psi = -\frac{\hbar^2}{2m}\sum_k \nabla_k\nabla_k\,\psi + eV\,\psi, \tag{2.32}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

where $\nabla_\alpha = (\nabla_0, \nabla_k)$ are covariant derivatives (with $x^0 = ct$) defined by

$$\nabla_0 = \frac{\partial}{\partial x^0} - \left(\frac{ie}{\hbar c}\right) A_0, \quad \nabla_k = \frac{\partial}{\partial x^k} - \left(\frac{ie}{\hbar c}\right) A_k, \qquad (A_0 = -\Phi). \tag{2.33}$$

The equation (2.32), equivalent to (2.31), is written compactly by using the covariant derivatives ∇_0 and ∇_k to represent the effect of electromagnetic field A_μ.

Weyl's principle of gauge invariance: If ψ satisfied the Schrödinger's equation (2.32) involving the potential A_μ, then the transformed wave function,

$$\psi' = \exp\left[i\gamma\, \chi(x^\mu) \right] \cdot \psi(x) \tag{2.34}$$

satisfies Schrödinger's equation when $\mathcal{A} = A_\nu \mathrm{d}x^\nu$ is replaced by $\mathcal{A} + \mathrm{d}\chi$. This is verified if the wave function ψ is represented as

$$\psi(x) = \Big(\exp[\, i\gamma \int A_\mu(x)\, \mathrm{d}x^\mu] \Big) \cdot \psi_0(x) \tag{2.35}$$

In fact, with a transformation $\mathcal{A} \to \mathcal{A} + \mathrm{d}\chi$. Then the new function $\psi^{(new)}$ is given by

$$\psi^{(new)}(x) = \exp[\, i\gamma \int (A_\mu(x) + \partial_\mu \chi)\, \mathrm{d}x^\mu] \cdot \psi_0(x) = \exp\left[i\gamma\, \chi(x^\mu) \right] \cdot \psi(x).$$

Thus the form (2.34) is obtained. In the gauge symmetry of QED, the key elements are summarized by the following set of *covariant* transformations (see the item (d) below):

$$A_\mu \to A_\mu + \partial_\mu \chi, \qquad \psi \to \exp[\, i\gamma\, \chi\,] \cdot \psi. \tag{2.36}$$

Here, the transformation of A_μ is equivalent to the pair of transformations $\boldsymbol{A} \to \boldsymbol{A} + \nabla\chi$ and $\Phi \to \Phi - \partial_t \chi$, which keep the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ invariant.‡

Thus, one can uphold the gauge principle to the following general guiding principle.

iii. *Generalized Gauge Principle*

Global gauge invariance:

This is defined by *invariance* under a constant change in the phase of wave function ψ. Writing it explicitly, instead of the added phase factor $\exp[i\gamma\chi(x^\mu)]$ of (2.34) depending on x^μ, the global transformation is given by

$$\psi(x^\mu) \to \psi'(x^\mu) = \exp[\, i\,\alpha\,]\, \psi(x^\mu), \qquad \alpha = const\,, \tag{2.37}$$

If this transformation does not cause any observable change, it is a *global invariance.*

Local gauge invariance:

This requires invariance with respect to the following local phase transformation:

$$\psi(x^\mu) \to \psi'(x^\mu) = \exp[\, i\,\alpha(x^\mu)\,]\, \psi(x^\mu), \quad \alpha : \text{ dependent on } x^\mu, \tag{2.38}$$

If our system is not invariant under the local transformation, it is understood to mean that a new field is required in order to satisfy the local invariance. By introducing such a new field interacting with the original field and transforming the system under

‡ The covariant vector-potential (downstairs) is $A_\mu = (-\Phi, A_k)$, while the upstairs vector-potential A^ν is $(A^0, A^k) = \eta^{\nu\mu} A_\mu$ where $A^0 = \Phi$, $(A^k) = \boldsymbol{A}$ and $A_k = A^k$. One-form $\mathcal{A}$ is defined by $\mathcal{A} = \eta_{\mu\nu} A^\nu dx^\mu = A_\mu dx^\mu = -\Phi dt + A_k \mathrm{d}x^k$, where $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. Note that $\nabla_0 = (\partial/\partial x^0) - (ie/\hbar c)\, A_0$.

investigation according to the local transformation, it is expected that local invariance is established. This is a general scenario to find a new physical field.

In fact, the previous item (*ii*) of Schrödinger's equation is a typical example. For the new field to be received to satisfy the local invariance, the system must be receptive, *i.e.* must have a potential capacity receptive to the new field. Firstly, one can say an elementary aspect of the complex function. Every complex function has a phase factor which absorbs the electromagnetic 4-potenial A_μ within the integral symbol as in (2.35).

Moreover, in the QED case, Schrödinger equation (2.29) represented with partial derivatives ∂'s was reformed and replaced by (2.32) represented with covariant derivatives ∇'s which are defined with (2.33) by taking account of the new field A_μ. Simultaneously the wave function ψ was transformed by (2.34). Thus, local invariance has been established.

In mathematical point of view, the global transformation $\psi \to e^{i\alpha}\psi$ appears to be a trivial transformation. But it is an important step to confirm a *capacity* which is receptive to the (harmless) phase modification. In the context of physics, however, it is understood to express the fact that once phase choice of α has been made at one spacetime point, the same change of phase must be adopted at all other spacetime points. This is unnatural from the view-point of causality.

It would be better if one can find other physically reasonable transformation. In §1.2, for electromagnetic 4-potential A_μ, we saw a particular A_μ-field defined by $\tilde{A}_\mu \equiv \partial_\mu \Theta$ with Θ an arbitrary scalar function. When the A_μ-field is introduced in the field. the wave function is transformed as $\psi \to \exp[i\gamma\Theta(x^\nu)] \cdot \psi$ instead of the uniform phase shift $e^{i\alpha}$. Nevertheless, the observable fields $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish identically, although there exists non-vanishing one-form $\tilde{A}$ in the background spacetime. This signifies that the system is receptive. It has a potential capacity receptive to the new field.

In the flow fields of a perfect fluid to be studied in the last section 5.2, there exists an analogous structure in the fluid-flow field. Hence, the global invariance of the flow field is strengthened by this property.

iv. *Covariance with respect to the gauge transformation*

Next, consider the transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$ from a different angle of mathematical viewpoint. Let us represent this operation as $g\circ$ with the symbol $\circ$ and an element g of a certain continuous differentiable group $\mathcal{G}$ (a Lie g), such that we write it as $A'_\mu = g \circ A_\mu$. Then the new wave function $\psi' \equiv \psi^{(new)}$ is written as $\psi' = g \circ \psi$, where ψ is given by (2.35). The operation g and ψ' are given by (2.36). Namely,

$$\psi' = g \circ \psi(x) = \exp[\, i\gamma\chi] \cdot \psi \tag{2.39}$$

Next, using the covariant derivative ∇_μ defined by (2.33), the covariant derivative of is given by

$$\nabla_\mu \psi = (\partial_\mu \psi_0) \cdot \exp\left[i\gamma \int A_\mu(x) \mathrm{d}x^\mu \right].$$

Its g-transformation is

$$\begin{aligned} g \circ \nabla_\mu \psi = &\ (\partial_\mu \psi_0) \cdot \exp\left[i\gamma \int g \circ A_\mu(x) \mathrm{d}x^\mu \right] = (\partial_\mu \psi_0) \cdot \exp\left[i\gamma \int (A_\mu + \partial_\mu \chi) \mathrm{d}x^\mu \right] \\ = &\ \exp[\, i\gamma\chi] \cdot \nabla_\mu \psi \end{aligned} \tag{2.40}$$

Comparing (2.39) and (2.40), it is seen that the "$g\,\circ$" operations on ψ and $\nabla_\mu \psi$ take the same form, that is, simple multiplication of the same phase factor $\exp[i\gamma\chi]$. In other words, the two functions ψ and $\nabla_\mu \psi$ are transformed covariantly by the operation g, that is by the gauge transformation $\mathcal{A} \to \mathcal{A} + \mathrm{d}\chi$. The *covariance* property of transformation shared by both of ψ and $\nabla_\mu \psi$ can be generalized to other transformations. We will see it later too.

v. *Transformation Group U(1)*

The invariance by the transformation (2.37) or (2.38) is said the gauge symmetry of the type of $U(1)$ group. Multiplication by a phase factor like $\exp[i\alpha]$ corresponds to a kind of rotation of the state vector $\psi = |\psi| \exp[i\theta]$ in the polar representation $(|\psi|, \theta)$ of ψ in the complex plane. The group $U(1)$ is an abelian group corresponding to the circle group, consisting of all complex numbers with absolute value 1 under multiplication.

Imagine doing two successive such transformations: $\psi \to \psi' \to \psi''$, where $\psi'' = \exp[i\beta]\, \psi'$, and the original one was $\psi' = \exp[i\alpha]\, \psi = U_\alpha \psi$ with $U_\alpha = \exp[i\alpha]$. So we have $\psi'' = \exp[i(\alpha+\beta)]\, \psi = \exp[i\delta]\, \psi$, where $\delta = \alpha + \beta$. This is a transformation of the same form as the original. The set of all such transformations forms a *group*, in this case called $U(1)$-group, meaning the group of all unitary ($|U_\alpha| = 1$) one-dimensional matrices (ψ, a single complex number). The transformations U_α and a subsequent transformation U_β are commutative. Namely,

$$U_\beta\, U_\alpha = U_{\alpha+\beta} = U_\alpha\, U_\beta .$$

Such a group $U(1)$ is called an *Abelian* group in mathematics where different transformations commute.

The Electro-Weak theory and Quantum Chromodynamics (QCD) are described by non-Abelian gauge symmetries of $SU(2) \times SU(1)$ group and $SU(3)$ group, respectively (see *e.g.* Aitchison & Hey (2013)). All of these theories form what is called today the Standard Model, which is the basis of the theoretical physics except for gravity.

As seen above, the gauge symmetry plays a fundamental role something like a *touchstone* of the theory, testing whether the theory is trustworthy or not. Gauge symmetry exists in other fields too. Geometrical theory of gravitation and Fluid Mechanics are considered below.

III. Geometric Theory of Gravitation

In this section we consider the geometric theory of gravity and the gauge symmetry existing within the theory. Amazingly, there are analogous structures between the quantum electrodynamics (QED) and the theory of gravity. It was known from the initial times of the gravity theory. Most obvious similarity resides in the covariant derivatives of both theories, the former QED including the connection term of the EM potential A_μ and the latter the connection term (Christoffel symbol) associated with the gravity field.

Concerning the theory of gravity at the classic times of Galileo and Newton in the 17th century, a flat Euclidean absolute 3d-space $x^k = (x^1, x^2, x^3)$ and an absolute time t are two distinct physical objects, which are unlinked. A physical object of a point-mass in free motion in an inertial frame in the absence of gravity moves uniformly along a straight line. In the presence of gravitational potential Φ, free motion of a particle takes curved trajectories in flat space. In Einstein's theory of gravitation, world lines of free particles (described by the geodesic equation) are a probe of structure of spacetime.

In Einstein's theory, gravitational field is represented as an object of four-dimensional continuum with curvature (Misner, Thorne & Wheeler (2017, §17.7)). In the equation of gravitation (Einstein, 1915), curvature-tensors are equated to tensors of source-term arising from material motion (mostly motion of fluids or gases), satisfying the conservation laws of energy and momentum of the source material. In this geometro-dynamics, geometry tells matter how to move, such as a free particle taking a curved trajectory, while the matter tells geometry how to curve. Suppose that the source material is a fluid. Being the source of gravity, the fluid tells geometry how to curve in the Einstein's theory. Time t and 3d-space (x^1, x^2, x^3) are two aspects of a single continuum entity, which is an inseparable object of curved *spacetime* $x^\mu = (x^0, x^1, x^2, x^3)$ with $x^0 = ct$. The $4d$-spacetime is not flat because of the presence of matter's energy and momentum of the fluid motion.

Squared interval between an event at x^μ and a nearby one at $x^\mu + \mathrm{d}x^\mu$ is given by

$$\mathrm{d}s^2 = g_{\mu\nu}(P)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu, \quad P = x^\mu = (x^0, x^1, x^2, x^3), \quad \mu, \nu = 0, 1, 2, 3\,. \tag{3.1}$$

where $g_{\mu\nu}$ is the metric tensor. The curved spacetime geometry of physical world is founded by the metric tensor $g_{\mu\nu}$. A special flat space is described by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. This is the space of *Special Relativity* which is a theory invariant under the Lorentz transformation. An important invariant object under the transformation is the proper time τ (the time of comoving frame) defined by

$$\mathrm{d}\tau^2 = -\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = (c\,\mathrm{d}t)^2 - |\mathrm{d}\boldsymbol{x}|^2 = c^2\,(1-\beta^2)\,(\mathrm{d}t)^2\,, \quad \beta \equiv |\boldsymbol{v}|/c\,. \tag{3.2}$$

where $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\mathrm{d}t$ with $\boldsymbol{v}$ being a particle velocity. The τ is the time of comoving frame with the particle (where $|\boldsymbol{v}|$ is zero, hence $\mathrm{d}t = c\,\mathrm{d}t$), while the time t is the time observed from other frame, which are related by $\mathrm{d}\tau = c\sqrt{1-\beta^2}\mathrm{d}t$. Appendix C supplemts certain aspects of mathematics of this section III for the Geometric Theory of Gravitation.

a) *An illustrative example: Free motion of a sigle particle and Equivalence Principle*

A free particle of mass m moves along a world line. Its trajectory is determined as an extremal of the action $S^{(\mathrm{m})} = -mc\int \mathrm{d}s$. The action principle is given by

$$\delta S^{(\mathrm{m})} = -mc\,\delta\int_a^b \mathrm{d}s = 0. \tag{3.3}$$

In the flat space of *Special Relativity* (Appendix B), the free motion takes a straight path, while in gravitational field it is curved. Let us consider a free motion taking a curved trajectory according to Newtonian mechanics.

Motion of a free particle in the Earth's gravity potential $\Phi_E(x^k)$ is described by

$$\frac{\mathrm{d}}{\mathrm{d}t}v^k + \frac{\partial\Phi_E}{\partial x^k} = 0, \qquad v^k \equiv \frac{\mathrm{d}x_p^k}{\mathrm{d}t}, \qquad k = 1, 2, 3\,, \tag{3.4}$$

yielding a curved trajectory for the particle path $x_p^k(t)$. In the modern view to take the space and time linked to form a 4d-continuum, the curved trajectory of a free particle is described as a *geodesic* curve in the linked space-time.

Let us take an illustrative example according to Utiyama (1987, §2.3), and consider a free-falling elevator in the Earth's gravitational field $\Phi_E(x^\nu)$. The free-falling elevator provides a particular *inertial* system of spacetime, in which free motion of a particle is described by

$$\mathrm{d}^2X^\mu/\mathrm{d}\tau^2 = 0, \tag{3.5}$$

where X^μ is the particle coordinates in the frame F_{el} fixed to the free-falling elevator. The gravity effect does not appear apparently because the acceleration owing to the gravity acting on both of the elevator and the particle are the same and cancel out in the free-falling frame F_{el}. Thus, the particle takes a straight path $X^\mu = a^\mu\tau + b^\mu$ with respect to F_{el} with a^μ and b^μ being constants.

Let us observe the same motion from another general frame, and as an example take the frame F_E fixed to the Earth surface, where the coordinates are given by x^μ. The squared interval $\mathrm{d}s^2$ in the frame F_E is given as (3.1). In the particular frame F_{el}, the metric is given by the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Suppose that the transformation between the two frames is connected according to $X^\mu = X^\mu(x^\nu)$. Under this transformation from X^μ to x^ν, the equation of free motion $\mathrm{d}^2X^\mu/\mathrm{d}\tau^2 = 0$ in the frame F_{el} is transformed to that of the frame F_E as follows,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\frac{\partial X^\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\Big] = A^\mu_\nu\Big[\frac{\mathrm{d}^2x^\nu}{\mathrm{d}\tau^2} + \Gamma^\nu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\Big] = 0$$

Using the inverse A^{-1} of $A^\mu_\nu = X^\mu_\nu$ and multiplying by $(A^{-1})^\lambda_\mu \equiv \partial x^\lambda/\partial X^\mu$, this becomes

$$\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0, \quad \text{where} \quad \Gamma^\lambda_{\alpha\beta} = \frac{\partial x^\lambda}{\partial X^\sigma}\,\frac{\partial X^\sigma}{\partial x^\alpha\,\partial x^\beta} = \Gamma^\lambda_{\beta\alpha}. \tag{3.6}$$

This states that the particle trajectory is curved in general when $\Gamma^\lambda_{\alpha\beta} \neq 0$.

The 4-velocity $u^\nu \equiv \mathrm{d}x^\nu/\mathrm{d}\tau$ of the particle is defined by

$$u^\nu = \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \Big(\frac{1}{\sqrt{1-\beta^2}}, \frac{\boldsymbol{v}}{c\,\sqrt{1-\beta^2}}\Big), \quad x^0 \equiv ct, \quad \boldsymbol{v} = (v^k) = (\mathrm{d}x^k/\mathrm{d}t). \tag{3.7}$$

In the non-relativistic limit as $\beta \ll 1$ for the particle velocity $|\boldsymbol{v}|$ is much less than the light velocity c, this leads to $u^\nu = \mathrm{d}x^\nu/\mathrm{d}\tau \to (1, \boldsymbol{v}/c)$ in the limit. In this case, the equation (3.6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}v^\lambda + c^2\,\Gamma^\lambda_{\alpha\beta}\,v^\alpha\,v^\beta = 0, \quad \text{in particular} \quad \frac{\mathrm{d}}{\mathrm{d}t}v^k + c^2\,\Gamma^k_{00}\cdot 1\cdot 1 = 0, \tag{3.8}$$

where the second equation is given for $\lambda = k = 1, 2, 3$, $(\alpha, \beta) = (0, 0)$, and the factors $\Gamma^\lambda_{\alpha\beta}$ other than Γ^k_{00} are set to zero. Compare this with (3.4). By assuming the following relation of equality,

$$c^2\,\Gamma^k_{00} = \partial\Phi_E/\partial x^k\,, \tag{3.9}$$

the second equation of (3.8) becomes equivalent to the equation (3.4). This implies an interesting relation between the gravitational potential Φ_E and the symbol $\Gamma^\lambda_{\alpha\beta}$ (called the Christoffel symbol). The equation (3.11) of the next part b) includes the same symbol Γ and expresses the geodesic equation of a free particle in curved spacetime. By replacing the proper time τ with an equivalent parameter λ, the equation (3.6) reduces to (3.11). We will come back to this point at the item (ii) given below.

In fact, the above simplified example illustrates the conceptual aspects of the geometrical theory of gravitation in three respects. (i) Any curved spacetime has a flat space (the freely-falling elevator in the above case) at any point (locally tangent to it). This is assured by a mathematical theorem, *i.e.* the *local flatness theorem* (Schutz, 1985, §6.2). One can always construct such a local *inertial* frame at any event.

(ii) Gravitational potential Φ_g is related to the metric tensor $g_{\mu\nu}$. In fact, Einstein had a view that there is a similarity between the gravitational field and Riemannian geometry. This is based on the particular feature of the gravity which is distinguished from other forces such as the electromagnetic force (say) and characterized by the fact that all bodies are given same acceleration. The potential Φ_g is related to the tensor $g_{\mu\nu}$, and covariant derivatives depending on $g_{\mu\nu}$ are defined in the curved spacetime.

In the above example of a free particle moving in a weak gravitational field of potential Φ_g, the squared interval $\mathrm{d}s^2$ defined by (3.1) is given by

$$\mathrm{d}s^2 = -(1 + 2\Phi_g/c^2)(c\,\mathrm{d}t)^2 + (1 + 2\Phi_g/c^2)^{-1}\,(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \tag{3.10}$$

as a leading order representation (Misner *et al.* , 2017, §16.2), where only diagonal elements $g_{\mu\nu}|_{\mu=\nu}$ are non-vanishing. Noting $x^0 = c\,\mathrm{d}t$, the metric tensor g_{00} is given by $-1-2\Phi_g/c^2$. In the theory of weak gravitational field ($\Phi_g/c^2 \ll 1$), the metric tensor $g_{\mu\nu}$ is set as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ by using the Minkowski metric $\eta_{\mu\nu}$ on the assumption $|h_{\mu\nu}| \ll 1$.

In the Earth's gravitational field, the potential Φ_g is replaced by $\Phi_E = -G_0 M/r$ and $h_{00} = -2\Phi_E/c^2$, where M is the Earth's mass and r the radial distance from its center.

Returning the equation (3.9): $\Gamma^k_{00} = c^{-2}\,\partial\Phi_E/\partial x^k$ again, the definition of the Christoffel symbol Γ is given by (3.12) of the next section, leading to $\Gamma^k_{00} = g^{k\mu}\Gamma_{\mu 00} \approx \eta^{k\mu}\Gamma_{\mu 00} = \Gamma^k{}_{00} = -\frac{1}{2}\,\partial_k h_{00} = c^{-2}\,\partial_k\Phi_E$. Thus, the the equation (3.9) was confirmed by the squared interval $\mathrm{d}s^2$ of (3.10).

(iii) *Cornerstone* of the Einstein's theory is the *Principle of equivalence* between gravity and acceleration. Consider a uniformly accelerating *rocket* moving in empty space free of gravity (Schutz, 1985, §5.1). Viewed from an observer inside, it appears that there is a gravitational field within the rocket. All objects released from the observer are subjected to uniformly accelerating motion, just as in gravity field. A frame falling freely within the ship is an inertial frame. It can be seen from this that frames accelerating uniformly in empty space are equivalent to uniform gravitational fields. This is a conceptual aspect of the equivalence principle.

Its technical aspect is stated as follows. Transition from the equation (3.5) in flat space-time to the equation (3.6) in a curved spacetime is enabled by the Equivalence Principle. The equation (3.5) can be written as $\mathrm{d}u^\mu/\mathrm{d}\tau = u^\mu{}_{,\tau} = 0$ where $u^\mu \equiv \mathrm{d}X^\mu/\mathrm{d}\tau$, while the equation (3.6) can be written as $\hat{\nabla}_\tau u^\mu \equiv \mathrm{d}u^\mu/\mathrm{d}\tau + \Gamma^\mu_{\alpha\beta}\, u^\alpha\, u^\beta \equiv u^\mu{}_{;\tau} = 0$. Hence, for the transition from *flat* spacetime to *curved* one, the *comma* of $u^\mu{}_{,\tau}$ is replaced by a *semicolon* like $u^\mu{}_{;\tau}$ (§3.2(c). This is the technical aspect of the Equivlence Principle.

The metric $g_{\mu\nu}$ describing the geometry of space-time is a symmetric tensor having ten independent components $g_{\mu\nu}(P)$ in 4-dimesional spacetime, functions of a world point P. Einstein's geometrodynamics is governed by ten tensor equations of the form: $G_{\mu\nu} = 8\pi k\, T_{\mu\nu}$. Among the ten equations, only six are effective. Its detailed account is given in §3.2(e).

The gravitational field considered in this paper is assumed to be weak so that the formulation can be compared with the electromagnetic field presented in the previous section and the fluid-flow field to be considered next in this paper.

b) *Review of Einstein's Theory*

Einstein's theory of gravitation (Einstein 1915) is founded on the Riemannian Geometry. Appendix A describes some of its basics.

i. *Geodesics and Covariant derivative*

In a gravitational field, its 4d-spacetime K_g is curved, and the line element $\mathrm{d}s$ is represented in terms of the metric tensor $g_{\mu\nu}(x^\alpha)$ of (3.1). A free particle in such a space moves along a geodesic line $x^\alpha(\lambda)$, governed by the following *geodesic* equation:

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\,\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\,\frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0. \tag{3.11}$$

where λ is an affine parameter defined as $\lambda = a\tau + b$ with τ the particle's proper time and a, b constants. The factors Γ's are the Christoffel symbol, defined by

$$\Gamma^\alpha{}_{\beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma}, \qquad \Gamma_{\mu\beta\gamma} = \tfrac{1}{2}\Big(\frac{\partial g_{\mu\beta}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu}\Big). \tag{3.12}$$

In such a curved space K_g, a *covariant derivative* of a vector field $v^\alpha(x^\mu)$ along a curve $P(\lambda)$ with its tangent $u^\beta = \mathrm{d}x^\beta/\mathrm{d}\lambda$ is defined by

$$(\hat{\nabla}_{\boldsymbol{u}}\boldsymbol{v})^\alpha \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda}v^\alpha + \Gamma^\alpha{}_{\beta\gamma}\, v^\beta u^\gamma \equiv \hat{\nabla}_\lambda. \tag{3.13}$$

where $\hat{\nabla}$ denotes the nabla-operator in the 4-d spacetime. Using this definition, the geodesic equation (3.11) can be written simply as

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\widehat{\nabla}_{\boldsymbol{u}}\boldsymbol{u}=0, \quad \text{or} \quad \widehat{\nabla}_{\lambda}\boldsymbol{u}=0, \quad \text{where} \quad u^{\alpha}\equiv \mathrm{d}x^{\alpha}(P)/\mathrm{d}\lambda. \tag{3.14}$$

According to the differential geometry (Misner *et al.* 2017, Chap.8), this states that the geodesic is a curve $P(\lambda)$ which parallel-transports its tangent $u^{\alpha}=\mathrm{d}x^{\alpha}(P)/\mathrm{d}\lambda$. In the flat space of special relativity where $g_{\mu\nu}$ is given by the metric $\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$, the geodesic takes a straight path $\mathrm{d}^2x^{\alpha}/\mathrm{d}\lambda^2=0$, since $\Gamma_{\mu\beta\gamma}=0$ by (3.12).

ii. *Geodesic deviation and Riemann curvature tensors* $R^{\alpha}_{\beta\gamma\delta}$

Equation of the geodesic deviation, that is now going to be presented, has a special term which represents the gravitation with *curvature tensors* mathematically. Consider a family of geodesics parameterized by λ, so that world points are expressed as $x^{\alpha}(\lambda,n)$, with each geodesic curve discriminated by a second parameter n.

Let us introduce the separation vector η^{α} defined by $\eta^{\alpha}=\partial x^{\alpha}/\partial n$, measuring the separation (deviation) $\Delta x^{\alpha}=\eta^{\alpha}\Delta n$ between the geodesic n and the nearby geodesic $n+\Delta n$ at the same value of λ. In curved spaces, parallel lines when extended do not necessarily remain parallel, which is formulated in terms of the Riemannian tensors.

To that end, we will make mathematical expressions more general than those of the previous section and define a general derivative form D for a general vector field $\boldsymbol{v}=v^{\alpha}\boldsymbol{e}_{\alpha}$ where $\boldsymbol{v}$ is expanded in terms of unit basis vectors $\boldsymbol{e}_{\alpha}$. Then the exterior derivative of the vector $\boldsymbol{v}$ is given one-form expression as

$$\mathrm{D}\,\boldsymbol{v}=(\mathrm{D}v^{\alpha})\,\boldsymbol{e}_{\alpha}+v^{\alpha}\,(\mathrm{D}\boldsymbol{e}_{\alpha}), \tag{3.15}$$

where $\mathrm{D}v^{\alpha}=(\partial_{\beta}v^{\alpha})\,\mathrm{d}x^{\beta}$ is a one-form, and the term $\mathrm{D}\boldsymbol{e}_{\alpha}$ is a vector-valued one-form which is expanded by using the connection coefficient (Christoffel symbol) in the form,

$$\mathrm{D}\,\boldsymbol{e}_{\alpha}=\boldsymbol{e}_{\nu}\,\Gamma^{\nu}_{\ \ \alpha\mu}\,\mathrm{d}x^{\mu}.$$

Thus, we have the expansion of $\mathrm{D}\,\boldsymbol{v}$ represented as

$$\mathrm{D}\,\boldsymbol{v}=\boldsymbol{e}_{\nu}\left(\frac{\partial v^{\nu}}{\partial x^{\beta}}+\Gamma^{\nu}_{\ \alpha\beta}v^{\alpha}\right)\mathrm{d}x^{\beta}\,, \ =\boldsymbol{e}_{\nu}\left(\frac{\mathrm{d}v^{\nu}}{\mathrm{d}\lambda}+\Gamma^{\nu}_{\ \alpha\beta}\,v^{\alpha}u^{\beta}\right)\mathrm{d}\lambda. \tag{3.16}$$

With these notations, we define

$$\mathrm{D}\eta^{\alpha}\equiv\left(\frac{\partial\eta^{\alpha}}{\partial\lambda}+\Gamma^{\alpha}_{\ \beta\gamma}\eta^{\beta}u^{\gamma}\right)\mathrm{d}\lambda, \qquad \frac{\mathrm{D}}{\mathrm{d}\lambda}\,\eta^{\alpha}\equiv\frac{\partial\eta^{\alpha}}{\partial\lambda}+\Gamma^{\alpha}_{\ \beta\gamma}\,\eta^{\beta}u^{\gamma}.$$

It is seen that the operator D is one-form expression of the covariant derivative ∇. Then, the separation vector η^{α} is governed by the following equation of *geodesic deviation*:

$$\frac{\mathrm{D}}{\mathrm{d}\lambda}\frac{\mathrm{D}}{\mathrm{d}\lambda}\,\eta^{\alpha}=R^{\alpha}_{\ \beta\gamma\delta}\,u^{\beta}\,u^{\gamma}\,\eta^{\delta}, \tag{3.17}$$

where $\eta^{\alpha}=\partial x^{\alpha}(\lambda,n)/\partial n$ is the separation vector and $u^{\beta}=\partial x^{\beta}/\partial\lambda$ the tangent vector.

The covariant derivative of $\boldsymbol{v}$ with respect to the coordinate x^{μ} is given by

$$(\widehat{\nabla}_{\mu}\boldsymbol{v})^{\nu}\,\left(=\frac{\mathrm{D}v^{\nu}}{\partial x^{\mu}}\right)=\partial_{\mu}v^{\nu}+\Gamma^{\nu}_{\ \alpha\mu}v^{\alpha}\equiv\widehat{\nabla}_{\mu}v^{\nu}, \qquad v^{\nu}_{\ ;\mu}=v^{\nu}_{\ ,\mu}+\Gamma^{\nu}_{\ \alpha\mu}v^{\alpha}. \tag{3.18}$$

(See next (c) for the notations of the second equation). The equation (3.17) serves as a definition of the Riemann curvature tensors $R^{\alpha}_{\ \beta\gamma\delta}$, which are defined by

$$R^{\alpha}_{\ \beta\gamma\delta}=\frac{\partial\Gamma^{\alpha}_{\ \beta\delta}}{\partial x^{\gamma}}-\frac{\partial\Gamma^{\alpha}_{\ \beta\gamma}}{\partial x^{\delta}}+\Gamma^{\alpha}_{\ \nu\gamma}\Gamma^{\nu}_{\ \beta\delta}-\Gamma^{\alpha}_{\ \nu\delta}\Gamma^{\nu}_{\ \beta\gamma}. \tag{3.19}$$

This can be represented in terms of the metric tensors $g_{\alpha\beta}$ and their derivatives (see (C.7)). According to (3.17), geodesics in flat space where $R^{\alpha}_{\ \beta\gamma\delta}=0$ maintain their separation, while those in curved spaces where $R^{\alpha}_{\ \beta\gamma\delta}\neq 0$ do not. This is said in the beginning that *geometry tells matter how to move.*

iii. *Equivalence Principle: Transition from flat spacetime to curved one*

How the matter influences the geometry for curving is the subject of subsequent sections. In the present theory of geometro-dynamics, the matter is a perfect fluid. Relativistic expressions of the stress-energy tensor of a perfect fluid are to be given in the section IV, d) by (4.25) and (4.26):

$$T_{\alpha\beta} = (\rho\, c^2 + \rho\, \epsilon(\rho) + p)\, u_\alpha u_\beta + p\, \eta_{\alpha\beta}, \tag{3.20}$$

where u^μ and $\eta^{\mu\nu}$ are defined in (3.7) and (4.21) respectively.§

Conservation law of energy-momentum given by (4.24) is cited here,

$$\partial_\beta T^{\alpha\beta} = T^{\alpha\beta}{}_{,\beta} = 0. \tag{3.21}$$

where the *comma* notation ', β' denotes the *partial* derivative with respect to x^β. This is an expression in global Lorentz (Minkowski) frame of flat spacetime. For the transition (to be considered next) from flat to curved spacetime, the *comma* is replaced by a *semicolon* such as $T^{\alpha\beta}{}_{;\beta}$, *i.e.* the covariant derivative of $T^{\alpha\beta}$.

From the equivalence principle explained in the section III, a), (iii) the same equation as (3.21) is given in local *Lorentz frame* (*Lf* in short) of curved spacetime as well by

$$T^{\alpha\beta}{}_{,\beta} = 0 \quad \text{at origin of local Lorentz frame.} \tag{3.22}$$

In such a frame of local *Lf*, free particles are viewed to move along straight lines at least locally. This means that the term $\Gamma^\alpha_{\beta\gamma}$ of (3.11) must vanish at the origin in the local *Lf*. Namely, all the laws of physics must take their forms known in the special relativity. This is the *Principle of Equivalence*.

Because the Christoffel symbols $\Gamma's$ vanish at the origin of local *Lf*, the equation (3.22) can be rewritten as

$$T^{\alpha\beta}{}_{;\beta} = 0 \quad \text{at origin of local Lorentz frame.}$$

Thus the conservation law given by the form $T^{\alpha\beta}{}_{,\beta} = 0$ at origin of local Lorentz frame is extended to curved spacetime of the form $T^{\alpha\beta}{}_{;\beta} = 0$ in any reference frame owing to the definitive character of tensor. Thus, we have

$$T^{\alpha\beta}{}_{;\beta} = 0 \qquad : \text{extended to any reference frame of curved spacetime.} \tag{3.23}$$

iv. *Einstein field equations*

Equations of the gravitational field are obtained from the principle of least action $\delta(S_g + S_m) = 0$, where S_g and S_m are the actions of the gravitational field and matter field respectively. Accordintg to the variational formulation of Appendix C.2, the variation of S_g with respect to the metric field $g^{\alpha\beta}$ is

$$\delta S_g = -A_g \int \Big(R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta} \hat{R}\Big) \delta g^{\alpha\beta} \sqrt{-g}\, \mathrm{d}\Omega, \quad A_g \equiv \frac{c^3}{16\pi G_0}, \tag{3.24}$$

where $\mathrm{d}\Omega = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$ and $\sqrt{-g}\, \mathrm{d}\Omega$ is the proper volume $[\mathrm{d}\Omega]_{prop}$ in a local Lorentz frame with $g = \det(g_{\mu\nu})$, and $R_{\alpha\beta}$ is the Ricci curvature tensor (C.11), and $\hat{R} \equiv g^{\alpha\nu} R_{\alpha\nu}$ is the scalar curvature, and G_0 is the gravitational constant.

§ The expression of stress-energy tensor $T_{\alpha\beta}$ given here is equivalent to the expression of (a) the equation (133.2) of §133 of "LL (1987)" and that of (b) Box 5.1 of §5.1 of "Gravitation (2017)", under the understanding that $\overline{\rho}\,(m_1 c^2 + \overline{\epsilon}) + \overline{p}$ (where $m_1 = 1$) is equivalent to $w = \rho\, e + p$ of (a) where $e = m_1 c^2 + \epsilon$, and to $\rho + p$ of (b) where ρ is defined by $\overline{\rho}\,(1 + \overline{\epsilon})$ since $m_1 c^2 = 1$ by the assumption $c = 1$ of the text (b). Note that the present Minkowski metric $\eta_{\alpha\beta}$ is equal to $-g_{\alpha\beta}$ of (a). Thus, all the stress-energy tensors $T_{\alpha\beta}$ of the three texts are equivalent under the above understanding.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

On the other hand, the variation of the action S_m of the matter field is

$$\delta S_m = \frac{1}{2c}\int T_{\alpha\beta}\,\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}\mathit{\Omega}. \tag{3.25}$$

where $T_{\alpha\beta}$ is the stress-energy tensor of the matter (*i.e.* the fluid in the present case). Note that variation of the coordinates from x^ν to $x'^\nu = x^\nu + \xi^\nu$ results in variation of the metric $\delta g^{\alpha\beta}$ ‖

From the action principle $\delta S_g + \delta S_m = 0$, we find the *Einstein field equation*:

$$G_{\alpha\beta} = 8\pi k\,T_{\alpha\beta}\,, \qquad k = G_0/c^4, \tag{3.26}$$

in view of the arbitrariness of the $\delta g^{\alpha\beta}$. (See Appendix C.2 for its derivation). The tensor $G_{\alpha\beta}$ is defined by

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\,g_{\alpha\beta}\hat{R}, \tag{3.27}$$

called the Einstein curvature tensor, while $T_{\alpha\beta}$ is the stress-energy tensor.

v. *Degree of freedom of geometro-dynamics*

Einstein's geometro-dynamics is governed by ten tensor equations (3.26): $G_{\alpha\beta} = 8\pi k\,T_{\alpha\beta}$. Among the ten equations, only six are effective. How can the ten equations be in reality only six ? This is because, owing to the four Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$, the equations $G_{\mu\nu} = 8\pi\,T_{\mu\nu}$ place four local conservation laws $T^{\mu\nu}{}_{;\nu} = 0$ of energy and momentum of the source fluid. Instead, four conditions become free, which enable four coordinates chosen arbitrarily. Hence the geometry is constrained by the six independent equations from (3.26).

It is worth emphasizing the ingenious composition of the theory by repeating the concept with other words. The ten equations of $G_{\alpha\beta} = 8\pi k\,T_{\alpha\beta}$ place four constraints on the source motion in the form of the four conservation equations $T^{\mu\nu}{}_{;\nu} = 0$, owing to the four Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$. This is exactly the meaning given in the beginning as "*the geometry tells the matter how to move*". The four conditions, instead, enable four coordinate frames chosen freely. Remaining six constraints from $G_{\mu\nu} = 8\pi\,T_{\mu\nu}$ are those meant by "*the matter tells geometry how to curve*".

The geometro-dynamics in vacuum space requires special attention. Because no matter exists in the vacuum, the six constraints to be imposed by matters mentioned above must be replaced by conditions of vacuum-space own. Here is the place where the Lorentz gauge condition comes into play. This is presented next.

c) Similarity between Gravity Theory and QED

There exist various similarities between the gravity field of the present section and the field of quantum electrodynamics (QED) considered in the section II. Those are reviewed with comparing corresponding mathematical expressions from three aspects here.

i. *Covariant derivatives*

The similarity is clearly seen in the form of the covariant derivatives of both fields. In the gravity, the covariant derivative of $\boldsymbol{v} = v^\nu \boldsymbol{e}_\nu$ with respect to x^μ is given by (3.18):

$$(\widehat{\nabla}_\mu \boldsymbol{v})^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\alpha\mu} v^\alpha. \tag{3.28}$$

In QED, according to (2.33) of the section II, b), (ii), corresponding form of its covariant derivative of wave function is given as

$$\nabla_\mu \boldsymbol{\psi} = \partial_\mu \boldsymbol{\psi} - i\gamma A_\mu\,\psi, \qquad \gamma = e/\hbar c. \tag{3.29}$$

‖ $\delta g^{\alpha\beta} = -\xi^\nu \partial_\nu g^{\alpha\beta} + g^{\alpha\nu}\partial_\nu \xi^\beta + g^{\beta\nu}\partial_\nu \xi^\alpha$. See LL (1975) §94.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

The coefficients of second connection term of each covariant derivative are directly connected to the source field of each case. The former ${\Gamma^{\nu}}_{\alpha\mu}$ are given by derivatives of metric tensors $g_{\mu\nu}$ including the gravity potential Φ_g (see (3.12) and (3.10)). The latter γA_μ is obvious since A_μ is the electromagnetic (EM) potential.

The covariant derivative $\widehat{\nabla}_\mu \boldsymbol{v}$ denotes the derivative in curved spacetime, leading to curved geodesic lines. Analogously, the latter derivative $\nabla_\mu \psi$ signifies curved motion of microscopic particles because the term $p_k\psi = -i\hbar\partial_k\psi$ of (2.28) denotes rectilinear momentum in the absence of the EM field A_μ.

ii. *Invariant variations*

Equations of the gravitational field are obtained from the principle of least action with total action defined by $S_{total} = S_g + S_m$, where S_g and S_m are the actions of gravitational field and matter field respectively. Variations of both actions δS_g and δS_m are given in Appendix C.2. From the action principle $\delta(S_g + S_m) = 0$, we obtain

$$\delta S_g + \delta S_m = -A_g \int \Big(G_{\alpha\beta} - 8\pi k\, T_{\alpha\beta} \Big)\, \delta g^{\alpha\beta}\, \sqrt{-g}\, \mathrm{d}\Omega = 0, \tag{3.30}$$

where $G_{\alpha\beta}$ is the Einstein's curvature tensor defined by (C.19), $A_g = c^3/(16\pi G_0)$ and $k = G_0/c^4$ with G_0 the gravitational constant. The action principle requires invariance of $S_g + S_m$, namely *vanishing* of $\delta(\, S_g + S_m\,)$ for arbitrary variations of the metric tensor $\delta g^{\alpha\beta}$. Thus, we obtain the Einstein equation,

$$G_{\alpha\beta} = 8\pi k\, T_{\alpha\beta}\,, \qquad k = G_0/c^4. \tag{3.31}$$

The action principle, *i.e.* the *invariant variation* described above, yields the Einstein field equation (3.31).

On the other hand, corresponding part of EM (electromagnetism) is the second pair of Maxwell equations presented in the section II a) (i). derived from the electromagnetic action composed of two components $S^{(\mathrm{em})}_{\mathrm{emA}}$ and $S^{(\mathrm{em})}_{\mathrm{int}}$ defined in the section II a) (i). Hence, from the action principle $\delta(\, S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}\,) = 0$, we obtain

$$\delta S^{(\mathrm{em})} \equiv \delta\Big(S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}\Big) = \int \Big(\frac{1}{c}\, j_e^{\,\nu} - \frac{1}{4\pi}\,\frac{\partial F^{\nu\lambda}}{\partial x^\lambda}\Big)\delta A_\nu \;\mathrm{d}\Omega = 0. \tag{3.32}$$

The action principle requires invariance of $S^{(\mathrm{em})} \equiv S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}$, namely *vanishing* of $\delta S^{(\mathrm{em})}$ for arbitrary variations of the potential δA_ν. Thus, we obtain

$$\partial_\lambda F^{\nu\lambda} = (4\pi/c)\, j_e^{\,\nu}\,. \tag{3.33}$$

This *invariant variation* yields the second pair of Maxwell equations (2.8).

Similarity between the gravity and the electromagnetism is seen not only in the form of the action principle by comparing (3.30) and (3.32), but also remarkable similarity is observed in the derived equations (3.31) and (3.33). Left-hand side of (3.31), $G_{\alpha\beta}$, denotes the spacetime structure of gravity, while that of (3.33), $\partial_\lambda F^{\nu\lambda}$, denotes the structure of electromagnetic field. Those are generated by the sources on the right-hand side: $T_{\alpha\beta}$ of (3.31) being the stress-energy tensor of the source perfect fluid, and $j_e^{\,\nu}$ of (3.33) being the source current flux.

iii. *Waves in vacuum space and gauge conditions*

In the section II a) (iii), we have seen electromagnetic waves governed by the wave equation (2.21) for the electromagnetic 4-potential A^α. In vacuum space, this reduces to

$$(\nabla^2 - c^{-2}\partial_t^2)A^\nu = 0. \tag{3.34}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

This can be derived from (3.33), which becomes, on substituting $F^{\nu\lambda} = \partial^\nu A^\lambda - \partial^\lambda A^\nu$,

$$-\partial_\lambda \partial^\lambda A^\nu + \partial^\nu (\partial_\lambda A^\lambda) = (4\pi/c)\, j_e^{\,\nu} \tag{3.35}$$

Imposing the Lorenz gauge condition (2.20),

$$\partial_\lambda A^\lambda = 0, \tag{3.36}$$

setting $j_e^{\,\nu} = 0$ in the vacuum space, and noting $-\partial_\lambda \partial^\lambda = c^{-2}\partial_t^2 - \nabla^2$, the equation (3.35) reduces to (3.34).

Similar structure is found in the gravitational waves as well to be presented in the next section d). In weak gravitational field, the metric tensor is represented as $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ under the condition $|h_{\alpha\beta}| \ll 1$. Linearizing the Einstein equation (3.31), the wave equation (3.47) is derived under the gauge condition (3.46), both of which are cited here in advance for comparison purpose:

$$(\nabla^2 - c^{-2}\partial_t^2)\, \overline{h}^{\mu\nu} = -16\pi k\, T^{\mu\nu}, \tag{3.37}$$

$$\partial_\nu \, \overline{h}^{\mu\nu} = 0\,, \tag{3.38}$$

where $\overline{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}(h^\alpha_{\ \alpha})$. One can recognize similar structures between EM and Gravity, although there is an obvious difference, vectorial fields of the former EM field and tensorial fields of the latter Gravity. Inspite of such difference, their similarity is remarkable.

Consider the EM wave equation (2.21) and apply the divergence operator ∂_ν on it, then we obtain

$$(\nabla^2 - c^{-2}\partial_t^2)(\partial_\nu A^\nu) = -(4\pi/c)\,(\partial_\nu j_e^{\,\nu}).$$

Hence, the gauge condition (3.36) requires the current conservation $\partial_\nu j_e^{\,\nu} = 0$.

Next, consider the gravitational wave equation (3.37) and apply the divergence operator ∂_ν on it, then we obtain

$$(\nabla^2 - c^{-2}\partial_t^2)\,(\partial_\nu \overline{h}^{\mu\nu}) = -16\pi k\,(\partial_\nu T^{\mu\nu})\,,$$

It is consistent with the formulation of the theory that the gauge condition (3.38) requires the conservation of stress-energy of dynamical motion of the source material (fluid) $\partial_\nu T^{\mu\nu} = 0$.

In vacuum space where both of the current flux $j_e^{\,\nu}$ and the stress-energy of material motion are absent. the gauge freedom resulting from the absence of materials is filled up by the gauge conditions $\partial_\nu A^\nu = 0$ or $\partial_\nu \overline{h}^{\mu\nu} = 0$. It is understood that the gauge conditions play the role of filling in the blanks of degrees of freedom.

d) *Gravitational waves (weak gravitational field)*

The spacetime is flat in the absence of gravity, and presence of a weak gravitational field is one in which spacetime is curved but close to flat. In the spacetime continuum object (*manifold* in mathematics), the metric components are represented as

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{3.39}$$

where

$$|h_{\alpha\beta}| \ll 1\,, \qquad \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1), \tag{3.40}$$

assuming small ripples in flat spacetime. Such spacetime is called nearly-Lorentz system and studied by a linearized theory. Merits of linearized theory lie not only in its manageability of analytic handling, but also in the fact that one can apply a *gauge* transformation to the weak gravitational field as well.

In fact, the weak field has a remarkable analogy with the electromagnetic field, as seen in the previous part c, evidenced by the similarity of corresponding wave equations (3.34) and (3.37). However, the difference is clearly recognized in the source terms on the right-hand sides of the two wave equations. In the former field, the source is the current density 4-vector $j_{\rm e}^{\mu}$. while in the latter, it is the stress-energy tensor $T^{\mu\nu}$ of fluid motion. Namely, the vector $j_{\rm e}^{\mu}$ and tensor $T^{\mu\nu}$ symbolize the difference of both fields. However it is more important to have an insight (and recognize) that they share a common physical mechanism for generation of each field despite their difference.

i. *Linearized theory and gravitational gauge transformation*

From the metric form (3.39) under the condition (3.40), one obtains a resulting form of the Christoffel symbol $\Gamma^{\alpha}_{\beta\gamma}$ from the definition (3.12), in which all three terms are linear without approximation: ${\Gamma^{\alpha}}_{\beta\gamma} = \frac{1}{2}\,({h^{\alpha}}_{\beta,\gamma} + {h^{\alpha}}_{\gamma,\beta} - {h_{\beta\gamma}}^{,\alpha})$. A linearized form of Riemann tensor is

$$R_{\alpha\mu\beta\nu} = \tfrac{1}{2}\Big(h_{\alpha\nu,\mu\beta} + h_{\mu\beta.\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}\Big), \tag{3.41}$$

and the Ricci tensor is given by $R_{\mu\nu} = {R^{\alpha}}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}_{\mu\alpha}$ from (C.12). Then, the linearized field equation is derived from the Einstein equation (3.26): $G_{\mu\nu} = 8\pi k\, T_{\mu\nu}$ as

$$-{\overline{h}_{\mu\nu,\alpha}}^{\alpha} - \eta_{\mu\nu}\, {\overline{h}_{\alpha\beta,}}^{\alpha\beta} + {\overline{h}_{\mu\alpha,}}^{\alpha}{}_{\nu} + {\overline{h}_{\nu\alpha,}}^{\alpha}{}_{\mu} = 16\pi k\, T_{\mu\nu}\,, \tag{3.42}$$

(Misner et al. (2017), Chap.18), where

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \tfrac{1}{2}\,\eta_{\mu\nu}\, h, \qquad h = h^{\alpha}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta}. \tag{3.43}$$

We are now in an important stage where one can conceive a *gravitational* gauge transformation, which is quite analogous to the electromagnetic one. Let us consider an infinitesimal transformation of the coordinates of a spacetime point $\mathcal{P}$ from old ones (x^{μ}) to new ones (x'^{μ}), expressed as

$$x'^{\mu}(\mathcal{P}) = x^{\mu}(\mathcal{P}) + \xi^{\mu}(\mathcal{P}), \tag{3.44}$$

where $x^{\mu}(\mathcal{P})$ and $x'^{\mu}(\mathcal{P})$ represent the same spacetime point $\mathcal{P}$, and only their reference frames are changed. Metric perturbations in the new (x'^{μ}) and old (x^{μ}) coordinate frames are related to first order in small quantities by¶

$$h^{\rm new}_{\mu\nu} = h^{\rm old}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}\ . \tag{3.45}$$

This is regarded as a gravitational gauge transformation since the Riemannian tensors are left unchanged by the transformation (3.45). This can be immediately verified by substituting the expression of $h^{\rm new}_{\mu\nu}$ into (3.41), finding $R^{\rm new}_{\alpha\mu\beta\nu} = R^{\rm old}_{\alpha\mu\beta\nu}$. This is reasonable because the change of reference frame only should not influence the physical world. Since the the curvature tensor $R_{\alpha\mu\beta\nu}$ is unchanged, the Ricci tensor $R_{\alpha\beta}$, scalar curvature $\hat{R}$, Einstein tensor $G_{\alpha\beta}$ are all unchanged. This is the gravitational gauge invariance, and the geometrical tensors are essentially the same whether calculated in an orthonormal frame $\eta_{\mu\nu}$, in the old frame $g^{\rm old}_{\mu\nu}$, or in the new frame $g^{\rm new}_{\mu\nu}$.

In general, one can impose the following *gauge condition*:

$$\overline{h}^{\mu\alpha}{}_{,\alpha} = 0, \tag{3.46}$$

¶ Defining matrix element of transformation by $\varLambda^{\alpha}_{\ \overline{\beta}} \equiv \partial x^{\alpha}/\partial x'^{\beta} = {\delta^{\alpha}}_{\beta} - {\xi^{\alpha}}_{,\beta}$, neglecting higher order terms of smallness, transformation of the metric tensor is given by $g^{\rm new}_{\alpha\beta} = \varLambda^{\mu}_{\ \overline{\alpha}}\varLambda^{\nu}_{\ \overline{\beta}}\, g^{\rm old}_{\mu\nu} = \varLambda^{\mu}_{\ \overline{\alpha}}\varLambda^{\nu}_{\ \overline{\beta}}\,\eta_{\mu\nu} + \varLambda^{\mu}_{\ \overline{\alpha}}\varLambda^{\nu}_{\ \overline{\beta}}\, h_{\mu\nu} = (\eta_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}) + h_{\mu\nu}$.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I 22 Year 2021

called the *Lorentz gauge* for gravitational waves. Under this Lorentz gauge condition, the linearized field equation (3.42) reduces to

$$-\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 16\pi k\, T_{\mu\nu}\,, \qquad \text{or equivalently} \quad \partial_\alpha \partial^\alpha\, \bar{h}_{\mu\nu} = -16\pi k\, T_{\mu\nu}\,, \tag{3.47}$$

since the second, third and forth terms on the left-hand side of (3.42) vanish, as follows:

$$\bar{h}_{\alpha\beta,}{}^{\alpha\beta} = \eta_{\alpha\mu}\eta_{\beta\nu}\bar{h}^{\mu\nu,\alpha\beta} = \bar{h}^{\mu\nu}{}_{,\mu\nu} = (\bar{h}^{\mu\nu}{}_{,\nu})_\mu = 0, \qquad \text{by (3.46)},$$

$$\bar{h}_{\mu\alpha,}{}^{\alpha}{}_{\nu} = \eta_{\mu\lambda}\bar{h}^{\lambda\beta}{}_{,\beta\nu} = \eta_{\mu\lambda}(\bar{h}^{\lambda\beta}{}_{,\beta})_\nu = 0, \qquad \bar{h}_{\nu\alpha,}{}^{\alpha}{}_{\mu} = \eta_{\nu\lambda}\bar{h}^{\lambda\beta}{}_{,\beta\mu} = 0.$$

The equation (3.47) represents gravitational wave-generation by the source term on the right-hand side, since the operator $\partial_\alpha\partial^\alpha$ is nothing but that of wave equation:

$$\partial_\alpha\partial^\alpha = -\partial_0{}^2 + \nabla^2 = \Box, \qquad \partial_\alpha = (\partial_0, \nabla), \quad \partial^\alpha = \eta^{\alpha\lambda}\partial_\lambda = (-\partial_0, \nabla).$$

Thus, we have found the gauge condition (3.46) and wave equation (3.47) for gravitational waves, which are equivalent to the equations (3.37) and (3.38) presented already in §3.3(c). Note that the indices of $\bar{h}_{\mu\nu}$ and $T_{\mu\nu}$ are raised with the Minkowski metrics $\eta^{\alpha\mu}\eta^{\beta\nu}$ multiplied on both sides of (3.47), obtaining $\bar{h}^{\alpha\beta}$ and $T^{\alpha\beta}$. Since the factors $\eta^{\alpha\mu}\eta^{\beta\nu}$ are constant, they enter through the diffrential operators.

ii. *Justification of Lorentz gauge*

Suppose that the tensors $\bar{h}_{\mu\nu}$ satisfy the equation (3.42), but do not satisfy the condition (3.46). Then, one can apply a gauge transformation (3.45) to obtain $(\bar{h}^{\text{new}})_{\mu\nu}$ from $(\bar{h}^{\text{old}})_{\mu\nu}$, and demand that $(\bar{h}^{\text{new}})_{\mu\nu}$ satisfies the gauge condition:

$$(\bar{h}^{\text{new}})^{\mu\alpha}{}_{,\alpha} = 0 \ = (\bar{h}^{\text{old}})^{\mu\alpha}{}_{,\alpha} - \partial_\alpha\partial^\alpha\,\xi^\mu - \partial^\mu(\partial_\alpha\xi^\alpha). \tag{3.48}$$

Under the condition $\partial_\alpha\xi^\alpha = 0$ (compatible with the transversality of the waves), one can find the perturbation ξ^μ satisfying the wave equation,

$$\partial_\alpha\partial^\alpha\,\xi^\mu \left[= (-c^{-2}\partial_t{}^2 + \nabla^2)\xi^\mu\right] = (\bar{h}^{\text{old}})^{\mu\alpha}{}_{,\alpha} \quad (\neq 0, \ \text{assumed}).$$

The new field $(\bar{h}^{\text{new}})^{\mu\nu}$ satisfies the Lorentz condition (3.48), $(\bar{h}^{\text{new}})^{\mu\alpha}{}_{,\alpha} = 0$ and the wave equation (3.47).

Even the new metric $(\bar{h}^{\text{new}})^{\mu\nu}$ satisfy the condition (3.48), there is arbitrariness. To fix it, consider a *restricted gauge transformation* $(\bar{h}^{\text{new}})_{\mu\nu} \to (\bar{h}^{\text{new}})'{}_{\mu\nu}$:

$$(\bar{h}^{\text{new}})'{}_{\mu\nu} = (\bar{h}^{\text{new}})_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha_{\ \alpha}, \tag{3.49}$$

derived from the form (3.43) and (3.45). Provided that ξ^μ satisfies the following wave equation,

$$\partial_\alpha\partial^\alpha\,\xi^\mu = (-c^{-2}\partial_t{}^2 + \nabla^2)\,\xi^\mu = 0 \quad [\textit{Restricting condition}], \tag{3.50}$$

the Lorentz condition $(\bar{h}^{\text{new}})'{}^{\mu\alpha}{}_{,\alpha} = 0$ is satisfied according to an equation equivalent to (3.48). Namely, the restricted gauge transformation preserves the Lorentz gauge condition. Therefore the Lorentz gauge is really a class of gauges.

iii. *Gravitational waves in vacuum*

Just as wavy deformations over sea surface propagate across the ocean, so small ripples of the gravitational metric tensor propagate across spacetime. Propagation of the latter gravitational wave in vacuum space (where $T_{\mu\nu} = 0$) is given by the wave equation (3.47) under the gauge condition (3.46):

$$\partial_\alpha\partial^\alpha\,\bar{h}_{\mu\nu} = (\nabla^2 - c^{-2}\partial_t{}^2)\,\bar{h}_{\mu\nu} = 0, \tag{3.51}$$

$$\partial_\alpha \overline{h}^{\mu\alpha} = 0, \qquad (\text{where} \quad \partial_\alpha \overline{h}^{\mu\alpha} = \eta^{\mu\nu}\, \partial^\alpha \overline{h}_{\nu\alpha}). \tag{3.52}$$

Plane Wave: For simplicity reason, let us consider a plane wave, described by the following monochromatic wave:

$$\overline{h}_{\mu\nu} = A_{\mu\nu} \exp[i\, k_\alpha x^\alpha], \qquad \Big(k_0 = -\omega/c, \quad \boldsymbol{k} = (k_1, k_2, k_3), \Big) \tag{3.53}$$

where $x^\alpha = (ct, x^1, x^2, x^3)$. Substituting this to the equation (3.51), one obtains

$$i^2\, k_\alpha\, k^\alpha = k_0^2 - |\boldsymbol{k}|^2 = 0, \qquad \therefore |\boldsymbol{k}|^2 = \omega^2/c^2, \tag{3.54}$$

which is referred to as the dispersion relation of the wave and k^α is called the null vector. The equation of gauge condition (3.52) requires the four (orthogonality) conditions:

$$k_\alpha\, A^{\mu\alpha} = 0. \tag{3.55}$$

iv. *Degree of freedom of gravitational waves*

Let us consider the degree of freedom of gravitational waves in vacuum space. Its degree of freedom is found to be *Two*. The reason is as follows. The metric perturbation $\overline{h}_{\mu\nu}$ of a plane wave is given by (3.53), which is a solution to the field equation (3.51) in the form of wave equation. Its wave amplitude $A_{\mu\nu}$ has ten independent components in general. The field equation (originally of the form $G_{\alpha\beta} = 8\pi k\, T_{\alpha\beta}$) is controlled by four constraints due to the four Bianchi identities $G^{\mu\nu}{}_{;\nu} = 0$, as mentioned at section III b) (v) The four conditions, instead, enable four frames of coordinate chosen freely. Those are provided by the orthogonality gauge-conditions (3.55): $k_\alpha A^{\mu\alpha} = 0$. Thus, the degree of freedom of $A_{\mu\nu}$ is reduced to six.

Wave propagation in vacuum space requires special attention. Because of absence of matters in the vacuum, the six constraints to be imposed by matters (if they existed) must be replaced by conditions of vacuum-space own. Here is the place where another gauge conditions come into play. However, even when the gauge condition (3.46) is satisfied, there is arbitrariness. Namely without violating the gauge condition (3.55), one can introduce the restricted gauge condition (3.50).

Let us express a solution to the restricted gauge condition (3.50) by another plane wave:

$$\xi_\alpha = B_\alpha \exp[i\, k_\mu x^\mu], \tag{3.56}$$

where B_α is a constant and k_μ is given by (3.54). Consequent change in $\overline{h}_{\alpha\beta}$ is given according to (3.49) as $(\overline{h}^{\,\text{new}})'_{\alpha\beta} = (\overline{h}^{\,\text{new}})_{\alpha\beta} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^\alpha{}_\alpha$. From (3.56), this gives

$$A'^{(\text{new})}_{\alpha\beta} = A^{(\text{new})}_{\alpha\beta} - iB_\alpha k_\beta - iB_\beta k_\alpha + i\, \eta_{\alpha\beta} B^\mu k_\mu, \tag{3.57}$$

by removing the exponential factor. One can show (Schutz 1985, Chap.9) that B_α can be chosen to impose two further restrictions on $A'^{(\text{new})}_{\alpha\beta}$:

$$A^\alpha{}_\alpha = 0 \qquad (\text{traceless}), \tag{3.58}$$

$$A_{\alpha\beta}\, u^\beta = 0 \qquad (\text{transverse}), \tag{3.59}$$

where u^β is any constant timelike unit vector.

Note that the condition (3.59) gives only three because $k^\alpha A_{\alpha\beta}\, u^\beta = 0$ is valid identically for any B_α. Hence, the constraints (3.55), (3.58) and (3.59) together give the eight conditions, which are called the *transverse-traceless* (TT) gauge conditions. The remaining two must be physically significant. Namely, the degree of freedom of the wave is found to be *Two*. The gravitational wave has two dynamic degrees of freedom, which is analogous to the electromagnetic waves propagating in vacuum space.

The TT-gauge is based on the vector u^β. Let us take the frame of background vacuum Minkowski spacetime (through which the wave is propagating) defined by the time basis set along the vector $u^\beta = \delta^\beta_{\ 0}$. Then, the condition (3.59) implies $A_{\alpha 0} = 0$ for all α. In this frame, we take the spatial x^3-axis parallel to the direction of wave propagation. Then we have $k_\alpha = (-k, 0, 0, k)$ from (3.54), and the equation (3.55) implies $A_{\alpha 0} + A_{\alpha 3} = 0$. Hence we have $A_{\alpha 3} = -A_{\alpha 0} = 0$ for all α.

Thus, using the symmetry of $A_{\alpha\beta}$ and the traceless condition $A_{11} + A_{22} = 0$, we can write the coefficient matrix $A_{\alpha\beta}$ in the TT-gauge (transverse-traceless gauge) as

$$A^{TT}_{\ \ \alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.60}$$

IV. Fluid Mechanics: Smooth Sequence of Non-Commutative Diffeomorphisms

In Fluid-Mechanics of a perfect fluid, the fluid *medium* is assumed as a continuum (*i.e.* a continuous distribution of mass) in the spacetime $x^\nu = (t, \boldsymbol{x}) = (ct, x^1, x^2, x^3)$. Flow variables such as the mass density ρ, pressure p or flow velocity $\boldsymbol{v}$ are represented by continuously differentiable functions of x^ν. Dynamical motion of fluid flows is characterized by the presence of *convective derivative* in the equation of motion. It is a derivative following the fluid motion, also called sometimes the advective derivative, Lagrange derivative or material derivative, but most importantly it is gauge-invariant *covariant derivative* under local gauge transformations. A fluid flow is a smooth sequence of diffeomophisms of particle configuration, which is a continuous sequence of transformations from one time to another, and two different sequences are not commutative. This is contrasted with the commutative $U(1)$ gauge transformations of QED, seen in §2.2.

a) *Euler's equation of motion*

To capture dynamical motion of fluids, we have two distinct kinds of specification: *Eulerian* type and *Lagrangian* type. With respect to each specification, one finds a gauge symmetry associated with the fluid mass in motion. In the first Eulerian type of specification, the mass density, pressure or flow velocity are represented by differentiable field variables of $\rho(t, \boldsymbol{x}), p(t, \boldsymbol{x})$ or $\boldsymbol{v}(t, \boldsymbol{x})$ respectively. Fluid motions are governed by two kinds of equations, the continuity equation and Euler's equation of motion:

$$\partial_t \rho + \nabla \cdot (\rho\, \boldsymbol{v}) = 0, \tag{4.1}$$

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\, \nabla\, p, \tag{4.2}$$

In the second Lagrangian type of specification, as in particle mechanics, flow variables such as mass-density ρ or velocity $\boldsymbol{v}$ are defined by functions of three parameters $\boldsymbol{a} = (a^1, a^2, a^3)$ identifying each fluid particle (a piece of material element of fluid) and time t_a, like $\rho(t_a, a^1, a^2, a^3)$ or $\boldsymbol{v}(t_a, a^1, a^2, a^3)$. In this specification, the quartet members (t_a, a^1, a^2, a^3) play as independent variables replacing the spacetime coordinates (x^0, x^1, x^2, x^3) of the Eulerian type. For example, the spatial position of a fluid particle at a time t_a

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

specified by the parameter $\boldsymbol{a} = (a^1, a^2, a^3)$ is described by $\boldsymbol{X}_a(t_a, a^1, a^2, a^3) = \boldsymbol{X}_a(t_a, \boldsymbol{a})$. However, to denote a point in Euclidian 3-space, we keep the symbols $(x^k) = (x^1, x^2, x^3)$.

In the case of Lagrangian type of specification, local gauge transformation (LGT) is considered with respect to the specification of position coordinate of a fluid particle identified with the Lagrange-parameter $\boldsymbol{a}$. To describe the particle motion, a *convective* derivative D_t is defined by

$$\mathrm{D}_t \equiv \partial_t + (\boldsymbol{v} \cdot \nabla)\,, \tag{4.3}$$

in addition to partial derivatives such as $\partial_t \equiv \partial/\partial t$ or $\partial_k \equiv \partial/\partial x^k$.

The convective derivative D_t is a generalization of the time derivative ∂_t having a remarkable property of invariance with respect to an LGT transformation defined below. This property is investigated in the section IV c) as another kind of gauge invariance, and also investigated as a *covariant* derivative in curved space-time. In fact, using D_t, the above Euler's equation of motion (4.2) can be rewritten as

$$\mathrm{D}_t \boldsymbol{v} + \rho^{-1}\,\nabla\, p = 0. \tag{4.4}$$

This is viewed as a generalization of Newton's equation of motion to a continuous matter of a perfect fluid, because the term $\mathrm{D}_t\boldsymbol{v}$ is regarded as an acceleration of a fluid particle of a unit mass identified with a fixed parameter $\boldsymbol{a}$.

b) Fluid flow: Sequence of non-commutative diffeomorphisms

i. *One-parameter sub-group of diffeomorphisms*

A fluid flow is a smooth sequence of diffeomorphisms of particle configuration on a spacetime manifold M^4 with a point $x = (x^\nu) = (t, \boldsymbol{x}) \in M^4$ ($\boldsymbol{x} = (x^k)$ with $k = 1, 2, 3$). Suppose that a vector field $U(x) = U^\nu\, e_\nu = \partial_t + U^k \partial_k$ is given at every point $x \in M^4$ (with $U^0 = 1$) as a vector operator U. With such a vector field, one can associate a particular flow, *i.e.* one-parameter sub-group of diffeomorphisms ξ_t with $\xi_0 = I$ (identity). This is a transformation of the particle configuration at the initial moment $\xi_0(x) = Ix = (0, \boldsymbol{X}_0)$ to the particle configuration $\xi_t(x) = (t, \boldsymbol{X}_t)$ at time t. The initial velocity field at $t = 0$ is given by $(\mathrm{d}/\mathrm{d}t)\,\xi_t(x)|_{t=0} = U\xi_t(x)|_{t=0} = U^\nu\, e_\nu = \partial_t + U^k \partial_k$. where U is an operator on $\xi_0(x)$. In this flow, the initial material point $\boldsymbol{X}_0 = \boldsymbol{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3)$ in 3-space is transformed to a 3-space point $\boldsymbol{X}_t(\sigma)$ at $t\,(>0)$. The transformation ξ_t is, as it were, an infinite-dimensional diffeomorphisms from $\boldsymbol{X}_0 = \boldsymbol{\sigma}$ to $\boldsymbol{X}_t(\boldsymbol{\sigma})$. (See, *e.g.* Kambe (2010) Chap.1 and its Appendix C).

On such a group (a Lie group) of diffeomorphisms, one-parameter subgroup with a tangent vector U at the origin I is represented by

$$\xi_t = I + tU + \frac{1}{2!}t^2U^2 + O(t^3). \tag{4.5}$$

With a second vector field $V(x) = V^k\, e_k$, a second flow of one-parameter subgroup $\eta_s(x)$ is generated analogously by V with $\eta_0 = I$. Noting that the composition $\eta_s \xi_t(x)$ is understood as $\eta_s(\xi_t(x))$, we have

$$\eta_s\, \xi_t - \xi_t\, \eta_s = st\,[U,\, V] + O(st^2, s^2t), \tag{4.6}$$

$$[U,\, V] \equiv \Big(U^k \partial_k V^i - V^k \partial_k U^i\Big)\, \partial_i \tag{4.7}$$

The commutator $[U,\, V]$ signifies the degree of non-commutativity of the two flows of diffeomorphisms represented by ξ_t and η_s. This non-commutativity signifies the spacetime being curved.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

(b) Geodesic equation of a fluid-flow

With respect to a flow $\xi(t)$, consider a trajectory $X^\mu(t)$ of a fluid particle on a Riemannian manifold M^4 with its tangent vector defined by $T(x^\mu) = \mathrm{d}\xi/\mathrm{d}t$. The curve is said to be a *geodesic* if its tangent is displaced parallel along the curve $\xi(t)$, *i.e.* if

$$\nabla_T T = 0\,. \tag{4.8}$$

See (A.17) of Appendix A, where general interpretation of geodesic equation and covariant derivative are given (*cf.* Kambe (2010) Chap.3, say). In local coordinates, we have $T \equiv \mathrm{d}\xi/\mathrm{d}t = T^\alpha e_\alpha = (\mathrm{d}X^\alpha/\mathrm{d}t)\, e_\alpha$.

Same geodesic equation $\nabla_T T = 0$ is also given by the action principle, *i.e.* by the equation deduced from the extremum of an action integral (*cf.* Appendix A.6). *Relativistic* form of the action integral of a perfect fluid is given as

$$S^{(\mathrm{pf})} = -c \int \rho\, \mathrm{d}\mathcal{V} \int \Big(1 + c^{-2}\,\overline{\epsilon}(\rho)\Big)\, \mathrm{d}\tau \tag{4.9}$$

This is an extended form of the relativistic action integral of a single particle of mass m, $S^{(\mathrm{m})} = -c\, m \int \mathrm{d}\tau$, to the perfect fluid, where the overlined value $\overline{\epsilon}$ denote proper value of the internal energy ϵ of the perfect fluid (the value in the *rest frame*, *i.e.* comoving frame where the fluid is at rest). Comparison of $S^{(\mathrm{pf})}$ with $S^{(\mathrm{m})}$ and considering $\int \rho\, \mathrm{d}\mathcal{V}$ equivalent to m of $S^{(\mathrm{m})}$, one can see that the term $c^{-2}\,\overline{\epsilon}$ is a small correction term to the fluid medium in non-relativistic case.

From the variation analysis, the geodesic equation of a perfect fluid is given as

$$\mathrm{D}_t v^k + \rho^{-1}\,\partial_k p = 0, \tag{4.10}$$

for non-relativistic limit of ordinary fluid flows (Kambe 2020, §2). This coincides with the Euler's equation of motion (4.4) of ordinary fluid mechanics.

c) Convective derivative D_t and its Gauge invariance

The convective derivative $\mathrm{D}_t = \partial_t + (\boldsymbol{v} \cdot \nabla)$ has a special property which is invariant with respect to a group of transformations like the gauge invariance of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Hence the following transformation may be a fluid version of the gauge transformation. The derivative D_t is also regarded as the covariant derivative analogously with the electromagnetic case. The operator D_t is also invariant with respect to the Lorentz transformation, i.e. a relativistic invariant (see Sec. I, d) of Kambe T (2021), Fluid Gauge Theory, GJSFR, vol. 21, iss.4).

i. *Local gauge transformation*

Suppose that we have two coordinate frames F and F' which are overlapping and each fluid particle is identified by the Lagrange-parameter $\boldsymbol{\alpha}$. Let us denote the position of the same particle $\boldsymbol{\alpha}$ with the coordinate $\boldsymbol{X}_a$ in the frame F and $\boldsymbol{X}'_a$ in the frame F'. Relaltive motion of the two frames is not assumed to be time-independent. Hence the frames are not necessarily inertial. We consider the relation between the two coordinates to be a transformation between $\boldsymbol{X}_a(t, \boldsymbol{\alpha})$ and $\boldsymbol{X}'_a(t', \boldsymbol{\alpha})$, which is given by the following *local gauge transformation* (dependent on $\boldsymbol{\alpha}$) at $t' = t$:

$$LGT: \qquad \boldsymbol{X}'_\alpha(t', \boldsymbol{\alpha}) = \boldsymbol{X}_\alpha(t, \boldsymbol{\alpha}) + \xi(t, \boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{X}_\alpha}, \qquad t' = t, \tag{4.11}$$

This is rewritten in the form of transformation acted by an element g of the group $\mathcal{G}$ defined by $\mathcal{G} = LGT$:

$$\boldsymbol{X}'_\alpha|_{t'=t} = g(t, \boldsymbol{\alpha}) \circ \boldsymbol{X}_\alpha, \qquad g \in \mathcal{G}. \tag{4.12}$$

This LGT is considered as a local transformation between two coordinates (of the same particle identified by $\boldsymbol{\alpha}$) specified in the two non-inertial reference frames F and F'. In fact, the same particle $\boldsymbol{\alpha}$ has a spatial position coordinate $\boldsymbol{X}_\alpha(t, \boldsymbol{\alpha})$ in the frame F and another one $\boldsymbol{X}'_\alpha = \boldsymbol{X}_\alpha + \xi(t, \boldsymbol{\alpha})$ in the frame F'. Therefore, its velocity at $\boldsymbol{x} \in F$,

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\boldsymbol{v}(t,\boldsymbol{x})|_{\boldsymbol{\alpha}} = \partial_t \boldsymbol{X}_\alpha, \tag{4.13}$$

is transformed to the velocity at $\boldsymbol{x}' = \boldsymbol{X}'_\alpha \in F'$ and $t' = t$:

$$\boldsymbol{v}'(t',\boldsymbol{x}')|_{\boldsymbol{\alpha}} = \partial_t \boldsymbol{X}'_\alpha(t,\boldsymbol{\alpha}) = \boldsymbol{v}(t,\boldsymbol{X}_\alpha) + (\mathrm{d}/\mathrm{d}t)\boldsymbol{\xi}_\alpha, \tag{4.14}$$

$$\boldsymbol{\xi}_\alpha = \boldsymbol{\xi}(t,\boldsymbol{X}_\alpha), \qquad (\mathrm{d}/\mathrm{d}t)\boldsymbol{\xi}_\alpha = \partial_t\boldsymbol{\xi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi}|_{\boldsymbol{x}=\boldsymbol{X}_\alpha}. \tag{4.15}$$

One may rewrite the equation (4.14) in a form analogous to (4.12) as

$$\boldsymbol{v}'_\alpha(\boldsymbol{X}'_\alpha) = g(t,\boldsymbol{\alpha}) \circ \boldsymbol{v}_\alpha(\boldsymbol{X}_\alpha). \tag{4.16}$$

This is a transformation of motion of the same particle between two different reference frames F and F'. Physically speaking, two vectors $\boldsymbol{X}_\alpha$ and $\boldsymbol{X}'_\alpha$ denote the same material point, represented by the common Lagrange parameter $\boldsymbol{\alpha}$. Namely, we are considering a gauge transformation between two reference frames.

According to the transformation (4.11), the time derivative ∂_t and space derivative $\partial_k = \partial/\partial x^k$ in the frame F are related to the derivatives ∂'_t and $\partial'_k = \partial/\partial x'^k$ of F' as follows:

$$\partial_t = \partial_{t'} + (\partial_t\boldsymbol{\xi})\cdot\nabla', \qquad \nabla' = (\partial'_k), \tag{4.17}$$

$$\partial_k = \partial'_k + (\partial_k\xi_l)\,\partial'_l, \qquad \partial'_k = \partial/\partial x'_k\,. \tag{4.18}$$

ii. *Gauge invariance of the convective derivative D_t*

The convective derivative $\mathrm{D}_t \equiv \partial_t + (\boldsymbol{v}\cdot\nabla)$ is invariant with respect to *LGT*: *i.e.* $\mathrm{D}_t = \mathrm{D}'_t$. In fact from (4.14) and (4.18), we have

$$\boldsymbol{v}\cdot\nabla = \boldsymbol{v}\cdot\nabla' + (\boldsymbol{v}\cdot\nabla\xi)\cdot\nabla' = \boldsymbol{v}'(\boldsymbol{x}')\cdot\nabla' + (\,-(\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t) + \boldsymbol{v}\cdot\nabla\xi)\cdot\nabla',$$

where $\boldsymbol{v} = \boldsymbol{v}' - \mathrm{d}\boldsymbol{\xi}/\mathrm{d}t$ is used. The last term is rewritten as

$$(\,-(\mathrm{d}\boldsymbol{\xi}/\mathrm{d}t) + \boldsymbol{v}\cdot\nabla\xi)\cdot\nabla' \;=\; -\partial_t\boldsymbol{\xi}\cdot\nabla' = \partial_{t'} - \partial_t, \tag{4.19}$$

by using (4.15) and (4.17). Hence, we have

$$\mathrm{D}_t \;=\; \partial_t + \boldsymbol{v}\cdot\nabla = \partial_{t'} + \boldsymbol{v}'\cdot\nabla' \;=\; \mathrm{D}'_t. \tag{4.20}$$

This means that the operator D_t satisfies the invariance with respect to *LGT*.

In addition, it can be shown that the operator D_t is a covariant derivative in the sense of gauge theory. As shown in (a), under the transformation by $g \in \mathcal{G}$, the expression (4.12) gives $\boldsymbol{X}_\alpha \to \boldsymbol{X}'_\alpha = g \circ \boldsymbol{X}_\alpha = \boldsymbol{X}_a + \boldsymbol{\xi}$, and its derivative (velocity) $\boldsymbol{v}(\boldsymbol{X}_a) = \mathrm{D}_t\boldsymbol{X}_a$ is transformed as

$$\boldsymbol{v}'(\boldsymbol{X}'_a) = \mathrm{D}'_t\boldsymbol{X}'_a = \mathrm{D}_t(\boldsymbol{X}_a + \boldsymbol{\xi}) = \boldsymbol{v}(\boldsymbol{X}_a) + \mathrm{D}_t\boldsymbol{\xi} = g\boldsymbol{v} = g\circ\mathrm{D}_t\boldsymbol{X}_a,$$

where the equality $\boldsymbol{v}+\mathrm{D}_t\boldsymbol{\xi} \equiv g\boldsymbol{v}$ is consistent with (4.13) and (4.14). The above sequence of equalities states that $\mathrm{D}_t\boldsymbol{X}_a$ is transformed to $g\circ\mathrm{D}_t\boldsymbol{X}_a$ in the same way as $\boldsymbol{X}_a$ is transformed to $g\circ\boldsymbol{X}_\alpha$. Therefore, the operator D_t has the covariance property and is reasonably called *Covariant Derivative*.

One can see that the equation of motion (4.4) of a perfect fluid is expressed in terms of the time derivative D_t. The fact that the covariant derivative D_t plays a role of time derivative in place of the partial time derivative ∂_t implies that the free motion according to (4.4) is like a motion in curved space. Rewriting it as $\mathrm{D}_t\boldsymbol{v} = -\rho^{-1}\nabla p$, the equation has a pressure force $-\rho^{-1}\nabla p$, which is not an external force, but an internal force. In fact, each fluid particle does not take a straight trajectory but a *curved* one, in general, owing to the internal pressure force.

d) Relativistic formulation of a perfect fluid

Let us investigate how the fluid mechanics of a perfect fluid is formulated according to the theory of special relativity, which is based on the Minkowski space equipped with

$$\text{Minkowski metric}: \quad \eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1). \tag{4.21}$$

In the space, a world element $\mathrm{d}s$ and an element of proper time $\mathrm{d}\tau/c$ are defined by

$$\begin{aligned} \mathrm{d}s^2 \quad &\equiv -\mathrm{d}\tau^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\mathrm{d}x^\mu x^\nu = -(1-\beta^2)\, c^2 \mathrm{d}t^2, \\ c^{-1}\,\mathrm{d}\tau &= \sqrt{1-\beta^2}\,\mathrm{d}t, \qquad \beta \equiv v/c, \quad v = |\boldsymbol{v}|, \end{aligned} \tag{4.22}$$

where $\mathrm{d}x^0 = c\,\mathrm{d}t$, and c the light speed, and $\boldsymbol{v} = (v^k)$ is the particle velocity, with its 3-space displacement $\mathrm{d}X^k = v^k \mathrm{d}t$ $(k=1,2,3)$. Relativistic 4-velocity u^ν is defined by

$$u^\nu = \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} = \Big(\frac{1}{\sqrt{1-\beta^2}}, \frac{\boldsymbol{v}}{c\,\sqrt{1-\beta^2}}\Big), \quad X^0 \equiv ct, \quad \boldsymbol{v} = (v^k) = (\mathrm{d}X^k/\mathrm{d}t). \tag{4.23}$$

Relativistic form of the action integral of a perfect fluid is already given by (4.9). Relativistic equations of conservation of energy-momentum are expressed in the form,

$$\frac{\partial}{\partial x^\nu} T^{\mu\nu} = 0 \qquad (\mu,\, \nu = 0,1,2,3), \tag{4.24}$$

where the stress-energy tensor $T^{\mu\nu}$ is given by Kambe (2020) for a *perfect fluid*$^+$ as

$$T^{\mu\nu} \equiv H\, u^\mu u^\nu + p\,\eta^{\mu\nu}. \quad H \equiv \rho\,\varepsilon + p = \rho\, c^2 + \rho\,\epsilon + p, \tag{4.25}$$

$$\varepsilon \equiv c^2 + \epsilon(\rho), \quad H \equiv \rho\, c^2 + \rho\, h, \quad h \equiv \epsilon(\rho) + p/\rho, \tag{4.26}$$

(*cf.* Landau & Lifshitz (1987) calling $T^{\mu\nu}$ as energy-momentum tensor), where $\varepsilon \equiv m_1 c^2 + \epsilon = c^2 + \epsilon$ (with $m_1 = 1$) is the relativistic internal energy per unit mass including the mass energy $m_1 c^2$. The thermodynamic variables like $\epsilon(\rho)$ (internal energy) denote the *proper* value (*i.e.* the value in the comoving frame where the fluid is at rest).† The term $\rho\, c^2$ in H denotes the relativistic energy of rest-mass ρ per unit volume.

The above stress-energy tensor $T^{\mu\nu}$ of (4.25) was derived from the Lagrangian density $L \equiv -c\,(\rho\,\mathrm{d}\mathcal{V})\,(1 + c^{-2}\,\bar{\epsilon}(\rho))$ in the action $S^{(\mathrm{pf})}$ of (4.9) under the mass conservation condition $\rho\,\mathrm{d}\mathcal{V} = const$ (see Kambe 2020, §2.2). Present study to be carried out below (and the accompanying paper) does not assume the mass conservation *a priori* (from the outset), but it is deduced from the formulation under a pertinent symmetry. Therefore, the stress-energy tensor $T^{\mu\nu}$ should be derived with taking a different way, which is given in Landau & Lifshitz (1987, §133) and presented here now.

The derivation is as follows. The momentum flux through a surface element $\mathrm{d}\sigma_k$ is just the force acting on the element. Hence $T^{ik}\mathrm{d}\sigma_k$ is the i-th component of the force acting on the surface element $(i, k = 1,2,3)$. Let us take a certain volume element within the fluid in which it is at rest (the local rest frame). In this frame, Pascal's law is valid, that is, the pressure force exerts independently of the direction of the surface element $\mathrm{d}\sigma_k$ and is everywhere perpendicular to the surface on which it acts. Therefore, one can write $T^{ik}\mathrm{d}\sigma_k = p\,\delta^{ik}\,\mathrm{d}\sigma_i$, whence $T^{ik} = p\,\delta^{ik}$.

$^+$ Note: There is no energy dissipation in the present case of perfect fluid, hence no entropy change. Assuming the entropy is uniform throughout the fluid, the internal energy ϵ depends only on ρ.

† Some textbooks such as Misner *et al.* (2017), *etc.* use the definition $T^{\mu\nu} \equiv (\rho + p)\, u^\mu u^\nu + p\eta^{\mu\nu}$ where ρ is understood to denote $\rho\varepsilon = \rho(c^2 + \epsilon)$ including the internal energy $\rho\epsilon$ with $c = 1$ in their unit.

In the local rest frame, then, the energy-momentum tensor has the form

$$T^{\mu\nu}_{\text{rest}} = \begin{pmatrix} \rho\varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{4.27}$$

where ε is the relativistic internal energy per unit mass including the mass energy $m_1 c^2$, hence $\rho\varepsilon$ denotes the energy per unit volume. In order to find the expression of the tensor $T^{\mu\nu}$ in arbitrary reference system, we introduce the 4-velocity u^ν defined by (4.23) for the motion of the fluid. In the rest frame of the particular fluid particle, we have $v^k = 0$ and $u^\nu = (1, 0, 0, 0)$. The expression to be sought for $T^{\mu\nu}$ must be such a form that it takes the form (4.27) when transformed to this rest frame. Such a second-rank tensor $T^{\mu\nu}$ must be

$$T^{\mu\nu} = (\rho\varepsilon + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}. \tag{4.28}$$

for the 4-velocity u^μ of (4.23) and the metric $\eta^{\mu\nu}$ of (4.21). This can be shown as follows, by using the Appendix B.

In the current unprimed frame x^μ, the particles are in motion with the velocity of (4.23). Lorentz transformation from this unprimed frame x^μ to the primed frame x'^{α} comoving with the particle P ($i.e.$ $\beta = |\boldsymbol{v}|/c$) is carried out by the transformation matrix $\Lambda^{\alpha'}_{\ \mu}$ defined with (B.6) and (B.7). By this transformation, the second rank tensor $T^{\mu\nu}$ in the unprimed frame x^μ is transformed to that in primed frame as follows:

$$\begin{aligned} T^{\mu\nu} \quad \Rightarrow \quad T^{\alpha'\beta'}_{\text{rest}} &= \Lambda^{\alpha'}_{\ \mu} \Lambda^{\beta'}_{\ \nu}\, T^{\mu\nu} = (\rho\varepsilon + p)\, (\Lambda^{\alpha'}_{\ \mu}\, u^\mu)\ (\Lambda^{\beta'}_{\ \nu} u^\nu) + p\, (\Lambda^{\alpha'}_{\ \mu}\, \Lambda^{\beta'}_{\ \nu})\ \eta^{\mu\nu} \\ &= \mathrm{diag}(\rho\varepsilon + p, 0, 0, 0) + \mathrm{diag}(-p, p, p, p). \end{aligned} \tag{4.29}$$

by using the transformation $u'^{\alpha} = \Lambda^{\alpha'}_{\ \nu}\, u^\nu$ and (B.9) of Appendix B. The last expression (4.29) reduces to the matrix of (4.27).

This is a wonderful derivation of $T^{\mu\nu}$ of (4.28) for a perfect fluid by Landau & Lifshitz (1987). From the point of view of the present study, however, there exists a crucial aspect to be remarked now. In regard to the momentum flux, the isotropic expression $p\,\delta^{ik}$ (Pascal's law) is taken at the rest frame and Lorentz-trandformed to arbitrary inertial systems of reference, $i.e.$ from the rest frame to frames of arbitrary high velocity, even turbulent, or close to the light velocity. If the medium is solid, then it may be one of choices. However, the fluid is receptive of diffeomorphic transformations among constituent fluid particles in infinitely different ways. Its degree of freedom is infinte (say). It is very likely that tensor form of momentum flux may be quite complex. The paper accompanying the present study, *Fluid Gauge Theory*, intends to present one of possible structures of a perfect fluid.

V. Motivations for Fluid Gauge Theory

A symmetry implies a conservation law (Noether's theorem). However it is shown below that, from a single relativistic energy equation of fluid motion, two conservation equations are obtained in the non-relativistic limit according to the current formulation of fluid mechanics: one is the mass conservation and the other is the traditional form of energy equation. This is a *riddle*. We are concerned particularly with the mass conservation equation and investigate what symmetry implies the mass conservation, and conversely what symmetry the mass conservation implies. A key to resolve this *Riddle* is hinted by the general representation of rotational flows of an ideal compressible fluid satisfying the Euler's equation, derived by Kambe (2013). This gives us a hint

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

of existence of a set of gauge fields, suggesting that our physical system should be a combined system consisting of a fluid flow field and a set of background gauge fields. The gauge symmetry of the latter ensures the law of mass conservation. Conversely as far as the mass conservation law is valid, the gauge invariance is ensured for the action representing interaction between the two components of the combined fields.

a) A riddle: By what symmetry the mass conservation law is implied?

It is well-known that the energy conservation is associated with the symmetry of time translation of mechanical systems. Main object of this section is to state motivation by raising a question of what physical symmetry implies the mass conservation law. This query is raised in regard to the relativistic equation of energy conservation of fluid flows when its non-relativistic limit is taken. In the ordinary fluid-mechanics valid in non-relativistic limit, the mass conservation law is given as valid *a priori*. However, let us see what happens in relativistic mechanics. It is reminded that the relativistic energy-momentum tensor has been given in the previous section IV, d).

The equation (4.24) represents four conservation equations. The *space* components of the equation (4.24) are given by $\partial_\nu T^{k\nu} = 0$ with $\mu = k = 1, 2, 3$, representing the momentum conservation of the k-th component.

On the other hand, its *time* component ($\partial_\nu T^{0\nu} = 0$) is the equation of energy conservation. In order to see its explicit representation in terms of flow variables in the non-relativistic limit ($\beta \equiv v/c \to 0$), the stress-energy tensors $T^{\mu\nu}$ are now written by leading-order terms of expansion with respect to small β in a matrix form:

$$T^{\alpha\beta} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}, \quad \left.\begin{array}{lcl} T^{00} & = & \underline{\rho c^2} + \frac{1}{2}\rho v^2 + \rho\epsilon \quad + \cdots, \\ T^{0k} & = & \underline{c\rho v^k} + c^{-1}\rho v^k(\frac{1}{2}v^2 + h) \quad + \cdots, \\ T^{k0} & = & c\rho v^k + c^{-1}\rho v^k(\frac{1}{2}v^2 + h) \quad + \cdots, \\ T^{ik} & = & \rho v^i v^k + p\,\delta_{ik} \quad + \cdots = T^{ki} \\ & & h \equiv \epsilon + p/\rho. \end{array}\right\} \quad (5.1)$$

where matrix elements are given together with flow variables on the right-hand part of the expression (5.1). The term T^{00} is the energy density, while T^{0k} ($k = 1, 2, 3$) is the energy flux density. The underlined terms ρc^2 in T^{00} and $c\rho v^k$ in T^{0k} came from the rest-mass energy part of the tensor $T^{\alpha\beta}$, which do not appear in the ordinary fluid mechanics. There exists the symmetry of $T^{0k} = T^{k0}$ in the relativistic expression of (5.1). This symmetry is lost in the non-relativistic ordinary fluid mechanics when the underlined terms are removed.

The equation $\partial_\nu T^{0\nu} = 0$ of energy conservation can be written down now as,

$$c^{-1}\partial_t \overline{T}^{00} + \partial_k \overline{T}^{0k} = c\left(\partial_t\rho + \partial_k(\rho v^k)\right) + \frac{1}{c}\left(\partial_t(\rho\hat{E}) + \partial_k(\rho v^k \hat{H})\right) + O(\beta^3) = 0, \quad (5.2)$$

$$\hat{E} = \frac{1}{2}v^2 + \epsilon, \quad \hat{H} = \frac{1}{2}v^2 + h. \quad (5.3)$$

(see (2.17) for ∂_ν). In the non-relativistic limit as $\beta \to 0$, we obtain the mass conservation equation from the first term,

$$\partial_t\rho + \partial_k(\rho v^k) = 0. \quad (5.4)$$

Then, deleting it, the remaining expression reduces to the energy equation of ordinary fluid mechanics in the non-relativistic limit. Thus, we obtain the energy conservation equation of fluid flow (Landau & Lifshitz (1987), Eq.(6.1)):

$$\partial_t(\rho\hat{E}) + \partial_k(\rho v^k \hat{H}) = 0. \quad (5.5)$$

Here we have obtained two conservation equations (5.4) and (5.5) from the single energy equation $\partial_\nu T^{0\nu} = 0$. However, the Noether's theorem (Noether 1918) of theoretical physics states 'A symmetry implies a conservation law', as noted in §1 (Introduction). Therefore, we must ask a question whether the above analysis is satisfactory, and we propose a resolution to this query in a separate paper. (Kambe 2021, "Fluid Gauge Theory",GJSFR).

b) *Hint to resolve the riddle: General solution of Euler's equation with helicity*

A hint to resolve the *Riddle* mentioned in the previous section is found in the general representation of rotational flows given by by Kambe (2013) for an ideal compressible flow solution satisfying the Euler's equation. Its expression in details is cited in Kambe (2020, §3). This solution was derived from the action principle with the action

$$S^{(\mathrm{Eul-rot})} = S^{(\mathrm{nR})} + S^{(\mathrm{Ga-inv})} = \int \rho \,\mathrm{d}\mathcal{V} \Big[\int \Lambda_{\mathrm{nR}} \,\mathrm{d}t + \int \Lambda_{\mathrm{Gi}} \,\mathrm{d}t \Big], \tag{5.6}$$

$$\Lambda_{\mathrm{nR}} = \tfrac{1}{2} v^2 - \epsilon, \qquad \Lambda_{\mathrm{Gi}} = -\mathrm{D}_t \quad - \mathrm{D}_t\langle \boldsymbol{U},\, \boldsymbol{Z}\rangle \tag{5.7}$$

$$\nabla \cdot (\rho \boldsymbol{Z}) = 0, \qquad \nabla \cdot \boldsymbol{U} = 0, \tag{5.8}$$

$$\mathcal{L}[\boldsymbol{Z}] \equiv \partial_t \boldsymbol{Z} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{Z} - (\boldsymbol{Z} \cdot \nabla)\boldsymbol{v} = 0, \tag{5.9}$$

for non-relativistic flow fields, where Λ_{nR} is nothing but the ordinary non-relativistic Lagrangian density, while Λ_{Gi} is a gauge-invariant Lagrangian newly introduced in the study of Kambe (2013). Actually, this study had double aims. One was an attempt to obtain general representation of rotational flow field with non-zero helicity (Kambe 2012). Second aim was more fundamental, striving to establish equivalence between two formulations of Eulerian and Lagrangian specifications under the action principle. Each term of the Lagrangian densities Λ_{nR} and Λ_{Gi} satisfies local gauge invariance with respect to translation and rotation, hence it is consistent with the gauge theory.

As discussed in details in Kambe (2020, §1 and 3.1), this new formulation introduced four independent fields. In fact, regarding the 3-vector potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$, each has three components. Those six fields have two invariance conditions of (5.8), *i.e.* divergence-free condition in 3-space. In addition, from (5.9) and the equation, $(\mathcal{L}_t^*[\boldsymbol{U}])_i \equiv \partial_t U_i + v^k \partial_k U_i + U_k \partial_i v^k = 0$ obtained from the variational analysis of Kambe (2013), we have the third invariance condition:

$$\mathrm{D}_t\langle \boldsymbol{U},\, \boldsymbol{Z}\rangle(t, \boldsymbol{x}) \equiv \langle \mathcal{L}_*[\boldsymbol{U}],\, \boldsymbol{Z}\rangle + \langle \boldsymbol{U},\, \mathcal{L}[\boldsymbol{Z}]\rangle = 0. \tag{5.10}$$

Hence, the value of scalar product $\langle \boldsymbol{U},\, \boldsymbol{Z}\rangle$ is invariant along the particle path $\boldsymbol{x} = \boldsymbol{X}_p(t, \boldsymbol{x})$, keeping its initial value along each trajectory. This is the third invariance imposed on the potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$. Therefore we have only three independent fields remaining among the six components of $\boldsymbol{U}$ and $\boldsymbol{Z}$. Furthermore, if we add the scalar field ψ which is also unconstrained, we have four independent fields in this solution.

Thus, four independent background fields are newly introduced in this solution. Those must be either given externally or determined internally within the framework of theory. In this paper, we take the latter approach, and the general solution given here is understood to predict existence of a new field, which is to be introduced according to the *fluid gauge theory* proposed in Kamle (2021). Hence, the present section describes a partial success, because we are lead to unavoidable circumstances which take us to a new step in two respects. First, owing to the existence of four components of background field, a set of new gauge fields must be introduced in the 4-spacetime according to the gauge-theoretic scenario. Second, it is understood that the newly introduced action $S^{(\mathrm{Ga-inv})} \equiv S^{(\mathrm{int})}$ of (5.12) given below represents interaction of the flow field with unknown background fields. Amazingly this action is analogous to the interaction form

$S^{em}_{\rm (int)}$ of (2.9) in the case of Electromagnetism section II, a). This implies a possible approach, by the formulation analogous to that of Electromagnetism.

What is the hint to resolve the riddle mentioned in section V, a)? It is as follows. We rewrite the part of action $S^{\rm (Ga-inv)}$ of gauge-invariant terms of (5.6) as $S^{\rm (int)}$, since this term is considered to describe interaction between the flow-current j^ν and background vector-potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$, and ψ. In addition to $S^{\rm (int)}$, we denote the scalar product $\langle \boldsymbol{U}, \boldsymbol{Z} \rangle$ by W, and define a 4-current j^ν as follows:

$$S^{\rm (int)} = \int \rho \, \mathrm{d}\mathcal{V} \int \varLambda_{\rm Gi} \mathrm{d}t, \qquad j^\nu \equiv (\rho\, c, \, \rho\, \boldsymbol{v}), \qquad W \equiv \langle \boldsymbol{U}, \, \boldsymbol{Z} \rangle. \tag{5.11}$$

Then the interaction part of action is expressed by

$$S^{\rm (int)} = -\int\int \Big(\rho \, \mathrm{D}_t \quad + \rho \, \mathrm{D}_t \, W \Big) \mathrm{d}\mathcal{V} \, \mathrm{d}t = \int\int \; j^\nu \, \tilde{a}_\nu \; \mathrm{d}\mathcal{V} \, \mathrm{d}t. \tag{5.12}$$

where $\tilde{a}_\nu = -\partial_\nu (\, \psi + W \,)$ and ∂_ν is the same as ∂_α of (2.17).

Note that the field $\tilde{a}_\nu = -\partial_\nu (\, \psi + W\,)$ is analogous to the particular field $\tilde{A}_\nu = \partial_\nu \varTheta$ considered in Section I, b) where all the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish identically, In other words, those fields are potentially existing, but vanish in this particular potential form of $\tilde{A}_\nu = \partial_\nu \varTheta$. Same can be said that new potential field $\tilde{a}_\nu$ can exist. But with the particular form $\tilde{a}_\nu = -\partial_\nu (\, \psi + W\,)$, the *potentially existing* new field does not show in observable world.

Based on this observation, new *Fluid Gauge Theory* is developed in the accompanying paper (Kambe 2021).

VI. Summary

Gauge invariance is one of the fundamental symmetries in modern theoretical physics. In this paper, the gauge symmetry is reviewed to see how it is working in fundamental physical fields: *Electromagnetism, Quantum ElectroDynamics* and *Geometric Theory of Gravity*. In the 19th century, the gauge invariance was recognized as a mathematicl non-uniqueness of the electromagnetic potentials, existing despite the uniqueness of observable electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. In the 20th century, physical significance of the gauge symmetry was recognized but in zigzag ways. Real recognition of its physical significance required two new fields: the relativity theory for recognizing the structure of linked 4d-spacetime $x^\mu = (ct, \boldsymbol{x})$ together with, say, a 4-potential $A^\nu = (\Phi, \boldsymbol{A})$ and a current 4-vector $j^{\,\nu} = (\rho c, \, \boldsymbol{j})$, and the quantum mechanics for the new dimension of a phase factor $\exp{[i\chi(x^\nu)]}$ (§2.2). Finally the gauge symmetry was understood to be very fundamental, and the gauge invariance played a role of guiding principle in the study of physical fields such as Quantum Electrodynamics, Particle Physics and Theory of Gravitation.

There exist similarities in mathematical formulation of physical fields between the quantum electrodynamics (QED, Section II, b) and the gravity theory section III, c) Those are consequences of gauge-invariance property of each field more or less. For example, the covariant derivative of wave function ψ is $\nabla_\mu \, \psi = \partial_\mu \psi - i\gamma A_\mu \psi$, while in the gravity the covariant derivative of a vector $\boldsymbol{v} = v^\nu \boldsymbol{e}_\nu$ is represented as $(\nabla_\mu \boldsymbol{v})^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\alpha\mu} v^\alpha$. Second terms in each expression represent the effects from the electromagnetic potential A_μ in the former and from the gravity through the factor $\Gamma^\nu{}_{\alpha\mu}$ in the latter.

Fundamental governing equations of both fields are derived from the action principle (*i.e.* the action should be invariant for arbitrary variations). A (second) pair of Maxwell equations (3.33) is the one for the electromagnetic field, while the Einstein equation (3.31) is the corresponding equation for the gravitational field, which are, respectively,

$$\partial_\lambda F^{\nu\lambda} = (4\pi/c) \, j_e^{\,\nu} \, . \tag{6.1}$$

$$G^{\alpha\beta} \; = 8\pi k \, T^{\alpha\beta} \, . \tag{6.2}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

The terms on the right hand side are the sources of each field. Taking 4-divergence ∂_ν of the first equation, the left hand side vanishes identically: $\partial_\nu \partial_\lambda F^{\nu\lambda} \equiv 0$, ensuring the current conservation: $\partial_\nu j_e^\nu = 0$. This is an outcome of the gauge symmetry of the field strength tensor $F^{\nu\lambda}$, which is anti-symmetric: $F^{\nu\lambda} = -F^{\lambda\nu}$. On the other hand, taking 4-divergence ∂_α of the second equation, the right hand side vanishes : $\partial_\alpha T^{\alpha\beta} = 0$ which is the conservation laws of stress-energy deduced as the Noether's theorem from the invariance of the action integral with respect to variations of 4-spacetime coordinates. Corresponding left hand side vanishes by the Bianchi identity of the gravitational field (Misner *et al.* (2017, Chap. 15)).

Waves in vacuum space and gauge conditions (there) are also seen to be similar between the two fields. Electromagnetic waves propagating in vacuum space are governed by the wave equation (3.34) for the potential A^ν under the gauge condition:

$$(\nabla^2 - c^{-2}\partial_t^2)A^\nu = 0. \qquad \partial_\nu A^\nu = 0. \tag{6.3}$$

In weak gravitational field, a linearized theory gives the wave equation (3.37) for the modified metric $\overline{h}^{\mu\nu}$ under the gauge condition (3.38). In vacuum space, we have

$$(\nabla^2 - c^{-2}\partial_t^2)\,\overline{h}^{\mu\nu} = 0\,, \qquad \partial_\nu\,\overline{h}^{\mu\nu} = 0\,, \tag{6.4}$$

In vacuum space where both of the current flux j_e^ν and the stress-energy tensor $T^{\alpha\beta}$ are absent. the gauge freedom resulting from the absence of materials is filled up by the gauge conditions $\partial_\nu A^\nu = 0$ or $\partial_\nu \overline{h}^{\mu\nu} = 0$. Namely, the gauge conditions play the role of filling in the blanks of degrees of freedom.

The section III, d), (iv) describes why the gravitational waves propagating in vacuum space have only two dynamic degrees of freedom, analogous to the electromagnetic waves, although in general, the metric perturbation $\overline{h}^{\mu\nu}$ has ten independent components.

Present review on the *gauge symmetry* is motivated from the previous study of Kambe (2020) having arrived at the conclusion that there exists a new gauge field within flow fields of a perfect fluid, and that the new field ensures the mass conservation. The gauge field is not recognized so far in the framework of mechanics of a perfect fluid.

This was an endeavor to resolve a *riddle*, which is presented in the section V a) as follows. *A symmetry implies a conservation law* (Noether 1918). However it can be shown that, from a single relativistic energy equation of fluid motion, two conservation equations are obtained in the non-relativistic limit according to the current formulation of fluid mechanics: one is the mass conservation and the other is the traditional form of energy equation. We are concerned particularly with the mass conservation equation and investigate what symmetry implies the mass conservation, and conversely what symmetry the mass conservation implies. A key to resolve this Riddle is hinted by the general representation of rotational flows (Kambe 2012, 2013) of an ideal compressible fluid satisfying the Euler's equation, described in the section V b). This gives us a hint of existence of a set of gauge fields, suggesting that our physical system should be a combined system consisting of a fluid flow field and a set of new gauge fields (Kambe 2017). From the gauge symmetry of the latter field, the law of mass conservation is deduced, rather than given *a priori*. As far as the mass conservation law is satisfied conversely, gauge invariance is ensured for the action representing interaction between the two components of the combined field.

Acknowledgments

For writing the present review paper motivated from the previous study (Kambe 2020), the author is grateful to Professor Yasuhide Fukumoto (Kyushu University) who is not

only interested in the subject itself, but encouraged to contemplate on writing a review paper covering the gauge symmetry lying in the background of Physics.

Appendix A. Riemannian Geometry

Gauge theory of physics is formulated on the basis of Riemannian geometry. To help the formulations in the main text, basics of Riemannian geometry are summarized here.

Appendix A.1. Tangent vectors and inner product

We consider the *inner* geometry of a Riemannian manifold M which is not a part of an Euclidean space. If a manifold M under consideration were a part of an Euclidean n-dimensional space E^n, it would inherit a local Euclidean geometry (such as the length) from the enveloping Euclidean space, as is the case of a 2-d surface in E^3. The manifold M^n under consideration is not a part of an Euclidean space, so the existence of a local geometry must be postulated. Let M^n be an n-dimensional manifold. The problem is how to define a tangent vector X when we are constrained to the manifold M^n. Let us introduce a local coordinate frame $(x^1, \cdots, x^n)$, and define a tangent vector $X \in T_x M^n$ at each point x of M^n by

$$X = X^i \frac{\partial}{\partial x^i} = X^i \, \partial_i \, ,$$

where $\partial_i = [\partial/\partial x^1, \cdots, \partial/\partial x^n]$ is a natural frame associated with the coordinate system. Furthermore, we define a vector-valued one-form by $\omega = \partial_i \otimes \mathrm{d}x^i$, where ∂_i and $\mathrm{d}x^i$ are bases of vector and covectors.† From the calculus of differential forms, we have $\omega[X] = \partial_i \otimes \mathrm{d}x^i[X] = X^i \partial_i = X$ where $\mathrm{d}x^i(X) = X^i$. By *eating* a vector X, the 1-form ω *yields* the same vector X, *i.e.* vector-valued one-form.

We consider *intrinsic* geometry of the manifold M^n. It is supposed that an inner product $\langle \cdot , \cdot \rangle$ is given in the tangent space $T_x M^n$. If X and Y are two smooth tangent vector fields of the tangent bundle $T_x M^n$, then $\langle X, \, Y \rangle$ is a smooth real function on M^n.

Appendix A.2. Riemannian metric

On a Riemannian manifold M^n, an inner product $\langle \cdot , \cdot \rangle$ is defined on the tangent space $T_x M^n$ at $x \in M$ and assumed to be differentiable. For two tangent fields $X = X^i(x)\partial_i$, $Y = Y^j(x)\partial_j \in T_x M^n$ (tangent bundle), the *Riemannian metric* is given by‡

$$\langle X, \, Y \rangle(x) \;=\; g_{ij} \, X^i(x) Y^j(x) \, ,$$

where the metric tensor, $g_{ij}(x) = \langle \partial_i, \, \partial_j \rangle = g_{ji}(x)$, is symmetric and differentiable with respect to x^i. This bilinear quadratic form is called the *first fundamental form*. In terms of differential 1-forms $\mathrm{d}x^i$, this is equivalent to $I \equiv g_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j$. *Eating* two vectors $X = X^i(x)\partial_i$ and $Y = Y^j(x)\partial_j$, this yields

$$I\,(X, Y) = g_{ij} \, \mathrm{d}x^i(X) \, \mathrm{d}x^j(Y) = \; g_{ij} \, X^i Y^j \, . \tag{A.1}$$

The inner product is said to be *non-degenerate*,

$$\text{if} \quad \langle X, \, Y \rangle = 0, \quad {}^\forall Y \in TM^n, \quad \text{only when} \quad X = 0 \, . \tag{A.2}$$

† These define symbols independent of local coordinate frames. If $u^1, \cdots, u^n$ is another frame, then we have transformation from ∂_i to $\partial/\partial u^i = (\partial x^l/\partial u^i)(\partial/\partial x^l)$ and from $\mathrm{d}x^i$ to $\mathrm{d}u^i = (\partial u^i/\partial x^k)\mathrm{d}x^k$, Then, their combination is $(\partial/\partial u^i)\otimes \mathrm{d}u^i = (\partial x^l/\partial u^i)(\partial/\partial x^l)\otimes(\partial u^i/\partial x^k)(\mathrm{d}x^k) = \delta^l_k \, (\partial/\partial x^l)\otimes \mathrm{d}x^k = \partial_k \otimes \mathrm{d}x^k$. Also, inner product is independent of frames: $U_i U^i = (\partial x^l/\partial u^i) X_l \, (\partial u^i/\partial x^k) X^k = \delta^l_k \, X_l X^k = X_k X^k$.

‡ If the inner product is only non-degenerate rather than positive definite, the resulting structure on M^n is called a *pseudo*-Riemannian.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

As an example, consider a manifold of one-sphere S^1 of continuous interval of real numbers, $S^1 \equiv M^{\infty}_{[0,2\pi]} : [0, 2\pi]$. Its dimension is *infinite*, because the real number $x \in M^{\infty}_{[0,2\pi]}$ distributes continuously within the section $[0, 2\pi]$. Suppose that two fields $X = u(x)\,\partial_x$ and $Y = v(x)\,\partial_x$ are given in the tangent space $T_x M^{\infty}_{[0,2\pi]}$ at a point $x \in S^1$. Their inner product is defined by

$$\langle X,\, Y\rangle \equiv \int_0^{2\pi} u(x)\, v(x)\; \mathrm{d}x\,.$$

This kind of metric is used for electromagnetic fields or flow fields of a fluid.

Appendix A.3. Covariant derivative (Connection)

We introduce an additional structure to the manifold M^n that allows to form a *covariant derivative*. In mathematics, general definition is given to a covariant derivative (called a connection) on a Riemannian *curved* manifold M^n. Let two vector fields $X,\ Y$ defined in the neighborhood of a point $p \in M^n$ and two vectors U and V defined at p. A *covariant derivative* (or *connection*) is an operator ∇. The operator ∇ assigns a vector $\nabla_U X$ at p to each pair $(U,\, X)$ and satisfies the following relations:

$$\left.\begin{array}{lrcl} \text{(i)} & \nabla_U\,(aX + bY) &=& a\nabla_U X + b\nabla_U Y\,, \\ \text{(ii)} & \nabla_{aU+bV}\, X &=& a\nabla_U X + b\nabla_V X\,, \\ \text{(iii)} & \nabla_U (f(x)X) &=& (Uf)X + f(x)\nabla_U X\,, \end{array}\right\} \tag{A.3}$$

for a smooth function $f(x)$ and $a,\ b \in \mathcal{R}$, where $U = U^j\partial_j$ and $Uf = U^j\partial_j f \equiv \mathrm{d}f[U]$. Using the representations, $X = X^i\,\partial_i$ and $Y = Y^j\,\partial_j$, and applying the above properties (i)∼(iii), we obtain

$$\begin{aligned} \nabla_X Y &= \nabla_{X^i\partial_i}(Y^j\partial_j) = X^i\nabla_{\partial_i}(Y^j\partial_j) \\ &= (X^i\partial_i Y^k)\partial_k + X^i Y^j\,\Gamma^k_{ij}\,\partial_k = (\nabla_X Y)^k\,\partial_k\,, \end{aligned} \tag{A.4}$$

$$\nabla_{\partial_i}\partial_j := \Gamma^k_{ij}\partial_k\,, \tag{A.5}$$

where Γ^k_{ij} is called the *Christoffel symbol*. The i-th component of $\nabla_X Y$ is

$$(\nabla_X Y)^i = X^j\,\frac{\partial Y^i}{\partial x^j} + \Gamma^i_{jk}X^jY^k = \mathrm{d}Y^i(X) + (\Gamma^i_{jk}Y^k)\,\mathrm{d}x^j(X) := \nabla Y^i(X), \tag{A.6}$$

$$\nabla Y^i = \mathrm{d}Y^i + \Gamma^i_{jk}\,Y^k\,\mathrm{d}x^j\,, \qquad \nabla_j Y^i = \partial_j Y^i + \Gamma^i_{jk}Y^k, \tag{A.7}$$

where ∇Y^i is called a *connection one-form*. On a manifold M^n, a coordinate frame consists of n vector fields $e_k = \partial_k$ $(k = 1, \cdots, n)$, which are linearly independent and furnish a basis of the tangent space at each point p. Writing (A.5) and (A.6) in the form of vector-valued one-forms, we have $\nabla \boldsymbol{e}_j = \boldsymbol{e}_k\,\Gamma^k_{ij}\mathrm{d}x^i$, and $\nabla Y = (\mathrm{d}Y^k)\,\boldsymbol{e}_k + Y^j\,\Gamma^k_{ij}\mathrm{d}x^i\,\boldsymbol{e}_k$. The operator ∇ is called the *affine connection*, and we have the following representation,

$$\nabla Y\,(X) = \nabla_X Y\,. \tag{A.8}$$

Appendix A.4. Riemannian connection

There is one connection that is of special significance, having the property that *parallel displacement* preserves inner products, and the connection is symmetric.

Definition: There is a unique connection ∇ on a Riemannian manifold M called the *Riemannian connection* or *Levi-Civita* connection that satisfies

$$\text{(i)} \qquad Z\,\langle X,\, Y\rangle \;= \langle \nabla_Z X,\, Y\rangle + \langle X,\, \nabla_Z Y\rangle \tag{A.9}$$

$$\text{(ii)} \quad \nabla_X Y - \nabla_Y X = [X,\, Y] \qquad \text{(torsion free)}, \tag{A.10}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

for vector fields $X, Y, Z \in TM$, where $Z\langle\cdot,\cdot\rangle = Z^j\partial_j\langle\cdot,\cdot\rangle$. The property (i) is a compatibility condition with the metric. The torsion-free property (ii) requires the following symmetry, $\Gamma^k_{ij} = \Gamma^k_{ji}$, with respect to i and j. In fact, writing as $X = X^i\,\partial_i$ and $Y = Y^j\,\partial_j$, the definitive expression (A.4) leads to

$$(\nabla_X Y - \nabla_Y X)^k = (XY - YX)^k + (\Gamma^k_{ij} - \Gamma^k_{ji})X^iY^j\,. \tag{A.11}$$

Christoffel symbol:

The Christoffel symbol Γ^k_{ij} can be represented in terms of the metric tensor $g = (g_{ij})$ by the following formula:

$$\Gamma^k_{ij} = g^{k\alpha}\Gamma_{ij,\alpha}, \qquad \Gamma_{ij,\alpha} = \tfrac{1}{2}\left(\partial_i\, g_{j\alpha} + \partial_j\, g_{\alpha i} - \partial_\alpha\, g_{ij}\right), \tag{A.12}$$

where $g^{k\alpha}$ denotes the inverse g^{-1}, $g^{k\alpha} = (g^{-1})^{k\alpha}$, satisfying $g^{k\alpha}g_{\alpha l} = g_{l\alpha}g^{\alpha k} = \delta^k_l$. The symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ follows immediately from (A.12) and $g_{ij} = g_{ji}$.

Appendix A.5. Covariant derivative along a curve

Consider a curve $x(t)$ on M^n passing through a point p whose tangent at p is given by

$$T = T^k\,\partial_k = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = \dot{x}^k\,\partial_k\,,$$

and let Y be a tangent vector field defined along the curve $x(t)$. According to (A.4) or (A.6), the covariant derivative $\nabla_T Y$ is given by

$$\nabla_T Y := \frac{\nabla Y}{\mathrm{d}t} = \left[\mathrm{d}Y^i(T) + \Gamma^i_{kj}\,T^kY^j\right]\partial_i = \left[\frac{\mathrm{d}}{\mathrm{d}t}Y^i + \Gamma^i_{kj}\,\dot{x}^kY^j\right]\partial_i\,, \tag{A.13}$$

since $T^k = \dot{x}^k$. When Y^i is a function of $x^k(t)$, then $(\mathrm{d}/\mathrm{d}t)Y^i = \dot{x}^k(\partial Y^i/\partial x^k)$. The expression $\nabla Y/\mathrm{d}t$ emphasizes the derivative along the curve $x(t)$ parameterized with t.

Parallel translation:

On the manifold M^n, one can define *parallel displacement* of a tangent vector $Y = Y^i\partial_i$ along a parameterized curve $x(t)$. Parallel displacement is given by (A.15) below. Mathematically, this is defined by

$$\frac{\nabla Y}{\mathrm{d}t} = \nabla_T Y = 0\,; \quad \text{namely,} \quad \dot{x}^k(\partial Y^i/\partial x^k) + \Gamma^i_{kj}\dot{x}^kY^j = 0\,. \tag{A.14}$$

For two vector fields X and Y translated parallel along the curve, we obtain

$$\langle\, X,\, Y\,\rangle = \text{constant} \quad \text{(under parallel translation)}, \tag{A.15}$$

because the scalar product is invariant by (A.9) and (A.14):

$$T\,\langle X,\, Y\rangle = \langle\nabla_T X,\, Y\rangle + \langle X,\, \nabla_T Y\rangle = 0\,. \tag{A.16}$$

Appendix A.6. Geodesic equation

One curve of special significance in a curved space is the geodesic curve. A curve $\gamma(t)$ on a Riemannian manifold M^n is said to be *geodesic* if its tangent $T = \mathrm{d}\gamma/\mathrm{d}t$ is displaced parallel along the curve $\gamma(t)$, *i.e.* if

$$\nabla_T T = \frac{\nabla}{\mathrm{d}t}\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right) = 0\,. \tag{A.17}$$

In local coordinates $\gamma(t) = (x^i(t))$, we have $\mathrm{d}\gamma/\mathrm{d}t = T = T^i\partial_i = (\mathrm{d}x^i/\mathrm{d}t)\partial_i$. By setting $Y = T$ in (A.13), we obtain

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\nabla_T T = \left[\frac{\mathrm{d}T^i}{\mathrm{d}t} + \Gamma^i_{jk} T^j T^k \right] \partial_i = 0\,, \quad \text{where} \quad T^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}. \tag{A.18}$$

Thus the *geodesic equation* $\nabla_T T = 0$ is expressed by local coordinates as

$$\frac{\mathrm{d}T^i}{\mathrm{d}t} + \Gamma^i_{jk} T^j T^k = 0, \qquad \text{or} \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} = 0. \tag{A.19}$$

Parallel translation again: Parallel translation of a tangent vector X along a geodesic $\gamma(s)$ with unit tangent T is defined by (A.14) as $\nabla_T X = 0$. By setting $Y = Z = T$ in the second property (A.9) of the Riemannian connection, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle X,\, T \rangle = T \langle X,\, T \rangle = \langle \nabla_T X,\, T \rangle\,, \tag{A.20}$$

since $\nabla_T T = 0$ by the definition of a geodesic. Hence, the inner product $\langle X,\, T \rangle$ is kept constant by the parallel translation.

Extremum of arc length : A geodesic curve denotes a path of shortest distance connecting two nearby points, or globally of an extremum for all variations with fixed end points. Let $C_0 : \gamma_0(s)$ be a geodesic curve with a length parameter $s \in [0, L]$. A varied curve is denoted by $C_\alpha : \gamma(s, \alpha)$ with $\gamma(s, 0) = \gamma_0(s)$, where $\alpha \in (-\varepsilon, +\varepsilon)$ is a variation parameter and s the arc length for $\gamma_0(s)$. The arc length of the curve C_α is

$$L(\alpha) = \int_0^L \left\| \frac{\partial \gamma(s, \alpha)}{\partial s} \right\| \mathrm{d}s = \int_0^L \langle\, T(s, \alpha),\, T(s, \alpha)\,\rangle^{1/2}\, \mathrm{d}s, \qquad T = \frac{\partial \gamma}{\partial s}\,.$$

Its variation is given by $L'(\alpha) = \int_0^L \partial_\alpha < \partial_s \gamma,\, \partial_s \gamma >^{1/2}\, \mathrm{d}s$. In case that the variation vanishes at both ends of $s = 0$ and L, the first variation $L'(0)$ at $\alpha = 0$ is given by

$$L'(0) = -\int_0^L \langle J,\, \nabla_T T \rangle\; \mathrm{d}s, \qquad \langle J,\, \nabla_T T\,\rangle = 0 \quad \text{for} \quad 0 < s < L\,, \tag{A.21}$$

where $J = \partial_\alpha \gamma((s, 0)$ is the variation vector. Thus, *the geodesic curve* $\nabla_T T = 0$ *takes the extremum of arc length among nearby curves having common endpoints*, in particular characterized by a path of the shortest distance if endpoints are sufficiently near.

Appendix B. Basics of Special Relativity

Suppose that a material particle or fluid particles are moving with high velocities in an inertial frame K: (ct, x^1, x^2, x^3) with c the light velocity. In a time interval $\mathrm{d}t$, the position of a particle changes with time and its displacement is given by a 4-vector:

$$\mathrm{d}x^\mu = (c\,\mathrm{d}t, \mathrm{d}X^1, \mathrm{d}X^2, \mathrm{d}X^3), \qquad \mathrm{d}X^k = v^k\, \mathrm{d}t \quad (k = 1, 2, 3), \tag{B.1}$$

where $\mu = 0, 1, 2, 3$, and the upper-case notation $\mathrm{d}X^k$ denotes material displacement with v^k being components of 3-velocity $\boldsymbol{v}$. In the relativity theory, an infinitesimal interval $\mathrm{d}s$ is defined by its squared form, $\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu$, which is a scalar product of a line-element 4-vector $\mathrm{d}x^\mu$ with its covariant version $\mathrm{d}x_\mu = \eta_{\mu\nu} \mathrm{d}x^\nu = (-c\,\mathrm{d}t, \mathrm{d}X^1, \mathrm{d}X^2, \mathrm{d}X^3)$, where $\eta_{\mu\nu}$ is the Minkowski metric, sometimes called the Lorentz metric, defined by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \tag{B.2}$$

Hence, we have $\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2 \mathrm{d}t^2 + |\mathrm{d}\boldsymbol{X}|^2$.†

† Note that the metric $g_{\mu\nu}$ used by Landau & Lifshitz (1975) is defined by $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) = -\eta_{\mu\nu}$. Hence, $\mathrm{d}\tau^2$ [present] $= -\eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}s^2$ [Landau & Lifshitz] $= -\mathrm{d}s^2$ [present].

The interval ds is a relativistic invariant, *i.e.* invariant under the Lorentz transformation now defined. Suppose that the coordinate transformation is expressed by $x^\mu \to x'^\alpha = \Lambda^{\alpha'}_{\ \mu} x^\mu$ with $\Lambda^{\alpha'}_{\ \mu}$ a matrix of Lorentz transformation. Then we have

$$\mathrm{d}s'^2 = \eta_{\alpha'\beta'}\,\mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta = \eta_{\alpha'\beta'}\,\Lambda^{\alpha'}_{\ \mu}\Lambda^{\beta'}_{\ \nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}s^2\,,$$

where $\Lambda^{\alpha'}_{\ \beta}\Lambda^{\beta}_{\ \gamma'} = \delta^{\alpha'}_{\ \gamma'}$ is required for the Lorentz transformation. The equalities,

$$\eta_{\mu\nu} = \eta_{\alpha'\beta'}\,\Lambda^{\alpha'}_{\ \mu}\Lambda^{\beta'}_{\ \nu} = (\Lambda^{\mathrm{T}})_{\mu}^{\ \alpha'}\,\eta_{\alpha'\beta'}\,\Lambda^{\beta'}_{\ \nu} = (\,\Lambda^{\mathrm{T}}\,\eta'\Lambda\,)_{\mu\nu},$$

define the *Lorentz invariance*, or relativistic invariance.

Another relativistic invariant is the *proper time* τ. Its increment dτ is defined by the time increment (multiplied by c) in the instantaneously rest frame where $\boldsymbol{v} = 0$. Squared interval of the proper time is defined by $\mathrm{d}\tau^2 = -\mathrm{d}x_\nu \mathrm{d}x^\nu = -\mathrm{d}s^2$. From this, noting $\mathrm{d}X^k = v^k\,\mathrm{d}t$, we obtain

$$\mathrm{d}\tau = c\,\mathrm{d}t\,\sqrt{1-\beta^2}\,, \qquad \beta \equiv v/c\,, \quad v = \sqrt{v_k v^k}. \tag{B.3}$$

Using the displacement $\mathrm{d}X^\nu$ of a fluid particle P, its relativistic 4-velocity is defined by

$$u^\nu = \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} = \Big(\frac{1}{\sqrt{1-\beta^2}}, \frac{\boldsymbol{v}}{c\,\sqrt{1-\beta^2}}\Big), \qquad \boldsymbol{v} = (v^k) = (\mathrm{d}X^k/\mathrm{d}t). \tag{B.4}$$

This fluid particle P is moving with the 4-velocity u^ν relative to the frame x^μ.

Consider the following useful transformation defined by the matrix components $\Lambda^{\alpha'}_{\ \mu}$:

$$v^1/c = \beta\,n^1, \quad v^2/c = \beta\,n^2, \quad v^3/c = \beta\,n^3, \quad \gamma \equiv 1/\sqrt{1-\beta^2}, \tag{B.5}$$

$$\Lambda^{0'}_{\ 0} = \gamma, \qquad \Lambda^{0'}_{\ j} = \Lambda^{j'}_{\ 0} = -\beta\,\gamma\,n^j, \tag{B.6}$$

$$\Lambda^{j'}_{\ k} = \Lambda^{k'}_{\ j} \;=\; (\gamma-1)\,n^j\,n^k + \delta^{jk}, \tag{B.7}$$

where the condition of unit 3-vector $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$ defines $\beta^2 = |\boldsymbol{v}|^2/c^2$.

With the matrix $\Lambda^{\alpha'}_{\ \mu}$ of (B.6) and (B.7), the unprimed frame x^μ is transformed to the primed frame x'^α by the coordinate transformation law: $x'^\alpha = \Lambda^{\alpha'}_{\ \mu} x^\mu$ at the instant when the origins of both frames coincide instantaneously. However, the primed frame x'^α is moving with the velocity $v^k/c = \beta\,n^k$ as seen in the unprimed frame x^μ.

It is remarkable that the 4-velocity u^ν is transformed by the same law: $u'^\alpha = \Lambda^{\alpha'}_{\ \nu} u^\nu$. Suppose that the particle P is comoving with the unprimed frame, hence its 4-velocity being $u^\nu = (1,0,0,0)$, and that the primed frame x'^α is moving with the velocity $-v^k = -|\boldsymbol{v}|\,n^k$ as seen in the unprimed frame x^μ (*i.e.* $\beta = |\boldsymbol{v}|/c$). It is not difficult to show that the 4-velocity $u'^\alpha = \Lambda^{\alpha'}_{\ \nu} u^\nu$ in the primed frame coincides with (B.4). Thus,

$$u^\nu = (1,0,0,0) \quad \Rightarrow \quad u'^\alpha = \gamma\Big(1,\,|\beta|\,n^j\Big) = \Big(\frac{1}{\sqrt{1-\beta^2}},\,\frac{\boldsymbol{v}}{c\,\sqrt{1-\beta^2}}\Big), \tag{B.8}$$

Conversely, suppose that the particle P is moving in the unprimed frame with the 4-velocity u^ν of (B.4), and that the primed frame x'^α is comoving with the particle P (*i.e.* $\beta = +|\boldsymbol{v}|/c$), hence moving with the velocity $v^k = |\boldsymbol{v}|\,n^k$ relative to the unprimed frame x^μ. Under the Lorentz transformation of (B.6) and (B.7), the 4-velocity $u'^\alpha = \Lambda^{\alpha'}_{\ \nu} u^\nu$ transformed from the u^ν of (B.4) is found as

$$u^\nu = \gamma\left(1,\, \beta\, n^j\right) \quad \Rightarrow \quad u'^{\,\alpha} = \Lambda^{\alpha'}{}_{\nu}\, u^\nu = (1,0,0,0), \tag{B.9}$$

where $\gamma \equiv 1/\sqrt{1-\beta^2}$, $\beta \equiv |\boldsymbol{v}|/c$ and $j = 1,2,3$.

Appendix C. Supplements to the Gravity Theory of Main Text

Appendix C.1. Useful formulae of gravity theory

- Covariant derivatives:

$$F\text{: scalar}: \qquad F_{;\gamma} = F_{,\gamma}\ , \tag{C.1}$$

$$V^\alpha\text{: vector}: \qquad V^\alpha_{\ ;\gamma} = V^\alpha_{\ ,\gamma} + \Gamma^\alpha{}_{\mu\gamma} V^\mu, \tag{C.2}$$

$$U_\alpha\text{: 1-form}: \qquad U_{\alpha\,;\gamma} = U_{\alpha\,,\gamma} - \Gamma^\mu{}_{\alpha\gamma} U_\mu, \tag{C.3}$$

$$T^\alpha_\beta\text{: tensor}: \qquad T^\alpha{}_{\beta\,;\gamma} = T^\alpha{}_{\beta\,,\gamma} + \Gamma^\alpha{}_{\mu\gamma} T^\mu{}_\beta - \Gamma^\mu{}_{\beta\gamma} T^\alpha{}_\mu \tag{C.4}$$

- Curvature tensors and symmetry properties:

$$\text{Riemann tensor}: \ R^\alpha{}_{\beta\gamma\delta} = \frac{\partial \Gamma^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial x^\delta} + \Gamma^\alpha_{\nu\gamma}\Gamma^\nu_{\beta\delta} - \Gamma^\alpha_{\nu\delta}\Gamma^\nu_{\beta\gamma}, \tag{C.5}$$

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\nu} R^\nu_{\beta\gamma\delta} \tag{C.6}$$

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \tfrac{1}{2}\left(\partial_\alpha\partial_\delta\, g_{\beta\gamma} + \partial_\beta\partial_\gamma\, g_{\alpha\delta} - \partial_\alpha\partial_\gamma\, g_{\beta\delta} - \partial_\beta\partial_\delta\, g_{\alpha\gamma}\right) \\ &\quad + g_{\mu\nu}(\Gamma^\mu_{\beta\gamma}\Gamma^\nu_{\alpha\delta} - \Gamma^\mu_{\beta\delta}\Gamma^\nu_{\alpha\gamma}), \end{aligned} \tag{C.7}$$

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} \tag{C.8}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{C.9}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0. \tag{C.10}$$

$$\text{Ricci tensor}: \qquad R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} \tag{C.11}$$

$$= \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{C.12}$$

$$\text{Scalar curvature}: \ \ R_{\rm sc} \equiv g^{\alpha\nu} R_{\alpha\nu} \tag{C.13}$$

Appendix C.2. Variational formulation

Equations of the gravitational field are obtained from the principle of least action $\delta(S_g + S_m) = 0$, where S_g and S_m are the actions of the gravitational field and matter field respectively. The action for the gravitational field is defined by

$$S_g = \, - A_g \int g^{\alpha\beta} R_{\alpha\beta} \sqrt{-g}\, \mathrm{d}\Omega, \quad A_g \equiv \frac{c^3}{16\pi G_0}, \quad \mathrm{d}\Omega = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3, \tag{C.14}$$

where $\sqrt{-g}\,\mathrm{d}\Omega$ is the proper volume $[\mathrm{d}\Omega]_{prop}$ in a local Lorentz frame with $g = \det(g_{\mu\nu})$, and $R_{\alpha\beta}$ is the Ricci curvature tensor (C.11), and $g^{\alpha\beta}R_{\alpha\beta} = R^\alpha_\alpha \equiv R_{\rm sc}$ is the scalar curvature. The variation of S_g with respect to the metric field $g^{\alpha\beta}$ is given by

$$\delta S_g = -A_g \int \left(R_{\alpha\beta} - \tfrac{1}{2}\, g_{\alpha\beta} R^\nu_\nu\right) \delta g^{\alpha\beta} \sqrt{-g}\, \mathrm{d}\Omega. \tag{C.15}$$

On the other hand, the action S_m of the matter field is

$$S_m = \frac{1}{c}\int \Lambda_m\Big(q, \frac{\partial q}{\partial x^\nu}\Big)\sqrt{-g}\,\mathrm{d}\Omega. \tag{C.16}$$

where the Lagrangian density Λ_m contains only the tensors $q = g_{\alpha\beta}$ and their first derivatives $\partial_\nu q = \partial_\nu g_{\alpha\beta}$. Noting that variation of the coordinate from x^ν to $x^\nu + \xi^\nu$ results in variation of the metric $\delta g^{\alpha\beta}$, we obtain the variation of action S_m given after some analyses as

$$\delta S_m = \frac{1}{2c}\int T_{\alpha\beta}\,\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}\Omega, \tag{C.17}$$

(Landau & Lifshitz (1975) Eq.(94.5)), where $T_{\alpha\beta}$ is the stress-energy tensor defined by

$$\tfrac{1}{2}\sqrt{-g}\,T_{\alpha\beta} = \frac{\partial\sqrt{-g}\,\Lambda}{\partial q} - \frac{\partial}{\partial x^\nu}\frac{\partial\sqrt{-g}\,\Lambda}{\partial(\partial_\nu q)}, \qquad q \equiv g^{\alpha\beta}. \tag{C.18}$$

From the action principle $\delta(S_g + S_m) = 0$, we find

$$-A_g\int\Big(R_{\alpha\beta} - \tfrac{1}{2}\,g_{\alpha\beta}R_{\rm sc} - 8\pi k\,T_{\alpha\beta}\Big)\,\delta g^{\alpha\beta}\sqrt{-g}\,\mathrm{d}\Omega = 0,$$

where $k = G_0/c^4$. In view of the arbitrariness of the $\delta g^{\alpha\beta}$, we obtain the Einstein field equation:

$$G_{\alpha\beta} = 8\pi k\,T_{\alpha\beta}\,, \qquad k = G_0/c^4, \qquad G_{\alpha\beta} \equiv R_{\alpha\beta} - \tfrac{1}{2}\,g_{\alpha\beta}R^\nu{}_\nu. \tag{C.19}$$

where $G_{\alpha\beta}$ is the Einstein curvature tensor.

Appendix C.3. Bianchi identity

The Bianchi identity is deeply rooted in geometrical structure of physical fields. But superficially, it is just expressed by a linear combination of three terms, each of which is given by covariant-derivative of a component of Riemann curvature tensor:

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0. \tag{C.20}$$

This can be easily verified in the local Lorentz frame by using the representation obtained from (C.7) with all Γ's (but not derivatives) set to 0. The equation thus obtained is the identity like (C.20) but the ";"-operator replaced by ",". Namely, the equation is verified only for the local Lorentz frame. Finally, transition to any frame of curved spacetime can be done just by replacing "comma" by "semicolon".

Physical significance of the Bianchi identity

From the viewpoint of physics, the set of curvature tensors $R_{\alpha\beta\mu\nu}$ has a remarkable geometrical property, and surprisingly shows a striking analogy to the electromagnetic field. First we spotlight the relevant part of *Electromagnetic field*

In terms of the electromagnetic four-potentials A_μ, one-form $\mathcal{A} = A_\mu \mathrm{d}x^\mu$ was defined (see §2.1 (a)). Out of this one-form, a two-form $\mathcal{F} = \mathrm{d}\mathcal{A}$ is derived by taking its exterior differentiation $\mathrm{d}\mathcal{A}$. The two-form field $\mathcal{F}$ satisfies the identity $\mathrm{d}\mathcal{F} \equiv 0$, because $\mathrm{d}^2\mathcal{A} \equiv 0$, *i.e.* $\partial\,\partial = 0$ by the language of differential geometry, in other words by the principle "*boundary of a boundary is zero*". This yields the identity equation (2.5): $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$, giving rise to a pair of Maxwell equations of (2.7). The last can be rewritten as

$$F_{\alpha\beta\,,\gamma} + F_{\beta\gamma\,,\alpha} + F_{\gamma\alpha\,,\beta} = 0, \quad \text{in short} \quad F_{[\alpha\beta,\gamma]} = 0. \tag{C.21}$$

The symbol $[\alpha\beta,\gamma]$ denotes cyclic permutation of the parameters of three anti-symmetric pairs $[\alpha\beta]$, $[\beta\gamma]$ and $[\gamma\alpha]$. It is amazing to find that the equation (C.20) can be written analogously as

$$R_{\alpha\beta[\mu\nu;\lambda]} = 0. \tag{C.22}$$

This is a startling *coincidence*. In fact, there exists a common structure in their backgrounds, which is now highlighted.

Using the exterior derivative D defined by (3.16), the vector-valued one-form $\mathrm{D}\boldsymbol{v}$ is

$$\mathrm{D}\,\boldsymbol{v} = \boldsymbol{e}_\mu \left(\frac{\mathrm{D}v^\mu}{\mathrm{d}x^\beta} + \Gamma^\mu_{\ \alpha\beta} v^\alpha \right) \mathrm{d}x^\beta\,. \tag{C.23}$$

Now differentiate this once again to get $\mathrm{D}^2\boldsymbol{v}$:

$$\mathrm{D}^2\boldsymbol{v} = \boldsymbol{e}_\mu\, \mathcal{R}^\mu_{\ \nu}\, v^\nu, \tag{C.24}$$

(Misner *et al.* (2017), §14.5, eq.(14.17)), where $\mathcal{R}^\mu_{\ \nu}$ is the curvature 2-form defined by

$$\mathcal{R}^\alpha_{\ \beta} \equiv \mathrm{d}(\Gamma^\alpha_{\ \beta\nu}\mathrm{d}x^\nu) + \Gamma^\alpha_{\ \lambda\mu}\,\Gamma^\lambda_{\ \beta\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \tag{C.25}$$

$$= R^\alpha_{\ \beta\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \quad (\mu<\nu).$$

where the summation of the last line is taken over μ, ν with $\mu<\nu$, and $R^\alpha_{\ \beta\mu\nu}$ is the Riemann curvature tensor of (C.5).

In order to take our last step, we consider the curvature two-form $\mathcal{R}^\alpha_{\ \beta}$ in the local Lorentz frame where the second term of (C.25) drops as is done in the proof of Bianchi-id. Then we have $\mathcal{R}^\alpha_{\ \beta} = \mathrm{d}(\Gamma^\alpha_{\ \beta\nu}\mathrm{d}x^\nu)$. Taking exterior differentiation again, we obtain

$$0 = \mathrm{d}\mathcal{R}^\alpha_{\ \beta} = \mathrm{d}^2(\Gamma^\alpha_{\ \beta\nu}\mathrm{d}x^\nu) = R^\alpha_{\ \beta\mu\nu,\lambda}\,\mathrm{d}x^\lambda\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu,$$

because $\mathrm{d}^2 = 0$. From this we find, with cyclic permutation of (λ,μ,ν):

$$R^\alpha_{\ \beta\mu\nu,\lambda} + R^\alpha_{\ \beta\lambda\mu,\nu} + R^\alpha_{\ \beta\nu\lambda,\mu} = 0.$$

in the local Lorentz frame. Final transition to any frame of curved spacetime can be done by replacing "comma" by "semicolon", obtaining the Bianchi identity (C.20).

Appendix D. Second Pair of Maxwell Equations

Second pair of Maxwell equations (2.8) for the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ can be derived from the action principle. The total action $S^{(\mathrm{em})}$ is expressed as $S^{(\mathrm{em})} = S^{(\mathrm{em})}_{\mathrm{emA}} + S^{(\mathrm{em})}_{\mathrm{int}}$, where $S^{(\mathrm{em})}_{\mathrm{emA}}$ is represented with a free-field Lagrangian of Lorentz-invariant quadrutic form of the field strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and $S^{(\mathrm{em})}_{\mathrm{int}}$ represents interaction between the field and 4-current $j_e^{\,\nu}$, defined by

$$S^{(\mathrm{em})}_{\mathrm{emA}} = -\frac{1}{16\pi c}\int F_{\mu\nu}\,F^{\mu\nu}\mathrm{d}\Omega, \qquad S^{(\mathrm{em})}_{\mathrm{int}} = \frac{1}{c^2}\int j_e^{\,\nu}\,A_\nu\,\mathrm{d}\Omega, \tag{D.1}$$

with $\mathrm{d}\Omega = c\,\mathrm{d}t\,\mathrm{d}\mathcal{V}$, and $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$. We vary only the A_ν (serving as the coordinates) with the material 4-current $j_e^{\,\nu}$ assumed given (Landau & Lifshitz, 1975).

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Thus, we have the action variation caused by the variation of A_ν:

$$\delta S^{(\mathrm{em})} = \frac{1}{c}\int \Big(\frac{1}{c}\, j^\nu\, \delta A_\nu - \frac{1}{8\pi} F^{\nu\lambda}\, \delta F_{\nu\lambda}\Big)\, \mathrm{d}\Omega = 0, \tag{D.2}$$

where we used the equality $F_{\nu\lambda}\, \delta F^{\nu\lambda} = F^{\nu\lambda}\, \delta F_{\nu\lambda}$. In $S^{(\mathrm{em})}_{\mathrm{int}}$, we must not vary j^ν which is a material current, not the field. Substituting $F_{\nu\lambda} = \partial A_\lambda/\partial x^\nu - \partial A_\nu/\partial x^\lambda$, we have

$$\delta S = \frac{1}{c}\int \Big(\frac{1}{c}\, j^\nu\, \delta A_\nu - \frac{1}{8\pi} F^{\nu\lambda}\, \frac{\partial}{\partial x^\nu}\delta A_\lambda + \frac{1}{8\pi} F^{\nu\lambda}\, \frac{\partial}{\partial x^\lambda}\delta A_\nu\Big)\, \mathrm{d}\Omega,$$

We interchange the indices ν and λ in the middle term. Using the antisymmetry of the matrix $F^{\lambda\nu}$, one can replace the factor $F^{\lambda\nu}$ by $-F^{\nu\lambda}$. Then we obtain

$$\delta S = \frac{1}{c}\int \Big(\frac{1}{c}\, j^\nu\, \delta A_\nu + \frac{1}{4\pi} F^{\nu\lambda}\, \frac{\partial}{\partial x^\lambda}\delta A_\nu\Big)\, \mathrm{d}\Omega,$$

To the second term, we perform integration by parts. Since the surface integral thus obtained vanishes by the imposed boundary conditions. Thus, the principle of least action leads to

$$\int \Big(\frac{1}{c}\, j^\nu - \frac{1}{4\pi}\, \frac{\partial F^{\nu\lambda}}{\partial x^\lambda}\Big)\delta A_\nu\, \mathrm{d}\Omega = 0. \tag{D.3}$$

Since the variation δA_ν is arbitrary, the coefficient of δA_ν must vanish:

$$\frac{\partial F^{\nu\lambda}}{\partial x^\lambda} = \frac{4\pi}{c}\, j^\nu, \tag{D.4}$$

where $j^\nu = (\rho_e c, \boldsymbol{j}_e)$ with $\boldsymbol{j}_e = \rho_e \boldsymbol{v}$, The field strength tensor $F_{\nu\lambda}$ is defined by (1.9), and its matrix representation by (1.10), while $F^{\nu\lambda}$ is defined by $g^{\nu\alpha}\, F_{\alpha\beta}\, g^{\beta\lambda}$.

References Références Referencias

1. Aitchison IJ R and Hey A J G 2013 *Gauge Theories in Particle Physics: A Practical Introduction*, Vol. 1 & 2, Fourth Edition. (CRC Press, Taylor & Francis Group).
2. Einstein A 1905 Zür elektrodynamik bewegter Köper, *Ann. Phys.* (Germany) 17, 891 –921.
3. Einstein A 1915 Zür Allgemeinen Relativitätstheorie, *Preuss. Akad. Wiss. Berlin, Sitzber.*, 778 - 786.
4. Fock, V., 1926, Über die invariante Form derWellen- und der Bewegungsgleichungen für einen geladenen Massenpunkt, *Zeit. für Physik* 39, 226-232.
5. Frankel T 1997 *The Geometry of Physics – An Introduction*, (Cambridge: Cambridge University Press).
6. Jackson J D 1999 *Classical Electrodynamics* (Third edition) (John Wiley & Sons).
7. Jackson J D and Okun L B 2001 Historical roots of gauge invariance, *Rev. Mod. Phys.* 73, 663 –680.
8. Kambe T 2010 *Geometrical Theory of Dynamical Systems and Fluid Flows*, (World Scientific).

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

9. Kambe T 2012 A new solution of Euler's equation of motion with explicit expression of helicity, *WSEAS Transactions on Fluid Mechanics* 7(2), 59-70.
10. Kambe T 2013 New representation of rotational flow fields satisfying Euler's equation of an ideal compressible fluid, *Fluid Dyn. Res.* 45, 015505 (16pp).
11. Kambe T 2017 New scenario of turbulence theory and wall-bounded turbulence: Theoretical significance, *Geophys. Astrophys. Fluid Dyn.* 111, 448-507.
12. Kambe T 2020 New perspectives on mass conservation law and waves in fluid mechanics, *Fluid Dyn. Res.* 52, 1 - 34 (34pp).
13. Kambe T 2021 Fluid Gauge Theory, *Global Journal of Science Frontier Research,* vol.21, iss.4.
14. Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* (Pergamon Press).
15. Landau L D and Lifshitz E M 1987 *Fluid Mechanics* (Pergamon Press, 2nd ed.).
16. Misner C W, Thorne KS and Wheeler J A 2017 *Gravitation* (W.H. Freeman and Co., San Francisco).
17. Noether E 1918 Invariant variations problem, *Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse*, 235-257.
18. O'Raifeartaigh L 1997 *The Dawning of Gauge Theory* (Princeton Univ. Press).
19. Schutz B E 1985 *A first course in general relativity* (Cambridge Univ. Press).
20. Utiyama R 1956 Invariant theoretical interpretation of interaction, *Phys. Rev.* 101, 1597.
21. Utiyama R 1987 *Ippan Gauge ba ron josetsu (Introduction to the Genaral Gauge Field Theory)* (Iwanami Shoten, Tokyo).
22. Weyl H 1918 Gravitation and Electricity, *Sitzungsber.* (Meeting Rep.) *Preuss. Akad. Berlin*, 465; a book, *Raum-Zeit-Materie* (Springer, Berlin, 1923) §14.
23. Weyl H 1929 Electron and Gravitation, *Zeit. f. Physik* 56, 330-352.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 21 Issue 4 Version 1.0 Year 2021
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

S-Existence: There is another Existence of Everything

By Sumeru Ray (Maharshi Maha Manas)

Abstract- 'S-Existence' is another existence of everything. Lack of appropriate word has caused to mark or name this unearthly existence– 'Sumeru-Existence', briefly– 'S-Existence'. This 'S-Existence' identity is too much subtle –unperceived –unfelt.

From fundamental particle to every element, – matter and energy have two different existences in its form. The formations of matter or element, components, quality, energy etc all are present in its 'S-Existence'. Generally, this 'S-Existence' is inseparable mixed with material existence.

Keywords: s-existence, another existence, contrary existence, sumeru-existence, theoretical physics.

GJSFR-A Classification: FOR Code: 029999

SEXISTENCETHEREISANOTHEREXISTENCEOFEVERYTHING

Strictly as per the compliance and regulations of:

S-Existence: There is another Existence of Everything

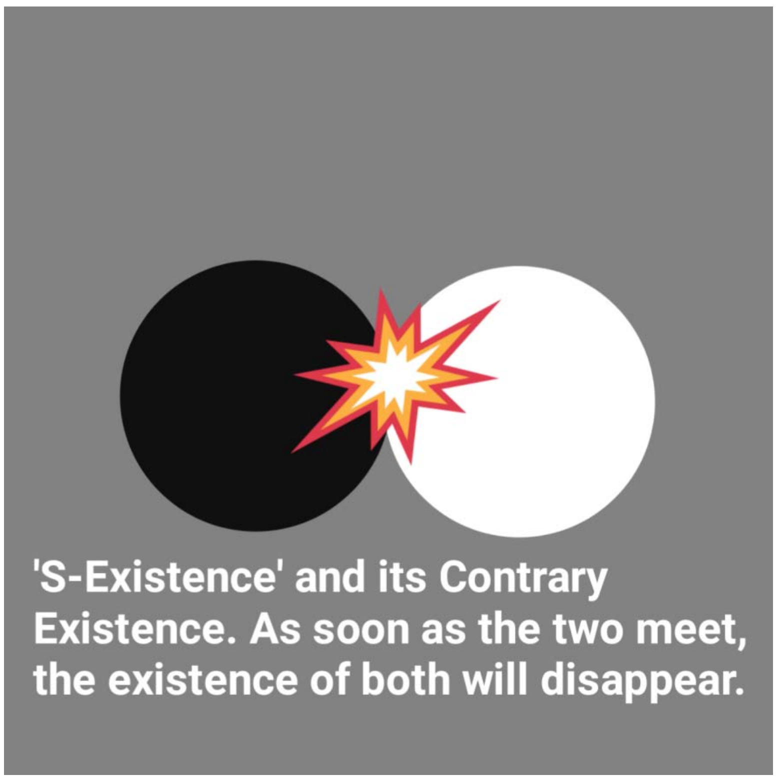
Sumeru Ray (Maharshi Maha Manas)

Abstract- 'S-Existence' is another existence of everything. Lack of appropriate word has caused to mark or name this unearthly existence– 'Sumeru-Existence', briefly– 'S-Existence'. This 'S-Existence' identity is too much subtle –unperceived –unfelt.

From fundamental particle to every element, – matter and energy have two different existences in its form. The formations of matter or element, components, quality, energy etc all are present in its 'S-Existence'. Generally, this 'S-Existence' is inseparable mixed with material existence.

Keywords: s-existence, another existence, contrary existence, sumeru-existence, theoretical physics.

I. S-Existence and its Contrary Existence



'S-Existence' is another existence of everything Lack of appropriate word has caused to mark or name this unearthly existence– 'S-Existence'. This 'S-Existence' identity is too much subtle –unperceived –unfelt.

From fundamental particle to every element, – matter and energy have two different existences in its form. The formations of matter or element, components, quality, energy etc all are present in its 'S-Existence'. Generally, this 'S-Existence' is inseparable mixed with material existence.

Author: Department of Versatile Science Research, MahaManan Kendra, MahaManan Kendra, Kalna, West Bengal, India.
e-mail: sumeru1235@gmail.com

In case of living body it is same; it also has 'S-Existence' –unearthly body or supernatural body. 'S-Existence' of mind, – vital energy and sense etc are also present in the supernatural body.

With matter and energy, their 'S-Existence' is also mixed inseparably. With support of material body or matter, mixed with it– its 'S-Existence' is present. Without 'S-Existence' any matter– anything cannot be there. If there is change in matter or body's shape –quality, similar change takes place in its 'S-Existence'. This supernatural body or matter is a true copy of material body or matter, which is not a known matter –is an unknown and different matter –is made of unearthly matter.

When there is an action and reaction between a matter and other matter, then there takes place similar action and reaction and change between their 'S-Existence'. When a matter comes into contact with its antimatter, –both the matters as soon as the union of their 'S-Existences' –both loses their independent existence. As a result of this union, can occur the dispersion of energy, and can be created different kinds of particles, new matters –elements and energy –depending on the condition and velocity of the union.

When any matter or particle of matter has repeated collisions with other matter or matter-particle, then their S-Existences will put a contrary impression* on each other's 'S-Existence'. Everyone's 'S-Existence' will carry the contrary impression of other 'S-Existence', till the time– the carrier matter or matter-particle and along with their S-Existence's disorder– deformity or change takes place.

* This contrary impression is not like two dimensional stamp-print or reverse-print. This contrary impression is a thorough –three (or more) dimensional impression of super (natural) existence which is externally inactive and just opposite impression existence of 'S-Existence' of any material thing.

From that contrary impression of 'S-Existence' (CISE) which is a burden to the matter's 'S-Existence', many more copy of that can be produced, –then only if repeated collisions takes place with the matter or element-particle carrying that impression and similar or other matters or element particles. But in this case, contrary impression of that impression of 'S-Existence' will not be made. Depending on other matter's 'S-Existence', this contrary impression of 'S-Existence' (CISE) is merely a (material or supernatural) bodiless impression. As it is not having its own body, it is unable to impress any contrary impression on others. As a result of collisions, from that– many similar copies can be made, but gradually those copies will get more and more fineness or subtlety. From one particle to other particle, from that to other –if in this way gradually that copy is made, then gradually the copy of 'CISE' will become finer to finest.

Usually this 'CISE' is inactive –quality less. But anyhow if this 'CISE' enters the living body, then by the help of the software of bio-system, is able to get a supernatural body.

The 'CISE' on entering the living body, reaches the particular center of brain with help of nervous system. There with the special work capability of the software of bio-system, that copy of impression (CISE) gets the supernatural body. That is to say, 'CISE' turns into contrary super (natural) body or existence. Then it is not merely inactive impression, an active supernatural (body or matter) existence –endowed with quality and character –properties. The birth that it took from matter or element as 'CISE', and then it becomes contrary S-Existence of that matter or element.

Then, that contrary S-Existences spread through the circulatory system and nervous system–throughout the creature's whole body –similar to defense force. In search of enemy– culprit or evildoer those who are causes of diseases, –thoroughly searching the whole body, meeting its contrary existence –immediately it jumps on that. As a result, disappear the identity of both.

The subject is absolutely new, so let's repeat in short, –when the CISEs come into contact with the body's nervous system, –in electrical speed, they reach a particular center of the brain. There, with the help of bio-system's software, the CISEs get supernatural bodies or 'S-Existence' –according to their quality –nature –property, everyone. From there, following the defense system of the living body, –they spread out the sickly parts of the body or throughout the body –in search of alike-contrary thing which are poisonous –harmful for body.

If the nervous system and the software of the body is weak, undisciplined or incapable, that 'CISE' may not be able to get a super (natural) body or existence and not be able to follow its working role properly.

After entering of 'CISE' (of anything) in the body, if that 'CISE' converts to 'S-Existence' with the help of body-mechanism and if that 'S-Existence' does not meet the contrary existence in the body, –in that case if the number of S-Existences are too many, then they are the contrary existence of which matter or element, who were able to create as like action of poisoning in the body, these (contrary) S-Existences are able to create contrary or opposite symptom of poisoning of that thing. For example, if main thing or matter is able to create constipation, then its contrary 'S-Existence' (about which is discussing) will create diarrhea.

But if these S-Existences' number or amount is less in poisonous limit, then they will do similar activity, but it will be not so acute –not be unhealthy. Rather relieving constipation –will help to clear the bowels. That will work like mild purgative.

This action will not be stable. The cause which created the constipation, that cause will not be stopped in this case.

If the number of (contrary) 'S-Existence' is too low, then any action may not be felt. Except it, cause behind the constipation (or other incident) if there is present any strong reason or poison, in that case, the 'CSE' may not be successful in its work. But if this 'CSE' comes in contact with a thing which is similar in every respect (form –quality –character etc) in the body, then action of poisoning can be created depending on their number or amount.

In the body, when 'CSE's are able to do their own work, at the end of work –end their longevity. But

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

those who do not get the opportunity of work, they slowly go out from the body through different paths or outlets. But everyone has its own life period.

Matter is material form of its 'S-Existence'. Any matter can never be without its 'S-Existence'. In case of energy, it is same. But the 'S-Existence' in some cases, without its material form, that is without the material body– able to remain, depending on some unearthly things. If any 'S-Existence' meets with its (exact) contrary 'S-Existence', the identity of both disappears. Along with that, if they have the material identity or existence, that also destroys.

Just like its material body, this 'S-Existence' is made by combination of quality-full unearthly particles. Atom and its inner energies and particles, fundamental particle and all other particles have 'S-Existence'. Every particle of matter or element has the similar 'S-Existence' –present in it.

The 'CISE' that entering the body– after getting the similar 'S-Existence', connecting with the harmful contrary existence, –both disappeared and destroyed. –This is the principle of 'MahaPathy' medicine's recovery procedure. I shall make an acquaintance with 'MahaPathy' next time. MahaPathy (briefly MPathy) –the super excellent system of medicine came into existence based on this theory.

If repeated collisions takes place between two molecules or particles of similar nature and quality –for some times, –does not produce contrary impression on each others 'S-Existence'. No particle accepts its contrary impression –does not accept its contrary impression on it. Here we have to keep in mind, (contrary) impression of 'S-Existence' (CISE) and 'S-Existence' –not same.

Now what will happen, if bodily enemy or poisonous matter or anything causing the action of poisoning –instead of being exact contrary of medicine if it is partly contrary –then?

Contrary 'S-Existence' (CSE) in medicine form (produced from CISE), after searching in the body, if does not meet its exact contrary existence, meets any partly (largely) contrary existence, then also it jumps to get united –in extreme attraction of union. In that case– depending on both's energy, amount, quality and strength along with body's condition of that time, can cause various incidents. Can occur partly alleviation or partly cure. If capable, the rest of part– body can manage oneself. Result of union can create– new matters –particles, and dispersion of rays can take place at different levels.

Except it, if any harmful poisonous germ is present in body and be attacked by that medicine, then it, for the self-defence, can assume very violent angry-looking, and can create indiscipline in bodily system. Or by escaping, can go exterior portion of the body like skin eruption. Or can try to self-hide, just like the snail.

Another thing is, if contrary impression (CISE) of any matter or element is applied repeatedly on any sensitive person, in that case, action of poisoning can take place and symptom of that poisoning can be seen. Those (medicine form) CISEs after becoming 'S-Existence' in the person's body, if they do not meet the contrary impression or contrary 'S-Existence', then also they (because of their majority) can create action of poisoning by themselves.

After any poison enters the body, medicine (CISE) prepared from that poison, at many times– is incapable of making that poison inactive. One of reasons of this is– many times, after any poison enters the body, connecting with other poisons or things present in the body, –that poison takes different form. For this reason, same poison takes different form in different bodies. As a result, the medicines (CISE) created from that poison is incapable of making inactive –the same poison. Then that has to be made inactive by any other medicine.

But anyhow, if the (mixed or unmixed) poison, is collected from body, and medicine (CISE) is made from that, –if that medicine is entered into the body, then that poison becomes inactive.

Except that, any poison, after entering the body, gradually changes take place in its form –quality –nature etc with the passing of time. The contrary 'S-Existence' of that changed condition's poison can make this changed poison– appropriately inactive (within the body).

The noticeable thing is– due to eating some amount of calcium –along with other symptoms, the secretion of bile from gall bladder will increase and with it– the temperature of the body will increase. But the medicine (CISE) prepared from that calcium, as a result of eating a lot of that medicine repeatedly, –the symptoms appeared in the body, that is opposite (symptoms) to material calcium. How it happens?

The happening is–, in the body, the particles of contrary 'S-Existence' (CSE) of calcium destroy the equal amount of calcium particles of the body. As a result, deficiency of calcium takes place, in the body. The contrary S-Existences (in a large amount) of calcium particles– create that kind of action of poisoning– in the body that is just opposite symptom of material calcium.

If that CSEs are not able to meet with similar or almost similar minute– contrary existence (matter or element), in the body, or remain some quantity –after meeting with contrary existence, then they create action or action of poisoning of different degrees –depending on their existing amount in the body.

If there is present –from before, any matter or element –endowed with similar action of poisoning, –that is to say, if similar action of poisoning is all ready present in the body, in that case, by their united action of poisoning– intense action of poisoning takes place –in the body. In such a condition, to remedy this intense action of poisoning, the defense system of the body–creates more quantity of antidotes or antitoxins. As a result of that, the poison produced from medicine, along with former poison –both are destroyed or disappeared. Besides, as medicine form 'CSE' has no material body, –the span of its life is finished in a short time.

II. Evidence

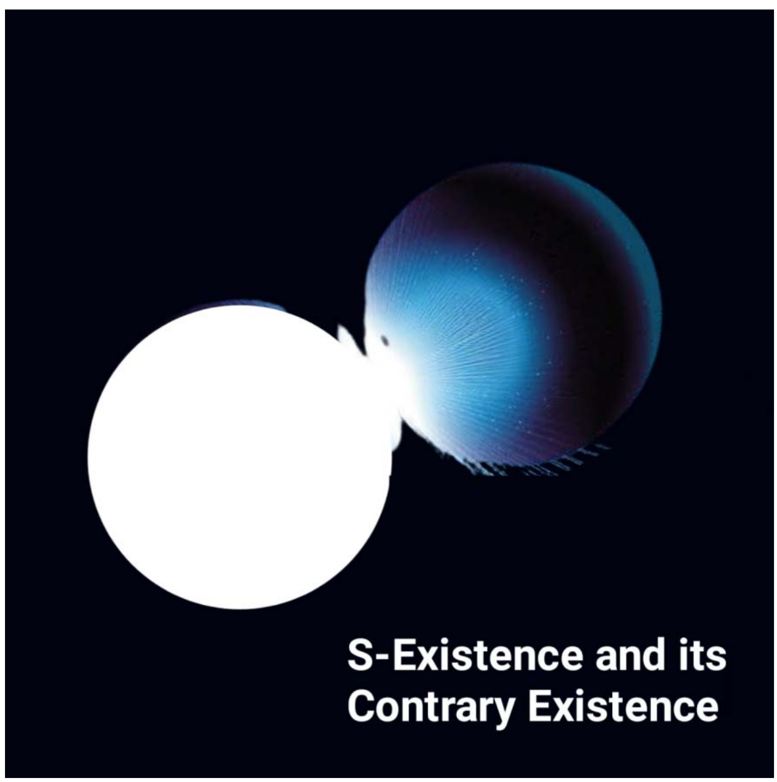
If Kirlian photography is true then evidence of this theory can be found largely in Kirlian photography. The whole picture of a leaf can be seen in Kirlian photography even after the leaf of a tree has been cut in half. Homeopathic medicine is another significant evidence of S-Existence.

III. Conclusion

The nanoparticle medicine or nanomedicine that is now in vogue, if the nanomedicine is prepared in a special process, by repeated mixing and friction then it becomes superior to ordinary nanomedicine. This method will create S-existence and it's contrary existence. If nanomedicine is made in this way from diseased cells or toxins in the body, the nanomedicine reaches the body and combines with the toxin or diseased cell, destroying both. In this way we can benefit through the practical application of S-existence. I am hopeful that the successful use of this S-existence in the future will enable people to benefit in many more ways.

References Références Referencias

1. The scientific thoughts... https://www.amazon.com/gp/aw/d/9354380417/ref=tmm_pap_swatch_0?ie=UTF8&qid=&sr=
2. MahaPathy: A new invented system of medical treatment. https://www.kobo.com/us/en/ebook/mahapathy
3. Kirlian Photography https://www.amazon.in/Kirlian-Photography-Hands-John-Iovine/dp/0830644571
4. 'The Right Way to Defeat Coronavirus!' A research paper by Sumeru Ray (on a new vaccine discovered for the coronavirus) https://www.worldwidejournals.com/international-journal-of-scientific-research-(IJSR)/article/the-right-way-to-defeat-coronavirus/MzY1ODM=/?is=1&b1=&k=



Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 21 Issue 4 Version 1.0 Year 2021
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Force-Mediating Particle of Coupling of Spin Angular Momenta

By ShaoXu Ren
Tongji University

Abstract- In this paper, a hypothesis is proposed, that something similar to what happen to the puzzle of the *energy losing* in β decay of neutron may also occur to the puzzle of the *sum losing* of the z-components of spin angular momenta in the synthetic course of spin coupling in Spin Topological Space. The former puzzle is related to hidden neutrial antineutrino that carries a small amount of energy away, the latter puzzle is related to hidden "constructive" zero-spin particle $0\hbar$ playing the role of a force-mediator that carries some amount of spin angular momentum, which *just offsets* the same amount of angular momentum *losing* in the formation of spin coupling.

Keywords: spin angular momentum coupling, Spin Topological Space, STS, spin-0 particle, Force-Mediating Particle, angular momentum losing, excited states of the C-G Coefficients.

GJSFR-A Classification: FOR Code: 029999

FORCEMEDIATINGPARTICLEOFCOUPLINGOFSPINANGULARMOMENTA

Strictly as per the compliance and regulations of:

Force-Mediating Particle of Coupling of Spin Angular Momenta

ShaoXu Ren

Abstract- In this paper, a hypothesis is proposed, that something similar to what happen to the puzzle of the *energy losing* in β decay of neutron may also occur to the puzzle of the *sum losing* of the z-components of spin angular momenta in the synthetic course of spin coupling in Spin Topological Space. The former puzzle is related to hidden neutrial antineutrino that carries a small amount of energy away, the latter puzzle is related to hidden "constructive" zero-spin particle $0\hbar$ playing the role of a force-mediator that carries some amount of spin angular momentum, which *just offsets* the same amount of angular momentum *losing* in the formation of spin coupling.
Keywords: spin angular momentum coupling, Spin Topological Space, STS, spin-0 particle, Force-Mediating Particle, angular momentum losing, excited states of the C-G Coefficients.

I. Introduction

In conventional spin theory [1],[2],[3],[4],[5], each spin particle has its own spin space; one particle possesses one spin space; two spin particles, two spin spaces,...... and n particles, n spin spaces. These spaces are independent each other. To couple two or more angular momentums and then obtain the angular momentum of the combined system. The spin space of the combined system can be expressed as the direct product of two or more single spin particles

$$V = V_1 \otimes V_2 \otimes V_3 \otimes \ldots \otimes V_n \tag{0.1}$$

The way the dimensionalities, an example of two spin-1/2 fermions, work out is as follows:

$$V = 1/2 \otimes 1/2 = 1 \oplus 0 \tag{0.2}$$

(V_1=1/2, V_2=1/2) The decomposition of the direct product $V_1 \otimes V_2$ space into a sum of space 1 and space 0 above. The dimensionality of each spin -1/2 is $2\cdot\frac{1}{2}+1=2$, spin-1 is $2\cdot1+1=3$ and spin-0 is $2\cdot0+1=1$. The total dimensionality is 2+2=3+1=4.

Now if suppose $V_1=V_2=V$, What will happen to their spin angular momentum couple (0.2)? The analysis shows: an amusing spin angular momentum picture, so-called Spin Topological Space, STS [6],[7],[8],[9] is introduced. The spin space dimensionalities of the two spin-1/2 fermions V_1, V_2 and V, mentioned before, all become to be infinite.

One of the achievements is that STS could invest spin-zero particle with math constructive constituent of angular momentum such as other bosons and fermions in the conventional spin theory. In STS math frame, spin-zero particle is no longer a " point-particle ".

Author: Tongji University. e-mail: shaoxu-ren@hotmail.com

But on the other hand, STS in itself always encounters with troubles for angular momentum z-component addtion of many single spin particles, some types of z-component addtion, in which the total z-components of the combined system are always less than the those that should be, that is so-called the puzzle of spin angular momentum *losing*, from the point of view of common sense for math and physice world today.

But if the spin-zero particle mentioned previously is supposed to be viewed as an invisible boson, a force-mediating mediator, which actually interacts with the two spin-1/2 fermions, is participanting in the synthetic course of z-component addition of the two fermioms, the problem of *losing* with z-component addition of angular momentums would be solved. It will be lucky, spin-zero boson particle will eliminate the faults of z-component addtion in STS.

Using the above peculiar ideas and new concepts, the matrix representations of C-G Coefficients of two spin-1/2 fermions coupling in STS are worked out, which predict many "excited states" of C-G Coefficients that have not been obverserd so far.

II. Spin Topological Space, STS

We work with systems made up of two angular momenta, $\vec{j_1}$ and $\vec{j_2}$

$$\vec{j_1} \times \vec{j_1} = i\,\vec{j_1}, \qquad \vec{j_2} \times \vec{j_2} = i\,\vec{j_2}, \tag{1}$$

If an interaction between $\vec{j_1}$ and $\vec{j_2}$ is such as to have the coupling of two angular momenta

$$\vec{j} = \vec{j_1} + \vec{j_2}$$

Then $$\vec{j} \times \vec{j} = (\vec{j_1} + \vec{j_2}) \times (\vec{j_1} + \vec{j_2}) = i\,(\vec{j_1} + \vec{j_2}) + (\vec{j_1} \times \vec{j_2} + \vec{j_2} \times \vec{j_1}) \tag{2}$$

Suppose $\overleftharpoon{j_1} \subset$ Space V_{j_1}, $\overleftharpoon{j_2} \subset$ Space V_{j_2}

It will allow the following two cases of (2) to happen:

【1】If *both* $\vec{j_1}$ *and* $\vec{j_2}$ attribute to *different* space

$$V_{j_1} \neq V_{j_2} \tag{3}$$

【2】If $\vec{j_1}$ *and* $\vec{j_2}$ all attribute to one same space

$$V_{j_1} = V_{j_2} = V, \tag{4}$$

Next we give the math construction of physics reality in cases of【1】and【2】

■【1】Obviously, $[\, j_{1,\alpha}\ ,\ j_{2,\beta}\ ]_- = 0, \qquad \alpha, \beta = 1, 2, 3$

we have $$\vec{j_1} \times \vec{j_2} + \vec{j_2} \times \vec{j_1} = 0 \tag{5}$$

And obtain angular momentum commutation rules:

$$(\vec{j_1} + \vec{j_2}) \times (\vec{j_1} + \vec{j_2}) = i\,(\vec{j_1} + \vec{j_2}) \tag{6}$$

where
$$\vec{j} \times \vec{j} = i\vec{j}, \quad \vec{j} = \vec{j_1} + \vec{j_2} \tag{7}$$

As $(j_1, m_1), (j_2, m_2)$ and (j, m) are the eigenvalues of j_1, j_2 and j respectively, by unitary transformation with Wigner or Clebsch-Gordan coefficients, we can have some matrix relationships among (j_1, m_1), (j_2, m_2) and (j, m).

Continue to discuss the example (0.2), let j_1 and j_2 be two spin-1/2 fermions, we can get formula below

$$\begin{array}{lccccccc} & 1/2 & \otimes & 1/2 & = & 1 & \oplus & 0 \\ \text{Casimir operator} & \frac{3\hbar^2}{4} & & \frac{3\hbar^2}{4} & & 2\hbar^2 & & 0\hbar^2 \end{array} \tag{8}$$

Here, reducible $(2j_1+1)(2j_2+1)=4$ dimensional representation of rotation group is denoted with multiplication 1/2 $\otimes$ 1/2, by unitary transformation, reduce it to its irreducible representation, a triplets, a 3 dimension of spin-1 and a singlet, one dimension, labelled addition 1 $\oplus$ 0. (8) called direct product of two spin-1/2 Hilbert spaces is a direct sum of a spin-1 space and a spin-0 space.

Here: spin-1 space is spanned by a triplet state of two symmetric fermions with Casimir operator eigenvalue $1(1+1)=2\hbar^2$. And spin-0 space, by a singlet state of two antisymmetric fermions with Casimir operator eigenvalue $0(0+1)=0\hbar^2$.

Formula (8), due to spin angular momentum couple, is based on the restriction (5) of case 【1】. We wonder what will happen to (8) in case 【2】

if
$$\vec{j_1} \times \vec{j_2} + \vec{j_2} \times \vec{j_1} \neq 0 \tag{9}$$

that is: how can we deal with the angular momentum coupling between the two spin-1/2 fermions if they *all attribute to one same spin space V* ?

■ 【2】 Call $(j, k), (r, s)$ Spin Topological Coordinate in real region of STS, and now sign $\vec{\pi}$ is used to represent spin angular momentum.

Giving following two definitions

$$\pi_{1;j,k} = \frac{1}{2}(\pi_j^+ + \pi_k^-) \tag{10.1}$$

$$\pi_{2;j,k} = \frac{1}{2i}(\pi_j^+ - \pi_k^-) \tag{10.2}$$

After calculations obtain

$$\pi_{3;j,k} = \pi_{1;j,k}\pi_{2;j,k} - \pi_{2;j,k}\pi_{1;j,k} = \frac{1}{2}(\pi_j^+\pi_k^- - \pi_k^-\pi_j^+) \tag{10.3}$$

From above three formulas, it can be shown $\pi_{1;j,k}, \pi_{2;j,k}$ and $\pi_{3;j,k}$ satisfy angular momentum rule.

$$\vec{\pi}_{j,k} \times \vec{\pi}_{j,k} = i\vec{\pi}_{j,k} \tag{10}$$

The similar results to

$$\pi_{1;\,r,s} = \frac{1}{2}(\pi_r^+ + \pi_s^-) \tag{11.1}$$

$$\pi_{2;\,r,s} = \frac{1}{2i}(\pi_r^+ - \pi_s^-) \tag{11.2}$$

$$\pi_{3;\,r,s} = \pi_{1;\,r,s}\pi_{2;\,r,s} - \pi_{2;\,r,s}\pi_{1;\,r,s} = \frac{1}{2}(\pi_r^+\pi_s^- - \pi_s^-\pi_r^+) \tag{11.3}$$

$$\vec{\pi}_{r,s} \times \vec{\pi}_{r,s} = i\vec{\pi}_{r,s} \tag{11}$$

Instesd of sign $\vec{j}$, making the substitutions

$$\vec{j}_1 \Rightarrow (\vec{j}_1)_{j,k} = \vec{\pi}_1 \equiv \vec{\pi}_{j,k} = \{\pi_{1;\,j,k}, \pi_{2;\,j,k}, \pi_{3;\,j,k}\} \tag{12}$$

$$\vec{j}_2 \Rightarrow (\vec{j}_2)_{r,s} = \vec{\pi}_2 \equiv \vec{\pi}_{r,s} = \{\pi_{1;\,r,s}, \pi_{2;\,r,s}, \pi_{3;\,r,s}\} \tag{13}$$

Then the lefthand of (9) becomes to

$$\vec{j}_1 \times \vec{j}_2 + \vec{j}_2 \times \vec{j}_1 \quad \Rightarrow \quad (\vec{\pi}_{j,k} \times \vec{\pi}_{r,s}) + (\vec{\pi}_{r,s} \times \vec{\pi}_{j,k}) \tag{14}$$

After the primordial representations introduced above, the raising operator π_j^+ can be defined by $\pi_j(0)$ and I_{+1} as (15). Similarly the lowering operator π_k^- defined by $\pi_k(0)$ and I_{-1} as (16)

$$\pi_j^+ = +\,\pi_j(0)I_{+1} \tag{15}$$

$$\pi_k^- = -\,I_{-1}\pi_k(0) \tag{16}$$

Where $$\pi_j(0) = \pi_0(0) + jI_0, \qquad \pi_k(0) = \pi_0(0) + kI_0 \tag{17}$$

Here: Subscript, " 0 " of I_0, represents the unit principle diagonal in STS. Subscripts, " $+1$ " and " -1 " the first up unit principle diagonal and the first down unit principle diagonal.

Substitute (15) and (16) into (10.3), and the same means into (11.3), we obtain

$$\pi_{3;\,j,k} = \pi_0(0) + \frac{1}{2}(j + k + 1) \tag{18.1}$$

$$\pi_{3;\,r,s} = \pi_0(0) + \frac{1}{2}(r + s + 1) \tag{18.2}$$

$$\pi_0(0) = \text{diag}\{\ ,\ +5, +4, +3, +2, +1, \underline{0}, -1, -2, -3, -4, -5,\ ,\ \}_0 \tag{19}$$

$\pi_0(0)$ is the spin basic state of all bosons, called the vacuum background of spin angular momentum.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Now back to (14), for convenience, we are referred to the third component of (14)

$$(\vec{\pi}_{j,k} \times \vec{\pi}_{r,s})_3 + (\vec{\pi}_{r,s} \times \vec{\pi}_{j,k})_3 \tag{20}$$

Firstly calculate

$$(\vec{\pi}_{j,k} \times \vec{\pi}_{r,s})_3 = \pi_{1;j,k}\pi_{2;r,s} - \pi_{2;j,k}\pi_{1;r,s} = \tfrac{i}{2}(\pi_j^+\pi_s^- - \pi_k^-\pi_r^+) \tag{21.1}$$

$$(\vec{\pi}_{r,s} \times \vec{\pi}_{j,k})_3 = \pi_{1;r,s}\pi_{2;j,k} - \pi_{2;r,s}\pi_{1;j,k} = \tfrac{i}{2}(\pi_r^+\pi_k^- - \pi_s^-\pi_j^+) \tag{21.2}$$

Then combine the two above expressions, yield

$$\begin{aligned}
&(\vec{\pi}_{j,k} \times \vec{\pi}_{r,s})_3 + (\vec{\pi}_{r,s} \times \vec{\pi}_{j,k})_3 \\
&= \tfrac{i}{2}(\pi_j^+\pi_s^- - \pi_k^-\pi_r^+ + \pi_r^+\pi_k^- - \pi_s^-\pi_j^+) = i(\pi_{3;j,s} + \pi_{3;r,k}) \\
&= i(\pi_{3;j,k} + \pi_{3;r,s}) = i(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_3
\end{aligned} \tag{22}$$

it allows the following to happen:

$$\begin{aligned}
&\{ (\vec{\pi}_{j,k} + \vec{\pi}_{r,s}) \times (\vec{\pi}_{j,k} + \vec{\pi}_{r,s}) \}_3 \\
&= (\vec{\pi}_{j,k} \times \vec{\pi}_{j,k})_3 + (\vec{\pi}_{r,s} \times \vec{\pi}_{r,s})_3 + (\vec{\pi}_{j,k} \times \vec{\pi}_{r,s})_3 + (\vec{\pi}_{r,s} \times \vec{\pi}_{j,k})_3 \\
&= i(\vec{\pi}_{j,k})_3 + i(\vec{\pi}_{r,s})_3 + i(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_3 = 2i(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_3
\end{aligned} \tag{23}$$

Finally approaching to commutation rule of the third component of (14)

$$\tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_3 \times \tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_3 = i\tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_3 \tag{24.3}$$

Proceeding similarly as the above discussion, we can get:

$$\tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_1 \times \tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_1 = i\,\tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_1 \tag{24.1}$$

$$\tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_2 \times \tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_2 = i\,\tfrac{1}{2}(\vec{\pi}_{j,k} + \vec{\pi}_{r,s})_2 \tag{24.2}$$

And $$\tfrac{1}{2}(\vec{\pi}_1 + \vec{\pi}_2) \times \tfrac{1}{2}(\vec{\pi}_1 + \vec{\pi}_2) = i\tfrac{1}{2}(\vec{\pi}_1 + \vec{\pi}_2) \tag{24}$$

★ Summary:

a▸ Angular momentum coupling between two spin particles $\vec{\pi}_1$ and $\vec{\pi}_2$ in STS

$$\vec{\Pi} \times \vec{\Pi} = i\vec{\Pi} \tag{25}$$

$$\vec{\Pi} = \tfrac{1}{2}(\vec{\pi}_1 + \vec{\pi}_2) \tag{26}$$

$$\Pi_3 = \tfrac{1}{2}(\pi_{3;\,j,k} + \pi_{3;\,r,s}) \tag{27}$$

$$= \pi_0(0) + \tfrac{1}{2}\{\ \tfrac{1}{2}(j + k + r + s) \ + 1\ \} \tag{28}$$

b▸ The extension of the spin coupling among three spin particles $\vec{\pi}_1, \vec{\pi}_2$ and $\vec{\pi}_3$ in STS

$$\vec{\Pi} \times \vec{\Pi} = i\vec{\Pi} \tag{28}$$

$$\vec{\Pi} = \tfrac{1}{3}(\vec{\pi}_1 + \vec{\pi}_2 + \vec{\pi}_3) \tag{29}$$

$$\Pi_3 = \tfrac{1}{3}(\pi_{3;\,j,k} + \pi_{3;\,r,s} + \pi_{3;\,u,v}) \tag{30}$$

$$= \pi_0(0) + \tfrac{1}{2}\{\ \tfrac{1}{3}(j + k + r + s + u + v) \ + 1\ \} \tag{31}$$

III. Puzzles of Spin Angular Momentum Addition of the Third Components

First using a▸ (27) (28) to discuss coupling of two spin-1/2 fermions $\vec{\pi}_1$ and $\vec{\pi}_2$ in STS.

(A) Put

$$j + k = r + s = 0 \tag{32}$$

Get

$$\Pi_3(\uparrow,\uparrow) = \tfrac{1}{2}(m_1 + m_2) = \tfrac{1}{2}\left(\tfrac{+\hbar}{2}\ \uparrow + \tfrac{+\hbar}{2}\ \uparrow\right) = \tfrac{+\hbar}{2}\ \uparrow \neq \tfrac{+\hbar}{2} + \tfrac{+\hbar}{2} = +1\hbar \uparrow \tag{33}$$

(B) Put

$$j + k = 0, \quad r + s = -2 \tag{34}$$

Get

$$\Pi_3(\uparrow,\downarrow) = \tfrac{1}{2}(m_1 + m_2) = \tfrac{1}{2}\left(\tfrac{+\hbar}{2}\ \uparrow + \tfrac{-\hbar}{2}\ \downarrow\right) = 0\hbar \tag{35}$$

(C) Put

$$j + k = -2, \quad r + s = 0 \tag{36}$$

Get

$$\Pi_3(\downarrow,\uparrow) = \tfrac{1}{2}(m_1 + m_2) = \tfrac{1}{2}\left(\tfrac{-\hbar}{2}\ \downarrow + \tfrac{+\hbar}{2}\ \uparrow\right) = 0\hbar \tag{37}$$

(D) Put

$$j + k = r + s = -2 \tag{38}$$

Get

$$\Pi_3(\downarrow,\downarrow) = \tfrac{1}{2}(m_1 + m_2) = \tfrac{1}{2}\left(\tfrac{-\hbar}{2}\ \downarrow + \tfrac{-\hbar}{2}\ \downarrow\right) = \tfrac{-\hbar}{2}\ \downarrow \neq \tfrac{-\hbar}{2} + \tfrac{-\hbar}{2} = -1\hbar \downarrow \tag{39}$$

Where m_1 and m_2 are the eigenvalues of $\pi_{3;\,j,k}$ and $\pi_{3;\,r,s}$.

We see there are some amusing phenomenons in the above spin addition of the third components of two spin-1/2 particles:

Results (35) and (37) are the those we expected: " positive $\frac{+\hbar}{2}$ plus negative $\frac{-\hbar}{2}$ equal to zero" that agree with current angular theory (8). Unfortunately, for caculation (33): " positive $\frac{+\hbar}{2}$ plus positive $\frac{+\hbar}{2}$ equal to positive $\frac{+\hbar}{2}$ ", this result conflicts with (8) and physics common sense in lab ! actually, (33) shoud be $+1\hbar$, which now loses its half value. Similar puzzles for (39), which should be $-1\hbar$. This type of puzzles also exist in three body coupling b▸ (30) (31). The sum values of the spin third components of coupled-spin particles would always be less than that should be, except when the sum values is zero, such as cases of (35) (37)

If we still want, in STS math world, to obtain the same results, which correspond with current angular momentum theory such as (8) do, some new concepts should be required to be introduced, even if the ideas of physics background of those new phenomenons are truly impossible to understand.

★ The purpose of this paper is to use three-body coupling b▸ (30) (31) to research the spin angular momentum coupling of two spin-1/2 particles $\vec{\pi}_1, \vec{\pi}_2$. We suggest there may exist a seclusive hidden spin particle $\vec{\pi}_3 \equiv \pi_{u,v}$, that actually and stealthily is participating the formation process of spin coupling between $\vec{\pi}_1$ and $\vec{\pi}_2$. $\vec{\pi}_3$ is a spin particle, a spin-force-mediating particle. $\vec{\pi}_3$ can interact with $\vec{\pi}_1$ and $\vec{\pi}_2$ through spin angular momentum coupling, then to solve the puzzles.

Obviously, zero spin particle, with Casimir operator eigenvalue $\pi_3^2 = 0 = 0(0{+}1)\hbar^2$, may seems to be the perfect candidate, because spin-0 particle possesses the following advantages of its properties of spin-dual-role: "nothing" and "everything".

In current spin angular momentum couple theory frame, zero spin particle is a trivial spin particle, it is a "point spin", no spin effect with any other spin particeles. Zero spin particle is "nothing", all for naught, superfluous in spin addition.

But on the other hand, in Spin Topological Space STS frame, zero spin particle turns to be "constructive", that is, has spin ability to interact with other spin particles, at this time the zero spin is "everything", a physical reality as an invisible mediator, actually is participating the spin couple between the two spin-1/2 fermions .

Next, we enter zero spin territory where no one has gone before. Apply the third component $\pi_{3;\,u,v}$ or m_3 of zero spin particle $\pi_{u,v}$ to explore the puzzles of (33),(39) and (35),(37). Using $\pi_{3;\,u,v}$ to offset the defects, to throw away what we dislike and obtain what we appreciate. Further the coupling of two-body of two spin-1/2 particles in fact turns into those of three-body spin particles.

From (30), write down (40)

$$\Pi_3 = \tfrac{1}{3}(\pi_{3;\,j,k} + \pi_{3;\,r,s} + \pi_{3;\,u,v}) = \tfrac{1}{3}\{\, m_1 + m_2 + m_3 \,\} \tag{40}$$

Hence, obtain next four expressions which are the extension of spin angular momentum coupling from the current math frame (8) to math STS frame. The realm of latter is much beyond that of the former.

$$\text{(a)}\ \Pi_3 = \mathbf{m} = \tfrac{1}{3}\{ m_1 + m_2 + m_3 \} = \tfrac{1}{3}\{ \tfrac{+\hbar}{2} + \tfrac{+\hbar}{2} + 2\hbar \} = +1\hbar \quad (40.1)$$

$$\text{(b)}\ \Pi_3 = \mathbf{m} = \tfrac{1}{3}\{ m_1 + m_2 + m_3 \} = \tfrac{1}{3}\{ \tfrac{+\hbar}{2} + \tfrac{-\hbar}{2} + 0\hbar \} = 0\hbar \quad (40.2)$$

$$\text{(c)}\ \Pi_3 = \mathbf{m} = \tfrac{1}{3}\{ m_1 + m_2 + m_3 \} = \tfrac{1}{3}\{ \tfrac{-\hbar}{2} + \tfrac{+\hbar}{2} + 0\hbar \} = 0\hbar \quad (40.3)$$

$$\text{(d)}\ \Pi_3 = \mathbf{m} = \tfrac{1}{3}\{ m_1 + m_2 + m_3 \} = \tfrac{1}{3}\{ \tfrac{-\hbar}{2} + \tfrac{-\hbar}{2} - 2\hbar \} = -1\hbar \quad (40.4)$$

Here $m_3 = 0\hbar$ in (40.2) and (40.3) are the eigenvalues of the ground state of zero spin particle, and $m_3 = +2\hbar$ (40.1), $-2\hbar$ (40.4) are the those of the positive-second excited state, the negative-second exicted state in STS.

IV. Fundamental Formulas in Spin Complex Region of STS

The following formulas are the essential in spin complex region of STS, we will use them and zero spin particle to explore the puzzle addition of the third components of two spin-1/2 particles mentioned previously. With this aim in mind.

1) Single-body spin particle

$$\pi_{3\,;\,j,b\,,\,k,d} = \pi_0(0) + \tfrac{1}{2}(j + k + 1) + \tfrac{1}{2} i(b - d) \quad (41)$$

$$\pi^2_{j,b\,,\,k,d} = \tfrac{1}{4}\{ (j - k)^2 - (b + d)^2 - 1 \} + i\tfrac{1}{2}(j - k)(b + d) \quad (42)$$

2) Two-body spin couple

$$\Pi_{3\,;\,j,b\,,\,r,a\,;\,k,d\,,\,s,c} = \pi_0(0) + \tfrac{1}{2}\{\tfrac{1}{2}(j + k + r + s) + 1\} + \tfrac{1}{4} i(b - d + a - c) \quad (43)$$

$$\Pi^2_{j,b\,,\,r,a\,;\,k,d\,,\,s,c} = \tfrac{1}{16}\{(j - k + r - s)^2 - (b + d + a + c)^2 - 4\} + i\tfrac{1}{8}(j - k + r - s)(b + d + a + c) \quad (44)$$

3) Three-body spin couple

$$\Pi_{3\,;\,j,b\,,\,r,a\,;\,k,d\,,\,s,c\,;\,u,e\,,\,v,f} = \pi_0(0) + \tfrac{1}{2}\{\tfrac{1}{3}(j + k + r + s + u + v) + 1\} + \tfrac{1}{6} i(b - d + a - c + e - f) \quad (45)$$

$$\Pi^2_{j,b\,,\,r,a\,;\,k,d\,,\,s,c\,;\,u,e\,,\,v,f} = \tfrac{1}{36}\{(j - k + r - s + u - v)^2 - (b + d + a + c + e + f)^2 - 9\} + i\tfrac{1}{18}(j - k + r - s + u - v)(b + d + a + c + e + f) \quad (46)$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Using Spin Topological Coordinates (α, β) (γ, δ) of a spin particle in complex region in STS, two arrays are given

$$(\mathrm{S}, \mathrm{T}) = (\alpha - \beta, \ \gamma + \delta) \tag{47}$$

$$(\mathrm{A}, \mathrm{B}) = (\alpha + \beta, \ \gamma - \delta) \tag{48}$$

array (S, T) is related to Casimir operator of spin particle, and array (A, B) to the spin third component respectively. apply (47) and (48) to discuss the three-body spin couple among two spin-1/2 fermions $\vec{\pi}_1(j, k \ ; b, d)$, $\vec{\pi}_2(r, s \ ; a, c)$ and spin-0 boson $\vec{\pi}_3(u, v \ ; e, f)$ below

$$\text{spin-1/2 } \pi_1 \qquad (\mathrm{S}_1, \mathrm{T}_1) = (j - k, \ b + d), \tag{47.1}$$

$$\text{spin-1/2 } \pi_2 \qquad (\mathrm{S}_2, \mathrm{T}_2) = (r - s, \ a + c), \tag{47.2}$$

$$\text{spin-0 } \pi_3 \qquad (\mathrm{S}_3, \mathrm{T}_3) = (u - v, \ e + f), \tag{47.3}$$

$$\text{spin-1/2 } \pi_1 \qquad (\mathrm{A}_1, \mathrm{B}_1) = (j + k, \ b - d), \tag{48.1}$$

$$\text{spin-1/2 } \pi_2 \qquad (\mathrm{A}_2, \mathrm{B}_2) = (r + s, \ a - c), \tag{48.2}$$

$$\text{spin-0 } \pi_3 \qquad (\mathrm{A}_3, \mathrm{B}_3) = (u + v, \ e - f), \tag{48.3}$$

Spin Topological Coordinates, STC

$$\text{spin-1/2 } \pi_1 \qquad (j, \ k) = \left(\tfrac{+1}{2}(\mathrm{A}_1+\mathrm{S}_1), \ \tfrac{+1}{2}(\mathrm{A}_1-\mathrm{S}_1) \right) \tag{49.1}$$

$$(b, \ d) = \left(\tfrac{+1}{2}(\mathrm{B}_1+\mathrm{T}_1), \ \tfrac{-1}{2}(\mathrm{B}_1-\mathrm{T}_1) \right) \tag{50.1}$$

$$\text{spin-1/2 } \pi_2 \qquad (r, \ s) = \left(\tfrac{+1}{2}(\mathrm{A}_2+\mathrm{S}_2), \ \tfrac{+1}{2}(\mathrm{A}_2-\mathrm{S}_2) \right) \tag{49.2}$$

$$(a, \ c) = \left(\tfrac{+1}{2}(\mathrm{B}_2+\mathrm{T}_2), \ \tfrac{-1}{2}(\mathrm{B}_2-\mathrm{T}_2) \right) \tag{50.2}$$

$$\text{spin-0 } \pi_3 \qquad (u, \ v) = \left(\tfrac{+1}{2}(\mathrm{A}_3+\mathrm{S}_3), \ \tfrac{+1}{2}(\mathrm{A}_3-\mathrm{S}_3) \right) \tag{49.3}$$

$$(e, \ f) = \left(\tfrac{+1}{2}(\mathrm{B}_3+\mathrm{T}_3), \ \tfrac{-1}{2}(\mathrm{B}_3-\mathrm{T}_3) \right) \tag{50.3}$$

V. Puzzles Solving

Now, for the implement of {(a),(b),(c),(d)} of (40), the above formulas mentioned in section **4.** need to be simplified, so we confine ourself to condition (51), for which the calculations are straightforward.

For z components, we take:

$$\mathrm{B}_1 = \mathrm{B}_2 = \mathrm{B}_3 = 0 \tag{51}$$

then the imaginaries of $\pi_{3;j,b,k,d}$, $\pi_{3;r,a,s,c}$ and $\pi_{3;u,e,v,f}$ all vanish, (41) become

$$\pi_{3;j,b,k,d} = \pi_0(0) + \tfrac{1}{2}(A_1 + 1) \tag{52.1}$$

$$\pi_{3;r,a,s,c} = \pi_0(0) + \tfrac{1}{2}(A_2 + 1) \tag{52.2}$$

$$\pi_{3;u,e,v,f} = \pi_0(0) + \tfrac{1}{2}(A_3 + 1) \tag{52.3}$$

further (45) turns into

$$\Pi_3 = \pi_0(0) + \tfrac{1}{2}\{ \tfrac{1}{3}(A_1 + A_2 + A_3) + 1 \} \tag{53}$$

For deepper understanding of the role of zero spin particle in the spin angular momentum coupling of two identical spin-1/2 fermions, more detailed processes of calculations (a),(b),(c),(d) of the third components π_3 and Π_3 are demostrated below.

Pay attention to the following correspondence

$$\{ (A),(B),(C),(D) \mid \subset \text{section } 5 \} \Rightarrow \{ (a),(b),(c),(d) \mid \subset \text{section } 3\}$$

(A)

$$(A_1, B_1) = (0, 0) \tag{54.1}$$

$$(A_2, B_2) = (0, 0) \tag{54.2}$$

$$(A_3, B_3) = (+3, 0) \tag{54.3}$$

$$\pi_3(A_1, B_1) = \pi_0(0) + \tfrac{1}{2}(0 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) + \tfrac{1}{2} \tag{55.1}$$

$$\pi_3(A_1, B_1) = \pi_0(0) + \tfrac{1}{2}(0 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) + \tfrac{1}{2} \tag{55.2}$$

$$\pi_3(A_3, B_3) = \pi_0(0) + \tfrac{1}{2}(+3 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) + 2 \tag{55.3}$$

$$\Pi_3 = \pi_0(0) + \tfrac{1}{2}\{ \tfrac{1}{3}(0 + 0 + 3) + 1 \} + \tfrac{1}{6}i(0 + 0 + 0) = \pi_0(0) + 1 \tag{56}$$

Formular (55.3) is the second positive excited eigenvalue of zero spin particle, which is like an invisible force-mediating particle, to make (56) to be in accord with (40.1).

Similar to (A), we get:

(B)

$$(A_1, B_1) = (0, 0) \tag{57.1}$$

$$(A_2, B_2) = (-2, 0) \tag{55.2}$$

$$(A_3, B_3) = (-1, 0) \tag{55.3}$$

$$\pi_3(A_1, B_1) = \pi_0(0) + \tfrac{1}{2}(0 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) + \tfrac{1}{2} \tag{58.1}$$

$$\pi_3(A_1, B_1) = \pi_0(0) + \tfrac{1}{2}(-2 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) - \tfrac{1}{2} \tag{58.2}$$

$$\pi_3(A_3, B_3) = \pi_0(0) + \tfrac{1}{2}(-1 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) + 0 \tag{58.3}$$

$$\Pi_3 = \pi_0(0) + \tfrac{1}{2}\{ \tfrac{1}{3}(0 - 2 - 1) + 1 \} + \tfrac{1}{6}i(0 + 0 + 0) = \pi_0(0) + 0 \tag{59}$$

Formular (58.3) is the lowest eigenvalue, based state, of zero spin particle, to make (59) to be in accord with (40.2).

The same one as (B), we get:

(C)

$$(A_1, B_1) = (-2, 0) \tag{60.1}$$

$$(A_2, B_2) = (0, 0) \tag{60.2}$$

$$(A_3, B_3) = (-1, 0) \tag{60.3}$$

$$\pi_3(A_1, B_1) = \pi_0(0) + \tfrac{1}{2}(-2 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) - \tfrac{1}{2} \tag{61.1}$$

$$\pi_3(A_2, B_2) = \pi_0(0) + \tfrac{1}{2}(0 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) + \tfrac{1}{2} \tag{61.2}$$

$$\pi_3(A_3, B_3) = \pi_0(0) + \tfrac{1}{2}(-1 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) + 0 \tag{61.3}$$

$$\Pi_3 = \pi_0(0) + \tfrac{1}{2}\{ \tfrac{1}{3}(-2 + 0 - 1) + 1 \} + \tfrac{1}{6}i(0_1 + 0 + 0) = \pi_0(0) + 0 \tag{62}$$

Formular (61.3) is the lowest eigenvalue, based state, of zero spin particle, to make (62) to be in accord with (40.3).

Almost same as (A), we get:

(D)

$$(A_1, B_1) = (-2, 0) \tag{63.1}$$

$$(A_2, B_2) = (-2, 0) \tag{63.2}$$

$$(A_3, B_3) = (-5, 0) \tag{63.3}$$

$$\pi_3(A_1, B_1) = \pi_0(0) + \tfrac{1}{2}(-2 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) - \tfrac{1}{2} \tag{64.1}$$

$$\pi_3(A_1, B_1) = \pi_0(0) + \tfrac{1}{2}(-2 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) - \tfrac{1}{2} \tag{64.2}$$

$$\pi_3(A_3, B_3) = \pi_0(0) + \tfrac{1}{2}(-5 + 1) + \tfrac{1}{2}i(0) = \pi_0(0) - 2 \tag{64.3}$$

$$\Pi_3 = \pi_0(0) + \tfrac{1}{2}\{ \tfrac{1}{3}(-2 - 2 - 5) + 1 \} + \tfrac{1}{6}i(0_1 + 0 + 0) = \pi_0(0) - 1 \tag{65}$$

Formular (64.3) is the second negative excited eigenvalue of zero spin particle to make (65) to be in accord with (40.4).

VI. Casimir Operators of Spin Particles

For Casimir operators, we take:

$$T_1 + T_2 + T_3 = 0 \tag{66}$$

further (46) turns into

$$\Pi^2 = \frac{1}{36}\{(S_1 + S_2 + S_3)^2 - 9\} \tag{67}$$

Considering two cases of (67)

IF $$S_1 + S_2 + S_3 = +9 \tag{68}$$

get $$\Pi^2 = \frac{1}{36}\{(+9)^2 - 9\} = \frac{72}{36} = 2 = 1(1+1)\hbar^2 \tag{69}$$

IF $$S_1 + S_2 + S_3 = +3 \tag{70}$$

get $$\Pi^2 = \frac{1}{36}\{(+3)^2 - 9\} = \frac{0}{36} = 0 = 0(0+1)\hbar^2 \tag{71}$$

Now, we present the explicit datas of two arrays (S, T), (72)♦1 and (77)♦2, which satisfy the math requirements of the above two cases of (67)

♦1 Array (72) for case (69), which construct the irreducible **j = 1** representation of (8)

$$(S_1, T_1) = (+2\sqrt{6}, -2\sqrt{5}) \tag{72.1}$$

$$(S_2, T_2) = (-2\sqrt{6}, -2\sqrt{5}) \tag{72.2}$$

$$(S_3, T_3) = (+9, +4\sqrt{5}) \tag{72.3}$$

Then apply (72) to (42), we get:

$$\pi^2_{S_1,T_1}(\text{fermion1}) = \frac{+3}{4} - i2\sqrt{30} \iff \mathbf{j}_1 = \frac{1}{2} \tag{73.1}$$

$$\pi^2_{S_2,T_2}(\text{fermion2}) = \frac{+3}{4} + i2\sqrt{30} \iff \mathbf{j}_2 = \frac{1}{2} \tag{73.2}$$

$$\pi^2_{S_3,T_3}(\text{spin-zero3}) = 0 + i18\sqrt{5} \iff \mathbf{j}_3 = 0 \tag{73.3}$$

Because

$$S_1 + S_2 + S_3 = +2\sqrt{6} - 2\sqrt{6} + 9 = +9 \tag{74}$$

$$T_1 + T_2 + T_3 = -2\sqrt{5} - 2\sqrt{5} + 4\sqrt{5} = 0 \tag{75}$$

Obtain Casimir operator

$$\Pi^2 = \frac{1}{36}\{81 - 0 - 9\} = 1(1+1)\hbar^2 \iff \mathbf{j = 1} \tag{76}$$

♦2 Array (77) for case (71), which construct the irreducible $\mathbf{j}=\mathbf{0}$ representation of (8)

$$(S_1, T_1) = (+\sqrt{6}\,,\ -\sqrt{2}\,) \tag{77.1}$$

$$(S_2, T_2) = (-\sqrt{6}\,,\ -\sqrt{2}\,) \tag{77.2}$$

$$(S_3, T_3) = (+3,\ +2\sqrt{2}\,) \tag{77.3}$$

Then apply (77) to (42), we get:

$$\pi^2_{S_1,T_1}(\text{fermion1}) = \tfrac{+3}{4} - i\sqrt{3} \quad \Leftrightarrow \quad \mathbf{j}_1 = \tfrac{1}{2} \tag{78.1}$$

$$\pi^2_{S_2,T_2}(\text{fermion2}) = \tfrac{+3}{4} + i\sqrt{3} \quad \Leftrightarrow \quad \mathbf{j}_2 = \tfrac{1}{2} \tag{78.2}$$

$$\pi^2_{S_3,T_3}(\text{spin-zero3}) = 0 + i3\sqrt{2} \quad \Leftrightarrow \quad \mathbf{j}_3 = 0 \tag{78.3}$$

Because

$$S_1 + S_2 + S_3 = +\sqrt{6} - \sqrt{6} + 3 = +3 \tag{79}$$

$$T_1 + T_2 + T_3 = -\sqrt{2} - \sqrt{2} + 2\sqrt{2} = 0 \tag{80}$$

Obtain Casimir operator

$$\Pi^2 = \tfrac{1}{36}\{\ 9 - 0 - 9\ \} = 0(0+1)\hbar^2 \quad \Leftrightarrow \quad \mathbf{j}=\mathbf{0} \tag{81}$$

VII. Matrix Represention of $\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$ in STS

Instead of using the matrices of C-G Coefficients in conventional spin theory to depict the spin couple, the results of section 4, 5 and section 6 could be used to build some other new matrix represention pictures of spin angular momentum addtion in STS. Some matrix tables are given below. Among those Table1 is so-called "based state represention", which is just the incarnation of matrices represention of $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ in Spin Topological Space. Table2 and Table3 are " excited state representions " of C-G Coefficients, which are the extensions of Table1.

Block ★1, Block ★2 and Block ★3 in Table2, which are the based states of C-G Coefficients with quantum number $\mathbf{j}=1$ in Table1, carry the based state z-components quantum numbers **m** whose values equal to $+$**1**, **0**, $+$**1** respectively. The rest blocks are excited states of $\mathbf{j}=1$

Block ★4 in Table3, is the based state of C-G Coefficients with quantum number $\mathbf{j}=0$ in Table1, carries the based state z-components quantum number **m** whose value equals to **0**. The rest blocks are excited states of $\mathbf{j}=0$

Arrays (A_i, B_i) and (S_i, T_i) or Spin Topological Coordinates (j, k) (b, d), (r, s) (a, c), and (u, v) (e, f) are the characteristic quantum numbers of schematics of spin angular momentum couple.

Table 1: Based State Representation of the C-G Coefficients of $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

j	=	**1**	**1**	**0**	**1**
m	=	+**1**	**0**	**0**	−**1**
$\mathbf{j}_1$=1/2, (S_1, T_1)	=	$(+2\sqrt{6}, -2\sqrt{5})$	$(+2\sqrt{6}, -2\sqrt{5})$	$(+\sqrt{6}, -\sqrt{2})$	$(+2\sqrt{6}, -2\sqrt{5})$
$\mathbf{j}_2$=1/2, (S_2, T_2)	=	$(+2\sqrt{6}, -2\sqrt{5})$	$(+2\sqrt{6}, -2\sqrt{5})$	$(+\sqrt{6}, -\sqrt{2})$	$(+2\sqrt{6}, -2\sqrt{5})$
$\mathbf{j}_3$= 0, (S_3, T_3)	=	$(+9, +4\sqrt{5})$	$(+9, +4\sqrt{5})$	$(+3, +2\sqrt{2})$	$(+9, +4\sqrt{5})$
m_i, (A_i, B_i) i = 1, 2, 3 ‖		$(j, k), (r, s), (u, v)$	$(j, k), (r, s), (u, v)$	$(j, k), (r, s), (u, v)$	$(j, k), (r, s), (u, v)$
$\frac{+1}{2}$, (0, 0)		$(0+\sqrt{6}, 0-\sqrt{6})$			
$\frac{+1}{2}$, (0, 0)		$(0-\sqrt{6}, 0+\sqrt{6})$	0	0	0
+2, (+3, 0)		(+6, −3)			
$\frac{+1}{2}$, (0, 0)			$(0+\sqrt{6}, 0-\sqrt{6})$	$\frac{0+\sqrt{6}}{2}, \frac{0-\sqrt{6}}{2}$	
$\frac{-1}{2}$, (−2, 0)		0	$(-1-\sqrt{6}, -1+\sqrt{6})$	$\frac{-2-\sqrt{6}}{2}, \frac{-2+\sqrt{6}}{2}$	0
0, (−1, 0)			(+4, −5)	(+1, −2)	
$\frac{-1}{2}$, (−2, 0)			$(-1+\sqrt{6}, -1-\sqrt{6})$	$\frac{-2+\sqrt{6}}{2}, \frac{-2-\sqrt{6}}{2}$	
$\frac{+1}{2}$, (0, 0)		0	$(0-\sqrt{6}, 0+\sqrt{6})$	$\frac{0-\sqrt{6}}{2}, \frac{0+\sqrt{6}}{2}$	0
0, (−1, 0)			(+4, −5)	(+1, −2)	
$\frac{-1}{2}$, (−2, 0)					$(-1+\sqrt{6}, -\ 1-\sqrt{6})$
$\frac{-1}{2}$, (−2, 0)		0	0	0	$(-1-\sqrt{6}, -\ 1+\sqrt{6})$
−2, (−5, 0)					(+2, −7)
m = $\frac{m_1+m_2+m_3}{3}$	=	+**1**	**0**	**0**	−**1**
$(b\ ,\ d)$	=	$(-\sqrt{5}, -\sqrt{5})$	$(-\sqrt{5}, -\sqrt{5})$	$(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$	$(-\sqrt{5}, -\sqrt{5})$
$(a\ ,\ c)$	=	$(-\sqrt{5}, -\sqrt{5})$	$(-\sqrt{5}, -\sqrt{5})$	$(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$	$(-\sqrt{5}, -\sqrt{5})$
$(e\ ,\ f)$	=	$(+2\sqrt{5}, +2\sqrt{5})$	$(+2\sqrt{5}, +2\sqrt{5})$	$(+\sqrt{2}, +\sqrt{2})$	$(+2\sqrt{5}, +2\sqrt{5})$

Table 2: **j** = 1, Based States & Excited States of $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

j	**1**	**1**	**1**	**1**	**1**
	excited state	excited state	based state	excited state	excited state
m	+**3**	+**2**	+**1**	**0**	−**1**
$m_1, (A_1, B_1)$	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)
$m_2, (A_2, B_2)$	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)
$m_3, (A_3, B_3)$	+8, (+15, 0)	+5, (+9, 0)	+2, (+3, 0)	−1, (−3, 0)	−4, (−9, 0)
			★1		
$(j\ ,\ k)_1$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$
$(r\ ,\ s)_2$	$0-\sqrt{6}, 0+\sqrt{6}$	$0-\sqrt{6}, 0+\sqrt{6}$	$0-\sqrt{6}, 0+\sqrt{6}$	$0-\sqrt{6}, 0+\sqrt{6}$	$0-\sqrt{6}, 0+\sqrt{6}$
$(u\ ,\ v)_3$	+12, +3	+9, 0	+6, −3	+3, −6	0, −9
– – –	– – – – – –	– – – – – –	– – – – – –	– – – – – –	– – – – – –
m	+**2**	+**1**	**0**	−**1**	−**2**
$m_1, (A_1, B_1)$	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)	$\frac{+1}{2}$, (0, 0)
$m_2, (A_2, B_2)$	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)
$m_3, (A_3, B_3)$	+6, (+11, 0)	+3, (+5, 0)	0, (−1, 0)	−3, (−7, 0)	−6, (−13, 0)
			★2		
$(j\ ,\ k)_1$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$	$0+\sqrt{6}, 0-\sqrt{6}$
$(r\ ,\ s)_2$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$
$(u\ ,\ v)_3$	(+10, +1)	(+7, −2)	(+4, −5)	(+1, −8)	(−2, −11)
– – –	– – – – – –	– – – – – –	– – – – – –	– – – – – –	– – – – – –
m	+**1**	**0**	−**1**	−**2**	−**3**
$m_1, (A_1, B_1)$	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)
$m_2, (A_2, B_2)$	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)	$\frac{-1}{2}$, (−2, 0)
$m_3, (A_3, B_3)$	+4, (+7, 0)	+1, (+1, 0)	−2, (−5, 0)	−5, (−11, 0)	−8, (−17, 0)
			★3		
$(j\ ,\ k)_1$	$-1+\sqrt{6}, -1-\sqrt{6}$	$-1+\sqrt{6}, -1-\sqrt{6}$	$-1+\sqrt{6}, -1-\sqrt{6}$	$-1+\sqrt{6}, -1-\sqrt{6}$	$-1+\sqrt{6}, -1-\sqrt{6}$
$(r\ ,\ s)_2$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$	$-1-\sqrt{6}, -1+\sqrt{6}$
$(u\ ,\ v)_3$	(+8, −1)	(+5, −4)	(+2, −7)	(−1, −10)	(−4, −13)
– – –	– – –	– – –	– – –	– – –	– – –
– – –	– – –	– – –	– – –	– – –	– – –
$(S_1, T_1)=$	$+2\sqrt{6}, -2\sqrt{5}$			$(b,\ d)\ =$	$-\sqrt{5}, -\sqrt{5}$
$(S_2, T_2)=$	$-2\sqrt{6}, -2\sqrt{5}$			$(a,\ c)\ =$	$-\sqrt{5}, -\sqrt{5}$
$(S_3, T_3)=$	$+9,\ +4\sqrt{5}$			$(e,\ f)\ =$	$+2\sqrt{5}, +2\sqrt{5}$

Table 3: $\mathbf{j} = 0$, Based States & Excited States of $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$

$\mathbf{j}$	**0**	**0**	**0**	**0**	**0**
	excited state	excited state	based state	excited state	excited state
$\mathbf{m}$	$+\mathbf{2}$	$+\mathbf{1}$	$\mathbf{0}$	$-\mathbf{1}$	$-\mathbf{2}$
$\mathbf{m}_1,(A_1,B_1)$	$\frac{+1}{2}$, $(0,\ 0)$	$\frac{+1}{2}$, $(0,\ 0)$	$\frac{+1}{2}$, $(0,\ 0)$	$\frac{+1}{2}$, $(0,\ 0)$	$\frac{+1}{2}$, $(0,\ 0)$
$\mathbf{m}_2,(A_2,B_2)$	$\frac{-1}{2}$, $(-2, 0)$	$\frac{-1}{2}$, $(-2, 0)$	$\frac{-1}{2}$, $(-2, 0)$	$\frac{-1}{2}$, $(-2, 0)$	$\frac{-1}{2}$, $(-2, 0)$
$\mathbf{m}_3,(A_3,B_3)$	$+6$, $(+11, 0)$	$+3$, $(+5, 0)$	0, $(-1, 0)$	-3, $(-7, 0)$	-6, $(-13, 0)$
			★4		
$(j\ ,\ k)_1$	$\frac{0+\sqrt{6}}{2}, \frac{0-\sqrt{6}}{2}$	$\frac{0+\sqrt{6}}{2}, \frac{0-\sqrt{6}}{2}$	$\frac{0+\sqrt{6}}{2}, \frac{0-\sqrt{6}}{2}$	$\frac{0+\sqrt{6}}{2}, \frac{0-\sqrt{6}}{2}$	$\frac{0+\sqrt{6}}{2}, \frac{0-\sqrt{6}}{2}$
$(r\ ,\ s)_2$	$\frac{-2-\sqrt{6}}{2}, \frac{-2+\sqrt{6}}{2}$	$\frac{-2-\sqrt{6}}{2}, \frac{-2+\sqrt{6}}{2}$	$\frac{-2-\sqrt{6}}{2}, \frac{-2+\sqrt{6}}{2}$	$\frac{-2-\sqrt{6}}{2}, \frac{-2+\sqrt{6}}{2}$	$\frac{-2-\sqrt{6}}{2}, \frac{-2+\sqrt{6}}{2}$
$(u\ ,\ v)_3$	$(+7,\ +4)$	$(+4,\ +1)$	$(+1,\ -2)$	$(-2,\ -5)$	$(-5,\ -8)$

$(S_1\ ,T_1) = (+\sqrt{6}\,, -\sqrt{2}\,)$	$(b\ ,\ d) = (\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$
$(S_2\ ,T_2) = (-\sqrt{6}\,, -\sqrt{2}\,)$	$(a\ ,\ c) = (\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2})$
$(S_3\ ,T_3) = (+3,\ \ +2\sqrt{2}\,)$	$(e\ ,\ f) = (+\sqrt{2}\,, +\sqrt{2}\,)$

VIII. Conclusions

In STS, Spin-zero particle possesses non-trivial angular momentum property, with which it could be thought as a force-mediating boson that holding the two spin-1/2 fermions to be coupled together each other, then to form a spin system as a whole. Subsequently, the matrix representions of $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ in STS are given.

The works that are the continuation of this paper about the matrix representions of $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$ and $1 \otimes 1 = 2 \oplus 1 \oplus 0$ in STS are accomplished.

The existence of so-called " based states " may be believed to be math reasonable, after all from Table1, we can obtain what the same informations of the C-G Coefficients just what from the conventional spin theory do. Are there any so-called "excited states" in nature, which appear in Table2 and Table3 ?, we are well not aware of as yet.

As an example of possible " excited states ": If the two spin-1/2 fermions all keep to be stay in based states, that is $m_{1,}(A_1, B_1)$ and $m_{2,}(A_2, B_2)$ stay in their own based states, when spin-zero particle is excited to jump out of its based state $m_{3,}(A_3, B_3) = 0, (-1, 0)$, then the based state quantum number $\mathbf{m} = 0$, of the spin combined system (for both **j =1** and **j = 0**), would turn to be the excited states $\mathbf{m} = \pm 1, \pm 2, \pm 3, \ldots$

References

1. D.M.BRINK & G.R.SATCHLER (1962) ANGULAR MOMENTUM, CLARENDON PRESS. OXFORD.
2. PAUL ROMAN (1965) Advanced Quantum Theory, AN OUTLINE OF THE FUNDAMENTAL IDEAS, ADDISON-WESLEY PUBLISHING COMPANY, INC.
3. Brian G. Wybourne (1974) Classical Group for Physicists, John Wiley & Sons. Inc.
4. A.W. JOSHI (1973) (1977) WILEY EASTERN LIMITED NEW DEIHi.
5. Ramamurti Shanlar (1980) Principles of Quantum Mechanics, Plenum Press, New York.
6. ShaoXu Ren, (2016) Journal of Modern Physics, 7, 2257-2265. https://doi.org/10.4236/jmp.2016.716194
7. ShaoXu Ren, 642. WE-Heraeus-Seminar, *Non-Hermitian Hamiltonians in Physics: Theory and Experiment.* May 15-19, 2017, Physikzentrum Bad Honnef, Germany.
8. ShaoXu Ren, (2021) Journal of Modern Physics, 12, 380-389. https://doi.org/10.4236/jmp.2021.123027
9. ShaoXu Ren, (2021) Global Journal of Science Frontier Research (A) Physics & Space Science, Volume 21, Issue 3, Version1.0, 17 https://doi.org/10.34257/GJSFRAVOL21IS3PG17

This page is intentionally left blank

Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 21 Issue 4 Version 1.0 Year 2021
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Non-Arbitrage Models of Financial Markets

By N. S. Gonchar

Abstract- In the first part of the paper, we construct the models of the complete non-arbitrage financial markets for a wide class of evolutions of risky assets. This construction is based on the observation that for a certain class of risky asset evolutions the martingale measure is invariant with respect to these evolutions. For such a financial market model the only martingale measure being equivalent to an initial measure is built. On such a financial market, formulas for the fair price of contingent liabilities are presented. A multi-parameter model of the financial market is proposed, the martingale measure of which does not depend on the parameters of the model of the evolution of risky assets and is the only one.

Keywords: random process; spot set of measures; optional doob decomposition; super-martingale; martingale; assessment of derivatives; non-arbitrage markets.

GJSFR-A Classification: *FOR Code: 240201*

Non-Arbitrage Models of Financial Markets

N. S. Gonchar

Abstract- In the first part of the paper, we construct the models of the complete non-arbitrage financial markets for a wide class of evolutions of risky assets. This construction is based on the observation that for a certain class of risky asset evolutions the martingale measure is invariant with respect to these evolutions. For such a financial market model the only martingale measure being equivalent to an initial measure is built. On such a financial market, formulas for the fair price of contingent liabilities are presented. A multi-parameter model of the financial market is proposed, the martingale measure of which does not depend on the parameters of the model of the evolution of risky assets and is the only one.

In the second part of the paper, a model of an incomplete non-arbitrage financial market is proposed. As in the first part of the paper, we use the fact that the family of spot martingale measures is invariant with respect to a certain class of evolutions of risky assets. The set of all martingale measures being equivalent to an initial measure is completely described. Each martingale measure is a linear convex combination of the finite number of spot measures whose structure is completely described. For a wide class of models for the evolution of risky assets, a formula is found for the fair price of a super-hedge, as well as an interval of non-arbitrage prices for any contingent liability. A multi-parameter model of the incomplete financial market is proposed, the martingale measures of which do not depend on the parameters of the model of the evolution of risky assets. For the parameters of the models of the evolution of risky assets, statistical estimates are found for both complete and incomplete non-arbitrage markets.

Keywords: random process; spot set of measures; optional doob decomposition; super-martingale; martingale; assessment of derivatives; non-arbitrage markets.

I. Introduction

In this paper, models of non-arbitrage markets are constructed on the basis of the invariance of a set of spot measures with respect to a certain class of evolution of risky assets. In the first part of the paper, models of complete non arbitrage markets are built on the basis of an analysis of conditions under which there is only one martingale measure. In the second part of the work, models of incomplete non-arbitrage realistic market models are built based on the same principles as in the first part of the work. For the introduced parametric models of the markets, estimates of parameters were obtained based on the observed real values of the evolution of risky assets. This opens up wide opportunities for hedging risks.

Historically the first model evolution of risky assets was suggested in Bachelier's work [4]. Then, in the famous works of Black F. and Scholes M. [5] and Merton R. S. [6] the formula was found for the fair price of the standard call option of European type. The absence of arbitrage in the financial market has a very transparent economic sense, since it can be considered reasonably arranged. The concept of non arbitrage in financial market is associated with the fact that one cannot earn money without risking, that is, to make money you need to invest in risky or risk-free assets. The exact mathematical substantiation of the concept of non arbitrage was first made in the papers [7], [8] for the finite probability space and in the general case in the paper [9]. In the continuous time evolution of risky asset, the proof of absence of arbitrage possibility see in [11]. The value of the established Theorems is that they make it possible to value assets. They got a special name "The First and The Second Fundamental Asset Pricing Theorems." Generalizations of these Theorems are contained in papers [12], [13], [14].

Author: Bogolyubov Institute for Theoretical Physics of NAS of Ukraine. e-mail: mhonchar@i.ua

[1] This work was partially supported by the Program of Fundamental Research of the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine (project No 0117U000240).

This work is a continuation of the works [1], [19], [20], [21]. In paper [1], a new method for constructing and describing a family of martingale measures was proposed. This made it possible to build models of non-arbitrage markets. The construction of a realistic model of non-arbitrage markets has been an urgent problem since the moment when the concept of the absence of arbitrage appeared in the scientific literature as the most equitable model of the functioning of financial markets. What could be more attractive than a realistic model that can be built on the basis of observations of the evolution of the financial market. The main obstacle to this was the limited possibilities of constructing a risk-neutral martingale measure for a given evolution of risky assets in the case of a complete market and their complete description in the case of incomplete markets. In the case of discrete evolution of risky assets, the theoretical possibility of the existence of such non-arbitrage markets was established in [7], [8], [9], [10], [11], [12], [13], [14]. But, there were no practically regular methods for constructing such non-arbitrage market models, although such attempts were made for some kind of models of the evolution of risky assets [13], [14]. With the appearance of the work [1], which proposes a regular method for describing all martingale measures for a wide class of evolutions of risky assets [22], [23], [24] that capture the phenomenon of price memory and clustering, it became possible to construct realistic models of non-arbitrage markets. Note that such efforts have been made in this direction, and more about this can be found in the monograph [13], [14]. Valuable is the fact that there is a wide range of models for the evolution of risky assets for which it is possible to build parametric models of non-arbitrage markets whose parameters can be estimated based on statistical data. Problems of risk estimates was considered in papers [15], [16], [17], [18].

This work is the first step in constructing parametric models of non-arbitrage markets whose parameters can be estimated based on empirical data. In this paper, models of the evolution of risk assets on a discrete probabilistic space are considered. Such models can be used to approximate realistic models of the evolution of risky assets. The value of this model is that in this case the structure of the set of martingale measures is relatively simple.

In the case of incomplete non-arbitrage markets, the set of equivalent martingale measures has the cardinality of the continuum, but since they are a linear convex combination of a set of spot measures whose number is finite, this allows calculating the required characteristics using a finite number of operations. This allows a computer to be used to simulate non-arbitrage markets.

In the third section of the work, the necessary and sufficient conditions for the uniqueness of a martingale measure are established in terms of the law of evolution of risky assets, and the only martingale measure is found. Using the results of Section 3 in Section 4, a multi-parameter model of the complete financial market is built and parameter estimates are obtained through empirical data of the financial market. This will allow the model to be adapted to realistic financial markets to estimate the fair price of European-type derivatives with different payment functions.

Section 5 establishes the general structure of the family of equivalent martingale measures for a wide class of risky asset evolutions. The structure of spot measures is completely described, the formulas for the fair price of the super hedge and the range of non-arbitrage prices are established. Based on the results of Section 5, Section 6 builds a multi-parameter model of the incomplete non-arbitrage market. The estimates of the parameters of the model are obtained through empirical observations of the financial market. This will allow the computer to be used to model the financial market.

II. Evolutions of Risky Assets

In this section, a class of evolutions of risky assets is described which is used in this paper. This class is fairly wide and includes well known in the literature evolutions of risky assets. Let $\{\Omega_N, \mathcal{F}_N, P_N\}$ be a direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $i = \overline{1, N}$, $\Omega_N = \prod\limits_{i=1}^{N} \Omega_i^0$, $P_N = \prod\limits_{i=1}^{N} P_i^0$, $\mathcal{F}_N = \prod\limits_{i=1}^{N} \mathcal{F}_i^0$, where the σ-

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

algebra $\mathcal{F}_N$ is a minimal σ-algebra, generated by the sets $\prod\limits_{i=1}^{N} G_i,\ G_i \in \mathcal{F}_i^0$. On the measurable space $\{\Omega_N, \mathcal{F}_N\}$, under the filtration $\mathcal{F}_n,\ n = \overline{1,N}$, we understand the minimal σ-algebra generated by the sets $\prod\limits_{i=1}^{N} G_i,\ G_i \in \mathcal{F}_i^0$, where $G_i = \Omega_i^0$ for $i > n$. We also introduce the probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1,N}$, where $\Omega_n = \prod\limits_{i=1}^{n} \Omega_i^0$, $\mathcal{F}_n = \prod\limits_{i=1}^{n} \mathcal{F}_i^0$, $P_n = \prod\limits_{i=1}^{n} P_i^0$. There is a one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n = \prod\limits_{i=1}^{n} \mathcal{F}_i^0$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$. Therefore, we don't introduce new denotation for the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}$, since it always will be clear the difference between the above introduced σ-algebra $\mathcal{F}_n$ of filtration on the measurable space $\{\Omega_N, \mathcal{F}_N\}$ and the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$.

We assume that the evolution of risky asset $\{S_n\}_{n=0}^N$, given on the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, is consistent with the filtration $\mathcal{F}_n$, that is, S_n is a $\mathcal{F}_n$-measurable. Due to the above one-to-one correspondence between the sets of the σ-algebra $\mathcal{F}_n$, belonging to the introduced filtration, and the sets of the σ-algebra $\mathcal{F}_n$ of the measurable space $\{\Omega_n, \mathcal{F}_n\}, n = \overline{1,N}$, we give the evolution of risky assets in the form $\{S_n(\omega_1, \ldots, \omega_n)\}_{n=0}^N$, where $S_n(\omega_1, \ldots, \omega_n)$ is an $\mathcal{F}_n$-measurable random variable, given on the measurable space $\{\Omega_n, \mathcal{F}_n\}$. It is evident that such evolution is consistent with the filtration $\mathcal{F}_n$ on the measurable space $\{\Omega_N, \mathcal{F}_N, P_N\}$.

Further, we assume that

$$P_n((\omega_1, \ldots, \omega_n) \in \Omega_n,\ \Delta S_n > 0) > 0,$$

$$P_n((\omega_1, \ldots, \omega_n) \in \Omega_n,\ \Delta S_n < 0) > 0, \quad n = \overline{1,N}, \tag{1}$$

where $\Delta S_n = S_n(\omega_1, \ldots, \omega_n) - S_{n-1}(\omega_1, \ldots, \omega_{n-1}),\ n = \overline{1,N}$.

Let us introduce the denotations

$$\Omega_n^- = \{(\omega_1, \ldots, \omega_n) \in \Omega_n,\ \Delta S_n \le 0\}, \quad \Omega_n^+ = \{(\omega_1, \ldots, \omega_n) \in \Omega_n,\ \Delta S_n > 0\}, \tag{2}$$

$$\Delta S_n^- = -\Delta S_n \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_n), \quad \Delta S_n^+ = \Delta S_n \chi_{\Omega_n^+}(\omega_1, \ldots, \omega_n), \tag{3}$$

$$V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2) = \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) + \Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2),$$

$$(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \in \Omega_n^-, \quad (\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \in \Omega_n^+. \tag{4}$$

In this paper we assume that

$$\Omega_n^+ = \Omega_{n-1} \times \Omega_n^{0+},\ \Omega_n^- = \Omega_{n-1} \times \Omega_n^{0-},\ \Omega_n^{0+},\ \Omega_n^{0-} \in \mathcal{F}_n^0,\ \Omega_n^{0-} \cup \Omega_n^{0+} = \Omega_n^0. \tag{5}$$

Further, in this paper, we assume that $P_n^0(\Omega_n^{0-}) > 0,\ P_n^0(\Omega_n^{0+}) > 0,\ n = \overline{1,N}$. We also assume some technical suppositions: there exist subsets $B_{n,i}^{0-} \in \mathcal{F}_n^0,\ i = \overline{1,I_n},\ I_n > 1$, and $B_{n,s}^{0+} \in \mathcal{F}_n^0,\ s = \overline{1,S_n},\ S_n > 1$, satisfying the conditions

$$B_{n,i}^{0-} \cap B_{n,j}^{0-} = \emptyset,\ i \ne j, \quad B_{n,s}^{0+} \cap B_{n,l}^{0+} = \emptyset,\ s \ne l, \quad n = \overline{1,N},$$

$$P_n^0(B_{n,i}^{0-}) > 0,\ i = \overline{1, I_n},\ P_n^0(B_{n,s}^{0+}) > 0,\ s = \overline{1, S_n}, \quad n = \overline{1, N},$$

$$\Omega_n^{0-} = \bigcup_{i=1}^{I_n} B_{n,i}^{0-}, \quad \Omega_n^{0+} = \bigcup_{s=1}^{S_n} B_{n,s}^{0+}, \quad n = \overline{1, N}. \tag{6}$$

Below, we give the examples of evolutions $\{S_n(\omega_1, \ldots, \omega_n)\}_{n=1}^N$ for which the representations (5) are true.

Suppose that the random values $a_i(\omega_1, \ldots, \omega_i)$, $\eta_i(\omega_i)$ satisfy the inequalities

$$a_i(\omega_1, \ldots, \omega_i) > 0, \qquad \sup_{\{\omega_1,\ldots,\omega_i\}\in\Omega_i} a_i(\omega_1, \ldots, \omega_i) < \frac{1}{\sup\limits_{\omega_i\in\Omega_i^0, \eta_i(\omega_i)<0} \eta_i^-(\omega_i)},$$

$$P_i^0(\eta_i(\omega_i) < 0) > 0, \quad P_i^0(\eta_i(\omega_i) > 0) > 0, \quad i = \overline{1, N}. \tag{7}$$

If $S_n(\omega_1, \ldots, \omega_n)$ is given by the formula

$$S_n(\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^n (1 + a_i(\omega_1, \ldots, \omega_i)\eta_i(\omega_i)), \quad n = \overline{1, N}, \tag{8}$$

then

$$\{\omega_i \in \Omega_i^0,\ \eta_i(\omega_i) \le 0\} = \Omega_i^{0-}, \quad \{\omega_i \in \Omega_i^0,\ \eta_i(\omega_i) > 0\} = \Omega_i^{0+},$$

$$\Omega_i^- = \Omega_{i-1} \times \Omega_i^{0-}, \quad \Omega_i^+ = \Omega_{i-1} \times \Omega_i^{0+}, \quad i = \overline{1, N}. \tag{9}$$

Let us note that not only the evolutions given by the formula (8) provide the representation (5). In this work, we use the evolutions of the kind (8). Below we give examples of the evolution of risky assets that have the form (8). For example, if

$$S_n(\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^n e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)}, \quad n = \overline{1, N}, \tag{10}$$

where the random values $\sigma_i(\omega_1, \ldots, \omega_{i-1}) \ge \sigma_i^0 > 0,\ i = \overline{1, N}$, and $P_i^0(\varepsilon_i(\omega_i) < 0) > 0,\ P_i^0(\varepsilon_i(\omega_i) > 0) > 0$, then such an evolution has the form (8) with

$$a_i(\omega_1, \ldots, \omega_i) = \frac{e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)} - 1}{e^{\sigma_i^0\varepsilon_i(\omega_i)} - 1}, \quad \eta_i(\omega_i) = e^{\sigma_i^0\varepsilon_i(\omega_i)} - 1, \quad i = \overline{1, N}.$$

satisfying needed conditions.

III. Uniqueness of the Martingale Measure

In this section, the necessary and sufficient conditions in terms of the evolution of risky assets are obtained relative to the uniqueness of martingale measure. Under the fairly wide assumptions about the evolution of risky assets, an expression for a single martingale measure is found . Based on the explicit construction of the martingale measure and its invariance with respect to a certain type of evolutions, it is possible to construct the models of non arbitrage markets, both complete and incomplete.

In this and section 4, we put that $\Omega_i^0=\{\omega_i^1,\omega_i^2\}$. Denote by $\mathcal{F}_i^0$ the σ-algebra of all subsets of the set Ω_i^0. Let P_i^0 be a probability measure on $\mathcal{F}_i^0$. We assume that $P_i^0(\omega_i^s)>0,\ i=\overline{1,N},\ s=\overline{1,2}$. As before, we put that the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$ is a direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\},\ i=\overline{1,N}$, and we put $N<\infty$. We also consider the probability spaces $\{\Omega_n,\mathcal{F}_n,P_n\}, n=\overline{1,N}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\},\ i=\overline{1,n}$. We assume that the evolution of a risky asset is given by the formula

$$S_n(\omega_1,\ldots,\omega_n)=$$

$$S_0\prod_{i=1}^{n}(1+a_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i)),\ \{\omega_1,\ldots,\omega_{n-1},\omega_n\}\in\Omega_n,\ n=\overline{1,N}, \tag{11}$$

where the random values $a_n(\omega_1,\ldots,\omega_{n-1},\omega_n),\ \eta_n(\omega_n),\quad n=\overline{1,N}$, given on the probability space $\{\Omega_n,\mathcal{F}_n,P_n\}$, satisfy the conditions

$$a_n(\omega_1,\ldots,\omega_{n-1},\omega_n)>0,\quad \max_{\{\omega_1,\ \ldots,\omega_{n-1}\}\in\Omega_{n-1}}a_n(\omega_1,\quad\ldots,\omega_{n-1},\omega_n^1)<\frac{1}{\eta_n^-(\omega_n^1)},$$

$$\eta_n(\omega_n^2)>0,\quad \eta_n(\omega_n^1)<0. \tag{12}$$

So, for $\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n),\ n=\overline{1,N}$, the representation

$$\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$S_{n-1}(\omega_1,\ldots,\omega_{n-1})a_n(\omega_1,\ldots,\omega_{n-1},\omega_n)\eta_n(\omega_n)=$$

$$d_n(\omega_1,\ldots,\omega_{n-1},\omega_n)\eta_n(\omega_n),\quad d_n(\omega_1,\ldots,\omega_{n-1},\omega_n)>0,\quad n=\overline{1,N},\quad S_0>0, \tag{13}$$

is true, From these conditions, we obtain $\Omega_n^-=\Omega_{n-1}\times\Omega_n^{0-},\ \Omega_n^+=\Omega_{n-1}\times\Omega_n^{0+}$, where $\Omega_n^{0-}=\{\omega_n\in\Omega_n^0,\eta_n(\omega_n)\le 0\},\ \Omega_n^{0+}=\{\omega_n\in\Omega_n^0,\eta_n(\omega_n)>0\}$.

From the suppositions above, it follows that $P_n^0(\Omega_n^{0-})>0,\ P_n^0(\Omega_n^{0+})>0$. The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-}=\Omega_n^{0-}\cap\mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+}=\Omega_n^{0+}\cap\mathcal{F}_n^0$.

Let us introduce the following denotation. For every point $\{\omega_1,\ldots,\omega_{n-1},\omega_n\}\in\Omega_n$, we introduce the set $A(\omega_1,\ldots,\omega_{n-1},\omega_n)\in\Omega_N$, where

$$A(\omega_1,\ldots,\omega_{n-1},\omega_n)=\bigcup_{i_{n+1}=1,\ldots,i_N=1}^{2}\{\omega_1,\ldots,\omega_{n-1},\omega_n,\omega_{n+1}^{i_{n+1}},\ldots,\omega_N^{i_N}\}.$$

For fixed indexes $i_1,\ldots,i_n$ we also use the denotation

$$A(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}},\omega_n^{i_n})=A^{i_1,\ldots,i_n}.$$

It is evident that every set $A^{i_1,\ldots,i_n}$ has the form

$$A^{i_1,\ldots,i_n}=\bigcup_{i_{n+1}=1,\ldots,i_N=1}^{2}\{\omega_1^{i_1},\ldots,\omega_n^{i_n},\omega_{n+1}^{i_{n+1}},\ldots,\omega_N^{i_N}\},$$

where indexes i_s takes only one value from the set $\{1,2\}$. Then, $A^{i_1,\dots,i_{n-1}} = A^{i_1,\dots,i_{n-1},1} \cup A^{i_1,\dots,i_{n-1},2} \in \mathcal{F}_{n-1}$, where

$$A^{i_1,\dots,i_{n-1},1} = \bigcup_{i_{n+1}=1,\dots,i_N=1}^{2} \{\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^1,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\} \in \mathcal{F}_n,$$

$$A^{i_1,\dots,i_{n-1},2} = \bigcup_{i_{n+1}=1,\dots,i_N=1}^{2} \{\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}},\omega_n^2,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\}\} \in \mathcal{F}_n.$$

If P_N is a measure on $\mathcal{F}_N$, then

$$P_N(A(\omega_1,\dots,\omega_{n-1},\omega_n)) = \sum_{i_{n+1}=1,\dots,i_N=1}^{2} P_N(\{\omega_1,\dots,\omega_{n-1},\omega_n,\omega_{n+1}^{i_{n+1}},\dots,\omega_N^{i_N}\}).$$

We give an evident construction of martingale measure for risky assets evolution, given by the formula (11). Below, we assume that measures P_n^0 is concentrated at points $\omega_n^1,\omega_n^2 \in \Omega_n^0$, where $\omega_n^1 \in \Omega_n^{0-}, \omega_n^2 \in \Omega_n^{0+}$ and we have the representation $\Omega_n^- = \Omega_{n-1}\times\Omega^{0-}$ and $\Omega_n^+ = \Omega_{n-1}\times\Omega^{0+}$. So, we have $\eta_n(\omega_n^1)<0, \eta_n(\omega_n^2)>0$.

Let us put $P_n^0(\omega_n^1)=p_n$, $P_n^0(\omega_n^2)=1-p_n$, where $0<p_n<1$. Then, to satisfy the conditions (14 - 16), (see [1]) we need to put

$$\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\}) = \frac{1}{p_n(1-p_n)}, \quad n=\overline{1,N}, \tag{14}$$

and to require that

$$\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)<\infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^1)\in\Omega_n^-,$$

$$\Delta S_n^+(\omega_1,\dots,\omega_{n-1},\omega_n^2)<\infty, \quad (\omega_1,\dots,\omega_{n-1},\omega_n^2)\in\Omega_n^+. \tag{15}$$

The next Lemma 1 is a consequence of results in [1].

Lemma 1. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, for the evolution of risky asset given by the formula (11) only one spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A)$ exists, where $\{\omega_i^1,\omega_i^2\}\in\Omega_i^0$, $i=\overline{1,N}$. For it the representation

$$\mu_0(A) = \mu_{\{\omega_1^1,\omega_1^2\},\dots,\{\omega_N^1,\omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad A\in\mathcal{F}_N, \tag{16}$$

is true. This measure is martingale measure for the considered evolution of risky asset, where

$$\psi_n(\omega_1,\dots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\dots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\dots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\dots,\omega_n), \tag{17}$$

$$\psi_n^1(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\frac{\Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \tag{18}$$

$$\psi_n^2(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}. \tag{19}$$

Next Theorem 1 appeared first in [2] (Theorem 1.4.1), where it was proved under the less general conditions.

Theorem 1. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *suppose that the evolution of risky asset* $\{S_n(\omega_1, \dots, \omega_n)\}_{n=1}^N$ *is given by the formula (11). The necessary and sufficient conditions of the uniqueness of martingale measure* $\mu_0(A)$, $A \in \mathcal{F}_N$, *are the inequalities*

$$S_n(\omega_1^{i_1}, \dots, \omega_{n-1}^{i_{n-1}}, \omega_n^1) \neq S_n(\omega_1^{i_1}, \dots, \omega_{n-1}^{i_{n-1}}, \omega_n^2), \quad n = \overline{1, N}, \tag{20}$$

for every set of indexes $i_1, \dots, i_{n-1}$. *For any martingale* $\{m_n(\omega_1, \dots, \omega_{n-1}, \omega_n)\}_{n=0}^N$ *relative to the unique measure* $\mu_0(A)$ *the representation*

$$m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\sum_{k=1}^{n} C_k(\omega_1, \dots, \omega_{k-1})[S_k(\omega_1, \dots, \omega_{k-1}, \omega_i) - S_{k-1}(\omega_1, \dots, \omega_{k-1})] +$$

$$m_0, \quad n = \overline{1, N}, \tag{21}$$

is true, where

$$C_k(\omega_1, \dots, \omega_{k-1}) = \sum_{i_1=1, \dots, i_{k-1}=1}^{2} d_{i_1, \dots, i_{k-1}} \chi_{A^{i_1, \dots, i_{k-1}}}(\omega_1, \dots, \omega_{k-1}). \tag{22}$$

$$d_{i_1, \dots, i_{k-1}} =$$

$$\frac{m_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^1) - m_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^2)}{S_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^1) - S_k(\omega_1^{i_1}, \dots, \omega_{k-1}^{i_{k-1}}, \omega_k^2)}, \quad k = \overline{1, N}. \tag{23}$$

Proof. The necessity. Suppose that the evolution $\{S_n(\omega_1, \dots, \omega_n)\}_{n=1}^N$ of the risky asset on the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$ is such that the martingale measure $\mu_0(A)$, $A \in \mathcal{F}_N$, being equivalent to the measure P_N, is unique. Then, for every attainable contingent liability $m_N(\omega_1, \dots, \omega_N)$ the representation (21) is true [11] for some $\mathcal{F}_{k-1}$-measurable finite valued random value $C_k(\omega_1, \dots, \omega_{k-1})$, $k = \overline{1, N}$, where $m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) = E^{\mu_0}\{m_N(\omega_1, \dots, \omega_N) | \mathcal{F}_n\}$. For $m_n(\omega_1, \dots, \omega_{n-1}, \omega_n)$ and $S_n(\omega_1, \dots, \omega_{n-1}, \omega_n)$ the representations

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$m_n(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\sum_{i_1=1,\ldots,i_n=1}^{2}\frac{\chi_{A^{i_1,\ldots,i_{n-1},i_n}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},i_n})}\int\limits_{A^{i_1,\ldots,i_{n-1},i_n}} m_N(\omega_1,\ldots,\omega_N)d\mu_0,\ n=\overline{1,N}, \quad (24)$$

$$S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\sum_{i_1=1,\ldots,i_n=1}^{2}\frac{\chi_{A^{i_1,\ldots,i_{n-1},i_n}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},i_n})}\int\limits_{A^{i_1,\ldots,i_{n-1},i_n}} S_N(\omega_1,\ldots,\omega_N)d\mu_0,\ n=\overline{1,N}, \quad (25)$$

are true. From the representation (21) and the equality (22) for $\{\omega_1,\ldots,\omega_{n-1}\}\in A^{i_1,\ldots,i_{n-1}}$ we obtain the equality

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},1})}\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0+$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})}\int\limits_{A^{i_1,\ldots,i_{n-1},2}} m(\omega_1,\ldots,\omega_N)d\mu_0-$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})}{\mu_0(A^{i_1,\ldots,i_{n-1}})}\int\limits_{A^{i_1,\ldots,i_{n-1}}} m(\omega_1,\ldots,\omega_N)d\mu_0=$$

$$d_{i_1,\ldots,i_{n-1}}\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})\times$$

$$\Bigg[\frac{\chi_{A^{i_1,\ldots,i_{n-1},1}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},1})}\int\limits_{A^{i_1,\ldots,i_{n-1},1}} S_N(\omega_1,\ldots,\omega_N)d\mu_0+$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1},2}}(\omega_1,\ldots,\omega_n)}{\mu_0(A^{i_1,\ldots,i_{n-1},2})}\int\limits_{A^{i_1,\ldots,i_{n-1},2}} S_N(\omega_1,\ldots,\omega_N)d\mu_0-$$

$$\frac{\chi_{A^{i_1,\ldots,i_{n-1}}}(\omega_1,\ldots,\omega_{n-1})}{\mu_0(A^{i_1,\ldots,i_{n-1}})}\int\limits_{A^{i_1,\ldots,i_{n-1}}} S_N(\omega_1,\ldots,\omega_N)d\mu_0\Bigg], \quad (26)$$

where $d_{i_1,\ldots,i_{n-1}}$ is finite. Since

$$\int\limits_{A^{i_1,\ldots,i_{n-1}}} m(\omega_1,\ldots,\omega_N)d\mu_0=$$

$$\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m(\omega_1,\ldots,\omega_N)d\mu_0, \tag{27}$$

we have

$$\mu_0(A^{i_1,\ldots,i_{n-1}}) \int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},1}) \int\limits_{A^{i_1,\ldots,i_{n-1}}} m(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$[\mu_0(A^{i_1,\ldots,i_{n-1},1}) + \mu_0(A^{i_1,\ldots,i_{n-1},2})] \int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},1}) \left[\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m(\omega_1,\ldots,\omega_N)d\mu_0\right] =$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},2}) \int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},1}) \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m(\omega_1,\ldots,\omega_N)d\mu_0. \tag{28}$$

Further,

$$\mu_0(A^{i_1,\ldots,i_{n-1}}) \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},2}) \int\limits_{A^{i_1,\ldots,i_{n-1}}} m(\omega_1,\ldots,\omega_N)d\mu_0 =$$

$$[\mu_0(A^{i_1,\ldots,i_{n-1},1}) + \mu_0(A^{i_1,\ldots,i_{n-1},2})] \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m(\omega_1,\ldots,\omega_N)d\mu_0 -$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},2}) \left[\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0 + \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m(\omega_1,\ldots,\omega_N)d\mu_0\right] =$$

$$-\left[\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m(\omega_1,\dots,\omega_N)d\mu_0-\right.$$

$$\left.\mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m(\omega_1,\dots,\omega_N)d\mu_0\right]. \tag{29}$$

If to put

$$R_1^m(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})=\mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} m(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} m(\omega_1,\dots,\omega_N)d\mu_0, \tag{30}$$

$$R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})=\mu_0(A^{i_1,\dots,i_{n-1},1})\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0. \tag{31}$$

Then, the equality (26) is transformed into the equality

$$R_1^m(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}})=d_{i_1,\dots,i_{n-1}}R_1^{S_N}(\omega_1^{i_1},\dots,\omega_{n-1}^{i_{n-1}}). \tag{32}$$

Due to that $S_n(\omega_1,\dots,\omega_n)$ and $m_n(\omega_1,\dots,\omega_n)$ are martingales relative to the measure μ_0 and $A^{i_1,\dots,i_{n-1},1}$, $A^{i_1,\dots,i_{n-1},2}\in\mathcal{F}_n$ we have

$$\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0=\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_n(\omega_1,\dots,\omega_n)d\mu_0=$$

$$\mu_0(A^{i_1,\dots,i_{n-1},1})S_n(\omega_1,\dots,\omega_n^1), \tag{33}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0=\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_n(\omega_1,\dots,\omega_n)d\mu_0=$$

$$\mu_0(A^{i_1,\dots,i_{n-1},2})S_n(\omega_1,\dots,\omega_n^2), \tag{34}$$

$$\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_N(\omega_1,\dots,\omega_N)d\mu_0=\int\limits_{A^{i_1,\dots,i_{n-1},1}} m_n(\omega_1,\dots,\omega_n)d\mu_0=$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},1})m_n(\omega_1,\ldots,\omega_n^1), \tag{35}$$

$$\int\limits_{A^{i_1,\ldots,i_{n-1},2}} m_N(\omega_1,\ldots,\omega_N)d\mu_0 = \int\limits_{A^{i_1,\ldots,i_{n-1},2}} m_n(\omega_1,\ldots,\omega_n)d\mu_0 =$$

$$\mu_0(A^{i_1,\ldots,i_{n-1},2})m_n(\omega_1,\ldots,\omega_n^2). \tag{36}$$

Since $d_{i_1,\ldots,i_{n-1}}$ is finite, then $R_1^{S_N}(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}}) \neq 0$. The last means that inequality (20) takes place. This proves the equality

$$d_{i_1,\ldots,i_{n-1}} = \tag{37}$$

$$\frac{m_n(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}},\omega_n^1) - m_n(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}},\omega_n^2)}{S_n(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}},\omega_n^1) - S_n(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}},\omega_n^2)},$$

$$n = \overline{1,N},$$

which means that (23) is true, where we introduced the denotation

$$m_n(\omega_1,\ldots,\omega_n) = E^{\mu_0}\{m(\omega_1,\ldots,\omega_N)|\mathcal{F}_n\} =$$

$$\sum_{i_{n+1}=1,\ldots,i_N=1}^{2} m(\omega_1,\ldots,\omega_n,\omega_n^{i_{n+1}},\ldots,\omega_N^{i_N})\mu_0(\{\omega_1,\ldots,\omega_n,\omega_n^{i_{n+1}},\ldots,\omega_N^{i_N}\}), \tag{38}$$

$$S_n(\omega_1,\ldots,\omega_n) = E^{\mu_0}\{S_N(\omega_1,\ldots,\omega_N)|\mathcal{F}_n\} =$$

$$\sum_{i_{n+1}=1,\ldots,i_N=1}^{2} S_N(\omega_1,\ldots,\omega_n,\omega_n^{i_{n+1}},\ldots,\omega_N^{i_N})\mu_0(\{\omega_1,\ldots,\omega_n,\omega_n^{i_{n+1}},\ldots,\omega_N^{i_N}\}). \tag{39}$$

This proves the necessity.

Proof of the sufficiency. Suppose that the inequalities (20) are true. Let us prove that the martingale measure μ_0 is unique. For this purpose, we prove that for every martingale the representation (21) is true with validity of equalities (22), (23).

Let us note that the equality (26) is true if for $d_{i_1,\ldots,i_{n-1}}$ to choose (37) since the equalities

$$\left[\frac{\int\limits_{A^{i_1,\ldots,i_{n-1},1}} m(\omega_1,\ldots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\ldots,i_{n-1},1})} - \frac{\int\limits_{A^{i_1,\ldots,i_{n-1}}} m(\omega_1,\ldots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\ldots,i_{n-1}})}\right] \times$$

$$\left[\frac{\int\limits_{A^{i_1,\ldots,i_{n-1},1}} S_N(\omega_1,\ldots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\ldots,i_{n-1},1})} - \frac{\int\limits_{A^{i_1,\ldots,i_{n-1}}} S_N(\omega_1,\ldots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\ldots,i_{n-1}})}\right]^{-1} =$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},2}} m(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},2})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} m(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right]\times$$

$$\left[\frac{\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1},2})} - \frac{\int\limits_{A^{i_1,\dots,i_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0}{\mu_0(A^{i_1,\dots,i_{n-1}})}\right]^{-1} =$$

$$d_{i_1,\dots,i_{n-1}} \tag{40}$$

are valid.

Taking into account the equality (26) and the equalities

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_{n-1})\times$$

$$\left[\frac{\chi_{A^{i_1,\dots,i_{n-1},1}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},1})}\int\limits_{A^{i_1,\dots,i_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0+\right.$$

$$\frac{\chi_{A^{i_1,\dots,i_{n-1},2}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},2})}\int\limits_{A^{i_1,\dots,i_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\left.\frac{\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1}})}\int\limits_{A^{i_1,\dots,i_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0\right] =$$

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_{n-1})\times$$

$$\sum_{j_1=1,\dots j_{n-1}=1}^{2}\left[\frac{\chi_{A^{j_1,\dots,j_{n-1},1}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{j_1,\dots,j_{n-1},1})}\int\limits_{A^{j_1,\dots,j_{n-1},1}} S_N(\omega_1,\dots,\omega_N)d\mu_0+\right.$$

$$\frac{\chi_{A^{j_1,\dots,j_{n-1},2}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{j_1,\dots,j_{n-1},2})}\int\limits_{A^{j_1,\dots,j_{n-1},2}} S_N(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\left.\frac{\chi_{A^{j_1,\dots,j_{n-1}}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{j_1,\dots,j_{n-1}})}\int\limits_{A^{j_1,\dots,j_{n-1}}} S_N(\omega_1,\dots,\omega_N)d\mu_0\right] = \tag{41}$$

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)[S_n(\omega_1,\dots,\omega_{n-1},\omega_n)-S_{n-1}(\omega_1,\dots,\omega_{n-1})],$$

we have

$$\frac{\chi_{A^{i_1,\dots,i_{n-1},1}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},1})}\int\limits_{A^{i_1,\dots,i_{n-1},1}} m(\omega_1,\dots,\omega_N)d\mu_0+$$

$$\frac{\chi_{A^{i_1,\dots,i_{n-1},2}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1},2})}\int\limits_{A^{i_1,\dots,i_{n-1},2}} m(\omega_1,\dots,\omega_N)d\mu_0-$$

$$\frac{\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)}{\mu_0(A^{i_1,\dots,i_{n-1}})}\int\limits_{A^{i_1,\dots,i_{n-1}}} m(\omega_1,\dots,\omega_N)d\mu_0=$$

$$d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_n)[S_n(\omega_1,\dots,\omega_{n-1},\omega_n)-S_{n-1}(\omega_1,\dots,\omega_{n-1})]. \quad (42)$$

Summing over all indexes $i_1,\dots,i_{n-1}$ left and right hand sides of the equality (42) we obtain the equality

$$m_n(\omega_1,\dots,\omega_n)-m_{n-1}(\omega_1,\dots,\omega_{n-1})=$$

$$C_n(\omega_1,\dots,\omega_{n-1})[S_n(\omega_1,\dots,\omega_{n-1},\omega_n)-S_{n-1}(\omega_1,\dots,\omega_{n-1})], \quad (43)$$

$$C_n(\omega_1,\dots,\omega_{n-1})=\sum_{i_1=1,\dots,i_{n-1}=1}^{2} d_{i_1,\dots,i_{n-1}}\chi_{A^{i_1,\dots,i_{n-1}}}(\omega_1,\dots,\omega_{n-1}). \quad (44)$$

We proved that for every martingale the representation (21) is true, due to the conditions (20). Let us prove that the martingale measure is unique. Suppose that there are at most two martingale measures μ_0^1 and μ_0^2. If to put $m(\omega_1,\dots,\omega_N)=\chi_A(\omega_1,\dots,\omega_N)$, then

$$\chi_A(\omega_1,\dots,\omega_N)=$$

$$\sum_{n=1}^{N} C_n(\omega_1,\dots,\omega_{n-1})[S_n(\omega_1,\dots,\omega_{n-1},\omega_n)-S_{n-1}(\omega_1,\dots,\omega_{n-1})]+c_0. \quad (45)$$

From this representation, we obtain the equalities $\mu_0^1(A)=\mu_0^2(A)=c_0,\ A\in\mathcal{F}_N$. Contradiction. The last proves Theorem 1.

Next Theorem is concerned the case as the set of martingale measures consists of one measure.

Theorem 2. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$*, being the direct product of the probability spaces* $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$*, suppose that the evolution of risky asset is given by the formula (11), then the set of martingale measures, being equivalent to the measure* P_N*, consists of one point*

$$\mu_0(A)=$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\dots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N. \quad (46)$$

The fair price of contract with option φ_0 *of European type with the payoff function* $\varphi(\omega_1,\dots,\omega_N)$ *is given by the formula*

$$\varphi_0=\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi(\omega_1^{i_1},\dots,\omega_N^{i_N}), \quad (47)$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

where

$$\psi_n(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^2(\omega_1,\ldots,\omega_n), \tag{48}$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \tag{49}$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}. \tag{50}$$

Proof. Since

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} > 0, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \tag{51}$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} > 0, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \tag{52}$$

we have

$$\psi_n(\omega_1,\ldots,\omega_n) = \chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\ldots,\omega_n) > 0, \quad (\omega_1,\ldots,\omega_n) \in \Omega_n. \tag{53}$$

From this, it follows that $\mu_0(A) > 0$ for every $A \in \mathcal{F}_N$. It means that $\mu_0(A)$ is equivalent to P_N. The inequality

$$S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1) = \prod_{i=1}^{n-1}(1+a_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i))(1+a_n(\omega_1,\ldots,\omega_n^1)\eta_i(\omega_n^1)) \neq$$

$$S_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2) =$$

$$\prod_{i=1}^{n-1}(1+a_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i))(1+a_n(\omega_1,\ldots,\omega_n^2)\eta_i(\omega_n^2)), \quad n=\overline{1,N}, \tag{54}$$

is true, since

$$(1+a_n(\omega_1,\ldots,\omega_n^1)\eta_i(\omega_n^1)) \neq$$

$$(1 + a_n(\omega_1, \dots, \omega_n^2)\eta_i(\omega_n^2)), \quad n = \overline{1, N}, \tag{55}$$

due to the suppositions relative to the evolutions of risky asset, given by the formula (11). Thanks to Theorem 1, the martingale measure μ_0 is unique.

To prove the rest statement of Theorem 2, we need to construct the self-financing strategy π such that the capital corresponding this strategy on (B, S) market satisfies the condition

$$X_N^\pi = \varphi(\omega_1, \dots, \omega_{n-1}, \omega_N).$$

Let us consider the martingale

$$m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) = E^{\mu_0}\{\varphi(\omega_1, \dots, \omega_{n-1}, \omega_N)|\mathcal{F}_n\}.$$

Due to Theorem 1, for the finite martingale $\{m_n(\omega_1, \dots, \omega_{n-1}, \omega_n)\}_{n=0}^N$ relative to the the measure $\mu_0(A)$ the representation

$$m_n(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\sum_{i=1}^n C_i(\omega_1, \dots, \omega_{i-1})[S_i(\omega_1, \dots, \omega_{i-1}, \omega_i) - S_{i-1}(\omega_1, \dots, \omega_{i-1})] +$$

$$m_0, \quad n = \overline{1, N}, \tag{56}$$

is true, where $C_i(\omega_1, \dots, \omega_{i-1})$ is $\mathcal{F}_{i-1}$ measurable random value, and $m_0 = E^{\mu_0}\varphi(\omega_1, \dots, \omega_{n-1}, \omega_N)$.

If to put $\pi = \{\beta_n, \gamma_n\}_{n=0}^N$, where

$$\gamma_n = C_n(\omega_1, \dots, \omega_{n-1}), \quad \beta_n = m_{n-1}(\omega_1, \dots, \omega_{n-1}) - \gamma_n S_{n-1}(\omega_1, \dots, \omega_{n-1}),$$

then it easy to see that π is self-financed strategy. Really,

$$\Delta\beta_n B_{n-1} + \gamma_n \Delta S_{n-1} = \Delta\beta_n + \Delta\gamma_n S_{n-1} =$$

$$m_{n-1} - \gamma_n S_{n-1} - m_{n-2} + \gamma_{n-1} S_{n-2} + (\gamma_n - \gamma_{n-1}) S_{n-1} =$$

$$m_{n-1} - m_{n-2} - \gamma_{n-1}(S_{n-1} - S_{n-2}) = 0.$$

$\mathcal{F}_{n-1}$-measurability of (β_n, γ_n) is evident.

It is easy to show that

$$X_n(\omega_1, \dots, \omega_n) = \beta_n B_n + \gamma_n S_n = m_n(\omega_1, \dots, \omega_n).$$

Therefore,

$$X_0 = m_0 = E^{\mu_0}\varphi(\omega_1, \dots, \omega_{n-1}, \omega_N), \quad X_N = \varphi(\omega_1, \dots, \omega_{n-1}, \omega_N).$$

IV. Complete Market Hedging

In this section, the securities market is constructed, the evolution of which occurs in accordance with Formula (11). Possible for this was the observation that with respect to a certain class of evolutions of risky assets, the family of martingale measures is invariant. This fact turned out to be crucial for the construction of models of non-arbitrage markets. In papers [10], [11], such a possibility of the existence of non-arbitrage markets is established on the basis of the Hahn-Banach Theorem. This beautiful result has the disadvantage that it does not provide an algorithm for constructing models of non-arbitrage markets. How to build them having the evolution of risky assets is practically a difficult problem.

In Proposition 1, we establish the form of measurable transformations relative to which the only measure is invariant. Using that, a model of the securities market is built, which is complete. This result is constructive in contrast to the existence theorem from [10], [11]. Our denotations in this section are the same as in the previous section. We consider the evolution of risky assets given by the formula (11) on the same probability space.

Proposition 1. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *let the evolution of risky asset be given by the formula (11), with* $a_i(\omega_1, \ldots, \omega_i) = b_i(\omega_1, \ldots, \omega_{i-1}) f_i(\omega_1, \ldots, \omega_i)$, *where the random variables* $f_i(\omega_1, \ldots, \omega_i)$, $b_i(\omega_1, \ldots, \omega_{i-1})$, *satisfy the inequalities*

$$f_i(\omega_1, \ldots, \omega_i) > 0, \quad b_i(\omega_1, \ldots, \omega_{i-1}) > 0, \quad \max_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} b_i(\omega_1, \ldots, \omega_{i-1}) <$$

$$\frac{1}{\max\limits_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} f_i(\omega_1, \ldots, \omega_{i-1}, \omega_i^1) \eta_i^-(\omega_i^1)}, \quad i = \overline{1, N}. \tag{57}$$

For such an evolution, the unique martingale measure μ_0 *does not depend on the random variables* $b_i(\omega_1, \ldots, \omega_{i-1})$, $i = \overline{1, N}$, *and it is given by the formula*

$$\mu_0(A) = \mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{58}$$

where

$$\psi_n(\omega_1, \ldots, \omega_n) = \chi_{\Omega_n^-}(\omega_1, \ldots, \omega_{n-1}, \omega_n) \psi_n^1(\omega_1, \ldots, \omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1, \ldots, \omega_{n-1}, \omega_n)\, \psi_n^2(\omega_1, \ldots, \omega_n), \tag{59}$$

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2)}{f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \tag{60}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) = \frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}. \tag{61}$$

Proof. Due to the representation (46) for the measure μ_0, to prove Proposition 1 it needs to prove that all $\boldsymbol{\psi}_n(\omega_1,\ldots,\omega_n)$, $n=\overline{1,N}$, do not depend on the random variables $b_i(\omega_1,\ldots,\omega_{i\_1})$, $i=\overline{1,N}$, where

$$\boldsymbol{\psi}_n(\omega_1,\ldots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\,\boldsymbol{\psi}_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\boldsymbol{\psi}_n^2(\omega_1,\ldots,\omega_n), \tag{62}$$

$$\boldsymbol{\psi}_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)},\quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \tag{63}$$

$$\boldsymbol{\psi}_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)},\quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}. \tag{64}$$

But,

$$\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)=$$

$$S_{n-1}(\omega_1,\ldots,\omega_{n-1})b_n(\omega_1,\ldots,\omega_{n-1})f_n(\omega_1,\ldots,\omega_n^2)\eta_n^+(\omega_n^2), \tag{65}$$

$$\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)=$$

$$S_{n-1}(\omega_1,\ldots,\omega_{n-1})b_n(\omega_1,\ldots,\omega_{n-1})f_n(\omega_1,\ldots,\omega_n^1)\eta_n^-(\omega_n^1). \tag{66}$$

Therefore,

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\frac{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)}{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \tag{67}$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}=$$

$$\frac{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \tag{68}$$

$$(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}.$$

The equalities (67), (68) prove Proposition 1.

Suppose that the market consists of d assets the evolutions of which are given by the law

$$S_n((\omega_1,\dots,\omega_n)=\{S_n^1((\omega_1,\dots,\omega_n),\dots,S_n^d((\omega_1,\dots,\omega_n)\},\quad n=\overline{1,N}, \tag{69}$$

where

$$S_n^k((\omega_1,\dots,\omega_n)=S_0^k\prod_{i=1}^n(1+b_i^k(\omega_1,\dots,\omega_{i-1})f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)),\quad k=\overline{1,d}, \tag{70}$$

and the random values $\eta_i(\omega_i)$, $f_i(\omega_1,\dots,\omega_i)$, $i=\overline{1,N}$, do not depend on k, and satisfy inequalities

$$f_i(\omega_1,\dots,\omega_i)>0,\quad b_i^k(\omega_1,\dots,\omega_{i-1})>0,\quad \max_{\{\omega_1,\dots,\omega_{i-1}\}\in\Omega_{i-1}} b_i^k(\omega_1,\dots,\omega_{i-1})<$$

$$\frac{1}{\max\limits_{\{\omega_1,\dots,\omega_{i-1}\}\in\Omega_{i-1}} f_i(\omega_1,\dots,\omega_{i-1},\omega_i^1)\eta_i^-(\omega_i^1)},\quad k=\overline{1,d},\quad i=\overline{1,N}. \tag{71}$$

Proposition 2. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, *if the evolution of* d *risky assets is given by the formula (69), (70), then such a market is complete non arbitrage one. The unique martingale measure does not depend on the random variables* $b_i^k(\omega_1,\dots,\omega_{i-1}), k=\overline{1,d},\quad i=\overline{1,N}$, *and it is determined by the formula (58). For the contingent claims* $\varphi_i(\omega_1,\dots,\omega_N), i=\overline{1,d}$, *the fair prices* φ_0^i *are given by the formulas*

$$\varphi_0^i=\sum_{i_1=1}^2\dots\sum_{i_N=1}^2\prod_{n=1}^N\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi_i(\omega_1^{i_1},\dots,\omega_N^{i_N}),\quad i=\overline{1,d}. \tag{72}$$

Corollary 1. (Cox, Ross, Rubinstein, see [3]) On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, *let the evolution of risky asset is given by the formula*

$$S_n^1((\omega_1,\dots,\omega_n)=S_0\prod_{i=1}^n(1+\rho_i(\omega_i)),\quad n=\overline{1,N}, \tag{73}$$

where the random values $\rho_i(\omega_i)$, $i=\overline{1,N}$, *are such that* $\rho_i(\omega_i^1)=a$, $\rho_i(\omega_i^2)=b$, *and let the bank account evolution be given by the formula*

$$B_n=B_0(1+r)^n,\quad r>0,\quad B_0>0\quad n=\overline{1,N}. \tag{74}$$

Then, for the discount evolution of risky asset

$$S_n((\omega_1,\dots,\omega_n)=\frac{S_0\prod\limits_{i=1}^n(1+\rho_i(\omega_i))}{B_0(1+r)^n},\quad n=\overline{1,N}, \tag{75}$$

the martingale measure μ_0 *is unique if* $a<r<b$. *It is a direct product of measures* $\mu_0^i(A)$, $A\in\mathcal{F}_i^0$, $i=\overline{1,N}$, *given on the measurable space* $\{\Omega_i^0,\mathcal{F}_i^0\}$, *where* $\mu_0^i(\omega_i^1)=$

$\frac{b-r}{b-a}$, $\mu_0^i(\omega_i^2) = \frac{r-a}{b-a}$. *The fair price* φ_0 *of the contingent liability* $\varphi_N(\omega_1, \ldots, \omega_N)$ *is given by the formula*

$$\varphi_0 = \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_0 =$$

$$\sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \varphi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \prod_{k=1}^{N} \mu_0^k(\omega_k^{i_k}). \tag{76}$$

Proof. For the discount evolution (75), the representation

$$S_n((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} (1 + \eta_i(\omega_i)), \quad n = \overline{1, N}, \tag{77}$$

is true, where $\eta_i(\omega_i) = \frac{\rho_i(\omega_i))-r}{(1+r)}$. Due to Theorems 1, 2, since $\eta_i(\omega_i^1) = \frac{a-r}{1+r} < 0$, $\eta_i(\omega_i^2) = \frac{b-r}{1+r} > 0$, then the measure μ_0 is unique.

Theorem 3. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i)), \quad n = \overline{1, N}, \tag{78}$$

where the random values $\rho_i(\omega_i)$, $i = \overline{1, N}$, *are such that* $\rho_i(\omega_i^1) = b_i^1$, $\rho_i(\omega_i^2) = b_i^2$, $i = \overline{1, N}$, *and let the bank account evolution be given by the formula*

$$B_n = B_0 \prod_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1, N}, \tag{79}$$

where the random values $r_i(\omega_i)$, $i = \overline{1, N-1}$, *are such that* $r_i(\omega_i^1) = r_i^1$, $r_i(\omega_i^2) = r_i^2$, $i = \overline{1, N-1}$, $r_0 > 0$. *Then, for the discount evolution of risky asset*

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod\limits_{i=1}^{n} (1 + \rho_i(\omega_i))}{B_0 \prod\limits_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1, N}, \tag{80}$$

the martingale measure μ_0 *is unique, if* $b_1^1 < r_0 < b_1^2$, $b_i^1 < r_{i-1}^1 < r_{i-1}^2 < b_i^2$, $i = \overline{2, N}$. *It is determined by the formula (58) with*

$$\eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \quad \eta_i(\omega_i) = \rho_i(\omega_i) - r_{i-1}^2, \quad i = \overline{2, N},$$

$$f_1(\omega_1) = \frac{1}{1 + r_0}, \quad f_i(\omega_1, \ldots, \omega_i) =$$

$$\frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - r_{i-1}^2)(1 + r_{i-1}(\omega_{i-1}))}, \quad i = \overline{2, N}. \tag{81}$$

The fair price φ_0 of the contingent liability $\varphi_N(\omega_1,\dots,\omega_N)$ is given by the formula

$$\varphi_0 = \int\limits_{\Omega_N} \varphi_N(\omega_1,\dots,\omega_N)d\mu_0 =$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi_N(\omega_1^{i_1},\dots,\omega_N^{i_N}). \tag{82}$$

Proof. To prove Theorem 3 it is necessary to prove the existence of unique spot measure. The discount evolution (80) can be represented in the form

$$S_n((\omega_1,\dots,\omega_n) =$$

$$\frac{S_0}{B_0}\prod_{i=1}^{n}\left(1+f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)\right), \quad n=\overline{1,N}, \tag{83}$$

where

$$\eta_1(\omega_1)=\rho_1(\omega_1)-r_0, \quad \eta_i(\omega_i)=\rho_i(\omega_i)-r_{i-1}^2, \quad i=\overline{2,N},$$

$$f_1(\omega_1)=\frac{1}{1+r_0}, \quad f_i(\omega_1,\dots,\omega_i)=$$

$$\frac{\rho_i(\omega_i)-r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i)-r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))}, \quad i=\overline{2,N}, \tag{84}$$

It is evident that $\eta_i(\omega_i^1)<0,\ \eta_i(\omega_i^2)>0,\ f_i(\omega_1,\dots,\omega_i)>0$. Therefore, from the representation (83), (84) it follows that we can construct only one spot measure, which is martingale measure being equivalent to the initial measure P_N. In accordance with Theorem 1, since $S_n(\omega_1,\dots,\omega_n^1)\neq S_n(\omega_1,\dots,\omega_n^2),\ \{\omega_1,\dots,\omega_{n-1}\}\in\Omega_{n-1}$ such a measure is unique. Theorem 3 is proved.

Theorem 4. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, let the evolution of risky asset be given by the formula

$$S_n^1((\omega_1,\dots,\omega_n)=S_0\prod_{i=1}^{n}e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}, \quad n=\overline{1,N}, \tag{85}$$

where the random values $\varepsilon_i(\omega_i),\ i=\overline{1,N}$, are such that $\varepsilon_i(\omega_i^1)<0,\ \varepsilon_i(\omega_i^2)>0$, $\sigma_i(\omega_1,\dots,\omega_{i-1})\geq\sigma_i^0>0,\ i=\overline{1,N}$, and let the bank account evolution be given by the formula

$$B_n=B_0\prod_{i=1}^{n}(1+r_{i-1}(\omega_{i-1})), \quad B_0>0, \quad n=\overline{1,N}, \tag{86}$$

where the random values $r_i(\omega_i),\ i=\overline{1,N-1}$, are such that $r_i(\omega_i^1)=r_i^1,\ r_i(\omega_i^2)=r_i^2,\ i=\overline{1,N-1},\ r_0>0$. Then, for the discount evolution of risky asset

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$S_n((\omega_1,\dots,\omega_n) = \frac{S_0\prod_{i=1}^{n} e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)}}{B_0\prod_{i=1}^{n}(1+r_{i-1}(\omega_{i-1}))}, \quad n=\overline{1,N}, \tag{87}$$

the martingale measure μ_0 *is unique, if*

$$\exp\{\sigma_1^0\varepsilon_1(\omega_1^1)\} < r_0 < \exp\{\sigma_1^0\varepsilon_1(\omega_1^2)\},$$

$$\exp\{\sigma_i^0\varepsilon_i(\omega_i^1)\} < r_{i-1}^1 < r_{i-1}^2 < \exp\{\sigma_i^0\varepsilon_i(\omega_i^2)\},\ i=\overline{2,N}. \tag{88}$$

It is determined by the formula (58) with

$$\eta_1(\omega_1) = \exp\{\sigma_1^0\varepsilon_1(\omega_1)\} - r_0, \quad f_1(\omega_1) = \frac{1}{1+r_0},$$

$$\eta_i(\omega_i) = \exp\{\sigma_i^0\varepsilon_i(\omega_i)\} - r_{i-1}^2, \quad f_i(\omega_1,\dots,\omega_i) =$$

$$\frac{e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)} - r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0\varepsilon_i(\omega_i)\} - r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))}, \quad \{\omega_1,\dots,\omega_i\}\in\Omega_n, \quad i=\overline{2,N}. \tag{89}$$

The fair price φ_0 *of the contingent liability* $\varphi_N(\omega_1,\dots,\omega_N)$ *is given by the formula*

$$\varphi_0 = \int_{\Omega_N}\varphi_N(\omega_1,\dots,\omega_N)d\mu_0 =$$

$$\sum_{i_1=1}^{2}\dots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\dots,\omega_n^{i_n})\varphi_N(\omega_1^{i_1},\dots,\omega_N^{i_N}). \tag{90}$$

Proof. For the discount evolution (87), the following representation

$$S_n((\omega_1,\dots,\omega_n) =$$

$$\frac{S_0}{B_0}\prod_{i=1}^{n}\left(1+f_i(\omega_1,\dots,\omega_i)\eta_i(\omega_i)\right), \quad n=\overline{1,N}, \tag{91}$$

is true, where

$$\eta_1(\omega_1) = \exp\{\sigma_1^0\varepsilon_1(\omega_1)\} - r_0, \quad f_1(\omega_1) = \frac{1}{1+r_0},$$

$$\eta_i(\omega_i) = \exp\{\sigma_i^0\varepsilon_i(\omega_i)\} - r_{i-1}^2, \quad f_i(\omega_1,\dots,\omega_i) =$$

$$\frac{e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)} - r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0\varepsilon_i(\omega_i)\} - r_{i-1}^2)(1+r_{i-1}(\omega_{i-1}))}, \quad \{\omega_1,\dots,\omega_i\}\in\Omega_n, \quad i=\overline{2,N}. \tag{92}$$

It is evident that $\eta_i(\omega_i^1) < 0,\ \eta_i(\omega_i^2) > 0,\ f_i(\omega_1,\dots,\omega_i) > 0$. From this, we obtain that the spot measure exists and it is unique. Theorem 4 is proved.

On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, suppose that the market consists of d assets the evolution of which is given by the law

$$S_n((\omega_1, \ldots, \omega_n) = \{S_n^1((\omega_1, \ldots, \omega_n), \ldots, S_n^d((\omega_1, \ldots, \omega_n)\}, \quad n = \overline{1, N}, \tag{93}$$

where

$$S_n^k((\omega_1, \ldots, \omega_n) = S_0^k \prod_{i=1}^{n}(1 + a_i^k f_i(\omega_1, \ldots, \omega_i)\eta_i(\omega_i)), \quad k = \overline{1, d}, \tag{94}$$

and the random values $\eta_i(\omega_i), \ f_i(\omega_1, \ldots, \omega_i), \ i = \overline{1, N}$, and constants a_i^k satisfy the inequalities

$$\eta_i(\omega_i^1) < 0, \quad \eta_i(\omega_i^2) > 0, \quad f_i(\omega_1, \ldots, \omega_i) > 0,$$

$$0 < a_i^k < \frac{1}{\max\limits_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} f_i(\omega_1, \ldots, \omega_i^1)\eta_i^-(\omega_i^1)}, \quad i = \overline{1, N}, \quad k = \overline{1, d}. \tag{95}$$

Proposition 3. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky assets be given by the formulas (93), (94), where constants a_i^k $i = \overline{1, N}, \ k = \overline{1, d}$, satisfy the inequalities (95). For such an evolution of risky asset the martingale measure μ_0 does not depend on a_i^k and is unique. It is determined by the formula (58). For the contingent claims $\varphi_N^i(\omega_1, \ldots, \omega_N), \ i = \overline{1, d}$, the fair prices φ_0^i are given by the formulas*

$$\varphi_0^i = \sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n})\varphi_N^i(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad i = \overline{1, d}. \tag{96}$$

If $f_i(\omega_1, \ldots, \omega_i) = 1, \ i = \overline{1, N}$, the unique martingale measure is a direct product of measures $\mu_0^i(A), \ A \in \mathcal{F}_i^0$, given on the measurable space $\{\Omega_i^0, \mathcal{F}_i^0\}, \ i = \overline{1, N}$, where

$$\mu_0^i(\omega_i^1) = \frac{\eta_i^+(\omega_i^2)}{(\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2))}, \quad \mu_0^i(\omega_i^2) = \frac{\eta_i^-(\omega_i^1)}{(\eta_i^-(\omega_i^1) + \eta_i^+(\omega_i^2))}. \tag{97}$$

The fair prices $\varphi_0^i, \ i = \overline{1, N}$, of the contingent liabilities $\varphi_N^i(\omega_1, \ldots, \omega_N), \ i = \overline{1, N}$, are given by the formula

$$\varphi_0^i = \int\limits_{\Omega_N} \varphi_N^i(\omega_1, \ldots, \omega_N) d\mu_0 =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \varphi_N^i(\omega_1^{i_1}, \ldots, \omega_N^{i_N}) \prod_{k=1}^{N} \mu_0^k(\omega_k^{i_k}). \tag{98}$$

Suppose that $\{g_k^i(X_N)\}_{k=1}^N, \ i = \overline{1, d}$, are the mappings from the set $[0, 1]^N$ into itself, where $X_N = \{x_1, \ldots, x_N\}, \ 0 \leq \ x_k \leq 1, \ k = \overline{1, N}$. If $S_0^i, S_1^i, \ldots, S_N^i, \ i = \overline{1, d}$, are the samples of the processes (93), (94) let us denote the order statistics $S_{(0)}^i, S_{(1)}^i, \ldots, S_{(N)}^i, \ i = \overline{1, d}$, of this samples. Introduce also the denotation

$$g_k^i([S^i]_N) = g_k^i\left(\frac{S_{(0)}^i}{S_{(N)}^i}, \ldots, \frac{S_{(N-1)}^i}{S_{(N)}^i}\right), \ k = \overline{1, N}, \ i = \overline{1, d}.$$

Proposition 4. Suppose that $S^i_0, S^i_1, \ldots, S^i_N$ *is a sample of the random processes (93), (94). Then, for the parameters* $a^i_1, \ldots, a^i_N$ *the estimation*

$$a^i_1 = \frac{\left[1 - \tau^i_0 \frac{S^i_{(0)}}{S^i_0} g^i_1\left([S^i]_N\right)\right]}{f_1 \eta^-_1(\omega^1_1)}, \quad 0 < \tau^i_0 \leq 1, \quad i = \overline{1,d},$$

$$a^i_k = \frac{\left[1 - \frac{g^i_k\left([S^i]_N\right)}{g^i_{k-1}([S^i]_N)}\right]}{f_k \eta^-_k(\omega^1_k)}, \quad k = \overline{2,N}, \quad i = \overline{1,d}, \tag{99}$$

is valid, if for $g^i_N([S^i]_N) > 0$, $[S^i]_N \in [0,1]^N$, *the inequalities* $g^i_1([S^i]_N) \geq g^i_2([S^i]_N) \geq \ldots \geq g^i_N([S^i]_N)$ *are true. If* $\tau^i_0 = 0$, *then* $a^i_k = 1,\ k = \overline{1,N},\ i = \overline{1,d}$.

In the formulas (99) we put that $f_k = \max_{\{\omega_1,\ldots,\omega_{k-1}\}\in\Omega_{k-1}} f_k(\omega_1,\ldots,\omega_{k-1},\omega^1_k), \quad k = \overline{1,N}$.

V. Martingale Measures on Discrete Probability Space

This section presents all the necessary results for constructing a non-arbitrage incomplete market on a discrete probability space. The conditions under which the entire family of martingale measures is described for the considered class of evolution of risky assets are minimal. In particular, conditions are presented under which the family of martingale measures considered is equivalent to the original measure. They are minimal. The entire set of equivalent martingale measures is a convex combination of a finite number of spot martingale measures. On this basis, new formulas were found for the fair price of the super hedge.

In this section, we put that $\Omega^0_i = \{\omega^1_i, \ldots, \omega^M_i\}$, $i = \overline{1,N}$, and we assume that $2 < M < \infty$, the σ-algebra $\mathcal{F}^0_i$ consists from all subsets of Ω^0_i. We suppose that $P^0_i(\omega^k_i) > 0, \omega^k_i \in \Omega^0_i, k = \overline{1,M}$. As before, the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$ is a direct product of probability spaces $\{\Omega^0_i, \mathcal{F}^0_i, P^0_i\}, i = \overline{1,N}$. Sometimes, any elementary event $\omega^k_i \in \Omega^0_i$ it is convenient to denote by ω_i not indicating the index k. Further, we use the both denotations. As in section 2, we introduce filtration $\mathcal{F}_n$ on the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$. As before, it is convenient to introduce the family of probability spaces $\{\Omega_n, \mathcal{F}_n, P_n\}, n = \overline{1,N}$, being a direct product of probability spaces $\{\Omega^0_i, \mathcal{F}^0_i, P^0_i\}, i = \overline{1,n}$.

The evolution of risky assets is given by the formula (8) with the assumptions given in the section 2. In this case

$$\Omega^-_n = \Omega_{n-1} \times \Omega^{0-}_n, \ \Omega^+_n = \Omega_{n-1} \times \Omega^{0+}_n, \tag{100}$$

where $\Omega^{0-}_n = \{\omega_n \in \Omega^0_n, \eta_n(\omega_n) \leq 0\}, \Omega^{0+}_n = \{\omega_n \in \Omega^0_n, \eta_n(\omega_n) > 0\}$, $P^0_n(\{\omega_n, \eta_n(\omega_n) > 0\}) > 0, P^0_n(\{\omega_n, \eta_n(\omega_n) < 0\}) > 0$. Further, we also use the measurable space with measure

$$\left\{\prod_{i=1}^N [\Omega^{0-}_i \times \Omega^{0+}_i], \prod_{i=1}^N [\mathcal{F}^{0-}_i \times \mathcal{F}^{0+}_i], \prod_{i=1}^N [P^{0-}_i \times P^{0+}_i]\right\}. \tag{101}$$

The measure P_n^{0-} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0-} = \Omega_n^{0-} \cap \mathcal{F}_n^0$, P_n^{0+} is a contraction of the measure P_n^0 on the σ-algebra $\mathcal{F}_n^{0+} = \Omega_n^{0+} \cap \mathcal{F}_n^0$. Additionally, we assume

$$P_n^0(\{\omega_n \in \Omega_n^0, |\eta_n(\omega_n)| < \infty\}) = 1. \tag{102}$$

In this case, Lemma 1 (see [1]) is formulated as follows

Lemma 2. Suppose that for $\Omega_n^a, a = -, +, \ n = \overline{1, N}$, the representations (100) are true. If the conditions

$$B_{n,i}^{0-} \cap B_{n,j}^{0-} = \emptyset, \ i \neq j,$$

$$B_{n,s}^{0+} \cap B_{n,l}^{0+} = \emptyset, \ s \neq l, \quad k = \overline{1, N_n},$$

$$\Omega_n^{0-} = \bigcup_{i=1}^{N_n} B_{n,i}^{0-}, \quad \Omega_n^{0+} = \bigcup_{i=1}^{N_n} B_{n,i}^{0+},$$

$$P_n^0(\Omega_n^{0-} \setminus B_{n,i}^{0,-}) > 0, \quad i = \overline{1, I_n}, \quad I_n > 1, \quad n = \overline{1, N},$$

$$P_n^0(\Omega_n^{0+} \setminus B_{n,s}^{0,+}) > 0, \quad s = \overline{1, S_n}, \quad S_n > 1, \quad n = \overline{1, N},$$

$$P_n^0(B_{n,i}^{0,-}) > 0, \quad i = \overline{1, I_n}, \quad I_n > 1, \quad n = \overline{1, N},$$

$$P_n^0(B_{n,s}^{0,+}) > 0, \quad s = \overline{1, S_n}, \quad S_n > 1, \quad n = \overline{1, N},$$

$$\int\limits_{\Omega_N} \Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n) dP_N < \infty, \qquad n = \overline{1, N}, \tag{103}$$

are true, then the set of bounded strictly positive random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2)$, satisfying the conditions (14) - (16),(see [1]) is a nonempty set.

Lemma 3. Suppose that the conditions of Lemma 2 are true. For the measure $\mu_0(A)$, $A \in \mathcal{F}_N$, constructed by the recurrent relations (23) - (25),(see [1]) the representation

$$\mu_0(A) = \int\limits_{\Omega_N} \prod_{n=1}^N \psi_n(\omega_1, \ldots, \omega_n) \chi_A(\omega_1, \ldots, \omega_N) \prod_{i=1}^N dP_i^0(\omega_i) \tag{104}$$

is true and $\mu_0(\Omega_N) = 1$, that is, the measure $\mu_0(A)$ is a probability measure being equivalent to the measure P_N, where we put

$$\psi_n(\omega_1, \ldots, \omega_n) = \chi_{\Omega_n^{0-}}(\omega_n) \psi_n^1(\omega_1, \ldots, \omega_n) +$$

$$\chi_{\Omega_n^{0+}}(\omega_n) \psi_n^2(\omega_1, \ldots, \omega_n), \tag{105}$$

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^{0+}}(\omega_n^2) \alpha_n(\{\omega_1, \ldots, \omega_{n-1}, \omega_n\}; \{\omega_1, \ldots, \omega_{n-1}, \omega_n^2\}) \times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2), \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}, \tag{106}$$

$$\boldsymbol{\psi}_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\int\limits_{\Omega_n^0}\chi_{\Omega_n^{0-}}(\omega_n^1)\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n^1\};\{\omega_1,\ldots,\omega_{n-1},\omega_n\})\times$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1), \quad (\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}. \tag{107}$$

Proof. We only need to prove that $\boldsymbol{\psi}_n(\omega_1,\ldots,\omega_n)>0,\ n=\overline{1,N}$. Suppose that

$$\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})=\alpha_n^1(\omega_n^1)\alpha_n^2(\omega_n^2),$$

where

$$\alpha_n^1(\omega_n^1)>0, \quad \omega_n^1\in\Omega_n^{0-}, \quad \alpha_n^2(\omega_n^2)>0, \quad \omega_n^2\in\Omega_n^{0+}.$$

Since

$$\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)=S_{n-1}(\omega_1,\ldots,\omega_{n-1})a_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1),$$

$$\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)=S_{n-1}(\omega_1,\ldots,\omega_{n-1})a_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2),$$

where

$$\eta_n^-(\omega_n)=-\chi_{\Omega_n^{0-}}(\omega_n)\eta_n(\omega_n), \quad \eta_n^+(\omega_n)=\chi_{\Omega_n^{0+}}(\omega_n)\eta_n(\omega_n),$$

$$a_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)>0, \quad a_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)>0.$$

Therefore,

$$\boldsymbol{\psi}_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\int\limits_{\Omega_n^0}\chi_{\Omega_n^{0+}}(\omega_n^2)\alpha_n(\{\omega_1,\ldots,\omega_{n-1},\omega_n\};\{\omega_1,\ldots,\omega_{n-1},\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2)=$$

$$S_{n-1}(\omega_1,\ldots,\omega_{n-1})\alpha_n^1(\omega_n)\int\limits_{\Omega_n^0}\chi_{\Omega_n^{0+}}(\omega_n^2)\alpha_n^2(\omega_n)\times$$

$$\frac{a_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^2)>0, \quad (\omega_1,\ldots,\omega_n)\in\Omega_{n-1}\times\Omega_n^{0-}. \tag{108}$$

Analogously,

$$\psi_n^2(\omega_1,\dots,\omega_{n-1},\omega_n) =$$

$$\int\limits_{\Omega_n^0} \chi_{\Omega_n^{0-}}(\omega_n^1)\alpha_n(\{\omega_1,\dots,\omega_{n-1},\omega_n^1\};\{\omega_1,\dots,\omega_{n-1},\omega_n\})\times$$

$$\frac{\Delta S_n^-(\omega_1,\dots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1) =$$

$$S_{n-1}(\omega_1,\dots,\omega_{n-1})\alpha_n^2(\omega_n)\int\limits_{\Omega_n^0} \chi_{\Omega_n^{0-}}(\omega_n^1)\alpha_n^1(\omega_n^1)\times$$

$$\frac{a_n(\omega_1,\dots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{V_n(\omega_1,\dots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1) > 0, \quad (\omega_1,\dots,\omega_n) \in \Omega_{n-1}\times\Omega_n^{0+}. \quad (109)$$

From these inequalities, we obtain

$$\psi_n(\omega_1,\dots,\omega_{n-1},\omega_n) > 0, \quad (\omega_1,\dots,\omega_n) \in \Omega_n. \quad (110)$$

This proves the equivalence of the measures P_N and μ_0.

Theorem 5. Suppose that the conditions of Lemma 2 are true. Then, the set of strictly positive random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2)$, $n=\overline{1,N}$, *satisfying the conditions*

$$E^{\mu_0}|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)| =$$

$$\int\limits_{\Omega_N}\prod_{i=1}^N \psi_i(\omega_1,\dots,\omega_i)|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|\prod_{i=1}^N dP_i^0(\omega_i) < \infty, \quad n=\overline{1,N}, \quad (111)$$

is a nonempty one and the convex linear span of the set of measures (104), defined by the random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, $n=\overline{1,N}$, *satisfying the conditions (111), is a set of martingale measures being equivalent to the measure* P_N.

Proof. All bounded random values $\alpha_n(\{\omega_1^1,\dots,\omega_n^1\};\{\omega_1^2,\dots,\omega_n^2\})$, $n=\overline{1,N}$, constructed in Lemma 2 satisfy the conditions (111), since $|\Delta S_n(\omega_1,\dots,\omega_{n-1},\omega_n)|$ takes only finite values. The fact that the measures (104) are martingale ones is proved as early (see [1]).

Lemma 4. Suppose that the conditions of Lemma 2 are valid. If, on the probability space $\{\Omega_{n-1},\mathcal{F}_{n-1},\mu_0^{n-1}\}$, *for each* $B\in\mathcal{F}_{n-1}$, $\mu_0^{n-1}(B)>0$, *the nonnegative random value* $f_n(\omega_1,\dots,\omega_{n-1},\omega_n)$ *satisfies the inequality*

$$\frac{1}{\mu_0^{n-1}(B)}\int\limits_B\int\limits_{\Omega_n^0}\prod_{i=1}^n \psi_i(\omega_1,\dots,\omega_i)f_n(\omega_1,\dots,\omega_n)\prod_{i=1}^n dP_i^0(\omega_i) \le 1, \quad B\in\mathcal{F}_{n-1}, \quad (112)$$

then the inequality

$$\int\limits_{\Omega_n^0} \psi_n(\omega_1,\ldots,\omega_n) f_n(\omega_1,\ldots,\omega_n) dP_n^0(\omega_n) \leq 1,$$

$$\{\omega_1,\ldots,\omega_{n-1}\} \in \Omega_{n-1}, \ \ n=\overline{1,N}, \tag{113}$$

is true.

Proof. The proof see in [1].

Theorem 6. Suppose that for $\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)$, $n=\overline{1,N}$, *the representation (13) is valid and Lemma 4 conditions are true. Then, for the nonnegative random value* $f_n(\omega_1,\ldots,\omega_{n-1},\omega_n)$ *the inequalities*

$$\chi_{\Omega_n^{0-}}(\omega_n^1)\chi_{\Omega_n^{0+}}(\omega_n^2)\left[\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1) + \right.$$

$$\left. \frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\right] \leq 1,$$

$$(\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \quad (\omega_n^1,\omega_n^2) \in \Omega_n^{0-}\times\Omega_n^{0+}, \quad n=\overline{1,N}, \tag{114}$$

are true.

Proof. The proof see in [1].

Theorem 7. Suppose that the conditions of Theorem 6 are true. Then, the non-negative random values $f_n(\omega_1,\ldots,\omega_{n-1},\omega_n)$, $n=\overline{1,N}$, *satisfy the inequalities*

$$f_n(\omega_1,\ldots,\omega_{n-1},\omega_n) \leq$$

$$(1+\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)), \quad n=\overline{1,N}, \tag{115}$$

where $\gamma_{n-1}(\omega_1,\ldots,\omega_{n-1})$ *is a bounded* $\mathcal{F}_{n-1}$*-measurable random value.*

Proof. It is evident that there exists $\omega_n^1 \in \Omega_n^{0-}$ and $\omega_n^2 \in \Omega_n^{0+}$ such that the inequalities

$$\max_{(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}} \frac{1}{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)} < \infty,$$

$$\max_{(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}} \frac{1}{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)} < \infty, \quad n=\overline{1,N}, \tag{116}$$

are true. This proves Theorem 7 (see [1]).

Theorem 8. Suppose that the evolution $\{S_n(\omega_1,\ldots,\omega_n)\}_{n=1}^N$ *of risky asset satisfies the conditions of Theorems 5, 6, 7, then for every nonnegative super-martingale* $\{f_n^1(\omega_1,\ldots,\omega_n)\}_{n=0}^N$ *relative to the set of martingale measure* M, *described in Theorem 5, the optional decomposition is true.*

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Proof. The proof see in [1]. More detail about optional decomposition see in [25], [26], [28] [27], [29].

Let us consider the random values

$$\psi_n(\omega_1, \ldots, \omega_n) = \chi_{\Omega_n^{0-}}(\omega_n)\, \psi_n^1(\omega_1, \ldots, \omega_n) +$$

$$\chi_{\Omega_n^{0+}}(\omega_n) \psi_n^2(\omega_1, \ldots, \omega_n), \tag{117}$$

where

$$\psi_n^1(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\frac{\Delta S_n^+(\omega_1, \ldots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \tag{118}$$

$$\psi_n^2(\omega_1, \ldots, \omega_{n-1}, \omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1, \ldots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \ldots, \omega_{n-1}) \in \Omega_{n-1}, \quad n = \overline{1, N}. \tag{119}$$

Definition 1. *Let the evolution of risky assets be given by the formula (8). On the measurable space $\{\Omega_N, \mathcal{F}_N\}$, being the direct product of the measurable spaces $\{\Omega_i^0, \mathcal{F}_i^0\}$, for every point $\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}\} \in \prod_{i=1}^{N} [\Omega_i^{0-} \times \Omega_i^{0+}]$ let us introduce the spot measure*

$$\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \ldots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{120}$$

where $\psi_n(\omega_1, \ldots, \omega_n)$ is determined by the formulas (117) - (119).

Lemma 5. *The spot measure $\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)$, given by the formula (120), is a martingale measure for the evolution of risky asset given by the formula (8) for every point $\{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}\} \in \prod_{i=1}^{N} [\Omega_i^{0-} \times \Omega_i^{0+}]$. If the point $\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}$ is such that $\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n^1) < 0,\ \Delta S_n(\omega_1, \ldots, \omega_{n-1} \omega_n^2) > 0,\ \{\omega_1, \ldots, \omega_{n-1}\} \in \Omega_{n-1},\ n = \overline{1, N}$, then the spot measure $\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)$ is a martingale measure being equivalent to the measure P_N.*

Proof. Let us prove that $\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)$ is a probability measure. Let us calculate

$$\sum_{i_j=1}^{2} \psi_j(\omega_1^{i_1}, \ldots, \omega_j^{i_j}) = \psi_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^1) + \psi_j(\omega_1^{i_1}, \ldots, \omega_{j-1}^{i_{j-1}}, \omega_j^2) =$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\boldsymbol{\psi}_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\boldsymbol{\psi}_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^1)\boldsymbol{\psi}_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^1)+$$

$$\chi_{\Omega_j^{0-}}(\omega_j^2)\boldsymbol{\psi}_j^1(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2)+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\boldsymbol{\psi}_j^2(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}}\omega_j^2) =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^{+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{\overset{i}{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^1)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^{-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}+$$

$$\chi_{\Omega_j^{0-}}(\omega_j^2)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^{+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^{-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\frac{\Delta S_j^{+}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}+$$

$$\chi_{\Omega_j^{0+}}(\omega_j^2)\chi_{\Omega_j^{0-}}(\omega_j^1)\frac{\Delta S_j^{-}(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)} = \chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2) = 1.$$

The last equalities prove that $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(\Omega_N) = 1$ for every point $\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\} \in \prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$. Further,

$$\sum_{i_j=1}^{2}\boldsymbol{\psi}_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\Delta S_j(\omega_1^{i_1},\ldots,\omega_j^{i_j}) =$$

$$\boldsymbol{\psi}_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)\Delta S_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+$$

$$\boldsymbol{\psi}_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\Delta S_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2) =$$

$$\chi_{\Omega_j^{0-}}(\omega_j^1)\chi_{\Omega_j^{0+}}(\omega_j^2)\times$$

$$\left[-\frac{\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^2)}\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)+\right.$$

$$\left.\frac{\Delta S_j^-(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1)}{V_j(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^1,\omega_j^1)}\Delta S_j^+(\omega_1^{i_1},\ldots,\omega_{j-1}^{i_{j-1}},\omega_j^2)\right]=0,\quad j=\overline{1,N}. \qquad (121)$$

Let us prove that the set of measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ is a set of martingale measures. Really, for A, belonging to the σ-algebra $\mathcal{F}_{n-1}$ of the filtration we have $A = B\times\prod_{i=n}^{N}\Omega_i^0$, where B belongs to σ-algebra $\mathcal{F}_{n-1}$ of the measurable space $\{\Omega_{n-1},\mathcal{F}_{n-1}\}$. Then,

$$\int\limits_A \Delta S_n(\omega_1,\ldots,\omega_n)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{j=1}^{N}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\ldots,\omega_n^{i_n}) =$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_n=1}^{2}\prod_{j=1}^{n}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}})\Delta S_n(\omega_1^{i_1},\ldots,\omega_n^{i_n}) =$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_{n-1}=1}^{2}\prod_{j=1}^{n-1}\psi_j(\omega_1^{i_1},\ldots,\omega_j^{i_j})\chi_B(\omega_1^{i_1},\ldots,\omega_{n-1}^{i_{n-1}})\times$$

$$\sum_{i_n=1}^{2}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})\Delta S_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})=0,\quad A\in\mathcal{F}_{n-1}. \qquad (122)$$

To prove the last statement it needs to prove that $\psi_n(\omega_1,\ldots,\omega_n)>0,\ n=\overline{1,N}$. But,

$$\psi_n(\omega_1,\ldots,\omega_n)=\chi_{\Omega_n^{0-}}(\omega_n)\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}+$$

$$\chi_{\Omega_n^{0+}}(\omega_n)\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}>0,\quad n=\overline{1,N}. \qquad (123)$$

The last means the needed statement.

We remind that the evolution of risky asset is given by the formula (8). Therefore, in this case the condition (16) (see [1]) is formulated, as follows:

$$\int\limits_{\Omega_n^0\times\Omega_n^0}\chi_{\Omega_n^{0-}}(\omega_n^1)\chi_{\Omega_n^{0+}}(\omega_n^2)\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$dP_n^0(\omega_n^1)dP_n^0(\omega_n^2) = 1, \quad n = \overline{1,N}. \tag{124}$$

Below, we describe the convex set of equivalent martingale measures.

Theorem 9. *The measure $\mu_0(A)$, constructed by the strictly positive finite valued random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2), n = \overline{1,N}$, satisfying the conditions (124), is a martingale measure for the evolution of risky asset, given by the formula (8). Every measure, belonging to the convex linear span of such measures, is also martingale measure for the considered evolution of risky asset. They are equivalent to the measure P_N.*

Proof. Since the set of strictly positive finite valued random values $\alpha_n(\{\omega\}_n^1; \{\omega\}_n^2), n = \overline{1,N}$, satisfies the conditions

$$E^{\mu_0}|\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)| =$$

$$\int\limits_{\Omega_N} \prod_{i=1}^{N} \psi_i(\omega_1, \ldots, \omega_i)|\Delta S_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)| \prod_{i=1}^{N} dP_i^0(\omega_i) < \infty, \quad n = \overline{1,N}, \tag{125}$$

then the set of measures $\mu_0(A)$, given by the formula (111), is a non empty one. This proves Theorem 9.

We use for $\alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\})$ the denotation $\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)$.

Theorem 10. *Let the evolution of risky asset be given by the formula (8). On the measurable space with measure $\{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}], \prod_{i=1}^{N}[\mathcal{F}_i^{0-} \times \mathcal{F}_i^{0+}], \prod_{i=1}^{N}[P_i^{0-} \times P_i^{0+}]\}$, suppose that the random value $\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)$, satisfies the conditions*

$$\alpha_N(\{\omega\}_N^1; \{\omega\}_N^2) > 0, \quad \{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\} \in \prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}], \tag{126}$$

$$\int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega_1^1, \ldots, \omega_N^1\}; \{\omega_1^2, \ldots, \omega_N^2\}) \prod_{i=1}^{N} dP_i^0(\omega_i^1)dP_i^0(\omega_i^2) = 1. \tag{127}$$

The measure $\mu_0(A)$, given by the formula

$$\mu_0(A) =$$

$$\int\limits_{\prod_{i=1}^{N}[\Omega_i^{0-} \times \Omega_i^{0+}]} \alpha_N(\{\omega\}_N^1; \{\omega\}_N^2)\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)d\prod_{i=1}^{N}[P_i^{0-} \times P_i^{0+}], \tag{128}$$

is a martingale measure, being equivalent to the measure P_N.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Proof. Let us note that $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(\Omega_N)=0$ if $\{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}\}$ does not belong to the set $\prod_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$. Let us introduce the denotations

$$\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\}) = \tag{129}$$

$$\frac{\int\limits_{\prod_{i=n+1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})\prod_{i=n+1}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)}{\int\limits_{\prod_{i=n}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]}\alpha_N(\{\omega_1^1,\ldots,\omega_N^1\};\{\omega_1^2,\ldots,\omega_N^2\})\prod_{i=n}^{N}dP_i^0(\omega_i^1)dP_i^0(\omega_i^2)},\ n=\overline{1,N}.$$

Since the random values $\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})$ are finite valued, then

$$\int\limits_{\Omega_n^{0-}\times\Omega_n^{0+}}\alpha_n(\{\omega_1^1,\ldots,\omega_{n-1}^1,\omega_n^1\};\{\omega_1^2,\ldots,\omega_{n-1}^2,\omega_n^2\})\times$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}dP_n^0(\omega_n^1)dP_n^0(\omega_n^2)<\infty,$$

$$(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}. \tag{130}$$

It is evident that the set of strictly positive finite valued random values $\alpha_n(\{\omega\}_n^1;\{\omega\}_n^2), n=\overline{1,N}$, given by the formula (129), satisfy the conditions

$$E^{\mu_0}|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)| =$$

$$\int\limits_{\Omega_N}\prod_{i=1}^{N}\psi_i(\omega_1,\ldots,\omega_i)|\Delta S_n(\omega_1,\ldots,\omega_{n-1},\omega_n)|\prod_{i=1}^{N}dP_i^0(\omega_i)<\infty,\quad n=\overline{1,N}. \tag{131}$$

Moreover, for the measure (128) the representation (104) is true, meaning that it is equivalent to the measure P_N. The last proves Theorem 10.

Let us define the integral for the random value $f_N(\omega_1,\ldots,\omega_{N-1},\omega_N)$ relative to the measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ by the formula

$$\int\limits_{\Omega_N} f_N(\omega_1,\ldots,\omega_{N-1},\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} =$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})f_N(\omega_1^{i_1},\ldots,\omega_N^{i_N}). \tag{132}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Theorem 11. *Let the evolution of risky asset be given by the formula (8). If the conditions of Theorem 10 are true, then the fair price of super-hedge for the non-negative payoff function $f(x)$ is given by the formula*

$$f_0 = \sup_{P \in M} E^P f(S_N) = \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}. \tag{133}$$

Proof. Let us prove the formula (133). Denote M the set of all martingale measures, being equivalent to P_N. If an equivalent martingale measure $P_0 \in M$, then $\alpha P_0 + (1-\alpha)\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \in M$ for arbitrary $0 < \alpha \leq 1$. We have the inequality

$$\alpha E^{P_0} f(S_N) + (1-\alpha) \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \leq \sup_{P \in M} E^P f(S_N).$$

Since $\alpha > 0$ is arbitrary, we obtain the inequality

$$\int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \leq \sup_{P \in M} E^P f(S_N).$$

From here, we obtain the inequality

$$\max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \leq \sup_{P \in M} E^P f(S_N).$$

The inverse inequality follows from the representation (128) for any martingale measure being equivalent to the measure P_N.

VI. Models of Non-Arbitrage Incomplete Financial Markets

Using the construction of the family of spot measures introduced in the previous section, this section presents the conditions under which the considered family of spot measures is invariant with respect to a certain class of evolutions of risky assets. For a certain class of contingent liabilities including a standard call option, the fair price of the super hedge is shown to be less than the spot price of the underlying asset. Specific applications of the results obtained for the previously known evolutions of risky assets are considered. New formulas are found for the non-arbitrage price range. A model of a non-arbitrage incomplete market is proposed and estimates are obtained in the case of a multi-parameter model of a non-arbitrage market.

On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, let us assume that the random values $b_i(\omega_1, \ldots, \omega_{i-1})$, $f_i(\omega_1, \ldots, \omega_i)$, $\eta_i(\omega_i)$, $i = \overline{1,N}$, satisfy the inequalities

$$b_i(\omega_1, \ldots, \omega_{i-1}) > 0, \quad f_i(\omega_1, \ldots, \omega_i) > 0,$$

$$\max_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} b_i(\omega_1, \ldots, \omega_{i-1}) <$$

$$\frac{1}{\max\limits_{\{\omega_1, \ldots, \omega_{i-1}\} \in \Omega_{i-1}} \max\limits_{\{\omega_i, \eta_i(\omega_i) < 0\}} f_i(\omega_1, \ldots, \omega_i) \eta_i^-(\omega_i)},$$

$$P_i^0(\eta_i(\omega_i) < 0) > 0, \quad P_i^0(\eta_i(\omega_i) > 0) > 0, \quad i = \overline{1,N}. \tag{134}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

As before, we put $\Omega_i^{0-}=\{\omega_i\in\Omega_i^0,\ \eta_i(\omega_i)\leq 0\},\ \ \Omega_i^{0+}=\{\omega_i\in\Omega_i^0,\ \eta_i(\omega_i)>0\}$. We assume that the evolution $S_n(\omega_1,\ldots,\omega_n)$ of risky asset is given by the formula

$$S_n(\omega_1,\ldots,\omega_n)=S_0\prod_{i=1}^{n}(1+b_i(\omega_1,\ldots,\omega_{i-1})f_i(\omega_1,\ldots,\omega_i)\eta_i(\omega_i)),\quad n=\overline{1,N}. \tag{135}$$

With every point $v=\{(\omega_1^1,\omega_1^2),\ldots,(\omega_N^1,\omega_N^2)\}\in\mathcal{V}$, where $\mathcal{V}=\prod\limits_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$, we connect the spot measure

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)=$$

$$\sum_{i_1=1}^{2}\ldots\sum_{i_N=1}^{2}\prod_{n=1}^{N}\psi_n(\omega_1^{i_1},\ldots,\omega_n^{i_n})\chi_A(\omega_1^{i_1},\ldots,\omega_N^{i_N}),\quad A\in\mathcal{F}_N. \tag{136}$$

Let us denote $\nu_v(A)=\prod\limits_{i=1}^{N}\nu_{\omega_i^1,\omega_i^2}(A_i),\ \ A=\prod\limits_{i=1}^{N}A_i,\in\mathcal{F}_N,$ the direct product of the measures $\nu_{\omega_i^1,\omega_i^2}(A_i),\ \ A_i\in\mathcal{F}_i^0,\ \ i=\overline{1,N}$, where $v=\{(\omega_1^1,\omega_1^2),\ldots,(\omega_N^1,\omega_N^2)\}\in\mathcal{V}$, $\mathcal{V}=\prod\limits_{i=1}^{N}[\Omega_i^{0-}\times\Omega_i^{0+}]$, and

$$\nu_{\omega_i^1,\omega_i^2}(A_i)=\chi_{A_i}(\omega_i^1)\frac{\eta_i^+(\omega_i^2)}{\eta_i^-(\omega_i^1)+\eta_i^+(\omega_i^2)}+\chi_{A_i}(\omega_i^2)\frac{\eta_i^-(\omega_i^1)}{\eta_i^-(\omega_i^1)+\eta_i^+(\omega_i^2)}, \tag{137}$$

for $\omega_i^1\in\Omega_i^{0-}$, $\omega_i^2\in\Omega_i^{0+}$, $A_i\in\mathcal{F}_i^0$. Then, there exists a countable additive function $\nu_v(A), A\in\mathcal{F}_N$, on the σ-algebra $\mathcal{F}_N$ for every $v\in\mathcal{V}$.

Theorem 12. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, let the evolution of risky asset be given by the formula (135). For every point $v=\{(\omega_1^1,\omega_1^2),\ldots,(\omega_N^1,\omega_N^2)\}\in\mathcal{V}$, the spot measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ given by the formula (136) does not depend on the random values $b_i(\omega_1,\ldots,\omega_{i-1}),\ \ i=\overline{1,N}$. In the case as $f_i(\omega_1,\ldots,\omega_i)=1,\ \ i=\overline{1,N}$, the formula

$$\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)=\nu_v(A) \tag{138}$$

is true. For the evolution of risky asset (135), the set of martingale measures being equivalent to the measure P_N does not depend on the random values $b_i(\omega_1,\ldots,\omega_{i-1}),\ \ i=\overline{1,N}$.

Proof. Since the spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ are given by the formula (136), to prove Theorem 12 it needs to prove that any $\psi_n(\omega_1,\ldots,\omega_n),\ n=\overline{1,N}$, does not depend on the random values $b_i(\omega_1,\ldots,\omega_{i-1}),\ i=\overline{1,N}$. Really,

$$\psi_n(\omega_1,\ldots,\omega_n)=\chi_{\Omega_n^-}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^1(\omega_1,\ldots,\omega_n)+$$

$$\chi_{\Omega_n^+}(\omega_1,\ldots,\omega_{n-1},\omega_n)\psi_n^2(\omega_1,\ldots,\omega_n), \tag{139}$$

$$\psi_n^1(\omega_1,\ldots,\omega_{n-1},\omega_n)=$$

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}, \tag{140}$$

$$\psi_n^2(\omega_1,\ldots,\omega_{n-1},\omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)}, \quad (\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}. \tag{141}$$

But,

$$\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2) =$$

$$S_{n-1}(\omega_1,\ldots,\omega_{n-1})b_n(\omega_1,\ldots,\omega_{n-1})f_n(\omega_1,\ldots,\omega_n^2)\eta_n^+(\omega_n^2), \tag{142}$$

$$\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1) =$$

$$S_{n-1}(\omega_1,\ldots,\omega_{n-1})b_n(\omega_1,\ldots,\omega_{n-1})f_n(\omega_1,\ldots,\omega_n^1)\eta_n^-(\omega_n^1). \tag{143}$$

Therefore,

$$\frac{\Delta S_n^+(\omega_1,\ldots,\omega_{n-1},\omega_n^2)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} =$$

$$\frac{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)}{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2) + f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \tag{144}$$

$$\frac{\Delta S_n^-(\omega_1,\ldots,\omega_{n-1},\omega_n^1)}{V_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1,\omega_n^2)} =$$

$$\frac{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2) + f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \tag{145}$$

$$(\omega_1,\ldots,\omega_{n-1}) \in \Omega_{n-1}.$$

From this, all the rest statements of Theorem 12 follow.

Theorem 13. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *let the evolution of risky asset be given by the formula (135). Suppose that the nonnegative convex down payoff function* $f(x)$ *on the set* $0 \leq x < \infty$ *satisfies the inequality* $0 \leq f(x) < x$. *Then, the inequalities*

$$f(S_0) \leq \sup_{P \in M} E^P f(S_N) =$$

$$\max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} < S_0 \tag{146}$$

are true.

Proof. Since the set of points $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\}$ in the set $\mathcal{V}$ is finite then the minimum in the formula

$$\min_{\omega_1,\ldots,\omega_N} [S_N(\omega_1, \ldots, \omega_N) - f(S_N(\omega_1, \ldots, \omega_N))] = d > 0 \tag{147}$$

is reached at a certain point $v_0 = \{(\omega_1^{1,0}, \omega_1^{2,0}), \ldots, (\omega_N^{1,0}, \omega_N^{2,0})\}$. Therefore, the inequality

$$S_N(\omega_1, \ldots, \omega_N) - f(S_N(\omega_1, \ldots, \omega_N)) \geq d, \quad \{\omega_1, \ldots, \omega_N\} \in \Omega_N, \tag{148}$$

is true

Integrating left and right parts of inequality over the measure $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$, we have

$$\int\limits_{\Omega_N} S_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} -$$

$$\int\limits_{\Omega_N} d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} f(S_N(\omega_1, \ldots, \omega_N)) \geq d. \tag{149}$$

Since

$$\int\limits_{\Omega_N} S_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} = S_0 \tag{150}$$

we obtain the needed. It is evident that from the convexity down of payoff function $f(x)$ and Jensen inequality we obtain the inequality

$$\int\limits_{\Omega_N} f(S_N(\omega_1, \ldots, \omega_N)) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \geq f(S_0). \tag{151}$$

Theorem 13 is proved.

Let us note that the interval of non arbitrage prices for a certain processes was found in the papers [30], [31].

Corollary 2. For the standard call option of European type with payoff function $f(x) = (x-K)^+, K > 0$, the conditions of Theorem 13 are true. Therefore, the inequalities (146) are valid.

Theorem 14. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula (135). Suppose that the nonnegative convex down payoff function $f(x)$ on the set $0 \leq x < \infty$ satisfies the inequality $0 \leq f(x) \leq K$, $K > 0$. Then, the inequalities

$$f(S_0) \leq \sup_{P \in M} E^P f(S_N) = \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int\limits_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}} \leq K \tag{152}$$

are true.

Proof. The proof is evident.

Corollary 3. *For the standard put option of European type with payoff function $f(x) = (K - x)^+, K > 0$, the conditions of Theorem 14 are true. Therefore, the inequalities (152) are valid.*

Corollary 4. *For the standard call option of European type with payoff function $f(x) = (x - K)^+, K > 0$, the interval of non arbitrage prices coincide with the interval*

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$
$$\left. \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right). \quad (153)$$

Corollary 5. *For the standard put option of European type with payoff function $f(x) = (K - x)^+, K > 0$, the interval of non arbitrage prices coincide with the interval*

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$
$$\left. \max_{\omega_i^1 \in \Omega_i^{0-}, \omega_i^2 \in \Omega_i^{0+}, i=\overline{1,N}} \int_{\Omega_N} f(S_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right). \quad (154)$$

Corollary 6. *On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset is given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i)), \quad n = \overline{1, N}, \quad (155)$$

where the random value $\rho_i(\omega_i)$ is given on the probability space $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1, N}$, and let the bank account evolution be given by the formula

$$B_n = B_0(1 + r)^n, \quad r > 0, \quad B_0 > 0, \quad n = \overline{1, N}. \quad (156)$$

Then, for the discount evolution of risky asset

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod_{i=1}^{n} (1 + \rho_i(\omega_i))}{B_0(1 + r)^n}, \quad n = \overline{1, N}, \quad (157)$$

the set of martingale measure is nonempty one if the following conditions are true

$$P_i^0(\rho_i(\omega_i) - r < 0) > 0, \; P_i^0(\rho_i(\omega_i) - r > 0) > 0,$$

$$P_i^0(\rho_i(\omega_i) - r < 0) + P_i^0(\rho_i(\omega_i) - r > 0) = 1, \quad i = \overline{1, N}.$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

For every point $v = \{(\omega_1^1, \omega_1^2), \ldots, (\omega_N^1, \omega_N^2)\}$ *in the set* $\mathcal{V}$ *the spot measure* $\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}(A)$ *is a direct product of measures* $\mu_0^i(A_i)$, $A_i \in \mathcal{F}_i^0$, $i = \overline{1, N}$, *given on the measurable space* $\{\Omega_i^0, \mathcal{F}_i^0\}$, *where* $\mu_0^i(A_i) = \nu_{\omega_i^1, \omega_i^2}(A_i)$, *and* $\nu_{\omega_i^1, \omega_i^2}(A_i)$ *is given by the formula (137) with* $\eta_i(\omega_i) = \frac{\rho_i(\omega_i) - r}{1+r}$, $i = \overline{1, N}$. *The fair price* φ_0 *of super-hedge of the nonnegative contingent liability* $\varphi_N(\omega_1, \ldots, \omega_N)$ *is given by the formula*

$$\varphi_0 = \max_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v, \ \max_{v \in \mathcal{V}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\nu_v \right).$$

Theorem 15. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$ *being the direct product of the probability spaces* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, *let the evolution of risky asset be given by the formula*

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^n (1 + \rho_i(\omega_i)), \quad n = \overline{1, N}, \tag{158}$$

where the random value $\rho_i(\omega_i)$, *is given on the probability space* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, $P_i^0(\{\rho_i(\omega_i) < 0\}) > 0$, $P_i^0(\{\rho_i(\omega_i) > 0\}) > 0$, $i = \overline{1, N}$, *and let the bank account evolution be given by the formula*

$$B_n = B_0 \prod_{i=1}^n (1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1, N}, \tag{159}$$

where the strictly positive random values $r_i(\omega_i)$ *are given on the probability* $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}, i = \overline{1, N}$. *Then, for the discount evolution of risky asset*

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod\limits_{i=1}^n (1 + \rho_i(\omega_i))}{B_0 \prod\limits_{i=1}^n (1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1, N}, \tag{160}$$

the set of martingale measure is nonempty one if the following conditions are true

$$\max_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) < \min_{\omega_i \in \Omega_i, \rho_i(\omega_i) > 0} \rho_i(\omega_i),$$

$$\min_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) > 0, \ i = \overline{2, N}$$

$$0 < r_0 < \min_{\omega_1 \in \Omega_1, \rho_1(\omega_1) > 0} \rho_1(\omega_1). \tag{161}$$

The fair price φ_0 *of super-hedge of the nonnegative contingent liability* $\varphi_N(\omega_1, \ldots, \omega_N)$ *is given by the formula*

$$\varphi_0 = \max_{\omega_i^1 \in \Omega_i^{0-}, \ \omega_i^2 \in \Omega_i^{0+}, \ i = \overline{1, N}} \int_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{\omega_i^1 \in \Omega_i^{0-},\ \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}} \int\limits_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}}, \right.$$

$$\left. \max_{\omega_i^1 \in \Omega_i^{0-},\ \omega_i^2 \in \Omega_i^{0+},\ i=\overline{1,N}} \int\limits_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} \right).$$

Proof. The discount evolution (160) can be represented in the form

$$S_n(\omega_1, \ldots, \omega_n) =$$

$$\frac{S_0}{B_0}\left(1 + \frac{(\rho_1(\omega_1) - r_0)}{1 + r_0}\right) \prod_{i=2}^{n} \left(1 + \frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{1 + r_{i-1}(\omega_{i-1})}\right) =$$

$$\frac{S_0}{B_0} \prod_{i=1}^{n} (1 + f_i(\omega_1, \ldots, \omega_i)\eta_i(\omega_i)), \tag{162}$$

where

$$f_1(\omega_1) = \frac{1}{1 + r_0}, \quad \eta_1(\omega_1) = \rho_1(\omega_1) - r_0, \tag{163}$$

$$f_i(\omega_1, \ldots, \omega_i) = \frac{\rho_i(\omega_i) - r_{i-1}(\omega_{i-1})}{(\rho_i(\omega_i) - \min\limits_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1})(1 + r_{i-1}(\omega_{i-1}))},$$

$$\eta_i(\omega_i) = \rho_i(\omega_i) - \min_{\omega_{i-1} \in \Omega_{i-1}} r_{i-1}(\omega_{i-1}) \quad i = \overline{2, N}. \tag{164}$$

Since

$$f_i(\omega_1, \ldots, \omega_i) > 0, \quad i = \overline{1, N}, \tag{165}$$

$$P_i^0(\eta_i(\omega_i) < 0) > 0, \quad P_i^0(\eta_i(\omega_i) > 0) > 0, \quad i = \overline{1, N}, \tag{166}$$

then it means that the set of martingale measures being equivalent to R_N is a nonempty set. Theorem 15 is proved.

Theorem 16. On the probability space $\{\Omega_N, \mathcal{F}_N, P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0, \mathcal{F}_i^0, P_i^0\}$, let the evolution of risky asset be given by the formula

$$S_n^1((\omega_1, \ldots, \omega_n) = S_0 \prod_{i=1}^{n} e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}, \quad n = \overline{1, N}, \tag{167}$$

where the random values $\varepsilon_i(\omega_i)$, $i = \overline{1, N}$, are such that

$$P_i^0(\varepsilon_i(\omega_i) < 0) > 0, \quad P_i^0(\varepsilon_i(\omega_i) > 0) > 0,$$

$$P_i^0(\varepsilon_i(\omega_i) < 0) + P_i^0(\varepsilon_i(\omega_i) > 0) = 1,$$

$$\sigma_i(\omega_1, \ldots, \omega_{i-1}) \geq \sigma_i^0 > 0, \quad i = \overline{1, N},$$

and let the bank account evolution be given by the formula

$$B_n = B_0 \prod_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1})), \quad B_0 > 0, \quad n = \overline{1, N}, \tag{168}$$

where the random values $r_i(\omega_i)$, $i = \overline{1, N-1}$, *are strictly positive ones,* $r_0 > 0$. *Then, for the discount evolution of risky asset*

$$S_n((\omega_1, \ldots, \omega_n) = \frac{S_0 \prod\limits_{i=1}^{n} e^{\sigma_i(\omega_1, \ldots, \omega_{i-1})\varepsilon_i(\omega_i)}}{B_0 \prod\limits_{i=1}^{n} (1 + r_{i-1}(\omega_{i-1}))}, \quad n = \overline{1, N}, \tag{169}$$

the set of martingale measure is nonempty one, if

$$\exp\{\sigma_1^0 \max_{\{\omega_1, \varepsilon_1(\omega_1) < 0\}} \varepsilon_1(\omega_1)\} < r_0 < \exp\{\sigma_1^0 \min_{\{\omega_1, \varepsilon_1(\omega_1) > 0\}} \varepsilon_1(\omega_1)\},$$

$$\exp\{\sigma_i^0 \max_{\{\omega_i, \varepsilon_i(\omega_i) < 0\}} \varepsilon_i(\omega_i)\} < \min_{\{\omega_{i-1} \in \Omega_{i-1}\}} r_{i-1}(\omega_{i-1}) <$$

$$\max_{\{\omega_{i-1} \in \Omega_{i-1}\}} r_{i-1}(\omega_{i-1}) < \exp\{\sigma_i^0 \min_{\{\omega_i, \varepsilon_i(\omega_i) > 0\}} \varepsilon_i(\omega_i)\}, \quad i = \overline{2, N}. \tag{170}$$

Then, the fair price of super-hedge φ_0 *of the nonnegative contingent liability* $\varphi_N(\omega_1, \ldots, \omega_N)$ *is given by the formula*

$$\varphi_0 = \max_{v \in \mathcal{V}} \int\limits_{\Omega_N} \varphi_N(\omega_1, \ldots, \omega_N) d\mu_{\{\omega_1^1, \omega_1^2\}, \ldots, \{\omega_N^1, \omega_N^2\}} =$$

$$\max_{v \in \mathcal{V}} \sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \ldots, \omega_n^{i_n}) \varphi_N(\omega_1^{i_1}, \ldots, \omega_N^{i_N}). \tag{171}$$

Proof. For the discount evolution (169), the following representation

$$S_n((\omega_1, \ldots, \omega_n) =$$

$$\frac{S_0}{B_0} \prod_{i=1}^{n} \left(1 + f_i(\omega_1, \ldots, \omega_i)\eta_i(\omega_i)\right), \quad n = \overline{1, N}, \tag{172}$$

is true, where

$$\eta_1(\omega_1) = \exp\{\sigma_1^0 \varepsilon_1(\omega_1)\} - r_0, \quad f_1(\omega_1) = \frac{1}{1+r_0},$$

$$\eta_i(\omega_i) = \exp\{\sigma_i^0 \varepsilon_i(\omega_i)\} - \max_{\{\omega_{i-1}\in\Omega_{i-1}\}} r_{i-1}(\omega_{i-1}),$$

$$f_i(\omega_1, \dots, \omega_i) =$$

$$\frac{e^{\sigma_i(\omega_1,\dots,\omega_{i-1})\varepsilon_i(\omega_i)} - r_{i-1}(\omega_{i-1})}{(\exp\{\sigma_i^0 \varepsilon_i(\omega_i)\} - \max_{\{\omega_{i-1}\in\Omega_{i-1}\}} r_{i-1}(\omega_{i-1}))(1 + r_{i-1}(\omega_{i-1}))} > 0,$$

$$\{\omega_1, \dots, \omega_i\} \in \Omega_i, \quad i = \overline{2, N}. \tag{173}$$

In this case, the spot measures

$$\mu_{\{\omega_1^1, \omega_1^2\}, \dots, \{\omega_N^1, \omega_N^2\}}(A) =$$

$$\sum_{i_1=1}^{2} \dots \sum_{i_N=1}^{2} \prod_{n=1}^{N} \psi_n(\omega_1^{i_1}, \dots, \omega_n^{i_n}) \chi_A(\omega_1^{i_1}, \dots, \omega_N^{i_N}), \quad A \in \mathcal{F}_N, \tag{174}$$

figuring in the formula (171), are determined by the formulas

$$\psi_n(\omega_1, \dots, \omega_n) = \chi_{\Omega_n^-}(\omega_1, \dots, \omega_{n-1}, \omega_n) \psi_n^1(\omega_1, \dots, \omega_n) +$$

$$\chi_{\Omega_n^+}(\omega_1, \dots, \omega_{n-1}, \omega_n) \psi_n^2(\omega_1, \dots, \omega_n), \tag{175}$$

$$\psi_n^1(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\frac{\Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \tag{176}$$

$$\psi_n^2(\omega_1, \dots, \omega_{n-1}, \omega_n) =$$

$$\frac{\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)}, \quad (\omega_1, \dots, \omega_{n-1}) \in \Omega_{n-1}, \tag{177}$$

where

$$\frac{\Delta S_n^+(\omega_1, \dots, \omega_{n-1}, \omega_n^2)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1, \dots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2)}{f_n(\omega_1, \dots, \omega_{n-1}, \omega_n^2) \eta_n^+(\omega_n^2) + f_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1) \eta_n^-(\omega_n^1)}, \tag{178}$$

$$\frac{\Delta S_n^-(\omega_1, \dots, \omega_{n-1}, \omega_n^1)}{V_n(\omega_1, \dots, \omega_{n-1}, \omega_n^1, \omega_n^2)} =$$

$$\frac{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}{f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^2)\eta_n^+(\omega_n^2)+f_n(\omega_1,\ldots,\omega_{n-1},\omega_n^1)\eta_n^-(\omega_n^1)}, \tag{179}$$

$$(\omega_1,\ldots,\omega_{n-1})\in\Omega_{n-1}.$$

and the random values $\eta_i(\omega_i)$, $f_i(\omega_1,\ldots,\omega_i)$, $i=\overline{1,N}$, are given by the formulas (173). The obtained representation (172) proves Theorem 16.

Suppose that the random values $\eta_k(\omega_k)$, $f_k(\omega_1,\ldots,\omega_k)$, $k=\overline{1,N}$, and constants a_k^i satisfy the inequalities

$$0<a_k^i<\frac{1}{\max\limits_{\{\omega_1,\ldots,\omega_{k-1}\}\in\Omega_{k-1}}\ \max\limits_{\{\omega_k,\eta_k(\omega_k)<0\}} f_k(\omega_1,\ldots,\omega_k)\eta_k^-(\omega_k)}, \quad k=\overline{1,N}, \quad i=\overline{1,d},$$

$$f_i(\omega_1,\ldots,\omega_i)>0, \quad P_i^0(\eta_i(\omega_i)<0)>0, \quad P_i^0(\eta_i(\omega_i)>0)>0, \quad i=\overline{1,N}. \tag{180}$$

We assume that the evolutions of d risky assets $S_n(\omega_1,\ldots,\omega_n)$ is given by the formula

$$S_n(\omega_1,\ldots,\omega_n)=\{S_n^i(\omega_1,\ldots,\omega_n)\}_{i=1}^d, \tag{181}$$

where

$$S_n^i(\omega_1,\ldots,\omega_n)=S_0^i\prod_{k=1}^n(1+a_k^i f_k(\omega_1,\ldots,\omega_k)\eta_k(\omega_k)), \quad n=\overline{1,N}, \quad i=\overline{1,d}. \tag{182}$$

Proposition 5. On the probability space $\{\Omega_N,\mathcal{F}_N,P_N\}$, being the direct product of the probability spaces $\{\Omega_i^0,\mathcal{F}_i^0,P_i^0\}$, let the evolution of risky assets be given by the formulas (181), (182), where the random values $\eta_k(\omega_k)$, $f_k(\omega_1,\ldots,\omega_k)$ and constants a_k^i, $k=\overline{1,N}$, $i=\overline{1,d}$ satisfy the inequalities (180). For such an evolution of risky assets the set of martingale measures μ_0 does not depend on a_k^i. The spot measures $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ are determined by the formulas (174) - (179). The fair price φ_0^i of super-hedge of the nonnegative contingent liability $\varphi_N^i(\omega_1,\ldots,\omega_N)$ is given by the formula

$$\varphi_0^i=\max_{v\in\mathcal{V}}\int\limits_{\Omega_N}\varphi_N^i(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}, \quad i=\overline{1,d}.$$

The interval of non-arbitrage prices is written in the form

$$\left(\min_{v\in\mathcal{V}}\int\limits_{\Omega_N}\varphi_N^i(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}\ ,\right.$$

$$\left.\max_{v\in\mathcal{V}}\int\limits_{\Omega_N}\varphi_N^i(\omega_1,\ldots,\omega_N)d\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}\right), \quad i=\overline{1,d}.$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

In the case $f_k(\omega_1,\ldots,\omega_k)=1, k=\overline{1,N}$, *for every point* $v=\{(\omega_1^1,\omega_1^2),\ldots,(\omega_N^1,\omega_N^2)\}$ *in the set* $\mathcal{V}$ *the spot measure* $\mu_{\{\omega_1^1,\omega_1^2\},\ldots,\{\omega_N^1,\omega_N^2\}}(A)$ *is a direct product of measures* $\mu_0^i(A_i),\ A_i\in\mathcal{F}_i^0,\ i=\overline{1,N}$, *given on the measurable space* $\{\Omega_i^0,\mathcal{F}_i^0\}$, *where* $\mu_0^i(A_i)=\nu_{\omega_i^1,\omega_i^2}(A_i)$, *and* $\nu_{\omega_i^1,\omega_i^2}(A_i)$ *is given by the formula (137).*

Suppose that $\{g_k^i(X_N)\}_{k=1}^N,\ i=\overline{1,d}$, are the mappings from the set $[0,1]^N$ into itself, where $X_N=\{x_1,\ldots,x_N\},\ 0\le\ x_k\le 1,\ k=\overline{1,N}$. If $S_0^i,S_1^i,\ldots,S_N^i,\ i=\overline{1,d}$, are the samples of the processes (181), (182), let us denote the order statistics $S_{(0)}^i,S_{(1)}^i,\ldots,S_{(N)}^i,\ i=\overline{1,d}$, of this samples. Introduce also the denotation

$$g_k^i\left([S^i]_N\right)=g_k^i\left(\frac{S_{(0)}^i}{S_{(N)}^i},\ldots,\frac{S_{(N-1)}^i}{S_{(N)}^i}\right),\ k=\overline{1,N},\ i=\overline{1,d}.$$

Let us introduce the denotations

$$f_k^1=\max_{\{\omega_1,\ldots,\omega_{k-1}\}\in\Omega_{k-1},\ \omega_k^1\in\Omega_1^{0-}}f_k(\omega_1,\ldots,\omega_{k-1},\omega_k^1),\ k=\overline{1,N}.$$

Proposition 6. *Suppose that* $S_0^i,S_1^i,\ldots,S_N^i$ *is a sample of the random processes (181), (182). Then, for the parameters* $a_1^i,\ldots,a_N^i$ *the estimation*

$$a_1^i=\frac{\left[1-\tau_0^i\frac{S_{(0)}^i}{S_0^i}g_1^i\left([S^i]_N\right)\right]}{f_1^1\max\limits_{\omega_1^1\in\Omega_1^{0-}}\eta_1^-(\omega_1^1)},\quad 0<\tau_0^i\le 1,\quad i=\overline{1,d},$$

$$a_k^i=\frac{\left[1-\frac{g_k^i\left([S^i]_N\right)}{g_{k-1}^i\left([S^i]_N\right)}\right]}{f_k^1\max\limits_{\omega_k^1\in\Omega_k^{0-}}\eta_k^-(\omega_k^1)},\quad k=\overline{2,N},\quad i=\overline{1,d},\tag{183}$$

is valid, if for $g_N^i([S^i]_N)>0,\ [S^i]_N\in[0,1]^N$, *the inequalities* $g_1^i([S^i]_N)\ge g_2^i([S^i]_N)\ge\ldots\ge g_N^i([S^i]_N)$ *are true. If* $\tau_0^i=0$, *then* $a_k^i=1,\ k=\overline{1,N},\ i=\overline{1,d}$.

VII. Applications

In this section, we discuss the issue of applying the results obtained to real calculations of the range of non-arbitrage prices in the case of incomplete non-arbitrage markets. The first question that arises is what should be the evolution of risky assets when describing non-arbitrage markets. In this case, we must rely on the study of the evolution of stock index proposed in [22], [23], [24], that is,

$$S_n(\omega_1,\ldots,\omega_n)=S_0\prod_{i=1}^n e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)},\quad n=\overline{1,N},\tag{184}$$

where the random values $\sigma_i(\omega_1,\ldots,\omega_{i-1})\ge\sigma_i^0>0,\ i=\overline{1,N}$, and $P_i^0(\varepsilon_i(\omega_i)<0)>0,\ P_i^0(\varepsilon_i(\omega_i)>0)>0$, then such an evolution has the form (8) with

$$f_i(\omega_1,\ldots,\omega_i)=\frac{e^{\sigma_i(\omega_1,\ldots,\omega_{i-1})\varepsilon_i(\omega_i)}-1}{e^{\sigma_i^0\varepsilon_i(\omega_i)}-1},\quad \eta_i(\omega_i)=e^{\sigma_i^0\varepsilon_i(\omega_i)}-1,\quad i=\overline{1,N}.$$

satisfying needed conditions. Here, the random values $\sigma_i(\omega_1,\ldots,\omega_{i-1}),\ i=\overline{1,N}$, are conditional volatilities, $\varepsilon_i(\omega_i),\ i=\overline{1,N}$, are identically distributed random values.

Therefore, when modeling non-arbitrage securities markets, the evolution of the stock index should be described by formula (167). The evolution of shares quoted on the exchange should be described by parametric processes described by formulas (181), (182). The parameters of such a process are determined in accordance with the Proposition 6.

VIII. Conclusions

Section 3 contains the results related to the uniqueness of the set of martingale measures. In Lemma 1 it is shown that in the case of evolution of risky assets given by formula (11) there is only one spot martingale measure for the considered class of risky assets. A wide class of risky asset evolutions has been identified for modeling real processes in the financial market. In Theorem 1, necessary and sufficient conditions are given for the evolution of risky assets under which the martingale measure is the only one, and in Theorem 2 it is shown that it coincides with a point martingale measure.

In section 4, Proposition 1 formulates the conditions for the evolution of risky assets under which the martingale measure is the same for a wide class of evolutions of risky assets. Proposition 2 states that the considered securities market in Proposition 1 is complete and non-arbitrage and provides formulas for the fair values of the contingent liabilities.

A direct consequence of the considered results is Corollary 1 known as the Cox, Ross, Rubinstein model and Theorem 3 being the direct generalization of the above mentioned model. In Theorem 4, the conditions are found under which the discounted evolution can be represented in the form considered in the work. A formula is found for the fair price of the super-hedge in this realistic case. In Proposition 3, a parametric model of the complete non-arbitrage market is proposed and formulas for the fair prices of contingent liabilities are written out. Proposition 4 provides an assessment of the parameters of a complete non-arbitrage market model, which opens up opportunities for modeling processes in financial markets.

Section 5 presents the theoretical foundations of the incomplete non-arbitrage market model. In Lemmas 2 and 3, conditions for the evolution of risky assets are formulated for which the family of martingale measures is equivalent to the original one. It is shown in Theorem 5 that the family of measures constructed in Lemma 2 is a family of martingale measures equivalent to the original measure. In Lemma 4 and Theorems 6 and 7, estimates are found for nonnegative random variables that ensure the validity of the optional decomposition for nonnegative super-martingales with respect to all martingale measures presented in Theorem 8. In contrast to earlier results, the optional decomposition can be found explicitly here. Lemma 5 contains a result that introduced in Definition 1 the spot measure is a martingale one (see also in [1]).

Theorems 9 and 10 describe all martingale measures equivalent to the original measure. In the case under consideration, the conditions of Theorems 9 and 10 are not restrictive. In Theorem 11, a formula is found for the fair price of the super-hedge for random claims, which allows it to be calculated using a finite number of operations.

Section 6 presents possible models of incomplete non-arbitrage markets. For this, Theorem 12 shows that the set of spot measures does not depend on a certain type of evolution of risky assets and is one and the same set. Under certain simplified conditions, each spot measure is a direct product of the spot measures indicated in the theorem. Due to the finiteness of the set of spot measures in Theorem 13, it was found that for a certain class of contingent liabilities the super-hedge price is less than the initial price of the underlying asset. The range of non-arbitrage prices is found. Among these contingent liabilities is the standard European call option. Non-arbitrage price interval is found. Corollaries 6 and 15 provide examples of the evolution of risky assets.

Theorems 15 and 16 consider realistic models of the evolution of both risky and non-risky assets for which there is a finite family of point measures.

Proposition 5 presents a realistic parametric model of an incomplete non-arbitrage market and also presents formulas for the fair price of the super hedge and the range of non-arbitrage prices. In Proposition 6, estimates of the parameters of the incomplete non-arbitrage market model are found.

References Références Referencias

1. Gonchar, N. S. (2020) Derivatives pricing in Non-arbitrage market, Global Journal of Science Frontier Research, V.20 issue 14, P. 32-101.
2. Gonchar, N. S. (2001) Fund market and economic grows. Bogolyubov Institute for Theoretical Physics, Kiev (in Ukrainian).
3. Cox J., Ross R.A., Rubinstein M. (1976) Option pricing: a simpli˙ed approach, Journal of Financial Economics, V. 7, P. 229 - 263.
4. Bachelier L.(1900) Theorie de la speculation. Annales de l'Ecole Normal Superieure. V. 17, P. 21-86.
5. Black F., Scholes M. (1973) The pricing of options and corporate liabilities. Journal of Political Economy, V. 81, N 3, P. 637 - 659.
6. Merton R. S. (1973) Theory of rational option pricing. Bell Journal of Economics and Management Science. N 4 (Spring), P. 141 - 183.
7. Harrison J.M., Kreps D.M. (1979) Martingales and Arbitrage in Multiperiod Securities Markets. Journal of Economic Theory, 20, 381– 408. https://doi.org/10.1016/0022-0531(79)90043-?
8. Harrison J.M., Pliska S.R. (1981) Martingales and Stochastic Integrals in the Theory of Continuous Trading. Stochastic Processes and their Applications. 11, 215.
9. Dalang R.C., Morton A., Willinger W. (1990): Equivalent Martingale measures and no-arbitrage in stochastic securities market model. Stochastics and Stochastic Reports. 29, 185–201. https://doi.org/10.1080/17442 509008833613
10. Schachermayer, W. (2003): Introduction to the Mathematics of Financial Markets. In: S. Albeverio, W. Schachermayer, M. Talagrand: Springer Lecture Notes in Mathematics 1816 - Lectures on Probability Theory and Statistics, Saint-Flour summer school 2000 (Pierre Bernard, editor), Springer Verlag, Hei-delberg, pp. 111-177.
11. Delbaen, F., and Schachermayer, W.(2006): The Mathematics and Arbitrage. Berlin: Springer.
12. Rogers L.C. G. (1995) Equivalent martingale measures and no-arbitrage. Stochastics and Stochastics Reports, V. 51, P. 41 - 50.
13. Shiryaev A. N. (1998) Foundations of Stochastic Financial Mathematics. V.1 Theory. PHAZIS, Moskow (in Russian).
14. Shiryaev A. N. (1998) Foundations of Stochastic Financial Mathematics. V.2 Theory. PHAZIS, Moskow (in Russian).
15. Gonchar, N. S. (2008) Mathematical foundations of information economics. Bogolyubov Institute for Theoretical Physics, Kiev.
16. Gonchar, N. S. (2015): Mathematical Model of Banking Operation. Cybernetics and System Analysis, 51, 378-399. https://doi.org/ 10.1007/s 10559-015-9730-0
17. Gonchar, N. S., and Terentieva, L.S.(2008) Default Risk Valuation of the Firm with the Special Process of Internal Yield. Journal of Automation and Information Sciences, 40, 57-71. https://doi.org/ 10.1615/ AutomatInfScien. v40.18.60
18. Gonchar N.S. (2017) Banking and Risk Assessment. In: Jerzy, K., Ed., Banking: Services, Opportunity and Risks, Chapter 8, Nova Science Publisher, Inc., New York.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

19. Gonchar N.S. (2020): Assessment of contingent liabilities for risk assets evolutions built on Brownian motion. Advances in Pure Mathematics, 9, 259 - 296. https://doi.org/10.4236/apm.2020.105016
20. Gonchar N.S. (2019): Description of incomplete financial markets for time evolution of risk assets. Advances in Pure Mathematics, 10, 567 - 610. https://doi.org/10.4236/apm.2019.96029
21. Gonchar N.S. (2018): Martingales and super-martingales relative to a convex set of equivalent measures. Advances in Pure Mathematics, 8, 428 - 462. https://doi.org/10.4236/apm.2018.84025
22. Engle R. F. (1982) Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom ination. Econometrica V. 50, N 4, P 987-1008.
23. Bollerslev T. (1986) Generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, V. 31, P. 307-327.
24. Engle R. F., Bollerslev T. (1986) Moddeling the persistence of conditional variance. Econometrics Reviews, V. 5, P. 1-50.
25. El Karoui, N., and Quenez, M.C. (1995): Dynamic programming and pricing of contingent claims in an incomplete market. SIAM J. Control Optimizat., 33, 27-66.
26. Kramkov, D. O. (1996) Optional decomposition of super-martingales and hedging in incomplete security markets. Probab. Theory Relat. Fields, 105, 459-479. https://doi.org/ 10.1007/BF01191909
27. Follmer, H., and Kramkov, D.O. (1997): Optional decomposition theorem under constraints. Probability Theory and Related Fields, 109, 1-25. https://doi.org/10.1007/s004400050122
28. Follmer, H., and Kabanov, Yu. M.(1996): 'Optional decomposition theorems in discrete time,' in Atti del convegno in onore di Oliviero Lessi, Padova, 25-26 marzo, 47-68.
29. Follmer, H., and Kabanov, Yu. M. (1998): Optional decomposition and Lagrange multipliers. Finance Stochast., 2, 69-81.https://doi.org/10.1007/s 007800050033
30. Eberlein, E., Jacod, J. (1997): On the range of option price. Finance Stoch. 1.
31. Bellamy, N., Jeanblanc, M. (1999): Incompleteness of markets driven by mixed diffusion. Finance Stoch. 4, 209 - 222.

Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 21 Issue 4 Version 1.0 Year 2021
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Fluid Gauge Theory

By Tsutomu Kambe

University of Tokyo

Abstract- According to the general gauge principle, *Fluid Gauge Theory* is presented to cover a wider class of flow fields of a perfect fluid without internal energy dissipation under anisotropic stress field. Thus, the theory of fluid mechanics is extended to cover time dependent rotational flows under *anisotropic* stress field of a compressible perfect fluid, including turbulent flows. Eulerian fluid mechanics is characterized with isotropic pressure stress fields. The study is motivated from three observations. First one is experimental observations reporting large-scale structures coexisting with turbulent flow fields. This encourages a study of how such structures observed experimentally are possible in turbulent shear flows, Second one is a theoretical and mathematical observation: the "General solution to Euler's equation of motion" (found by Kambe in 2013) predicts a new set of four background-fields, existing in the linked $4d$-spacetime. Third one is a physical query, "what symmetry implies the current conservation law ?". The latter two observations encourage a gauge-theoretic formulation by defining a differential one-form representing the interaction between the fluid-current field $j\mu$ and a background field $a\mu$.

Keywords: anisotropic stress field, gauge principle, fluid gauge theory, current conservation, euler equation.

GJSFR-A Classification: FOR Code: 020399

Strictly as per the compliance and regulations of:

Fluid Gauge Theory

Tsutomu Kambe

Abstract- According to the general gauge principle, *Fluid Gauge Theory* is presented to cover a wider class of flow fields of a perfect fluid without internal energy dissipation under anisotropic stress field. Thus, the theory of fluid mechanics is extended to cover time dependent rotational flows under *anisotropic* stress field of a compressible perfect fluid, including turbulent flows. Eulerian fluid mechanics is characterized with isotropic pressure stress fields. The study is motivated from three observations. First one is experimental observations reporting large-scale structures coexisting with turbulent flow fields. This encourages a study of how such structures observed experimentally are possible in turbulent shear flows, Second one is a theoretical and mathematical observation: the "General solution to Euler's equation of motion" (found by Kambe in 2013) predicts a new set of four background-fields, existing in the linked 4*d*-spacetime. Third one is a physical query, "what symmetry implies the current conservation law ?". The latter two observations encourage a gauge-theoretic formulation by defining a differential one-form representing the interaction between the fluid-current field j^μ and a background field a_μ. A known relativistic action of a perfect fluid is introduced together with the interaction action just mentioned, and furthermore, a third gauge invariant action is defined to govern the field a_μ linearly in its free-state. The general gauge principle is applied to the combined system of the three actions to describe general time-dependent rotational flow fields of an ideal compressible fluid. The combined system can be shown to be invariant under both global and local gauge transformations of variations of a_μ. The global gauge transformation is a diagnostic test whether the system is receptive to a new field $a_\mu(x^\nu)$. Since the test is cleared, a new internal stress field $M_{ik}(x^\nu)$ is introduced into the flow field of a perfect fluid, together with the current conservation $\partial_\mu j^\mu = 0$, where the stress M_{ik} is an anisotropic stress field which is an extension added to the Eulerian isotropic pressure-stress field $p\,\delta_{ik}$.

Keywords: anisotropic stress field, gauge principle, fluid gauge theory, current conservation, Euler equation.

I. Introduction

a) Background of present research

Gauge invariance is one of the fundamental symmetries in modern theoretical physics. It took almost a century for transition from the 19th-century recognition of a mathematical invariance existing in classical electromagnetic theory to the 20th-century recognition of its fundamental physical significance. Real recognition of the gauge symmetry and its physical significance required two new fields developed in the 20th century: the relativity theory for physics of the world structure of linked 4d-spacetime and the quantum mechanics for the new dimension of a phase factor in complex representation of wave function. The 20th-century recognition resulted in the naming of the invariance as *gauge invariance* and in successful formulation of the Gauge Principle. Its historical development is reviewed by Kambe (2021a) concerning its gradual and zigzag developmental processes in quantum electrodynamics (QED). The *gauge theory* played vital roles in modern particle physics which was revolutionary (*e.g.* Aitchison & Hey (2013), Utiyama (1956)). The same paper (Kambe 2021a) presents also reviews of the gauge invariances existing in the two theories of the weak gravitational field and the electromagnetic theory with emphasizing the similarity between them. In addition, its last section 5 presents "*Motivations for Fluid Gauge Theory*". Thus, on the basis of these backgrounds of gauge theories reviewed in the article (Kambe 2021a), the present paper proposes possible application of the gauge theory to fluid flows although the field of fluid-flow is not listed in the literatures reviewed.

Author: (Former Professor at) Department of Physics, University of Tokyo, Tokyo, Japan, (Home) Higashi-yama 2-11-3, Meguro-ku, Tokyo, Japan. e-mail: kambe@ruby.dti.ne.jp

From the aspects of fluid flows, recent studies of wall turbulence and shear flow turbulence recognize existence of large scale structures. It is known from experimental studies that periodic waves exist *robustly* in the background irregularly fluctuating flow field, and streak structures are observed to coexist with turbulent field. The streaks in turbulence are wavy streamwise vortices surrounded by a sea of incoherent turbulent motions. The observed large scale structures are characterized by streamwise streaks and long meandering vortical structures. Concerning these experimental studies, see Reynolds & Hussain (1972); Kim & Adrian (1999); Del Álamo & Jiménez (2006); Monty *et al.* (2007); Hutchins & Marusic (2007); Smits *et al.* (2011).

Related mathematical aspect was investigated by Scofield and Huq (2014), stating that the conservation law of current flux implies existence of a wavy field governed by Maxwell-type equations, proposing that this might be applied to transverse travelling waves observed in turbulent internal flows along a spiral pipe. The theory is based on four conservation equations of energy and momentum of the whole system.

In order to highlight such aspects of large-scale structures coexisting with irregularly fluctuating flow field and contemplate how such structures observed experimentally are possible in turbulent shear flows, Kambe (2017) proposed a new scenario of turbulence theory, based on the view that the entire physical system is composed of two fields: fluid-flow field and transverse-wave field. General formalism of theoretical physics is applied to the study of whole system consisting of a flow field and a wave field, with two Lagrangian densities corresponding to each component and additional Lagrangian expressing their interaction. This approach yielded good results that are consistent with observations.

In particular, it is remarked that the "*fluid gauge theory* to be presented here supports the above theoretical approach proposed by Kambe (2017). In addition, the most recent research of Kambe (2020) sheds light on physical mechanism how and why the entire physical system has a structure composed of two fields: a flow field and a wavy *gauge* field. The underlying key is the inseparable relation between the mass conservation law and a gauge symmetry. Namely, the current conservation must be satisfied at every point and every time by the flow field, which is to be ensured by a background gauge field. This is the physical idea requiring the system to be composed of two fields of a flow and a background gauge field.

The new system consisting of a fluid-flow and a background field ensuring current conservation is powerful. It is likely that this enables to resolve an issue historically unresolved, that is the problem of "Dust striations observed in the resonance-tube experiment" by August Kundt (1866), where there exists two different length scales with their ratio more than fifty. The larger one correponds to the wavelength of the resonant acoustic wave. A recent numerical test study based on the present new system gives encouraging outputs on the smaller scales (see the section §4 *Summary and discussions*).

As investigated in Kambe (2020), according to the current formulation of fluid mechanics, from a single relativistic energy equation of fluid motion, two conservation equations are obtained in the non-relativistic limit : one is the mass conservation and the other is the traditional form of energy equation. This is a riddle (see below at the part c) from the physics point of view. We are particularly concerned with the mass conservation equation and investigate what symmetry implies the mass conservation. A key to resolve this *Riddle* is provided by the general representation of rotational flows of an ideal compressible fluid satisfying the Euler's equation (the part d), derived by Kambe (2013). This gives us a hint of existence of a set of gauge fields, giving rise to anisotropic stress fields within the flows which are time-dependent and rotational flow fields.

In the present study, the Euler's equation of motion is still valid, but it is remarked that the equation is characterized intrinsically by the pressure stress which is represented

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021 114

by *isotropic* stress fields. The present study of Fluid Gauge Theory predicts that the perfect fluid (*i.e.* a fluid without dissipative internal mechanism) can accommodate *anisotropic* stress field as well which can exist in unsteady shear flow fields.

This suggests that our physical system should be a combined system consisting of a fluid flow field and a set of background gauge fields. This aspect and its significance have been already investigated by Kambe (2017, 2020). The gauge symmetry of the new background gauge fields ensures the law of mass conservation.

b) Section composition of the paper and mathematical structure of the theory

In the present paper, *Fluid Gauge Theory* is presented for a perfect fluid according to the general gauge principle. Section II is a preparative section collecting necessary articles for the theory with the section title: *An approach aiming at a fluid gauge theory*. Section III presents the main theory with the title, *Fluid Gauge Theory* and adds a remark section supplementing insufficient parts of the presentation.

On a mathematical point of view, more must be commented on the present approach of the *Fluid Gauge Theory*. When new fields are taken into consideration, those should be implemented (or absorbed) into the structure of covariant derivatives as connection terms like the terms of Christoffel symbol in the gravity theory or the gauge potentials in the electromagnetic theory. The concept of *connection* in the mathematics is an essential ingredient of the physical gauge theory. It is a challenging work to implement connection terms in the structure of fluid gauge theory. This is left to the Appendices B and C, because sufficient mathematical expressions and concepts must be presented to arrive at the goal.

Another aspect of the present system of a perfect fluid must be noted. Our system is free from external forcing and in addition free from any internal mechanism of energy dissipation. From mechanical point of view, free fluid motions are not always described by straight trajectories of time evolution of the mechanical system as a whole, namely their geodesics describing the whole system are curved in general.

In fact, we will see in Appendix C.2 that the free motion of a perfect fluid under a background field a_ν can be described by a geodesic equation representing a curved free dynamics. This is derived by the variational principle that makes the action integrals invariant. Namely the new field of the fluid gauge theory has been taken into the structure of covariant derivatives as connection terms and the free dynamics of a perfect fluid under a background field a_ν is described by the geodesic equation.

These mathematical concepts would make the structure of theory complicated easily. In order to make the storyline of the theory clear and as much as simple, those complicating mathematical factors are left to Appendices B and C. However, the mathematical concepts such as *geodesic, covariant derivative, connection, etc.* . are absolutely necessary for the theory of Fluid Gauge Theory. It is the reason why the appendices to the present paper get massive.

c) By what symmetry the mass conservation law is implied ?

It is well-known that the energy conservation is associated with the symmetry of time translation of mechanical systems. One of the motivations for proposing a fluid gauge theory is stated by the following question: "What physical symmetry implies the mass conservation law ?" This query is raised in regard to the relativistic equation of energy conservation of fluid flows when its non-relativistic limit is taken (Kambe 2020). In the ordinary fluid-mechanics valid in non-relativistic limit, the mass conservation law is given as valid *a priori*. However, in the fluid-mechanics of relativity theory, fluid motions are governed by four relativistic conservation equations of energy and

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

momentum $\partial_\nu T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is the stress-energy tensor for $\mu, \nu = 0, 1, 2, 3$, $\partial_\nu \equiv \partial/\partial x^\nu$ and $x^\nu = (x^0, x^k)$ with $x^0 \equiv ct$ for t the time. (see [Kambe (2020) §2.2, 2.3] or [Landau & Lifshitz, 1987, §133]). Its *space* components for $\mu = 1, 2, 3$ represent momentum conservation of three components.

On the other hand, its *time* component $\partial_\nu T^{0\nu} = 0$ represents an energy conservation equation. In the non-relativistic limit as a representative flow velocity v is much less than the light velocity c ($\beta \equiv v/c \to 0$), the equation for a perfect fluid of mass density ρ and specific internal energy ϵ‡ can be written in the following form:

$$0 = c^{-1}\partial_t \overline{T}^{00} + \partial_k \overline{T}^{0k} = c\Big(\partial_t\rho + \partial_k(\rho v^k)\Big) + \frac{1}{c}\Big(\partial_t(\rho\hat{E}) + \partial_k(\rho v^k \hat{H})\Big) + O(\beta^3), \quad (1.1)$$

$$\hat{E} = \frac{1}{2}v^2 + \epsilon, \quad \hat{H} = \frac{1}{2}v^2 + h. \quad (1.2)$$

where v^k is the k-th component of fluid velocity for $k = 1, 2, 3$. In the non-relativistic limit as $\beta \to 0$, we obtain the mass conservation equation from the first term:

$$\partial_t\rho + \partial_k(\rho v^k) = 0. \quad (1.3)$$

Then, deleting it, the remaining expression reduces to the energy equation of ordinary fluid mechanics in the limit as $\beta \to 0$. Thus, we obtain the energy conservation equation of fluid flow (Landau & Lifshitz (1987), Eq.(6.1)):

$$\partial_t(\rho\hat{E}) + \partial_k(\rho v^k \hat{H}) = 0. \quad (1.4)$$

Here we have obtained two conservation equations from the single energy equation $\partial_\nu T^{0\nu} = 0$. But, the Noether's theorem (Noether 1918) of theoretical physics states 'A symmetry implies a conservation law'. This is a *riddle*. We must ask a question whether the above is satisfactory. In this paper, we try to propose a resolution to this query.

d) A hint to resolve the riddle: General solution of Euler's equation with helicity

A hint to resolve the *Riddle* mentioned above is found in the general representation of rotational flows given by Kambe (2013) for an ideal compressible flow solution satisfying the Euler's equation. This solution was derived from the action principle for the action $S^{(\mathrm{Eul-rot})}$ of non-relativistic flow fields:

$$S^{(\mathrm{Eul-rot})} = S^{(\mathrm{nR})} + S^{(\mathrm{g-inv})} = \int \rho \,\mathrm{d}\mathcal{V} \Big[\int \Lambda_{\mathrm{nR}} \,\mathrm{d}t + \int \Lambda_{\mathrm{Gi}} \,\mathrm{d}t \Big], \quad (1.5)$$

$$\Lambda_{\mathrm{nR}} = \tfrac{1}{2}v^2 - \epsilon, \qquad \Lambda_{\mathrm{Gi}} = -\mathrm{D}_t\boldsymbol{\psi} - \mathrm{D}_t\langle \boldsymbol{U}, \boldsymbol{Z}\rangle \quad (1.6)$$

$$\nabla\cdot(\rho\boldsymbol{Z}) = 0, \qquad \nabla\cdot\boldsymbol{U} = 0, \qquad \mathrm{D}_t \equiv \partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}, \quad (1.7)$$

$$\mathcal{L}[\boldsymbol{Z}] \equiv \partial_t\boldsymbol{Z} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{Z} - (\boldsymbol{Z}\cdot\nabla)\boldsymbol{v} = 0, \quad (1.8)$$

where $\boldsymbol{v} = (v^k)$ is the 3-velocity vector, $\boldsymbol{\psi}$ a scalar function to be determined, and Λ_{nR} is nothing but the ordinary non-relativistic Lagrangian density, while Λ_{Gi} is a gauge-invariant Lagrangian newly introduced in the study (Kambe, 2013). Regarding the two 3-vectors $\boldsymbol{Z}$ and $\boldsymbol{U}$, see the paragraph below. Actually, this study had double aims. One was an attempt to obtain general representation of rotational flow with non-zero helicity (Kambe 2011). Second aim was more fundamental, striving to establish

‡ There is no energy dissipation in the perfect fluid, hence no entropy change. Assuming uniform entropy throughout, the internal energy ϵ depends only on ρ. Hence $\epsilon = \epsilon(\rho)$, and $h \equiv \epsilon(\rho) + p/\rho$.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

equivalence between two formulations of Eulerian specification of field variables and the Lagrangian specification under the action principle. Each term of the Lagrangians $\Lambda_{\rm nR}$ and $\Lambda_{\rm Gi}$ satisfies local gauge invariance with respect to translation and rotation, hence it is consistent with the gauge theory.

As discussed in details in Kambe (2020, §1 and 3.1), this new formulation introduced four independent fields. In fact, regarding the 3-vector potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$, each has three components. Those six fields have two invariance conditions of (1.7), *i.e.* two divergence-free conditions in 3-space. In addition, from (1.8) and the equation, $(\mathcal{L}_t^*[\boldsymbol{U}])_i \equiv \partial_t U_i + v^k \partial_k U_i + U_k \partial_i v^k = 0$ obtained from the variational analysis of Kambe (2013), we have the third invariance condition:

$$\mathrm{D}_t \langle \boldsymbol{U},\, \boldsymbol{Z} \rangle (t, \boldsymbol{x}) \equiv \langle \mathcal{L}_*[\boldsymbol{U}],\, \boldsymbol{Z} \rangle + \langle \boldsymbol{U},\, \mathcal{L}[\boldsymbol{Z}] \rangle = 0. \tag{1.9}$$

Hence, the value of scalar product $\langle \boldsymbol{U},\, \boldsymbol{Z} \rangle$ is invariant along the particle path $\boldsymbol{x} = \boldsymbol{X}_p(t, \boldsymbol{x})$, keeping its initial value along each trajectory. This is the third invariant imposed on the potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$. Therefore we have only three independent fields remaining free among the six components of $\boldsymbol{U}$ and $\boldsymbol{Z}$. Furthermore, if we add the scalar field $\boldsymbol{\psi}$ which is also unconstrained, we have four independent fields in this solution.

Thus, four independent background fields are newly introduced in this solution. Those must be either given externally or determined internally within the framework of theory. In the recent study Kambe (2020), the latter approach was taken, and the general solution of Kambe (2013) is understood to predict existence of new fields $\tilde{a}_\nu$. Four independent fields $\tilde{a}_\nu$ existing in the $4d$-spacetime enables a gauge-theoretic formulation in terms of one-form. On the basis of this perspective, the present study proposes a set of new fields to be introduced according to the gauge principle, which may be called a *fluid gauge theory*.

Another perspective is as follows. What is the hint to resolve the riddle mentioned in the part c) is as follows. We *rewrite* the part of action $S^{(\mathrm{g-inv})}$ of (1.5) as $S^{(\mathrm{int})} \equiv \int \rho \mathrm{d}\mathcal{V} \int \Lambda_{\rm Gi} \mathrm{d}t$, since this term is considered to describe interaction between the flow-current j^ν and background vector-potentials $\boldsymbol{U}$ and $\boldsymbol{Z}$, and $\boldsymbol{\psi}$. Corresponding to the new name $S^{(\mathrm{int})}$, we rename the scalar product $\langle \boldsymbol{U},\, \boldsymbol{Z} \rangle$ with W, and newly define a 4-current j^ν and a background field $\tilde{a}_\nu$ as

$$j^\nu \equiv (\rho\, c,\, \rho\, \boldsymbol{v}) = c\, \overline{\rho}\, u^\nu, \qquad \tilde{a}_\nu \equiv -\partial_\nu (\,\boldsymbol{\psi} + W\,). \tag{1.10}$$

(See (A.3), (B.9) for the definition of $\overline{\rho}$, u^ν.) The interaction part $S^{(\mathrm{int})}$ is expressed by

$$S^{(\mathrm{int})} = -\int\!\!\int \Big(\rho\, \mathrm{D}_t \boldsymbol{\psi} + \rho\, \mathrm{D}_t\, W\Big) \mathrm{d}\mathcal{V}\, \mathrm{d}t = \int\!\!\int\, j^\nu\, \tilde{a}_\nu\, \mathrm{d}\mathcal{V}\, \mathrm{d}t, \tag{1.11}$$

$$j^\nu \partial_\nu = \rho\, (\partial_t + \boldsymbol{v} \cdot \boldsymbol{\nabla}) = \rho\, \mathrm{D}_t\,, \tag{1.12}$$

where $\partial_\nu \equiv \partial / \partial x^\nu = (c^{-1} \partial_t,\ \boldsymbol{\nabla})$. The 4-current j^ν is defined by

$$j^\nu = \rho\, v^\nu = \rho\, (c,\, \boldsymbol{v}) = \rho\, (\mathrm{d}X_p^\nu / \mathrm{d}t), \qquad \mathrm{d}X_p^\nu = (c\, \mathrm{d}t, \mathrm{d}\boldsymbol{X}_p) = (c, \boldsymbol{v})\, \mathrm{d}t. \tag{1.13}$$

where $\mathrm{d}X_p^\mu$ is 4-spacetime notation of displacement of a fluid particle p, and $\mathrm{d}\boldsymbol{X}_p = \boldsymbol{v}\, \mathrm{d}t$ is 3-space displacement of the particle p moving with 3-velocity $\boldsymbol{v}$ during an infinitesimal time interval $\mathrm{d}t$. Denoting $\Psi \equiv \boldsymbol{\psi} + W$, the field $\tilde{a}_\nu$ is given by

$$\tilde{a}_\nu = -\partial_\nu \Psi. \tag{1.14}$$

is analogous to the particular field $\tilde{A}_\mu = \partial_\mu \Theta$ considered in the recent review paper (Kambe 2021a, Section I b), where all the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish identically. In other words, those fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are potentially existing, but vanish in the particular form of $\tilde{A}_\mu = \partial_\mu \Theta$. Same can be said that our new potential field $\tilde{a}_\nu$ does exist. But with the particular form $\tilde{a}_\nu = -\partial_\nu \Psi$, the *potentially existing* new field does not show in observable world. From this observation, new *Fluid Gauge Theory* is proposed in this paper, according to the theory of general gauge fields proposed by Utiyama (1956, 1987).

In particular, the following is important in the context of our problem. Gauge invariance applied to the action $S^{(\mathrm{int})}$ yields the *current conservation law*:

$$\partial_\nu j^\nu = \partial_\nu (c\,\overline{\rho}\,u^\nu) = (c^{-1}\partial_t,\ \boldsymbol{\nabla}) \cdot (\rho\,c,\ \rho\,\boldsymbol{v}) = \partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0. \qquad (1.15)$$

(See (2.24) in §2.5.) This equation is understood showing a potentiality of the fluid gauge theory. It is remarkable that the two scalar products, $\partial_\nu j^\nu$ and

$$j^\nu \partial_\nu = c\,\overline{\rho}\,u^\nu \partial_\nu = (\rho\,c,\ \rho\,\boldsymbol{v}) \cdot (c^{-1}\partial_t,\ \boldsymbol{\nabla}) = \rho\,(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}) \equiv \rho\,\mathrm{D}_t, \qquad (1.16)$$

are represented relativistically, and that both of them are invariant with respect to the Lorentz transformation. Thus the above two expressions (1.15) and (1.16) show us a glimpse of structures of linked 4d-spacetime existing in fluid mechanics.

e) A motivation for fluid gauge theory

From these observations, we set out toward a new approach of *fluid gauge theory.*

i. *Gauge transformations and gauge principle:* In the gauge principle, the system under investigation is examined whether the system is invariant with respect to both global and local transformations. The *global* gauge transformation is defined by the transformation: $\tilde{a}_\mu \to a_\mu = \tilde{a}_\mu + \epsilon_\mu$ for $\tilde{a}_\mu$ of (1.14) and ϵ_μ being 4 arbitrary constants. Firstly, the system must be shown to be invariant with this global transformation. This is the *first* step toward the fluid gauge theory, examining whether the system under consideration is equipped with desirable receptive property.

A next essential step of the gauge principle lies in requiring *local* gauge invariance. This is defined by $\tilde{a}_\mu \to a_\mu = \tilde{a}_\mu + \alpha_\mu(x^\nu)$ for 4 arbitrary differentiable fields $\alpha_\mu(x^\nu)$ depending on spacetime coordinates x^ν. Once the local invariance is established, the so-called *gauge-potential* a_μ is taken into the system which represents a new interaction force. This is the scenario of the gauge principle to introduce a new force into the system under consideration by the local gauge invariance.

ii. *Important factor for the gauge theory:* From the gauge principle and reflecting on the form of the interaction action $S^{(\mathrm{int})}$ of (1.11), one realizes that an important factor is the linked $4d$-spacetime structure. In order to see it, let us remind of the general solution of Euler's equation of motion considered in the part d). This corresponds to a *vanishing*-field state, because, considering one-form $\tilde{A}$ defined by $\tilde{A} = \tilde{a}_\mu \mathrm{d}x^\mu$ with $\tilde{a}_\nu = -\partial_\nu \Psi$ of (1.14), one obtains

$$\tilde{A} = \tilde{a}_\mu \mathrm{d}x^\mu = -\partial_\mu \Psi\, \mathrm{d}x^\mu = -\mathrm{d}\Psi.$$

This represents the vanishing-field state since $\mathrm{d}\tilde{A} = -\mathrm{d}^2\Psi \equiv 0$. This is the case of the general solution to the Euler's equation of the part d) The field $\tilde{a}_\mu$ itself exists, but does not show in the observable world (see Section II, e) ii). Only the Euler's flow field is observed. In this case, the stress field is given by the isotropic pressure field $p\,\delta_{ik}$, On the other hand, existence of the new field $a_\mu(x^\nu) = \tilde{a}_\mu + \alpha_\mu(x^\nu)$ changes the flow field drastically. Consider the 4d-spacetime (x^ν) of fluid flows that is structured with the one-form $\mathcal{A} \equiv a_\mu dx^\mu$, from which one obtains non-vanishing field strength, $\mathcal{F} = \mathrm{d}\mathcal{A} \neq 0$. This gives rise to anisotropic stress field within the flow field, as given in later sections (see Sections III, c) ii.). Thus, the main factor is the one-form defined by

$$\mathcal{A} = a_\mu dx^\mu, \tag{1.17}$$

that plays the role of a game-changer from vanishing-field state of $\tilde{a}_\mu$ to non-vanishing state of the new field $a_\mu(x^\nu)$. With this fact, the gauge principle is rooted on the fundamental of Physics. This is the central theme of the present paper.

II. An Approach Aiming at a Fluid Gauge Theory

a) Euler's equation of a perfect fluid in the absence of background field

Relativistic form of the action integral of a perfect fluid is given in Appendix B.2 as

$$S^{(\mathrm{pf})} = -c \int \rho \, \mathrm{d}\mathcal{V} \int \Big(1 + c^{-2}\,\bar{\epsilon}(\rho)\Big)\, \mathrm{d}\tau\,. \tag{2.1}$$

where τ the *proper time*. Its increment $\mathrm{d}\tau$ is defined by the time increment (multiplied by c) in the instantaneously rest frame where $\boldsymbol{v} = 0$. The relativistic action $S^{(\mathrm{pf})}$ is defined as an extension to the perfect fluid from that of a single particle of mass m represented by $S^{(\mathrm{m})} = -c\,m \int \mathrm{d}\tau$, which is given in Appendix B.1. The overlined $\bar{\epsilon}$ in (2.1) denotes proper value of the internal energy ϵ (the value in the *rest frame*, *i.e.* comoving frame where the fluid is at rest). Comparing $S^{(\mathrm{pf})}$ with $S^{(\mathrm{m})}$ and considering the quantity $\int \rho \, \mathrm{d}\mathcal{V}$ corresponding to the mass m of $S^{(\mathrm{m})}$, one can see that the second correction term $c^{-2}\,\bar{\epsilon}$ in the parenthesis is a small correction for the fluid medium in non-relativistic case.

Non-relativistic limit (as $\beta \to 0$, Section I, c) of the integrand $\Lambda_{(\mathrm{pf})}$ of $S^{(\mathrm{pf})}$(multiplied by c) per unit mass is given as $\Lambda^{(\mathrm{pf})} = -m_1 c^2 + \frac{1}{2}\, m_1 v^2 - \epsilon + \cdots$ with $m_1 = 1$. Neglecting the first term $m_1 c^2$ of the rest-mass energy, the Lagrangian density $\Lambda_{(\mathrm{pf})}$ reduces to the non-relativistic form of $\Lambda_{(\mathrm{nR})}$ of (1.6). Hence it is seen that the action $S^{(\mathrm{pf})}$ is a relativistic version extended from the classic non-relativistic action $S^{(\mathrm{nR})}$ of (1.5).

From the variation analysis of $S^{(\mathrm{pf})}$ carried out in Appendix B.2, the action principle yields the following Euler's equation of motion (B.17) as a geodesic equation:

$$\mathrm{D}_t v^k + \rho^{-1}\, \partial_k p = 0, \qquad k = 1, 2, 3 \tag{2.2}$$

in the non-relativistic limit of ordinary fluid flows. Noting that the factor $\partial_k p$ of the second term can be replaced by $\partial_j (p\, \delta_{jk})$, one can rewrite the equation (2.2) as

$$\rho\, \mathrm{D}_t v^k = -\partial_j (p\, \delta_{jk}) \qquad k = 1, 2, 3 \tag{2.3}$$

where

$$\mathrm{D}_t \equiv \partial_t + \boldsymbol{v} \cdot \nabla \qquad k = 1, 2, 3 \tag{2.4}$$

is the convective derivative, which is invariant with respect to local gauge transformation (Kambe (2020), Appendix A2).

b) New action $S^{(\mathrm{int})}$ including a field a_μ ensuring current conservation

In the local gauge transformation considered in Section I, e) i. the field α_μ is assumed to take a general form not limited to the particular gradient form $\partial_\mu \Psi$, *i.e.* not like (1.10), the interaction action $S^{(\mathrm{int})}$ of (1.11) should be extended to general 4-potential a_μ by replacing the particular field $\tilde{a}_\mu$. Hence now, the action $S^{(\mathrm{int})}$ is re-defined by

$$S^{(\mathrm{int})} = \int\!\!\int j^\mu\, a_\mu\, \mathrm{d}\mathcal{V}\, \mathrm{d}t = \int \rho\, \mathrm{d}\mathcal{V} \int v^\mu\, a_\mu\, \mathrm{d}t = \int \rho\, \mathrm{d}\mathcal{V} \int a_\mu\, \mathrm{d}X^\mu\,, \tag{2.5}$$

where $j^\nu = \rho\, v^\mu$ and see (1.13). This action $S^{(\mathrm{int})}$ was already introduced in Kambe (2020) at its §4.2. This is rewritten here as an additional action to be added to the main part $S^{(\mathrm{pf})}$, in order to constrain the conservation of current j^μ:

$$S^{(\mathrm{int})} = \int\!\!\int j^\nu\, a_\nu\, \mathrm{d}\mathcal{V}\, \mathrm{d}t = \int \rho\, \mathrm{d}\mathcal{V} \int a_\nu\, \mathrm{d}x^\nu\,. \tag{2.6}$$

The one-form structure $a_\nu\, \mathrm{d}x^\nu$ in the last integral reminds us of the similar structure considered in the previous section I, e) ii. Similar structure is known in quantum electrodynamics (Section II, b) of Kambe (2021a)). There, the wave function is required to undergo the transformation $\psi = \psi_0(x^\nu)\ \exp[\, i\gamma \int A_\nu(x^\nu)\, dx^\nu]$ in the presence of electromagnetic field of 4-potential A_ν from the zero-field wave function ψ_0, where the A_ν field is the *gauge-potential* representing a new interaction force of electromagnetism.

Our case is based on the gauge principle such that the action $S^{(\mathrm{int})}$ thus introduced represents the interaction between the current field $j^\nu(x^\lambda)$ and a background (*gauge-potential*) field $a_\nu(x^\lambda)$ and is receptive to the gauge principle requiring local gauge invariance. The new field $a_\nu(x^\nu)$ thus introduced ensures the mass conservation (2.25) shown in Section II, e) iii. under the requirement of *gauge invariance* of the new field.

In addition, in the Maxwell system described in Section II, a) i. of the review article (Kambe 2021a), the interaction action is given by $S^{(\mathrm{em})}_{\mathrm{int}} = c^{-2} \int j_e^\nu\, A_\nu\, \mathrm{d}\Omega$ of eq.(2.9) of the same article. Amazingly, the analogy with the present system is obvious.

c) Composite action S_c and modified Euler's equation of motion

According to the previous sections II a) and b), one can define a composite action S_c by using the action $S^{(\mathrm{pf})}$ of a perfect fluid of (2.1) and the action $S^{(\mathrm{int})}$ of (2.6) for the interaction of j^ν and a_ν. Let us define

$$S_c \quad \equiv S^{(\mathrm{pf})} + S^{(\mathrm{int})}, \qquad \mathrm{d}\Omega \equiv \mathrm{d}^4 x = \mathrm{d}\mathcal{V}\, \mathrm{d}t_c, \quad t_c = c\, t, \tag{2.7}$$

$$S^{(\mathrm{pf})} \equiv -c \int \rho\, \mathrm{d}\mathcal{V} \int \Big(1 + c^{-2}\, \bar{\epsilon}(\rho)\Big)\, \mathrm{d}\tau = \int \mathcal{L}^{(\mathrm{pf})}\, \mathrm{d}^4 x, \tag{2.8}$$

$$S^{(\mathrm{int})} \equiv \int \mathcal{L}^{(\mathrm{int})}\, \mathrm{d}^4 x, \qquad\qquad \mathcal{L}^{(\mathrm{int})} \equiv c^{-1}\, j^\nu\, a_\nu, \tag{2.9}$$

where $\mathcal{L}^{(\mathrm{pf})} \equiv -c\, \rho\, (1 + c^{-2}\, \bar{\epsilon})\, \sqrt{1-\beta^2}$. To find the equations of motion, the action principle is applied to the composite action S_c, by assuming the gauge potential a_ν given and vary only the position coordinate X_p^k of fluid particles moving with the velocity $\mathrm{D}_t X_p^k$ along their trajectories. On the other hand, to find the equations governing the a_ν, we vary only the gauge-potential a_ν with assuming the fluid motion given and fixed. However, to carry out the latter variation, we have to define a third action to characterize the background field a_ν and add it to S_c. Here, we carry out the former variation (in which the third action is kept fixed), then the action principle applied to the varied S_c should yield the equation of fluid motion.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Note that, under the requirement of invariance of $S^{(\mathrm{int})}$ to the gauge transformation of potential a_ν considered in Section II, e), the current conservation law $\partial_\nu j^\nu = 0$ is deduced. Therefore, when variations are taken with respect to the particle position X_p^k, the invariance of the mass $\mathrm{d}m \equiv \rho\,\mathrm{d}\mathcal{V}$ is assumed for a fluid particle during the motion along its trajectory.

Modified Euler's equation of motion in the presence of new field a_ν

The variational analysis of the composite action S_c is given in Appendix B.3. The equation (B.26) summarizes the variation analyses with respect to the particle coordinate δX^ν of fluid element Δm,

$$\delta\mathcal{J}^{(\mathrm{fl+a})} \equiv \delta\,\mathcal{J}^{(\mathrm{pf})} + \delta\mathcal{J}^{(\mathrm{int})} = -c\,\overline{\Delta m}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}u_\nu + c^{-2}\frac{1}{\overline{\rho}}\partial_\nu\overline{p}\, - c^{-1}\,f_{\nu\mu}\,u^\mu\,\right]\mathrm{d}\tau\,\delta X^\nu \quad (2.10)$$

as leading order terms in the expansion with respect to the very small parameter $\beta = v/c$. The action principle requires $\delta\mathcal{J}^{(\mathrm{fl+a})} = 0$ for arbitrary variation δX^ν. This leads to the equation: $(\mathrm{d}u_\nu/\mathrm{d}\tau) + c^{-2}\,(1/\overline{\rho})\partial_\nu\overline{p} - c^{-1}\,f_{\nu\mu}u^\mu = 0$. Its non-relativistic limit (as $\beta \to 0$) is expressed by the equations:

$$\mathrm{D}_t v^k = -\rho^{-1}\,\partial_k p + f_{k\nu}\,v^\nu, \qquad (k = 1,2,3;\quad \nu = 0,1,2,3), \quad (2.11)$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \quad (2.12)$$

This is the Euler's equation (2.2) *modified* by the effect of a background field a^ν expressed by the third term $f_{k\nu}\,v^\nu$. Section III c), ii. gives details concerning the significance of the new tensor field $f_{\mu\nu}$ by the section title, *Background field: Fluid Maxwell fields.*

d) Scenario of general gauge principle according to Utiyama

In the previous section II c), we have derived the modified Euler equation (2.11) from the composite action S_c $(= S^{(\mathrm{pf})} + S^{(\mathrm{int})})$ of (2.7). The first term $-\rho^{-1}\,\partial_k p$ on the right-hand side of (2.11) represents the pressure force caused by the isotropic stress tensor $-p\,\delta_{kl}$, while the second term $f_{k\nu}\,v^\nu$ represents a new force caused by an anisotropic stress field, as explained in a later section III c), iii. It is remarkable that the tensor components $f_{\mu\nu}$ of (2.12) all are linear with respect to space-time derivatives of the background potential a_μ. This is essential for the gauge principle to be given now.

Suppose that our basic undisturbed state is described by the equation (2.11) and by the composite action S_c of (2.7), and in addition that the background field a_μ is given a special form, *i.e.* a potential field expressed by $\tilde{a}_\mu = \partial_\mu\Psi$ with $\Psi(x^\nu)$ a scalar field. Note that this field form $\tilde{a}_\mu = \partial_\mu\Psi$ is a special class among general background fields.

As described in the next section II e), ii. in details, all the components $f_{\mu\nu}$ associated with the background potential $\tilde{a}_\mu = \partial_\mu\Psi$ vanish identically. Hence the equation (2.11) reduces to the Euler's equation (2.2). Namely, the basic undisturbed state is assumed to be governed by the Euler's equation. In addition, from the action S^{int}, the continuity equation (2.14) was derived in II e), iii.

In this case, the action $S^{(\mathrm{int})}$ does not give any mechanical effect on the system (see II e), ii.). Therefore, the mechanical effect of the composite action S_{c} is equivalent to that of the term $S^{(\mathrm{pf})}$ of perfect fluid only, not different from the state without the field a_μ. Therefore, for the potential fields of $\tilde{a}_\mu = \partial_\mu\Psi$, the basic undisturbed state is represented by that of the perfect-fluid action S^{pf} only. Namely, the state is nothing but the Euler field.

According to Utiyama (1956, 1987), the general gauge principle states as follows. "*If both of the composite action S_c and the equation of motion (2.11) are invariant under* *a global transformation defined by* $\tilde{a}_\mu \to \tilde{a}'_\mu = \tilde{a}_\mu + \delta a_\mu$ *for uniformly constant value of* $\delta a_\mu = \epsilon_\mu$*, then the system is said invariant globally for the* a^μ*-transformation.*"

The invariance of the governing equation (2.11) is almost trivial because the field $\tilde{a}_\mu$ is included only in $f_{\mu\nu}$ where all the components $\tilde{a}_\mu$ are expressed in derivative forms, as seen from (2.12). Hence, constant variation ϵ_μ of $\tilde{a}_\mu$ does not give any effect on the equation (2.11). Concerning the action $S^{(\mathrm{int})}$, its invariance by the global transformation $\tilde{a}_\mu \to \tilde{a}_\mu + \epsilon_\mu$ is investigated in the section III b) below the line (3.11), and the system is invariant mechanically with this global transformation. Thus, the system is *globally* invariant for the uniform $\tilde{a}_\mu$-variation.

The gauge principle reads furthermore, "*Even if the global invariance of the system is satisfied, one may consider local transformation with* $\delta a_\mu(x^\nu)$ *varying with the space-time coordinates* x^ν*, for which neither the action integral nor the equation of motion are invariant locally under such a local gauge transformation.*"

Here, we have to remind that the interaction action $S^{(\mathrm{int})}$ of (2.9) is already defined by using general $a_\mu(x^\nu)$ field depending on the space-time coordinates x^ν. In the previous Section II c) using this $S^{(\mathrm{int})}$, we have derived the equation of motion (2.11). For the particular form of potential $\tilde{a}_\mu = \partial_\mu \Psi$ considered above in the global transformation, the new tensor field $f_{\mu\nu}$ vanishes identically (verified immediately by substitution). Then, the equation (2.11) reduces to the Euler's equation (2.2). Under the local transformation, however, the tensor field $f_{\mu\nu}$ does not vanish in general, then the equation (2.11) deviates from the Euler's equation (2.2). In other words, the equation (2.11) is not invariant for the local transformation: $a_\mu \to a_\mu + \delta a_\mu(x^\nu)$. This fact is interpreted as follows.

There may exist a background field a_μ in the flow field $v^\mu(x^\nu)$, which interacts with the flow under the action of the stress field $f_{\mu\nu}v^\nu$. The last is a new stress field. Thus, the general gauge principle predicts existence of a certain background field a_μ and an internal stress field $f_{\mu\nu}v^\nu$ generated by a_μ. The original basic state was the one governed by the Euler's equation. The equation (2.3) states that its stress field is given by an isotropic stress tensor $-p\,\delta_{ik}$. In the present context, corresponding equation (2.11) can be rewritten as

$$\rho\, \mathrm{D}_t v^k = -\partial_j (p\, \delta^{jk}) + \rho\, f^k_{\ \nu}\, v^\nu, \ \ (k = 1,2,3; \ \ \nu = 0,1,2,3), \tag{2.13}$$

Later in section III c) ii. the second term on the right $\rho\, f^k_{\ \nu}\, v^\nu$ is rewritten by (3.24) as $-\partial_\nu M^{\nu k}$, which represents anisotropic stress field.

It is essential in the scenario of the general gauge principle of Utiyama that the background field a_μ ensures the current conservation $\partial_\mu j^\mu = 0$, by the *gauge invariance* property of the background field a_μ itself. In the author's previous paper (Kambe, 2020), this action $S^{(\mathrm{int})}$ was already introduced at its section 4.2 "*Gauge invariance and mass conservation*", where invariance of $S^{(\mathrm{int})}$ was required to the gauge transformation $a_\mu \ \to \ a_\mu - \partial_\mu \Psi_*$ for arbitrary scalar field of $\Psi_*(x^\nu)$. The close connection between the gauge invariance and the law of mass conservation has been established there.

In the next section II e) iii. requiring the invariance of the action $S^{(\mathrm{int})}$ under the gauge transformation for arbitrary scalar field of $\Psi(x^\nu)$, the mass conservation equation is deduced:

$$\partial_\mu j^\mu = \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{2.14}$$

This is the scenario of the general gauge principle of Utiyama.

According to the scenario of Utiyama's general gauge principle, we have to show one more, which is as follows. The new field a_μ should be incorporated as a connection term in a covariant derivative. This is done in the Appendix B.4 where physical and mathematical formulations for curved space are presented and applied to flow fields of a perfect fluid.

This enables to introduce a general notion of covariant derivatives (connections). Its mathematical formulation is applied to free evolution of physical systems (free from external actions) but exhibiting non-straight motion even in flat space. Mathematical formulation by geometrical language enables us to formulate this generalization, namely enabling to conclude the fluid gauge theory.

e) Background field a_ν ensuring current conservation $\partial_\mu j^\mu = 0$

i. *Background field a_ν represented by $\mathcal{A}$, $\mathcal{F}$ and $\boldsymbol{f}_a$*

Using the background field a_ν, one can define a one-form $\mathcal{A} \equiv a_\nu \, \mathsf{d}x^\nu$. Taking its exterior differential d, a field strength two-form $\mathcal{F}$ is given by

$$\mathcal{F} \equiv \mathsf{d}\mathcal{A} = \sum_{\nu<\lambda} f_{\nu\lambda} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda, \qquad f_{\nu\lambda} \equiv \partial_\nu a_\lambda - \partial_\lambda a_\nu = -f_{\lambda\nu}. \tag{2.15}$$

where $\partial_\nu \equiv \partial/\partial x^\nu = (c^{-1}\partial_t, \ \boldsymbol{\nabla})$, the tensor $f_{\nu\lambda}$ is anti-symmetric.

Representing the 4-potential a_ν with a (1+3)-expression given by $a_\nu = (-\phi/c, \boldsymbol{a})$ where $a_0 = -\phi/c$ and $\boldsymbol{a} = (a_k) = (a_1, a_2, a_3)$, one can define

$$\boldsymbol{b} = (b_k) \equiv \nabla \times \boldsymbol{a}; \qquad f_{ij} = \partial_i a_j - \partial_j a_i = \varepsilon_{ijk} b_k, \quad b_k = \varepsilon_{klm}\partial_l a_m, \tag{2.16}$$

$$\boldsymbol{e} = (e_k) \equiv -\partial_t \boldsymbol{a} - \nabla\phi; \qquad f_{k0} = \partial_k a_0 - \partial_0 a_k = c^{-1} e_k \equiv \overline{e}_k. \tag{2.17}$$

where each of i, j and k takes the number of either 1, 2, or 3.

The modified Euler equation (2.11) includes the term of *internal stress* field $f_{\mu\nu}v^\nu$, which can be given a 3-vector form by using 3-vectors $\boldsymbol{e}$, $\boldsymbol{b}$ and $\boldsymbol{v}$ as

$$\boldsymbol{f}_a = (f_{a,i}) \equiv \boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b}, \qquad f_{a,i} = f_{i\nu}v^\nu = f_{i0}v^0 + f_{ik}v^k. \tag{2.18}$$

$$f_{a,i} = f_{i\nu}v^\nu = f_{i0}v^0 + f_{ij}v^j = c^{-1}e_i \, c + (\varepsilon_{ijk}b_k)\, v^j = (\boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b})_i, \tag{2.19}$$

where $v^\nu = (v^0, \boldsymbol{v}) = (c, \ \boldsymbol{v})$ from (1.13).

ii. *Special background field $\tilde{a}_\mu = \partial_\mu \Psi$ implies Eulerian field*

Using $\tilde{a}_\mu = \partial_\mu \Psi$, let us define a one-form by $\tilde{A} = \tilde{a}_\nu \, \mathsf{d}x^\nu = (\partial_\nu \Psi)\, \mathsf{d}x^\nu = \mathsf{d}\Psi$. Then the field strength two-form $\tilde{\mathcal{F}} \equiv \mathsf{d}\tilde{A}$ vanishes identically since $\mathsf{d}^2\Psi \equiv 0$, which can be shown alternatively by using the vanishing components $f_{\nu\lambda}$:

$$\tilde{\mathcal{F}} = \mathsf{d}\tilde{A} = \sum_{\nu<\lambda} f_{\nu\lambda} \, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda \equiv 0, \quad i.e. \ \ f_{\nu\lambda} = \partial_\nu\partial_\lambda\Psi - \partial_\lambda\partial_\nu\Psi = 0.$$

Hence, the background field of the type $\tilde{a}_\mu = \partial_\mu \Psi$ constitutes a special class in the flow field, and that, for this type of field, the modified Euler equation (2.11) reduces to the original Euler's equation (2.2), because all the fields $f_{\mu\nu}$ derived from $\tilde{a}_\mu = \partial_\mu \Psi$ vanish identically. In other words, the tensor fields $f_{\mu\nu}$ are potentially existing, but vanish for the particular form $\tilde{a}_\mu = \partial_\mu \Psi$, where $\Psi(x^\nu)$ is an arbitrary twice differentiable scalar field in the spacetime x^ν. The $\tilde{a}_\mu$ field does exist, but with the particular form $\tilde{a}_\nu = \partial_\nu \Psi$, the *potentially existing* fields do not show in observable world. Thus, the potential form, $\tilde{a}_\mu = \partial_\mu \Psi$, constitutes a special class of background field.

Hence, the modified Euler equation (2.11) reduces to the original Euler equation (2.2). In addition, regarding the composite action $S_c = S^{(\mathrm{pf})} + S^{(\mathrm{int})}$, the interaction part $S^{(\mathrm{int})}$ does not give any mechanical effect on the system with the special field $\tilde{a}_\mu = \partial_\mu \Psi$. The field $\tilde{a}_\mu$ is included only in the interaction action $S^{(\mathrm{int})}$ of (2.9).

The variational principle requires that variation of the action S_c must vanish with respect to the variation $\tilde{a}_\mu$ given by $\delta\tilde{a}_\mu = \partial_\mu(\delta\Psi)$ where $\delta\Psi$ is the variation of Ψ. Substituting the $\tilde{a}_\mu$-variation into $S^{(\text{int})}$, its resulting variation $\delta S^{(\text{int})}$ owing to $\delta\tilde{a}_\mu = \partial_\mu(\delta\Psi)$ is given by

$$\delta S^{(\text{int})} = \int j^\mu\,(\partial_\mu \delta\Psi)\,\mathrm{d}\Omega = -\int (\partial_\mu j^\mu)\,\delta\Psi\,\mathrm{d}\Omega + \int \partial_\mu\Big(j^\mu\,\delta\Psi\Big)\,\mathrm{d}\Omega, \tag{2.20}$$

where $\mathrm{d}\Omega = \mathrm{d}\mathcal{V}\,\mathrm{d}t$. The invariance requires the integral on the left-hand side to vanish. The last integral of 4-divergence $\partial(\Psi j^\nu)/\partial x^\nu$ on the right-hand side is transformed to vanishing integrals over bounding hypersurfaces (where the imposed function Ψ is assumed to vanish, or irrelevant because the variational analysis is carried out only at internal points). Vanishing of the first integral for arbitrary variation $\delta\Psi$ (at internal points) leads to the following equation of the mass conservation:

$$\partial_\nu j^\nu = \partial_\tau(\rho c_*) + \partial_k(\rho\boldsymbol{v})_k = \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{2.21}$$

Thus, the mechanical effect of the composite action S_c reduces to that of the term $S^{(\text{pf})}$ of perfect fluid without the background field a_μ. Therefore, for potential fields of $\tilde{a}_\mu = \partial_\mu\Psi$, the basic undisturbed state is equivalent to the Eulerian field constrained with the mass conservation equation.

iii. *Gauge invariance of $S^{(\text{int})}$ requires the mass conservation*

It is essential in the scenario of the general gauge principle of Utiyama that the background field a_μ ensures the current conservation $\partial_\mu j^\mu = 0$, by the *gauge invariance* property of the background field a_μ itself. Likewise the transformation done in Kambe (2021a; section II a), we define a one-form $\mathcal{A} \equiv a_\nu\,\mathsf{d}x^\nu$ and introduce an arbitrary scalar field $\Theta(x^\nu)$. Then, we carry out a *gauge transformation*, $\mathcal{G}$: $a_\nu \to a'_\nu = a_\nu - \partial_\nu\Theta$, and we have

$$\mathcal{A}' \equiv a'_\nu\,\mathsf{d}x^\nu = (a_\nu - \partial_\nu\Theta)\,\mathsf{d}x^\nu = a_\nu\mathsf{d}x^\nu - \partial_\nu\Theta\;\mathsf{d}x^\nu = \mathcal{A} - \mathsf{d}\Theta. \tag{2.22}$$

From this, we find the invariance of the field strength two-form $\mathcal{F} \equiv \mathsf{d}\mathcal{A}$ as follows:

$$\mathcal{F}' \equiv \mathsf{d}\mathcal{A}' = \mathsf{d}\mathcal{A} + \mathsf{d}^2\Theta = \mathsf{d}\mathcal{A} \equiv \mathcal{F}, \tag{2.23}$$

since $\mathsf{d}^2\Theta = 0$ identically. Thus it is found that the two-form $\mathcal{F}$ is invariant with respect to the gauge transformation $\mathcal{G}$.

Matrix elements of $\mathcal{F}$ represented by $f_{\nu\lambda} = \partial_\nu a_\lambda - \partial_\lambda a_\nu$ are also *gauge-invariant*. The expressions of (3.15) give matrix-form representations of $f_{\nu\lambda}$ and its contravariant form $f^{\nu\lambda}$. In these matrix forms, the 4-potential a_ν is expressed by $(-\phi/c, \boldsymbol{a})$, together with $\boldsymbol{b} = (b_k) \equiv \nabla\times\boldsymbol{a}$ and $\boldsymbol{e} = (e_k) \equiv -\partial_t\boldsymbol{a} - \nabla\phi$. Thus, the background field matrix $f_{\nu\lambda}$ can be represented by components of $\boldsymbol{b}$ and $\boldsymbol{e}$, which are also *gauge-invariant*.

Next, let us require invariance of the action $S^{(\text{int})}$ under the gauge transformation $\mathcal{G}$ for arbitrary scalar field of $\Theta(x^\nu)$. By replacing a_μ with $a_\mu - \partial_\mu\Theta$. Then, the action $S^{(\text{int})}$ of (2.9) has an additional term (which is required to vanish),

$$-\int j^\mu\,(\partial_\mu\Theta)\,\mathrm{d}\Omega = \int (\partial_\mu j^\mu)\,\Theta\,\mathrm{d}\Omega - \int \partial_\mu\Big(j^\mu\,\Theta\Big)\,\mathrm{d}\Omega, \tag{2.24}$$

The gauge invariance requires the integral on the left-hand side to vanish. The last integral of 4-divergence $\partial(\Theta j^\nu)/\partial x^\nu$ is transformed to vanishing integrals over bounding surfaces where the imposed function Θ is assumed to vanish. Vanishing of the above integral for arbitrary Θ at internal points leads to the current conservation $\partial_\mu j^\mu = 0$. Namely, the following mass conservation equation must be satisfied:

$$\partial_\mu j^\mu = \partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{2.25}$$

Hence, the invariance of $S^{(\text{int})}$ under the transformation $\mathcal{G}$ requires the mass conservation equation to be satisfied.

iv. *Action $S^{(\mathrm{F})}$ of the background Field a_ν*

Up to now, the action formulation on our fluid system is not completed. To make the fluid system self-contained, we need a third action $S^{(\mathrm{F})} = \int \mathcal{L}^{(\mathrm{F})} \mathrm{d}\Omega$ governing free-state of the background field a_ν, describing only on the property of the field itself. To establish the form of the Lagrangian density $\mathcal{L}^{(\mathrm{F})}$ of the field, we start from the following observation and requirements:

(i) The tensor field $f_{\nu\lambda}$ should be ensured to vanish when the background field a_ν takes the special form $\tilde{a}_\nu = \partial_\mu \Psi$ with $\Psi(x^\nu)$ a twice differentiable scalar field. This means the following. According to the item $(i.)$ of this section, the original Euler's equation of motion (2.2) is valid in spite of the existence of the field a_ν.

(ii) The Lagrangian density $\mathcal{L}^{(\mathrm{F})}$ is a Lorentz scalar, *i.e.* invariant with respect to the Lorentz transformation (Appendix).

(iii) In the subsections $(i.) \sim (iii.)$ of this section II e) we have already defined the field strength tensor $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, which is *gauge-invariant* and also satisfies the condition (i), namely $f_{\mu\nu} = 0$ for a_μ given by $\tilde{a}_\nu = \partial_\mu \Psi$. This representation of $f_{\mu\nu}$ was derived from the one-form $\mathcal{A} = a_\mu dx^\mu$ defined by (1.17) in Sections I e) ii.

Under these conditions, we expect the free-Lagrangian $\mathcal{L}^{(\mathrm{F})}$ to be quadratic in $\partial_\mu a_\nu$ or $f_{\mu\nu}$, because the variation of $S^{(\mathrm{F})}$ reduces the degree by one with resulting equation becoming linear to $\partial_\mu a_\nu$. The only Lorentz-invariant quadratic form is a multiple of $f_{\mu\nu} f^{\mu\nu}$ (see Landau & Lifshitz (1975, §27), or Jackson (1999, §12.7), for the corresponding Lagrangian of electromagnetic field).

Thus, the Lagrangian density $\mathcal{L}^{(\mathrm{F})}$ for the background Field a_ν should be represented as

$$S^{(\mathrm{F})} = \int \mathcal{L}^{(\mathrm{F})} \mathrm{d}\Omega, \qquad \mathcal{L}^{(\mathrm{F})} \equiv C\, f_{\mu\nu} f^{\mu\nu}. \qquad C : \text{a constant}. \tag{2.26}$$

f) New field equations re-ensuring current conservation

According to the observations of Sections II c) $\sim$ e) on the actions of the present fluid system, it is proposed that total Lagrangian density $\mathcal{L}$ consists of three terms: Lagrangians of (i) a perfect fluid $\mathcal{L}^{(\mathrm{pf})}$, (ii) a background field $\mathcal{L}^{(\mathrm{F})}$ and (iii) their mutual interaction $\mathcal{L}^{(\mathrm{int})}$. Hence, the total Lagrangian is $\mathcal{L} = \mathcal{L}^{(\mathrm{pf})} + \mathcal{L}^{(\mathrm{int})} + \mathcal{L}^{(\mathrm{F})}$:

$$\mathcal{L}^{(\mathrm{pf})} = -c\,\overline{\rho}\,(1 + c^{-2}\,\overline{\epsilon}); \quad \mathcal{L}^{(\mathrm{int})} = c^{-1}\, j^\mu\, a_\mu; \quad \mathcal{L}^{(\mathrm{F})} = -\frac{1}{4\mu c}\, f_{\mu\nu} f^{\mu\nu}. \tag{2.27}$$

where the constant C is rewritten as $C = -(4\mu c)^{-1}$ with using another constant μ for later convenience.

i. *Action principle*

We define the total action $S^{(\mathrm{total})}$ by

$$S^{(\mathrm{total})} = S_c + S^{(\mathrm{F})} = \int \left[\int \left(\mathcal{L}^{(\mathrm{pf})} + \mathcal{L}^{(\mathrm{int})} + \mathcal{L}^{(\mathrm{F})} \right) \mathrm{d}\mathcal{V} \right] c\, \mathrm{d}t, \tag{2.28}$$

where the Lagrangian densities $\mathcal{L}^{(\mathrm{pf})}$, $\mathcal{L}^{(\mathrm{int})}$ and $\mathcal{L}^{(\mathrm{F})}$ are defined by (2.27), $\mathrm{d}\Omega = c\, \mathrm{d}t\, \mathrm{d}\mathcal{V}$, $\mathrm{d}\mathcal{V} = \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3$ and $\overline{\rho}\, \overline{\mathrm{d}\mathcal{V}} = \rho\, \mathrm{d}\mathcal{V}$.† Relativistic 4-current is defined by $j^\nu \equiv \overline{\rho} v^\nu$, in addition by

$$j^\nu = \rho\, \frac{\mathrm{d}X^\nu}{\mathrm{d}t} = \rho\, v^\nu \sqrt{1 - \beta^2} = \rho\, (\, c,\, \boldsymbol{v}\,). \tag{2.29}$$

† $\mathrm{d}\tau = c\mathrm{d}t \sqrt{1 - \beta^2}$, $\overline{\mathrm{d}\mathcal{V}} = \mathrm{d}\mathcal{V} / \sqrt{1 - \beta^2}$ and $\overline{\rho} = \rho\, \sqrt{1 - \beta^2}$. Hence $\overline{\mathrm{d}\mathcal{V}}\, \mathrm{d}\tau = \mathrm{d}\mathcal{V}\, c\, \mathrm{d}t$.

The tensor $f_{\nu\lambda} \equiv \partial_\nu a_\lambda - \partial_\lambda a_\nu$ in the expression $\mathcal{L}^{(\mathrm{F})}$ is a field strength tensor, and $a_\nu \equiv (-\phi/c, \boldsymbol{a})$ is a 4-potential.

To find the equations governing the background field a_ν, the principle of least action is applied to the action $S^{(\mathrm{total})}$. We must assume the fluid motion a given field, hence fixed. We vary only the potential field a_ν. In regard to the fluid motion, its equation of motion is already found from the composite action S_c in the section II c) where a_ν is assumed to be a given field. The equation of motion is given by (2.11). Citing it,

$$\mathrm{D}_t v^k + \rho^{-1}\,\partial_k p - f_{k\nu}\,v^\nu = 0, \qquad (k = 1, 2, 3; \;\; \nu = 0, 1, 2, 3). \tag{2.30}$$

We aim that the continuity equation of fluid flows is deduced also from this variational analysis. Since the first Lagrangian $\mathcal{L}^{(\mathrm{pf})}$ does not include the field a_ν to be varied, we consider variations of the other two Lagrangians $\mathcal{L}^{(\mathrm{int})}$ and $\mathcal{L}^{(\mathrm{F})}$.

Variation with respect to a_ν

The two Lagrangians $\mathcal{L}^{(\mathrm{int})}$ and $\mathcal{L}^{(\mathrm{F})}$ include the background field a_ν. First, we note $\delta\Big(f^{\nu\lambda}\,f_{\nu\lambda}\Big) = 2f^{\nu\lambda}\,(\delta f_{\nu\lambda})$. This is because

$$(\delta f^{\nu\lambda})\,f_{\nu\lambda} = (\delta f^{\nu\lambda})\,\eta_{\nu\alpha}\eta_{\lambda\beta}f^{\alpha\beta} = f^{\alpha\beta}\,(\delta f_{\alpha\beta}).$$

See Appendix A for the Minkowski metric $\eta_{\nu\alpha}$.

Therefore, variation of $\mathcal{L}^{(\mathrm{int})} + \mathcal{L}^{(\mathrm{F})}$ is given by

$$c\,(\delta\mathcal{L}^{(\mathrm{int})} + \delta\mathcal{L}^{(\mathrm{F})}\,) = j^\nu\,\delta a_\nu - \frac{1}{2\mu} f^{\nu\lambda}\,\delta f_{\nu\lambda} = j^\nu\,\delta a_\nu - \frac{1}{2\mu}\,f^{\nu\lambda}\frac{\partial}{\partial x^\nu}\delta a_\lambda$$

$$+ \frac{1}{2\mu}\,f^{\nu\lambda}\frac{\partial}{\partial x^\lambda}\delta a_\nu \;=\; \left(j^\nu \;-\frac{1}{\mu}\frac{\partial}{\partial x^\lambda}f^{\nu\lambda}\right)\delta a_\nu. \tag{2.31}$$

where the term $-(1/2\mu)f^{\nu\lambda}\partial_\nu(\delta a_\lambda)$ of the last term on the upper line can be equated to $(1/2\mu)f^{\nu\lambda}\partial_\lambda(\delta a_\nu)$ by using the anti-symmetry, $-f^{\nu\lambda} = f^{\lambda\nu}$. On interchanging the indices ν and λ, this term can be combined with its next term to give $(1/\mu)f^{\nu\lambda}\partial_\lambda(\delta a_\nu)$. Finally carrying out integration-by-parts leads to the second term of (2.31), with omitting the term of divergence-form $\partial_\lambda[(1/\mu)f^{\nu\lambda}\,\delta a_\nu]$ which vanishes on integration.

Requiring vanishing of $\delta\mathcal{L}^{(\mathrm{int})} + \delta\mathcal{L}^{(\mathrm{F})} = 0$ for arbitrary variation δa_ν, we obtain

$$\frac{\partial}{\partial x^\lambda}f^{\nu\lambda} = \mu\,j^\nu. \tag{2.32}$$

where the 4-current j^ν is defined by (2.29). This is the equation governing the background field $f^{\nu\lambda}$ derived from the principle of least action. Thus,

the system of field equations (2.30) and (2.32) have been derived by the invariant variation of the total action $S^{(\mathrm{total})}$ of (2.28).

ii. *Current conservation*

The equation of current conservation can be derived from this, which is directly connected with the gauge-invariant property of the Lagrangian $\mathcal{L}^{(\mathrm{F})}$. This is analogous to the electromagnetic fields (Kambe (2021a) Section II a) ii). In fact, applying the divergence operator ∂_ν on the equation (2.32), one obtains

$$0 = \partial_\nu\partial_\lambda f^{\nu\lambda} = \mu\,\partial_\nu j^\nu. \tag{2.33}$$

The left-hand side vanishes because of the anti-symmetry of $f^{\nu\lambda}$ and the symmetry of $\partial_\nu\partial_\lambda$. Total sum with respect to ν and λ (taking indices $0, 1, 2, 3$) vanishes identically. Hence, we find the current conservation equation:

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\partial_\nu j^\nu = 0 \quad \Rightarrow \quad \partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{2.34}$$

for $j^\nu = (\rho c, \, \rho \boldsymbol{v})$. Thus it is found that the newly added Lagrangian $\mathcal{L}^{(\mathrm{F})}$ ensures the mass conservation.

All the analyses, concerning the field potential a_ν, the differential forms and the variations, are exactly analogous to the electromagnetic case (Kambe (2021a) §2.1 (a)). Only differences are the letters used, whether those are lower-case or upper-case, and the material constants are different between the two cases. Thus for our fluid system, we obtain the same form of Maxwell-type equations with the field vectors $\boldsymbol{e} \equiv -\partial_t \boldsymbol{a} - \nabla \phi$ and $\boldsymbol{b} \equiv \nabla \times \boldsymbol{a}$ with two field constants, μ and $\varepsilon = 1/(c^2 \mu)$:

$$\partial_t \boldsymbol{b} + \nabla \times \boldsymbol{e} = 0, \qquad \nabla \cdot \boldsymbol{b} = 0. \tag{2.35}$$

$$\varepsilon \, \nabla \cdot \boldsymbol{e} = \rho, \qquad -\varepsilon \, \partial_t \boldsymbol{e} + \mu^{-1} \nabla \times \boldsymbol{b} = \boldsymbol{j}. \tag{2.36}$$

III. Fluid Gauge Theory

a) A road to fluid gauge theory

More than sixty years ago, Utiyama (1956) proposed a general approach to the gauge theory and called it *General Gauge Theory*. He extended the Weyl's gauge principle (described in Kambe (2021a)) to general Lie groups and included the theory of gravity (O'Raifeartaigh 1997, Chap.10). He realized already the broad analogy between the two theories of gravitational field and electromagnetic field, which is reviewed in Kambe (2021a) too. According to the scenario of the general gauge theory, new fields are introduced in the systems under investigation which are carrying interaction forces such as gravity force or electromagnetic force.

How the fluid-flow field is required to be improved or reformed by the fluid gauge theory ? The answer is that the isotropic pressure stress field is extended to general anisotropic stress field in the flow of perfect fluid if its motion is time-dependent and rotational. This is required by the constraint to the current conservation under background gauge fields. However, the Euler's equation of motion is still valid as far as the stress field is constrained to be isotropic.

On a mathematical point of view, more must be added according to the Utiyama's approach of the *General Gauge Theory*. The new fields should be taken (or absorbed) into the structure of covariant derivatives as connection terms like the terms of Christoffel symbols in the gravity theory or the gauge potentials in the electromagnetic theory. The concept of *connection* in the mathematics of Riemannian geometry is an essential ingredient in the physical gauge theory. It is a challenging work to implement connection terms in the structure of fluid gauge theory. This is left to Appendix B (*Relativistic formulation of three mechanical systems*) and Appendix C (*Free motion of physical systems and curved geodesics*), because sufficient mathematical expressions and concepts must be presented to arrive at the goal.

We are going to propose and present a new formulation of *Fluid Gauge Theory* in this section §3 by the help of Appendix B and Appendix C. In fact, before giving the final conclusion, we have to examine that free motions are not always described by straight trajectories of time evolution of mechanical systems, namely their geodesics are curved in general. In Appendix C.2, we will see that the free motion of a perfect fluid under a background field a_ν can be described by the geodesic equation representing a curved free dynamics. This is derived by the variational principle that makes the action integrals invariant. Namely the new field of the fluid gauge theory has been taken into the structure of covariant derivatives as connection terms and the free dynamics of a perfect fluid under a background field a_ν is described by the geodesic equation.

Thus, the Fluid Gauge Theory is concluded now.

b) Fluid Gauge Theory summarized

According to the scenario of the general gauge principle, the last section II has concluded that the flow field of a perfect fluid is supported with a background field a_μ ensuring current conservation. Now our quest for the fluid gauge theory has come to the final stage.

Statement of the fluid gauge theory:

Collecting main results obtained in the last section II *(An approach aiming at a fluid gauge theory)*, the fluid gauge theory is presented by the following set of expressions:

$$S^{(\text{total})} = S^{(\text{pf})} + S^{(\text{int})} + S^{(\text{F})} = \int \Big[\int \Big(\mathcal{L}^{(\text{pf})} + \mathcal{L}^{(\text{int})} + \mathcal{L}^{(\text{F})} \Big) \, \mathrm{d}\mathcal{V} \Big] \, c \, \mathrm{d}t, \qquad (3.1)$$

$$\mathcal{L}^{(\text{pf})} = -c^{-1}(c^2 + \overline{\epsilon}(\overline{\rho}))\,\overline{\rho}\,, \qquad \mathcal{L}^{(\text{int})} = c^{-1} j^\nu \, a_\nu\,, \qquad (3.2)$$

$$\mathcal{L}^{(\text{F})} = -\frac{1}{4\mu c} f^{\nu\lambda} \, f_{\nu\lambda}, \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \qquad (3.3)$$

$$a_\mu = (a_0, a_1, a_2, a_3) = (-\phi/c, \boldsymbol{a}), \qquad (3.4)$$

$$\mathrm{D}_t \boldsymbol{v} + \rho^{-1} \, \nabla p \qquad = \boldsymbol{f}_a\,, \qquad (3.5)$$

$$\partial_\nu j^\nu = \partial_t \rho + \nabla \cdot \boldsymbol{j} = 0, \qquad j^\nu = (\rho c, \boldsymbol{j}), \quad \boldsymbol{j} = \rho\, \boldsymbol{v}, \qquad (3.6)$$

$$\boldsymbol{f}_a \quad = \boldsymbol{e} + \boldsymbol{v} \times \boldsymbol{b} \qquad (3.7)$$

$$\boldsymbol{b} \quad = \nabla \times \boldsymbol{a}, \qquad \boldsymbol{e} = -\partial_t \boldsymbol{a} - \nabla\phi. \qquad (3.8)$$

$$\nabla \cdot \boldsymbol{b} = 0, \qquad \partial_t \boldsymbol{b} + \nabla \times \boldsymbol{e} = 0, \qquad (3.9)$$

$$\nabla \cdot (\varepsilon\, \boldsymbol{e}) = \rho, \qquad -\partial_t(\varepsilon\, \boldsymbol{e}) + \mu^{-1} \nabla \times \boldsymbol{b} = \boldsymbol{j}. \qquad (3.10)$$

where c is the light velocity and a_μ the background field. Two parameters μ and $\varepsilon = 1/(c^2\,\mu)$ are field constants.

According to the scenario of Utiyama (1956), to begin with, we have to check the global invariance of the system under consideration. Namely, with respect to the global transformation $a_\mu \to a_\mu + \epsilon_\mu$ for constants ϵ_μ independent of coordinates x^ν, we ask whether the action integral $S^{(\text{total})}$ and the governing equation (3.5) derived from it are invariant or not.

In regard to the modified Euler equation (3.5), the field a_μ is included only in the additional term $\boldsymbol{f}_a$ which includes all the components a_μ linearly and in derivative forms, as seen from (3.7) and (3.8). Hence, constant variation ϵ_μ of a_μ does not give any effect on the equation (3.5). Not only the equation of motion (3.5), but the action $S^{(\text{total})}$ of (3.1) must be invariant. Since the third Lagrangian $\mathcal{L}^{(\text{F})}$ of (3.3) includes only derivative forms of a_μ, the above global transformation causes no variation. However, regarding the second interaction Lagrangian $\mathcal{L}^{(\text{int})}$, its integrand $\mathcal{J}^{(\text{int})}$ of $S^{(\text{int})}$ associated with the part $\mathrm{d}m = \rho\,\mathrm{d}\mathcal{V}$ is given by

$$\mathcal{J}^{(\text{int})} \equiv \mathcal{L}^{(\text{int})} \, \mathrm{d}\mathcal{V} \; c\,\mathrm{d}t = c^{-1} \, (\rho\, \mathrm{d}\mathcal{V}) \; a_\mu \, (\mathrm{d}X^\mu/\mathrm{d}t) \, c\,\mathrm{d}t = (\mathrm{d}m) \, a_\mu \, \mathrm{d}X^\mu. \qquad (3.11)$$

With respect to the global transformation $a_\mu \to a_\mu + \epsilon_\mu$, the action variation is

$$\mathsf{d}S^{\text{int}} = \int \mathcal{J}^{(\text{int})} = (\mathrm{d}m)\, \epsilon_\mu \int_a^b \mathrm{d}X^\mu = (\mathrm{d}m)\, \epsilon_\mu \, [X^\mu]_a^b\,,$$

for a fixed mass element $\mathrm{d}m$ of a fluid particle. It is seen that the variation $\mathsf{d}S^{\text{int}}$ does not depend on internal values of X^μ, but depends only on its boundary values. Hence the constant variation ϵ_μ does not give any mechanical effect on the system. Thus, the present system described by $S^{(\text{total})}$ is *globally* invariant for the uniform a_μ-variation.

Next, even if the global invariance of the system $S^{(\text{total})}$ is satisfied, one may consider local transformation by $\delta a_\mu(x^\nu)$ varying with space-time coordinates x^ν. By substituting the transformed variable $a_\mu + \delta a_\mu(x^\nu)$ into a_μ of the action $S^{(\text{total})}$ of (2.28), the system of field equations (2.30) and (2.32) have been deduced by the action principle in section II f) In other words, *Invariant Variation* of the total action $S^{(\text{total})}$ of (3.1) yields the system of field equations (3.5) and (3.10). The two equations of (3.9) are immediately derived from the identity $\mathsf{d}^2\mathcal{A} \equiv 0$ satisfied by the one-form $\mathcal{A}$, which is given in section III c) ii.

The present system is a *genuine* mechanical system described by the action (3.1) and the equation of motion (3.5) derived by the action principle. Then, in this case, there must be a certain background field $a_\mu(x^\nu)$ that is interacting with the flow field $v^\mu(x^\nu)$, and the interaction force is the fluid Lorentz force $\boldsymbol{f}_a$ of (3.7). Concerning the last point, more detailed account is given at the second half of the last section of Section IV *(Summary and discussions)*. The background field $a_\mu(x^\nu)$ ensures the current conservation $\partial_\nu j^\nu = 0$ of (3.6), which is verified in section II f) ii. based on the gauge invariance. Hence, our riddle mentioned in section I c) is resolved.

Thus, the above fluid gauge theory is proposed, according to the gauge principle of Utiyama (1956). The field of fluid-flow is required to be improved or reformed as follows. The flow field of Eulerian system is characterized by the isotropic pressure stress field. The stress field is extended to general anisotropic stress field in the flow of a perfect fluid if its motion is time-dependent and rotational. This is required by the constraint of the current conservation driven by the background gauge fields.

c) Remarks on the fluid gauge theory

Whole structure of the present theory is founded on the *Gauge Principle* which worked successfully in theoretical physics, particularly in the particles physics. The analyses presented so far in the present paper verifies that the scenario of the *Gauge Principle* works successfully in the flow field of a perfect fluid too. Here in this section some remarks on the theory are presented in order to supplement insufficient parts of the above presentation.

i. Euler's equation $\mathrm{D}_t\boldsymbol{v} + \rho^{-1}\nabla p = 0$ is still valid

Euler's equation of motion is still valid as a family member. There is a special class of background field $\tilde{a}_\nu$, for which the equation (3.5) reduces to the Euler's equation of motion (2.2) for $\tilde{a}_\nu = \partial_\nu \Psi$, because, with this field, the fluid Maxwell fields $\boldsymbol{e}$ and $\boldsymbol{b}$ vanish by (3.8) (where $a_k \to +\partial_k\Psi$ and $\phi \to -\partial_t\Psi$ since $a_\mu = (-\phi/c, \boldsymbol{a})$ and $\partial_0 = c^{-1}\partial_t$), and hence $\boldsymbol{f}_a$ of (3.7) vanishes as well.

Namely, the Euler's equation of motion (2.2), $\mathrm{D}_t\boldsymbol{v} + \rho^{-1}\ \nabla p = 0$, is valid for the background field $\tilde{a}_\nu = \partial_\nu\Psi$ and the continuity equation $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{v}) = 0$ is deduced from the action principle as shown in section II e) ii. Accordingly, one may say that the field $\tilde{a}_\nu$ itself exists, but does not show in the observable world. Hence, the rotational flow solution (Kambe 2013) mentioned in section I d) and Appendix C is yet valid as a general solution to the Euler's equation of motion $\mathrm{D}_t\boldsymbol{v} + \rho^{-1}\ \nabla p = -\nabla\Phi_E$, where the term $-\nabla\Phi_E$ on the right hand side (due to the gravitational potential Φ_E) does not cause any problem in the present context. Anyway, if Φ_E is set to a constant, this equation reduces to (2.2). The class of flow fields governed by the Euler's equation of motion, $\mathrm{D}_t\boldsymbol{v} + \rho^{-1}\ \nabla p = 0$, may be called a *ground* flow-state.

ii. Significance of the new force $\boldsymbol{f}_a$ represented with anisotropic stress tensor

a. *Background field: Fluid Maxwell fields*

In section II e) i, we have defined the 4-potential a_ν by $(a_0, \boldsymbol{a})$ with $a_0 = -\phi/c$ and $\boldsymbol{a} = (a_1, a_2, a_3)$ and the one form $\mathcal{A} = a_\nu\,\mathsf{d}x^\nu$ with $x^0 \equiv ct$. Taking exterior differential, the *field strength* two-form $\mathcal{F}$ is given by

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$\mathcal{F} = \mathsf{d}\mathcal{A} = \sum_{\nu<\lambda} f_{\nu\lambda}\, \mathrm{d}x^\nu \wedge \mathrm{d}x^\lambda\,, \tag{3.12}$$

$$f_{\nu\lambda} = \partial_\nu a_\lambda - \partial_\lambda a_\nu = -f_{\lambda\nu}. \tag{3.13}$$

Then, a pair of *fluid* Maxwell fields $\boldsymbol{e}$ and $\boldsymbol{b}$ are defined:

$$\boldsymbol{e} \equiv -\partial_t \boldsymbol{a} - \nabla\phi, \qquad \boldsymbol{b} \equiv \nabla\times\boldsymbol{a}\,, \qquad \text{and} \quad \bar{\boldsymbol{e}} \equiv \boldsymbol{e}/c\,. \tag{3.14}$$

For the tensor $f_{\nu\lambda}$, the diagonal elements are all zero, and the element f_{01} is given by $\partial_0 a_1 - \partial_1 a_0 = (\partial_t a_1 + \partial_1\phi)/c = -e_1/c = -\bar{e}_1$ and the element f_{12} given by $\partial_1 a_2 - \partial_2 a_1 = (\nabla\times\boldsymbol{a})_3 = b_3$. The field strength tensor of covariant (downstairs) indices $f_{\mu\nu}$ and that of contravariant (upstairs) form $f^{\mu\nu} = \eta^{\mu\alpha}F_{\alpha\beta}\eta^{\beta\nu}$ are given by matrix forms as follows:‡

$$(\,f_{\nu\lambda}\,) = \begin{pmatrix} 0 & -\bar{e}_1 & -\bar{e}_2 & -\bar{e}_3 \\ \bar{e}_1 & 0 & b_3 & -b_2 \\ \bar{e}_2 & -b_3 & 0 & b_1 \\ \bar{e}_3 & b_2 & -b_1 & 0 \end{pmatrix}, \quad (\,f^{\nu\lambda}\,) = \begin{pmatrix} 0 & \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ -\bar{e}_1 & 0 & b_3 & -b_2 \\ -\bar{e}_2 & -b_3 & 0 & b_1 \\ -\bar{e}_3 & b_2 & -b_1 & 0 \end{pmatrix}. \tag{3.15}$$

Taking non-relativistic limit of the definition of $v^\nu = cu^\nu$ of (A.3), we have $v^\nu = (c, \boldsymbol{v})$. Then, from (3.15) with noting the index of $f_{k\nu}$, the equation (2.30), equivalently (2.11), can be represented in 3-vector form as follows:

$$\mathrm{D}_t \boldsymbol{v} + \rho^{-1}\,\nabla p - \boldsymbol{f}_a = 0, \tag{3.16}$$

$$\boldsymbol{f}_a \equiv \boldsymbol{e} + \boldsymbol{v}\times\boldsymbol{b} = -\partial_t\boldsymbol{a} - \nabla\phi + \boldsymbol{v}\times(\nabla\times\boldsymbol{a}). \tag{3.17}$$

The equation (3.16) is the modified Euler's equation of motion of a perfect fluid with additional force term $\boldsymbol{f}_a$ depending on the assumed background gauge-potential a_μ. Note that the background potential $a_\mu = (-\phi/c, \boldsymbol{a})$ is analogous to the electromagnetic gauge potential A_μ of eq.(2.1) of Kambe (2021a), where Φ corresponds to ϕ/c.

Using the definition (3.14) of the fields $\boldsymbol{e}$ and $\boldsymbol{b}$ and the matrix representation (3.15) of the field strength tensor $f^{\nu\lambda}$, the equation (2.32) represents the followings:

$$\varepsilon\,\nabla\cdot\boldsymbol{e} = \rho, \qquad -\varepsilon\,\partial_t\boldsymbol{e} + \mu^{-1}\nabla\times\boldsymbol{b} = \boldsymbol{j}. \tag{3.18}$$

where $\varepsilon = 1/(c^2\,\mu)$. Another pair of fluid Maxwell equations is given by (3.20),

b. *Second pair of fluid Maxwell equations for the background field $a_\nu = (-\phi/c, \boldsymbol{a})$*

Using the field tensor $(f_{\nu\lambda})$ defined by (3.13), one can derive *fluid Maxwell equations* of source-free type. In fact, taking one more exterior differential of $\mathcal{F}$ of (3.12), we obtain

$$\mathsf{d}\mathcal{F} = \mathsf{d}^2\mathcal{A} = \sum_{\alpha<\beta<\gamma} \Big(\partial_\alpha f_{\beta\gamma} + \partial_\beta f_{\gamma\alpha} + \partial_\gamma f_{\alpha\beta}\Big)\, \mathrm{d}x^\alpha\wedge\mathrm{d}x^\beta\wedge\mathrm{d}x^\gamma \;=\; 0, \tag{3.19}$$

because $\mathsf{d}^2\mathcal{A} \equiv 0$. This leads to the equation, $\partial_\alpha f_{\beta\gamma} + \partial_\beta f_{\gamma\alpha} + \partial_\gamma f_{\alpha\beta} = 0$, yielding the following pair of *fluid* Maxwell equations for $\boldsymbol{b}$ and $\boldsymbol{e}$ of (3.14):

$$\nabla\cdot\boldsymbol{b} = 0, \qquad \partial_t\boldsymbol{b} + \nabla\times\boldsymbol{e} = 0, \tag{3.20}$$

where the first is obtained with $(\alpha,\,\beta,\,\gamma) = (1,2,3)$, while the second is obtained when one of α, β and γ takes the suffix number 0.

‡ The latter matrix $f^{\nu\lambda}$ is to be used later to derive field equations. Practically, the matrix $f^{\nu\lambda}$ is obtained from $f_{\nu\lambda}$ with simply replacing $\bar{\boldsymbol{e}}$ by $-\bar{\boldsymbol{e}}$.

c. *Significance of $\boldsymbol{f}_a$ in the representation with anisotropic stress tensor*

Because the formulation described so far is based on logical reasonings both physically and mathematically, the present theory is valid solidly. The present fluid gauge theory for a perfect fluid represents a broader class of flow fields than the current Eulerian field, although the Eulerian field too is valid, as stated in the part i. of this section. The current Eulerian flow field is governed by the Euler's equation of motion: $\mathrm{D}_t\boldsymbol{v} + \rho^{-1}\,\nabla p = 0$. However, the present gauge theory extends the current Eulerian flow field to a wider class, covering a broader family of flow fields of a perfect fluid (an inviscid fluid).

In the presence of background field a^ν, the governing equations are given by (3.5), (3.7) and (3.8):

$$\rho\,\mathrm{D}_t\boldsymbol{v} = -\nabla p + \rho\,\boldsymbol{f}_a. \tag{3.21}$$

$$\boldsymbol{f}_a = \boldsymbol{e} + \boldsymbol{v}\times\boldsymbol{b} \quad = -\nabla\phi - \partial_t\boldsymbol{a} + \boldsymbol{v}\times(\nabla\times\boldsymbol{a}). \tag{3.22}$$

At first sight, it is surprising to see the Lorentz-type force $\boldsymbol{f}_a$ (*acceleration* correctly) in fluid-flow field which is electrically neutral. The role of charge density in the electromagnetism is played by the mass density ρ. Significance of the fluid Lorentz acceleration $\boldsymbol{f}_a$ is interpreted from the following two aspects.

Firstly, the acceleration $\boldsymbol{f}_a$ is independent of the mass density ρ as obviously seen in (3.7), but depends on the velocity $\boldsymbol{v}$ unlike the gravity acceleration, in addition depending on the time derivative term $\partial_t\boldsymbol{a}$ and rotational term $\nabla\times\boldsymbol{a}$. In other words, the acceleration term $\boldsymbol{f}_a$ would become significant in turbulent flow fields in which flow fields are time-dependent and rotational.

It is emphasized that the fluid Lorentz acceleration $\boldsymbol{f}_a$ is considered to be a generalization of the pressure force $-\nabla p$. In fact, citing the equation (2.13) again:

$$\rho\,\mathrm{D}_t v^k = -\partial_j(p\,\delta_{jk}) + \rho\,f_{k\nu}\,v^\nu, \qquad (k=1,2,3;\;\; \nu=0,1,2,3), \tag{3.23}$$

this is equivalent to (3.21), but represented in component form.

Secondly, physical meaning of $\boldsymbol{f}_a$ may be given as follows. It is remarkable to find that the force field $\boldsymbol{F}_a \equiv \rho\boldsymbol{f}_a$ can be represented by the stress field $M^{\nu k}$ (where $\boldsymbol{F}_a$ may be called the fluid Lorentz force). In fact, for spatial components $(i,k=1,2,3)$, the k-th component of the force $\boldsymbol{F}_a \equiv \rho\boldsymbol{f}_a$ can be rewritten as follows:

$$(\boldsymbol{F}_a)_k = (\rho\boldsymbol{e} + \rho\boldsymbol{v}\times\boldsymbol{b})_k = -\,\partial_\nu M^{\nu k}, \qquad \partial_\nu = (c^{-1}\partial_t, \partial_k), \tag{3.24}$$

$$M^{0k} = c\epsilon\,(\boldsymbol{e}\times\boldsymbol{b})_k, \qquad w_e \equiv \tfrac{1}{2}\,\epsilon\,|\boldsymbol{e}|^2 + \tfrac{1}{2}\,\mu^{-1}\,|\boldsymbol{b}|^2 = M^{00},$$

$$M^{ik} = -\epsilon\,e_i e_k - \mu^{-1}\,b_i b_k + w_e\delta_{ik}, \tag{3.25}$$

[$M^{\alpha\beta} \equiv \Theta_{\mathrm{w}}^{\alpha\beta}$ with $\Theta_{\mathrm{w}}^{\alpha\beta}$ defined by Eq.(33) of Kambe (2017)], where μ and $\epsilon = 1/(\mu\,c^2)$ are parameters of flow fields, and the equality $(\rho\boldsymbol{e} + \rho\boldsymbol{v}\times\boldsymbol{b})_k = -\partial_\nu M^{\nu k}$ can be shown by using (3.9) and (3.10).

The stress tensor M^{ik} of (3.25) as well as the parameters ϵ and μ are analogous to those (Maxwell stress) of electromagnetism. The term $(-\nabla p)_k$ on the right-hand side of (3.21) can be written as $-\partial_\nu(p\,\delta^{\nu k})$, a force from the isotropic pressure stress $-p\,\delta^{\nu k}$.

According to the *present* fluid gauge theory, the state of isotropic pressure stress $p\,\delta^{\nu k}$ of Eulerian system is extended to the state of combined anisotropic stress $p\,\delta^{\nu k} + M^{\nu k}$. Namely the isotropic pressure stress $p\,\delta^{\nu k}$ valid at the Eulerian system is modified and augmented by an anisotropic stress $M^{\nu k}$ depending on the velocity v^k and the time change $\partial_t a^k$, to ensure the current conservation.

The flow field described by the Euler's equation $\rho\,\mathrm{D}_t\boldsymbol{v} + \nabla p = 0$ may be called a *ground* flow-state. Then, another flow states governed by (3.21) may be an excited flow-state. The terms "ground" and "excited" are used in analogy with quantum states, although the states here are not discrete.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

IV. Summary and Discussions

Theory of fluid mechanics is extended to cover time-dependent rotational flows under *anisotropic* stress field of a compressible perfect fluid, including turbulent flows. The Eulerian fluid mechanics is characterized with isotropic pressure stress fields. According to the general gauge principle, the current theoretical structure of fluid mechanics can be extended to a wider class of flow fields of a perfect fluid under anisotropic stress field by the *Fluid Gauge Theory* presented in the present paper.

Motivation of the present study is based on three observations. First one is the experimental evidence of observation of large-scale structures coexisting with turbulent flow fields; second one is a physical query of what symmetry implies the current conservation law; and third motivation is posed by a mathematical representation of the field of fluid flow, described in section I d) (Introduction).

The third one is based on the general representation of rotational flows of an ideal compressible fluid satisfying the Euler's equation presented by Kambe (2013), in which "*four independent fields are newly introduced in the general solution to the Euler's equation of motion.* Those fields must be either given externally or to be determined internally within the framework of theory. Present study has taken the latter approach on the understanding that the general solution predicts existence of new fields. Importantly, the very fact that the four independent fields exist in the 4-dimesional linked spacetime encourages a gauge-theoretic formulation on the basis of differential forms or one-form $\mathcal{A}$ existing in the liked $4d$-spacetime.

The last point is essential in the present study in the sense that even the fluid mechanics from the Euler's point of view has a glimpse of structures of linked $4d$-spacetime. As an obvious example, this is seen in the equation of current conservation, $\partial j^\nu/\partial x^\nu$, represented in terms of the current 4-vector $j^\nu = (\rho\, c, \rho v^k)$ and the 4-differential operator $\partial/\partial x^\nu = (c^{-1}\partial_t,\, \partial_k)$. We have $\partial j^\nu/\partial x^\nu = \partial_t \rho + \partial_k(\rho v^k)$. It is remarkable that the scalar product $\partial_\nu j^\nu$ is invariant by the Lorentz transformation.

In the 4d-spacetime x^ν, by introducing a set of four fields $a_\mu(x^\nu)$, a one-form structure is defined by $\mathcal{A} \equiv a_\mu \mathrm{d}x^\mu = \rho^{-1} j^\mu a_\mu\, \mathrm{d}t$. Using it, an interaction action $S^{(\mathrm{int})} = c^{-1}\int j^\mu\, a_\mu \;\mathrm{d}^4x^\nu$ was defined for the combined field of a 4-current field $j^\mu = \rho\,(\mathrm{d}x^\mu/\mathrm{d}t)$ and a background 4-field a_μ. Correspondingly, a combined action $S_c = S^{(\mathrm{pf})} + S^{(\mathrm{int})}$ is defined by incorporating the known action of a perfect fluid $S^{(\mathrm{pf})}$, both expressed relativistically. The action principle applied to S_c yields the equation of motion (2.11):

$$\mathrm{D}_t v^k + \rho^{-1}\, \partial_k p = f_{k\nu}\, v^\nu. \qquad f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu \tag{4.1}$$

($v^k = \eta^{kl} v_l = v_k$). The term $f_{k\nu}\, v^\nu$ on the right-hand side is a new term (owing to interaction) added to the Euler's equation of motion given on the left-hand side. However, if the field a_ν takes a particular form $\tilde{a}_\nu = \partial_\nu \Psi$, then the field tensor $f_{\mu\nu}$ vanishes identically since $f_{\mu\nu} = \partial_\mu\partial_\nu\Psi - \partial_\nu\partial_\mu\Psi = 0$. Then the equation (4.1) reduces to the original Euler's equation, and in addition the equation of current conservation (2.21), $\partial_\nu j^\nu = 0$, is deduced for this case (§2.1).

According to Utiyama (1956, 1987), the general gauge principle states as follows. "*If both of the composite action S_c and the equation of motion (4.1) are invariant under a global transformation of $\tilde{a}_\mu$ defined by $\tilde{a}_\mu \to \tilde{a}_\mu + \delta a_\mu$ for a uniform value of $\delta a_\mu = \epsilon_\mu$ (constant), then the system is said invariant globally for the a_μ-transformation*".

The constant variation ϵ_μ of $\tilde{a}_\mu$ does not give any effect on the equation (4.1). Invariance of the action $S^{(\mathrm{int})}$ (therefore invariance of S_c) is also verified in §3.2.† Hence,

† In §3.2, the transformation was $\tilde{a}_\mu \to a_\mu = \tilde{a}_\mu + \epsilon_\mu$. However, as far as the variation part of the action $S^{(\mathrm{int})}$ is concerned, there is no difference from that of the transformation $a_\mu \to a'_\mu = a_\mu + \epsilon_\mu$.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

the constant variation ϵ_μ does not give any mechanical effect on the system, and the system is *globally* invariant for the uniform a_μ-variation.

The gauge principle reads furthermore, "*Even if the global invariance of S_c is satisfied, one may consider local transformation of $\tilde{a}_\mu \to a_\mu(x^\nu) = \tilde{a}_\mu + \delta a_\mu(x^\nu)$ with $\delta a_\mu(x^\nu)$ varying with the space-time coordinates x^ν.*" The problem is now reduced to whether one can construct a physical system which is invariant under such a local gauge transformation of a_μ field.

The last point is interpreted as follows. There may exist a background field a_μ in the flow field $j^\mu = \rho\, v^\mu$, which interacts with the flow by the force $f_{k\nu}\, v^\nu$ (which vanished in the test of global transformation for $\tilde{a}_\mu$). Existence of a background field a_μ causes drastic change of our battle field. Not only the term $f_{k\nu}\, v^\nu$ is non-vanishing, but also the equations governing the new field a_μ should be given a physically reasonable form.

This is done by introducing the third action $S^{(\mathrm{F})}$ with the total action given by $S^{(\mathrm{total})} = S_c + S^{(\mathrm{F})}$. The system is still free from external forcing. The Appendix C.2 (c) investigates the dynamics: "*Free dynamical systems and the action principle of invariant variations*" based on the *invariant variations* in the presence of *background* gauge field ensuring mass conservation. The background gauge field is an agent to make the dynamical system curved. This is carried out in the present formulation by implementing the connection term (*i.e.* the background gauge field) in the covariant derivative to make up the structure of fluid gauge theory. Finally, the free motion of physical system is described by *curved geodesics*. This scenario of investigating physical systems of "Free motions described by *curved geodesics*" is presented in Appendix C in details. In this way, the geodesic equation governing our physical system in curved motion is given by the equation (C.23) of Appendix C.2 (c), which reduces finally to the modified Euler equation (2.11) obtained in Section II c) or (3.16) of Section III c) ii.

This is the scenario of *Gauge Principle.* Thus, the local invariance is established, and the *gauge - potential* a_μ is taken into the system which represents a new interaction force, and a new force field $\boldsymbol{f}_a(x^\nu)$ has been introduced into our physical system by the gauge principle. Significance of the fluid Lorentz acceleration $(\boldsymbol{f}_a)_k = f_{k\nu}\, v^\nu$ is given in Section III c) ii. The force field $f_{k\nu} v^\nu$ is considered to be a generalization of the pressure acceleration $-\rho^{-1}\partial_i(p\,\delta_{ik})$, as follows.

In the ground flow-state where $a_\nu = \partial_\nu\Psi$, the Euler's equation of motion (2.2) is valid. The equation can be rewritten in the form, $\rho\,\mathrm{D}_t v_k = \partial_i\sigma_{ik}\,(\equiv F_k)$, where the stress field σ_{ik} is isotropic: $\sigma_{ik}[\mathrm{iso}] \equiv -p(x^\nu)\,\delta_{ik}$. One can say that the background field $\tilde{a}_\nu \equiv \partial_\nu\Psi$ itself exists, but it does not show in the observable world.

However, transition of the stress field can occur from the isotropic state $\sigma_{ik}[\mathrm{iso}]$ to states of anisotropic stress $\sigma_{ik}[\mathrm{aniso}]$ when the flow field (velocity field) becomes non-uniform and time-dependent. In other words, in addition to the isotropic pressure stress $\sigma_{ik}[\mathrm{iso}] = -p\,\delta_{ik}$ valid at the rest frame $v^k = 0$, an anisotropic stress field $\sigma_{ik}[\mathrm{aniso}]$ begins to grow, which depends on the velocity v^k and the time change $\partial_t a_k$, to ensure the current conservation. To be more precise, using the fluid Maxwell stress $M^{\nu k}$ of (3.25), the force F_k from the anisotropic stress is given by (3.24), as follows:

$$F_k[\mathrm{aniso}] = -\partial_\nu M^{\nu k} = -\rho\,\partial_t a_k - \rho\,\partial_k\phi + \rho\,(\boldsymbol{v}\times\boldsymbol{b})_k\,.$$

Thus, outcomes of the Fluid Gauge Theory are summarized in this concluding section.

Based on the present fluid gauge theory, a test study has been carried out recently by M. Hashiguchi (Former iCFD researcher, Tokyo) with solving numerically the new system, finding an encouraging result on the problem: "*Dust striations observed in the resonance-tube experiment*" by August Kundt (1866). Brief result is shown in the presentation of Kambe (2021b). In this system there exist two different length scales with their ratio more than fifty, consistent with the photo observation of Kundt. The

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

larger scale corresponds to the wavelength of the resonant acoustic wave, while the smaller one corresponds to an eddy structure generated by the background gauge field.

Acknowledgments

The author expresses his deep thanks to late Professor Ryoyu Utiyama, the author of the book in Japanese "一般ゲージ場論序説 (Ippan Gauge ba ron josetsu) *Introduction to the general gauge field theory*". The present study aims to apply the theory to fluid systems according to the scenario of the *theory of general gauge fields* of Utiyama (1956, 1987). One of the motivations of the present study is that Fluid Mechanics of a perfect fluid can join in the circles of the physical theory. Also, the present author benefitted from informal discussions with Professor Yasuhide Fukumoto (Kyushu University) on the theory of general gauge fields of Utiyama.

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Appendices

Appendix A. Some basics of linked 4d-spacetime in relativity theory

Here, some basics of the relativity theory are presented for expressions of linked 4-dimensional space-time.

Suppose that a material particle or fluid particles are moving with high velocities in an inertial frame K: (x^0, x^1, x^2, x^3) with $x^0 = ct$ and c the light velocity. In a time interval $\mathrm{d}t$, the position of the particle changes with time and its displacement is given by a 4-vector:

$$\mathrm{d}x^\mu = (c\,\mathrm{d}t, \mathrm{d}X^1, \mathrm{d}X^2, \mathrm{d}X^3), \qquad \mathrm{d}X^k = v^k\,\mathrm{d}t \quad (k = 1, 2, 3), \tag{A.1}$$

where $\mu = 0, 1, 2, 3$, and the upper-case notation $\mathrm{d}X^k$ denotes displacement of a material (fluid) particle with v^k components of 3-velocity $\boldsymbol{v}$. In the relativity theory, an infinitesimal interval $\mathrm{d}s$ is defined by its squared form, $\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu$, which is a scalar product of a line-element 4-vector $\mathrm{d}x^\mu$ with its covariant version $\mathrm{d}x_\mu = \eta_{\mu\nu}\mathrm{d}x^\nu = (-c\,\mathrm{d}t, \mathrm{d}X^1, \mathrm{d}X^2, \mathrm{d}X^3)$, where $\eta_{\mu\nu}$ is the Minkowski metric, sometimes called the Lorentz metric, defined by $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Hence, we have $\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + |\mathrm{d}\boldsymbol{X}|^2$. The interval $\mathrm{d}s$ is a relativistic invariant, *i.e.* invariant under the Lorentz transformation (see Appendix B of Kambe (2021a)).

Another relativistic invariant is the *proper time* τ. Its increment $\mathrm{d}\tau$ is defined by the time increment (multiplied by c) in the instantaneously rest frame where $\boldsymbol{v} = 0$. Squared interval of the proper time is defined by $\mathrm{d}\tau^2 = -\mathrm{d}x_\nu \mathrm{d}x^\nu = -\mathrm{d}s^2$. From this, noting $\mathrm{d}X^k = v^k\,\mathrm{d}t$, we obtain

$$\mathrm{d}\tau = c\,\mathrm{d}t\,\sqrt{1-\beta^2}\,, \qquad \beta \equiv v/c\,, \quad v = \sqrt{v_k v^k}. \tag{A.2}$$

Using the displacement $\mathrm{d}X^\nu$ of a fluid particle P, its relativistic 4-velocity is defined by

$$u^\nu = \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} = \Big(\frac{1}{\sqrt{1-\beta^2}}, \frac{\boldsymbol{v}}{c\sqrt{1-\beta^2}}\Big), \qquad \boldsymbol{v} = (v^k) = (\mathrm{d}X^k/\mathrm{d}t). \tag{A.3}$$

This fluid particle P is moving with the 4-velocity u^ν relative to the frame x^μ.

Appendix B. Relativistic formulation of action principle: three mechanical systems

Relativistic formulation of the variational principle is presented for three fundamental mechanical systems. We consider how covariant derivatives or gauge invariant equations are deduced from the principle of least action, *i.e.* from the invariant variations.

Appendix B.1. **Free motion of a point mass**

The action of a free material particle of rest mass m is given by

$$S^{(\mathrm{m})} = -mc\int_a^b \mathrm{d}\tau \equiv \int \mathcal{L}^{(\mathrm{m})}\,\mathrm{d}t, \tag{B.1}$$

(its derivation, see Landau & Lifshitz (1975, §8)), where the present $\mathrm{d}\tau$ is equivalent to the $\mathrm{d}s$ of Landau & Lifshitz owing to the difference of the metric tensor definitions.‡ Since $\mathrm{d}\tau = c\,\mathrm{d}t\,\sqrt{1-\beta^2}$ from (A.2) of the main text, the Lagrangian $\mathcal{L}^{(\mathrm{m})}$ is

$$\mathcal{L}^{(\mathrm{m})} = -mc^2\sqrt{1-\beta^2}. \tag{B.2}$$

Principle of least action requires vanishing of the variation $\delta\,S^{(\mathrm{m})}$:

$$\delta\,S^{(\mathrm{m})} = -mc\,\delta\int \mathrm{d}\tau = 0. \tag{B.3}$$

Since $\mathrm{d}\tau^2 = -\eta_{\mu\nu}\mathrm{d}x^\nu\mathrm{d}x^\nu$ (Appendix A), we obtain $\delta(\mathrm{d}\tau^2) = 2\,\mathrm{d}\tau\,\delta\mathrm{d}\tau = -2\,\eta_{\mu\nu}\mathrm{d}x^\nu\,\delta\mathrm{d}x^\nu$. Hence, we obtain the followings:

$$\delta\mathrm{d}\tau = -\eta_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\,\delta\mathrm{d}x^\nu = -u_\nu\,\delta\mathrm{d}x^\nu = -u_\nu\,\mathrm{d}(\delta x^\nu). \tag{B.4}$$

$$\delta\,S^{(\mathrm{m})} = mc\int_a^b u_\nu\,\mathrm{d}(\delta x^\nu) = mc\Big[u_\nu\,\delta x^\nu\Big]_a^b - mc\int_a^b \delta x^\nu\,\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\mathrm{d}\tau\,. \tag{B.5}$$

where the integration limits of lower a and upper b are added.

To get the equation of motion, different trajectories are compared by assuming that the variation δx^ν is arbitrary within the interval $[a,b]$, but vanishes at a and b. Then, the principle of least action determines the trajectory by $\delta S^{(\mathrm{m})} = 0$. Thus we obtain

$$\nabla_\tau u^\nu \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\,u^\nu = 0. \tag{B.6}$$

Namely, the 4-velocity u^ν of the free particle is constant in time, as well-known.

Appendix B.2. **Free motion of a perfect fluid**

A perfect fluid is defined as a continuum object (a continuous matter) in the 4d-spacetime $x^\mu = (ct,\boldsymbol{x}) = (x^0,x^1,x^2,x^3)$, characterized with a mass density $\rho(x^\mu)$ in motion with 3-velocity $\boldsymbol{v} = (v^1,v^2,v^3)$ and without any internal mechanism of energy dissipation. During its motion, the entropy change $\Delta s = T^{-1}(\Delta\epsilon + p\,\Delta\mathcal{V}_1)$ is assumed to vanish, where s and ϵ are thermodynamic variables of entropy and internal energy per unit mass with the volume element $\Delta\mathcal{V}_1$ defined by $1/\rho$. The pressure and temperature are denoted by p and T. Flow variables such as ρ, p , $\boldsymbol{v}$, *etc.* are represented by continuous differentiable functions of the coordinates $x^\mu = (ct,\boldsymbol{x})$.

The action for free motion of a perfect fluid is given by

$$S^{(\mathrm{pf})} = \int\Big[\int L^{(\mathrm{pf})}\,\overline{\rho}\,\overline{\mathrm{d}\mathcal{V}}\Big]\,\mathrm{d}\tau = \int\Big[\int \mathcal{L}_{(\mathrm{pf})}(x^\nu)\;\mathrm{d}\mathcal{V}\Big]\,c\mathrm{d}t, \tag{B.7}$$

‡ The negative sign is added in front of the integral $\int_a^b \mathrm{d}\tau$, because it takes its maximum value along a straight geodesic line (see the text cited).

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

$$L^{(\mathrm{pf})} = -c^{-1}(m_1c^2 + \overline{\epsilon(\rho)}\,) = -c\,(1 + c^{-2}\,\overline{\epsilon(\rho)}\,) \quad \text{since} \quad m_1 = 1, \tag{B.8}$$

$$\mathrm{d}\tau = c\,\mathrm{d}t\,\sqrt{1-\beta^2}, \quad \overline{\rho}\,\overline{\mathrm{d}\mathcal{V}} = \rho\,\mathrm{d}\mathcal{V} = \mathrm{d}m\,, \quad \overline{\rho} = \rho\sqrt{1-\beta^2}\,, \tag{B.9}$$

$$\mathcal{L}_{\mathrm{pf}}\,\mathrm{d}\mathcal{V} \equiv L^{(\mathrm{pf})}\,\overline{\rho}\,\overline{\mathrm{d}\mathcal{V}}\,\sqrt{1-\beta^2} = -c^{-1}(m_1c^2 + \overline{\epsilon}(\overline{\rho}))\,\sqrt{1-\beta^2}\,[\rho\,\mathrm{d}\mathcal{V}], \tag{B.10}$$

where the integration within the first bracket [ ] of (B.7) is done with respect to $\mathrm{d}\mathcal{V}$ of material location $(\mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3)$, and *overlined* values denote proper values (*i.e.* the values in the comoving frame where the fluid is at rest). The terms $\overline{\epsilon}$ and $\overline{\rho}$ denote the *proper* internal energy and density, with $\mathrm{d}\tau = c\,\mathrm{d}t\,\sqrt{1-\beta^2}$ the proper time interval, $\overline{\rho}\,\overline{\mathrm{d}\mathcal{V}}$ denotes the proper mass element $\mathrm{d}m$, and $\mathrm{d}\mathcal{V} = \mathrm{d}X^1\mathrm{d}X^2\mathrm{d}X^3$ is a volume element associated with the mass $\mathrm{d}m$. The $m_1 = 1$ (unit mass) is added to clarify physical meaning of the term $(m_1c^2 + \overline{\epsilon}(\overline{\rho}))$ as relativistic proper internal energy per unit mass including the rest-mass energy m_1c^2. (For the form of $L^{(\mathrm{pf})}$, see Kambe (2020), Dewar (1977), or Salmon (1988b).)

The Lagrangian per a volume-element $\mathrm{d}\mathcal{V}$ is given by $\mathcal{L}_{\mathrm{pf}} = -c^{-1}\sqrt{1-\beta^2}\,(m_1c^2 + \epsilon(\rho)\,)\,\rho$. This is the proper Lagrangian with respect to an inertial frame. Its non-relativistic limit (as $\beta \to 0$) per unit mass is given as $c^{-1}\left(-m_1c^2 + \frac{1}{2}\,m_1v^2 - \epsilon + \cdots\right)$. The first term m_1c^2 denotes the mass energy (with negative sign attached) and neglected in the non-relativistic limit. Subsequent two terms $(\frac{1}{2}\,m_1v^2 - \epsilon)$ per unit mass is equivalent to the traditional non-Relativistic Lagrangian Λ_{nR}:

$$\Lambda_{\mathrm{nR}} = \tfrac{1}{2}\,v^2 - \epsilon, \qquad S^{(\mathrm{nR})} = \int\Big[\int \Lambda_{\mathrm{nR}}\,\rho\,\mathrm{d}\mathcal{V}\,\Big]\,\mathrm{d}t, \tag{B.11}$$

where the front factor c^{-1} and the integration element $c\,\mathrm{d}t$ make the $\mathrm{d}t$ in the above integral. Let us take variation of $S^{(\mathrm{pf})}$ of (B.7):

$$\delta\,S^{(\mathrm{pf})} = \int\!\!\int\Big[\overline{L}^{(\mathrm{pf})}\,\delta\mathrm{d}\tau + \delta\overline{L}^{(\mathrm{pf})}\,\mathrm{d}\tau\Big]\,\mathrm{d}m = \delta\,S_1{}^{(\mathrm{pf})} + \delta\,S_2{}^{(\mathrm{pf})}\,. \tag{B.12}$$

Variation is taken keeping the mass element $\mathrm{d}m = \overline{\rho\,\mathrm{d}\mathcal{V}}$ fixed and written as $\Delta m = \overline{\rho}\,\overline{\Delta\mathcal{V}}$. Then the second term is, under the thermodynamic condition $\delta\overline{\epsilon}|_{s:\mathrm{fixed}} = -\overline{p}\,\delta(1/\overline{\rho})$,

$$\begin{aligned}(\Delta m)\,\delta\overline{L}^{(\mathrm{pf})}\,\mathrm{d}\tau &= -\,c^{-1}\,\Delta m\,\,\delta\overline{\epsilon(\rho)}\,\mathrm{d}\tau = -c^{-1}\,\Delta m\,\Big((\overline{\rho})^{-1}\delta\overline{p} - \delta(\overline{p}/\overline{\rho})\Big)\,\mathrm{d}\tau \\ &= -\,c^{-1}\,\Delta m\,\frac{1}{\overline{\rho}}\partial_\nu\overline{p}\;\delta x^\nu\,\mathrm{d}\tau + c^{-1}\,\delta\left(\overline{\rho}\;\overline{\Delta\mathcal{V}}\;\frac{\overline{p}}{\overline{\rho}}\right)\mathrm{d}\tau\,,\end{aligned}$$

while for the first term, using the definition $\mathrm{d}\tau = \sqrt{-\eta_{\mu\nu}\mathrm{d}x^\nu\mathrm{d}x^\nu}$ together with (B.4),

$$\begin{aligned}(\Delta m)\,\overline{L}^{(\mathrm{pf})}\,\delta\mathrm{d}\tau &= c^{-1}(c^2 + \overline{\epsilon}(\overline{\rho}))\,\Delta m\;u_\nu\,\mathrm{d}_\tau(\delta x^\nu)\,\mathrm{d}\tau \\ &= c^{-1}(c^2 + \overline{\epsilon}(\overline{\rho}))\,\Delta m\,\,\Big(-\,(\frac{\mathrm{d}}{\mathrm{d}\tau}u_\nu)\;\delta x^\nu + \frac{\mathrm{d}}{\mathrm{d}\tau}\,(u_\nu\,\delta x^\nu)\Big)\,\mathrm{d}\tau.\end{aligned} \tag{B.13}$$

Thus, summing up both terms, we obtain

$$\begin{aligned}\Delta m\,\Big[\overline{L}^{(\mathrm{pf})}\,\delta\mathrm{d}\tau + \delta\overline{L}^{(\mathrm{pf})}\,\mathrm{d}\tau\,\Big] &= -\,c\,\Delta m\,\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}u_\nu + c^{-2}\,\frac{1}{\overline{\rho}}\partial_\nu\overline{p}\Big)\,\delta x^\nu\,\mathrm{d}\tau \\ &\quad + c^{-1}\,\delta\,\Big(\,\overline{p}\;\overline{\Delta\mathcal{V}}\Big)\,\mathrm{d}\tau + c\,(1 + O(\beta^2))\,\Delta m\,\;\mathrm{d}_\tau(u_\nu\,\delta x^\nu)\,\mathrm{d}\tau\,.\end{aligned} \tag{B.14}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

Since the two terms of the second line (B.14) do not give any contribution to the variation by the reasons explained below, we are concerned with the first line (B.14) only for the variational analysis. For arbitrary variations δx^k ($k = 1, 2, 3$ with $\delta x^0 = 0$), vanishing of the total variations requires the following equation, to the leading order of the series with respect to the $\beta^2 (\ll 1)$ expansion:

$$c^2 \frac{\mathrm{d}}{\mathrm{d}\tau} u_k + \frac{1}{\rho}\, \partial_k p = 0, \tag{B.15}$$

where $\mathrm{d}\tau = c\mathrm{d}t\, \sqrt{1-\beta^2}$. We note that $\overline{\rho} = \rho\sqrt{1-\beta^2}$ (Kambe (2020) Appendix B.1) and $\overline{p} = p$ (Agmon 1977). Hence using $u_k = (v_k/[c\sqrt{1-\beta^2}])$ from (A.3), the above becomes

$$\frac{1}{\sqrt{1-\beta^2}} \frac{\mathrm{D}}{\mathrm{D}t} \frac{v_k}{\sqrt{1-\beta^2}} + \frac{1}{\rho}\, \partial_k p = 0, \tag{B.16}$$

where $(\mathrm{D}/\mathrm{D}t) \equiv \mathrm{D}_t = \partial_t + \boldsymbol{v} \cdot \nabla$ is the convective derivative defined by (2.4). Transforming this into a contravariant form by multiplying η^{ik} (no change except the change of indices from lower to upper), the leading order form of the equation becomes

$$\nabla_\tau^{(\mathrm{pf})} v^k \equiv \mathrm{D}_t v^k + \rho^{-1}\, \partial_k p = 0, \tag{B.17}$$

since $1/\sqrt{1-\beta^2} = 1 + O(\beta^2)$ and $\partial^k = \partial_k$. This is nothing but the Euler's equation of motion in the form of (2.2).

The equation (C.10) is a geodesic equation of free motion of perfect fluid of a constant density ρ_*, expressed as $\widehat{\nabla}_t \boldsymbol{u} \equiv \partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla(p/\rho_*) = 0$. Here we used the symbol $\nabla_\tau^{(\mathrm{pf})}$ to denote the covariant τ-derivative of perfect fluid, because the leading term of (B.14) can be written as

$$\Delta m \left[\overline{L}^{(\mathrm{pf})}\, \delta \mathrm{d}\tau + \delta \overline{L}^{(\mathrm{pf})} \right] = -\frac{1}{c\sqrt{1-\beta^2}}\, \Delta m \left(\nabla_\tau^{(\mathrm{pf})} v^k \right) \mathrm{d}\tau\, \delta x^k = 0, \tag{B.18}$$

with the second line of (B.14) deleted.

In regard to the second line of (B.14), the factor $\mathrm{d}_\tau(u_\nu\, \delta x^\nu)\, \mathrm{d}\tau$ of the last term can be integrated with respect to τ. Hence the last term does not give any contribution to the variation, while the remaining first term leads to total variation of the integration $I \equiv \int p\, \mathrm{d}\mathcal{V}$ in the rest frame, which is fixed for the variation, *i.e.* $\delta I = 0$. In fact, the kinetic theory of statistical mechanics implies that $\int p\, \mathrm{d}\mathcal{V}$ denotes 2/3 of total kinetic energy E of particles composing an ideal gas, which is invariant in free state. Thus, the second line of (B.14) does not give any contribution to the variation,

Appendix B.3. **Free motion of a perfect fluid under interaction action S^{int}**

According to the sections III a) and b) in the main text, one can define a composite action S_c by using the action $S^{(\mathrm{pf})}$ of a perfect fluid of (B.7) and the action $S^{(\mathrm{int})}$ of (2.6) for interaction of current j^ν and the gauge field a_ν. Let us define

$$S_c \equiv S^{(\mathrm{pf})} + S^{(\mathrm{int})}, \qquad \mathrm{d}\Omega \equiv \mathrm{d}^4 x = \mathrm{d}\mathcal{V}\, \mathrm{d}t_c \tag{B.19}$$

$$S^{(\mathrm{pf})} \equiv -c \int \rho\, \mathrm{d}\mathcal{V} \int \Big(1 + c^{-2}\, \overline{\epsilon}(\rho)\Big)\, \mathrm{d}\tau = \int \mathcal{L}^{(\mathrm{pf})}\, \mathrm{d}^4 x, \tag{B.20}$$

$$S^{(\mathrm{int})} \equiv \int \mathcal{L}^{(\mathrm{int})}\, \mathrm{d}^4 x, \qquad \mathcal{L}^{(\mathrm{int})} \equiv c^{-1}\, j^\mu\, a_\mu, \qquad j^\mu = \rho\, v^\mu\,, \tag{B.21}$$

where $\mathcal{L}^{(\mathrm{pf})} \equiv -c\,\rho\,(1 + c^{-2}\,\overline{\epsilon})\,\sqrt{1-\beta^2}$, and $v^\mu = \mathrm{d}X^\mu/\mathrm{d}t$.

To find the equations of motion, the action principle is applied to the composite action S_c, by assuming the gauge potential a_ν given and vary only the position coordinate X_p^k of fluid particles moving with the velocity $\mathrm{D}_t X_p^k$ along their trajectories. On the other hand, to find the equations governing the a_ν, we vary only the gauge-potential a_ν with assuming the fluid motion given and fixed. However, to carry out the latter variation, we have to define a third action to characterize the background field a_ν and add it to S_c, (which is postponed to the next Appendix B.4), Here, we carry out the former variation, then the action principle applied to the varied S_c should yield the equation of fluid motion.

By the way, under the requirement of invariance of $S^{(\mathrm{int})}$ to the gauge transformation of potential a_ν, the current conservation law $\partial_\nu j^\nu = 0$ is deduced in section II a) Hence, we assume the invariance of the mass element $\mathrm{d}m \equiv \rho\,\mathrm{d}\mathcal{V}$ of a fluid particle during the motion along its trajectory when variations are taken with respect to the particle position X_p^k.

Regarding the action $S^{(\mathrm{pf})}$ of (B.7). Its variation is given by (B.14), and its variation-integrand $\delta\mathcal{J}^{(\mathrm{pf})}$ is found as follows:

$$\delta\,\mathcal{J}^{(\mathrm{pf})} = -c\,\Delta m\left(\frac{\mathrm{d}}{\mathrm{d}\tau}u_\nu + c^{-2}\frac{1}{\bar{\rho}}\partial_\nu\bar{p}\right)\delta x^\nu\,\mathrm{d}\tau + \text{higher order terms of } O(\beta^2). \quad \text{(B.22)}$$

Before taking variation of the interaction action $S^{(\mathrm{int})} = \int\int\mathcal{J}^{(\mathrm{int})}$, we rewrite its integrand $\mathcal{J}^{(\mathrm{int})}$ as follows (since $\mathrm{d}^4x = c\,\mathrm{d}t\,\mathrm{d}\mathcal{V}$):

$$\mathcal{J}^{(\mathrm{int})} = (\rho\,\mathrm{d}\mathcal{V})\ v^\nu\,a_\nu\,\mathrm{d}t = (\mathrm{d}m)\,\frac{\mathrm{d}X^\nu}{\mathrm{d}t}\,a_\nu\,\mathrm{d}t = (\mathrm{d}m)\,a_\nu\,\mathrm{d}X^\nu\,, \quad \text{(B.23)}$$

where (2.9) and $v^\mu = \mathrm{d}X^\mu/\mathrm{d}t$ are used with $\mathrm{d}m = \rho\,\mathrm{d}\mathcal{V}$. Its variation is given by

$$\begin{aligned}
\delta\mathcal{J}^{(\mathrm{int})} &= (\Delta m)\Big(a_\nu\,\mathrm{d}(\delta x^\nu) + \delta a_\nu\,\mathrm{d}x^\nu\Big) = \overline{\Delta m}\Big(-\mathrm{d}a_\nu\,\delta x^\nu + \delta a_\nu\,\mathrm{d}x^\nu + \mathrm{d}(a_\nu\,\delta x^\nu)\Big) \\
&= \overline{\Delta m}\left[-\frac{\partial a_\nu}{\partial x^\mu}\mathrm{d}x^\mu\delta x^\nu + \frac{\partial a_\nu}{\partial x^\mu}\delta x^\mu\mathrm{d}x^\nu + \mathrm{d}(a_\nu\,\delta x^\nu)\right] \\
&= \overline{\Delta m}\left(\frac{\partial a_\mu}{\partial x^\nu} - \frac{\partial a_\nu}{\partial x^\mu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\ \delta x^\nu\ \mathrm{d}\tau + c\,\overline{\Delta m}\,\frac{\mathrm{d}(a_\nu\,\delta x^\nu)}{\mathrm{d}\tau}\ \mathrm{d}\tau \\
&= \overline{\Delta m}\,f_{\nu\mu}\,u^\mu\ \delta x^\nu\ \mathrm{d}\tau + \overline{\Delta m}\,\frac{\mathrm{d}(a_\nu\,\delta x^\nu)}{\mathrm{d}\tau}\,\mathrm{d}\tau,
\end{aligned} \quad \text{(B.24)}$$

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu = -f_{\nu\mu}\,. \quad \text{(B.25)}$$

Thus, summing up the two variations (B.22) and (B.24), we obtain

$$\delta\mathcal{J}^{(\mathrm{fl+a})} \equiv \delta\,\mathcal{J}^{(\mathrm{pf})} + \delta\mathcal{J}^{(\mathrm{int})} = -c^{-1}\,\overline{\Delta m}\left[c^2\,\frac{\mathrm{d}}{\mathrm{d}\tau}u_\nu + \frac{1}{\bar{\rho}}\partial_\nu\bar{p} - c\,f_{\nu\mu}\,u^\mu\right]\mathrm{d}\tau\,\delta x^\nu \quad \text{(B.26)}$$

by neglecting higher order terms and vanishing integrals with respect to τ. Requiring $\delta\mathcal{J}^{(\mathrm{fl+a})} = 0$ for arbitrary variation δx^ν, this leads to

$$c^2\,\frac{\mathrm{d}}{\mathrm{d}\tau}u_\nu + \frac{1}{\bar{\rho}}\partial_\nu\bar{p} - c\,f_{\nu\mu}\,u^\mu = 0. \quad \text{(B.27)}$$

Remembering that the first two terms reduced to (B.16) in the Appendix B.2, we find that the above equation reduces to the following ($\nu = 0,1,2,3;\ k = 1,2,3$):

$$\frac{1}{\sqrt{1-\beta^2}}\,\frac{\mathrm{D}}{\mathrm{D}t}\frac{v_k}{\sqrt{1-\beta^2}} + \frac{1}{\rho}\,\partial_k p - f_{k\nu}\,v^\nu = 0, \quad \text{(B.28)}$$

where $u_k = (v_k/[c\sqrt{1-\beta^2}])$ is used from (A.3), Extending the reasoning of Appendix B.2, leading order form of this equation becomes

$$\nabla_\tau^{(\mathrm{pfa})} v^k \equiv \mathrm{D}_t v^k + \rho^{-1}\,\partial_k p - f_{k\nu}\,v^\nu = 0, \qquad (\nu = 0,1,2,3). \tag{B.29}$$

since $1/\sqrt{1-\beta^2} = 1 + O(\beta^2)$. Thus we have found an extended equation of Euler's equation in the presence of background field a_μ giving rise to new third term.

Appendix B.4. Free motion of a perfect fluid : in the presence of gauge field re-ensuring mass conservation

To make the fluid system self-contained, we need a third action $S^{(\mathrm{F})} = \int \mathcal{L}^{(\mathrm{F})}\,\mathrm{d}\Omega$ in addition to $S^{(\mathrm{pf})}$ and $S^{(\mathrm{int})}$ of Appendix B.3, to govern free-state of the background field a_ν, describing only on the property of the field itself. According to section II e) iv. a possible form of the free-Lagrangian $\mathcal{L}^{(\mathrm{F})}$ is proposed to be quadratic in $\partial_\mu a_\nu$ or $f_{\mu\nu}$, because the variation of $S^{(\mathrm{F})}$ reduces the degree by one with resulting equation becoming linear to $\partial_\mu a_\nu$. The only Lorentz-invariant quadratic form is a multiple of $f_{\mu\nu}f^{\mu\nu}$. This satisfies the requirement (i) of section II e) iv. namely, the fluid Maxwell fields $\boldsymbol{e}$ and $\boldsymbol{b}$ should be ensured to vanish when the background field a_ν takes the special form $\tilde{a}_\nu = \partial_\mu\Psi$. This means that the original Euler's equation of motion (2.2) is valid in spite of the existence of the field a_ν.

Following the propositions of Kambe (2017, 2020), our fluid system is a combined system of two fields: a fluid-current field $\jmath^\nu$ and a background field a_ν ensuring the continuity equation. Accordingly, the Lagrangian density $\mathcal{L}$ consists of three terms: Lagrangians of (i) perfect fluid $\mathcal{L}^{(\mathrm{pf})}$, (ii) back-ground field $\mathcal{L}^{(\mathrm{F})}$ and (iii) their mutual interaction $\mathcal{L}^{(\mathrm{int})}$. Total Lagrangian is expressed as $\mathcal{L}^{(\mathrm{total})} = \mathcal{L}^{(\mathrm{pf})} + \mathcal{L}^{(\mathrm{int})} + \mathcal{L}^{(\mathrm{F})}$.

(a) *Total action*

The total action $S^{(\mathrm{total})}$ is given by

$$S^{(\mathrm{total})} = \int\!\!\int \Big(\mathcal{L}^{(\mathrm{pf})} + \mathcal{L}^{(\mathrm{int})} + \mathcal{L}^{(\mathrm{F})}\Big)\,c\,\mathrm{d}t\,\mathrm{d}\mathcal{V}, \tag{B.30}$$

where $\mathrm{d}^4x = \mathrm{d}\Omega = c\,\mathrm{d}t\,\mathrm{d}\mathcal{V}$, and $\mathrm{d}\mathcal{V} = \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \overline{\mathrm{d}\mathcal{V}}\,\sqrt{1-\beta^2}$. Since $\rho\,\sqrt{1-\beta^2} = \overline{\rho}$, the mass element $\mathrm{d}m = \rho\,\mathrm{d}\mathcal{V}$ is invariant, *i.e.* $\overline{\rho}\,\overline{\mathrm{d}\mathcal{V}} = \rho\,\mathrm{d}\mathcal{V}$.

The Lagrangian densities are defined by

$$\mathcal{L}^{(\mathrm{pf})} = -c^{-1}\,\overline{\rho}\,(c^2 + \overline{\epsilon(\rho)}), \qquad \mathcal{L}^{(\mathrm{int})} = c^{-1} j^\nu\,a_\nu\,, \qquad \mathcal{L}^{(\mathrm{F})} = -\frac{1}{4\mu c} f^{\nu\lambda}\,f_{\nu\lambda}, \tag{B.31}$$

where $j^\nu \equiv \overline{\rho} v^\nu = \rho\sqrt{1-\beta^2}\;v^\nu = \rho\,\mathrm{d}X^\nu/\mathrm{d}t$ is the 4-current density and v^ν is the relativistic 4-velocity, defined by

$$v^\nu = c\,\frac{\mathrm{d}X^\nu}{\mathrm{d}\tau} = \Big(\frac{c}{\sqrt{1-\beta^2}},\ \frac{\boldsymbol{v}}{\sqrt{1-\beta^2}}\Big) \equiv c\,u^\nu, \quad j^\nu = \rho\,\frac{\mathrm{d}X^\nu}{\mathrm{d}t} = \rho\,(c,\ \boldsymbol{v}). \tag{B.32}$$

The tensor $f_{\nu\lambda}$ is field-strength tensor of background field, defined by

$$f_{\nu\lambda} = \partial_\nu a_\lambda - \partial_\lambda a_\nu, \tag{B.33}$$

where $a_\nu = (-\phi/c, a_k)$ is a 4-potential of the background field. Here in the present fluid system, we use lower-case letters to denote field variables corresponding to Electromagnetic variables where upper-case letters are used in section II a) of Kambe (2021a).

To find the equations governing the background field a_ν, we apply the principle of least action to the action $S^{(\mathrm{total})}$, by assuming the fluid motion given and vary the potential a_ν only. On the other hand, to find the equations of fluid motion, we assume the field potential a_ν given and vary only the trajectory of the fluid particle ($\boldsymbol{X}$ or $\boldsymbol{v} = \partial_t \boldsymbol{X}$). The latter variation is equivalent to what is done in Appendix B.3. The equation (B.29) derived there is cited as the equation (B.36) at the end of this section.

We carry out the former variation, anticipating to deduce the current conservation $\partial_\nu j^\nu = 0$ of fluid flows.

(*b*) *Variation with respect to* a_ν

We have two Lagrangian densities which include the field a_ν: $\mathcal{L}^{(\mathrm{int})} = c^{-1} v^\nu\, a_\nu$ and $\mathcal{L}^{(\mathrm{F})} = -\frac{1}{4\mu c} f^{\nu\lambda}\, f_{\nu\lambda}$. First, we note $\delta\Big(f^{\nu\lambda}\, f_{\nu\lambda}\Big) = 2 f^{\nu\lambda}\,(\delta f_{\nu\lambda})$. This is because

$$(\delta f^{\nu\lambda})\, f_{\nu\lambda} = (\delta f^{\nu\lambda})\, \eta_{\nu\alpha}\eta_{\lambda\beta} f^{\alpha\beta} = f^{\alpha\beta}\,(\delta f_{\alpha\beta}).$$

Therefore, variation of $\mathcal{L}^{(\mathrm{int})} + \mathcal{L}^{(\mathrm{F})}$ is given by

$$\begin{aligned} c\,\delta\mathcal{L}^{(\mathrm{int})} + c\,\delta\mathcal{L}^{(\mathrm{F})} &= j^\nu\,\delta a_\nu - \frac{1}{2\mu} f^{\nu\lambda}\,\delta f_{\nu\lambda} = j^\nu\,\delta a_\nu - \frac{1}{2\mu}\, f^{\nu\lambda}\frac{\partial}{\partial x^\nu}\delta a_\lambda + \frac{1}{2\mu}\, f^{\nu\lambda}\frac{\partial}{\partial x^\lambda}\delta a_\nu \\ &= \Big(j^\nu \;-\frac{1}{\mu}\frac{\partial}{\partial x^\lambda} f^{\nu\lambda}\Big)\,\delta a_\nu. \end{aligned} \tag{B.34}$$

where the term $-(1/2\mu) f^{\nu\lambda}\partial_\nu(\delta a_\lambda)$ next to the last on the upper line can be equated to the last term $(1/2\mu) f^{\nu\lambda}\partial_\lambda(\delta a_\nu)$ by using the anti-symmetric property, $-f^{\nu\lambda} = f^{\lambda\nu}$, and interchanging the indices ν and λ, and the last two terms on the upper line are combined to give $(1/\mu) f^{\nu\lambda}\partial_\lambda(\delta a_\nu)$, and finally carrying out integration-by-parts leads to the second line of (B.34) with omitting the term of the form $\partial_\lambda[(1/\mu) f^{\nu\lambda}\,\delta a_\nu]$, which is transformed to vanishing boundary integrals in the original action integral.

Requiring vanishing of the varied Lagrangian $\delta\mathcal{L}^{(\mathrm{int})} + \delta\mathcal{L}^{(\mathrm{F})} = 0$ for arbitrary variation δa_ν, we obtain

$$\frac{\partial}{\partial x^\lambda} f^{\nu\lambda} = \mu\, j^\nu. \tag{B.35}$$

From this, the current conservation equation can be derived, that is directly connected with the gauge invariance of the system section II f) ii. of main text).

(*c*) *Variation with respect to* x^ν

To find the equations of fluid motion, we assume the field potential a_ν given and vary only the trajectory of the fluid particle ($\boldsymbol{X}$ or $\boldsymbol{v} = \partial_t \boldsymbol{X}$). In this case, the third Lagrangian $\mathcal{L}^{(\mathrm{F})}$ of (B.31) is kept unchanged because it depends only on the field a_ν. Therefore, the variation under consideration is equivalent to what is done in Appendix B.3. The equation (B.29) derived there is rewritten here:

$$\nabla_\tau^{(\mathrm{pfa})} v^k \equiv \mathrm{D}_t v^k + \rho^{-1}\,\partial_k p - f_{kl}\, v^l = 0. \tag{B.36}$$

Thus we find an extended equation of Euler's equation in the presence of background field a_μ giving rise to new third term.

In Appendix C, we see that the equation (C.10) is a geodesic equation of free motion of perfect fluid of a constant density ρ_*, expressed as $\widehat{\nabla}_t\, \boldsymbol{u} \equiv \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla(p/\rho_*) = 0$. An analogous equation (B.17) is given for a perfect fluid in Appendix B.2. Just above, we have obtained another analogous equation (B.36). Now we have introduced a new symbol $\nabla_\tau^{(\mathrm{pfa})}$ to denote the covariant τ-derivative of perfect fluid in the presence of background field a_μ. In fact, the leading term of (B.26) can be written as

$$\delta\,\mathcal{J}^{(\mathrm{PFA})} = -\frac{1}{c\sqrt{1-\beta^2}}\,\Delta m\,\Big(\,\nabla_\tau^{(\mathrm{pfa})} v^k\,\Big)\,\mathrm{d}\tau\,\delta x^k = 0. \tag{B.37}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

for the action principle concerned with the geodesic equation of free motion of a perfect fluid in the presence of the background field a_μ.

Appendix C. Free motion of physical systems and curved geodesics

Appendix C.1. Two free physical systems described by curved geodesics

(*a*) Free motion of a particle in gravity field by Newtonian mechanics

To begin with, consider free motion of a test particle of a unit mass in the Galilean rectangular space (x^k) with a universal absolute time t. The particle in free motion in a gravitational field takes a curved trajectory in general according to Newtonian mechanics. By the equation of motion, the particle motion in the Earth's gravity potential $\Phi_E(x^k)$ is described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,v^k + \frac{\partial \Phi_E}{\partial x^k} = 0, \qquad v^k \equiv \frac{\mathrm{d}x_p^k}{\mathrm{d}t}, \qquad k = 1,2,3\,, \tag{C.1}$$

where the particle takes a curved trajectory $x_p^k(t)$ and v^k is the k-th component of its velocity. In the modern view to take the space and time linked to form a 4d-continuum, the curved trajectory of a free particle is described as a *geodesic* curve in the linked space-time. Let us take an illustrative example according to Utiyama (1987, §2.3), and consider a free-falling elevator in the Earth's gravitational field $\Phi_E(x^\nu)$. The free-falling elevator provides a particular *inertial* system of spacetime, in which free motion of a particle is described by

$$\mathrm{d}^2 X^\mu / \mathrm{d}\tau^2 = 0, \tag{C.2}$$

where X^μ is the particle coordinates in the frame F_{el} fixed to the free-falling elevator. The gravity effect does not appear apparently because the acceleration owing to the gravity acting on both of the elevator and the particle are the same and cancel out in the free-falling frame F_{el}. Thus, the particle takes a straight path $X^\mu = a^\mu \tau + b^\mu$ with respect to F_{el} with a^μ and b^μ being constants.

According to the section §3 of Part I, let us observe the same motion from another frame, which is the frame F_E fixed to the Earth surface, where the coordinates are given by x^μ. Suppose that the relation between the two frames F_{el} and F_E is given by the transformation function $X^\mu = X^\mu(x^\nu)$. Under this transformation from X^μ to x^ν, the equation of free motion $\mathrm{d}^2X^\mu/\mathrm{d}\tau^2 = 0$ in the free-falling frame F_{el} (where τ is the proper time defined by (A.2)) is transformed to that of the frame F_E as follows,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big[\frac{\partial X^\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\Big] = A^\mu_\nu\Big[\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \Gamma^\nu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\Big] = 0$$

where $A = (A^\mu_\nu)$ is a transformation matrix. Using the inverse A^{-1} of A^μ_ν and multiplying by $(A^{-1})^\sigma_\mu \equiv \partial x^\sigma/\partial X^\mu$, the above equation becomes

$$\frac{\mathrm{d}^2 x^\sigma}{\mathrm{d}\tau^2} + \Gamma^\sigma_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0, \quad \text{where} \quad \Gamma^\sigma_{\alpha\beta} = \frac{\partial x^\sigma}{\partial X^\sigma}\,\frac{\partial X^\sigma}{\partial x^\alpha\,\partial x^\beta} = \Gamma^\sigma_{\beta\alpha}. \tag{C.3}$$

This states that the particle trajectory is curved in general when $\Gamma^\sigma_{\alpha\beta} \neq 0$, where $\Gamma^\sigma_{\alpha\beta}$ is playing the role of Christoffel symbol of covariant derivatives of Riemannian geometry.

The 4-velocity $u^\nu \equiv \mathrm{d}x^\nu/\mathrm{d}\tau$ of the particle is defined by (A.3) as $u^\nu = (1, \boldsymbol{v}/c)/\sqrt{1-\beta^2}$. In the non-relativistic limit as $\beta \ll 1$ for the particle velocity $|\boldsymbol{v}|$ much less than the light velocity c, this yields $u^\nu = \mathrm{d}x^\nu/\mathrm{d}\tau \to (1, \boldsymbol{v}/c) = v^\nu/c$ in the limit $(\mathrm{d}\tau \to c\,\mathrm{d}t)$. In this case, the equation (C.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\,v^\sigma + \Gamma^\sigma_{\alpha\beta}\,v^\alpha\,v^\beta = 0, \quad \text{in particular} \quad \frac{\mathrm{d}}{\mathrm{d}t}\,v^k + \Gamma^k_{00}\cdot 1\cdot 1 = 0, \tag{C.4}$$

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

where $v^\sigma = \mathrm{d}x^\sigma/\mathrm{d}t$, the second equation is given for $\sigma = k = 1,2,3$, $(\alpha,\beta) = (0,0)$, and the factors $\Gamma^\sigma_{\alpha\beta}$ other than Γ^k_{00} are set to zero. Compare this with (C.1). By assuming the following relation of equality: $\Gamma^k_{00} = \partial\Phi_E/\partial x^k$, the second equation of (C.4) becomes equivalent to the equation (C.1). From the context of physics of the gravity theory, this is very important because it implies a relation which equates the geometrical term Γ^k_{00} (called the Christoffel symbol) to a space derivative of the gravity potential $\partial_k\Phi_E$.

More precisely in mathematics, it is known section III b) of Kambe (2021a)) that a free particle moving in curved spacetime is governed by the geodesic equation of the form

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha_{\ \beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\lambda} = 0. \tag{C.5}$$

Replacing the affine parameter λ with an equivalent proper time τ, the equation (C.5) reduces to (C.3). Since the equations (C.1) and (C.4) have the form of the geodesic equation (C.5), one can read off

$$\Gamma^k_{00} = \partial\Phi_E/\partial x^k \quad (k=1,2,3); \qquad \text{all other } \Gamma^\alpha_{\ \beta\gamma} \text{ vanish.} \tag{C.6}$$

According to section III b) i. of Part I, in a curved 4d spacetime, a *covariant derivative* of a vector field $v^\alpha(x^\mu)$ along a curve $P(\lambda)$ with its tangent $u^\beta = \mathrm{d}x^\beta/\mathrm{d}\lambda$ is defined by

$$(\widehat{\nabla}_{\boldsymbol{u}}\boldsymbol{v})^\alpha \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda}v^\alpha + \Gamma^\alpha_{\ \beta\gamma}\,v^\beta u^\gamma \equiv \widehat{\nabla}_\lambda. \tag{C.7}$$

where $\widehat{\nabla}$ denotes the nabla-operator in the 4d spacetime. Using this definition, the *geodesic equation* (C.5) can be written simply as

$$\widehat{\nabla}_{\boldsymbol{u}}\boldsymbol{u} = 0, \quad \text{or} \quad \widehat{\nabla}_\lambda \boldsymbol{u} = 0, \quad \text{where} \quad u^\alpha \equiv \mathrm{d}x^\alpha(P)/\mathrm{d}\lambda. \tag{C.8}$$

According to the differential geometry (Misner *et al.* 2017, Chap.8), this states that the geodesic is a curve $P(\lambda)$ which parallel-transports its tangent $u^\alpha = \mathrm{d}x^\alpha(P)/\mathrm{d}\lambda$. In the flat space of special relativity where $g_{\mu\nu}$ is given by the metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, the geodesic takes a straight path $\mathrm{d}^2x^\alpha/\mathrm{d}\lambda^2 = 0$, since $\Gamma^\alpha_{\beta\gamma} = 0$ (see the footnote†).

The equation (C.5) can be written in the form of a geodesic equation, by taking the specification of $\lambda = t$, $u^\alpha = (1, v^k)$ with $v^k = \mathrm{d}x^k(P)/\mathrm{d}t$ and the equation (C.6):

$$(\widehat{\nabla}_{\boldsymbol{u}}\boldsymbol{u})^k \equiv \frac{\mathrm{d}}{\mathrm{d}t}u^k + \Gamma^k_{\ 00}\,u^0u^0 = 0, \qquad u^0 = 1, \quad u^k \equiv \frac{\mathrm{d}x^k(P)}{\mathrm{d}t} = v^k, \tag{C.9}$$

where the operator $\widehat{\nabla}$ is defined by (C.7). This implies that the free motion of a test particle in a gravitational potential field $\Phi_E(x^k)$ is described by a geodesic trajectory which is curved in a linked space-time. A geodesic curve is a generalization of a straight line in flat spacetime to a curved spacetime.

With a simplified potential $\Phi = gz$ of a uniform value g of gravity in a cartesian (x,y,z)-space, one finds a trajectory of a point-mass: $z(t) = \frac{1}{2}gt^2 + w_0t$, $x(t) = u_0t$ and $y(t) = v_0t$ where (u_0, v_0, w_0) denote the initial velocity. Thus it is seen that the parabolic trajectory of free motion of a point-mass in the gravity field $\Phi = gz$ is a *geodesic* in the linked spacetime (t,x,y,z), which is curved in a geometrical sense.

(*b*) Free motion of a perfect fluid in a flat space

Let us consider free motion of a perfect fluid of constant density under pressure field in a flat space. This case is worth given a particular remark, because this is a *free motion* characterized with *curvature* tensors that occurs in a flat 3-space for a perfect

† The equation (3.12) of Part I paper: $\Gamma^\alpha_{\ \beta\gamma} = g^{\alpha\mu}\Gamma_{\mu\beta\gamma}$, $\Gamma_{\mu\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\mu\beta}}{\partial x^\gamma} + \frac{\partial g_{\mu\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu}\right)$.

fluid under pressure field. This is formulated mathematically with an exact analysis by Arnold (1966).

In §2.1, we have seen that free motion of a perfect fluid can be described by the Euler's equation of motion of the form (2.2), which is analogous to the Newton's equation of motion (C.4) rewritten in the geometrical language, transformed from the original Newton equation (C.1).

This implies that the Euler's equation of motion too may be further transformed to the form of geodesic equation (C.9), because both equations are descriptions of *free motion*. This is true. In fact, it is already down by the mathematician V.I. Arnold (1966) for a perfect fluid of constant density satisfying the condition of incompressibility by applying the differential geometry of Lie groups of infinite dimensions.

According to Arnold, it is found on the basis of Riemannian geometry and Lie group theory that the Euler's equation of motion for flows of a perfect fluid of uniform density on a bounded flat space-time M is a geodesic equation on a group of volume-preserving diffeomorphisms with the metric of the kinetic energy (see also Kambe (2010, Chap. 8) for some details, in addition to Arnold (1966)). Here, we consider the fluid motion in flat space.

Defining $\boldsymbol{u}(\boldsymbol{x})$ as the 3-velocity field for $\boldsymbol{x} \in M$ (a bounded 3-space) satisfying the divergence-free condition $\mathrm{div}_x \boldsymbol{u}(\boldsymbol{x}) = 0$ for a constant density ρ_*, the geodesic equation is given by

$$\widehat{\nabla}_t\, \boldsymbol{u} \equiv \partial_t \boldsymbol{u} + \overline{\nabla}_{\boldsymbol{u}} \boldsymbol{u} = 0, \qquad \overline{\nabla}_{\boldsymbol{u}} \boldsymbol{u} \equiv (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla(p/\rho_*), \tag{C.10}$$

(eq. (8.42) of Kambe (2010)), where $\overline{\nabla}$ is a divergence-free connection satisfying

$$\mathrm{div}\Big(\overline{\nabla}_{\boldsymbol{u}} \boldsymbol{u}\Big) = \mathrm{div}\Big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla(p/\rho_*)\Big) = 0. \tag{C.11}$$

This ensures $\mathrm{div}\, \boldsymbol{u} = 0$ at any time from (C.10) if it is satisfied initially. The equation (C.11) constrains the pressure field p to satisfy $\nabla^2 p = -\rho_*\, \partial_j \partial_k (u^j u^k)$. The geodesic equation (C.10) is nothing but the Euler's equation of a perfect fluid of constant density:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla(p/\rho_*) = 0. \tag{C.12}$$

Using the operator D_t of (2.4), this can be rewritten in the form of equation-of-motion of a fluid particle analogous to (2.2) as $\mathrm{D}_t \boldsymbol{u} + \nabla P = 0$, where $P \equiv p/\rho_*$. Since this is analogous to (C.1), it is suggested that the geodesic equation (C.12) can be rewritten in the form of an equation using a covariant derivative $\widehat{\nabla}$ of 4-spacetime in terms of Christoffel symbol Γ's.

For that purpose, we define the 4-velocity by $v^\nu = \mathrm{d}X^\nu/\mathrm{d}t = (\, c,\, \boldsymbol{u}\,)$, in the non-relativistic limit $(\beta \to 0)$ with $\mathrm{d}X^\nu = (c\,\mathrm{d}t, \mathrm{d}\boldsymbol{X})$. Then, the equation (C.12) can be rewritten in the following form of geodesic equation:

$$(\widehat{\nabla}_v v)^k \equiv \frac{\mathrm{D}}{\mathrm{D}t}\, v^k + \Gamma^k{}_{00}\, v^0 v^0 = 0, \qquad \text{for} \quad k = 1, 2, 3\,, \tag{C.13}$$

$$v^0 = c, \quad \Gamma^k_{00} = c^{-2} (\partial/\partial x^k)\, P\,, \qquad \text{all other } \Gamma^\alpha{}_{\beta\gamma} \text{ vanish.}$$

Appendix C.2. **Free dynamics and action principle of invariant variations**
Free motion of a perfect fluid under a pressure field $p(x^\nu)$ and a background field $a_\mu(x^\nu)$ in a flat space was studied in section II c) and f), where modified Euler's equation of motion (2.11) was derived in the former section II c) and the equation (2.32) governing the background field was derived in the latter. In particular, the set of latter equations re-ensures the continuity equation. The same continuity equation was required already in section II a) from the invariance of the interaction action $S^{(\mathrm{int})}$ to the gauge transformation of $a_\mu(x^\nu)$.

By the scenario of Utiyama's gauge principle, the new field a_μ should be incorporated as a connection term in a covariant derivative. To that end, physical and mathematical formulation are presented by geometrical language in this section enabling the generalization. As a matter of fact, this section aims to conclude the *fluid gauge theory*.

The Euler's equation (2.2) can be written as $\mathrm{D}_t\boldsymbol{v} + \rho^{-1}\nabla p = 0$ in the form of an equation of motion of a fluid particle and describes *free motion* of a perfect fluid. In principle, free motion is given by a geodesic equation. Appendix B.2 derives the same equation (B.17) as a geodesic equation deduced from the action principle. In mechanical systems, the variational principle of action integrals yields a geodesic equation.

Naively speaking, a geodesic is a curve representing the shortest (or extremum) path between two points in a Riemannian manifold. More generally, a geodesic is defined to be a curve whose tangent vector T remain parallel if they are transported along it, *i.e.* if $\widehat{\nabla}_T T = 0$ (see Kambe (2021a): Eq.(4.8) and Appendix A.6). This recovers the statement mentioned above that the arc length between two points in a Riemannian manifold takes the extremum length when $\widehat{\nabla}_T T = 0$. This fact is seen transparently in the definition of action integral of a point mass of Eq.(B.1) of Appendix B.1:

$$S^{(\mathrm{m})} = -c\,m\int \mathrm{d}\tau, \qquad \mathrm{d}\tau \equiv \sqrt{-\mathrm{d}x_\nu \mathrm{d}x^\nu} = c\,\mathrm{d}t\,\sqrt{1-\beta^2}, \quad \beta \equiv \frac{v}{c}, \tag{C.14}$$

where $\mathrm{d}\tau$ is the relativistic infinitesimal time-like interval, invariant by Lorentz transformation. Vanishing variation of the integral, $\delta\int_a^b \mathrm{d}\tau = 0$, signifies the extremum of the time-like interval between the end points a and b. Hence there is no doubt that a geodesic equation such as $\widehat{\nabla}_T T = 0$ plays a role in the variation.

(a) **Free particle of mass** m

In the Appendix B, it is found that the variation of $S^{(\mathrm{m})}$ is given by (B.5):

$$\delta\,S^{(\mathrm{m})} = -c\,m\int_a^b \frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\,\delta x^\nu\,\mathrm{d}\tau\,,$$

under the condition that the variation δx^ν vanishes at end points. Requiring $\delta\,S^{(\mathrm{m})} = 0$ for arbitrary variation δx^ν, we obtain the geodesic equation:

$$\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} = \nabla_\tau u^\nu = u^\nu\partial_\nu u^\nu = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} = \nabla_\tau = u^\nu\partial_\nu, \tag{C.15}$$

Note that a tangent vector T is defined as $T = T^\nu\partial_\nu = u^\nu\partial_\nu = (\mathrm{d}x^\nu/\mathrm{d}\tau)(\partial/\partial x^\nu) = \mathrm{d}/\mathrm{d}\tau$, and that the above equation can be written also in the form, $\widehat{\nabla}_T T = 0$ with $\widehat{\nabla}_T = T^\nu\partial_\nu$. Thus, it is found that a free particle is governed by the geodesic equation: $\nabla_\tau u^\nu = 0$.

(b) **Free motion of a perfect fluid**

The action of a perfect fluid is given by (B.7) and (B.8) as

$$S^{(\mathrm{pf})} = -c\int\!\!\int(\rho\,\mathrm{d}\mathcal{V})\left(1 + c^{-2}\,\bar{\epsilon}(\bar{\rho})\right)\mathrm{d}\tau, \qquad \rho\mathrm{d}\mathcal{V} = \mathrm{d}m. \tag{C.16}$$

Comparing this action for a perfect fluid (of continuum material of density ρ) with the action $S^{(\mathrm{m})}$ of (C.14) for a single particle of mass m, one finds that the mass energy $m\,c^2$ is replaced by an integral of the energy $\rho\,(c^2 + \bar{\epsilon})$ per a volume element $\mathrm{d}\mathcal{V}$, where $\bar{\epsilon}$ is the specific internal energy in the rest frame of the fluid. Namely, the internal energy is added to the mass energy because the fluid has its own thermal energy in addition to the rest-mass energy. This is the difference of the two systems.

In the Appendix B.2, the integrand of variation $\delta S^{(\mathrm{pf})}$ is given by (B.14). Deleting vanishing terms, its dominant leading order term is given by

$$\delta\,\mathcal{J}^{(\mathrm{pf})} = -c\,(\Delta m)\left(\frac{\mathrm{d}}{\mathrm{d}\tau}u_\nu + c^{-2}\frac{1}{\rho}\partial_\nu\,p\right)\delta x^\nu\;\mathrm{d}\tau, \qquad \Delta m = \rho\,\Delta\mathcal{V}.$$

For the invariance of the action $S^{(\mathrm{pf})}$, it is required that $\delta\mathcal{J}^{(\mathrm{pf})} = 0$ is satisfied for arbitrary variation δx^ν. Thus, we obtain the following geodesic equation:

$$\nabla_\tau^{(\mathrm{pf})} u^\nu \equiv \frac{\mathrm{d}}{\mathrm{d}\tau} u^\nu + c^{-2}\frac{1}{\rho}\partial_\nu p = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\tau} = u^\nu \partial_\nu = \frac{1}{c}\Big((\partial_t + v^k \partial_k) + O(\beta^2)\Big). \quad \text{(C.17)}$$

This reduces to the Euler's equation of motion of (2.2), which is the same as Eq. (B.17) deduced in Appendix B.2. As the above derivation shows clearly, the equation (C.17) defines the covariant derivative $\nabla_\tau^{(\mathrm{pf})} u^\nu$ of the perfect fluid. The derivative may be called more appropriately as an *invariant derivative* (Utiyama (1987) Chap.11), because the invariance of the action $S^{(\mathrm{pf})}$ is ensured by (C.17).

(c) **Free motion of perfect fluid under a background field** a_ν

The action of this system is given in section II f) by (2.28):

$$S^{(\mathrm{total})} = S^{(\mathrm{pf})} + S^{(\mathrm{int})} + S^{(\mathrm{F})} = \int \Big[\int \Big(\mathcal{L}^{(\mathrm{pf})} + \mathcal{L}^{(\mathrm{int})} + \mathcal{L}^{(\mathrm{F})} \Big)\, \mathrm{d}\mathcal{V} \Big]\, c\, \mathrm{d}t, \quad \text{(C.18)}$$

where $\mathcal{L}^{(\mathrm{pf})}$, $\mathcal{L}^{(\mathrm{int})}$ and $\mathcal{L}^{(\mathrm{F})}$ are defined by (2.27).

(i) To find the equations governing the background field a_ν, we take variation of the total action $S^{(\mathrm{total})}$ by assuming the fluid motion given and vary the potential a_ν only. Expressing the integrand of variation of $S^{(\mathrm{int})}$ and $S^{(\mathrm{F})}$ by $\mathcal{J}^{(\mathrm{int})}$ and $\mathcal{J}^{(\mathrm{F})}$ respectively, and using (2.31) in §2.6, their variations are given by

$$\delta\mathcal{J}^{(\mathrm{int})} + \delta\mathcal{J}^{(\mathrm{F})} = \Big(j^\nu - \frac{1}{\mu}\frac{\partial}{\partial x^\lambda} f^{\nu\lambda} \Big)\, \delta a_\nu\, \mathrm{d}\tau\, c^{-1} \mathrm{d}\overline{\mathcal{V}}, \qquad \mathrm{d}\mathcal{V} = \mathrm{d}\overline{\mathcal{V}}\, \sqrt{1-\beta^2}. \quad \text{(C.19)}$$

Vanishing of $\delta\mathcal{J}^{(\mathrm{int})} + \delta\mathcal{J}^{(\mathrm{F})} = 0$ for arbitrary variation δa_ν is given in §2.6 as

$$\frac{\partial}{\partial x^\lambda} f^{\nu\lambda} = \mu\, j^\nu, \quad \text{(C.20)}$$

(see (2.29) for the 4-current j^ν). This includes two important messages inside.

Firstly, the equation yields the law of current conservation $\partial_\nu j^\nu = 0$ of (2.33), which is rewritten in the following equation of continuity

$$\partial_\nu j^\nu = 0 \quad \Rightarrow \quad \partial_t \rho + \nabla\cdot\boldsymbol{j} = 0, \quad \text{(C.21)}$$

for $j^\nu = (\rho c,\, \rho\boldsymbol{v})$ with $\boldsymbol{j} = \rho\boldsymbol{v}$. Secondly, the equation (C.20) represents a pair of fluid Maxwell equations. Using the definition (3.14) of the fields $\boldsymbol{e}$ and $\boldsymbol{b}$ and the matrix representation (3.15) of $f^{\nu\lambda}$, the equation (C.20) represents the followings:

$$\varepsilon\,\nabla\cdot\boldsymbol{e} = \rho, \quad -\varepsilon\,\partial_t \boldsymbol{e} + \mu^{-1}\nabla\times\boldsymbol{b} = \boldsymbol{j}, \quad \text{where} \ \ \varepsilon = 1/(c^2\mu). \quad \text{(C.22)}$$

(ii) Next, to find the equations of fluid motion, we assume the field potential a_ν given and vary only the trajectory of the fluid particle ($\boldsymbol{X}$ or $\boldsymbol{v} = \partial_t \boldsymbol{X}$). This is done in Appendix B.3 or in section III c) with the integrand variation $\delta\mathcal{J}^{(\mathrm{fl+a})}$ given by (2.10) or by (B.26). For the invariance of the action, vanishing of $\delta\mathcal{J}^{(\mathrm{fl+a})} = 0$ is required for arbitrary variation δx^ν. Thus, we obtain the following geodesic equation:

$$\nabla_\tau^{(\mathrm{fl+a})} u^\nu \equiv \frac{\mathrm{d}}{\mathrm{d}\tau} u_\nu + c^{-2}\,\frac{1}{\rho}\partial_\nu \overline{p} - c^{-1} f_{\nu\mu}\, u^\mu = 0. \quad \text{(C.23)}$$

For the definition of $\mathrm{d}/\mathrm{d}\tau$, see (C.17). In section III c), it is already shown that this reduces to the modified Euler equation (3.16) with additional term of fluid Lorentz force $\boldsymbol{f}_a$:

$$\mathrm{D}_t \boldsymbol{v} + \rho^{-1}\,\nabla p - \boldsymbol{f}_a = 0, \qquad \boldsymbol{f}_a \equiv \boldsymbol{e} + \boldsymbol{v}\times\boldsymbol{b}. \quad (96)$$

References Références Referencias

1. Aitchison IJ R and Hey A J G 2013 *Gauge Theories in Particle Physics: A Practical Introduction,* Vol. 1 & 2, Fourth Edition. (CRC Press, Taylor & Francis Group).
2. Arnold V I 1966 Sur la gèomètrie diffèrentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier* (Grenoble) 16, 319-361.
3. Del ȷAlamo, J. C. and Jimȷenez, J., Linear energy amplification in turbulent channels. *J. Fluid Mech.* 2006, 559, 205-213.
4. Hutchins, N. and Marusic, I., Evidence of very long meandering features in the logarithmic region of turbulent boundary layers. *J. Fluid Mech.* 2007, 579, 1-28.
5. Jackson J D 1999 *Classical Electrodynamics* (Third edition) (John Wiley & Sons).
6. Kambe T 2010 *Geometrical Theory of Dynamical Systems and Fluid Flows,* (World Scientific).
7. Kambe T 2011 A new solution of Euler's equation of motion with explicit expression of helicity, *WSEAS Special Issue "Mathematical Fluid Dynamics* 53-716; or WSEAS Transactions on Fluid Mechanics 7(2), 59-70.
8. Kambe T 2013 New representation of rotational flow fields satisfying Euler's equation of an ideal compressible fluid, *Fluid Dyn. Res.* 45, 015505 (16pp).
9. Kambe T 2017 New scenario of turbulence theory and wall-bounded turbulence: Theoretical significance, *Geophys. Astrophys. Fluid Dyn.* 111, 448-507.
10. Kambe T 2020 New perspectives on mass conservation law and waves in fluid mechanics, *Fluid Dyn. Res.* 52, 1 - 34 (34pp).
11. Kambe T 2021a: Gauge symmetries in physical fields (review), *Global Journal of Science Frontier Research,* vol.21, iss.4.
12. Kambe T 2021b: "Fluid Gauge Theory (updating title): New Perspective on Mass Conservation and Rotational Waves", presented at ICTAM Milano 2020+1 virtual, Aug. 2021: DOI: 10.13140/RG.2.2.28629.35047. In "Tsutomu Kambe The University of Tokyo — Todai -ResearchGate"
13. Kim, K.C. and Adrian, R.J., Very large-scale motion in the outer layer. *Phys. Fluids* 1999, 11, 417–422.
14. Kundt, A. 1866: Ueber eine neue Art Akustischer Staubfiguren und uber die Anwendung derselben zur Bestimmung der Shallgeschwindigkeit in festen Korpern und Gasen, *Annalen der Physik und Chemie,* vol.127 (4): 497?523. doi:10.1002/andp.18662030402.
15. Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields* (Pergamon Press).
16. Landau L D and Lifshitz E M 1987 *Fluid Mechanics* (Pergamon Press, 2nd ed.).
17. Misner C W, Thorne KS and Wheeler J A 2017 *Gravitation* (W.H. Freeman and Co., San Francisco).
18. Monty, J.P., Stewart, J.A., Williams, R.C. and Chong, M.S., Large-scale features in turbulent pipe and channel flow. *J. Fluid Mech.* 2007, 589, 147–156.
19. Noether E 1918 Invariant variations problem, *Nachr. d. König.* Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse, 235-257.
20. Reynolds, W.C. and Hussain, A.K.M.F., The mechanics of an organized wave in turbulent shear flow. Part 3. Theoretical models and comparisons with experiments. *J. Fluid Mech.* 1972, 54, 263 - 288.
21. Scofield, D.F. and Huq, P., Fluid dynamical Lorentz force law and Poynting theorem ?derivation and implications. *Fluid Dyn. Res.* 2014, 46, 055514 (22 pp).

22. Smits, A. J., McKeon B. J. & Marusic, I., High-Reynolds number wall turbulence. *Annu. Rev. Fluid Mech.* 2011, 43, 353-375.
23. Utiyama R 1956 Invariant theoretical interpretation of interaction, *Phys. Rev.* 101, 1597 - 1607.
24. Utiyama R 1987 *Ippan Gauge ba ron josetsu (Introduction to the General Gauge Field Theory)* (Iwanami Shoten, Tokyo).

This page is intentionally left blank

Global Journal of Science Frontier Research: A
Physics and Space Science
Volume 21 Issue 4 Version 1.0 Year 2021
Type: Double Blind Peer Reviewed International Research Journal
Publisher: Global Journals
Online ISSN: 2249-4626 & Print ISSN: 0975-5896

Application of Ammonium Carbonate Solutions in Electrochemical Technology of Heavy Alloy Waste Processing

By O G Kuznetsova, A M Levin, M A Sevostyanov, O I Tsybin & A O Bolshikh

Abstract- The electrochemical behavior of the VNZhK alloy waste (wt.%: W 90, Ni 7,2, Fe 1,8, Co 1) in a solution of $(NH_4)_2CO_3$ 1M was investigated by using the methods of cyclic voltammetry in the potentiodynamic mode, potentiostatic electrolysis and electrolysis under the action of alternating current (industrial frequency 50 Hz). It was found, that the highest oxidation rate of VNZhK alloy waste (1700 mg /$cm^2 \cdot h$) is achieved by using alternating current, and the highest current efficiency (about 100%) - by using direct current. A technological scheme was proposed for recovery of tungsten from the heavy tungsten alloy waste in the form of ammonium paratungstate.

GJSFR-A Classification: FOR Code: 091499

Strictly as per the compliance and regulations of:

Application of Ammonium Carbonate Solutions in Electrochemical Technology of Heavy Alloy Waste Processing

O G Kuznetsova α, A M Levin σ, M A Sevostyanov ρ, O I Tsybin Ѡ & A O Bolshikh ¥

Abstract- The electrochemical behavior of the VNZhK alloy waste (wt.%: W 90, Ni 7,2, Fe 1,8, Co 1) in a solution of $(NH_4)_2CO_3$ 1M was investigated by using the methods of cyclic voltammetry in the potentiodynamic mode, potentiostatic electrolysis and electrolysis under the action of alternating current (industrial frequency 50 Hz). It was found, that the highest oxidation rate of VNZhK alloy waste (1700 mg /cm^2·h) is achieved by using alternating current, and the highest current efficiency (about 100%) - by using direct current. A technological scheme was proposed for recovery of tungsten from the heavy tungsten alloy waste in the form of ammonium paratungstate.

I. Introduction

Research in the field of tungsten-containing secondary raw materials processing is now widespread [1-10]. Electrochemical technologies for recycling tungsten from waste of metalized tungsten-containing raw materials, including heavy tungsten alloys (WHAs), are often based on the use of alkaline solutions [11-15]. Despite the fact, that alkaline electrolytes have a higher electrical conductivity compared to ammonium carbonate solutions [16, 17], the use of the latter in the electrochemical processing of WHAs can significantly simplify the production of a commercial product - ammonium paratungstate (APM). The extraction of tungsten from traditional alkaline solutions, as well as ammonium alkaline tungsten-containing solutions, is based on their neutralization with acids and precipitation of tungstic acid. Tungstic acid is further purified to achieve the necessary requirements. In this work, we propose the use of an ammonium carbonate solution as an electrolyte for processing of WHAs waste of the VNZhK type. It will simplify the separation of tungsten from the rest of the alloy components (metals of the iron subgroup), and at the same time, the use of significant volumes of acids will be excluded from the technological process. Also there will be no need to dispose of concentrated salt solutions. The proposed approach prevents a decrease in the purity of the APM by impurities from the electrolyte. The use of ammonium carbonate solutions in the WHAs electrochemical processing contributes to an increase in the environmental safety of production and also reduces the number of technological operations when obtaining the final product - APM.

Author α σ ρ Ѡ ¥: A.A. Baikov Institute of Metallurgy and Materials Science of Russian Academy of Sciences, 49 Leninsky pr., Moscow, Russian Federation. e-mail: olyakolya@mail.ru

II. Experimental Part

The anodic behavior of the VNZhK alloy waste (wt.%: W 90, Ni 7,2, Fe 1,8, Co 1) was studied by linear voltammetry in a potentiodynamic mode using an IPC-Pro potentiostat. The VNZhK alloy waste was applied as a working electrode. The measurements were carried out relative to a saturated chlorine-silver reference electrode with an auxiliary glassy-carbon electrode. The potential sweep speed was 1 mV/s. The samples were preliminarily washed with hydrochloric acid (4M) and distilled water. All studies were carried out in a ammonium carbonate solution 1M. This is due, on the one hand, to sufficient electrical conductivity of the electrolyte, and on the other hand, to the possibility of achieving a high concentration of tungsten in the electrolyte [17]. The solution temperature 20°C was maintained using a thermostat TW2-02. Potentiostatic dissolution of VNZhK alloy waste under the action of direct current (DC) in a solution of ammonium carbonate 1M was carried out at a potential of +0,25 V, using an auxiliary glassy-carbon electrode at a temperature of 20°C. Dissolution of the VNZhK alloy waste under the influence of alternating current (AC) was carried out at the industrial frequency 50 Hz, using two electrodes made of the processed material at a temperature of 20 °C. The completeness of the tungsten leaching from the VNZhK alloy surface during its electrochemical processing under the action of DC in a potentiostatic mode was determined using X-ray spectral analysis (ISM-6380LV equipped with an Energy 250 analyser and X-ray diffractometry (ARL X'TRA). The content of tungsten oxide in ammonium paratungstate was determined by the gravimetric method [18].

The anodic polarization of the VNZhK alloy in the $(NH_4)_2CO_3$ 1M solution is shown in Figure 1. Curves 1 - 3 in Figure 1 represent three cycles of volt-ampere curves of the VNZhK alloy waste in the potential range from -0,25 to +1,25 V, scanned sequentially one after the other. Curve 4 in Figure 1 is the last cycle of the volt-ampere curve, after which its appearance practically does not change. It is associated with the complete

Global Journal of Science Frontier Research (A) Volume XXI Issue IV Version I Year 2021

leaching of tungsten from the surface of the alloy. Figure 1 shows that the polarization curves have a shape, which is typical for materials, that tend to passivation. The initial part of the polarization curves (in the potential range from -0.25 to +0.25 V) is associated with the oxidation of tungsten and its transition into solution. It can be seen, that with each subsequent cycle the value of the maximum anodic current density decreases during alloy dissolution. It drops from 170 mA/cm^2 for a fresh alloy surface to 40 mA/cm^2 for alloy surface completely leached with tungsten and enriched with iron, nickel and cobalt.

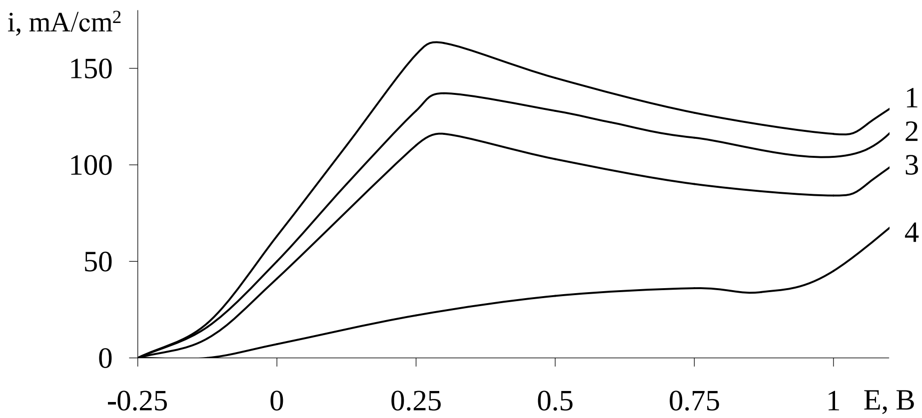


Figure 1: Anode polarization curves of VNZhK alloy waste in solution $(NH_4)_2CO_3$ 1M. 1- the first cycle, 2 - the second cycle, 3 - the last cycle after complete leaching of tungsten from the surface of the alloy

Electrochemical dissolution of VNZhK alloy waste under the action of a DC was carried out in a potentiostatic mode at a potential of +0,25 V, which corresponds to the maximum anodic current density of the investigated alloy in electrolyte $(NH_4)_2CO_3$ 1M. It was found, that the dissolution of the VNZhK alloy waste is carried out at a rate of 150 mg/cm^2·h and current efficiency close to 100% (based on the ionization of tungsten in the oxidation state +6) with a degree of tungsten extraction into the solution - 99,5%. At the same time, it was found that, the use of ammonium carbonate electrolytes, as well as alkaline and ammonium alkaline solutions, leads to decrease in the dissolution rate of investigated alloy in the process of electrochemical leaching of tungsten from the surface of the WHAs under the action of DC [19]. The use of AC makes it possible to overcome this obstacle [14].

Figure 2 shows the dependence of the oxidation rate of the VNZhK alloy waste and its current efficiency on the alternating current density in the $(NH_4)_2CO_3$ 1M solution. It can be seen, that current density growth from 1 to 7 A/cm^2 leads to a manifold increase in the oxidation rate of the alloy from 100 to 1700 mg/cm^2·h. The process is accompanied by an increase in the current efficiency only from 3 to 33%. It is important to note the similarity of the electrochemical behavior of the VNZhK alloy in ammonium carbonate and ammonium alkaline solutions under the action of AC [14, 20]. In both cases, the transition of tungsten into solution is accompanied by the concentration of the iron subgroup metals in the finely dispersed electrolysis residue.

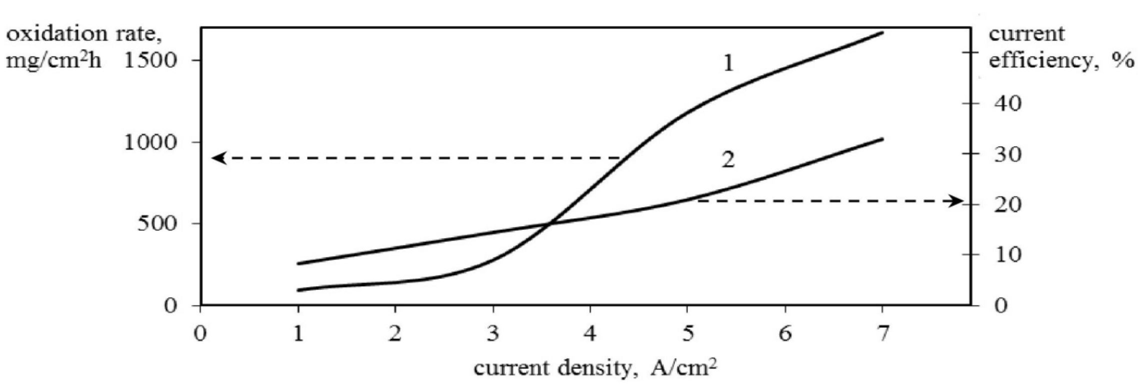


Figure 2: Dependences of oxidation rate (1) of VNZhK alloy waste and its current efficiency (2) on the alternating current density in solution $(NH_4)_2CO_3$ 1M.

In order to optimize the electrolysis energy parameters, it is advisable to carry out the processing of WHAs by using both DC and AC. In this case, DC is most effective in cases, where the surface of the processed alloy is enriched with tungsten, and AC - when tungsten is already leached from the alloy surface and it needs to be renewed through the formation of a residue of the iron subgroup metals oxides in the form micro-dispersed powder [14, 20].

When the concentration of tungsten in the electrolyte reached 50 g/l, the latter was evaporated at a temperature of 70-80˚C until the odor of ammonia was removed in order to obtain APM. The precipitated APM was washed and dried at 90˚C. The content of WO_3 in the obtained APM was 87,4 wt.%.

A schematic diagram for the processing of tungsten alloy waste of the VNZhK type under the action of DC and AC in ammonium carbonate solutions is shown in Figure 3. Tungsten passes from the alloy into the electrolyte as a result of the electrochemical dissolution of the alloy under the action of AC and DC. The electrolyte is evaporated, and excess of ammonium carbonate decomposes into ammonia and carbon dioxide, released in the form of gases. The process is accompanied by crystallization of APM. Nickel, iron and cobalt are concentrated in the oxide form as micro-dispersed electrolysis residue, which is filtered before the electrolyte evaporation stage. Traces of tungsten oxides in micro-dispersed residue are leached with an ammonium carbonate solution and returned to the stage of electrochemical dissolution of VNZhK alloy waste.

III. Conclusions

The electrochemical behavior of heavy tungsten alloy waste of the VNZhK type in a 1 M ammonium carbonate solution was investigated. It was found, that the optimal value of the alloy dissolution potential was + 0,25 V in the investigated electrolyte. It was shown, that the dissolution rate of the alloy was 150 mg/cm^2·h at its current efficiency of ~ 100% in the process of potentiostatic electrolysis.

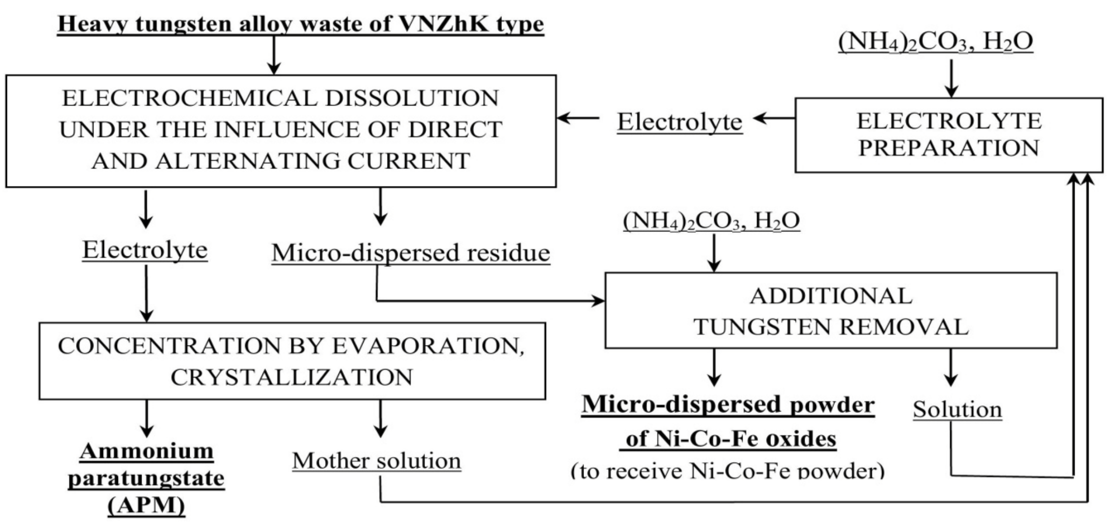


Figure 3: Schematic diagram for the recycling of heavy tungsten alloys of VNZhK type in ammonium carbonate solutions

The change in the oxidation rate of heavy tungsten alloy waste of the VNZhK type and its current efficiency depending on the alternating current density of the industrial frequency in the range from 1 to 7 A/cm^2 was investigated. It was revealed, that the oxidation rate of alloy increased to 1700 mg/cm^2·h at a current density of 7 A/cm^2 but its current efficiency was about 30%.

A technological scheme for processing heavy tungsten alloy waste of the VNZhK type in ammonium carbonate solutions using direct and alternating electric current was proposed.

The work was carried out according to the state ask 075-00328-21-00.

References Références Referencias

1. Han Z, Golev A, Edraki M 2021 A Review of tungsten resources and potential extraction from mine waste *Minerals* 11 701.
2. Zeiler B, Bartl A, Schubert W 2021 Recycling of tungsten: Current share, economic limitations, technologies and future potential *Int Journal of Refractory Metals and Hard Materials* 98 105546.
3. Shen L, Li X, Linberg D, Taskinen P 2019 Tungsten extractive metallurgy: A review of processes and their challenges for sustainability *Minerals Engineering* 142 105934.
4. Kamal S S, Sahoo P K, Vimala J, Shanker B, Ghosal P, Durai L 2016 Synthesis of high purity tungsten nanoparticles from tungsten heavy alloy scrap by

selective precipitation and reduction route *Journal of Alloys and Compounds* 678 403–409.

5. Shemi A, Magumise A, Ndlovu S, Sacks N 2018 Recycling of tungsten carbide scrap metal: A review of recycling methods and future prospects *Mineral Engineering* 122 195-205.
6. Ye L, Ouyang Z, Chen Y, Liu S 2020 Recovery of rhenium from tungsten-rhenium wire by alkali fusion in KOH-K_2CO_3 binary molten salt electrolysis *Int Journal of Refractory Metals and Hard Materials* 87 1-7.
7. Ming Li, Xiaoli Xi, Zuiren Nie, Liwen Ma 2019 Recovery of tungsten from WC-Co hard metal scraps using molten salts electrolysis *Journal of Material Research and Technology* 8 1 1440-1450.
8. Ageeva E V, Selutin V L, Andreeva L P 2020 Research of the influence of the parameters of electric erosion dispersion of VNZh alloy on the average size of the obtained particles *Electrometallurgy* 6 32-40.
9. Kovalenko V, Kotoc V 2020 Investigation of the anodic behavior of W-based superalloy, for electrochemical selective treatment *Eastern-European Journal of Enterprise Technologies* 6/12 108 61-66.
10. Kovalenko V, Kotoc V 2017 Selective anodic treatment or W(WC)-based superalloy scrap *Eastern-European Journal of Enterprise Technologies* 1/5 85 3-58.
11. Parshutin V V 2008 Corrosion and electrochemical behavior of pseudo-alloys based on tungsten and their components *Electronic processing of materials* 6 27-45.
12. Palant A A, Levin A M , Palant L A 2013 The method of electrochemical processing of metal tungsten-copper alloy *Patent RU2479652.*
13. Hairunnisha S, Sendil GK, Prabhakar Rethinaraj J, Srinivasan G N, Adaikkalam P, Kulandaisamy S 2007 Studies on the preparation of pure ammonium paratungstate from tungsten alloy scrap *Hydrometallurgy* 85 67-71.
14. Kuznetsova O G, Levin A M, Sevastyanov M A, Tsybin O I, Bolshikh A O 2021 Electrochemical processing of a heavy W-Ni-Fe alloy by direct and alternating current in ammonia-alkali solutions *Russian Metallurgy (Metally)* 5 586-593.
15. Luo L, Kejun L, Shibayama A, Yen W, Fujita T, Shindo O, Katai 2004 A Recovery of tungsten and vanadium from tungsten alloy scrap *Hydrometallurgy* 7 1-8.
16. Kuznetsova O G, Levin A M, Sevastyanov M A, Tsybin O I, Bolshikh A O 2020 Electrical conductivity of tungsten-containing ammonium hydroxide solutions *Russian Metallurgy (Metally)* 9 971-975.
17. Kuznetsova O G, Levin A M, Sevastyanov M A 2021 Specific electrical conductivity of tungsten-containing aqueous solutions of ammonium carbonate *Conf "Theory and practice of scientific activity modernization in the context of digitalization"* 25-28.
18. Tungsten, molybdenum. General requirements for methods of chemical and spectral analysis GOST 29103-91 Moscow: Publishing house of standards 1991.
19. Kuznetsova O G, Levin A M, Sevastyanov M A, Tsybin O I, Bolshikh A O 2019 Electrochemical oxidation of a heavy tungsten-containing VNZh-type alloy and its components in ammonia–alkali electrolytes *Russian Metallurgy (Metally)* 5 507–510.
20. Kuznetsova O G, Levin A M, Sevastyanov M A, Tsybin O I, Bolshikh A O 2021 Electrochemical processing of heavy tungsten alloy wastes for obtaining a microdispersed iron-nickel base powder by using alternating current *IOP Conf Ser: Journal of Physics* 1942 012056.

Global Journals Guidelines Handbook 2021

www.GlobalJournals.org

Memberships

FELLOWS/ASSOCIATES OF SCIENCE FRONTIER RESEARCH COUNCIL

FSFRC/ASFRC MEMBERSHIPS

Introduction

FSFRC/ASFRC is the most prestigious membership of Global Journals accredited by Open Association of Research Society, U.S.A (OARS). The credentials of Fellow and Associate designations signify that the researcher has gained the knowledge of the fundamental and high-level concepts, and is a subject matter expert, proficient in an expertise course covering the professional code of conduct, and follows recognized standards of practice. The credentials are designated only to the researchers, scientists, and professionals that have been selected by a rigorous process by our Editorial Board and Management Board.

Associates of FSFRC/ASFRC are scientists and researchers from around the world are working on projects/researches that have huge potentials. Members support Global Journals' mission to advance technology for humanity and the profession.

FSFRC

FELLOW OF SCIENCE FRONTIER RESEARCH COUNCIL

FELLOW OF SCIENCE FRONTIER RESEARCH COUNCIL is the most prestigious membership of Global Journals. It is an award and membership granted to individuals that the Open Association of Research Society judges to have made a 'substantial contribution to the improvement of computer science, technology, and electronics engineering.

The primary objective is to recognize the leaders in research and scientific fields of the current era with a global perspective and to create a channel between them and other researchers for better exposure and knowledge sharing. Members are most eminent scientists, engineers, and technologists from all across the world. Fellows are elected for life through a peer review process on the basis of excellence in the respective domain. There is no limit on the number of new nominations made in any year. Each year, the Open Association of Research Society elect up to 12 new Fellow Members.

Benefit

To the institution

Get letter of appreciation

Global Journals sends a letter of appreciation of author to the Dean or CEO of the University or Company of which author is a part, signed by editor in chief or chief author.

Exclusive Network

Get access to a closed network

A FSFRC member gets access to a closed network of Tier 1 researchers and scientists with direct communication channel through our website. Fellows can reach out to other members or researchers directly. They should also be open to reaching out by other.

Career | Credibility | Exclusive | Reputation

Certificate

Receive a print ed copy of a certificate

Fellows receive a printed copy of a certificate signed by our Chief Author that may be used for academic purposes and a personal recommendation letter to the dean of member's university.

Career | Credibility | Exclusive | Reputation

Designation

Get honored title of membership

Fellows can use the honored title of membership. The “FSFRC” is an honored title which is accorded to a person’s name viz. Dr. John E. Hall, Ph.D., FSFRC or William Walldroff, M.S., FSFRC.

Career | Credibility | Exclusive | Reputation

Recognition on the Platform

Better visibility and citation

All the Fellow members of FSFRC get a badge of "Leading Member of Global Journals" on the Research Community that distinguishes them from others. Additionally, the profile is also partially maintained by our team for better visibility and citation. All fellows get a dedicated page on the website with their biography.

Career | Credibility | Reputation

Future Work

Get discounts on the future publications

Fellows receive discounts on future publications with Global Journals up to 60%. Through our recommendation programs, members also receive discounts on publications made with OARS affiliated organizations.

Career | Financial

GJ Internal Account

Unlimited forward of Emails

Fellows get secure and fast GJ work emails with unlimited forward of emails that they may use them as their primary email. For example, john [AT] globaljournals [DOT] org.

Career | Credibility | Reputation

Premium Tools

Access to all the premium tools

To take future researches to the zenith, fellows and associates receive access to all the premium tools that Global Journals have to offer along with the partnership with some of the best marketing leading tools out there.

Financial

Conferences & Events

Organize seminar/conference

Fellows are authorized to organize symposium/seminar/conference on behalf of Global Journal Incorporation (USA). They can also participate in the same organized by another institution as representative of Global Journal. In both the cases, it is mandatory for him to discuss with us and obtain our consent. Additionally, they get free research conferences (and others) alerts.

Career | Credibility | Financial

Early Invitations

Early invitations to all the symposiums, seminars, conferences

All fellows receive the early invitations to all the symposiums, seminars, conferences and webinars hosted by Global Journals in their subject.

Exclusive

Publishing Articles & Books

Earn 60% of sales proceeds

Fellows can publish articles (limited) without any fees. Also, they can earn up to 60% of sales proceeds from the sale of reference/review books/literature/ publishing of research paper. The FSFRC member can decide its price and we can help in making the right decision.

Exclusive Financial

Reviewers

Get a remuneration of 15% of author fees

Fellow members are eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get a remuneration of 15% of author fees, taken from the author of a respective paper.

Financial

Access to Editorial Board

Become a member of the Editorial Board

Fellows may join as a member of the Editorial Board of Global Journals Incorporation (USA) after successful completion of three years as Fellow and as Peer Reviewer. Additionally, Fellows get a chance to nominate other members for Editorial Board.

Career Credibility Exclusive Reputation

And Much More

Get access to scientific museums and observatories across the globe

All members get access to 5 selected scientific museums and observatories across the globe. All researches published with Global Journals will be kept under deep archival facilities across regions for future protections and disaster recovery. They get 10 GB free secure cloud access for storing research files.

ASSOCIATE OF SCIENCE FRONTIER RESEARCH COUNCIL

ASSOCIATE OF SCIENCE FRONTIER RESEARCH COUNCIL is the membership of Global Journals awarded to individuals that the Open Association of Research Society judges to have made a 'substantial contribution to the improvement of computer science, technology, and electronics engineering.

> The primary objective is to recognize the leaders in research and scientific fields of the current era with a global perspective and to create a channel between them and other researchers for better exposure and knowledge sharing. Members are most eminent scientists, engineers, and technologists from all across the world. Associate membership can later be promoted to Fellow Membership. Associates are elected for life through a peer review process on the basis of excellence in the respective domain. There is no limit on the number of new nominations made in any year. Each year, the Open Association of Research Society elect up to 12 new Associate Members.

Benefit

To the Institution

Get letter of appreciation

Global Journals sends a letter of appreciation of author to the Dean or CEO of the University or Company of which author is a part, signed by editor in chief or chief author.

Exclusive Network

Get access to a closed network

A ASFRC member gets access to a closed network of Tier 1 researchers and scientists with direct communication channel through our website. Associates can reach out to other members or researchers directly. They should also be open to reaching out by other.

Career | Credibility | Exclusive | Reputation

Certificate

Receive a print ed copy of a certificate

Associates receive a printed copy of a certificate signed by our Chief Author that may be used for academic purposes and a personal recommendation letter to the dean of member's university.

Career | Credibility | Exclusive | Reputation

Designation

Get honored title of membership

Associates can use the honored title of membership. The “ASFRC” is an honored title which is accorded to a person’s name viz. Dr. John E. Hall, Ph.D., ASFRC or William Walldroff, M.S., ASFRC.

Career | Credibility | Exclusive | Reputation

Recognition on the Platform

Better visibility and citation

All the Associate members of ASFRC get a badge of "Leading Member of Global Journals" on the Research Community that distinguishes them from others. Additionally, the profile is also partially maintained by our team for better visibility and citation. All associates get a dedicated page on the website with their biography.

Career | Credibility | Reputation

Future Work

Get discounts on the future publications

Associates receive discounts on the future publications with Global Journals up to 60%. Through our recommendation programs, members also receive discounts on publications made with OARS affiliated organizations.

Career Financial

GJ Internal Account

Unlimited forward of Emails

Associates get secure and fast GJ work emails with unlimited forward of emails that they may use them as their primary email. For example,
john [AT] globaljournals [DOT] org.

Career Credibility Reputation

Premium Tools

Access to all the premium tools

To take future researches to the zenith, fellows receive access to almost all the premium tools that Global Journals have to offer along with the partnership with some of the best marketing leading tools out there.

Financial

Conferences & Events

Organize seminar/conference

Associates are authorized to organize symposium/seminar/conference on behalf of Global Journal Incorporation (USA). They can also participate in the same organized by another institution as representative of Global Journal. In both the cases, it is mandatory for him to discuss with us and obtain our consent. Additionally, they get free research conferences (and others) alerts.

Career Credibility Financial

Early Invitations

Early invitations to all the symposiums, seminars, conferences

All associates receive the early invitations to all the symposiums, seminars, conferences and webinars hosted by Global Journals in their subject.

Exclusive

Publishing Articles & Books

Earn 30-40% of sales proceeds

Associates can publish articles (limited) without any fees. Also, they can earn up to 30-40% of sales proceeds from the sale of reference/review books/literature/publishing of research paper.

Exclusive Financial

Reviewers

Get a remuneration of 15% of author fees

Associate members are eligible to join as a paid peer reviewer at Global Journals Incorporation (USA) and can get a remuneration of 15% of author fees, taken from the author of a respective paper.

Financial

And Much More

Get access to scientific museums and observatories across the globe

All members get access to 2 selected scientific museums and observatories across the globe. All researches published with Global Journals will be kept under deep archival facilities across regions for future protections and disaster recovery. They get 5 GB free secure cloud access for storing research files.

Associate	Fellow	Research Group	Basic
$4800 lifetime designation	$6800 lifetime designation	$12500.00 organizational	APC per article
Certificate, LoR and Momento 2 discounted publishing/year **Gradation** of Research 10 research contacts/day 1 **GB** Cloud Storage **GJ** Community Access	**Certificate,** LoR and Momento **Unlimited** discounted publishing/year **Gradation** of Research **Unlimited** research contacts/day 5 **GB** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**Certificates,** LoRs and Momentos **Unlimited** free publishing/year **Gradation** of Research **Unlimited** research contacts/day **Unlimited** Cloud Storage **Online Presense** Assistance **GJ** Community Access	**GJ** Community Access

Preferred Author Guidelines

We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

Authors should submit their complete paper/article, including text illustrations, graphics, conclusions, artwork, and tables. Authors who are not able to submit manuscript using the form above can email the manuscript department at submit@globaljournals.org or get in touch with chiefeditor@globaljournals.org if they wish to send the abstract before submission.

Before and during Submission

Authors must ensure the information provided during the submission of a paper is authentic. Please go through the following checklist before submitting:

1. Authors must go through the complete author guideline and understand and *agree to Global Journals' ethics and code of conduct,* along with author responsibilities.
2. Authors must accept the privacy policy, terms, and conditions of Global Journals.
3. Ensure corresponding author's email address and postal address are accurate and reachable.
4. Manuscript to be submitted must include keywords, an abstract, a paper title, co-author(s') names and details (email address, name, phone number, and institution), figures and illustrations in vector format including appropriate captions, tables, including titles and footnotes, a conclusion, results, acknowledgments and references.
5. Authors should submit paper in a ZIP archive if any supplementary files are required along with the paper.
6. Proper permissions must be acquired for the use of any copyrighted material.
7. Manuscript submitted *must not have been submitted or published elsewhere* and all authors must be aware of the submission.

Declaration of Conflicts of Interest

It is required for authors to declare all financial, institutional, and personal relationships with other individuals and organizations that could influence (bias) their research.

Policy on Plagiarism

Plagiarism is not acceptable in Global Journals submissions at all.

Plagiarized content will not be considered for publication. We reserve the right to inform authors' institutions about plagiarism detected either before or after publication. If plagiarism is identified, we will follow COPE guidelines:

Authors are solely responsible for all the plagiarism that is found. The author must not fabricate, falsify or plagiarize existing research data. The following, if copied, will be considered plagiarism:

- Words (language)
- Ideas
- Findings
- Writings
- Diagrams
- Graphs
- Illustrations
- Lectures

- Printed material
- Graphic representations
- Computer programs
- Electronic material
- Any other original work

Authorship Policies

Global Journals follows the definition of authorship set up by the Open Association of Research Society, USA. According to its guidelines, authorship criteria must be based on:

1. Substantial contributions to the conception and acquisition of data, analysis, and interpretation of findings.
2. Drafting the paper and revising it critically regarding important academic content.
3. Final approval of the version of the paper to be published.

Changes in Authorship

The corresponding author should mention the name and complete details of all co-authors during submission and in manuscript. We support addition, rearrangement, manipulation, and deletions in authors list till the early view publication of the journal. We expect that corresponding author will notify all co-authors of submission. We follow COPE guidelines for changes in authorship.

Copyright

During submission of the manuscript, the author is confirming an exclusive license agreement with Global Journals which gives Global Journals the authority to reproduce, reuse, and republish authors' research. We also believe in flexible copyright terms where copyright may remain with authors/employers/institutions as well. Contact your editor after acceptance to choose your copyright policy. You may follow this form for copyright transfers.

Appealing Decisions

Unless specified in the notification, the Editorial Board's decision on publication of the paper is final and cannot be appealed before making the major change in the manuscript.

Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

Declaration of funding sources

Global Journals is in partnership with various universities, laboratories, and other institutions worldwide in the research domain. Authors are requested to disclose their source of funding during every stage of their research, such as making analysis, performing laboratory operations, computing data, and using institutional resources, from writing an article to its submission. This will also help authors to get reimbursements by requesting an open access publication letter from Global Journals and submitting to the respective funding source.

Preparing your Manuscript

Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

Segment draft and final research paper: You have to strictly follow the template of a research paper, failing which your paper may get rejected. You are expected to write each part of the paper wholly on your own. The peer reviewers need to identify your own perspective of the concepts in your own terms. Please do not extract straight from any other source, and do not rephrase someone else's analysis. Do not allow anyone else to proofread your manuscript.

Written material: You may discuss this with your guides and key sources. Do not copy anyone else's paper, even if this is only imitation, otherwise it will be rejected on the grounds of plagiarism, which is illegal. Various methods to avoid plagiarism are strictly applied by us to every paper, and, if found guilty, you may be blacklisted, which could affect your career adversely. To guard yourself and others from possible illegal use, please do not permit anyone to use or even read your paper and file.

CRITERION FOR GRADING A RESEARCH PAPER (COMPILATION) BY GLOBAL JOURNALS

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

INDEX

save our planet

Global Journal of Science Frontier Research

Visit us on the Web at www.GlobalJournals.org | www.JournalofScience.org
or email us at helpdesk@globaljournals.org

ISSN 9755896