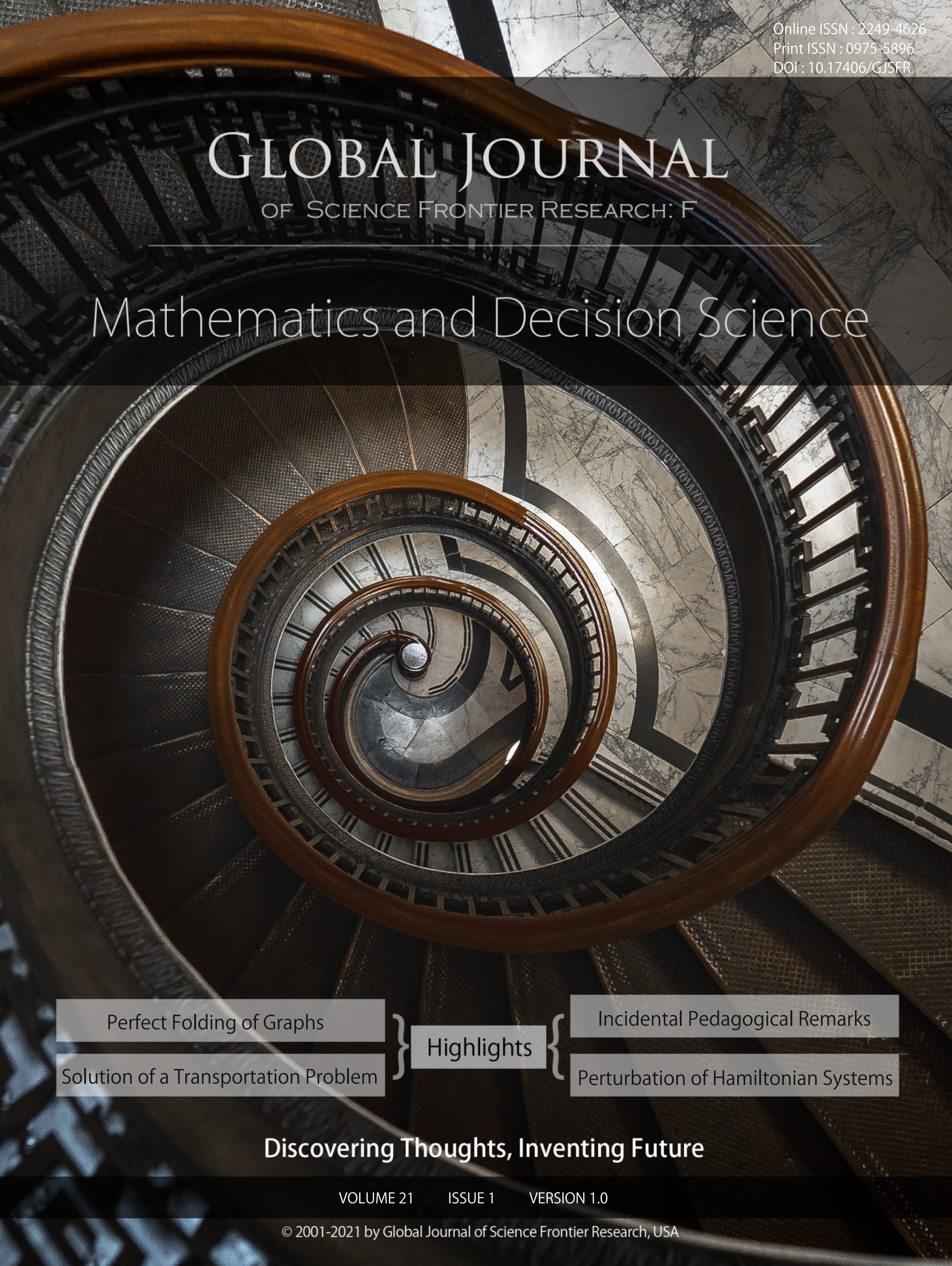

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Highlights

- Perfect Folding of Graphs
- Solution of a Transportation Problem
- Incidental Pedagogical Remarks
- Perturbation of Hamiltonian Systems

Discovering Thoughts, Inventing Future

VOLUME 21 ISSUE 1 VERSION 1.0

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A Note on Identifying Critical Activities in Project Scheduling Via Linear Programming on Spreadsheets, with Incidental Pedagogical Remarks

By Gregory L. Light

Abstract- This note presents a speedy resolution of the critical activities for the critical path method (CPM) in project management by first running Excel Solver to obtain the minimized time of the completion of the project in question and next perturbing the required times of all the involved activities concomitantly to reveal the critical activities by observing the difference in the minimized times. We use extensions of decimal places for the classroom demonstration of the above-said perturbation, and consider additions of log(prime numbers) to the required times of all the activities to serve any large-scale professional analyses without using tailored-made software. As a separate incidental pedagogical note, we show a heuristic approach to constructing exactly three constraints to yield positive optimal values for all the three decision variables in linear programming.

Keywords: CPM critical perturbation, CPM sensitivity, CPM by LP, critical/slack identification, 3-D LP examples.

GJSFR-F Classification: MSC 2010: 91G50

ANOTEONIDENTIFYINGCRITICALACTIVITIESINPROJECTSCHEDULINGVIALINEARPROGRAMMINGONSPREADSHEETSWITHINCIDENTALPEDAGOGICALREMARKS

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Keywords: CPM critical perturbation, CPM sensitivity, CPM by LP, critical/slack identification, 3-D LP examples.

I. Introduction

Critical Path Method (CPM) has wide-ranging applications from business operations [1], to medical procedures [2],[3], to engineering constructions [4], to electric circuitry, computer soft and hard wares [5]. There are multitudes of computer programs to conduct these analyses. As such, the topic has been considered a standard teaching material in many a college curriculum. While industries benefit from efficient computing packages, students of education need to have a fundamental understanding of this theme. Two prevalent pedagogical treatments have been that of drawing a network flow chart to consider "forward/backward passes" [6] and that of conducting linear programming [7], [8]. In either approach, the "slack time" for a non-critical activity is subject to ambiguities. Consider a linear predecessor-successor relation from activity E to F to G, with no "Y-shaped" lateral bifurcation; then if E is not critical, its released time can be passed on to F and/or G. Thus, we limit our scope here to the identification of acritical path of activities without going into any detailed analyses of slack times. We also will not pursue the possibility of the existence of more than one critical path.

From an extensive literature research, we did not find any similar techniques to ours to identify critical activities, with the closest being [9]. We propose a perturbation of the required times of all the activities and subsequent observation of how the

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Ref

6. Heizer, J., Render, B., and Munson, C., *Operations Management*, Boston, Pearson (2020).

minimized finishing time has changed. Clearly, one can engage in this tack one activity at a time, but we will demonstrate ways to make the perturbation all at once. We contend that our method here will not only help students quickly identify the critical activities of a project but also contribute to their appreciation of the distinction between "critical" and "non-critical" activities.

In the following, we will illustrate our procedure by an example, which can nevertheless be generalized. As this paper has a pedagogical intent, we will also add a note in the matter of constructing examples of linear programming in class.

II. Analysis

The objective of scheduling an ensemble of activities as contained in a project is to minimize its final completion time but subject to two sets of constraints: [1] the time point to start any activity j equals the time point for all its immediate predecessors to deliver "their torches" to j(by an analogy to a marathon), and [2] the delivery time point of the "torch" by any activity k(to all its immediate successor(s)) minus k's starting time point is greater than or equal to k's required time (interval) of completion. Accordingly, one can have the following spreadsheet presentation (as an example):

Activities/contacting times	0	?	?	?	?	?	?
A	-1	1					
B	-1		1				
C		-1	1				
D			-1	1			
E			-1		1		
F				-1		1	
G					-1	1	
H						-1	1

for a project with

Activities	Immediate Predecessor(s)
A	-
B	-
C	A
D	B, C
E	B, C
F	D
G	E
H	F, G

where "-1" refers to the starting time point of an activity, and "1," the contacting time to its immediate successor(s), thereof addressing constraint set [1]. For constraint set [2], along with the consequent optimal solution, we may have:

	0	73	110	148	152	188	297	=H23		required time
A	-1	1						=SUMPRODUCT($B$23:$H$23,B24:H24)	>=	73
B	-1		1					=SUMPRODUCT($B$23:$H$23,B25:H25)	>=	41
C		-1	1					=SUMPRODUCT($B$23:$H$23,B26:H26)	>=	37
D			-1	1				=SUMPRODUCT($B$23:$H$23,B27:H27)	>=	38
E			-1		1			=SUMPRODUCT($B$23:$H$23,B28:H28)	>=	37
F				-1		1		=SUMPRODUCT($B$23:$H$23,B29:H29)	>=	40
G					-1	1		=SUMPRODUCT($B$23:$H$23,B30:H30)	>=	36
H						-1	1	=SUMPRODUCT($B$23:$H$23,B31:H31)	>=	109

Ref

1. Ahmad, H. H., Analysis of business networks in project management by using critical path method (CPM), *J. Interdisc. Math.*, 23(4), 2020, 851-855.

Notes

or

	0	73	110	148	152	188	297	297		required time
A	-1	1						73	>=	73
B	-1		1					110	>=	41
C		-1	1					37	>=	37
D			-1	1				38	>=	38
E			-1		1			42	>=	37
F				-1		1		40	>=	40
G					-1	1		36	>=	36
H						-1	1	109	>=	109

That is, the least amount of time is 297 units of time (in cell "H3"). A special note that is worth mentioning here is that the six decision variables, with their optimal time points: 73, 110, 148, 152, 188, and 297, do not need to be in increasing order in general; consider an interchange between the two columns headed by 73 and 110; one would nevertheless obtain the identical solution:

	0	110	73	148	152	188	297	297		required time
A	-1		1					73	>=	73
B	-1	1						110	>=	41
C		1	-1					37	>=	37
D		-1		1				38	>=	38
E		-1			1			42	>=	37
F				-1		1		40	>=	40
G					-1	1		36	>=	36
H						-1	1	109	>=	109

A more demanding task now is to identify the critical path associated with the optimal objective function's value of 297. This can be accomplished by a perturbation of the required times as follows:

	0	110	73.1	148	152	188	297	297.10110101		required time
A	-1		1					73.1	>=	73.1
B	-1	1						110.101	>=	41.01
C		1	-1					37.001	>=	37.001
D		-1		1				38.0001	>=	38.0001
E		-1			1			42.0001009	>=	37.00001
F				-1		1		40.000001	>=	40.000001
G					-1	1		36.0000001	>=	36.0000001
H						-1	1	109.00000001	>=	109.00000001

so that the critical activities are identified to be: A, C, D, F, and H from the decimal extension of 297 by 0.1, 0.001, 0.0001, 0.000001, and 0.00000001. We observe from the original solution that the slack times have been known to be integers; hence adding fractional values to the required times does not alter the identification of the critical activities. In principle, this technique can be applied to much greater number of activities by multiplying the required times by a common multiple of a power of 10 and perturbing successively by lower and lower power of 10 across the required times - - provided that one ensures the sum of the time increments is less than any slack time as solved from the original optimization. Otherwise, one may consider adding 0.01ln(2), 0.01ln(3), ..., 0.01ln(19, the eighth prime number) to the required times of A, B, ..., H respectively, so that the perturbed objective optimal value minus the pre-perturbed value = 0.01ln(2*5*7*13*19) = "d" and exp(100*d) = 2*5*7*13*19 recovers the critical activities, A, C, D, F, and H by the unique factorization theorem. In this regard, one can readily obtain 200 prime numbers from the Internet; dividing such a number as the above exp(100*d) by each of the prime numbers as having been assigned to all the activities, one then identifies a critical activity when the quotient is an integer.

As a separate matter of teaching linear programming by examples of 3 decision variables with exactly three constraints (in addition to non-negativity), one often finds not all the decision variables ending in positive values (which may be considered undesirable from a pedagogical perspective), e.g.,

Max 2x + 3y + 4z

(x,y,z) >= 0

s.t. [1] 5x + 6y + 7z <= 400,

[2] 30x + 20y +10z <= 500, and

[3] x + y + z <= 600.

Then one has the optimal solution: x = y = 0 and z = 50, with the objective function's value = 200. The crux of the problem here is that the z-direction yields the greatest ascent to the objective value so that x and y are necessarily zero. Of course, one would quickly think of altering the signs of the coefficients; yet rather than by a haphazard trial and error, we propose a minimization over an unbounded region as constrained by three planes that intersect at a point of (x*, y*, z*) > 0, as the optimal solution, such as P* = (10, 10, 10). Any pair of the three planes intersect into a line, which is to intersect the (x, y)-plane, the (y, z)-plane, or the (z, x)-plane at a point, such as Q = (5, 5, 0), R = (5, 0, 5), or S = (0, 5, 5); then P*Q, P*R, and P*S yield three lines in R^3 and any two of these three lines form a plane. Elementary algebra then leads to the following three equations:

[1] 3x − y − z = 10,

[2] − x + 3y − z = 10, and

[3] − x − y + 3z = 10.

Then the solution to, say,

Min x + 2y + 3z

subject to

[1] 3x − y − z >= 10,

[2] − x + 3y − z >= 10, and

[3] − x − y + 3z >= 10

is x* = y* = z* = 10 with the objective function of value 60, as expected. By the duality theorem of linear programming, we have the following dual:

Max 10u + 10v + 10w

s.t.

[1] 3u − v − w <= 1,

[2] − u + 3v − w <= 2, and

Notes

[3] $-u - v + 3w <= 3$,

which has the solution: $u^* = 1.75$, $v^* = 2$, and $w^* = 2.25$, necessarily yielding the identical objective function's value of 60.

In this way, an instructor can construct examples of linear programming with the optimal solutions for the decision variables all positive, naturally with the leeway of perturbation of the data without affecting the said qualitative outcome.

Notes

III. Summary

In this note, we have presented (1) a speedy way of identifying critical activities in CPM and (2) a procedure to construct examples of linear programming that may be more illuminating to students. Although the undertone in our exposition here leans toward teaching, we contend that even for professional research/analysis, our aforementioned "perturbation via prime numbers" can bring about a quick resolution of the critical path by simple spreadsheet operations.

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Perfect Folding of Graphs

By E. M. El-Kholy & H. Ahmed
Tanta University

Abstract- In this paper we introduced the definition of perfect folding of graphs and we proved that cycle graphs of even number of edges can be perfectly folded while that of odd number of edges can be perfectly folded to $C3$. Also we proved that wheel graphs of odd number of vertices can be perfectly folded to $C3$. Finally we proved that if G is a graph of n vertices such that $2 >$ clique number $=$ chromatic number $= k > n$, then the graph can be perfectly folded to a clique of order k.

Keywords: *clique number, chromatic number, perfect graphs, graph folding.*

GJSFR-F Classification: *MSC 2010: 05C17*

PERFECTFOLDINGOFGRAPHS

Notes

Perfect Folding of Graphs

E. M. El-Kholy [α] & H. Ahmed [σ]

Abstract- In this paper we introduced the definition of perfect folding of graphs and we proved that cycle graphs of even number of edges can be perfectly folded while that of odd number of edges can be perfectly folded to $C3$. Also we proved that wheel graphs of odd number of vertices can be perfectly folded to $C3$. Finally we proved that if G is a graph of n vertices such that 2 > clique number = chromatic number = $k > n$, then the graph can be perfectly folded to a clique of order k.

Keywords: *clique number, chromatic number, perfect graphs, graph folding.*

I. Introduction

Let $G = (V, E)$ be a graph , where V is the set of its vertices and E is the set of its edges. Two distinct vertices $u, v \in V$ are called independent if $\{u, v\}$ is not an edge in G. Two vertices u, v are called neighbors (adjacent) if $\{u,v\}$ is an edge in G. The degree (valency) of a vertex is the number of edges with the vertex as an end point. A graph with no loops or multiple edges is called a simple graph. A graph is said to be connected if every pair of vertices has a path connecting them otherwise the graph is disconnected. A graph $H =(V',E')$ is called induced subgraph of $G=(V,E)$ if $V' \subseteq V$ and $\{ u,v \}$ is an edge in H wherever u and v are distinct vertices in V ' and $\{ u,v \}$ is an edge in G, H is called proper if $H \neq G$. A cycle graph is a graph that consists of a single cycle, or in other words, some number of distinct vertices connected in a closed chain. The cycle graph with n vertices is denoted by C_n. The number of vertices in C_n equals the number of edges , and every vertex has degree 2. The wheel graph W_n or n-wheel is a graph that contains a cycle of order n-1, and for which every graph vertex in the cycle is connected to one other graph vertex which is called the hub. A bipartite graph is a graph whose vertex set can be split into two sets A and B in such a way that each edge of the graph joins a vertex in A to a vertex in B. A vertex coloring of a

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graph $G=(V,E)$ is a way of coloring the vertices of the graph such that no two adjacent vertices share the same color. A clique of a graph G is a maximal complete subgraph. In this case each pair of vertices of the clique are adjacent. The clique number $W(G)$ of a graph is the number of graph vertices in the largest clique of G , [8]. The clique number of a cycle graph C_n , n odd is 3 and 2 otherwise. For a wheel graph W_n , n is even the clique number is 4 and is 3 otherwise. The chromatic number of a graph G is the smallest number of colors needed to color the vertices of a graph G so that no two adjacent vertices share the same color, and is often denoted by χ(G). A graph G is called perfect if for every induced subgraph H of G, $\chi(H) = W(H)$. Note that if G is a perfect graph, then every induced subgraph of G is also perfect,[2].

II. Perfect Folding

Definition (2-1)

Let G_1 and G_2 be two simple graphs and $f: G_1 \to G_2$ be continuous map. Then f is called a graph map, if

(i) For each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$.

(ii) For each edge $e \in E(G_1)$, $dim(f(e)) \leq dim(e)$, [3] .

Definition (2-2)

A graph map $f: G_1 \to G_2$ is called a graph folding if and only if f maps vertices to vertices and edges to edges ,i.e., if

(i) For each vertex $v \in V(G_1)$, $f(v)$ is a vertex in $V(G_2)$.

(ii) For each edge $e \in E(G_1)$, $f(e)$ is an edge in $E(G_2)$,[4] .

Note that if the vertices of an edge $e=(u,v) \in E(G_1)$ are mapped to the same vertex , then the edge e will collapse to this vertex and hence we cannot get a graph folding. In other words, any graph folding cannot maps edges to loops but it may maps loops, if there is any , to loops.

Definition(2-3)

Let G and H be simple connected graphs. We call a graph folding $f: G \to H$ perfect folding if its image $f(G)$ is a perfect subgraph of H.

In general the image of a graph folding $f: G \to H$ is not a perfect graph e.g., if G_1 is the imperfect graph shown in Fig.(1-a), where $V(G_1)=\{v_1,v_2,v_3, v_4, v_5, v_6, v_7\}$ and $E(G_1)=\{ e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Then the graph folding $f: G_1 \to G_1$ defined by $f\{v_6 ,v_7\}=\{v_5 ,v_4\}$ and $f\{ e_6 ,e_7\}=\{ e_2,e_4\}$ is not a perfect folding. While if we consider the imperfect graph G_2 shown in Fig.(1-b), where $V(G_2)=\{ u_1,\dots,u_7 \}$ and $E(G_2)=\{ e_1,\dots, e_7 \}$.

Ref

8. Lowell, W. and Robin, J. "Selected topics in graph theory2" Academic Press Inc. (London) LTD, (1983).

Then the graph folding $g\colon\ G_2 \to G_2$ defined by $g\{u_1, u_4\} = \{u_6, u_6\}$ *and* $g\{e_4, e_7\} = \{e_5, e_6\}$ is a perfect folding. The omitted vertices and edges in this example and through the paper will be mapped to themselves.

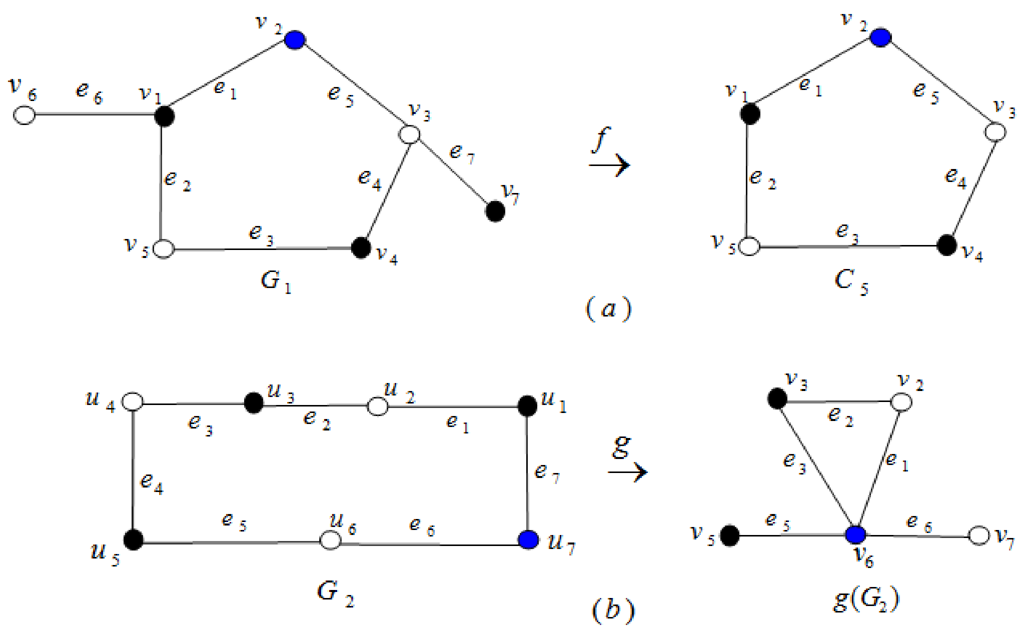

Figure 1

Theorem (2-4)

Let G be a simple connected graph such that the number of $E(G) \geq 2$. If the chromatic number $\chi(G)$ is equal to two, then G can be perfectly folded.

Proof

From [5], any simple connected graph G such that $E(G) \geq 2$ and $\chi(G)=2$ can be folded to an edge. In this case $\chi(f(G)) = W(f(G)) = 2$, and thus the graph G can be perfectly folded to an edge.

Example (2-5)

The cubic graph G with $\chi(G) = W(G) = 2$, shown in Fig. (2) can be folded to an edge by the graph folding $f(v_1, \ldots, v_8) = (v_1, v_2, v_1, v_2, v_1, v_2, v_1, v_2)$. This folding can be done by the composition of a sequence of foldings f_1, f_2, f_3 *and* f_4, see Fig.(2). And hence the graph folding is a perfect.

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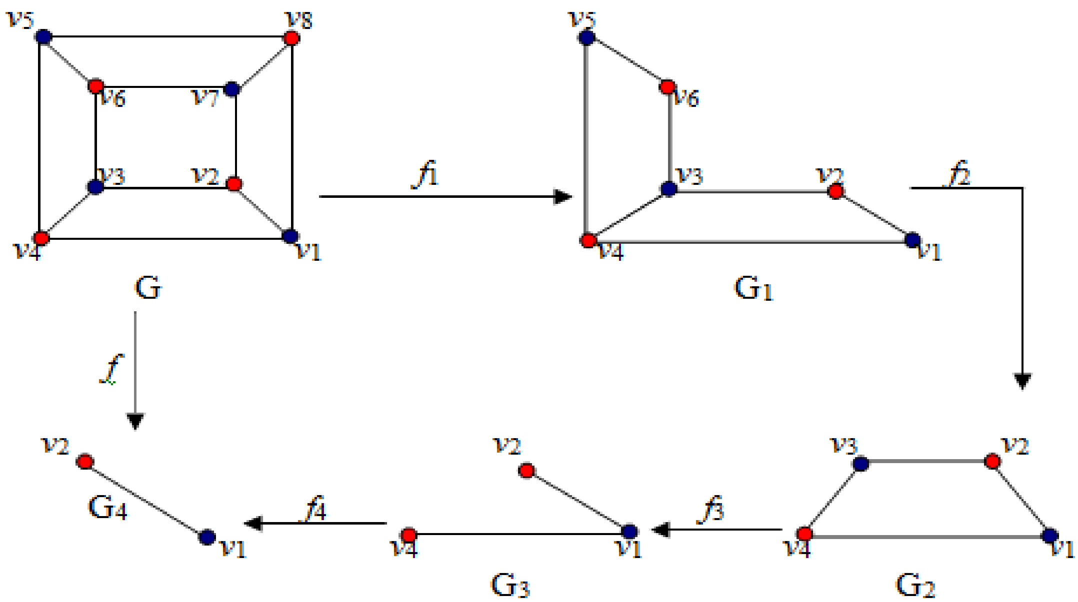


Figure 2

Lemma (2-6)

Any folding of a bipartite graph (complete) is a perfect folding.

Proof

This follows from the fact that the chromatic number of a bipartite graph is equal to two, and thus it can be perfectly folded.

Example (2-7)

Consider the bipartite graph G shown in Fig.(3). A graph folding $f: G \rightarrow G$ defined by $f\{v_1, v_3\}=\{v_2\}$ and $f\{e_1, e_4\}=\{e_2, e_3\}$ is a perfect folding.

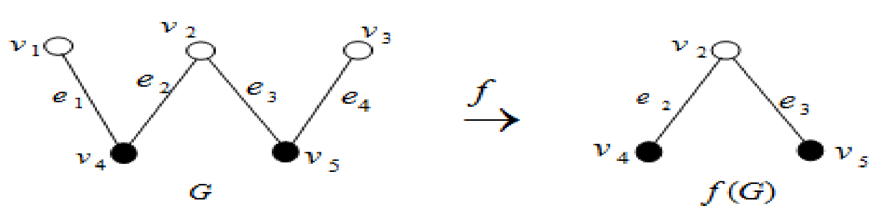


Figure 3

III. Perfect Folding of Cycle Graphs

The chromatic number of a cycle graph C_n, $n > 2$ where n is odd is 3 while that for n even is 2 , [1].

Notes

Theorem (3-1)

Any folding of a cycle graph C_n of an even number of edges is a perfect folding.

Proof

This follows from the fact that $\chi(C_n)$, n is an even number is equal to two. Thus C_n can be perfectly folded.

Example (3-2)

Consider the cycle graph C_4 where $\chi(C_4) = W(C_4) = 2$, the graph folding $f: C_4 \rightarrow C_4$ defined by $f\{v_1, v_4\} = \{v_3, v_2\}$ and $f\{e_i\} = \{e_3\}$, $i=1,2,4$ is a perfect folding , see Fig.(4).

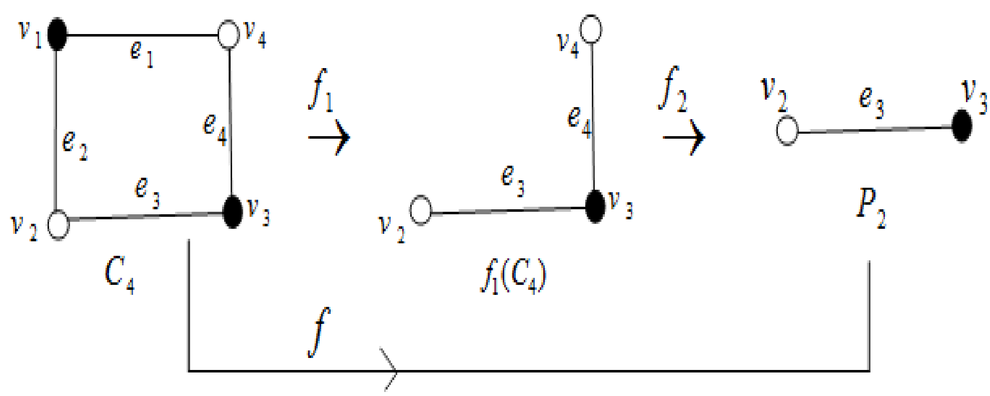


Figure 4

It should be noted that the cycle graph C_3 cannot be folded , [4].

Theorem (3-3)

Let $G = C_n$, $n > 3$ be a cycle graph of an odd number of edges (vertices). Then *G* can be perfectly folded to C_3.

Proof

Since $G = C_n$ has an odd number of edges (vertices). Thus the graph C_n has three color classes, say V_1, V_2 *and* V_3. We can color the vertices of C_n alternatively with the two colors of V_1 and V_2 except the last two edges one will join a vertex colored by the color of V_2 and a vertex colored by the color of V_3 and the other edge will join a vertex colored by the color of V_3 and a vertex colored by the color of V_1. Thus the number of vertices of color class V_1 = the number of vertices color class

$V_2=(n-1)/2$, but V_3 has only one vertex w. We can define a graph folding $f: C_n \rightarrow C_n$, n is odd , by mapping vertices of V_1 to a vertex of V_1 , say u, and mapping the vertices of V_2 to a vertex of V_2 , say v, finally mapping w into itself. Thus we have three vertices u, v, w and hence three edges in the image i.e., we have C_3. But $\chi(C_3) = W(C_3) = 3$, i.e., the graph folding f is perfect .

Example (3-4)

Let $G=C_5$ *and* $h: G \rightarrow G$ be the graph folding defined by $h\{v_5,v_4\}=\{v_3, v_1\}$ and $h\{e_i\} =\{e_2\}$, $i=3,4$ is a perfect folding, see Fig.(5). This can be done by the composition of the two graph foldings $h_1 : C_5 \rightarrow C_5$ defined by $h_1\{v_5\}=\{v_3\}$, $h_1\{e_3\}=\{e_4\}$ *and* $h_2: h_1(C_5) \rightarrow h_1(C_5)$ defined by $h_2\{v_4\}=\{v_1\}$, $h_2\{e_4\} =\{e_2\}$.

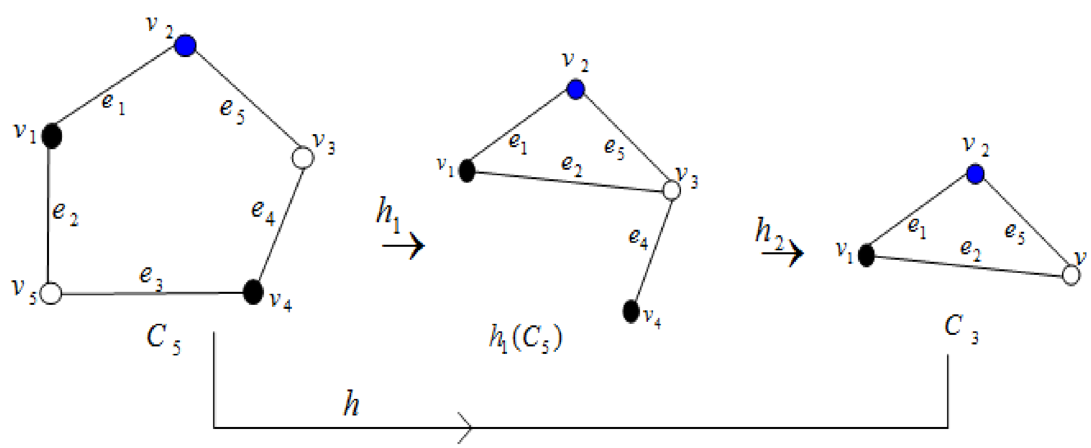


Figure 5

IV. Perfect Folding of Wheel Graphs

The chromatic number of a wheel graph W_n if n is odd is 3 and 4 if n is even, [1].

Theorem (4-1)

Any wheel graph W_n of an odd number of vertices can be perfectly folded to C_3.

Proof

A wheel graph W_n of order n , n is an odd number, is a graph that contains a cycle of even order $n-1$, and each vertex in the cycle is

Ref

1. Bollobas, B. and West, D. B. "A note on generalized chromatic number and generalized girth" Disc. Math. 213, 29-34, (2000).

connected to the hub. In this case the chromatic number $\chi(W_n) = 3$, thus the graph W_n can be colored by using three colors A, B and C. One color for the hub, say A, and the vertices of the even cycle C_{n-1} can be colored alternatively with two colors B and C , i.e., if the set of vertices of the cycle C_{n-1} is $V(C_{n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$, then the colors B and C have the following vertices, $B=\{v_1, v_3, \ldots, v_{n-2}\}$ and $C=\{v_2, v_4, \ldots, v_{n-1}\}$. Now we can define a graph folding by mapping the vertices of B to a vertex of B, the vertices of C to a vertex of C and the hub onto itself. The image of this map will contains three vertices, three edges and thus we have C_3, i.e., the graph folding is perfect.

Example (4-2)

Consider the wheel graph W_7 and the graph folding $f: W_7 \to W_7$ defined by $f\{v_i\} = \{v_1\}$, $i=3,5$, $f\{v_j\}=\{v_2\}$, j=4,6 and $f\{e_k\}=\{e_1, e_1, e_1, e_1, e_1, e_1, e_7, e_8, e_7, e_8, e_7, e_8\}$, $k=1,\ldots,12$. This graph folding is perfect, see Fig.(6).

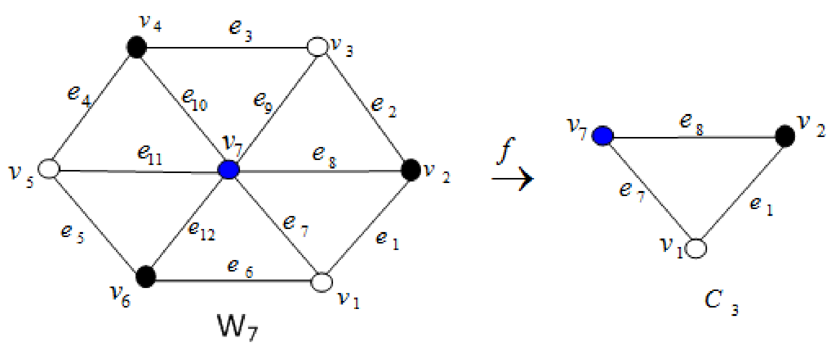


Figure 6

It should be noted that the wheel graph of an even number of vertices cannot be folded, [4], and hence cannot be perfectly folded.

V. The Clique Number and Perfect Folding

The chromatic number of any graph is equal to or greater than its clique number, i.e., $\chi(G) \geq W(G)$. For connected graphs $2 \leq W(G) \leq \chi(G) \leq n$, where n is the number of vertices of the graph G , [7].

Theorem (5-1)

Let G be a simple connected graph, if the clique number $W(G)$ equal to the chromatic number $\chi(G)$ equal to 2 and $E(G) \geq 2$, then the graph G can be perfectly folded .

Ref

4. El-Kholy, E.M. and Al-Esawy, A. "Graph folding of some special graphs" J. Math. & Statistics 1(1), 66-70, (2005).

Proof

It is immediately follows from Theorem (2-4) and since $\chi(G)=2$, then G can be perfectly folded.

Example (5-2)

Consider the cycle graph C_6 shown in Fig.(7). A graph folding $f: C_6 \to C_6$ defined by $f\{ v_2, v_3, v_4, v_5\} = \{v_6, v_1, v_6, v_1\}$ and $f\{e_i\}=\{e_6\}$, $i=1,\ldots,5$ is a perfect folding.

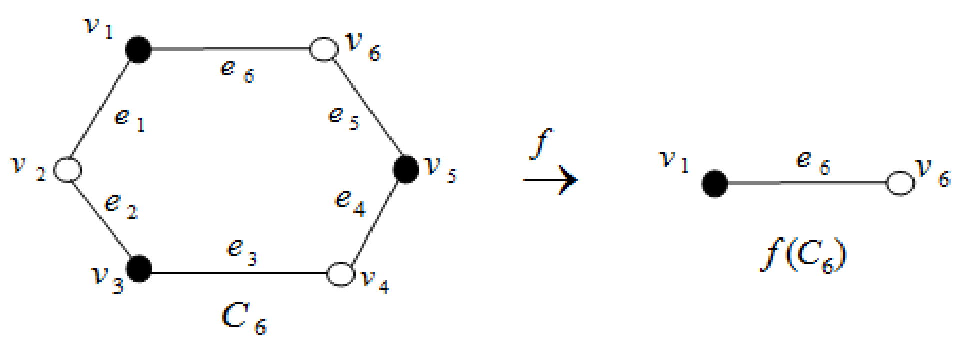


Figure 7

Theorem (5-3)

Let G be a simple connected graph such that $no.\ V(G) = n$. If $2 < W(G) = \chi(G) = k < n$, then the graph can be perfectly folded to a clique of order k.

Proof

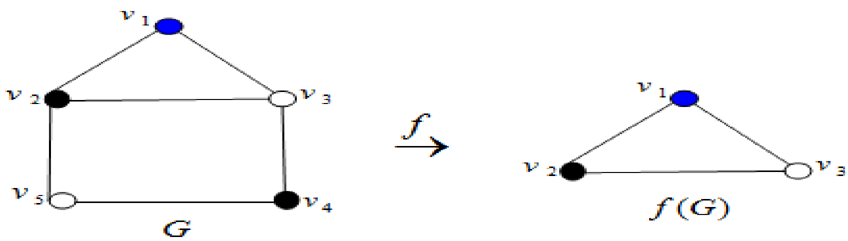
Let $W(G) = \chi(G) = k$, then we have a maximal complete subgraph of k vertices. This complete graph cannot be folded, [3].These vertices must be colored by different colors $A_1, A_2, \ldots, A_k$. Now the other $(n-k)$ vertices of G, will be colored by the colors $A_1, \ldots, A_m$, $m \le k$ in such a way that any edge will joins two vertices of different colors. So we can define a sequence of graph folding $f_i: G \to G_i$, where $G_i = f_i(G_{i-1})$, $i = 1,\ldots,m$, $G_0 = G$, by mapping the $(n-k)$ vertices to other vertices but of the same color, until we get the k-clique which cannot be folded any more. And hence $W(f_i(G_i)) = \chi(f_i(G_i)) = k$, i.e., the graph folding is a perfect.

Example (5-4)

Consider the house graph G *with* 5 *vertices and* 6 *edges* shown in Fig.(8), where $2 < W(G) = \chi(G) = 3 < n = 5$. This graph can be folded to a triangle by the graph folding $f: G \to G$ defined by $f\{ v_4, v_5\} = \{v_2, v_3\}$ which is a perfect folding.

Ref

3. Erdos, P. "Graph theory and probability" Canad. J. Math. 11, 34-38, (1959).



Notes

Figure 8

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Generic Rank-2 Perturbation of Hamiltonian Systems with Periodic Coefficients

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Abstract- In this paper, it is about a theory of double rank-one perturbation of a Hamiltonian system with periodic coefficients. Some reminders of the rank-one perturbation and an adaptation of a theorem given in [C. Mehl, et al., Linear Algebra Appl. J., 435(2011), 687-716] to the cases of symplectic matrices have been made.

Keywords: *hamiltonian system, rank-one perturbation, symplectic matrix.*

GJSFR-F Classification: *MSC 2010: MSC 2010: 15A21, 47A55, 93B10, 93C73*

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Strictly as per the compliance and regulations of:

Generic Rank-2 Perturbation of Hamiltonian Systems with Periodic Coefficients

Mouhamadou DOSSO

Abstract- In this paper, it is about a theory of double rank-one perturbation of a Hamiltonian system with periodic coefficients. Some reminders of the rank-one perturbation and an adaptation of a theorem given in [C. Mehl, et al., Linear Algebra Appl. J., 435(2011), 687-716] to the cases of symplectic matrices have been made.

Keywords: *hamiltonian system, rank-one perturbation, symplectic matrix.*

I. Introduction

Let $J \in \mathbb{R}^{2n\times 2n}$ (n is a fixed non-zero positive integer) be skew-symmetric and nonsingular (i.e. $J^T = -J$) and τ be a positive real. Consider the following Hamiltonian system with τ-periodic coefficients

$$\left\{\begin{array}{rcl} J\dfrac{dX(t)}{dt} & = & H(t)X(t) \\ X(0) & = & I \end{array}\right. \tag{1.1}$$

where $t \mapsto H(t) \in \mathbb{R}^{2n\times 2n}$ is a piecewise continuous matrix function on $[0, \tau]$ such that

$$H(t+\tau) = H(t) = (H(t))^T, \ \ \forall t \in \mathbb{R}.$$

Throughout this paper, the identity and zero matrices of order k are denoted by I_k and 0_k or just I and 0 whenever the order is clear from the context.

The solution of the system (1.1) is called the fundamental solution of the Hamiltonian system with τ-periodic coefficients $J\dfrac{dX(t)}{dt} = H(t)X(t), \ \forall t \in \mathbb{R}$. It verifies $\forall t \in \mathbb{R}, \ X(t)^T JX(t) = J$ and satisfies the following relationship

$$X(t+p\tau) = X(t)X^p(\tau) \neq X^p(\tau)X(t), \ \ \forall (t,p) \in \mathbb{R}\times\mathbb{N}.$$

We say that the solution of (1.1) has a symplectic structure. Recall that a matrix $W \in \mathbb{R}^{2n\times 2n}$ has a J-symplectic structure or W is J-symplectic (or J-orthogonal) if it verifies $W^T JW = J$.

The symplectic matrices come very often from Hamiltonian differential systems with periodic coefficients (see [14, Chapter 3]). Besides many problems in physics and engineering lead to systems of linear differential equations with periodic coefficients consequently to Hamiltonian systems with periodic coefficients. This gives an important place to the study of these systems ; particularly to the study of the stability of Hamiltonian systems which is closely related to the analysis of their perturbations. Regarding stability (strong stability) of system (1.1), we have the following definition

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Definition 1.1 1. *System (1.1) is stable if its solution $X(t)$ remains bounded for all $t \in \mathbb{R}$.*

2. *System (1.1) is strongly stable if any Hamiltonian system with τ-periodic coefficients sufficiently close to (1.1) is stable.*

Specifically, system (1.1) is strongly stable if there exists $\varepsilon > 0$ such that any Hamiltonian system with τ-periodic coefficients of the form $J\dfrac{\widetilde{X}(t)}{dt} = \widetilde{H}(t)\widetilde{X}(t)$ and satisfying $\|H - \widetilde{H}\| \equiv \displaystyle\int_0^\tau \|H(t) - \widetilde{H}(t)\| dt < \varepsilon$, is stable. Therefore, we focus our study in this paper to study of a type of perturbation of Hamiltonian system with periodic coefficients called rank-one pertubation studied by Mehl, et al. in [11, 12] but within the framework of a structured matrix such as a symplectic matrix. In some of our work, we have defined from the work of Mehl, et al. the rank-one perturbation of a Hamiltonian system with τ-periodic coefficients [1, 2, 5].

In this paper, we consider the case of generic structure-preserving rank-2 perturbation of system (1.1). Let us recall the meta-conjecture resulting from a numerical experiment with random perturbations [4].

Meta -Conjecture 1 *Let $W \in \mathbb{R}^{p\times p}$ be a structured matrix with respect to some indefinite inner product and $E \in \mathbb{R}^{p\times p}$ be a matrix of rank k so that $W + E$ is of the same structure class as W. Then generically the Jordan structure and sign characteristic of $W + E$ are the same that one would obtain by performing a sequence of k generic structure-preserving rank-one perturbations on W.*

Let us give some reminders on generic sets [3, 11]

Definition 1.2 1. *A set $\Omega \subseteq \mathbb{R}^{2n}$ is said to be algebraic if there exists a finite set of polynomials $p_1(x_1, ..., s_{2n}), ..., p_k(x_1, ..., x_{2n})$ with real coefficients such that $(\alpha_1, \alpha_2, ..., \alpha_{2n})^T \in \Omega$ if and only if $p_j(\alpha_1, ..., \alpha_{2n}) = 0, \ \forall j = 1, ..., k$.*

2. *An algebraic set $\Omega \subset \mathbb{R}^{2n}$ is said non-trivial if $\Omega \neq \mathbb{R}^{2n}$.*

3. *A non-trivial set $\Omega \in \mathbb{R}^{2n}$ is said to be generic if Ω is not empty and $\mathbb{R}^{2n} \setminus \Omega$ is contained in a finite union of non-trivial algebraic sets.*

Recall a result of [12] to the case of unstructured generic rank-one perturbation theory

Theorem 1.1 *Let $W \in \mathbb{C}^{\ell\times\ell}$ be a matrix having the pairwise distinct eigenvalues $\lambda_1, ..., \lambda_p$ with geometric multiplicities $r_1, ..., r_p$ and having the Jordan canonical form*

$$\bigoplus_{k=1}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_p} \mathcal{J}_{\ell_{p,k}}(\lambda_p),$$

where $l_{j,1} \geq \cdots \geq l_{j,r_j}$, $j = 1, ..., p$. Consider the rank one matrix $E = uv^T$, with $u, v \in \mathbb{C}^\ell$. The generically (with respect to the entries of u and v) the Jordan blocks of $W + E$ with eigenvalue λ_j are just the $r_j - 1$ smallest Jordan blocks of W with eigenvalue λ_j, and all other eigenvalues of $W + E$ are simple ; if $r_j = 1$, then generically λ_j is not an eigenvalue $W + E$.
More precisely, there is a generic set $\Omega \subseteq \mathbb{C}^\ell \times \mathbb{C}^\ell$ such that for every $(u, v) \in \Omega$, the Jordan structure of $W + uv^T$ is described in (a) and (b) bellow :

(a) the Jordan structure of W+ uv^T for the eigenvalues $\lambda_1, ..., \lambda_p$ is given by

$$\bigoplus_{k=2}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=2}^{r_p} \mathcal{J}_{\ell_{p,k}}(\lambda_p) ;$$

(b) the eigenvalues of $W + uv^T$ that are different from any of $\lambda_1, ..., \lambda_p$, are all simple.

In the rest of the paper, we will recall, in section 2, the rank-one perturbation of symplectic matrices and the Hamiltonian system with periodic coefficients. In this part, an adaptation of Theorem 1.1 to the case of symplectic matrices will be given. As for section 3, it defines the rank-2 perturbation as a double rank-one perturbation of a Hamiltinian system with periodic coefficients.

Ref

4. L. Batzke, C. Mehl, A. C.M. Ran and L. Rodman, Generic rank-k Perturbations of Structured Matrices. Operator Theory, Function Spaces, and Applications Birkhuser, Cham. (2016), p. 27-48.

II. Generality on a Rank-One Perturbation Theory

a) Generic rank-one perturbation of a symplectic matrix

Consider a symplectic matrix $W \in \mathbb{R}^{2n\times 2n}$ and an anti-symmetric matrix $J \in \mathbb{R}^{2n\times 2n}$. Recall that the spectrum of any symplectic matrix of order $2n$ is divided into three groups of eigenvalues : n_0 eigenvalues inside the unit circle, $n_\infty = n_0$ eigenvalues outside the unit circle and symmetrically placed with respect to the first group, and $n_1 = 2(n - n_0)$ eigenvalues on the unit circle [8, 9, 10]. These types of matrices which belong to a group of so-called structured matrices, have simple and useful spectral properties that we recall in the following theorem ([8])

Theorem 2.1 Let $W \in \mathbb{R}^{2n\times 2n}$ be a symplectique matrice. Then any eigenvalue of W verifies : .for all eigenvalue λ of W,

1. *if $\lambda \in \mathbb{C}^*$, with $|\lambda| \neq 1$, then $\bar{\lambda}$, $1/\lambda$ and $1/\bar{\lambda}$ are eigenvalues of W;*
2. *if $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ then $\bar{\lambda}$ is an eigenvalue of W;*
3. *if $\lambda \in \mathbb{R}^*$, $1/\lambda$ is an eigenvalue of W.*

Regarding a rank-one perturbation treated by Mel, et al. in [11], we have the following lemma :

Lemma 2.1 If W and $\widetilde{W} \in \mathbb{R}^{2n\times 2n}$ are J-symplectiques such that

$$rg\left(\widetilde{W} - W\right) = 1,$$

then there exists a vector $u \in \mathbb{R}^{2n}$ verifying

$$\widetilde{W} = (I + cuu^T J)W, \tag{2.1}$$

where $c = \pm 1$. Moreover for all $u \in \mathbb{R}^{2n}$, the matrix $\widetilde{W}$ is J-symplectic.

Proof

The hypothesis

$$rg\left(\widetilde{W} - W\right) = 1,$$

implies that there exists two non-zero vectors $\widehat{u}$ and $v \in \mathbb{R}^{2n}$ such that

$$\widetilde{W} = W + \widehat{u}v^T.$$

Thus, $\widetilde{W}$ J-symplectic implies $\widetilde{W}^T J\widetilde{W} = J$ and we have

$$(W + \widehat{u}v^T)^T J(W + \widehat{u}v^T) = J$$

$$\underbrace{W^T JW}_{=J} + v\widehat{u}^T JW + W^T J\widehat{u}v^T + v\underbrace{\widehat{u}^T J\widehat{u}}_{=0}v^T = J$$

this implies

$$v\widehat{u}^T JW + W^T J\widehat{u}v^T = 0, \tag{2.2}$$

which gives, by multiplying on the right by v,

$$v\widehat{u}^T JWv + W^T J\widehat{u}v^T v = 0.$$

Ref

8. M. Dosso, Sur quelques algorithms d'analyse de stabilit'e forte de matrices symplectiques, PHD Thesis (September 2006), Université de Bretagne Occidentale. Ecole Doctorale SMIS, Laboratoire de Mathématiques, UFR Sciences et Techniques.

We deduce

$$W^T J\widehat{u} = -v\frac{\widehat{u}^T JWv}{v^T v}, \quad (\text{since} \;\; v \neq 0)$$

$$= -v\frac{\widehat{u}^T w}{v^T v}, \quad \text{where} \;\; w = JWv.$$

By setting

$$n = \frac{\widehat{u}^T w}{v^T v},$$

We get $W^T J\widehat{u} = -nv$ which shows that v and $W^T J\widehat{u}$ are collinear. Thus, there is a non-zero real constant α such as $v = -\alpha W^T J\widehat{u}$. Therefore

$$\widetilde{W} = \left(I + \alpha\widehat{u}\widehat{u}^T J\right) W, \quad \text{where} \;\; \alpha = \pm 1.$$

Since the matrix $\alpha\widehat{u}\widehat{u}^T J$ is J-Hamiltonian $\left(\left(J\alpha\widehat{u}\widehat{u}^T J\right)^T = J\alpha\widehat{u}\widehat{u}^T\right)$, according to point 3) of Lemma 2.3 of [4], there is a vector $u \in \mathbb{R}^{2n}$ and a constant $c = \pm 1$ such that $\alpha\widehat{u}\widehat{u}^T J = cuu^T J$. Therefore

$$\widehat{W} = \left(I + c\, uu^T J\right) W, \quad \text{where} \;\; c = \pm 1.$$

Moreover, for any $u \in \mathbb{R}^{2n}$, we easily show that $\widetilde{W} J \widetilde{W} = J$.

We have the following general definitions

Definition 2.1 *Let* $W \in \mathbb{R}^{2n\times 2n}$ *be a symplectic matrix. We call rank-one perturbation of W, any symplectic matrix* $\widetilde{W}$ *of the form*

$$\widetilde{W} = (I + c\, uu^T)W, \tag{2.3}$$

where $c = \pm 1$ *and* $u \in \mathbb{R}^{2n}$.

In Theorem (1.1), if we consider a J-symplectic matrix $W \in \mathbb{R}^{2n\times 2n}$ and take $v = W^T J^T u$, we get the following theorem

Theorem 2.2 *Let* $W \in \mathbb{R}^{2n\times 2n}$ *be a matrix having the pairwise distinct eigenvalues* $\lambda_1, ..., \lambda_{2p}$ *with geometric multiplicities* $r_1, ..., r_{2p}$ *and having the Jordan canonical form*

$$\bigoplus_{k=1}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_{2p}} \mathcal{J}_{\ell_{2p,k}}(\lambda_{2p}), \tag{2.4}$$

where $l_{j,1} \geq \cdots \geq l_{j,r_j}$, $j = 1, ..., 2p$. *Consider the rank one matrix* $E = uu^T JW$, *with* $u \in \mathbb{R}^{2n}$. *Then generically (with respect to the entries of u) the Jordan blocks of* $W + E$ *with eigenvalue* λ_j *are just the* $r_j - 1$ *smallest Jordan blocks of* W *with eigenvalue* λ_j, *and all other eigenvalues of* $W + E$ *are simple ; if* $r_j = 1$, *then generically* λ_j *is not an eigenvalue* $W + E$.
More precisely, there is a generic set $\Omega \subseteq \mathbb{C}^{2n}$ *such that for every* $u \in \Omega$, *the Jordan structure of* $\left(I + uu^T J\right) W$ *is described in (a) and (b) bellow :*

(a) the Jordan structure of $\left(I + uu^T J\right) W$ *for the eigenvalues* $\lambda_1, ..., \lambda_{2p}$ *is given by*

$$\bigoplus_{k=2}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_{2p}} \mathcal{J}_{\ell_{2p,k}}(\lambda_{2p})$$

(b) the eigenvalues of $\left(I + uu^T J\right) W$ *that are different from any of* $\lambda_1, ..., \lambda_{2p}$, *are all simple.*

Ref

4. L. Batzke, C. Mehl, A. C.M. Ran and L. Rodman, Generic rank-k Perturbations of Structured Matrices. Operator Theory, Function Spaces, and Applications Birkhuser, Cham. (2016), p. 27-48.

Proof
Note that, λ being an eigenvalue of W, $1/\lambda$, $\bar{\lambda}$ et $1/\bar{\lambda}$ are also eigenvalues of W. So the number of eigenvalues W is even. Thus, W will have the Jordan canonical form (2.4).

According to (a) of Theorem (1.1), the structure of W by the eigenvalues λ_1, λ_2,...,λ_{2p} is given by

$$\bigoplus_{k=2}^{r_1} \mathcal{J}_{\ell_{1,k}}(\lambda_1) \oplus \cdots \oplus \bigoplus_{k=1}^{r_{2p}} \mathcal{J}_{\ell_{2p,k}}(\lambda_{2p})\ ;$$

Notes

and from point (b) of the same Theorem, the eigenvalues of $(I+uu^TJ)W$ which are different from $\lambda_1, ..., \lambda_{2p}$ are all simple.

b) Generic rank-one perturbation of the Hamiltonian system with periodic coefficients

Let u be a vector of a generic set $\Omega \subset \mathbb{R}^{2n}$. Consider the Hamiltonian systems with τ-periodic coefficients

$$J\frac{d\widetilde{X}(t)}{dt} = [H(t)+E(t)]\,\widetilde{X}(t), \tag{2.5}$$

where $t \mapsto H(t)$ and $t \mapsto E(t)$ are piecewise continuous matrix functions on $[0,\tau]$ such that for all $t \in \mathbb{R}$ and $\tau > 0$,

$$H(t+\tau) = H(t) = H^T(t) \in \mathbb{R}^{2n\times 2n} \ \text{ and } \ E(t+\tau) = E(t) = E^T(t) \in \mathbb{R}^{2n\times 2n}.$$

Definition 2.2 We call a rank-one perturbation of the fundamental solution $(X(t))_{t\in\mathbb{R}}$ of (1.1) any matrix function of the form

$$\widetilde{X}(t) = (I + cuu^TJ)X(t), \forall t \in \mathbb{R} \tag{2.6}$$

where $c = \pm 1$.

The rank-one perturbations of fundamental solution of (1.1) are J-symplectic [2, 5]. Therefore, we collect some properties of Hamiltonian systems with periodic coefficients of the type (2.5) in Proposition 2.1

Proposition 2.1 1. *Let $t \in \mathbb{R}$ and $(X(t))_{t\in\mathbb{R}}$ be the fundamental solution of (1.1). If a solution $\left(\widetilde{X}(t)\right)_{t\in\mathbb{R}}$ of (2.5) is of the form*

$$\widetilde{X}(t) = (I + c(t)u(t)u^T(t)J)X(t), \tag{2.7}$$

where $t \mapsto u(t) \in \mathbb{R}^{2n}$ is a vector function and $t \mapsto c(t)$ is a function with value in $\{-1,+1\}$. Then there exists a constant vector u such that $u(t) = u$, $c(t) = c = \pm 1$ is a real constant and $E(t)$ is of the form

$$E(t) = (cJuu^TH(t))^T + cJuu^TH(t) + c^2(uu^T)^TH(t)(uu^TJ). \tag{2.8}$$

2. *Let u be a non-zero vector of $\mathbb{R}^{2n}$. Consider the perturbed Hamiltonian equation of (1.1)*

$$J\frac{d\widetilde{X}(t)}{dt} = [H(t)+E(t)]\,\widetilde{X}(t) \tag{2.9}$$

where $t \mapsto H(t)$ is piecewise continuous and

$$E(t) = (cJuu^TH(t))^T + cJuu^TH(t) + c^2(uu^T)^TH(t)(uu^TJ).$$

Then $\widetilde{X}(t) = (I + cuu^T)X(t)$ is a solution of (2.9)

3. System (2.9) can be put in the form

$$\left\{\begin{array}{rcl} J\dfrac{\widetilde{X}(t)}{dt} & = & (I-cuu^T)^T H(t)(I-cuu^T)\widetilde{X}(t), \forall t\in\mathbb{R} \\ \widetilde{X}(0) & = & I+cuu^T J. \end{array}\right. \tag{2.10}$$

Proof

1. Suppose $u(t)$ is not constant. Then $u(t)u(t)^T$ is not also constant. We have

$$J\frac{d\widetilde{X}(t)}{dt} = J(I+c(t)u(t)u^T(t)J)\frac{X(t)}{dt} + J\left[\frac{d(c(t)u(t)u^T(t))}{dt}\right]JX(t)$$

$$= J(I+c(t)u(t)u^T(t)J)J^{-1}H(t)X(t) + J\left[\frac{d(c(t)u(t)u^T(t))}{dt}\right]JX(t)$$

$$= \big[(I-c(t)u(t)u^T(t)J)^T H(t)(I-c(t)u(t)u^T(t)J)+$$

$$J\frac{d(c(t)u(t)u^T(t))}{dt}J(I-c(t)u(t)u^T(t)J)\Big]\widetilde{X}(t) \quad \text{with } \widetilde{X}(t) = (I+c(t)u(t)u^T(t)J)X(t)$$

$$= [H(t)+$$

$$\left.\underbrace{(c(t)Ju(t)u^(t)H(t))^T + c(t)Ju(t)u^T(t)H(t) + c(t)^2(u(t)u^T(t)J)^T H(t)(u(t)u^T(t)J)}_{E(t)}\right]\widetilde{X}(t)$$

$$+\left[\underbrace{J\frac{d(c(t)u(t)u^T(t))}{dt}J(I-c(t)u(t)u^T(t)J)}_{F(t)}\right]\widetilde{X}(t)$$

$$= [H(t)+E(t)+F(t)]\,\widetilde{X}(t),$$

We note that $E(t)+F(t)$ is not symmetric because $E(t)$ is symmetric and $F(t)$ is not symmetric. Which gives us a contradiction. To have $H(t)+E(t)+F(t)$ symmetric, we must have $F(t) = 0,\ \forall t\in\mathbb{R}$. Then $c(t)u(t)u^T(t)$ is constant. In particular, $c(t)u(t)u^T(t) = c(0)u(0)u(0)^T \quad \forall t\in\mathbb{R}$. We deduce that there is a constant vector $u = u(0)$ and a real constant $c = c(0)\in\{-1,+1\}$ such that $E(t)$ is of the form (2.8).

2. By deriving $\widetilde{X}(t)$, we get

$$\begin{aligned} J\frac{d\widetilde{X}(t)}{dt} &= J(I+cuu^TJ)J^{-1}J\frac{dX(t)}{dt} \\ &= J(I+cuu^TJ)J^{-1}H(t)X(t), \ \text{ from } (1.1) \\ &= \left[H(t)+cJuu^TH(t)\right]X(t) \\ &= \left[H(t)+cJuu^TH(t)\right](I-cuu^TJ)\widetilde{X}(t) \end{aligned}$$

because the matrix $(I-cuu^TJ)$ is the inverse of $(\ \ I+cuu^TJ)$ see [14]

$$= \left[H(t)+\underbrace{(cJuu^TH(t))^T + cJuu^TH(t) + c^2(uu^TJ)^TH(t)(uu^TJ)}_{E(t)}\right]\widetilde{X}(t)$$

Ref

14. YAN Qing-you. The properties of a kind of random symplectic matrices. Applied mathematics and Mechanics. Vol 23, No 5, May 2002.

We then obtain equation (2.9) with

$$E(t) = c(Juu^T H(t))^T + cJuu^T H(t) + c^2(uu^T J)^T H(t)(uu^T J).$$

This shows that $\widetilde{X}(t) = (I + cuu^T J)X(t)$ is a solution of (2.9).

3. Indeed, it suffices to develop $(I - uu^T J)^T H(t)(I - uu^t J)$ to obtain

$$(I - cuu^T J)^T H(t)(I - cuu^t J) = H(t) + \underbrace{(cJ^T uu^T H(t))^T + cJ^T uu^T H(t) + c^2(uu^T J)^T H(t)(uu^T J)}_{E(t)}$$

Ref

4. L. Batzke, C. Mehl, A. C.M. Ran and L. Rodman, Generic rank-k Perturbations of Structured Matrices. Operator Theory, Function Spaces, and Applications Birkhuser, Cham. (2016), p. 27-48.

III. Generic Double Rank-One Perturbation of Hamiltonian System with Periodic Coefficients

In this section, we consider two vectors u_1 and u_2 taken in a generic set Ω of $\mathbb{R}^{2n}$ such that $u_1^T J u_2 = 0$. Note that this holds if $u_1 = u_2$. On the other hand, we can consider generic vectors belonging to isotropic (or Lagrangian) subspaces. A subspace $\mathcal{X} \subseteq \mathbb{R}^{2n}$ is called isotropic if $\mathcal{X} \perp J\mathcal{X}$. The maximum isotropic subspaces containing $\mathcal{X}$ are of dimension n [4]. Hence we have this definition

Definition 3.1 *A subspace $\mathcal{L}$ of $\mathbb{R}^{2n}$ is called a Lagrangian subspace if it is of the dimension n and*

$$x^T J y = 0, \qquad \forall x, y \in \mathcal{L}.$$

We first consider the rank-one pertubation

$$X_1(t) = (I + c_1 u_1 u_1^T J)X(t), \ \ \forall t \in \mathbb{R} \ \text{ and } \ c = \pm 1$$

of the solution $(X(t))_{t\in\mathbb{R}}$ of system (1.1) using the vector u. Then we perturb the solution a second time using the second vector u_2. We get

$$X_2(t) = (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J)X(t), \forall t \in \mathbb{R} \ \text{ and } \ c_1, c_2 \in \{-1, +1\} \tag{3.1}$$

We have the following Proposition

Proposition 3.1 *The double rank-one perturbation of the solution of (1.1) is the solution of the following system*

$$\begin{cases} J\dfrac{\widehat{X}(t)}{dt} &= (I - c_2 u_2 u_2^T J)^T (I - c_1 u_1 u_1^T J)^T H(t)(I - c_1 u_1 u_1^T J)(I - c_2 u_2 u_2^T J)\widehat{X}(t) \\ \widehat{X}(0) &= (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J) \end{cases} \tag{3.2}$$

Proof
The double rank-one perturbation of the solution of (1.1) is given by (3.1). Then $\forall t \in \mathbb{R}$ and $c_1, c_2 \in \{-1, +1\}$, we have

$$\begin{aligned} \frac{dX_2(t)}{dt} &= (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J)\frac{X(t)}{dt}, \\ &= (I + c_2 u_2 u_2^T J)(I + c_1 u_1 u_1^T J)J^{-1} H(t)X(t), \\ &= \left[(I + c_2 u_2 u_2^T J)(J^{-1} + c_1 u_1 u_1^T)H(t)(I + c_1 u_1 u_1^T J)^{-1}(I + c_2 u_2 u_2^T J)^{-1}\right] \times \end{aligned}$$

$$\underbrace{(I + c_2u_2u_2^TJ)(I + c_1u_1u_1^TJ)X(t)}_{X_2(t)},$$

$$= \left[(I + c_2u_2u_2^TJ)J^{-1}(I - c_1u_1u_1^TJ)^TH(t)(I + c_1u_1u_1^TJ)^{-1}(I + c_2u_2u_2^TJ)^{-1}\right]X_2(t)$$

$$= J^{-1}\left[(I - c_2u_2u_2^TJ)^T(I - c_1u_1u_1^TJ)^TH(t)(I - c_1u_1u_1^TJ)(I - c_2u_2u_2^TJ)\right]X_2(t),$$

because for all vector u and $c \in \{-1, +1\}$, we have $(I + cuu^TJ)^{-1} = (I - cuu^TJ)$. Moreover $X_2(0) = (I + c_2u_2u_2^TJ)(I + c_1u_1u_1^TJ)X(0)$.

Notes

Remark 3.1 *The double rank-one perturbation $X_2(t)$ (or $\widehat{X}(t)$) of $X(t)$ can be put in the form*

$$X_2(t) = X(t) + c_2u_2u_2^TJX(t) + c_1u_1u_1JX(t), \quad \forall t \in \mathbb{R} \ \ c_1, c_2 \in \{-1, +1\}; \tag{3.3}$$

or by putting the vectors u_1 and u_2 as column of a matrix $U = [u_1\ u_2] \in \mathbb{R}^{2n\times 2}$

$$X_2(t) = (I + U\Sigma_2U^TJ)X(t) = X(t) + U\Sigma_2U^TJX(t) \tag{3.4}$$

where $\Sigma_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ is a diagonal matrix with $c_1, c_2 \in \{-1, +1\}$.

We have the following Definition

Definition 3.2 *We call a generic rank-2 perturbation of system (1.1), any system given by (3.2).*

From (3.2), the following corollary gives another writing of system (3.2)

Corollary 3.1 *System (3.2) can be put in the form below*

$$\begin{cases} J\dfrac{\widehat{X}(t)}{dt} &= (I - U\Sigma_2U^TJ)^TH(t)(I - U\Sigma_2U^TJ)\widehat{X}(t) \\ \widehat{X}(0) &= (I + U\Sigma_2U^TJ) \end{cases} \tag{3.5}$$

or in a following simple form

$$\begin{cases} J\dfrac{\widehat{X}(t)}{dt} &= (H(t) + E(t))\,\widehat{X}(t) \\ \widehat{X}(0) &= I + U\Sigma_2U^TJ \end{cases} \tag{3.6}$$

where

$$E(t) = JU\Sigma_2U^TH(t) + (JU\Sigma_2U^TH(t))^T + (U\Sigma_2U^TJ)^TH(t)(U\Sigma_2U^TJ)$$

Proof

To have (3.5), It suffices to notice that

$$(I - c_1u_1u_1^TJ)(I - c_2u_2u_2J) = I - c_1u_1u_1^TJ - c_2u_2u_2^TJ$$

$$= I - [u_1\ u_2]\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}[u_1\ u_2]^TJ$$

$$= I - U\Sigma_2U^TJ$$

Similarly, we have $(I + c_1 u_1 u_1^T J)(I + c_2 u_2 u_2 J) = I + U\Sigma_2 U^T J$.

Next, by developing $(I - U\Sigma_2 U^T J)^T H(t)(I - U\Sigma_2 U^T J)$ in (3.5), we get

$$(I - U\Sigma_2 U^T J)^T H(t)(I - U\Sigma_2 U^T J) = H(t) + E(t)$$

where $E(t) = JU\Sigma_2 U^T H(t) + (JU\Sigma_2 U^T H(t))^T + (U\Sigma_2 U^T J)^T H(t)(U\Sigma_2 U^T J)$. Hence we have (3.6).

Notes

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Global Existence and Intrinsic Decay Rates for the Energy of a Kirchhoff Type in a Nonlinear Viscoelastic Equation

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Abstract- In this work we consider a nonlinear hyperbolic equations of Kirch-hoff type in viscoelasticity. By using the potential well theory we obtain the existence of a global solution. Then, we prove the intrinsic decays for the energy of the nonlinear hyperbolic equations of Kirchhoff type in viscoelasticity of relaxation kernels described by the inequality $h'(t) \leq -H(h(t))$ for all $t \geq 0$, with *H* convex.

Keywords and phrases: *global existence, exponential decay, polynomial decay, viscoelastic damping, intrinsic decay rates.*

GJSFR-F Classification: *MSC 2010: 47N70*

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Global Existence and Intrinsic Decay Rates for the Energy of a Kirchhoff Type in a Nonlinear Viscoelastic Equation

Draifia Alaeddine

Abstract- In this work we consider a nonlinear hyperbolic equations of Kirch-hoff type in viscoelasticity. By using the potential well theory we obtain the existence of a global solution. Then, we prove the intrinsic decays for the energy of the nonlinear hyperbolic equations of Kirchhoff type in viscoelasticity of relaxation kernels described by the inequality $h'(t) \leq -H(h(t))$ for all $t \geq 0$, with *H* convex.

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I. Introduction

In this works, we study the global existence and intrinsic decay rates for the energy of a kirchhoff type in a nonlinear viscoelastic equation

$$\begin{cases} u_{tt}(x,t) - \Phi(x)\left[\mu\left(\|\nabla u(t)\|^2_{L^2(\Omega)}\right)\Delta u(x,t) - \int_0^t h(t-s)\,\Delta u(x,s)ds\right] \\ +bu_t(x,t) = 0, \quad x \in \Omega \times \mathbb{R}_+^*, \end{cases} \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $h(.) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are given functions which will be spaced later and $u_0(x)$, $u_1(x)$ are given initial data belonging to appropriate space. All the parameter b are assumed to be positive constants. The function $\Phi(x)$ is the density, $(\rho(x))^{-1} = \Phi(x)$, $\Phi(x) > 0$, for all $x \in \Omega$, and

$$\mu(s) := \xi_0 + \xi_1 s^\gamma,$$

where $s > 0$, $\xi_0 > 0$, $\xi_1 > 0$ and $\gamma \geq 1$. For more information on using Kirchhoff type, see $[4-6]$.

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Also, a result of local existence for problem $(1.1)-(1.3)$ for $\xi_1 = 0$ has been proved in [1], for $\xi_1 \neq 0$, in the same way as [1], we get the same basic results for the local existence of problem $(1.1)-(1.3)$ with a slight change in some calculations that do not affect the basic results.

The motivation of our work is due to some results regarding the following research papers: Boumaza, N and Boulaaras, S. [2] studied the general decay for Kirchhoff type in viscoelasticity with not necessarily decreasing kernel of $(1.1)-(1.3)$. Marcelo M. Irena Lasiecka and Claudete M. Webler. [3] studied the intrinsic decay rates for the energy of a nonlinear viscoelastic equation modeling the vibrations of thin rods with variable density. M. M. Cavalcanti, V. N. Domingos Cavalcanti, I. Lasiecka and F. A. Falcao Nascimento. [7] studied the intrinsic decay rate estimates for the wave equation with competing viscoelastic and frictional dissipative effects. I. Lasiecka, S. A. Messaoudi and M. I. Mustafa. [8] studied the note on intrinsic decay rates for abstract wave equations with memory. I. Lasiecka and X. Wang. [9] studied the intrinsic decay rate estimates for semilinear abstract second order equations with memory. Cavalcanti M. Filho VND. Cavalcanti JSP. Soriano JA. [10] studied the existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping. For more results in this direction, see $[11-15]$.

However, $[1-3]$, $[4-6]$ and $[11-15]$ did not study the intrinsic decay rates for the energy of problem $(1.1)-(1.3)$ of relaxation kernels described by the inequality $h'(t) \leq -H(h(t))$ for all $t \geq 0$, with H convex. Motivated by the above research, we will consider the intrinsic decay rates for the energy of relaxation kernels described by the inequality $h'(t) \leq -H(h(t))$ for all $t \geq 0$ of the model $(1.1)-(1.3)$ in this paper.

The outline of the paper is as follows. In the second section we define the energy $E(t)$ associated to $(1.1)-(1.3)$ and show that it is a non-increasing function of t. In section 3, we prove global existence of solution of $(1.1)-(1.3)$. Finally, in section 4, we prove the intrinsic decay rates for the energy of the posed problem.

II. Assumptions and Main Results

In this section, we define the energy $E(t)$ associated to $(1.1)-(1.3)$ and show that it is a non-increasing function of t. We suppose that the kernel $h(t)$ is a function satisfying

Assumptions 1

The relaxation function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a $C^1 \cap L^1$ decreasing function and satisfies

$$h(0) > 0 \text{ and } \int_0^t h(s)\, ds < \xi_0.$$

Assumptions 2

(**i**) In addition to **Assumption 1**, we require

$$h'(t) \leq -H(h(t)) \text{ for all } t \geq 0,$$

where $H \in C^1(R_+)$ which $H(0) = 0$ is a given strictly increasing and convex function. Moreover,

$$H \in C^2(0,\infty) \text{ and } \lim \inf_{x \to 0^+} \left\{x^2 H''(x) - xH'(x) + H(x)\right\} \geq 0.$$

(**ii**) With reference to the function H introduced above, let $y(t)$ be the solution of the ODE

$$y'(t) + H(y(t)) = 0, \; y(0) = h(0) > 0.$$

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(**iii**) We assume that there exists $\alpha_0 \in [0,1)$ such that $y^{1-\alpha_0} \in L_1(1,\infty)$.
In order to formulate the long-time behavior results, we recall the binary notation

$$\begin{cases} (h * w)(t) := \int_0^t h(t-s)\, w(s)\, ds, \\ \int_\Omega (h \circ w)(t)\, dx := \int_0^t h(t-s) \|w(x,s) - w(x,t)\|^2_{L^2(\Omega)}\, ds, \\ (h \diamond w)(t) := \rho(x) \int_0^t h(t-s)\,(w(t) - w(s))\, ds. \end{cases} \tag{2.1}$$

We define the corresponding energy functional by

$$\begin{aligned} E(t) \quad : \quad &= \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\left(\xi_0 - \int_0^t h(s)\, ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)} \\ &+ \frac{\xi_1}{2(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)}_{L^2(\Omega)} + \frac{1}{2}\int_\Omega (h \circ \nabla u)(t)\, dx. \end{aligned} \tag{2.2}$$

Note that, in view of (2.1), we have that

$$0 < l := \left(\xi_0 - \int_0^\infty h(s)\, ds\right) \leq \xi_0 \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}_+. \tag{2.3}$$

The energy satisfies the following identity

Lemma 1. We have the identity

$$\begin{aligned} \frac{d}{dt}\{E(t)\} \quad &= \quad \frac{1}{2}\int_\Omega (h' \circ \nabla u)(t)\, dx - \frac{1}{2} h(t) \|\nabla u(t)\|^2_{L^2(\Omega)} - b\|u_t(t)\|^2_{L^2_\rho(\Omega)} \\ &\leq \quad 0. \end{aligned} \tag{2.4}$$

Proof. Multiplying (1.1) by $\rho(x)\, u_t$ and integration over Ω, we have

$$\begin{aligned} &\int_\Omega \rho(x)\, u_{tt} u_t dx - \int_\Omega \left(\xi_0 + \xi_1 \|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right) \Delta u(x,t) u_t(x,t)\, dx \\ &+ \int_\Omega u_t(x,t) \left[\int_0^t h(t-s)\, \Delta u(x,s)\, ds\right] dx + b\int_\Omega \rho(x)\, u_t^2(x,t) dx \\ &= \quad 0. \end{aligned} \tag{2.5}$$

We have

$$\int_\Omega \rho(x)\, u_{tt} u_t dx = \frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2_{L^2_\rho(\Omega)}\right\}. \tag{2.6}$$

And by using integration by parts, we have

$$
\begin{aligned}
& -\int_{\Omega}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\Delta u(x,t)u_t(x,t)\,dx \\
= & -\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\int_{\Omega}\Delta u(x,t)u_t(x,t)\,dx \\
= & \left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\int_{\Omega}\nabla u(x,t)\,.\nabla u_t(x,t)\,dx \\
= & \left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\frac{1}{2}\frac{d}{dt}\left\{\int_{\Omega}|\nabla u(x,t)|^2\,dx\right\} \\
= & \frac{\xi_0}{2}\frac{d}{dt}\left\{\|\nabla u(t)\|_{L^2(\Omega)}^2\right\}+\frac{\xi_1}{2}\|\nabla u(t)\|_{L^2(\Omega)}^{2\gamma}\frac{d}{dt}\left\{\|\nabla u(t)\|_{L^2(\Omega)}^2\right\} \\
= & \frac{\xi_0}{2}\frac{d}{dt}\left\{\|\nabla u(t)\|_{L^2(\Omega)}^2\right\}+\frac{\xi_1}{2(\gamma+1)}\frac{d}{dt}\left\{\|\nabla u(t)\|_{L^2(\Omega)}^{2(\gamma+1)}\right\} \\
= & \frac{d}{dt}\left\{\frac{1}{2}\left(\xi_0+\frac{\xi_1}{(\gamma+1)}\|\nabla u(t)\|_{L^2(\Omega)}^{2\gamma}\right)\|\nabla u(t)\|_{L^2(\Omega)}^2\right\}. \qquad (2.7)
\end{aligned}
$$

And by using integration by parts, we have

$$
\begin{aligned}
& \int_{\Omega}u_t(x,t)\left[\int_0^t h(t-s)\,\Delta u(x,s)\,ds\right]dx \\
= & -\int_0^t h(t-s)\left[\int_{\Omega}\nabla u(x,s)\,.\nabla u_t(x,t)\,dx\right]ds,
\end{aligned}
$$

and using

$$
-\nabla u(x,s)\,.\nabla u_t(x,t)=\frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\}-\frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\},
$$

then

$$
\begin{aligned}
& \int_{\Omega}u_t(x,t)\left[\int_0^t h(t-s)\,\Delta u(x,s)\,ds\right]dx \\
= & \int_0^t h(t-s)\int_{\Omega}\left(\frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,s)-\nabla u(x,t)|^2\right\}\right)dxds \\
& -\int_0^t h(t-s)\int_{\Omega}\left(\frac{1}{2}\frac{d}{dt}\left\{|\nabla u(x,t)|^2\right\}\right)dxds \\
= & \frac{1}{2}\int_0^t h(t-s)\left(\frac{d}{dt}\left\{\int_{\Omega}|\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right\}\right)ds \\
& -\frac{1}{2}\int_0^t h(t-s)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|_{L^2(\Omega)}^2\right\}\right)ds, \qquad (2.8)
\end{aligned}
$$

Notes

Notes

by using (2.1) , we get

$$\begin{aligned}
&\frac{1}{2}\int_0^t h(t-s)\frac{d}{dt}\left\{\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right\}ds\\
=&\ \frac{1}{2}\int_0^t \frac{d}{dt}\left\{h(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)\right\}ds\\
&-\frac{1}{2}\int_0^t h'(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds\\
=&\ \frac{1}{2}\frac{d}{dt}\left\{\int_0^t h(t-s)\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dxds\right\}\\
&-\frac{1}{2}\int_0^t h'(t-s)\left(\int_\Omega |\nabla u(x,s)-\nabla u(x,t)|^2\,dx\right)ds\\
=&\ \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx\right\}-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx, \qquad (2.9)
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{2}\int_0^t h(t-s)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\right)ds\\
=&\ -\frac{1}{2}\left(\int_0^t h(t-s)\,ds\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\right)\\
=&\ -\frac{1}{2}\left(\int_0^t h(s)\,ds\right)\left(\frac{d}{dt}\left\{\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\right)\\
=&\ -\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}. \quad (2.10)
\end{aligned}$$

By replacement (2.9) and (2.10) into (2.8) , we get

$$\begin{aligned}
&\int_\Omega u_t(x,t)\left[\int_0^t h(t-s)\,\Delta u(x,s)\,ds\right]dx\\
=&\ \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx\right\}-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx\\
&-\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}\\
=&\ \frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx-\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\\
&-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}. \qquad (2.11)
\end{aligned}$$

By combining (2.6), (2.7) and (2.11) into (2.5), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2_{L^2_\rho(\Omega)}\right\}\\
&+\frac{d}{dt}\left\{\frac{1}{2}\left(\xi_0+\frac{\xi_1}{(\gamma+1)}\|\nabla u(t)\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\\
&+\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega (h\circ\nabla u)(t)\,dx-\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right\}\\
&-\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx+\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}+b\|u_t(t)\|^2_{L^2_\rho(\Omega)}\\
=\;& 0,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\frac{1}{2}\left(\xi_0-\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\right.\\
&\left.+\frac{\xi_1}{2(\gamma+1)}\|\nabla u(t)\|^{2(\gamma+1)}_{L^2(\Omega)}+\frac{1}{2}\int_\Omega (h\circ\nabla u)(t)\,dx\right\}\\
=\;&\frac{1}{2}\int_\Omega (h'\circ\nabla u)(t)\,dx-\frac{1}{2}h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}-b\|u_t(t)\|^2_{L^2_\rho(\Omega)}, \quad (2.12)
\end{aligned}$$

by using (2.2) into (2.12), we get (2.4).
The proof of **Lemma 1** is completes.

III. Global Existence

In this section we show that any solution of (1.1) – (1.3) is bounded and global, provided that $E(0)$ is positive and small enough.

Theorem 1. Assume that (2.3) *holds. Then the solution to problem* (1.1) – (1.3) *is bounded and global.*

Proof. It suffices to show that $\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\|\nabla u(t)\|^2_{L^2(\Omega)}$ is bounded independently of t.
By using (2.3) and (**A1**) into (2.12), we get

$$\omega_1\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\omega_2\|\nabla u(t)\|^2_{L^2(\Omega)}\le E(t)\le E(0),$$

where $\omega_1>0$ and $\omega_2>0$, then

$$\|u_t(t)\|^2_{L^2_\rho(\Omega)}+\|\nabla u(t)\|^2_{L^2(\Omega)}\le \omega_3 E(0),$$

where $\omega_3>0$.
Then the solution to problem (1.1) – (1.3) is bounded and global.
The proof of **Theorem 1** is completes.

IV. Decay of Solutions

Now, we are in a position to state our main result.

Lemma 2. Let us assume that ***Assumption 1*** *and* ***Assumption 2*** *are the place. Then, there exists a positive constant* $T_0>0$ *such that*

$$E((n+1)T)+\tilde{H}\left(C_9^{-1}E((n+1)T)\right)\le E(nT), \quad n=1,2,3...,$$

for all $T>T_0$ *and all* $n\in N$*, where* $\tilde{H}$ *is given in* (4.49) *and* C_9 *is given in* (4.52).

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Proof. For this purpose, a by now standard procedure is to multiply (1.1) by the viscoelastic multiplier

$$(h \diamond u)(t) = \rho(x) \int_0^t h(t-s)(u(t)-u(s))\,ds,$$

and integrating over $\Omega \times (nT,(n+1)T)$, we infer that

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left(u_{tt}(t), (h \diamond u)(t)\right)_{L^2(\Omega)} dt \\
&- \int_{nT}^{(n+1)T} \left(\Phi(x)\left(\xi_0 + \xi_1 \|\nabla u(t)\|_{L^2(\Omega)}^{2\gamma}\right)\Delta u(t), (h \diamond u)(t)\right)_{L^2(\Omega)} dt \\
&+ \int_{nT}^{(n+1)T} \left(\Phi(x)\left(\int_0^t h(t-s)\Delta u(s)\,ds\right), (h \diamond u)(t)\right)_{L^2(\Omega)} dt \\
&+ b\int_{nT}^{(n+1)T} \left(u_t(t), (h \diamond u)(t)\right)_{L^2(\Omega)} dt \\
&= 0. \qquad (4.1)
\end{aligned}$$

We shall analyze the above terms separately.
Direct calculation give

$$\begin{aligned}
&\left(u_{tt}(t), (h \diamond u)(t)\right)_{L^2(\Omega)} \\
&= \frac{d}{dt}\left\{\left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\right\} \\
&\quad - \left(u_t(t), \frac{d}{dt}\left(\int_0^t h(t-s)(u(t)-u(s))\,ds\right)\right)_{L^2_\rho(\Omega)} \\
&= \frac{d}{dt}\left\{\left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\right\} \\
&\quad - \left(u_t(t), \left(\int_0^t h'(t-s)(u(t)-u(s))\,ds\right)\right)_{L^2_\rho(\Omega)} \\
&\quad - \left(u_t(t), \left(\int_0^t h(t-s)\,u_t(t)\,ds\right)\right)_{L^2_\rho(\Omega)},
\end{aligned}$$

then

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left(u_{tt}(t), (h \diamond u)(t)\right)_{L^2(\Omega)} dt \\
&= \left.\left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\right|_{nT}^{(n+1)T} \\
&\quad - \int_{nT}^{(n+1)T} \left(u_t(t), \int_0^t h'(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)} dt \\
&\quad - \int_{nT}^{(n+1)T} \left(\int_0^t h(s)\,ds\right)\|u_t(x,t)\|_{L^2_\rho(\Omega)}^2\,dt. \qquad (4.2)
\end{aligned}$$

For the second term, by using integration by parts, we have

$$
\begin{aligned}
&-\int_{nT}^{(n+1)T}\left(\Phi(x)\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\Delta u(t),(h\diamond u)(t)\right)_{L^2(\Omega)}dt\\
=&-\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\left(\Delta u(t),\int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2(\Omega)}dt\\
=&\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\left(\nabla u(t),\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt.
\end{aligned}
\tag{4.3}
$$

For the third term, by using integration by parts, we have

$$
\begin{aligned}
&\int_{nT}^{(n+1)T}\left(\Phi(x)\int_0^t h(t-s)\Delta u(s)\,ds,(h\diamond u)(t)\right)_{L^2(\Omega)}dt\\
=&-\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(s)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt\\
=&\int_{nT}^{(n+1)T}\left\|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|_{L^2(\Omega)}^2dt\\
&-\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(t)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt.
\end{aligned}
\tag{4.4}
$$

Combining $(4.2)-(4.4)$ into (4.1), we arrive at

$$
\begin{aligned}
&\left(u_t(t),\int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\Bigg|_{nT}^{(n+1)T}\\
&-\int_{nT}^{(n+1)T}\left(u_t(t),\int_0^t h'(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}dt\\
&-\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|u_t(x,t)\|_{L^2_\rho(\Omega)}^2dt\\
&+\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|_{L^2(\Omega)}^{2\gamma}\right)\left(\nabla u(t),\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt\\
&+\int_{nT}^{(n+1)T}\left\|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|_{L^2(\Omega)}^2dt\\
&-\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(t)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt\\
&+b\int_{nT}^{(n+1)T}(u_t(t),(h\diamond u)(t))_{L^2(\Omega)}dt\\
=&\ 0,
\end{aligned}
\tag{4.5}
$$

Notes

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then (4.5) is equivalent

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
=\;& \left(u_t(t),\int_0^t h(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}\Bigg|_{nT}^{(n+1)T} \\
&-\int_{nT}^{(n+1)T}\left(u_t(t),\int_0^t h'(t-s)(u(t)-u(s))\,ds\right)_{L^2_\rho(\Omega)}dt \\
&+\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\left(\nabla u(t),\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt \\
&+\int_{nT}^{(n+1)T}\left\|\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right\|^2_{L^2(\Omega)}dt \\
&-\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\nabla u(t)\,ds,\int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\,ds\right)_{L^2(\Omega)}dt \\
&+b\int_{nT}^{(n+1)T}\left(u_t(t),(h\diamond u)(t)\right)_{L^2(\Omega)}dt \\
=\;& J_1+J_2+J_3+J_4+J_5+J_6. \qquad (4.6)
\end{aligned}$$

Estimate for $|J_1|$, where

$$\begin{aligned}
J_1:\;=\;& u_t((n+1)T),\int_0^{(n+1)T} h((n+1)T-s)(u((n+1)T)-u(s))\,ds\Bigg)_{L^2_\rho(\Omega)} \\
-\;& u_t(nT),\int_0^{nT} h(nT-s)(u(nT)-u(s))\,ds\Bigg)_{L^2_\rho(\Omega)}.
\end{aligned}$$

Now, let $m\in N$ be an arbitrary, natural number.
By using Young's inequality (for $\varepsilon=1$), we get

$$\begin{aligned}
& u_t(mT),\int_0^{mT} h(mT-s)(u(mT)-u(s))\,ds\Bigg)_{L^2_\rho(\Omega)} \\
=\;& \int_0^{mT} h(mT-s)\left(u_t(mT),(u(mT)-u(s))\right)_{L^2_\rho(\Omega)}ds \\
\le\;& \int_0^{mT} h(mT-s)\left[\frac12\|u_t(mT)\|^2_{L^2_\rho(\Omega)}+\frac12\|u(mT)-u(s)\|^2_{L^2_\rho(\Omega)}\right]ds \\
=\;& \frac12\;\int_0^{mT} h(mT-s)\,ds\Bigg)\|u_t(mT)\|^2_{L^2_\rho(\Omega)} \\
&+\frac12\int_0^{mT} h(mT-s)\|u(mT)-u(s)\|^2_{L^2_\rho(\Omega)}\,ds, \qquad (4.7)
\end{aligned}$$

by using

$$\|u(t)\|^2_{L^2_\rho(\Omega)}\le\|\rho\|^2_{L^2(\Omega)}\|\nabla u(t)\|^2_{L^2_\rho(\Omega)}, \qquad (4.8)$$

we get

$$\begin{aligned}
&\frac{1}{2}\int_0^{mT} h\left(mT-s\right)\|u(mT)-u\left(s\right)\|_{L^2_\rho(\Omega)}^2\,ds \\
\leq\ & \frac{1}{2}\|\rho\|_{L^2(\Omega)}^2\int_0^{mT} h\left(mT-s\right)\|\nabla u(mT)-\nabla u\left(s\right)\|_{L^2(\Omega)}^2\,ds \\
=\ & \frac{1}{2}\|\rho\|_{L^2(\Omega)}^2\int_\Omega \left(h\circ\nabla u\right)\left(mT\right)dx, \qquad (4.9)
\end{aligned}$$

by replacement (4.9) into (4.7) and using $\int_0^{mT} h\left(mT-s\right)ds=\int_0^{mT} h\left(s\right)ds$, we get

$$\begin{aligned}
& \left(u_t\left(mT\right),\int_0^{mT} h\left(mT-s\right)\left(u\left(mT\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(\Omega)} \\
\leq\ & \frac{1}{2}\left(\int_0^{mT} h\left(s\right)ds\right)\|u_t(mT)\|_{L^2_\rho(\Omega)}^2 \\
& +\frac{1}{2}\|\rho\|_{L^2(\Omega)}^2\int_\Omega\left(h\circ\nabla u\right)\left(mT\right)dx, \qquad (4.10)
\end{aligned}$$

by using (2.2), we get

$$\begin{cases}
\frac{1}{2}\|u_t(mT)\|_{L^2_\rho(\Omega)}^2\leq E\left(mT\right), \\
\text{and} \\
\frac{1}{2}\int_\Omega\left(h\circ\nabla u\right)\left(mT\right)dx\leq E\left(mT\right),
\end{cases} \qquad (4.11)$$

then, by combining (4.11) into (4.10), we get

$$\begin{aligned}
& \left(u_t\left(mT\right),\int_0^{mT} h\left(mT-s\right)\left(u\left(mT\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(\Omega)} \\
\leq\ & \left(\int_0^{mT} h\left(s\right)ds\right)E\left(mT\right)+\|\rho\|_{L^2(\Omega)}^2 E\left(mT\right) \\
\leq\ & \left\{\|h\|_{L^1(0,\infty)}+\|\rho\|_{L^2(\Omega)}^2\right\}E\left(mT\right),
\end{aligned}$$

then

$$|J_1|\leq C_1\left[E((n+1)T)+E(nT)\right], \qquad (4.12)$$

where

$$C_1:=\|h\|_{L^1(0,\infty)}+\|\rho\|_{L^2(\Omega)}^2\,.$$

Estimate for $|J_2|$, where

$$J_2:=-\int_{nT}^{(n+1)T}\left(u_t\left(t\right),\int_0^t h'\left(t-s\right)\left(u\left(t\right)-u\left(s\right)\right)ds\right)_{L^2_\rho(\Omega)}dt.$$

Notes

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By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_1}{2}\right)$, we get

$$\begin{aligned} |J_2| &\leq \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)} dt \\ &+ \frac{1}{4\varepsilon_1} \int_{nT}^{(n+1)T} \left\| \int_0^t h'(t-s)(u(t)-u(s))\, ds \right\|^2_{L^2_\rho(\Omega)} dt. \end{aligned} \tag{4.13}$$

By using (4.8), Cauchy-Schwarz inequality and (2.1), we get

$$\begin{aligned} & \int_{nT}^{(n+1)T} \left\| \int_0^t h'(t-s)(u(t)-u(s))\, ds \right\|^2_{L^2_\rho(\Omega)} dt \\ \leq\ & \|\rho\|^2_{L^2(\Omega)} \int_{nT}^{(n+1)T} \left\| \int_0^t h'(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|^2_{L^2(\Omega)} dt \\ \leq\ & -\|\rho\|^2_{L^2(\Omega)} h(0) \int_{nT}^{(n+1)T} \int_0^t h'(t-s) \|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)} dsdt \\ =\ & -\|\rho\|^2_{L^2(\Omega)} h(0) \int_{nT}^{(n+1)T} \int_\Omega (h' \circ \nabla u)(t)\, dxdt. \end{aligned} \tag{4.14}$$

By replacement (4.14) into (4.13), we get

$$\begin{aligned} |J_2| &\leq \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)} dt \\ &- \frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} h(0) \int_{nT}^{(n+1)T} \int_\Omega (h' \circ \nabla u)(t)\, dxdt. \end{aligned} \tag{4.15}$$

Estimate $|J_3|$, where

$$J_3 := \int_{nT}^{(n+1)T} \left(\xi_0 + \xi_1 \|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right) \left(\nabla u(t), \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds\right)_{L^2(\Omega)} dt.$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_2}{2}\right)$, we get

$$\begin{aligned} |J_3| &\leq \varepsilon_2 \int_{nT}^{(n+1)T} \left(\xi_0 + \xi_1 \|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)^2 \|\nabla u(t)\|^2_{L^2(\Omega)} dt \\ &+ \frac{1}{4\varepsilon_2} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|^2_{L^2(\Omega)} dt, \end{aligned} \tag{4.16}$$

by using $\|\nabla u(t)\|^{2\gamma}_{L^2(\Omega)} \leq \left(\frac{2(\gamma+1)}{\xi_1} E(0)\right)^{\dfrac{2\gamma}{2(\gamma+1)}}$, we get

$$\begin{aligned} & \varepsilon_2 \int_{nT}^{(n+1)T} \left(\xi_0 + \xi_1 \|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)^2 \|\nabla u(t)\|^2_{L^2(\Omega)} dt \\ \leq\ & \varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1} E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}} \Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2_{L^2(\Omega)} dt, \end{aligned} \tag{4.17}$$

by using Cauchy-Schwarz inequality and (2.1), we get

$$\left\| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2$$

$$\leq \left(\int_0^t h(t-s)\, ds \right) \int_0^t h(t-s) \|\nabla u(t) - \nabla u(s)\|_{L^2(\Omega)}^2\, ds$$

$$\leq \|h\|_{L^1(0,\infty)} \int_\Omega (h \circ \nabla u)(t)\, dx, \tag{4.18}$$

by replacement (4.17) and (4.18) into (4.16), we get

$$|J_3| \leq \varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1} E(0) \right)^{\frac{2\gamma}{2(\gamma+1)}} \Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|_{L^2(\Omega)}^2\, dt$$

$$+ \frac{1}{4\varepsilon_2} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dx dt. \tag{4.19}$$

Estimate $|J_4|$, where

$$J_4 := \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2 dt.$$

By using (4.18), we get

$$|J_4| \leq \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dx dt. \tag{4.20}$$

Now, estimate $|J_5|$, where

$$J_5 := -\int_{nT}^{(n+1)T} \left(\int_0^t h(t-s) \nabla u(t)\, ds, \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right)_{L^2(\Omega)} dt.$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_3}{2} \right)$, Cauchy-Schwarz inequality and (4.18), we get

$$|J_5| \leq \varepsilon_3 \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s) \nabla u(t)\, ds \right\|_{L^2(\Omega)}^2 dt$$

$$+ \frac{1}{4\varepsilon_3} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(\nabla u(t) - \nabla u(s))\, ds \right\|_{L^2(\Omega)}^2 dt$$

Notes

Notes

$$\begin{aligned}
&\leq \varepsilon_3 \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_0^t h(t-s) \|\nabla u(t)\|^2_{L^2(\Omega)} dsdt \\
&\quad + \frac{1}{4\varepsilon_3} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt \\
&= \varepsilon_3 \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds\right) \|\nabla u(t)\|^2_{L^2(\Omega)} dt \\
&\quad + \frac{1}{4\varepsilon_3} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt. \qquad (4.21)
\end{aligned}$$

Now, estimate $|J_6|$, where

$$\begin{aligned}
J_6 \;:&= b \int_{nT}^{(n+1)T} (u_t(t), (h \diamond u)(t))_{L^2(\Omega)}\, dt \\
:&= b \int_{nT}^{(n+1)T} \left(u_t(t), \int_0^t h(t-s)(u(t)-u(s))\, ds\right)_{L^2_\rho(\Omega)} dt.
\end{aligned}$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_1}{2}\right)$, (4.8) and (4.18), we get

$$\begin{aligned}
|J_6| &\leq b^2 \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)} dt \\
&\quad + \frac{1}{4\varepsilon_1} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(u(t)-u(s))\, ds \right\|^2_{L^2_\rho(\Omega)} dt \\
&\leq b^2 \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)} dt \\
&\quad + \frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} \int_{nT}^{(n+1)T} \left\| \int_0^t h(t-s)(\nabla u(t)-\nabla u(s))\, ds \right\|^2_{L^2(\Omega)} dt \\
&\leq b^2 \varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)} dt \\
&\quad + \frac{1}{4\varepsilon_1} \|\rho\|^2_{L^2(\Omega)} \|h\|_{L^1(0,\infty)} \int_{nT}^{(n+1)T} \int_\Omega (h \circ \nabla u)(t)\, dxdt. \qquad (4.22)
\end{aligned}$$

Combining (4.12), (4.15) and (4.19)−(4.22) into (4.6), and recalling that $\|h\|_{L^1(0,\infty)} < \xi_0$, we write

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds\right) \|u_t(x,t)\|^2_{L^2_\rho(\Omega)} dt \\
&\leq C_1 [E((n+1)T) + E(nT)]
\end{aligned}$$

$$
+\varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\, dt
$$

$$
-\frac{1}{4\varepsilon_1}\|\rho\|^2_{L^2(\Omega)} h(0) \int_{nT}^{(n+1)T}\int_\Omega (h' \circ \nabla u)(t)\, dxdt
$$

$$
+\varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1}E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}}\Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt
$$

$$
+\frac{1}{4\varepsilon_2}\xi_0 \int_{nT}^{(n+1)T}\int_\Omega (h \circ \nabla u)(t)\, dxdt
$$

$$
+\xi_0 \int_{nT}^{(n+1)T}\int_\Omega (h \circ \nabla u)(t)\, dxdt
$$

$$
+\varepsilon_3\xi_0 \int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds\right) \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt
$$

$$
+\frac{1}{4\varepsilon_3}\xi_0 \int_{nT}^{(n+1)T}\int_\Omega (h \circ \nabla u)(t)\, dxdt
$$

$$
+b^2\varepsilon_1 \int_{nT}^{(n+1)T} \|u_t(t)\|^2_{L^2_\rho(\Omega)}\, dt
$$

$$
+\frac{1}{4\varepsilon_1}\xi_0 \|\rho\|^2_{L^2(\Omega)} \int_{nT}^{(n+1)T}\int_\Omega (h \circ \nabla u)(t)\, dxdt. \tag{4.23}
$$

Since $h(0) > 0$, we can select a points $t_1 < T$ with t_1 close to zero such that for all $t \geq t_1$

$$
\int_0^t h(s)\, ds \geq t_1 h(t_1) := c_0.
$$

Then (4.23) is equivalent

$$
\int_{nT}^{(n+1)T} \left\{t_1 h(t_1) - \varepsilon_1\left(1+b^2\right)\right\} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\, dt
$$

$$
\leq \quad C_1\left[E((n+1)T) + E(nT)\right]
$$

$$
-\frac{1}{4\varepsilon_1}\|\rho\|^2_{L^2(\Omega)} h(0) \int_{nT}^{(n+1)T}\int_\Omega (h' \circ \nabla u)(t)\, dxdt
$$

$$
+\varepsilon_2 \quad \xi_0 + \xi_1 \left(\frac{2(\gamma+1)}{\xi_1}E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}}\Bigg)^2 \int_{nT}^{(n+1)T} \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt
$$

$$
+\xi_0 \left\{\frac{1}{4\varepsilon_2} + 1 + \frac{1}{4\varepsilon_3} + \frac{1}{4\varepsilon_1}\|\rho\|^2_{L^2(\Omega)}\right\} \int_{nT}^{(n+1)T}\int_\Omega (h \circ \nabla u)(t)\, dxdt
$$

$$
+\varepsilon_3\xi_0 \int_{nT}^{(n+1)T} \left(\int_0^t h(s)\, ds\right) \|\nabla u(t)\|^2_{L^2(\Omega)}\, dt. \tag{4.24}
$$

Notes

Notes

Now, multiplying (1.1) by $\rho(x)\,u(x,t)$ and integrating over $\Omega\times(nT,(n+1)T)$, we infer that

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(u_{tt}(t),u(t)\right)_{L^2_\rho(\Omega)}dt\\
&-\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\Delta u(t),u(t)\right)_{L^2(\Omega)}dt\\
&+\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\,\Delta u(s)\,ds,u(t)\right)_{L^2(\Omega)}dt\\
&+b\int_{nT}^{(n+1)T}\left(u_t(x,t),u(x,t)\right)_{L^2_\rho(\Omega)}dt\\
=\;&0. \qquad (4.25)
\end{aligned}$$

By using

$$u_{tt}(t)\,u(t)=\frac{d}{dt}\{u_t(t)\,u(t)\}-u_t^2(t),$$

we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(u_{tt}(x,t),u(x,t)\right)_{L^2_\rho(\Omega)}dt\\
=\;&\left(u_t(t),u(t)\right)_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T}-\int_{nT}^{(n+1)T}\|u_t(x,t)\|^2_{L^2_\rho(\Omega)}dt. \qquad (4.26)
\end{aligned}$$

By using integration by parts, we get

$$\begin{aligned}
&-\int_{nT}^{(n+1)T}\left(\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\Delta u(t),u(t)\right)_{L^2(\Omega)}dt\\
=\;&\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\left(\nabla u(t),\nabla u(t)\right)_{L^2(\Omega)}dt\\
=\;&\int_{nT}^{(n+1)T}\left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}dt. \qquad (4.27)
\end{aligned}$$

By using integration by parts, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T}\left(\int_0^t h(t-s)\,\Delta u(s)\,ds,u(t)\right)_{L^2(\Omega)}dt\\
=\;&-\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
=\;&\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t)-\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
&-\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
=\;&\int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t)-\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt\\
&-\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}dt. \qquad (4.28)
\end{aligned}$$

And

$$
\begin{aligned}
& b\int_{nT}^{(n+1)T} (u_t(x,t),u(x,t))_{L^2_\rho(\Omega)}\,dt \\
= \; & \frac{b}{2}\int_{nT}^{(n+1)T} \frac{d}{dt}\left\{\|u(x,t)\|^2_{L^2_\rho(\Omega)}\right\}dt \\
= \; & \frac{b}{2}\left\{\|u((n+1)T)\|^2_{L^2_\rho(\Omega)} - \|u(nT)\|^2_{L^2_\rho(\Omega)}\right\}. \qquad (4.29)
\end{aligned}
$$

By combining (4.26) – (4.29) into (4.25) , we get

$$
\begin{aligned}
& (u_t(t),u(t))_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T} - \int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
& + \int_{nT}^{(n+1)T} \left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
& + \int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t)-\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt \\
& - \int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
& + \frac{b}{2}\left\{\|u((n+1)T)\|^2_{L^2_\rho(\Omega)} - \|u(nT)\|^2_{L^2_\rho(\Omega)}\right\} \\
= \; & 0. \qquad (4.30)
\end{aligned}
$$

Then (4.30) is equivalent

$$
\begin{aligned}
& -\int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
& + \int_{nT}^{(n+1)T} \left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
= \; & -(u_t(t),u(t))_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T} \\
& - \int_{nT}^{(n+1)T}\int_0^t h(t-s)\left(\nabla u(t)-\nabla u(s),\nabla u(t)\right)_{L^2(\Omega)}dsdt \\
& + \int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
& - \frac{b}{2}\left\{\|u((n+1)T)\|^2_{L^2_\rho(\Omega)} - \|u(nT)\|^2_{L^2_\rho(\Omega)}\right\}. \qquad (4.31)
\end{aligned}
$$

To estimate the term

$$
\begin{aligned}
I_1 \; : & = -(u_t(t),u(t))_{L^2_\rho(\Omega)}\Big|_{nT}^{(n+1)T} \\
: & = -(u_t((n+1)T),u((n+1)T))_{L^2_\rho(\Omega)} + (u_t(nT),u(nT))_{L^2_\rho(\Omega)}.
\end{aligned}
$$

Notes

Notes

By using Young's inequality (for $\varepsilon = 1$), (4.8), $\frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)} \le E(t)$ and $\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} \le l^{-1}E(t)$, we get

$$\begin{aligned}
& (u_t(t), u(t))_{L^2_\rho(\Omega)} \\
\le\ & \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|u(t)\|^2_{L^2_\rho(\Omega)} \\
\le\ & \frac{1}{2}\|u_t(t)\|^2_{L^2_\rho(\Omega)} + \frac{1}{2}\|\rho\|^2_{L^2(\Omega)}\|\nabla u(t)\|^2_{L^2(\Omega)} \\
\le\ & E(t) + \|\rho\|^2_{L^2(\Omega)} l^{-1}E(t) \\
=\ & \left\{1 + \|\rho\|^2_{L^2(\Omega)} l^{-1}\right\}E(t),
\end{aligned}$$

then

$$|I_1| \le C_2\{E((n+1)T) + E(nT)\}, \tag{4.32}$$

where

$$C_2 := 1 + \|\rho\|^2_{L^2(\Omega)} l^{-1}.$$

To estimate the term

$$I_2 := -\int_{nT}^{(n+1)T}\int_0^t h(t-s)(\nabla u(t) - \nabla u(s), \nabla u(t))_{L^2(\Omega)}\,dsdt.$$

By using Young's inequality $\left(\text{for } \varepsilon = \frac{\varepsilon_4}{2}\right)$ and (2.1), we get

$$\begin{aligned}
|I_2| \le\ & \frac{1}{4\varepsilon_4}\int_{nT}^{(n+1)T}\int_0^t h(t-s)\|\nabla u(t) - \nabla u(s)\|^2_{L^2(\Omega)}\,dsdt \\
& +\varepsilon_4\int_{nT}^{(n+1)T}\int_0^t h(t-s)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dsdt \\
=\ & \frac{1}{4\varepsilon_4}\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt \\
& +\varepsilon_4\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt.
\end{aligned} \tag{4.33}$$

By using (4.8) and $\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\Omega)} \le l^{-1}E(t)$, we get

$$\begin{aligned}
& -\frac{b}{2}\left\{\|u((n+1)T)\|^2_{L^2_\rho(\Omega)} - \|u(nT)\|^2_{L^2_\rho(\Omega)}\right\} \\
\le\ & \frac{b}{2}\|\rho\|^2_{L^2(\Omega)}\left\{\|\nabla u((n+1)T)\|^2_{L^2(\Omega)} + \|\nabla u(nT)\|^2_{L^2(\Omega)}\right\} \\
\le\ & b\|\rho\|^2_{L^2(\Omega)} l^{-1}\{E((n+1)T) + E(nT)\} \\
=\ & C_2'\{E((n+1)T) + E(nT)\},
\end{aligned} \tag{4.34}$$

where

$$C_2' := b\|\rho\|^2_{L^2(\Omega)} l^{-1} > 0.$$

By combining (4.32) – (4.34) into (4.31) , we can write

$$
\begin{aligned}
&-\int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
&+\int_{nT}^{(n+1)T} \left(\xi_0+\xi_1\|\nabla u\|^{2\gamma}_{L^2(\Omega)}\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
\leq\ & [C_2+C_2']\{E((n+1)T)+E(nT)\} \\
&+\frac{1}{4\varepsilon_4}\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt \\
&+(\varepsilon_4+1)\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt. \qquad (4.35)
\end{aligned}
$$

On multiplied (4.24) by γ_1 and multiplied (4.35) by γ_2 and combining suitably, we get

$$
\begin{aligned}
&\left[\gamma_1\left\{t_1h(t_1)-\varepsilon_1\left(1+b^2\right)\right\}-\gamma_2\right]\int_{nT}^{(n+1)T}\|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
&+\gamma_2\xi_0\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt+\gamma_2\xi_1\int_{nT}^{(n+1)T}\|\nabla u\|^{2(\gamma+1)}_{L^2(\Omega)}\,dt \\
\leq\ & \{\gamma_1C_1+\gamma_2[C_2+C_2']\}[E((n+1)T)+E(nT)] \\
&-\gamma_1\frac{1}{4\varepsilon_1}\|\rho\|^2_{L^2(\Omega)}h(0)\int_{nT}^{(n+1)T}\int_\Omega (h'\circ\nabla u)(t)\,dxdt \\
&+\gamma_1\varepsilon_2\quad \xi_0+\xi_1\left(\frac{2(\gamma+1)}{\xi_1}E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}}\Bigg)^2\int_{nT}^{(n+1)T}\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt \\
&+\left\{\gamma_1\xi_0\left\{\frac{1}{4\varepsilon_2}+1+\frac{1}{4\varepsilon_3}+\frac{1}{4\varepsilon_1}\|\rho\|^2_{L^2(\Omega)}\right\}+\gamma_2\frac{1}{4\varepsilon_4}\right\}\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt \\
&+\{\gamma_1\varepsilon_3\xi_0+\gamma_2(\varepsilon_4+1)\}\int_{nT}^{(n+1)T}\left(\int_0^t h(s)\,ds\right)\|\nabla u(t)\|^2_{L^2(\Omega)}\,dt. \qquad (4.36)
\end{aligned}
$$

Let

$$
\begin{cases}
\varepsilon_1 := \dfrac{t_1h(t_1)}{2(1+b^2)}, \\[2ex]
\varepsilon_2 := \dfrac{3\varepsilon\xi_0}{\gamma_1\left(\xi_0+\xi_1\left(\frac{2(\gamma+1)}{\xi_1}E(0)\right)^{\frac{2\gamma}{2(\gamma+1)}}\right)^2}, \\[2ex]
\varepsilon_3 := \dfrac{\varepsilon}{\gamma_1\xi_0}, \\[2ex]
\varepsilon_4 := 2\varepsilon,
\end{cases} \qquad (4.37)
$$

Notes

Notes

and

$$\left\{\begin{array}{l} \gamma_1 := \dfrac{4}{t_1 h\left(t_1\right)}, \\ \\ \quad \gamma_2 := 1, \end{array}\right. \tag{4.38}$$

by using (4.37) and (4.38) into (4.36), we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T} \left\|u_t\left(x,t\right)\right\|_{L_\rho^2(\Omega)}^2 dt \\
&+\xi_0 \int_{nT}^{(n+1)T} \left\|\nabla u\left(t\right)\right\|_{L^2(\Omega)}^2 dt + \xi_1 \int_{nT}^{(n+1)T} \left\|\nabla u\left(t\right)\right\|_{L^2(\Omega)}^{2(\gamma+1)} dt \\
\leq\ & C_3\left[E((n+1)T)+E(nT)\right] \\
&-C_4 \int_{nT}^{(n+1)T} \int_\Omega \left(h' \circ \nabla u\right)\left(t\right) dxdt \\
&+3\varepsilon\xi_0 \int_{nT}^{(n+1)T} \left\|\nabla u\left(t\right)\right\|_{L^2(\Omega)}^2 dt \\
&+C_5 \int_{nT}^{(n+1)T} \int_\Omega \left(h \circ \nabla u\right)\left(t\right) dxdt \\
&+\left(3\varepsilon+1\right) \int_{nT}^{(n+1)T} \left(\int_0^t h\left(s\right) ds\right) \left\|\nabla u\left(t\right)\right\|_{L^2(\Omega)}^2 dt,
\end{aligned} \tag{4.39}$$

where

$$C_3 := \gamma_1 C_1 + C_2 + C_2',$$

$$\left\{\begin{array}{l} C_4 := \gamma_1 \dfrac{\left(1+b^2\right)}{2t_1 h\left(t_1\right)} \left\|\rho\right\|_{L^2(\Omega)}^2 h\left(0\right), \\ \\ C_5 := \frac{4\xi_0}{t_1 h(t_1)} \left\{ \dfrac{\gamma_1 \left. \xi_0+\xi_1 \left(\frac{2(\gamma+1)E(0)}{\xi_1}\right)^{\frac{2\gamma}{2(\gamma+1)}} \right)^2}{12\varepsilon\xi_0} + 1 + \frac{\gamma_1\xi_0}{4\varepsilon} + \frac{\left(1+b^2\right)}{2t_1 h(t_1)} \left\|\rho\right\|_{L^2(\Omega)}^2 \right\} + \frac{1}{8\varepsilon}. \end{array}\right.$$

Adding and subtracting in (4.39) the term

$$-\int_{nT}^{(n+1)T} \int_\Omega \left(\int_0^t h\left(s\right) ds\right) \left|\nabla u\right|^2 dxdt \quad \text{and} \quad \int_{nT}^{(n+1)T} \int_\Omega a\left(x\right)\left(h \circ \nabla u\right)\left(t\right) dxdt,$$

in order to recover the energy $E\left(t\right)$,we obtain

$$\left(1-3\varepsilon\right) \int_{nT}^{(n+1)T} \int_\Omega \left(\xi_0 - \int_0^t h\left(s\right) ds\right) \left|\nabla u\left(t\right)\right|^2 dxdt$$

$$
\begin{aligned}
&+\int_{nT}^{(n+1)T} \|u_t(x,t)\|^2_{L^2_\rho(\Omega)}\,dt \\
&+\xi_1 \int_{nT}^{(n+1)T} \|\nabla u(t)\|^{2(\gamma+1)}_{L^2(\Omega)}\,dt + \int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt \\
&\leq \quad C_3\left[E((n+1)T)+E(nT)\right] \\
&+C_5\int_{nT}^{(n+1)T}\int_\Omega k_1(-h'\circ\nabla u)(t)\,dxdt \\
&+C_5\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt, \qquad (4.40)
\end{aligned}
$$

where

$$k_1 := \frac{C_4}{C_5}.$$

From (4·40), choosing ε sufficiently small, $k_1 > 0$ and T large enough and using

$$
\begin{aligned}
&\alpha_1\left\{\|u_t(t)\|^2_{L^2_\rho(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^{2(\gamma+1)}_{L^2(\Omega)} + \int_\Omega (h\circ\nabla u)(t)\,dx\right\} \\
\leq \quad & E(t) \leq \alpha_2\left\{\|u_t(t)\|^2_{L^2_\rho(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^{2(\gamma+1)}_{L^2(\Omega)} + \int_\Omega (h\circ\nabla u)(t)\,dx\right\},
\end{aligned}
$$

where

$$
\begin{cases}
\alpha_1 := \dfrac{1}{2}\min\left\{1, l, \dfrac{\xi_1}{(\gamma+1)}\right\}, \\
\qquad\qquad \text{and} \\
\alpha_2 := \dfrac{1}{2}\max\left\{1, \xi_0, \dfrac{\xi_1}{(\gamma+1)}\right\},
\end{cases}
$$

we get

$$
\begin{aligned}
\int_{nT}^{(n+1)T} E(t)\,dt \quad \leq \quad & C_6\left[E((n+1)T)+E(nT)\right] \\
& +C_7\int_{nT}^{(n+1)T}\int_\Omega (h\circ\nabla u)(t)\,dxdt \\
& +C_7\int_{nT}^{(n+1)T}\int_\Omega k_1(-h'\circ\nabla u)(t)\,dxdt, \qquad (4.41)
\end{aligned}
$$

where

$$
\begin{cases}
C_6 := \dfrac{\alpha_2 C_3}{\min\{(1-3\varepsilon)l, 1, \xi_1\}}, \\
\qquad\qquad \text{and} \\
C_7 := \dfrac{\alpha_2 C_5}{\min\{(1-3\varepsilon)l, 1, \xi_1\}}.
\end{cases}
$$

Notes

Notes

In the last step, we need to relate the viscoelastic energy to the viscoelastic damping. In the case when the relaxation function obeys a linear equation, this relation is straightforward and is expressed by a suitable multiplication. However, in the case of general decays, additional arguments are used. Here, we follow [17]. From the **assumption 2** made on the viscoelastic kernel h and from [17, **Lemma 4**] we obtain

$$(h \circ \nabla u)(t) \leq \hat{H}_{\alpha}^{-1}(-h' \circ \nabla u)(t), \quad t \in [nT, (n+1)T], \tag{4.42}$$

where $\hat{H}_{\alpha}$ is a rescaling of H_{α} with

$$H_{\alpha}(s) = \alpha s^{1-\frac{1}{\alpha}} H\left(s^{\frac{1}{\alpha}}\right),$$

and $\alpha \in (0,1)$ is such that

$$\sup_{t>0} \int_0^t h^{1-\alpha}(t-s) \|\nabla u(t) - \nabla u(s)\|^2 ds < \infty.$$

From **Assumption 2** it is clear that $\alpha \geq \alpha_0$. The main point, however, is that the argument can be reiterated (based on [16, **Lemma 8**] leading to $\alpha = 1$). This allows us to replace H_{α}, the function in (4.42), by the original function $\hat{H}$ which is a rescaling of $H(s)$. This means that $\hat{H} = cH\left(\frac{C}{s}\right)$ for some c, $C > 0$. Now, from (4.42) and taking (4.41) into account, we deduce that

$$\begin{aligned}\int_{nT}^{(n+1)T} E(t)\,dt \quad &\leq \quad C_6\left[E((n+1)T) + E(nT)\right] \\ &+ C_7 \int_{nT}^{(n+1)T} \int_{\Omega} \left[\hat{H}^{-1} + k_1\right](-h' \circ \nabla u)(t)\,dx dt. \end{aligned}\tag{4.43}$$

Next, we shall employ the following version of **Jensen's inequality** applied to measures and convex functions F. Let F be a convex increasing function on $[\alpha, b]$, let $f : \Omega \longmapsto [\alpha, b]$, and let h be an integrable function such that $h(x) \geq 0$ and

$$\int_{\Omega} h(x)\,dx = h_0 > 0.$$

Then, we have

$$\int_{\Omega} F^{-1}(f(x))\,h(x)\,dx \leq h_0 F^{-1}\left[h_0^{-1} \int_{\Omega} f(x)\,h(x)\,dx\right]. \tag{4.44}$$

We shall use (4.44) in order to bring the functions H in front of the integrals. Let us denote

$$\alpha_0 := meas(\Omega).$$

We note that the function $\hat{H}^{-1}+k_1$ is concave.
Let

$$\begin{cases} F^{-1}=\hat{H}^{-1}+k_1, \\ f(x)=(-h'\circ\nabla u)(t), \\ h(x)=T, \\ h_0=T\alpha_0, \\ h_0^{-1}=\alpha_0^{-1}T^{-1}, \end{cases}$$

thus, we have

$$\begin{aligned} &\int_{nT}^{(n+1)T}\int_\Omega\left[\hat{H}^{-1}+k_1\right](-h'\circ\nabla u)(t)\,dxdt \\ \leq\ &\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_\Omega(-h'\circ\nabla u)(t)\,dxdt\right]. \end{aligned} \tag{4.45}$$

On the other hand, from the identity (2.4) for the energy, we can write

$$\begin{aligned} &E((n+1)T)-E(nT) \\ =\ &\frac{1}{2}\int_{nT}^{(n+1)T}\left\{\int_\Omega(h'\circ\nabla u)(t)\,dx-h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}-2b\|u_t(t)\|^2_{L^2_\rho(\Omega)}\right\}dt \\ =\ &-\int_{nT}^{(n+1)T}D(t)\,dt, \end{aligned}$$

where

$$D(t):=\frac{1}{2}\left\{\int_\Omega(-h'\circ\nabla u)(t)\,dx+h(t)\|\nabla u(t)\|^2_{L^2(\Omega)}+2b\|u_t(t)\|^2_{L^2_\rho(\Omega)}\right\}. \tag{4.46}$$

En replacement (4.45) into (4.43) and using

$$E(nT)=E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt, \tag{4.47}$$

we get

$$\begin{aligned} &\int_{nT}^{(n+1)T}E(t)\,dt \\ \leq\ &C_6\left\{2E((n+1)T)+\int_{nT}^{(n+1)T}D(t)\,dt\right\} \\ &+C_7\alpha_0T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T}\int_\Omega(-h'\circ\nabla u)(t)\,dxdt\right], \end{aligned}$$

and using (4.46) we get

$$\int_{nT}^{(n+1)T}\int_\Omega(-h'\circ\nabla u)(t)\,dxdt\leq 2\int_{nT}^{(n+1)T}D(t)\,dt,$$

Notes

thus, we get

$$\begin{aligned}
&\int_{nT}^{(n+1)T} E(t)\,dt \\
\leq\;& C_6\left\{2E((n+1)T)+\int_{nT}^{(n+1)T} D(t)\,dt\right\} \\
&+2C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T} D(t)\,dt\right] \\
=\;& 2C_6E((n+1)T)+C_6\int_{nT}^{(n+1)T} D(t)\,dt \\
&+2C_7\alpha_0 T\left[\hat{H}^{-1}+k_1\right]\left[\alpha_0^{-1}T^{-1}\int_{nT}^{(n+1)T} D(t)\,dt\right] \\
\leq\;& 2C_6E((n+1)T)+C_8\left[\hat{H}^{-1}+k_2\right]\left[\int_{nT}^{(n+1)T} D(t)\,dt\right] \\
=\;& 2C_6E((n+1)T)+C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right], \qquad (4.48)
\end{aligned}$$

where

$$\begin{cases}
C_8 := \max\{2C_7, 1\}, \\
\tilde{H} := \left[\hat{H}^{-1}+k_2\right]^{-1}, \\
k_2 := (C_6+2C_7k_1).
\end{cases} \qquad (4.49)$$

By integrating t to $(n+1)T$ on both sides of the inequality $\frac{d}{dt}\{E(t)\} \leq 0$ yields

$$E((n+1)T) \leq E(t) \quad \text{for all } (n+1)T \geq t, \qquad (4.50)$$

integrating (4.50) from nT to $(n+1)T$ yields

$$\begin{aligned}
\int_{nT}^{(n+1)T} E(t)\,dt \;&\geq\; \int_{nT}^{(n+1)T} E((n+1)T)\,dt \\
&=\; \int_{nT}^{(n+1)T} dt E((n+1)T) \\
&=\; TE((n+1)T), \qquad (4.51)
\end{aligned}$$

by replacement (4.51) into (4.48), we get

$$TE((n+1)T) \leq 2C_6E((n+1)T)+C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right],$$

then

$$(T-2C_6)E((n+1)T) \leq C_8\tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D(t)\,dt\right].$$

For T large enough, where C_6 is a positive constant, which implies that

$$E\left((n+1)T\right) \leq C_9 \tilde{H}^{-1}\left[\int_{nT}^{(n+1)T} D\left(t\right)dt\right],$$

where

$$C_9 := \frac{C_8}{(T-2C_6)}, \tag{4.52}$$

which gives that

$$\tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq \int_{nT}^{(n+1)T} D\left(t\right)dt, \tag{4.53}$$

by using (4.47) into (4.53), we get

$$\tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq E\left(nT\right) - E\left((n+1)T\right),$$

from the above we have

$$E\left((n+1)T\right) + \tilde{H}\left(C_9^{-1}E\left((n+1)T\right)\right) \leq E\left(nT\right), \quad n = 1,2,3....$$

Then the Proof of **Lemma 2** is complete.

Lemma 3. Let p be a positive, increasing function such that $p(0) = 0$. Since p is increasing, we can define an increasing function q, $q(x) \equiv x-(I+p)^{-1}(x)$. Consider a sequence F_n of positive numbers which satisfies

$$F_{m+1} + p\left(F_{m+1}\right) \leq F_m. \tag{4.54}$$

Then $F_m \leq S(m)$ where $S(t)$ is a solution of the differential equation

$$\frac{d}{dt}\left\{S\left(t\right)\right\} + q\left(S\left(t\right)\right) = 0, \quad S\left(0\right) = F_0. \tag{4.55}$$

Moreover, if $p(x) > 0$ for $x > 0$ then $\lim_{t\longrightarrow\infty} S(t) = 0$.

Proof. Proof of the Lemma use the proof retraction. Assume $F_m \leq S(m)$ and prove that $F_{m+1} \leq S(m+1)$.
Inequality (4.54) is equivalent to

$$(I+p)\,F_{m+1} \leq F_m,$$

and since $(I+p)^{-1}$ is monotone increasing, $F_{m+1} \leq (I+p)^{-1} F_m$, and using

$$(I+p)^{-1} F_m = (I-q)\,F_m,$$

we get

$$\begin{aligned} F_{m+1} &\leq (I-q)\,F_m \\ &= F_m - q\left(F_m\right). \end{aligned} \tag{4.56}$$

On the other hand, since q is an increasing function, the solution $S(t)$ of equation (4.55) is described by a nonlinear contraction.

In particular integrating $\frac{d}{dt}\{S(t)\} \leq 0$ from m to τ yields

$$S\left(\tau\right) \leq S\left(m\right) \quad \text{for all } t \geq \tau. \tag{4.57}$$

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Integrating equation (4.55) from m to $(m+1)$ yields

$$S(m+1) - S(m) + \int_m^{m+1} q(S(\tau))\, d\tau = 0. \tag{4.58}$$

Since q is increasing, by using (4.57) we obtain for all $m \leq \tau \leq m+1$

$$\begin{aligned} \int_m^{m+1} q(S(\tau))\, d\tau &\leq \int_m^{m+1} q(S(m))\, d\tau \\ &= q(S(m)) \int_m^{m+1} d\tau \\ &= q(S(m)), \end{aligned}$$

then

$$-\int_m^{m+1} q(S(\tau))\, d\tau \geq -q(S(m)), \quad \text{for all } m \leq \tau \leq m+1, \tag{4.59}$$

by replacement (4.59) into (4.58) and using the inductive assumption $F_m \leq S(m)$, we get

$$\begin{aligned} S(m+1) &\geq S(m) - q(S(m)) \\ &= (I-q)\, S(m) \\ &\geq (I-q)\, F_m \\ &= F_m - q(F(m)), \end{aligned} \tag{4.60}$$

comparing (4.60) with (4.56) yields

$$S(m+1) \geq F_{m+1}.$$

Then the Proof of **Lemma 3** is complete.

*Theorem 2. Let us assume that **Assumption 1** and **Assumption 2** ar the place. Then there exist positive constants c_1, c_2 and T_0 such that the solution of problem $(1.1)-(1.3)$ satisfies $E(t) \leq s(t)$, where $s(t)$ verifies the ODE*

$$s_t + \hat{H}(s) = 0, \quad s(0) = E(0), \ t \geq T_0 > 0,$$

with $\hat{H}(s) = c_1 H(c_2 s)$.

Proof. Thus, we are in a position to apply the result of **Lemma 2** with

$$F_m \equiv E(mt), \quad F_0 \equiv E(0).$$

This yields

$$E(mT) \leq S(m), \quad m = 0, 1, 2, 3....$$

Setting $t = mT + \tau$ and recalling the evolution property gives

$$E(t) \leq E(mT) \leq S(m) \leq S\left(\frac{t-\tau}{T}\right) \leq S\left(\frac{t}{T} - 1\right),$$

which completes the proof of **Theorem 2.**

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Solution of a Transportation Problem using Bipartite Graph

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Abstract- The transportation problem is also one of the important problems in the field of optimization in which the goal is to minimize the total transportation cost of distributing to a specific number of sources to a specific number of destinations. Different techniques have been developed in the literature for solving the transportation problem. Specific methodologies concentrated on finding an initial basic feasible solution and the other to find the optimal solution. This manuscript analyses method of the optimal solution for the transportation problem utilizing a Bipartite graph. This procedure contains topological spaces, graphs, and transportation problems. Initially, it converts the transportation problem into a graphical demonstration then transforms into a new graphical image. Afterward using the proposed algorithmic rule we've obtained the optimal cost of transporting quantities from providing vertices to supply vertices.

Keywords: *transportation problem, bipartite graph, balanced and unbalanced, VAM and MODI methods.*

GJSFR-F Classification: *MSC 2010: 00A79*

SOLUTIONOFATRANSPORTATIONPROBLEMUSINGBIPARTITEGRAPH

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Solution of a Transportation Problem using Bipartite Graph

Ekanayake E. M. U. S. B α, Daundasekara W. B σ & Perera S. P. C ρ

Abstract- The transportation problem is also one of the important problems in the field of optimization in which the goal is to minimize the total transportation cost of distributing to a specific number of sources to a specific number of destinations. Different techniques have been developed in the literature for solving the transportation problem. Specific methodologies concentrated on finding an initial basic feasible solution and the other to find the optimal solution. This manuscript analyses method of the optimal solution for the transportation problem utilizing a Bipartite graph. This procedure contains topological spaces, graphs, and transportation problems. Initially, it converts the transportation problem into a graphical demonstration then transforms into a new graphical image. Afterward using the proposed algorithmic rule we've obtained the optimal cost of transporting quantities from providing vertices to supply vertices. The above approach shows that the relation between the transportation problem and graph theory and it initiates to search out the various kind of solutions to the transportation problem. This method is also to be noticed that, requires the least number of steps to reach optimality as compare the obtained results with other well-known meta-heuristic algorithms. In the end, this method is illustrated with a numerical example.

Keywords: transportation problem, bipartite graph, balanced and unbalanced, VAM and MODI methods.

I. Introduction

Network models are one in every of the most effective studies that apply to a vast type of decision problems that can be modeled as networks optimization problems and solved with efficiency and effectiveness. The family of network optimization problems includes the; max flow, transportation problem, and min-cost flow problems. These problems are simply expressed by using a network of edges, and vertices. Transportation Network and Graph theory are the two major elementary application areas of Mathematics. Transportation Network models and graphs play a very important role in Optimizing techniques, Network analysis, Network-flow theory is one of the best-studied and developed fields of optimization, and has important relations to quit completely different fields of science and technology such as combinatorial mathematics, algebraical topology, circuit theory, geographic info systems(GIS), VLSI design, and so forth, etc, besides standard applications to transportation, scheduling, etc. in operations research.

In 2005, Antonievella[1] initiate and introduced the foundations of topological properties on graph theory. Consequently, Vimala and Kalpana [5] developed the concept named Bipartite Graph and applied it in Matching and Coloring. In recently, 2019, Introduced Topological solution of a Transportation problem using Topologized

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Graph by Santhi et al. In 2015, Kadhim et al. An Approach for solving Transportation Problem Using Modified Kruskal's Algorithm. In a network with unit transportation cost on the edges, the problem is to determine the maximum possible flow from the source to the demand. Also, Transportation problems link along with the factors of production during an advanced net of relationships between producers and consumers. The result is usually a more effective division of production by the exploitation of comparative geographical ideal conditions, similarly as the best approach to make economies of scale and scope.

The productivity of space, capital, and labor is therefore increased with the efficiency of distribution and personal mobility. The economic process is progressively connected with transport developments, namely infrastructures, but also with managerial expertise, which is crucial for logistics.

The Transportation Problem (TP) is also one of the highly regarded problems in the field of optimization in which the objective is to minimize the total transportation cost of distributing resources from several sources to some destinations. It has numerous applications in the real world. Hitchcock is responsible for formulating the TP as a mathematical model. The Hitchcock-Koopmans transportation problem, or basically the transportation problem is to compute an assignment with a minimum possible cost. To handle a transportation problem, the decision parameters, for example, availability, requirement, and therefore the unit transportation cost of the model. Many of the researchers mentioned and introduced so many methods to find the optimal solution to a Transportation problem.

Many researchers have made numerous attempts to find an IBFS such as Northwest Corner Method, Minimum Cost Method, VAM -Vogel's Approximation Method, MODI Method, and Stepping Stone Method which are all heuristic in nature. In this study, we attempted to solve the TP using a Bipartite graph to enhance the convergence rate to reach a promising optimal solution. This algorithm is also heuristic in nature but less complicated in the implementation compared to many existing heuristic algorithms.

II. Mathematical Formulation of the Transportation Problem

Let us assume that in general that a particular product is manufactured in m production plants known as sources denoted by $S_1, S_2, \dots, S_m$ with respective capacities $a_1, a_2, \dots, a_m$, and total distributed to n distribution centers known as sinks denoted by $D_1, D_2, \dots, D_m$ with respective demands $b_1, b_2, \dots, b_n$. Also, assume that the transportation cost from i[th] - source to the j[th] - sink is unit transportation cost e_{ij} and the amount shipped is X_{ij}, where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Mathematical Model:

The total transportation cost is

Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}\, e_{ij}$

Subject to the constraints

i. $\sum_{j=1}^{n} X_{ij} = a_i, \qquad i = 1,2,\dots,m$

ii. $\sum_{i=1}^{m} X_{ij} = b_j, \qquad j = 1,2,\dots,n$ and

iii. $X_{ij} \geq 0$ for all $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$

Notes

Note that here the sum of the supplies equals the sum of the demands. i.e. $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. Such problems are called balanced transportation problems and otherwise, i.e. $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, known as unbalanced transportation problems.

i. $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$

ii. $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$

Introduce a dummy origin in the transportation table; the cost associated with this origin is set equal to zero. The availability at this origin is: $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j = 0$.

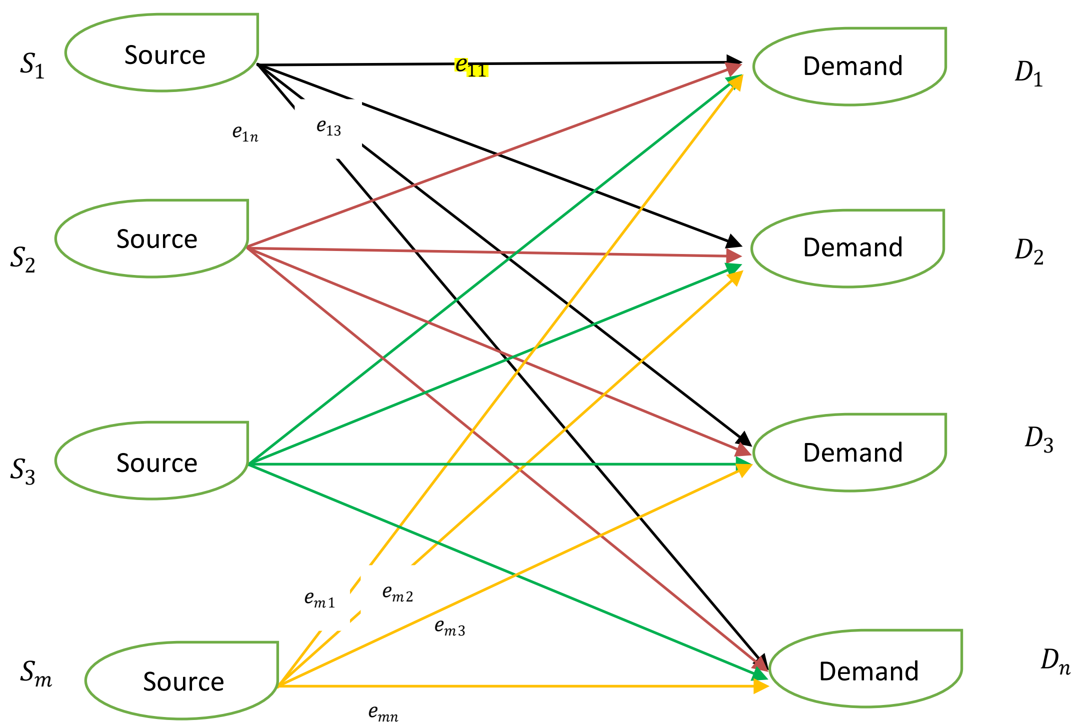
III. Proposed Algorithm to Solve the TP

The proposed method can be applied to solve balanced and unbalanced TPs.

Step 1: Verify the given transportation problem is balanced or unbalance.

Step 2: If the problem is unbalanced transportation problem by introducing dummy row(s) or dummy column(s) with zero transportation cost.

Step 3: Draw the graph of the transportation problem dependent on the situation of the supplies and demands for the graphical representation of the transportation problem.



Step 4: Now selected bipartite graph which every Supply and demand of the graph has two minimum unit cost.

Step 5: Identify edges should have the minimum unit cost e_{ij} (unit transportation cost) in the above step and first allocated mini(a_i, b_j) most least unit cost edge.

Step 6: Start the allocation from which edge has the minimum transportation cost and reduce the minimum value from the supply vertex and demand vertex with satisfies boundary condition of the bipartite graph.

Step 7: If it satisfies the two conditions of graph go to the next step.

Step 8: Identify edges should have the minimum unit cost e_{ij} (unit transportation cost) in above step and first allocated mini(a_i, b_j) most least unit cost edge of above step, and reduce the minimum value from the supply vertex and demand vertex with satisfies boundary condition of the bipartite graph.

IV. A Comparison of the Methods

The comparisons of the results are studied in this research to measure the effectiveness of the proposed method. The detailed representation of the numerical data of Table I. is provided in Appendix I.[4].

Table 1: Comparative results of NWCM, LCM, VAM, IAM and New Approach (NEWA) for 10 benchmark instances

	TCIFS						% increase	from	the	minimal	total cost
Ahamd et al..(2016)	NWCM	LCM	VAM	IAM	BA	OPTIMAL	NWCM	LCM	VAM	IAM	NEWA
BTP-1	1,500	1,450	1,500	1,390	1,390	1,390	7.91	4.31	7.91	0.00	0.00
BTP-2	226	156	156	156	156	156	44.87	0.00	0.00	0.00	0.00
BTP-3	234	191	187	186	183	183	27.87	4.37	2.18	1.64	0.00
BTP-4	4,285	2,455	2,310	2,365	2,170	2,170	97.46	13.13	6.45	8.99	0.00
BTP-5	3,180	2,080	1,930	1,900	1,900	1,900	67.37	9.47	1.58	0.00	0.00
UTP-1	1,815	1,885	1,745	1,695	1,655	1,650	10.0	14.24	5.76	2.73	0.30
UTP-2	18,800	8,800	8,350	8,400	7,100	7,100	142.6	13.55	7.74	8.39	0.00
UTP-3	14,725	14,625	13,225,	13,075	12,475	12,475	18.04	17.23	6.01	4.80	0.00
UTP-4	13,100	9,800	9,200	9,200	9,200	9,200	42.39	6.52	0.00	0.00	0.00
UTP-5	8,150	6,450	6,000	5,850	5,600	5,600	45.53	15.18	7.14	4.46	0.00

The comparative results obtained in Table I are also depicted using bar graphs and the results are given in Figure 1.

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Notes

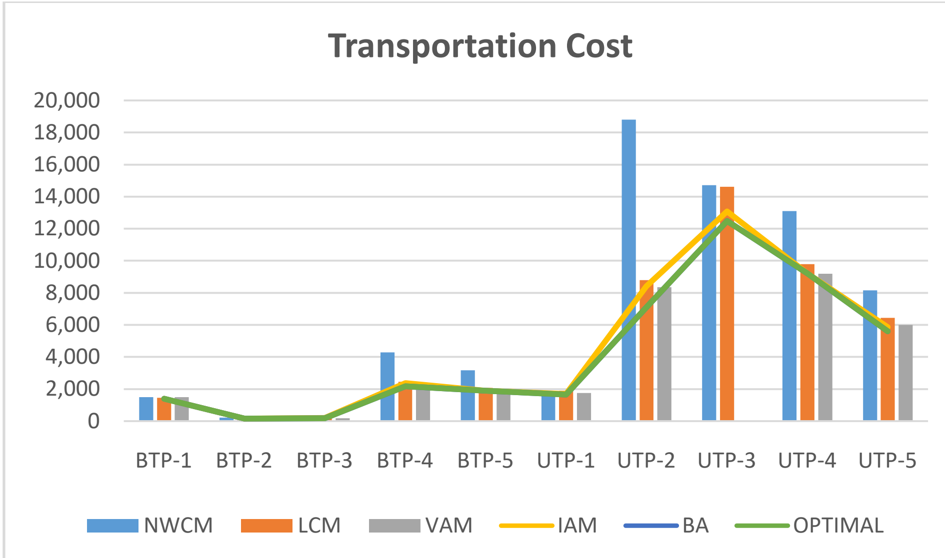


Figure 1: Comparative Stud of the Result obtained by NWCM, LCM, VAM, IAM and BA method

Radar graphs for the percentage deviation (of the NWCM, MC, VAM,TDM,TDSM, VAM) with New method(BA) from minimal total cost solution) obtained in Table I are presented in Figure 2.

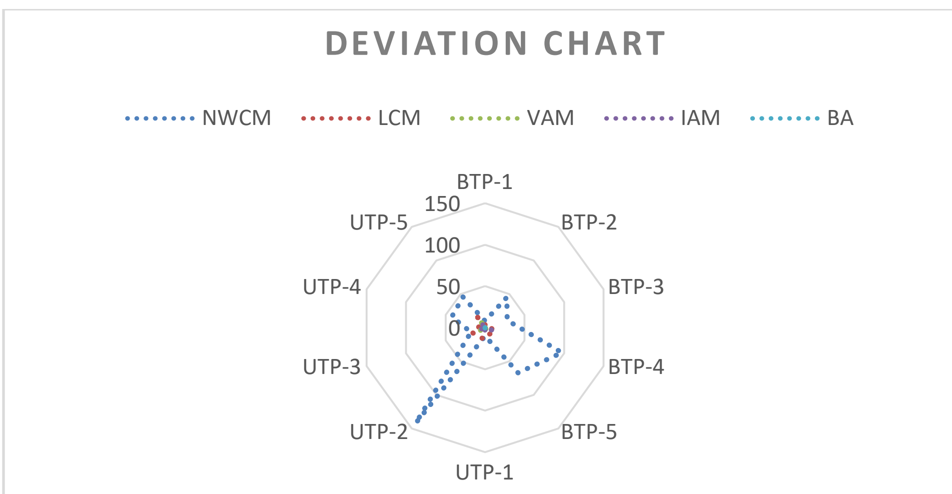


Figure 2: Percentage of Deviation of the Results obtained by NWCM,LCM, VAM,IAM and BA method

It can easily be observed the above results (Table 1, Figure 2 and Figure 3), new method yields better results to all the problems in Table 1 compared with NWCM, LCM, VAM and IAM.

Next comparative results obtained by NWCM, LCM, VAM, MODI and New method for the one benchmark instances is shown in the following Table II. (*Kenan Karagul and Yusuf Sahin).*

Destination/ Sources	D_1	D_2	D_3	D_4	D_5	Su.
S_1	73	40	9	79	20	8
S_2	62	93	96	8	13	7
S_3	96	65	80	50	65	9
S_4	57	58	29	12	87	3
S_5	56	23	87	18	12	5
Dem.	6	8	10	4	4	

The comparisons of the results are studied in this research to measure the effectiveness of the proposed method. The detailed representation of the numerical data of Table II. is provided in Appendix I.[4].

Table 2: Comparative results of NWCM, LCM, VAM, IAM and New Approach (NEWA) for 10 benchmark instances

Solution Method	Values	Deviation from optimal solution(%)
KSAM	1,102	0.00
RAM	1.104	0.18
	1,104	0.18
RM	1.123	1.90
MM	1,123	1.90
CLM	1,491	35.29
TCM	1,927	74.86
NWC	1,994	80.94
BA	1,102	0.00
OPTIMAL	1,102	-

The comparative results obtained in Table II are also depicted using bar graphs and the results are given in

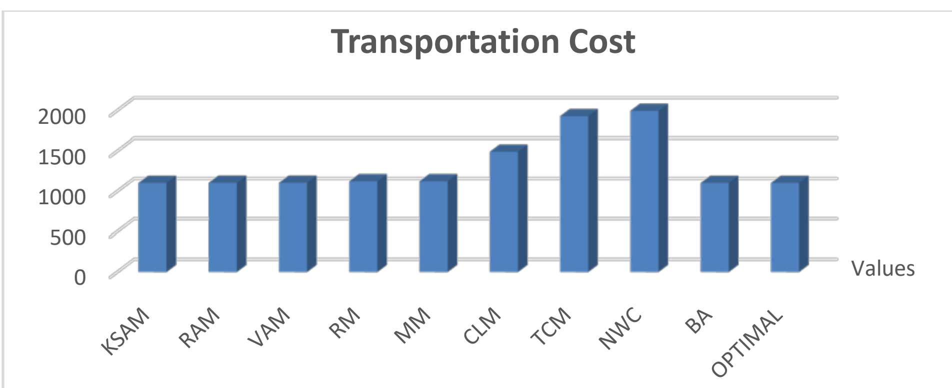


Figure 1

V. Computational Results (Problem Choose from Santhi Et Al)

Example1.

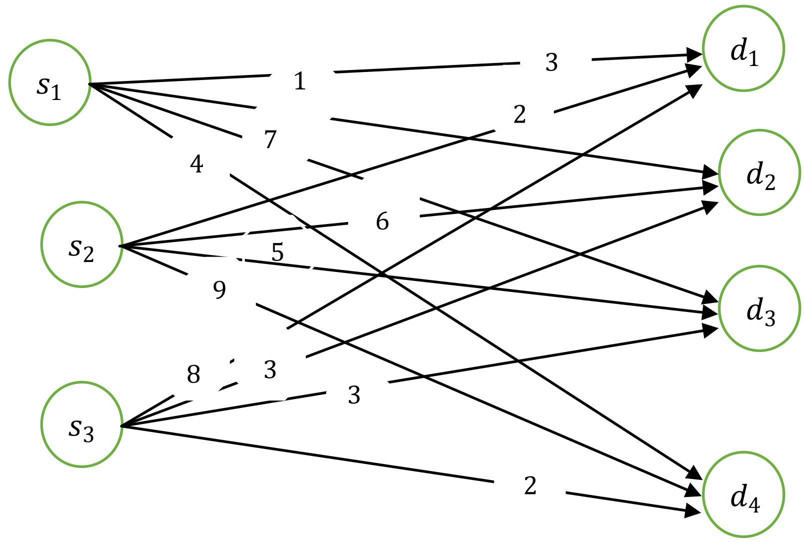
A transport company is planning to allocate owned vehicles to cities A, B and C. Here are the transport tables that have been prepared by managers of the company which gives the transportation cost from warehouses (Supply Points) to the cities(Demand Points).

	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	

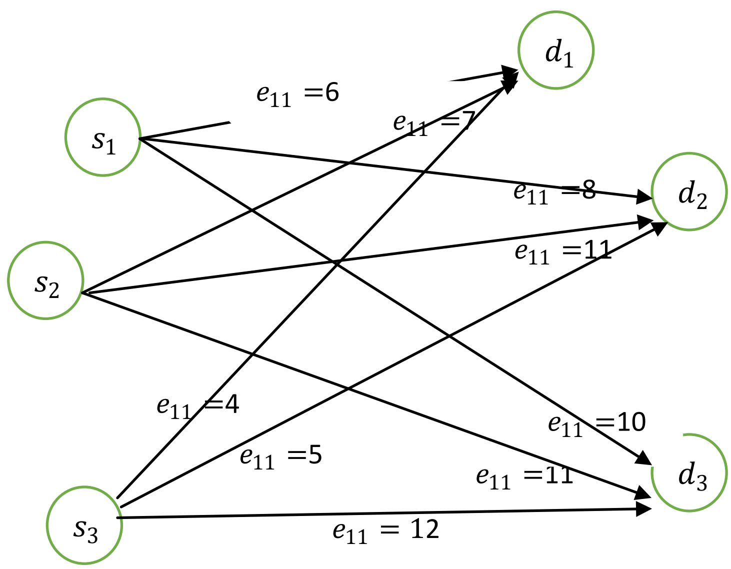
Ref

4. Ahmed, M. M.; Khan, A. R.; Uddin, S.; and Ahme, F.(2016). A New Approach to Solve Transportation Problems", Open Journal of Optimization, 5, pp. 22-30.

Step 2



Step 3.



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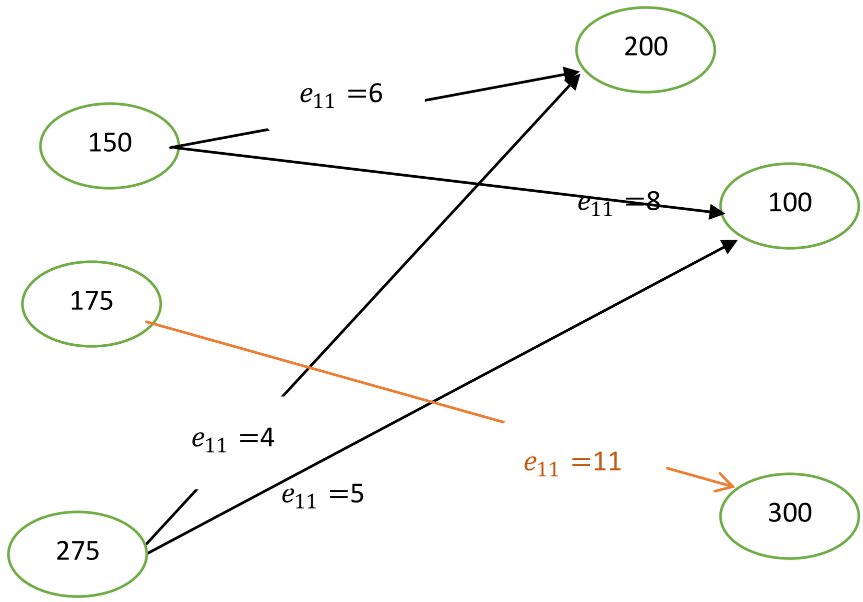
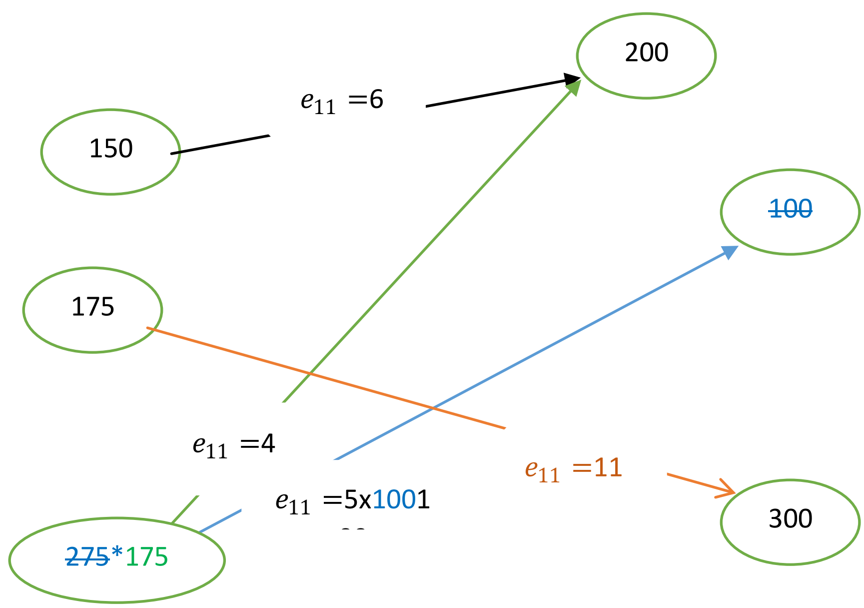
Step 4.



Step 5



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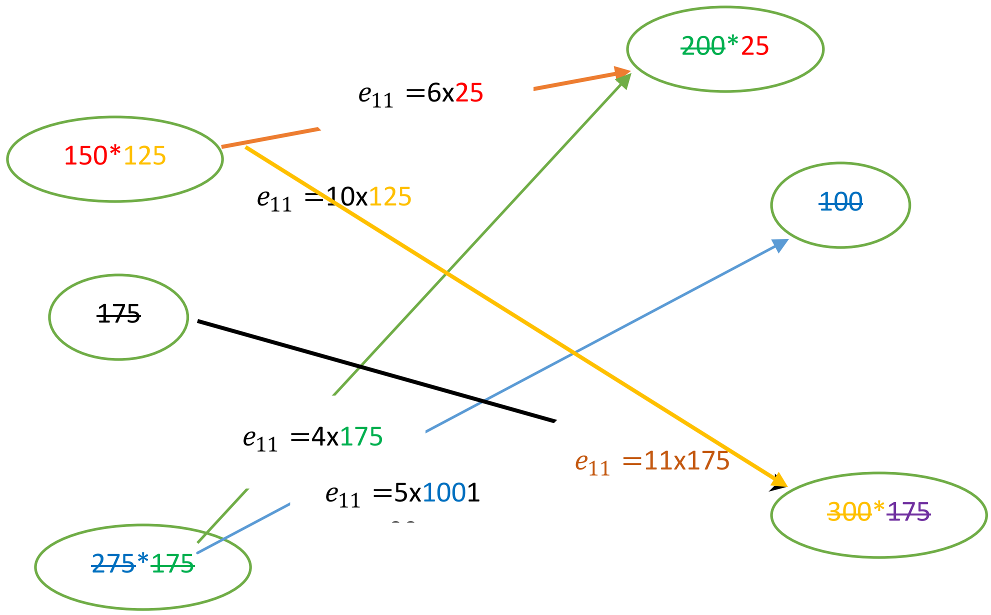
Step 6



Minimum cost = 5x100+4x175+6x25+10x125+11x175=4,525

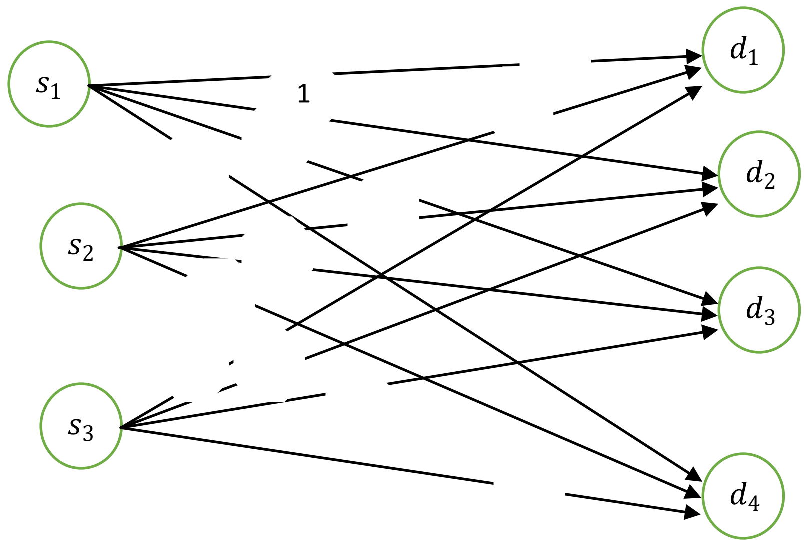
Santhi method = 4,550

Optimal solution = 4,525

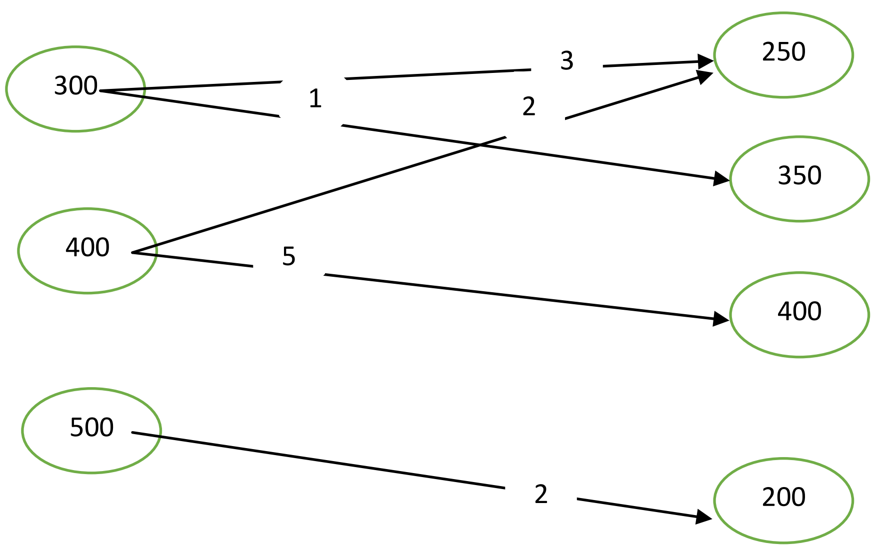
Example 2. A company manufactures motor cars and it has three factories F1, F2 and F3 whose weekly production capacities are 300, 400 and 500 pieces of cars respectively. The company supplies motor cars to its four showrooms located at d1, d2, d3 and d4 whose weekly demands are 250, 350, 400 and 200 pieces of cars respectively. The transportation costs per piece of motor cars are given in the following transportation Table. Find out the schedule of shifting of motor cars from factories to showrooms with minimum cost:

	d_1	d_2	d_3	d_4	
s_1	3	1	7	4	300
s_2	2	6	5	9	400
s_3	8	3	3	2	500
	250	350	400	200	

Step 2



Step 3



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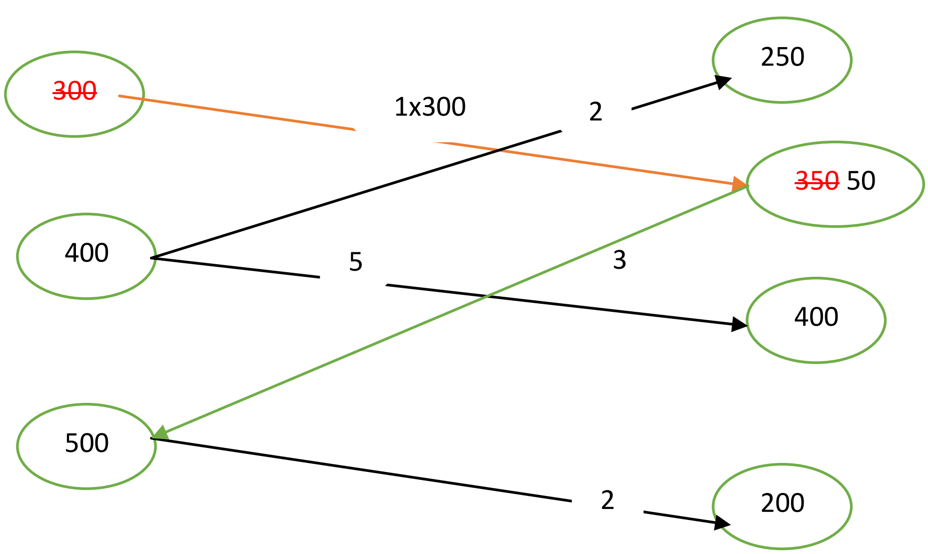

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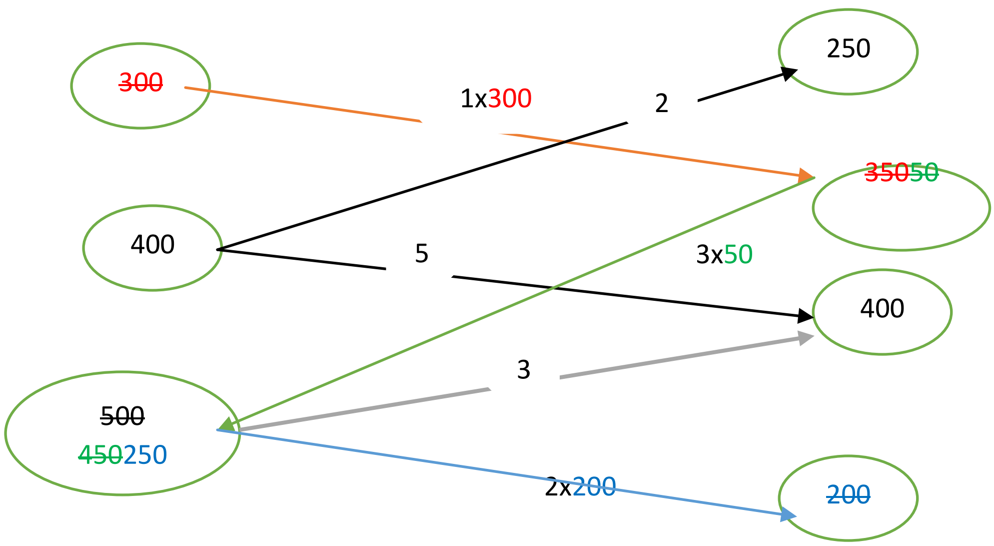

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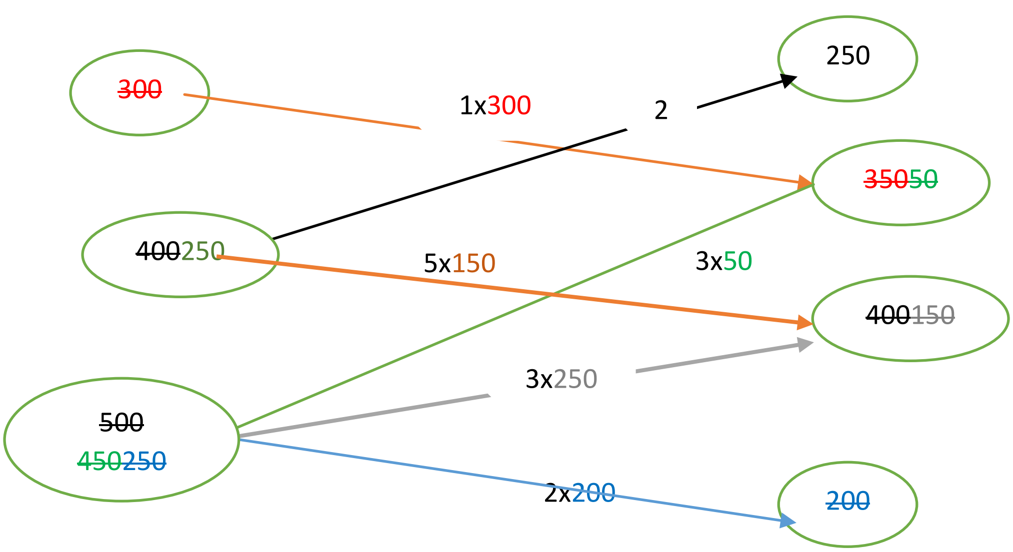

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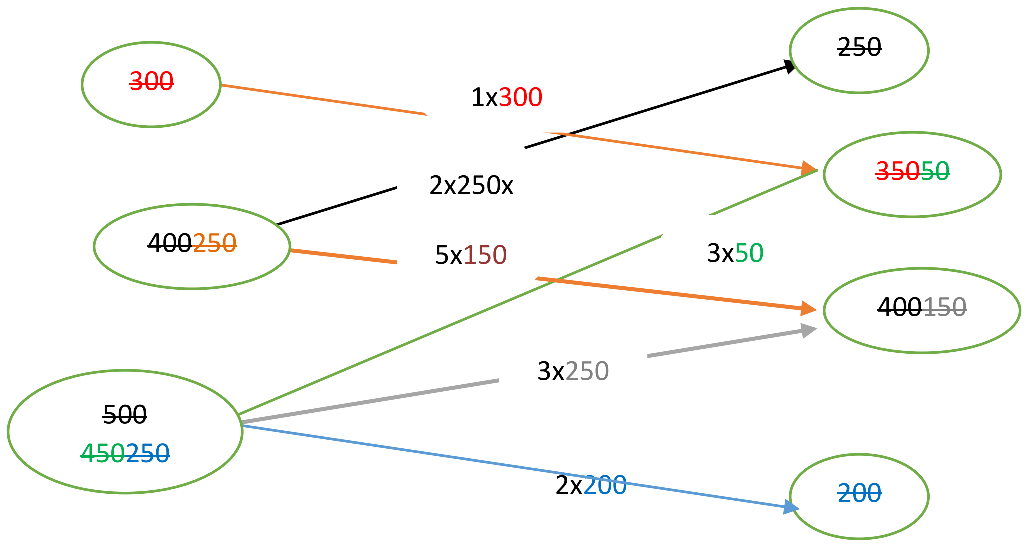
Step 7



Minimum cost=1x300+3x50+2x200+3x250+5x150+2x250=2,850

Santhi method=2,850

Optimal solution=2,850

Based on the above results new method (BA) better than other approaches.

VI. Conclusion

In this study, a new approach for attaining the optimal solution of a transportation problem using the Bipartite graph. Different techniques have been developed in the literature for solving the transportation problem but this approach plays an important role among topology, transportation, and graph. The comparative assessments of the above different cases show that the bipartite algorithm is efficient as compared to the studied approaches of this paper in terms of the quality of the solution. This innovative approach consumes less computational time and minimum steps to find the optimal solution to the transportation problem compared with the existing methods. However, This new method is based on the allocation of transportation costs in the transportation matrix and can be applied to all balance and unbalance transportation problems, using more variables. Hence, the comparative assessments of the above different cases show that the bipartite algorithm is efficient as compared to the studied approaches of this paper in terms of the quality of the solution. Therefore, perhaps this method will be interested in future works in real topological transportation problems, and graph and topological transportation problems are interrelationships.

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Notes

Appendix I

Problem	Data of the problem
BTP-1	$c_{ij} = [4,3,5; 6,5,4; 8,10,7], s_i = [90,80,100], d_j = [70,120,80]$
BTP-2	$c_{ij} = [4,6,9,5; 2,6,4,1; 5,7,2,9], s_i = [16,12,15], d_j = [12,14,9,8]$
BTP-3	$c_{ij} = [5,7,10,5,3; 8,6,9,12,14; 10,9,8,10,15], s_i = [5,10,10], d_j = [3,3,10,5,4]$
BTP-4	$c_{ij} = [12,4,13,18,9,2; 9,16,10,7,15,11; 4,9,10,8,9,7; 9,3,12,6,4,5; 7,11,15,18,2,7; 16,8,4,5,1,10]$, $s_i = [120,80,50,90,100,60], d_j = [75,85,140,40,95,65]$
BTP-5	$c_{ij} = [12,7,3,8,10,6,6; 6,9,7,12,8,12,4; 10,12,8,4,9,9,3; 8,5,11,6,7,9,3; 7,6,8,11,9,5,6,]$ $s_i = [60,80,70,100,90], d_j = [20,30,40,70,60,80,100]$
UTP-1	$c_{ij} = [6,10,14; 12,19,21; 15,14,17], s_i = [50,50,50], d_j = [30,40,55]$
UTP-2	$c_{ij} = [10,8,4,3; 12,14,20,2; 6,9,23,25], s_i = [500,400,300], d_j = [250,350,600,150]$
UTP-3	$c_{ij} = [12,10,6,13; 19,8,16,25; 17,15,15,20; 23,22,26,12], s_i = [150,200,600,225], d_j = [300,500,75,100]$
UTP-4	$c_{ij} = [5,8,6,6,3; 4,7,7,6,5; 8,4,6,6,4], s_i = [800,500,900], d_j = [400,400,500,400,800]$
UTP-5	$c_{ij} = [5,4,8,6,5; 4,5,4,3,2; 3,6,5,8,4], s_i = [600,400,1,000], d_j = [450,400,200,250,300]$

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Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

Format Structure

It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

Title

The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

Abbreviations

Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

For scanned images, the scanning resolution at final image size ought to be as follows to ensure good reproduction: line art: >650 dpi; halftones (including gel photographs): >350 dpi; figures containing both halftone and line images: >650 dpi.

Color charges: Authors are advised to pay the full cost for the reproduction of their color artwork. Hence, please note that if there is color artwork in your manuscript when it is accepted for publication, we would require you to complete and return a Color Work Agreement form before your paper can be published. Also, you can email your editor to remove the color fee after acceptance of the paper.

Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

4. Use of computer is recommended: As you are doing research in the field of science frontier then this point is quite obvious. Use right software: Always use good quality software packages. If you are not capable of judging good software, then you can lose the quality of your paper unknowingly. There are various programs available to help you which you can get through the internet.

5. Use the internet for help: An excellent start for your paper is using Google. It is a wondrous search engine, where you can have your doubts resolved. You may also read some answers for the frequent question of how to write your research paper or find a model research paper. You can download books from the internet. If you have all the required books, place importance on reading, selecting, and analyzing the specified information. Then sketch out your research paper. Use big pictures: You may use encyclopedias like Wikipedia to get pictures with the best resolution. At Global Journals, you should strictly follow here.

6. Bookmarks are useful: When you read any book or magazine, you generally use bookmarks, right? It is a good habit which helps to not lose your continuity. You should always use bookmarks while searching on the internet also, which will make your search easier.

7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- o You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- o Give details of all of your remarks as much as possible, focusing on mechanisms.
- o Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- o One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- o Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

The Administration Rules

Administration Rules to Be Strictly Followed before Submitting Your Research Paper to Global Journals Inc.

Please read the following rules and regulations carefully before submitting your research paper to Global Journals Inc. to avoid rejection.

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Criterion for Grading a Research Paper (Compilation) by Global Journals

Please note that following table is only a Grading of "Paper Compilation" and not on "Performed/Stated Research" whose grading solely depends on Individual Assigned Peer Reviewer and Editorial Board Member. These can be available only on request and after decision of Paper. This report will be the property of Global Journals.

Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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