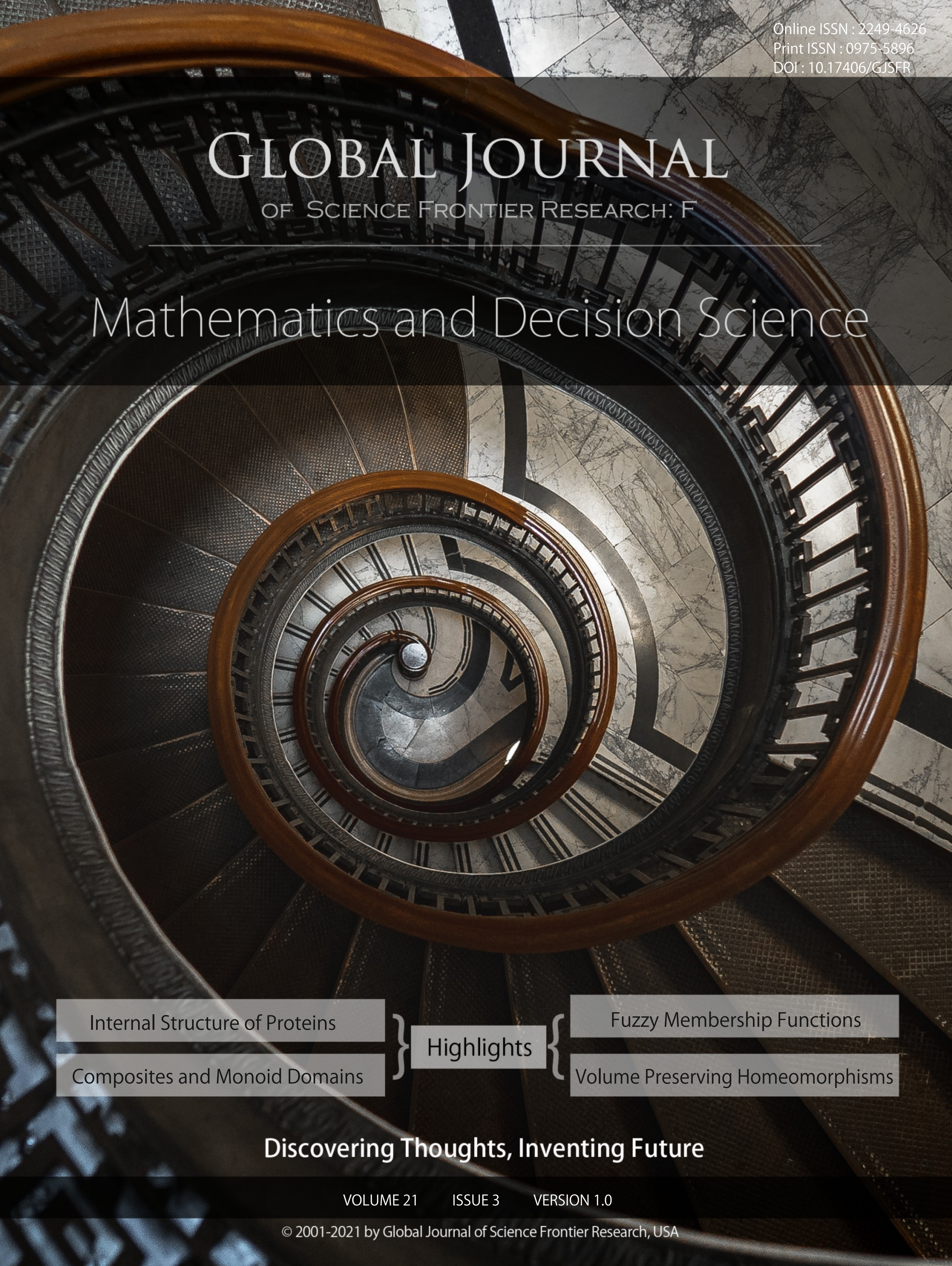

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Mathematics and Decision Science

Highlights

- Internal Structure of Proteins
- Composites and Monoid Domains
- Fuzzy Membership Functions
- Volume Preserving Homeomorphisms

Discovering Thoughts, Inventing Future

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Contents of the Issue

i. Copyright Notice
ii. Editorial Board Members
iii. Chief Author and Dean
iv. Contents of the Issue

1. Two Mathematical Approaches to Inferring the Internal Structure of Proteins from their Shape. *1-25*
2. Type II Half Logisitic Exponentiated Lomax Distribution. *27-49*
3. Development of Fuzzy Membership Functions and Meteorological Drought Classification as Per SPI. *51-58*
4. A Modified Taylor Series Expansion Method for Solving Fredholm Integro-Differential Equations. *59-69*
5. A Polynomial Composites and Monoid Domains as Algebraic Structures and their Applications. *71-89*

v. Fellows
vi. Auxiliary Memberships
vii. Preferred Author Guidelines
viii. Index

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Two Mathematical Approaches to Inferring the Internal Structure of Proteins from their Shape

By Naoto Morikawa

Abstract- Using a simple mathematical model, we propose two approaches to externally infer how the amino-acid sequence is folded in a protein. One is the previously proposed differential geometric approach. The other is a new category theoretical approach proposed in this paper. As an example, we consider detecting the presence of internal singularities from the outside. Knowledge of Category theory is not required. Proteins are represented as a loop of triangles. In both approaches, the outer contour of the loop is examined to detect the presence of singular triangles (such as isolated triangles) inside. By considering the interaction between loops, the new approach allows us to detect more singular triangles than the previous approach. We hope that this research will provide a new perspective on protein structure analysis and promote further collaboration between mathematics and biology.

Keywords: *protein structure; protein-protein interactions; protein condensates; differential geometry; category theory; discrete mathematics; loops of triangles; triangular flow; singular points.*

GJSFR-F Classification: *DDC: 516, 570*

Two Mathematical Approaches to Inferring the Internal Structure of Proteins from their Shape

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Keywords: protein structure; protein-protein interactions; protein condensates; differential geometry; category theory; discrete mathematics; loops of triangles; triangular flow; singular points.

I. Introduction

a) *The problem considered and the motivation for the research*

Since proteins are obtained by folding a chain of basic blocks (i.e., amino acids), there are restrictions on the shapes they can take. This means that, by observing their shapes from the outside, we should be able to make some guesses as to how the chains are folded inside. In this paper, we consider the problem using a simple mathematical model proposed in [1], and present two approaches: the previous differential geometric approach and a new category theoretical approach. The author is unaware of similar studies by other researchers. Using the same category theoretical approach, the author has considered the defining equations of proteins in [2]. Knowledge of category theory is not required.

Proteins often interact with other molecules in the concave areas of their surface. On the other hand, the shape of the areas depend on their internal structure as mentioned above. Therefore, it is important to investigate the dependence between the surface and the internal structure of proteins.

Until now, the structural analysis of proteins has been carried out mainly by biologists who are familiar with the structures of various molecules. This reminds me of the history of cryptography. That is, many of the cryptographers of the past were linguists who knew a lot of languages. Then, William Friedman realized that mathematics would be useful in cryptography and hired many mathematicians [4]. The

Ref

1. Morikawa, N. (2014). Discrete Differential Geometry of n Simplices and Protein Structure Analysis. *Applied Mathematics*, 5, 2458-2463. https://doi.org/10.4236/am.2014.516237

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motivation for this study is to explore the usefulness of mathematical approaches in the field of protein structure analysis.

b) The previous differential geometric approach

In our model, protein molecules are represented as closed trajectories of n-simplices. For simplicity, we will only consider the case where $n = 2$. Closed trajectories of 2-simplices are then referred to as *loops of triangles*.

Figure 1 illustrates the two approaches using a simple example (Figure 1 (a)). As you can see, there is an "isolated" triangle inside the loop. In the previous approach proposed in [1] and [3], we examine the outer contour of a loop to detect the presence of singular triangles inside. Specifically, the "pitch" of a loop is calculated as follows. First, divide the outer contour into a set of edges of triangles. Then, moving clockwise, assign either "+1" or "−1" to each edge of the contour. The rule of assignment is "change the sign if the direction of the edge changes". The "pitch" of a loop is then defined as the sum of all the "+1"s and "−1"s assigned.

In this example, the outer contour of the loop is made up of 9 edges. First, select the starting edge (S in Figure 1 (a)) and assign it "+1". Then, move down and assign "+1" again to the adjacent edge. This is because the direction of the edge is the same as the previous one. Move down further and assign "−1" to the next edge. This is because the direction of the edge is different from the previous one. In this way, we get a sequence of "+1"s and "−1"s, as shown in Figure 1 (a). Adding up all the "+1"s and "−1"s, we obtain the "pitch" of the loop:

$$+3 = +1 + 1 - 1 + 1 + 1 - 1 + 1 + 1 - 1.$$

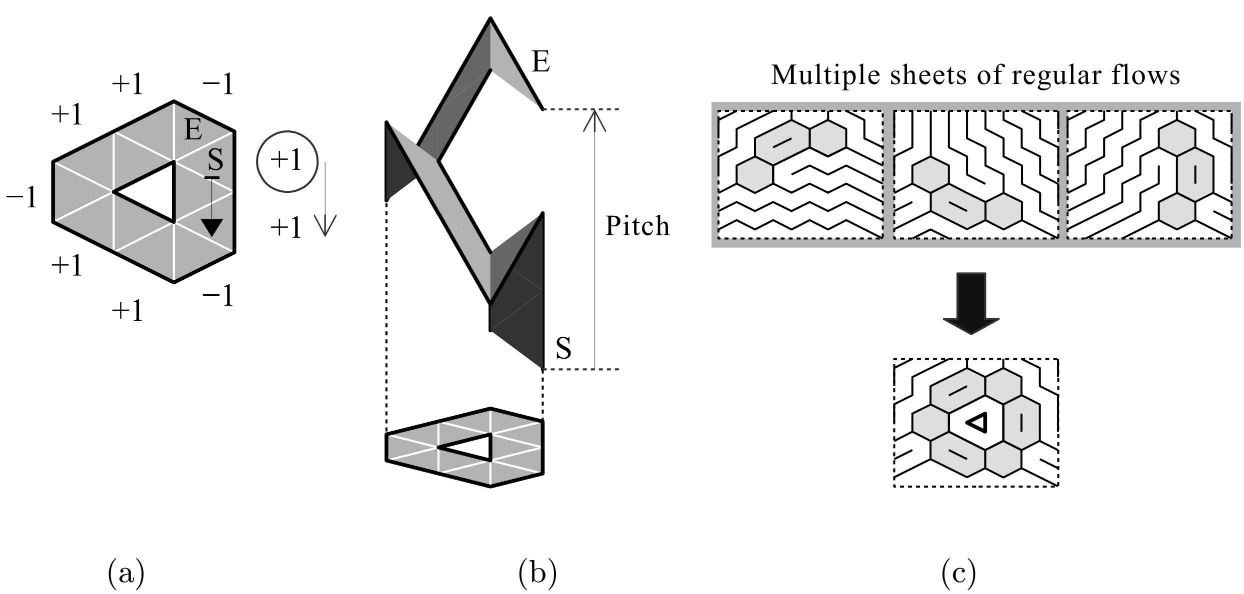

Figure 1: Comparison of the previous approach and the new approach. (a) A loop of triangles around an "isolated" triangle. (b) A schematic view of the previous differential geometric approach. (c) A schematic view of the new category theoretical approach.

Ref
3. Morikawa, N (2016). Discrete Differential Geometry of Triangles and Escher-Style Trick Art. *Open Journal of Discrete Mathematics*, 6, 161-166. http://dx.doi.org/10.4236/ojdm.2016.63013.

Notes

It is easy to show that the pitch will be zero if there are no singular triangles (such as "isolated" triangles) inside. Therefore, we now know that there are singular triangles inside the loop.

Shown in Figure 1 (b) is the mechanism of the calculation. Note that each "flat" triangle in the loop has three different directions (the lower part of the figure). Correspondingly, there are three types of "slant" triangles with different tilt directions (the upper part of the figure). Each "flat" triangle is assigned a "slant" triangle, depending on the direction of the loop at the triangle. The loop is then lifted to a trajectory of "slant" triangles in a three dimensional space. As you can see, "+1"s (resp. "−1"s) assigned to the loop correspond to the ascent (resp. descent) along the lifted trajectory. In particular, the pitch of a loop is nothing but the pitch of the lifted trajectory, i.e., the pitch of a spiral of "slant" triangles.

c) The new category theoretical approach

On the other hand, in the new approach proposed in this paper, we consider a set of loops that encloses a given loop (Figure 1(c) below). In this example, the given loop is enclosed by six loops. We then embed as many loops of the six enclosing loops as possible in a "regular" flow (i.e., a flow without "singular" triangles). However, it is not possible to embed all of them into a "regular" flow at once if there is a "singular" triangle inside. That is, we need multiple sheets of "regular" flows.

The "multiplicity" of a loop is the minimum number of sheets of "regular" flows required to embed all enclosing loops. In this case, three "regular" flows are required to embed the six enclosing loops (Figure 1 (c) above). On the other hand, it is easy to show that the "multiplicity" of a loop will be one if there are no singular triangles inside. Therefore, we see again that there are singular triangles inside.

Remark. In a flow of triangles, a singular loop causes a turbulence in the flow around itself, which can be detected by considering the "pitch" or the "multiplicity" of the loop. In physics, on the other hand, a particle causes a distortion in the space-time around itself. That is, the "pitch" measures the "mass" of a loop, and the "multiplicity" measures the "distortion" of the flow.

d) About this paper

In what follows, the author tries to present the two approaches outlined above in a self-contained manner using simple examples. The rest of the paper is organized as follows. Section 2 gives a brief description of Category theory. Section 3 gives a brief review of the previous approach. Section 4 gives an introduction to the new approach proposed. Section 5 summarizes our main results. Finally, Section 6 presents discussion and some suggestions for future research.

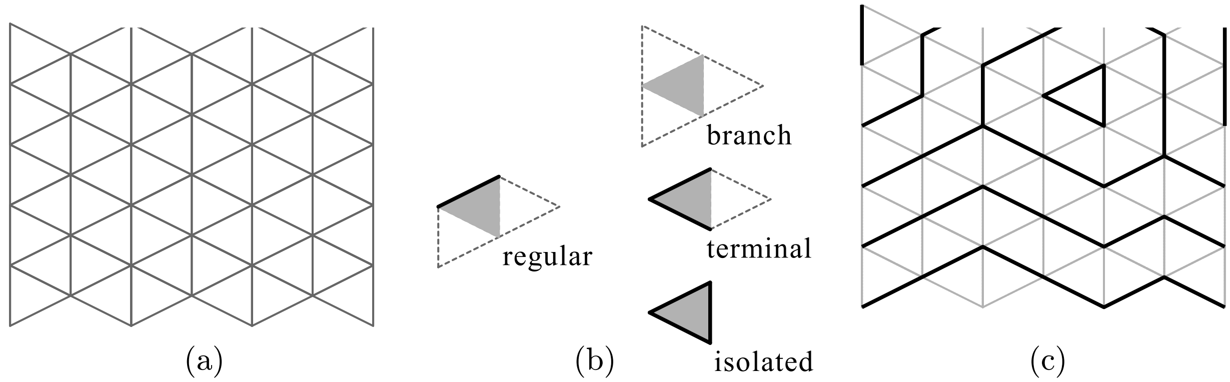

Figure 2: Flow of triangles: (a) The base space B. (b) Regular and singular triangles. (c) A singular vector field on B. (Normal edges are shown as thick black lines.)

II. About Category Theory

Category theory is the language of mathematics, appearing almost everywhere and often being a natural approach to a deeper understanding of mathematics [5, 6, 7]. Before the advent of categories, we were used to dealing with sets that had a given structure and studying their properties. On the other hand, in Category theory, the stress is placed not upon the structure of objects, but on the relations between objects within the category. In our case, the focus is on relations between proteins rather than structures of proteins.

A "category" is an embarrassingly simple concept [7]. In category theory, a mathematical system (i.e., a "category") is represented by a diagram of arrows. Each vertex represents an "object" of the category. Each arrow represents a "relation" between two objects. The properties of the objects of the category are then represented as properties of the diagram. The strength of this language lies in its ability to unify various branches of mathematics and to create unexpected links between seemingly different subjects.

III. The Differential Geometric Approach

a) *Flow of triangles*

To define flows of triangles, we first define trajectories of triangles. Roughly speaking, trajectories of triangles are obtained by connecting adjacent triangles by their common edge (Figure 1 (a)).

We start with the definition of the space of triangles, on which we define flows of triangles. We denote by E^n the n-dimensional Euclidean space.

Definition 3.1 (The base space B of triangles). *The base space B is the set of the triangles obtained by dividing E^2 into triangles. As shown in Figure 2 (a), the partitioning is done along an equilateral triangular lattice. Note that the vertices of each triangle are not contained in the interior of the edges of other triangles.*

Triangles of B have a relative positional relationship due to the underlying lattice structure. To specify connections between triangles at a given triangle, we define a discrete version of the "normal vector" at the triangle.

Definition 3.2 (Normal edge). *Given a triangle $b \in B$. A normal edge of b is an edge of b through which b is not connected to any other triangle. In other words, adjacent triangles are connected if their common edge is not a normal edge. In the figures, normal edges are shown as thick black lines (Figure 2 (b)). Note that triangles of B*

Ref
7. Milewski, B. (2014, Oct 28). Category Theory for Programmers: The Preface. https://bartoszmilewski.com/2014/10/28/category-theory-for-programmers-the-preface/

Notes

may have more than one normal edge. We denote by $N(b)$ the set of normal edges assigned to $b \in B$.

Using normal edges, trajectories of triangles are defined as follows.

Definition 3.3 (Trajectory of triangles). *Let $I = [0, l]$ be an integer interval. Let*

$$T := \{b[i] \mid i \in I\} \subset B.$$

be a series of triangles. Suppose that triangles $b[i]\ (i \in I)$ are assigned normal edges. Then, T is called a trajectory in B if each successive pair has a common edge that is not a normal edge, i.e.,

$$b[i] \cap b[i+1] \notin N(b[i]) \cup N(b[i+1])$$

for all i, $i+1 \in I$. A trajectory is called closed if $b[l]$ and $b[0]$ have a common edge that is not a normal edge. Closed trajectories are simply referred to as loops. A trajectory is called open if it is not closed. A trajectory is called maximal if it cannot be made any longer. Loops are maximal.

Definition 3.4 (Regular and singular triangles). *Triangles are called regular if they have exactly one normal edge. A regular triangles is connected to two adjacent base triangles (Figure 2 (b)). Triangles are called singular if they are not regular. Triangles with no normal edges are called branch triangle. They are connected to all three adjacent triangles. Triangles with two normal edges are called terminal triangle. They are connected to only one adjacent triangle. Triangles with three normal edges are called isolated triangle. They are not connected to any other triangles.*

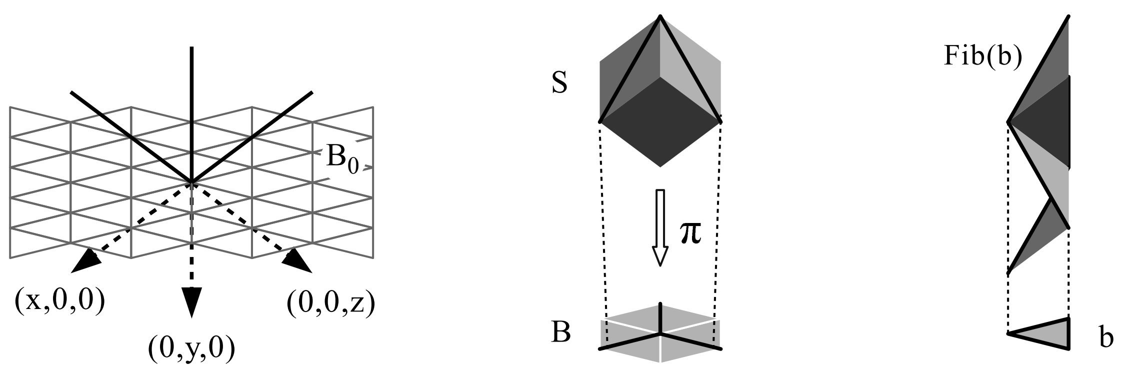

Figure 3: Flow of slant triangles. (a) Spatial arrangement of plane B_0 in E^3. (b) Projection of slant triangles onto B by π. (c) Fiber space $Fib(b)$ over a base triangle b.

Example 3.5. Figure 1 (a) shows a loop of length 12 *around an isolated triangle.*

Flows of triangles are obtained by assigning normal edges to all triangles of B. We denote by PQR the triangle with vertices P, Q, and R. The three edges of PQR are then denoted by PQ (or QP), QR (or RQ), and RP (or PR).

Definition 3.6 (Vector field of triangles). *A vector field V on B is an assignment of normal edges to all triangles of B, i.e., for any $PQR \in B$,*

$$V(PQR) \subset \{PQ, QR, RP\}.$$

A vector field V is called singular if there is $b \in B$ such that $V(b)$ contains more than one normal edge. A vector field is called regular if it is not singular.

Example 3.7. A singular vector field is shown in Figure 2 (c).

Definition 3.8 (Flows of triamgles). *Let V be a vector field on B. By connecting all pairs of adjacent triangles by their common edges not in V, we obtain a set of trajectories in B. We call them the flow (of triangles) in B defined by V. A flow is called singular if the corresponding vector field is singular. A flow is called regular if the corresponding vector field is regular.*

Example 3.9. The vector field of Figure 2 (c) defines a flow that includes the loop of Figure 1 (a).

b) Flow of lifted triangles

By lifting trajectories in B to trajectories in a three-dimansional space, we can compute flows in B instantly [1]. Moreover, we can characterize loops in B using the pitch of the corresponding spiral trajectory in the three-dimensional space. To lift trajectories upward, we first place B on the plane

$$B_0 := \{(x,\ y,\ z) \mid x + y + z = 0\} \subset E^3$$

as shown in Figure 3 (a). We then stack three-dimensional unit cubes diagonally over B (i.e. in the direction from $(+\infty, +\infty, +\infty)$ to $(-\infty, -\infty, -\infty)$). Note that only three faces of a unit cube are visible from $(-\infty, -\infty, -\infty)$ (Figure 3 (b)). We call the three faces the upper faces of the unit cube. We denote by π the diagonal projection from E^3 onto B_0, i.e.,

$$\pi(x,\ y,\ z) := \Big((2x - y - z)/3,\quad (-x + 2y - z)/3,\quad (-x - x + 2z)/3 \Big) \in B_0.$$

The three-dimensional integer lattice Z^3 are projected onto the set of all vertices of triangles in B by π.

Definition 3.10 (Slant triangles). *Slant triangles are the triangles obtained by dividing the upper faces of a unit cube by their vertical diagonal lines. That is, a slant triangle is composed of two edges of a unit cube and the vertical diagonal of an upper face of the unit cube (Figure 3 (b)). We denote by S the set of all slant triangles. S is projected on B by π, i.e.,*

$$\pi : S \to B,\ \pi(PQR) := \pi(P)\pi(Q)\pi(R).$$

Ref

1. Morikawa, N. (2014). Discrete Differential Geometry of n Simplices and Protein Structure Analysis. *Applied Mathematics*, 5, 2458-2463. https://doi.org/10.4236/am.2014.516237

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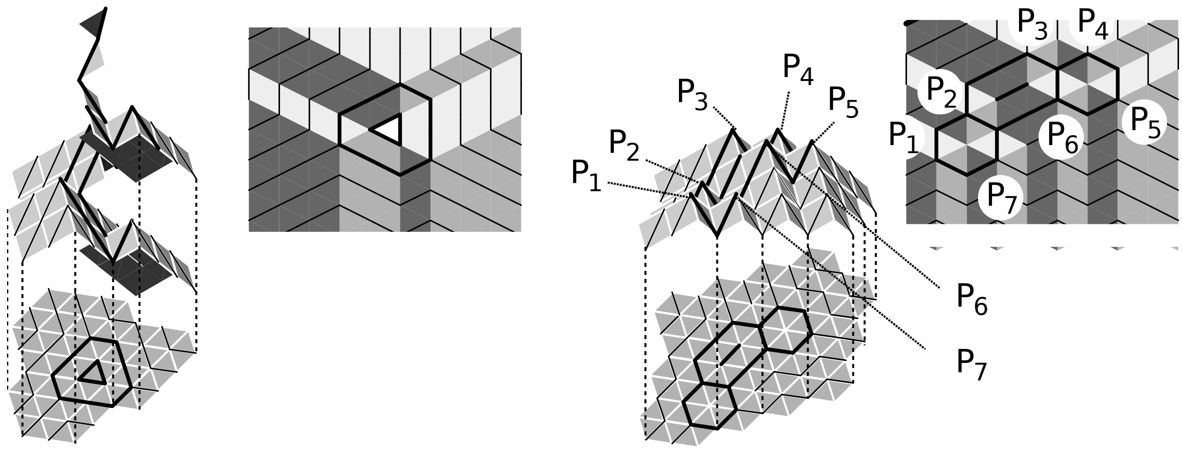


(a) (b)

Figure 4: Flows in S defined by vector fileds on B. (a) A singular flow in S (left) defined by the vector field on B (right). In the right figure, triangles are painted according to the direction of their normal edges. (b) A regular flow in S (left) defined by the vector field on B (right).

Remark. To distinguish between the triangles of B and the lifted triangles of S, we often refer to the former as base triangles or flat triangles.

Example 3.11. By dividing the three upper faces of a cube by the vertical diagonals, we obtain six slant triangles. The six slant triangles are then projected onto a hexagonal region of B_0 consisting of six base triangles (Figure 3 (b)).

Definition 3.12 (Fiber space of slant triangles over a base triangle). *Let $b \in B$. The fiber space $Fib(b)$ of slant triangles over b is defined by*

$$Fib(b) := \{s \in S \mid \pi(s) = b\}$$

(Figure 3 (c)).

Definition 3.13 (The normal edge of slant triangles). *Let $s \in S$. The normal edge of s is the edge that corresponds to the vertical diagonal. In the figures, normal edges are shown as thick black lines. Unlike triangles of B, the normal edge of s is uniquely determined by its slope (Figure 3 (b)). We denote by N(s) the normal edge of s. We denote by $\pi(N(s))$ the corresponding edge of $\pi(s) \in B$ i.e.,*

$$\pi(N(s)) := \pi(P)\pi(Q),$$

where $s = PQR \in S$ and $N(s) = PQ$.

Remark. Let $s \in S$ and $b \in B$. $N(s)$ is an edge, but $N(b)$ is a set of edges.
Using normal edges, trajectories in S are defined in the same way as trajectories in B.

Definition 3.14 (Trajectory of slant triangles). *Let $I = [0, l]$ be an integer interval. Let*

$$T := \{s[i] \mid i \in I\} \subset S$$

be a series of slant triangles. Then, T is called a trajectory in S if each successive pair has a common edge that is not a normal edge, i.e.,

$$s[i] \cap s[i+1] \notin \{N(s[i]),\ N(s[i+1])\}$$

for all $i, i+1 \in I$. *A trajectory is called closed if* $s[0]$ *and* $s[l]$ *have a common edge that is not a normal edge. Closed trajectories are simply referred to as loops. A trajectory is called maximal if it cannot be made any longer. Loops are maximal.*

Let $T = \{s[i] \| i \in I\}$ be a trajectory in S, where I is an integer interval. Let $\pi(T)$ be the image of T by π, i.e.,

$$\pi(T) = \{\pi(s[i]) \mid i \in I\} \subset B.$$

Normal edges are then assigned on $\pi(T)$ by

$$N(\pi(s[i]) := \{\pi(N(s[i]))\} \quad (i \in I).$$

Example 3.15. In Figure 1 (b), an open trajectory in S (above) is projected on a loop in B (below).

Example 3.16. Let $PQR \in B$ *be an isolated triangle.* $Fib(PQR)$ *gives a maximal open trajectory in S over PQR (Figure 3 (c)), i.e.,*

$$\begin{cases} \pi^{-1}(PQR) = Fib(b), \\ \{\pi(N(s)) \mid s \in Fib(b)\} = \{PQ,\ QR,\ RP\}. \end{cases}$$

Definition 3.17 (Flows of slant triamgles). *Let* F_S *be a set of maximal trajectories in S, i.e.,*

$$\begin{cases} F_S := \{T[k] \subset S \mid k \in K\}, \\ T[k] := \{s[k][i] \in S \mid i \in I_k\}, \end{cases}$$

where K and I_k $(k \in K)$ *are integer intervals. The fiber* $Fib_{F_S}(b)$ *of* F_S *over* $b \in B$ *is defined by*

$$Fib_{F_S}(b) := Fib(b) \cap \{s[k][i] \mid k \in K, i \in I_k\}.$$

F_S *is called a flow in S if* $Fib_{F_S}(b)$ *is defined for all* $b \in B$.

Definition 3.18 (Vector field induced by flows in S). *Let* F_S *be a flow in S. The vector field induced on B by* F_S *is defined by*

$$V(b) := \{\pi(N(s)) \mid s \in Fib_{F_S}(b)\}$$

for $b \in B$.

Definition 3.19 (Flow in S defined by a vector field on B). *Let V be a vector field on B. Let* F_B *be the flow in B defined by V. By lifting the trajectories of* F_B*, we obtain a flow* F_S *in S.* F_S *is called a flow in S defined by V.*

Example 3.20. In Figure 4 (a), a flow in S (left above) is defined by the vector field in B on the right. The flow is spiraling around the fiber over the isolated triangle in B. The flow in S has no loop.

Example 3.21. In Figure 4 (b), a flow in S (left above) is defined by the vector field in B on the right. The flow in S has three loops. As you can see, the flow in S forms a

Notes

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

"mountain range" like shape. In particular, by piling up unit cubes diagonally in E^3, we obtain a flow in S.

Definition 3.22 (Affine flow of slant triangles). *Let M be a union of triangular cones obtained by piling up unit cubes in E^3 in the direction from $(+\infty, +\infty, +\infty)$ to $(-\infty, -\infty, -\infty)$. If we give the set of top vertices of the cones, M is uniquely determined. Note that only the slant triangles on the surfaces of M are visible from $(-\infty, -\infty, -\infty)$. The Flow in S defined on the surfaces of M are called the affine flow in S defined by M.*

Notes

Example 3.23. Figure 4 (b) above is an affine flow in S defined by seven triangular cones.

Definition 3.24 (Affine flow of base triangles). *Let M be a union of triangular cones. A regular flow in B is obtained by projecting the affine flow in B defined by M by π. The regular flow in B is called the affine flow in S defined by M. The corresponding regular vector field is called the affine vector field on B defined by M. A flow in B is called locally affine if it is obtained by pasting together regions of affine flows in B.*

Proposition 3.25. For each loop L_B in an affine flow in B, there is a loop L_S in S such that $\pi(L_S) = L_B$.

Proof. It follows immediately from the definition.
The author does not have a proof of the following assertion.

Assertion 3.26. Every regular flow in B is an affine flow.

Remark. It is easy to show that every regular flow in B is locally affine.

c) The pitch of loops in B.

Finally, we define the "pitch" of a loop in B. In the previous approach, loops of flows in B are characterized by their "pitch" (Figure 1 (b)).
Let V be a vector field on B. Let

$$T_B = \{b[i] \in B \mid i \in [0,\ l)\}$$

be a loop of the flow in B defined by V, where $l \in \mathbb{Z}$. Let

$$T_S = \{s[i] \in S \mid i \in [0,\ l)\}$$

be a trajectory in S such that

$$\pi(s[i]) = b[i] \quad \text{for } i \in [0,\ l).$$

In general, T_S is not a loop.

Remark. Let V be an affine vector field. Then, T_S is a loop by Proposition 3.25.

Remark. Let V be a singular vector field. Then, it depends on the case whether or not T_S is a loop.

Definition 3.27 (Height of a slant triangle). *Let $s = PQR \in S$, where $P = (x_0,\ y_0,\ z_0)$, $Q = (x_1,\ y_1,\ z_1)$, and $R = (x_2,\ y_2,\ z_2)$. The height function h on S is defined by*

$$\begin{aligned} ht(s) := \max\{ \quad & -(x_0 + y_0 + z_0)/2, \\ & -(x_1 + y_1 + z_1)/2, \\ & -(x_2 + y_2 + z_2)/2 \ \}. \end{aligned}$$

Note that $|x_0 + y_0 + z_0|$ *is the distance from point P to the plane* B_0 *defined above.*

Definition 3.28 (Pitch of a loop in B). *Let*

$$T_B = \{b[i] \in B \mid i \in [0,\ l)\}$$

be a loop of a flow in B, where $l \in \mathbb{Z}$. *Let*

$$T_S = \{s[i] \in S \mid i \in [0,\ +\infty)\}$$

be a trajectory in S such that

$$\pi(s[i]) = b[i \mod l] \quad \textit{for}\ i \in [0,\ +\infty).$$

The pitch of T_B *is defined by*

$$ptch(T_B) = |ht(s[l]) - ht(s[0])|.$$

The "pitch" of loop T_B *is nothing but the pitch of the spiral* T_S *in S. The value of* $ptch(T_B)$ *dose not depend on the choice of* T_S*. Note that* $ptch(T_B) = 0$ *if and only if* T_S *is a loop (i.e., closed trajectory) in S.*

Example 3.29. The loop of Figure 1 (a) is lifted to the trajectory of Figure 1 (b) above. Because of the isolated triangle at the center, the lifted trajectory in S is not closed. We have calculated the pitch of the loop in the introduction, i.e., $ptch(T_B) = 3$*. See subsection 3.4 below.*

Proposition 3.30. Let T_B *be a loop of a regular flow in B. Then,*

$$ptch(T_B) = 0.$$

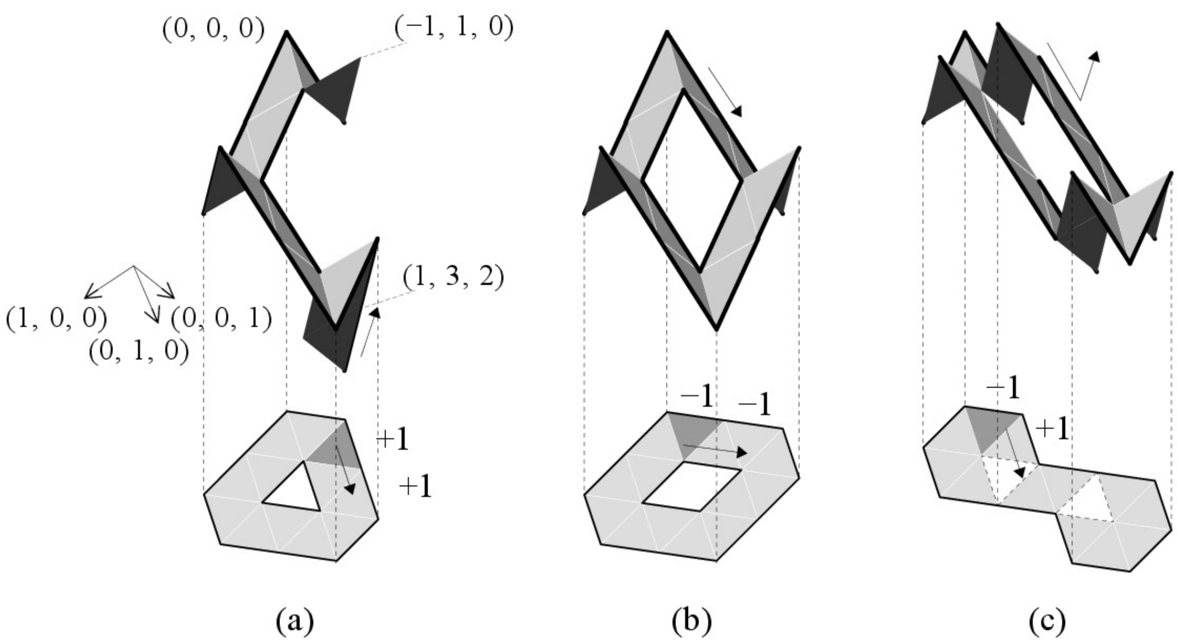


Figure 5: Calculation of the pitch of loops in B. (a) A loop enclosing an isolated triangle. (b) A loop enclosing two terminal triangles. (c) A loop enclosing two branch triangles. Starting from the dark grey triangle in the direction of the arrow, two first values assigned are shown. Note that "+1" (resp. "-1") corresponds to upstream (resp. downstream) of the lifted trajectory in S.

Notes

Proof. (Sketch of the proof) Let F be a regular flow in B. Let T_B be a loop of F. Let T_S be the slant trajectory on T_B such that $\pi(T_S) = T_B$. Suppose that the pitch of T_B is not zero. This would create a tear in the "cover" formed by the triangle of T_S. However, the starting point of the tear gives a singular base triangle.

d) Singular triangle detection by the pitch.

In the introduction, we examined the contour of a given loop from outside, and detected the presence of a singular triangle in the loop. Here we show that what we have calculated is nothing but the pitch of the loop.

Notes

Lemma 3.31. Let I be an integer interval. Let

$$T_B = \{b[i] \in B \mid i \in I\}$$

be a loop in B. Let $b[j]$, $b[k] \in T_B$ be adjacent triangles. Suppose that $b[j]$ and $b[k]$ are assigned the same normal edge (i.e., the edge shared by $b[j]$ abd $b[k]$). Then, $b[j-1] - b[j] - b[j+1]$ and $b[k-1] - b[k] - b[k+1]$ go in opposite directions.

Proof. Suppose that the two local flows go in the same direction. Then, the triangles in T_B are connected in the following order:

$$\cdots - b[j] - b[j+1] - \cdots - b[k-1] - b[k] - \cdots .$$

In particular, two triangles $b[j-1]$ and $b[k+1]$ are sepa rated by the closed chain

$$b[j] - b[j+1] - \cdots - b[k-1] - b[k].$$

This means that the loop T_B intersects with itself, which is a contradiction.

Proposition 3.32. A loop in B goes in the same direction on its contour. In particular, if you follow its contour along the loop stream, you will be going around the region occupied by the loop.

Proof. It follows from lemma 3.31 immediately.

Corollary 3.33. The pitch of a loop is obtained as follows:

Step 1 Divide the outer contour into a set of edges of triangles,

Step 2 Moving clockwise, assign either "+1" or "−1" to each edge of the contour. The rule of assign ment is "change the sign if the direction of the edge changes".
The pitch of a loop is then obtained as the sum of all the"+1"s and "−1"s assigned.

Example 3.34. Let

$$T_B = \{b[i] \in B \mid i \in [0,\ 12)\}$$

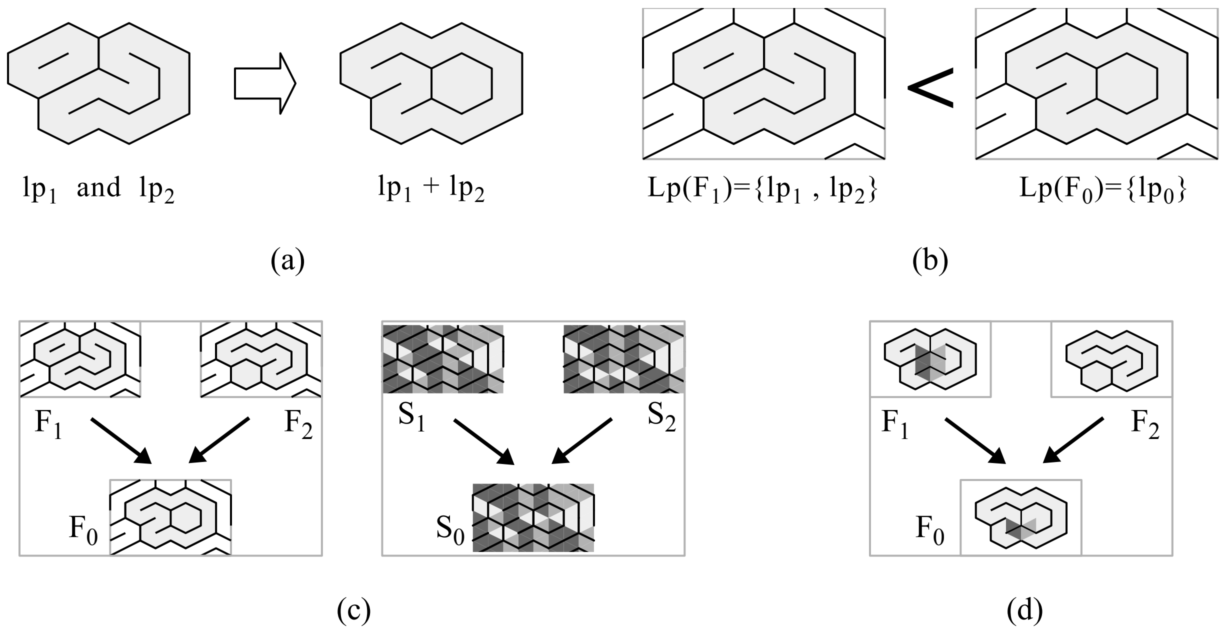


Figure 6: Relation between flows. (a) Interaction "+" between loops. (b) Relation "<" between flows. (c) An upper bound F_0 of $\{F_1, F_2\} \subset FW_B$ (left) and the corresponding flows in S (right). (d) An upper bound F_0 of $\{F_1, F_2\}$ shown in a hybrid diagram

be the loop in Figure 5 (a) below. This is the example considered in the introduction. Starting from the dark grey triangle, we obtain a $\{+1, -1\}$*-valued sequence of length* 9:

$$+1, +1, -1, +1, +1, -1, +1, +1, -1.$$

Summing them up, we obtain 3*. On the other hand, let*

$$T_S = \{s[i] \in S \mid i \in [0, +\infty)\}$$

be the lifted trajectory of T_B *(Figure 5 (a) above). Then,*

$$\begin{cases} ht(s[0]) = -(-1+1+0)/2 = 0 \\ ht(s[12]) = -(1+3+2)/2 = -3. \end{cases}$$

Therefore,

$$ptch(T_B) = |-3-0| = 3.$$

Example 3.35. Let T_B *be the loop shown in Figure 5 (b) below. Starting fron the dark grey triangle, we obtain a* $\{+1, -1\}$*-valued sequence of length* 12*:*

$$-1, -1, +1, -1, -1, +1, +1, -1, +1, +1.$$

Summing them up, we obtain 0*. On the other hand, the lifted trajectory of* T_B *is a loop (Figure 5 (b) above). Therefore,* $ptch(T_B) = 0$.

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

Example 3.36. Let T_B be the loop shown in Figure 5 (c) below. Starting fron the dark grey triangle, we obtain a $\{+1, -1\}$*-valued sequence of length* 10*:*

$$-1, +1, -1, -1, +1, -1, +1, -1, +1, +1, 1, +1.$$

Summing them up, we obtain 0*. On the other hand, the lifted trajectory of T_B is a loop (Figure 5 (b) above). Therefore,* $ptch(T_B) = 0$.

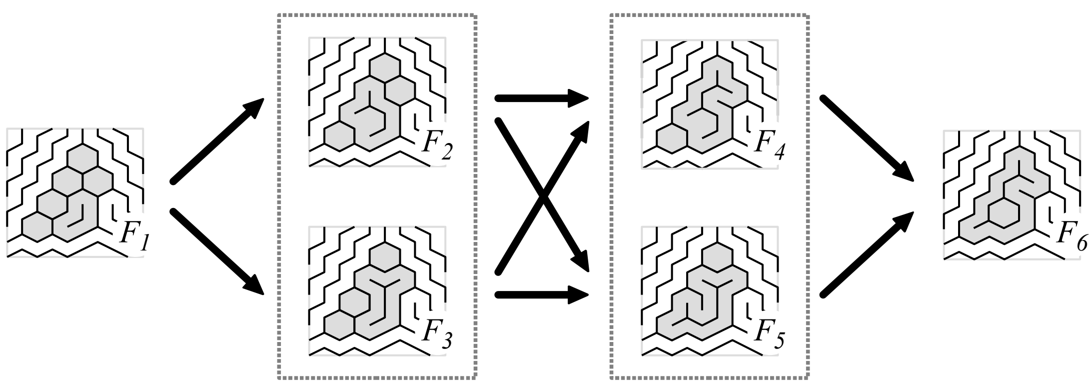
In the last two examples, we were not able to detect the presence of singular triangles inside using the differential geometric approach. In the next section, we will show that a new approach can be used to detect both of the singular triangles.

IV. The Category Theoretical Approach

In the previous approach, we examine the shape of the contour of a given loop. In the new approach, we will consider the interactions between a given loop and other loops in order to infer its internal structure. "Relations" between flows are then defined using the interactions between loops. The interactions between loops are determined only by their shape.

a) *Relation between flows*

Proteins are known to form protein-protein complexes as they perform their tasks. Also recognized recently is the importance of droplets of proteins (i.e., transient liquid-like assemblies of proteins) called "protein condensates" in protein-protein interactions [8, 9]. In our mathematical toy model, proteins are represented as loops of triangles. Protein condensates then corresponds to a set of coexisting loops, i.e., (a region of) a flow of triangles. In order to model the interactions between proteins, we first define the interaction between loops as follows.



(a)

Ref

8. Hyman, A.A., Weber C.A., & Julicher, F. (2014). Liquid-Liquid Phase Separation in Biology. *Annual Review of Cell and De velopmental Biology,* 30, 39-58. https://doi.org/10.1146/annurev cellbio-100913-013325

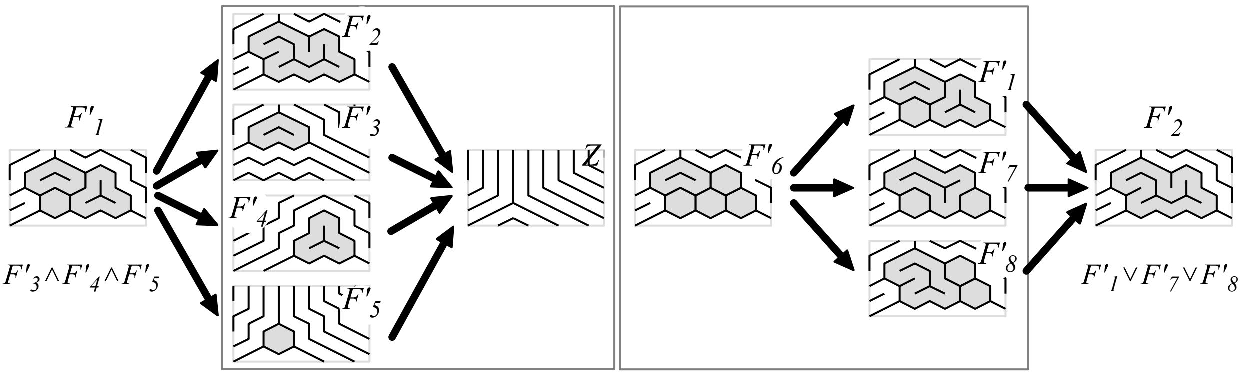


Figure 7: Suprema $\vee F_i$ and infima $\wedge F_i$ of flows in B. (a) Lower bounds and infima of $\{F_4, F_5\}$, upper bounds and suprema of $\{F_2, F_3\}$. (b) $\wedge$-decomposition of F_1'. (c) $\vee$-decomposition of F_2'. ($F_i \rightarrow F_j$ denotes the relation $F_i \leqslant F_j$.)

Definition 4.1 (Interaction between loops). *Let*

$$lp_0,\ lp_1,\ \ldots,\ lp_n$$

be loops of B ($n \in \mathbb{Z}$). We say that $lp_1, lp_2, \ldots, lp_n$ interact to form lp_0 if lp_0 contains all the triangles contained in $lp_1, lp_2, \ldots, lp_n$. Then, we denote the interactions using "+", i.e,

$$lp_1 + \cdots + lp_n = lp_0.$$

We often write $\sum_{i=1,\,n} lp_i$ as an abbreviation for $lp_1 + \ldots + lp_n$.

Remark. Loops that are contained inside another loop are considered a part of the surrounding loop.

Remark. Since there may be multiple loops with the same contour, loop $\sum_{i=1,\,n} lp_i$ is not uniquely determined. However, the contour of $\sum_{i=1,\,n} lp_i$ is uniquely determined.

Example 4.2. In Figure 6 (a), two loops lp_1 and lp_2 interact to form a loop. Note that $lp_1 + lp_2$ contains a loop of length 6 inside. In this case, $lp_1 + lp_2$ is uniquely determined.

In order to characterize a protein by its interaction with other proteins, it is necessary to take into account a droplet of proteins that contains the protein. In our model, we need to consider flows that contain the given loop. The "category" of flows is defined as follows.

Definition 4.3 (The set FW_B of all flows in B). *We de note by FW_B the set of all flows in B. FW_B is the "object" we consider in the new approach. Let $F \in FW_B$. We denote by $Lp(F)$ the set of all loops of F. In the following, we will identify F with $Lp(F)$ since we do not consider interactions between loops (i.e., closed trajectories) and open trajectories. Let $F_1, F_2 \in FW_B$. We write*

$$F_1 \equiv F_2$$

if $Lp(F_1) = Lp(F_2)$. "≡" gives an equivalence relation on FW_B. We denoted by Z the flow with no loop. Z is unique up to "≡".

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

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Notes

Using the interaction "+" between loops, we define a relation on FW_B.

Definition 4.4 (Relation "$\leqslant$" on FW_B). *Let* $F_1, F_2 \in FW_B$. *We define a binary relation* "$\leqslant$" *on* FW_B *by*

$$F_1 \leqslant F_2$$

if and only if, for any $lp' \in Lp(F_2)$, *there is a set*

$$\{lp_1, lp_2, \ldots, lp_n\} \subset Lp(F_1)$$

($n \in \mathbb{Z}$) *such that*

$$lp' = \sum_{i=1,\, n} lp_i.$$

We write $F_1 < F_2$ *if* $F_1 \leqslant F_2$ *and* $F_1 \neq F_2$. *By the conventions of Category theory, we often write* $F_1 \to F_2$ *instead of* $F_1 \leqslant F_2$ *(especially in the figures).*

Example 4.5. In Figure 6 (b), $F_1 < F_0$ *since* $lp_0 = lp_1 + lp_2$.

b) Covering flow of regular flows.

"Suprema" (i.e., least upper bounds) and "infima" (i.e., greatest lower bounds) are defined as follows.

Definition 4.6 (Upper bound and lower bound). *Let* $C \subset FW_B$. *Let* $M \in FW_B$. *M is called an upper bound of C if* $F < M$ *for all* $F \in C$. *M is called a lower bound of C if* $M < F$ *for all* $F \in C$. *An upper (resp. lower) bound M is called regular if M is a regular flow.*

Remark. An upper bound (resp. lower bound) M of C is not containd in C.

Example 4.7. In figure 6 (c) left, F_0 *is an upper bound of* $\{F_1, F_2\}$. *In figure 6 (c) right, the flow* S_0 *(resp.* S_1, S_2*) in S corresponds to the flow* F_0 *(resp.* F_1, F_2*) in B. As you can see,* S_0 *is obtained by putting a cube on* S_2. S_1 *is then obtained by putting one more cube on* S_0. *In this way, we can compute fusion and fission of loops immediately. In Figure 6 (d), the added cubes are drawn to show the process of computation.*

Remark. In the following, "Figure 6 (d)"-style figures will be often used instead of "Figure 6 (c)"-style figures when describing the relation between flows.

Definition 4.8 (Supremum $\vee$). *Let* $C \subset FW_B$. *Let* $M \in FW_B$ *be an upper bound of C. M is called a supremum of C if* $M \leqslant M'$ *for any upper bound of C. We denote by* $\vee C$ *the set of all suprema of C. If* $C = \{F_1, F_2, \ldots, F_n\}$, *we often write* $F_1 \vee F_2 \vee \cdots \vee F_n$ *or* $\bigvee_{i=1,\, n} F_i$ *instead of* $\vee\{F_1, F_2, \ldots, F_n\}$. *C often has more than one supremum.*

Definition 4.9 (Infimum $\wedge$). *Let* $C \subset FW_B$. *Let* $N \in FW_B$ *be a lower bound of C. N is called an infimum of C if* $N' \leqslant N$ *for any lower bound* N' *of C. We denote by* $\wedge C$ *the set of all infima of C. If* $C = \{F_1, F_2, \ldots, F_n\}$, *we often write* $F_1 \wedge F_2 \wedge \cdots \wedge F_n$ *or* $\wedge_{i=1,\, n} F_i$ *instead of* $\wedge\{F_1, F_2, \ldots, F_n\}$. *C often has more than one infimum.*

Example 4.10. In figure 7 (a), $F_4 \wedge F_5 = \{F_2, F_3\}$ *and* $F_2 \vee F_3 = \{F_4, F_5\}$. *Since* F_2 F_1, F_1 $\notin F_4 \wedge F_5$. *Since* $F_6 \not\leqslant F_4$, $F_6 \notin F_2 \wedge F_3$.

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

Example 4.11. *($\wedge$-decomposition) In Figure 7 (b), all upper bounds of F'_1 are shown in the rectangle. Using three of them, we have $\{F'_1\} = F'_3 \wedge F'_4 \wedge F'_5$. (Let F_a and F_b be flows with one loop. Then, $F_a \vee F_b = \{Z\}$. For example, $F'_2 \vee F'_3 = \{Z\}$.)*

Example 4.12. *($\vee$-decomposition) In Figure 6(c), all lower bounds of F'_2 are shown in the rectangle. Using three of them, we have $\{F'_2\} = F'_1 \vee F'_7 \vee F'_8$.*

Notes

In order to characterize the contour of a given loop *lp*, we consider the set of all lower bounds of a flow F such that $Lp(F) = \{lp\}$. (Note that F has no upper bound other than Z.)

Definition 4.13 (Covering flow of a flow in B). *Let $F \in FW_B$. We denote by $Cv(F)$ the set of all regular lower bounds of F, i.e.,*

$$Cv(F) := \{F' \in FW_B \mid F' \leqslant F,\ F' \text{ is regular}\}.$$

Flows of $Cv(F)$ are called covering flows of F. The minimum elements of $Cv(F)$ are called generators of $Cv(F)$. The dimension of $Cv(F)$ is defined as the number of its generators.

Lemma 4.14 ($\vee$-decomposition). *Let $F \in FW_B$ be a regular flow. Then, $\{F\} = \vee Cv(F)$.*

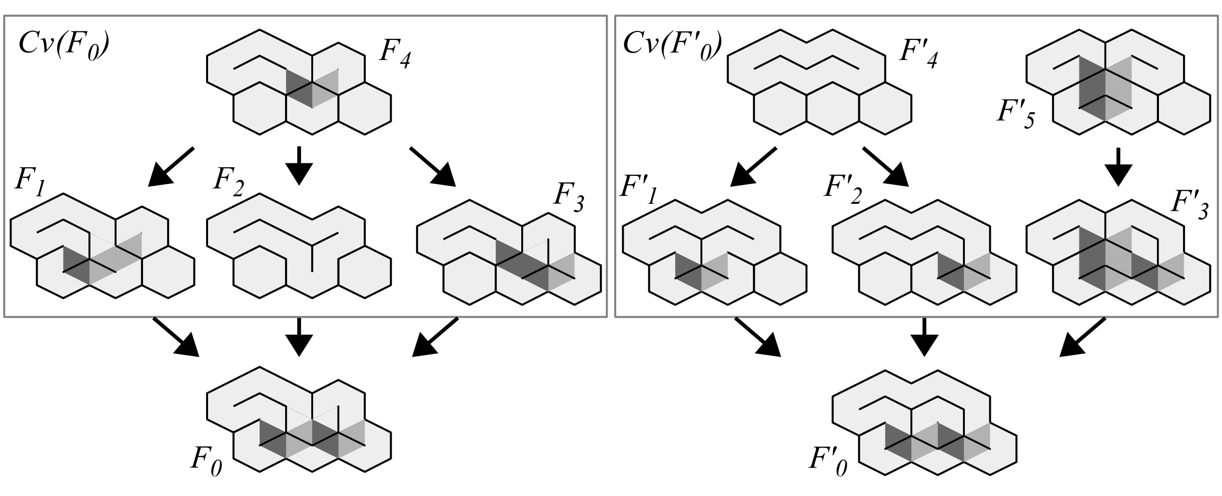
Proof. It follows immediately from the definition.

Remark. If F is a singular flow, then $Cv(F) = \emptyset$. We will consider covering flows of a singular flow in section 4.4 below.

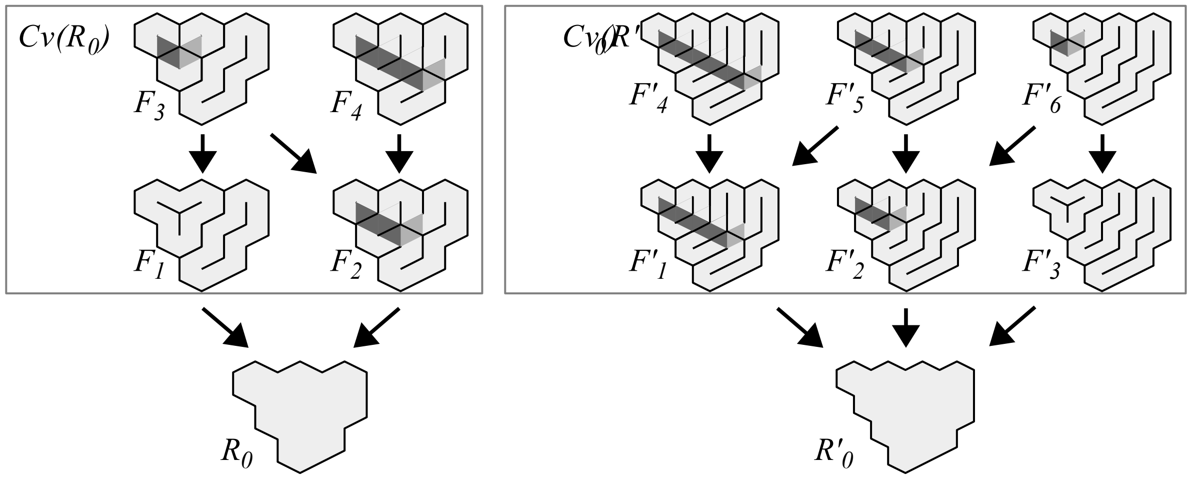
By considering the set of all covering flows, we can distinguish loops with similar contours, as the following examples show.

Example 4.15. In Figure 8 (a), F_0 is a flow such that $Lp(F_0)$ consists of one loop. $Cv(F_1)$ consists of four flows. $Cv(F_1)$ has one generator F_4. The dimension of $Cv(F_1)$ is one. Moreover,

$$\{F_0\} = \vee Cv(F_0) = F_1 \vee F_2 \vee F_3.$$



(a) (b)



(c) (d)

Figure 8: Covering flows. (a) and (b) Covering flows of flows in B. (c) and (d) Covering flows of regions of B.

Example 4.16. In Figure 8 (b), F'_0 is a flow such that $Lp(F'_0)$ consisting of one loop. $Cv(F_1)$ consists of five flows. $Cv(F_1)$ has two generators F'_4 and F'_5. The dimension of $Cv(F'_1)$ is two. Moreover,

$$\{F'_0\} = \vee Cv(F'_0) = F'_1 \vee F'_2 \vee F'_3.$$

c) Closure of FW_B with respect to "∨".

Let $C \subset FW_B$. Then, $\vee C$ often contains no flows other than Z, i.e., $\vee C = \{Z\}$. For example, let C be the set of all flows such that their loops sweep a given region collectively. Then, $\vee C = \{Z\}$ if there is no loop with the contour of the region. However, since we want to treat both protein complexes and proteins seamlessly, we treat the "contour made up of multiple loop contours" in the same way as the "contour of a single loop".

Definition 4.17 (Covering flow of a region of B). *Let R be a region of B. We denote by $Cv(R)$ the set of all regular flows such that the region swept by their loops matches the region R. Flows of $Cv(R)$ are called covering flows of R. The minimum elements of $Cv(R)$ are called generators of $Cv(R)$. The dimension of $Cv(R)$ is defined as the number of its generators.*

Example 4.18. In Figure 8 (c), $Cv(R_0)$ consists of four flows. Since it has two generators, its dimension is two. Since there is no loop whose contour matches the contour of R_0, $\vee Cv(R_0) = \{Z\}$.

Example 4.19. In Figure 8 (d), $Cv(R'_0)$ consists of six flows. Since it has three generators, its dimension is three. Since there is no loop whose contour matches the contour of R_1, $\vee Cv(R'_0) = \{Z\}$.

Now let us extend FW_B to include ''virtual" loops with contours made up of multiple contours of loops.

Notes

First, we identify the set FW_B of flows in B with the set of $\{0, 1\}$-valued functions on FW_B.

Definition 4.20 (Hom-Sets). *Let $F_1, F_2 \in FW_B$. In the language of Categories, the set of all the possible relations between F_1 and F_2 is denoted by $Hom(F_1, F_2)$, i.e.,*

$$Hom(F_1,\ F_2) := \begin{cases} 1 & \text{if } F_1 \leqslant F_2, \\ 0, & \text{otherwise}, \end{cases}$$

where 1 *denotes a set with one element and* 0 *denotes a set with no element. We define a binary relation on* $\{0,\ 1\}$ *by*

$$0 < 1,\ 0 = 0,\ \text{and } 1 = 1.$$

We write $a \leqslant b$ if $a < b$ or $a = b$. We define multiplication "$\cdot$" on $\{0,\ 1\}$ *by*

$$1 \cdot 1 = 1 \text{ and } 1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 1.$$

We often write ab instead of $a \cdot b$ when there is no risk of confusion.

Remark. Let $F, X \in FW_B$ Roughly speaking, X is a "refinement" of F if *Hom* $(X, F) = 1$. X is a "component" of F if $Hom(F,\ X) = 1$.

Remark. $\{0,\ 1\}$ is a category equipped with two objects $\{0,\ 1\}$ and three relations $0 < 1, 0 = 0$, and $1 = 1$.

Hom-sets provide two types of $\{0,\ 1\}$-valued functions on FW_B: one is order-preserving function and the other is order-reversing function.

Definition 4.21 (*Hom* functions on FW_B). Let $F, X, Y \in FW_B$.
(1) Order-preserving function $k(F)$ on FW_B is defined by

$$k(F)(X) := Hom(F,\ X)$$

Note that $X \leqslant Y$ implies $k(F)(X) \leqslant k(F)(Y)$.
(2) Order-reversing function $h(F)$ on FW_B is defined by

$$h(F)(X) := Hom(X, F).$$

Note that $X \leqslant Y$ implies $h(F)(X) \geqslant h(F)(Y)$.

Definition 4.22 ($FW_B^{\vee}$and $FW_B^{\wedge}$). *(1) We denote by $FW_B^{\vee}$ the set of all order-preserving functions from FW_B to $\{0, 1\}$. Let $h_0, h_1 \in FW_B^{\vee}$. We then define a binary relation "$\leqslant$" on $FW_B^{\vee}$ by*

$$h_0 \leqslant h_1 \quad \text{if and only if} \quad h_0(X) \geqslant h_1(X)$$

for all $X \in FW_B$.

(2) We denote by $FW_B^{\wedge}$ the set of all order-reversing functions from FW_B to $\{0, 1\}$. We then define a binary relation "$\leqslant$" on $FW_B^{\wedge}$ by

$$h_0 \leqslant h_1 \quad \text{if and only if} \quad h_0(X) \leqslant h_1(X)$$

for all $X \in FW_B$.

Notes

Note that k is a function defined by

$$F \in FW_B \mapsto k(F) \in FW_B{}^{\vee}.$$

h is a function defined by

$$F \in FW_B \mapsto h(F) \in FW_B{}^{\wedge}.$$

We can then identify FW_B as a subcategory of $FW_B{}^{\vee}$ (resp. $FW_B{}^{\wedge}$) by k (resp. h). Recall that $F \equiv G$ if and only if $Lp(F) = Lp(G)$.

Proposition 4.23. (1) k is a one-to-one function (up to "≡"). (2) h is a one-to-one function (up to "≡").

Proof. It follows from the the Yoneda lemma ([6]).

Functions in $FW_B{}^{\vee}$(or $FW_B{}^{\wedge}$) are called "representable" if they correspond to "actual" flows in B. That is,

Definition 4.24 (Representable function on FW_B). *(1) Let $c \in FW_B{}^{\vee}$. c is called representable if there is a flow $F \in FW_B$ such that*

$$c(X) = Hom(F,\ X).$$

F is called a representation of c.

(2) Let $c \in FW_B{}^{\wedge}$. c is called representable if there is a flow $F \in FW_B$ such that

$$c(X) = Hom(X, F).$$

F is called a representation of c.

Example 4.25. In the case of Figure 8 (c), define $h_0 \in FW_B{}^{\wedge}$ by

$$h_0(X) := \begin{cases} 1 & \textit{if } X \in Cv(R_0), \\ 0, & \textit{otherwise}, \end{cases}$$

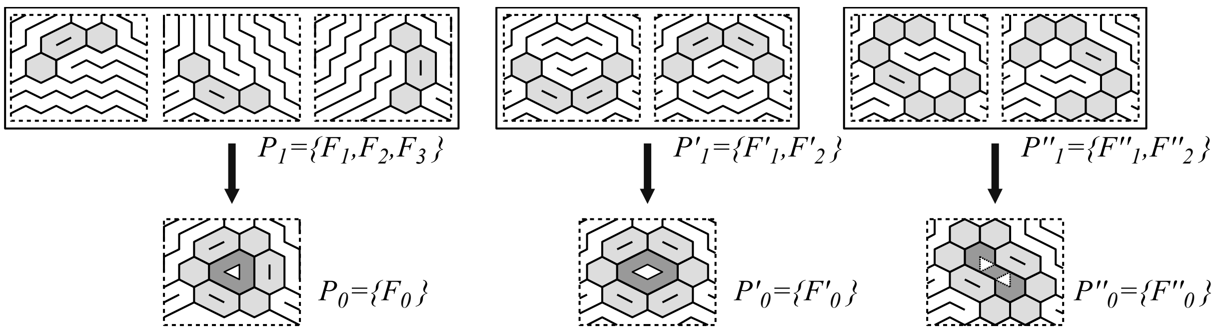
Then,

$$h_0 = \vee Cv(R_0) \quad \textit{in } FW_B{}^{\wedge}.$$

Since $h(F_1)(F_2) = 0$, $h_0 \neq h(F_1)$. Since $h(F_2)(F_1) = 0$, $h_0 \neq h(F_2)$. Since $h(Z)(X) = 1$ for all $X \in FW_B$, $h_0 \neq h(Z)$. Moreover, h_0 is not representable. Thus, we can regard h_0 as the "virtual" loop whose contour matches the contour of R_0.

Ref

6. Kashiwara, M., & Schapira, P. (2006). Categories and Sheaves. Springer-Verlag Berlin Heidelberg.



(a) (b) (c)

Figure 9: Branched covering of singular flows. (a) A branched covering P_1 of F_0. P_1 has tbree sheets F_1, F_2, and F_3. P_0 consists of one singular flow F_0, where F_0 has an isolated triangle. (b) A branched covering P'_1 of F'_0. P'_1 has two sheets F'_1 and F'_2. P'_0 consists of one singular flow F'_0, where F'_0 has two terminal triangles. (c) A branched covering P''_1 of F''_0. P''_1 has two sheets F''_1 and F''_2. P''_0 consists of one singular flow F''_0, where F''_0 has two branch triangles.

Example 4.26. In the case of Figure 8 (d), define $h_1 \in FW_B{}^{\wedge}$ *by*

$$h_1(X) := \begin{cases} 1 & \text{if } X \in Cv(R_1), \\ 0, & \text{otherwise}, \end{cases}$$

Then,

$$h_1 = \vee Cv(R_1) \in FW_B{}^{\wedge}.$$

Since $h(F'_1)(F'_2) = 0, h_1 \neq h(F'_1)$. *Since* $h(F'_2)(F'_1) = 0$, $h_1 \neq h(F'_2)$. *Since* $h(F'_3)(F'_1) = 0$, $h_1 \neq h(F'_3)$. *Since* $h(Z)(X) = 1$ *for all* $X \in FW_B$, $h_1 \neq h(Z)$. *Moreover,* h_1 *is not representable. Thus, we can regard* h_1 *as the "virtual" loop whose contour matches the contour of* R_1.

Example 4.27 (Heyting algebra [10]). *Let A be the flow obtained by dividing B into a hexagonal lattice. That is,* $Lp(A)$ *consists of an infinite number of hexagonal shaped loops of length 6.* Let $X, Y \in FW_B$. *We define the negation* $\neg X$ *of X and an exponential* Y^X *by*

$$\begin{cases} \neg X(C) := Hom(C \wedge X,\ A), \\ Y^X(C) := Hom(C \wedge X,\ Y). \end{cases}$$

In general, neither is representable.

d) Branched covering of singular flows.

Here we define "extended covering flows" of a singular flow. For simplicity, we will only consider the flows of FW_B, not the flows of $FW_B{}^{\vee}$ or $FW_B{}^{\wedge}$

Ref

10. MacLane, S., & Moerdijk, I.(1992). Sheaves in Geometry and Logic: A First Introduction to Topos Theory. Springer-Verlag New York, Inc.

Notes

Definition 4.28 (Regular loop). *A loop is called regular if it contains no singular triangles inside.*

Remark. We can embed a regular loop in a regular flow.

Definition 4.29 (Relation "$\leqslant_X$" on FW_B). Let $F_1, F_2 \in FW_B$. We define a binary relation "$\leqslant_X$" on FW_B by

$$F_1 \leqslant_X F_2$$

if and only if, for any regular loop $lp' \in Lp(F_2)$ there is a set

$$\{lp_1, lp_2, \ldots, lp_n\} \subset Lp(F_1)$$

$(n \in \mathbb{Z})$ *such that*

$$lp' = \sum_{i=1,\, n} lp_i.$$

We write $F_1 <_X F_2$ if $F_1 \leqslant_X F_2$ and $F_1 \neq F_2$.

We then extend the relation "$\leqslant_X$" on the set $P(FW_B)$ of all subsets of FW_B.

Definition 4.30 (Relation "$\leqslant_X$" on $P(FW_B)$). *Let $P_1, P_2 \in P(FLW_B)$. We define a binary relation "$\leqslant_X$" on $P(FW_B)$ by*

$$P_1 \leqslant_X P_2$$

if and only if, for any regular loop $lp' \in Lp(F')$ of $F' \in P_2$, there are $F \in P_1$ and

$$\{lp_1, lp_2, \ldots, lp_n\} \subset Lp(F)$$

$(n \in \mathbb{Z})$ *such that*

$$lp' = \sum_{i=1,\, n} lp_i.$$

We write $P_1 <_X P_2$ if $P_1 \leqslant_X P_2$ and $P_1 \neq P_2$.

Example 4.31. In Figure 9, $P_1 <_X P_0$, $P'_1 <_X P'_0$, and $P''_1 <_X P''_0$.

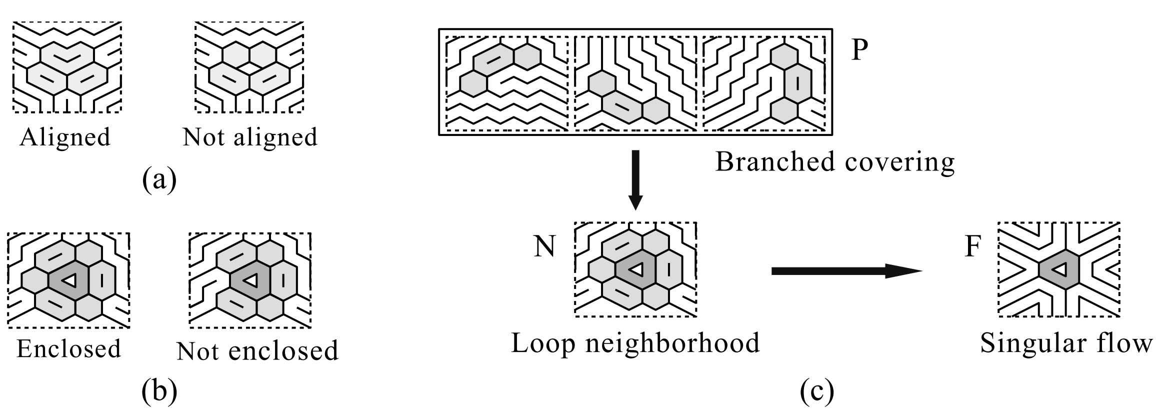


Figure 10: Multiplicity of signular flows. (a) Aligned flow. (b) Enclosing neighborhood of a loop. (c) Computation of the multiplicity of a loop

Definition 4.32 (Branched lower bound). *Let $C \subset P(FW_B)$. Let $P \in P(FW_B)$. P is called a branched lower bound of C if*

$$P <_X P' \quad \text{for all } P' \in C.$$

Flows contained in a branched lower bound P are called the sheets of P. A branched lower bound P is called regular if P consists only of regular flows.

Example 4.33. In figure 9, P_1 (resp. P'_1, P''_1) is a branched lower bound of one-element set P_0 (resp. P'_0, P''_0). P_1 contains three sheets. P'_1 and P''_1 contains two sheets.

Definition 4.34 (Branched covering of a flow). *Let $F \in FW_B$. We denote by $BCv(F)$ the set of all regular branched lower bounds of $\{F\}$, i.e.,*

$$BCv(F) := \{P \in P(FW_B) \mid \quad P \leqslant_X \{F\}, \; P \text{ is regular}\}.$$

Elements of $BCv(F)$ are called branched covering of F. Roughly speaking, the sheets of a branched covering collectively cover F. The multiplicity $m(P)$ of a branched covering P is defined as the number of sheets of P.

Example 4.35. In Figure 9, P_1 (resp. P'_1, P''_1) is a branched covering of F_0 (resp. F'_0, F''_0). Then, $m(P_1) = 3$, $m(P'_1) = 2$, and $m(P''_1) = 2$.

e) Multiplicity of loops.

Let $F \in FW_B$. Let $lp \in Lp(F)$. To capture the "turbulence" of the flow F around the loop lp, we consider a set of loops surrounding lp.

Definition 4.36 (Aligned flow). *Let $F \in FW_B$. F is called aligned if, for any pair lp_1, $lp_2 \in Lp(F)$, the vertices of the contour of lp_1 are not contained in the interior of the edges of the contour of lp_2 (Figure 10 (a)).*

Definition 4.37 (Enclosing neighborhood of a flow). *Let $N, F \in FW_B$. N is called an enclosing neighborhood of F if $Lp(F) \subset Lp(N)$ and the contour of the region swept by $Lp(N)$ dose not contain the vertices of the contour of $Lp(F)$.*

Example 4.38. In Figure 10 (b), F consists of one flow (colored dark grey). In Figure 10 (b) left, the dark grey loop is encolsed by a set of six loops. On the other hand, in Figure 10 (b) right, a part of the grey loop is exposed outside.

Remark. $N \leqslant_X F$ if N is an enclosing neighborhood of F.

Definition 4.39 (Loop neighborhood of a loop). *Let $F \in FW_B$. Let N be an enclosing neighborhood of F. N is called a loop neighborhood of F if N is aligned.*

Definition 4.40 (Multiplicity of a flow). *Let $F \in FW_B$. The multiplicity $mul(F)$ of flow F is defined by*

$$mul(F) := \min\{ \quad m(P) \mid P \in BCv(N), \; N \text{ is a loop neighborhood of } F \}.$$

If $Lp(F) = \{lp\}$, $mul(F)$ is called the multiplicity of loop lp and denoted by $mul(lp)$.

Remark. The multiplicity of an affine flow is one.

Notes

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

Notes

Figure 10 (c) shows the computation process of the singular loop considered in the introduction (Figure 1). Suppose that we are given a flow F consisting of the singular loop (lower right).

First, find a loop neighborhood N of the flow (lower left). In this case, N contains three regular loops of length six, three regular loops of length 10, and the singular loop.

Next, find a branched covering P of N (upper left). In this case, P has three sheets. Because of the overlap of the lifted trajectories in S, two of the three regular loops of length 10 cannot be embedded in one sheet at the same time. That is,

$$mul_{bc}(N) := \min\{m(P) \mid P \in BCv(N)\} = 3.$$

With a little consideration, we can see that N give the minimum multiplicity, i.e.,

$$mul(F) = 3.$$

Remark. Let N, N' be two loop neighborhoods of the same flow F. The author dose not know whether

$$mul_{bc}(N) = mul_{bc}(N')$$

or not.

f) Singular triangle detection by the multiplicity

Recall that the aim of this paper is to detect the presence of singular triangles inside by examining the outer contour of a given loop. Since we can compute the multiplicity of a loop from the outside, we can use the multiplicity of the loop for that porpose.

Proposition 4.41. Let $F \in FW_B$.
(1) If $mul(F) > 1$, then F is a singular flow.
(2) If $mul(F) = 1$, then $ptch(lp) = 0$ for all $lp \in Lp(F)$.

Proof. It follows immediately from the definition.

Corollary 4.42. Let lp be a loop of B. If $mul(lp) > 1$, then there are singular triangles inside lp.

Example 4.43. In Figure 9 (a), the dark grey loop lp of F_0 contains singular loops inside since $mul(lp) = 3 > 1$. Note that F_0 is a loop neighborhood of a flow consisting only of lp.

Example 4.44. In Figure 9 (b), the dark grey loop lp' of F'_0 contains singular loops inside since $mul(lp') = 2 > 1$. Note that F'_0 is a loop neighborhood of a flow consisting only of lp'.

Example 4.45. In Figure 9 (c), the dark grey loop lp'' of F''_0 contains singular loops inside since $mul(lp'') = 2 > 1$. Note that F''_0 is a loop neighborhood of a flow consisting only of lp''.

Recall that in the last two examples, the differential geometry approach failed to detect the presence of singular triangles inside (Example 3.35 and 3.36).

V. Conclusion

Using a simple mathematical model, we have presented two approaches to inferring the internal structure of proteins from the outside. One is the differential geometric approach proposed in [1]. The other is a new category theoretical approach proposed in this paper.

In the former approach, we calculate the pitch of a given loop. In the latter approach, we compute the multiplicity of a neighborhood of a given loop. We then showed that the new approach can detect more singular triangles inside than the previous approach.

VI. Discussion

This research is intended to be applied to the structural study of proteins. First, we represented proteins as a loop of triangles (i.e., 2-simplices). Second, we proposed a new method to infer the internal structure of a protein (i.e., a loop) from the turbulence of a droplet (i.e., a loop neighborhood) surrounding the protein. Third, as an example, we considered the detection of singularities (i.e., singular triangles) in a protein.

In relation to these three points, three issues come to mind for discussion: first, how to approximate the shape of a protein using the loops of n-simplices; second, how to measure the turbulence (multiplicity) of a droplet of biomolecules surrounding a protein; and third, why we considered singular triangle detection. Let us discuss these issues in turn.

(1) Due to strong constraints on the geometry of loops of n-simplices, it is not straightforward to approximate the folded structure of proteins using a loop of n-simplices. However, it is this simplification that allowed us to obtain a simple mathematical model of the relation between the internal structure and the external shape of proteins. The author hopes that this research will serve as a stepping stone to obtain better mathematical models of proteins in the future, which can handle the internal structure and the external shape simultaneously.

(2) In recent years, droplets of biomolecules have been witnessed everywhere in cells. In particular, the idea that their functions emerge from the collective behaviors of the molecules has become the central concept in condensate biology ([9]). However, since droplets are often formed transiently, it is difficult to measure their movement. Nev ertheless, the author believes that even ridiculous needs can lead to the development of novel measurement techniques for droplets.

(3) In physics, particles correspond to singular points of the function representing their interaction when we consider the interaction between them. In this sense, it is natural to consider the influence of singular triangles on their surroundings when we consider the interaction between loops of triangles. However, actual measurements are required to determine whether proteins have "internal singularities" or not (in addition to the definition of "internal singularities" of proteins).

Finally, the author would like to mention some future research topics.

(1) Change of the base space. For example, a loop of tetrahedra induces a flow of triangles on its surface. It is interesting to consider what kind of triangular flow can be obtained if the base space is the surface of a tetrahedral loop. The author is also curious as to whether there is any flow that cannot be obtained as a surface flow of a tetrahedral loop.

(2) Classification of covering flows. Sometimes a loop (L_1) of triangles will interact with another loop (L_2) only after it has interacted with a third loop (L_3). In other words, the interaction of L_1 and L_2 is regulated by the presence of L_3 (a long distance interaction between L_2 and L_3). Then, L_1 is called an "allosteric" loop [11]. It will be

Ref

1. Morikawa, N. (2014). Discrete Differential Geometry of n Simplices and Protein Structure Analysis. *Applied Mathematics*, 5, 2458-2463. https://doi.org/10.4236/am.2014.516237

interesting to see if we can characterize “allosteric” loops simply by considering their covering flows.

(3) Flows in higher dimensions. When applied to protein structure analysis, we need to consider loops of tetrahedra. There seems to be a large gap in difficulty between the study of flows of triangles and the study of flows of n-simplices ($n > 2$). However, because of the simplicity of the model, the author believes that we can jump over the gap and think about flows in higher dimensions. Even with this simple model, the gap may produce interesting results in higher dimensions.

Notes

Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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Type II Half Logisitic Exponentiated Lomax Distribution

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Abstract- In this paper, we introduces a new distribution called Type II half logistic exponentiated lomax (TIIHLEE) distribution established from Type II half logistic G family of distribution We investigate some of its mathematical and statistical properties such as the explicit form of the ordinary moments, moment generating function, conditional moments, mean deviations, residual life and Renyi entropy. The maximum likelihood method is used to estimate the model parameters. Simulation studies were conducted to assess the finite sample behavior of the maximum likelihood estimators. Finally, we illustrate the importance and applicability of the model by the study of two real data sets.

Keywords: *type I half logistic distribution, lomax distribution, quantile function, moments, maximum likelihood estimate.*

GJSFR-F Classification: *MSC 2010: 11D61*

TYPEIIHALFLOGISITICEXPONENTIATEDLOMAXDISTRIBUTION

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Type II Half Logisitic Exponentiated Lomax Distribution

Bukoye, Abdulwasiu [α], Bada Olatunbosn [σ], Muritala Abdul Kabir [ρ] & Ajadi, A. Akeem [Ѡ]

Abstract- In this paper, we introduces a new distribution called Type II half logistic exponentiated lomax (TIIHLEE) distribution established from Type II half logistic G family of distribution We investigate some of its mathematical and statistical properties such as the explicit form of the ordinary moments, moment generating function, conditional moments, mean deviations, residual life and Renyi entropy. The maximum likelihood method is used to estimate the model parameters. Simulation studies were conducted to assess the finite sample behavior of the maximum likelihood estimators. Finally, we illustrate the importance and applicability of the model by the study of two real data sets.

Keywords: type I half logistic distribution, lomax distribution, quantile function, moments, maximum likelihood estimate.

I. Introduction

Lomax distribution can also be called Pareto Type II distribution and its application can be found in many fields like actuarial science, economics, and so on Al-Zahrani and Al-Sobhi (2013). The distribution was defined by Lomax (1954) and it is a heavy-tailed distribution. It has also been considered to be useful in reliability and life testing problems in engineering and in survival analysis as an alternative distribution Kilany (2016), Hassan and Al-Ghamdi (2009).

Modified and extended versions of the Lomax distribution have been studied; examples include the weighted Lomax distribution by Kilany (2016), exponential Lomax distribution by El-Bassiouny et al (2015), exponentiated Lomax distribution by Salem (2014), gamma Lomax distribution Cordeiro et al (2015), transmuted Lomax distribution by Ashour and Eltehiwy (2013), Poisson Lomax distribution by Al-Zahrani and Sagor (2014), McDonald Lomax distribution by Lemonte and Cordeiro (2013), Weibull Lomax distribution by Tahir et al (2015), and power Lomax distribution [12]. Besides, estimation of the parameters of Lomax distribution under general progressive censoring has also been considered by Al-Zahrani and Al-Sobhi by Al-Zahrani and Al-Sobhi (2013).

The half logistic distribution is a member of the family of logistic distributions which is introduced by (Balakrishnan, 1985) which has the following cumulative distribution function (CDF)

$$F(t) = \frac{1-e^{\lambda t}}{1+e^{\lambda t}} \qquad t>0, \quad \lambda>0 \, . \tag{1}$$

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Ref

12. Law, A. & Kelton, W. (1991). *Simulation Modeling and Analysis.* McGraw-Hill, Inc. New York.

The associated probability density function (pdf) corresponding

$$f(t) = \frac{2e^{-\lambda t}}{(1+e^{\lambda t})^2} \qquad t > 0, \quad \lambda > 0 \tag{2}$$

Hassan et al. (2017) use the half logistic generator instead of the gamma generator to obtain type II half logistic family, which is denoted by TIIHL-*G*. Then the CDF and pdf of TIIHL-G is defined as follows

$$F(x;\lambda) = 1 - \int_0^{-\log G(x)} \frac{2\lambda e^{-\lambda}}{\left(1+e^{-\lambda t}\right)^2} dt = \frac{2[G(x)]^{\lambda}}{1+[G(x)]^{\lambda}} \qquad x > 0, \lambda > 0 \tag{3}$$

$$f(x;\lambda) = \frac{2\lambda g(x)[G(x)]^{\lambda-1}}{\left[1+[G(x)]^{\lambda}\right]^2} \qquad x > 0, \lambda > 0 \tag{4}$$

Where λ is the shape parameter and g(x;λ), and G(x;λ) is the baseline distribution, respectively.

However, the TIIHLEE has tractable properties, especially for simulation, since its quantile function takes a simple form.

$$Q(u) = G^{-1}\left[\frac{u}{2-u}\right]^{\frac{1}{\lambda}} \tag{5}$$

Where u is a uniform distribution on the interval (0,1) and G^{-1} (.) is the inverse function of G(.)

Our aim in this work is to study a modified statistical distribution that will be suitable to fit positively skewed and unimodal data and to check the flexibility of the existing and proposed distribution.

II. Type II Half Logistic Exponentiated Lomax Distribution

The cumulative density function (CDF) and probability density function (pdf) of exponentiated Lomax distribution (EE) are defined as follows:

$$G(x) = \left(1-(1+\beta x)^{-\theta}\right)^{\alpha} \tag{6}$$

$$g(x) = \alpha\beta\theta\left(1-(1+\beta x)^{-\theta}\right)^{\alpha-1}(1+\beta x)^{-(\theta+1)} \tag{7}$$

Substituting 6 and 7 in 4 and 3 then we define probability density function (pdf) and cumulative density function (CDF) Type II Half Logistic Exponentiated Lomax Distribution (TIIHLEL) as follows:

$$f(x;\alpha,\beta,\lambda,\theta) = \frac{2\alpha\beta\lambda\theta\left[1-(1+\beta x)^{-\theta}\right]^{\alpha-1}(1+\beta x)^{-(\theta+1)}\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^2} \tag{8}$$

Notes

$$F(x;\alpha,\beta,\lambda,\theta)=\frac{2\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}}{1+\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}} \tag{9}$$

Henceforth, a random variable with probability density function (pdf) is denoted by $X \sim TIIHLEL(\alpha,\beta,\lambda,\theta)$

Notes

a) Hazard Function of TIIHLEL

$$h(x;\alpha,\beta,\lambda,\theta)=\frac{f(x)}{F(x)}=\frac{2\lambda g(x)[G(x)]^{\lambda-1}}{1-[G(x)]^{2\lambda}}$$

$$=\frac{2\alpha\beta\lambda\theta\left(1-(1+\beta x)^{-\theta}\right)^{\alpha-1}(1+\beta x)^{-(\theta+1)}\left[\left(1-(1+\beta x)^{\theta}\right)^{\alpha}\right]^{\lambda-1}}{\left(1-\left(\left(1-(1+\beta x)^{\theta}\right)^{\alpha}\right)^{2\lambda}\right)} \tag{10}$$

b) Survival Function of TIIHLEL

$$\overline{F}(x;\alpha,\beta,\lambda,\theta)=1-F(x;\alpha,\beta,\lambda,\theta)$$

$$=1-\left[\frac{1-[G(x;\alpha,\beta,\lambda,\theta)]^{\lambda}}{1+[G(x;\alpha,\beta,\lambda,\theta)]^{\lambda}}\right]=1-\left[\frac{1-\left(\left(1-(1+\beta x)^{-\theta}\right)^{\alpha}\right)^{\lambda}}{1+\left(\left(1-(1+\beta x)^{-\theta}\right)^{\alpha}\right)^{\lambda}}\right] \tag{11}$$

Plots of the density function (8) can be represented through Fig. 1. As it seems from Fig. 1 that the pdf of TIIHLEL can take different shapes according to different values of θ and λ. It can be symmetric, right-skewed, unimodal

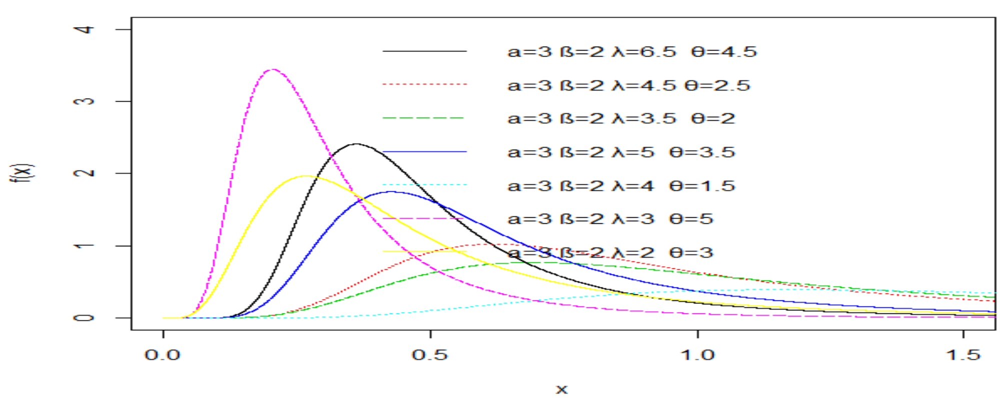

Figure 1: Plot of the Density function of TIIHLEL

Plots of cumulative distribution function (9) for the TIIHLEL distribution are displayed in Fig. 2. The CDF graph tend to move at a constant value from 0 to1 on the x-axis before projecting to one on the y-axis

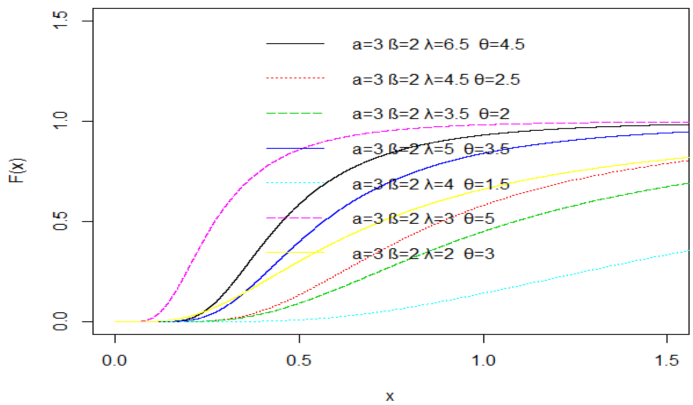

Figure 2: Plot of the Distribution function (CDF) of TIIHLEL

Plots of hazard function (10) for the TIIHLEL distribution are displayed in Fig. 3 It was examined from Fig. 3 show that the shape of the h(x) of the TIIHLEL shows the hazard functions can be constant, increasing or decreases, depending on the shape and scale parameters.

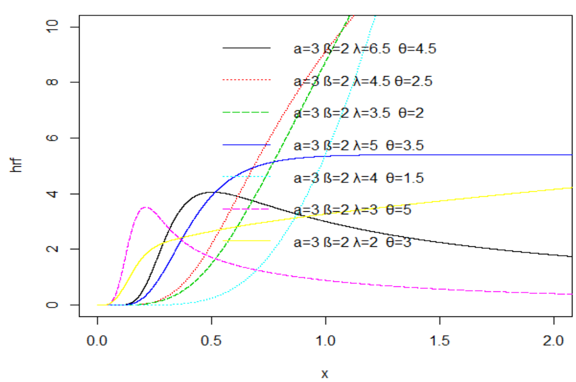

Figure 3: Plot of the Hazard function of TIIHLEL

Plots of survival function (11) for the TIIHLEL distribution are displayed in Fig. 4. It was deduced from Fig. 4 that the shape parameters have a strong effect on the survival plot. A decrease in the shape parameter tends to reduce the movement of the line on the y-axis, and no plot is exceeding the benchmark of 1

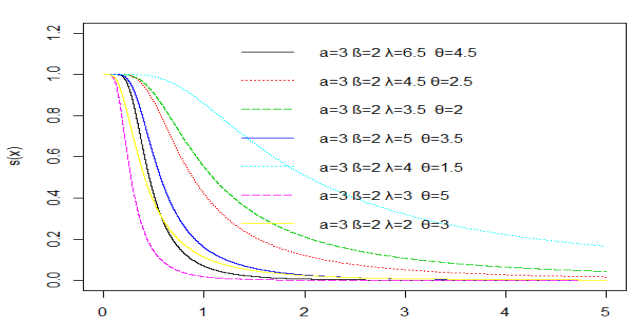

Figure 4: Plot of the Survival function of TIIHLEL

Notes

Notes

c) Some Special case of TIIHLEL

1. When α=1, the TIIHLEL model reduces to Type II half Logistic Lomax (TIIHLL) model with λ and θ to shape parameter and β scale parameters.

$$f(x;\beta,\lambda,\theta)=\frac{2\beta\lambda\theta(1+\beta x)^{-(\theta+1)}\left[1-(1+\beta x)^{-\theta}\right]^{\lambda-1}}{\left[1+\left[1-(1+\beta x)^{-\theta}\right]^{\lambda}\right]^{2}}$$

2. When β=1 the TIIHLEL model reduces to Type II half Logistic Exponentiated Pareto (TIIHLEP) model with λ, θ and α are shape parameters.

$$f(x;\alpha,\lambda,\theta)=\frac{2\alpha\lambda\theta\left[1-(1+x)^{-\theta}\right]^{\alpha-1}(1+x)^{-(\theta+1)}\left[\left[1-(1+x)^{-\theta}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left[1-(1+x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^{2}}$$

3. When λ=1, the TIIHLEL model reduces to Half Logistic Exponentiated Lomax (HLEL) model with α and θ are shape parameters λ is scale parameter.

$$f(x;\alpha,\beta,\theta)=\frac{2\alpha\beta\theta\left[1-(1+\beta x)^{-\theta}\right]^{\alpha-1}(1+\beta x)^{-(\theta+1)}}{\left[1+\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]\right]^{2}}$$

4. When α=β=1, the TIIHLEL model reduces to the Type II Half Logistic Pareto (TIIHLP) model with λ and θ are shape parameters.

$$f(x;\lambda,\theta)=\frac{2\lambda\theta(1+x)^{-(\theta+1)}\left[(1+x)^{-\theta}\right]^{\lambda-1}}{\left[1+\left[1-(1+x)^{-\theta}\right]^{\lambda}\right]^{2}}$$

5. When α=λ=1, the TIIHLEL model reduces to Half Logistic Lomax (HLL) model with λ and θ and α are shape parameters.

$$f(x;\beta,\theta)=\frac{2\beta\theta(1+\beta x)^{-(\theta+1)}}{\left[1+\left[1-(1+\beta x)^{-\theta}\right]\right]^{2}}$$

6. When α=λ=β=1, the TIIHLEL model reduces to Half Logistic Pareto (HLP) model, with θ scale parameter.

$$f(x;\beta,\theta)=\frac{2\theta(1+x)^{-(\theta+1)}}{\left[1+\left[1-(1+x)^{-\theta}\right]\right]^{2}}$$

Table 1: Summary of Sub-Models from the TIHLEL distribution

Distribution	α	β	λ	θ
TIIHLL	1	β	λ	θ
TIIHLEP	α	1	λ	θ
HLEL	α	β	1	θ
TIIHLP	1	1	λ	θ
HLL	1	β	1	θ
TIIHLEL	1	1	1	θ

Table 1 depict some existing model obtained when some parameters of the proposed distribution are relaxed.

III. Mathematical Properties for TIIHLEL

a) Infinite Linear Combination

$$f(x) = 2\lambda\alpha\beta\theta\left(1-(1+\beta x)^{-\theta}\right)^{\alpha-1}(1+\beta x)^{-(\theta+1)}\left(1-(1+\beta x)^{-\theta}\right)^{\alpha(\lambda-1)} \times \left(\left(1+1-(1+\beta x)^{-\theta}\right)^{\alpha\lambda}\right)^{-2}$$

$$f(x) = 2\lambda\alpha\beta\theta\left(1-(1+\beta x)^{-\theta}\right)^{\alpha\lambda-1}(1+\beta x)^{-(\theta+1)}\left(\left(1+1-(1+\beta x)^{-\theta}\right)^{\alpha\lambda}\right)^{-2}$$

Let v=λα

$$f(x) = 2v\beta\theta\left(1-(1+\beta x)^{-\theta}\right)^{v-1}(1+\beta x)^{-(\theta+1)}\left(\left(1+1-(1+\beta x)^{-\theta}\right)^{v}\right)^{-2}$$

$$= 2v\beta\theta\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(v)}{\Gamma(v-k)k!}(1+\beta x)^{-\theta_k-\theta-1}\sum_{w=0}^{\infty}\binom{-2}{w}\binom{wv}{i}(-1)^i(1+\beta x)^{-\theta_i}$$

$$= 2v\beta\theta\sum_{k=0}^{\infty}\sum_{w=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{k+i}\frac{\Gamma(v)}{(v-k)k!}\binom{-2}{w}\binom{wv}{i}(1+\beta x)^{-\theta_k-\theta_i-\theta-1}$$

$$f(x) = 2v\theta\sum_{k=0}^{\infty}\sum_{w=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{k+i}\frac{\Gamma(v)}{(v-k)k!}\binom{-2}{w}\binom{wv}{i}\beta(1+\beta x)^{-\theta_k-\theta_i-\theta-1}$$

Let

$$\varphi_{i,k,w} = 2v\theta\sum_{k=0}^{\infty}\sum_{w=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{k+i}\frac{\Gamma(v)}{(v-k)k!}\binom{-2}{w}\binom{wv}{i}$$

$$f(x) = \varphi_{i,k,w}\beta(1+\beta x)^{-\theta_k-\theta_i-\theta-1} \tag{12}$$

b) Moment of Proposed TIIHLEL Distribution

It is imperative to derive the moments when a new distribution is proposed. They play a significant role in statistical analysis, particularly in applications. Moments are used in computing measures of central tendency, dispersion, and shapes, among others.

Suppose the rth moment for the TIIIHL-*G* family is derived.

$$E(X^r) = \mu^r = \int_0^\infty x^r f(x)dx \tag{13}$$

$$= \int_0^\infty x^r \phi_{i,k,w}\beta(1+\beta x)^{-\theta_k-\theta_i-\theta-1}dx$$

$$= \phi_{i,k,w}\int_0^\infty x^r \beta(1+\beta x)^{-(\theta_k+\theta_i+\theta+1)}dx$$

Let

$$\beta x = a \qquad x = \frac{a}{\beta} \qquad dx = \frac{da}{\beta}$$

Notes

Notes

$$= \phi_{i,k,w}\beta\int_0^\infty \frac{\left(\frac{a}{\beta}\right)^r}{(1+a)^{-(\theta_k+\theta_i+\theta+1)}}\frac{da}{\beta}$$

$$= \phi_{i,k,w}\beta^{-r}\beta\int_0^\infty \frac{a^r}{(1+a)^{-(\theta_k+\theta_i+\theta+1)}}da \tag{14}$$

$$\int_0^\infty \frac{Z^{x-1}}{(1+Z)^{x+y}}dz = B(x,y)$$

Beta function is given

$$x\text{-}1=r \qquad x=r+1 \qquad x+y=\theta_k+\theta_i+\theta+1 \tag{15}$$

Therefore y= $\theta_k+\theta_i+\theta-r$

Comparing 3.79 and 3.80 we have

$$\mu_1^r = \phi_{i,k,w}\beta^{-r+1}\int_0^\infty \frac{a^r}{(1+a)^{-(\theta_k+\theta_i+\theta+1)}}da = \phi_{i,k,w}\beta^{-r+1}B(r+1,\theta_k+\theta_i+\theta-r)$$

$$\mu_1^r = \phi_{i,k,w}\beta^{-r+1}B(r+1,\theta_k+\theta_i+\theta-r) \tag{16}$$

The mean of the proposed TIIHLEL distribution is gotten by making μ^r moment equal to one (—r=1)

$$\mu_1^1 = \phi_{i,k,w}B(2,\theta_k+\theta_i+\theta-1) \tag{17}$$

When r=2

$$\mu_1^2 = \phi_{i,k,w}\beta^{-1}B(3,\theta_k+\theta_i+\theta-2) \tag{18}$$

When r=3

$$\mu_1^3 = \phi_{i,k,w}\beta^{-2}B(4,\theta_k+\theta_i+\theta-3) \tag{19}$$

When r=4

$$\mu_1^4 = \phi_{i,k,w}\beta^{-3}B(4,\theta_k+\theta_i+\theta-4) \tag{20}$$

The variance of TIIHLEL can is obtain

$$\mathrm{var}(x) = \sigma^2 = E(x^2) - [E(x)]^2 = \mu_j^2 - \left(\mu_j^1\right)^2 \tag{21}$$

Substituting equation 17 and 18 into 21 to obtain the variance of TIIHLEL

$$\sigma^2 = \phi_{i,k,w}\beta^{-1}B(3,\theta_k+\theta_i+\theta-2) - \left(\phi_{i,k,w}B(2,\theta_k+\theta_i+\theta-1)\right)^2 \tag{22}$$

Standard Deviation of TIIHLEE

$$\sigma = Std(x) = \sqrt{Var(x)}$$

$$\sigma = \sqrt{\phi_{i,k,w}\beta^{-1}B(3,\theta_k+\theta_i+\theta-2)-\left(\phi_{i,k,w}B(2,\theta_k+\theta_i+\theta-1)\right)^2} \tag{23}$$

Coefficient of Variation of TIIHLEE

$$CV = \frac{S\tan dard\ deviation(x)}{E(x)}$$

$$CV = \frac{\sqrt{\phi_{i,k,w}\beta^{-1}B(3,\theta_k+\theta_i+\theta-2)-\left(\phi_{i,k,w}B(2,\theta_k+\theta_i+\theta-1)\right)^2}}{\phi_{i,k,w}B(2,\theta_k+\theta_i+\theta-1)} \tag{24}$$

c) Moment Generating Function of TIIHLEE Distribution

A random variable x with pdf f(x) is defined as a

$$M_x(t) = E(e^{tx})$$

$$E(e^{tx}) = \int_0^\infty e^{tx} f(x)dx$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^1(x)$$

$$M_x(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\phi_{i,k,w}\beta^{-r+1}B(r+1,\theta_k+\theta_i+\theta-r) \tag{25}$$

d) Incomplete Moment

The incomplete moment plays an important role in computing the mean deviation, median deviation.

The r^th incomplete moment of the TIIHLEL is given by

$$v_s(t) = \int_0^\infty x^s f(x)dx$$

$$= 2v\theta\sum_{k=0}^{\infty}\sum_{w=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{k+i}\frac{\Gamma(v)}{(v-k)k!}\binom{-2}{w}\binom{wv}{i}\beta^{-r+1}B(r+1,\theta_k+\theta_i+\theta-r) \tag{26}$$

Where B(.,.) is the incomplete beta function and r=1,2,3…..

IV. Mean Deviation

The variation in a population can be measured to some degree by the totality of deviations from the mean and the median. If the random variable X follows the TIIHLEL distribution, then the mean and the median deviations are given as

Notes

$$\delta_1(x) = \int_0^\infty |x-\mu| g(x)dx$$

$$= \int_0^\mu (\mu - x) g(x)dx + \int_\mu^\infty (\mu - x) g(x)dx$$

$$= 2\mu G(\mu) - 2\int_0^\mu x g(x)dx$$

Notes

$$\delta_1(x) = 2\mu G(\mu) - 4v\theta \sum_{k=0}^\infty \sum_{w=0}^\infty \sum_{i=0}^\infty (-1)^{k+i} \frac{\Gamma(v)}{(v-k)k!} \binom{-2}{w} \binom{wv}{i} \beta^{-r+1} B(r+1, \theta_k + \theta_i + \theta - r) \qquad \theta > 1$$

Where $\int_0^\mu xg(x)dx$ is simplified using the first incomplete moment

$$\delta_2(x) = \mu - 4v\theta \sum_{k=0}^\infty \sum_{w=0}^\infty \sum_{i=0}^\infty (-1)^{k+i} \frac{\Gamma(v)}{(v-k)k!} \binom{-2}{w} \binom{wv}{i} \beta^{-r+1} B(r+1, \theta_k + \theta_i + \theta - r) \quad \theta > 1 \qquad (27)$$

a) Renyi Entropy

Entropy plays a vital role in science, engineering, and probability theory, and has been used in various situations as a measure of variation or uncertainty of a random variable (Renyi, 1961). The Renyi entropy of TIIHLEL is given as

b) Quantile and Median

Simulation methods utilize quantile function to produce simulated random variables for classical and new continuous distributions.

The inverse of the CDF in (6) yields the quantile function of the

$$x = \frac{1}{\beta}\left(1 - \left(\left(\left(\frac{p}{2-p}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\theta}} - 1\right) \qquad (28)$$

When $p = 0.25, 0.5\ and\ 0.75$ we obtain the first quantile, median and third quartile of TIIHLEL distribution

In particular $Q(0.5) = F^{-1}(u)$ is the median of the probability distribution given as

$$F(x) = p_r(X \leq m) = \int_0^m f(x)dx = 0.5$$

The median of TIIHLEL Distribution can be obtained by equating equation 28 to 0.5

$$x = \frac{1}{\beta}\left(1 - \left(\left(\left(\frac{1}{3}\right)^{\frac{1}{\lambda}}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\theta}} - 1\right) \qquad (29)$$

c) Order of Statistics

Let $Y_1, Y_2, \ldots\ldots\ldots Y_n$ be independently and identically distributed (i .i.d) random variable with their corresponding cumulative function (cdf) $F(x)$. Let $X_{(1)}$ be the smallest (first-order statistic) of the $Y_1, Y_2, \ldots\ldots\ldots Y_n$, $X_{(2)}$ is the second-order statistics greater than $X_{(1)}$, and $X_{(n)}$ is the largest order statistics. Hence, $X_{(1)} < X_{(2)} \ldots\ldots.. < X_{(n)}$ is the order statistics corresponding to random variable $Y_1, Y_2, \ldots\ldots\ldots Y_n$ (see Kapur and Sexena (1960)).

The density $f_{n:i}(x)$ of the ith order statistics, for $i = 1, \ldots\ldots n$, from independent identical distribution random variable $Y_1 \ldots\ldots Y_n$ is given by

$$f_{n:i}(x) = \frac{f(x)}{B(i, n-i+1)} F(x)^{i-1}\left(1 - F(x)\right)^{n-i}$$

i. *Order Statistic of TIIHLEL*

$$f(x;\alpha,\beta,\lambda) = \frac{f(x;\alpha,\beta,\lambda,\theta)}{B(r,n-r-1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x;\alpha,\beta,\lambda,\theta)^{v+r-1} \tag{30}$$

$B(.,.)$is the beta function. The pdf of the r[th] order statistic for TIIHLEL distribution is derived by substituting (6) and (7) in (30) as follows.

$$f(x;\alpha,\beta,\lambda,\theta) = \frac{1}{B(r,n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} \frac{\alpha\beta\lambda\theta\left[1-(1+\beta x)^{-\theta}\right]^{\alpha-1}(1+\beta x)^{-(\theta+1)}\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda(v+r)-1}}{\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda+r+1}} \tag{31}$$

Applying the binomial expansion (12) in (31), then we have

$$f(x;\alpha,\beta,\lambda) = \frac{1}{B(r,n-r+1)} \sum_{v=0}^{n-r} (-1)^{v+i} \binom{n-r}{v}\binom{v+r+i}{i}$$

$$\times \alpha\beta\lambda\theta\left[1-(1+\beta x)^{-\theta}\right]^{\alpha-1}(1+\beta x)^{-(\theta+1)}\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda(v+r+i)-1}$$

Again, using the binomial expansion (12) in the previous equation, then the pdf of the r[th] order statistic for TIIHLEE distribution is obtained as follows.

$$f(x;\alpha,\beta,\lambda) = \frac{1}{B(r,n-r+1)} \sum_{v=0}^{n-r} \sum_{I=0}^{v+r-1} \sum_{J=0}^{\infty} \tag{32}$$

$$\eta^{*} = (-1)^{v+i+j} \binom{n-r}{v}\binom{v+r+i}{i}\binom{\lambda(v+r+i)-1}{j}$$

The distribution of the smallest and largest order statistics are obtained by putting $r = 1$ and $r = n$ in (32) respectively as follows.

$$f_{(1)}(x;\alpha,\beta,\lambda) = \frac{1}{B(r,n-r+1)} \sum_{k=0}^{\infty} \sum_{w=0}^{\infty} \sum_{i=0}^{\infty} \beta(1+\beta x)^{-\theta_k-\theta_i-\theta-1} \tag{33}$$

$$\eta^{**} = (-1)^{v+i+j} \binom{n-1}{v}\binom{v+1+i}{i}\binom{\lambda(v+1+i)-1}{j}$$

Notes

Notes

However

ii. *Skewness and kurtosis of the TIIHLEL distribution*

Skewness is used to measure the asymmetry, and kurtosis is used to measure the peakedness of probabilistic models. Both measures are the descriptive measures of the shape of the probability distribution.

The skewness and Kurtosis of TIIHLEE are given as follows:

$$\text{Skewness} = \frac{E(x-\mu)^3}{\sigma^3} = \frac{\mu_3}{\sigma^3} = \frac{\mu_3^1 - 3\mu_1^1\mu_2^1 + 2\mu_1^3}{\left(\mu_2^1 - \mu^2\right)^{3/2}}$$

$$\text{Skewness}=\frac{\phi_{i,k,w}\beta^{-2}B(4,\theta_k+\theta_i+\theta-3)}{\left(\sqrt{\phi_{i,k,w}\beta^{-1}B(3,\theta_k+\theta_i+\theta-2)-\left(\phi_{i,k,w}B(2,\theta_k+\theta_i+\theta-1)\right)^2}\right)^3} \quad (34)$$

And

$$\text{Kurtosis}=\frac{E(x-\mu)^4}{\sigma^4} = \frac{\mu_4}{\sigma^4} = \frac{\mu_4^1 - 4\mu_1^1\mu_3^1 + 6\mu_1^2\mu_3^1 - 3\mu_3}{\left(\mu_2^1 - \mu^2\right)^2}$$

$$\text{Kurtosis}=\frac{\phi_{i,k,w}\beta^{-3}B(4,\theta_k+\theta_i+\theta-4)}{\left(\sqrt{\phi_{i,k,w}\beta^{-1}B(3,\theta_k+\theta_i+\theta-2)-\left(\phi_{i,k,w}B(2,\theta_k+\theta_i+\theta-1)\right)^2}\right)^4} \quad (35)$$

V. Parameter Estimation

a) *Maximum Likelihood Estimator (MLE) for TIIHLEL*

Let $x_1; x_2; \ldots, x_n$ be a random sample of size *n* from the TIIHL family of distributions $(\alpha; \beta; \lambda, \theta)$. The log-likelihood function for the vector of parameters $L = (\alpha; \beta; \lambda, \theta)^T$ can be expressed as

Let $f\left(x_1, x_2, x_3, \ldots, x_n; \alpha, \beta, \lambda, \theta\right) = \prod_{i=1}^{n} f(x)$

$$\prod_{i=1}^{n} \frac{2\alpha\beta\lambda\theta\left[1-(1-\beta x)^{-\theta}\right]^{\alpha-1}(1+\beta x)^{-(\theta+1)}\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[\left[1-(1+\beta x)^{-\theta}\right]^{\alpha}\right]^{\lambda}\right]^2}$$

The log-likelihood function is expressed as and let $w=(1+\beta x)^{-\theta}$

$$l = n\log 2 + n\log\lambda + n\log\beta + n\log\theta + \alpha - 1\sum_{i=1}^{n}\log(1-w) -$$
$$(\theta+1)\sum_{i=1}^{n}\log(1+\beta x) + (\lambda+1)\sum_{i=1}^{n}\log(1+w^{\alpha}) - 2(\lambda-1)\log\sum_{i=1}^{n}(1-w^{\alpha}) \quad (36)$$

Taking the first partial derivatives of $\ell(x; \alpha, \beta, \lambda, \theta)$ of (36) with respect to α, β, and λ and letting them equal zero, we obtain a nonlinear system of equations.

$$U_n(\varphi) = \left[\frac{\delta \log l}{\delta \alpha}, \frac{\delta \log l}{\delta \beta}, \frac{\delta \log l}{\delta \lambda}, \frac{\delta \log l}{\delta \theta}\right]$$

$$\frac{\delta \log l}{\delta \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1-w) + (\lambda - 1)\sum_{i=1}^{n} \frac{w^{\alpha} \log w}{1-w} + 2\lambda \sum_{i=1}^{n} \frac{w^{\alpha} \log w(1-w^{\alpha})^{\lambda-1}}{\left[1+\left(1+w^{\alpha}\right)^{\lambda}\right]}$$

$$\frac{\delta \log l}{\delta \beta} = \frac{n}{\beta} + (\alpha - 1)\sum_{i=1}^{n} \frac{\theta(1+\beta x)^{\theta-1}}{(1+\beta x)^{\theta}} + (\theta - 1)\sum_{i=1}^{n} \frac{x}{1-\beta x} + (\lambda - 1)\sum_{i=1}^{n} \frac{w^{\alpha} \log w}{\left(1+w^{\alpha}\right)} - 2\lambda\beta \sum \frac{\log w(1-w^{\alpha})^{\lambda-1}}{\left[1+\left(1-w^{\alpha}\right)^{\lambda}\right]}$$

$$\frac{\delta \log l}{\delta \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \frac{\alpha - 1}{(1-w)} - \sum_{i=1}^{n} (1-\beta x) + (\lambda - 1)\sum_{i=1}^{n} \frac{w^{\alpha} \log w}{\left(1+w^{\alpha}\right)} - 2\lambda \sum_{I=1}^{n} \frac{\log w^{\alpha}(1-w^{\alpha})^{\lambda-1}}{\left[1+\left(1-w^{\alpha}\right)^{\lambda}\right]}$$

The above-derived equations are in the complex form; therefore, the exact solution of ML estimator for unknown parameters is not possible. So it is convenient to use nonlinear Newton Raphson algorithm for exact numerically solution to maximize the above likelihood function.

However, the above equation cannot be solved analytically, and statistical software can be used to solve them numerically using iterative methods.

Solving the nonlinear system of the equation of $\frac{\delta l}{\delta \alpha} = 0$ $\frac{\delta l}{\delta \beta} = 0$ $\frac{\delta l}{\delta \lambda} = 0$, and $\frac{\delta l}{\delta \theta} = 0$ we then obtain the Maximum likelihood estimate α, β λ, and θ respectively. Information matrix of 4x4 will be obtained through

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \\ \hat{\theta} \end{pmatrix} = \left[\begin{pmatrix} \alpha \\ \beta \\ \lambda \\ \theta \end{pmatrix} \begin{pmatrix} \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha\lambda} & \hat{V}_{\alpha\theta} \\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta\lambda} & \hat{V}_{\beta\theta} \\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\lambda} & \hat{V}_{\lambda\theta} \\ \hat{V}_{\theta\alpha} & \hat{V}_{\theta\beta} & \hat{V}_{\theta\lambda} & \hat{V}_{\theta\theta} \end{pmatrix}\right]$$

$$V^{-1} = -E \begin{pmatrix} \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha\lambda} & \hat{V}_{\alpha\theta} \\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta\lambda} & \hat{V}_{\beta\theta} \\ \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\lambda} & \hat{V}_{\lambda\theta} \\ \hat{V}_{\theta\alpha} & \hat{V}_{\theta\beta} & \hat{V}_{\theta\lambda} & \hat{V}_{\theta\theta} \end{pmatrix}$$

$$V_{\alpha\alpha} = \frac{\delta^2 l}{\delta \alpha^2} \qquad V_{\beta\alpha} = V_{\alpha\beta} = \frac{\delta^2 l}{\delta \alpha\beta} \qquad V_{\alpha\lambda} = V_{\alpha\lambda} = \frac{\delta^2 l}{\delta \alpha\lambda}$$

$$V_{\beta\beta} = \frac{\delta^2 l}{\delta \beta^2} \qquad V_{\lambda\beta} = V_{\beta\lambda} = \frac{\delta^2 l}{\delta \beta\lambda} \qquad V_{\lambda\theta} = V_{\theta\lambda} = \frac{\delta^2 l}{\delta \theta\lambda}$$

$$V_{\lambda\lambda} = \frac{\delta^2 l}{\delta \lambda^2} \qquad V_{\theta\beta} = V_{\beta\theta} = \frac{\delta^2 l}{\delta \beta\theta} \qquad V_{\theta\alpha} = V_{\alpha\theta} = \frac{\delta^2 l}{\delta \alpha\theta}$$

$$V_{\theta\theta} = \frac{\delta^2 l}{\delta \theta^2} \qquad V_{\alpha\lambda} = V_{\lambda\alpha} = \frac{\delta^2 l}{\delta \alpha\lambda} \qquad V_{\lambda\beta} = V_{\beta\lambda} = \frac{\delta^2 l}{\delta \beta\lambda}$$

The solution to the above equation inverse dispersion matrix yields the asymptotic variance and covariance of the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ *and* θ

Notes

Notes

b) *Confidence Interval for the Parameters*

The approximate confidence intervals for 100(α-1)% for α, β and λ is given respectively by

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{V_{\alpha\alpha}}, \quad \hat{\beta} \pm Z_{\frac{\alpha}{2}} \sqrt{V_{\beta\beta}} \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}} \sqrt{V_{\lambda\lambda}} \quad \text{and} \quad \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{V_{\theta\theta}}$$

Where the $Z_{\frac{\alpha}{2}}$ is the α^{th} percentiles of the standard normal distribution.

VI. Simulation Study for TIIHLEL Distribution

A simulation study was carried out to check MLE of TIIHLEL for different sample size n=10 and 100 in order to obtain Mean, Median, Standard Deviation, Skewness, and Kurtosis of the parameters. The quantile function of TIIHLEL used for simulation studies is given in (28) with α, θ, and λ as shape parameters while β is a scale parameter.

Table 2: The Mean, Median and Standard of TIIHLEL Distribution for β = 2, 4 and 6 for n=10

α	λ	θ	β=2			β=4			β=6		
			Mean	Median	SD(σ)	Mean	Median	SD(σ)	Mean	Median	SD(σ)
0.5	0.5	0.5	0.4568	0.1146	1.1677	-0.0612	-0.0378	0.0769	-0.1373	-0.0853	0.1412
		0.8	72.4466	-0.000	321.5256	36.1165	-0.0098	160.6742	24.006	-0.0201	107.0571
		2.5	3.6512	-0.0056	15.774	1.7188	-0.0401	7.7977	-0.1627	-0.0956	0.1772
		5	0.0246	-0.0332	0.3320	-0.1373	-0.0853	0.1412	-0.1341	-0.0942	0.1322
	0.8	0.5	-0.1341	-0.0942	0.1322	-0.1088	-0.0976	0.0949	-0.2155	-0.1901	0.1574
		0.8	73.5713	-0.0064	325.9628	36.6245	-0.04936	162.9039	24.3089	-0.0681	108.551
		2.5	187.2195	0.1497	829.0332	1.6962	-0.1042	7.9176	1.0233	-0.1337	5.2270
		5	-0.0145	-0.0783	0.3475	-0.1684	-0.1486	0.1606	-0.2197	-0.1763	0.1596
	2.5	0.5	-0.2416	-0.2967	0.1691	-0.4400	-0.5243	0.1643	-0.4400	-0.5243	0.1643
		0.8	0.6782	0.2158	1.8055	-0.1262	-0.1435	0.1140	-0.1810	-0.3735	0.5405
		2.5	0.0746	-0.0577	0.5156	-0.2693	-0.3163	0.0931	-0.3823	-0.4944	0.1715
		5	-0.2266	-0.2808	0.0830	-0.2680	-0.3727	0.2282	-0.4827	-0.5773	0.1776
	5	0.5	-0.4969	-0.5906	0.1889	-0.3401	-0.3798	0.0684	-0.5510	-0.6165	0.1233
		0.8	0.7013	0.2536	1.9227	-0.0302	-0.2838	0.9234	-0.2741	-0.4728	0.5955
		2.5	0.0424	-0.0785	0.5637	-0.3597	-0.4515	0.2536	-0.4937	-0.5756	0.1713
		5	-0.2921	-0.3274	0.0760	-0.5270	-0.5864	0.1100	-0.6053	-0.6763	0.1326
5	0.5	0.5	40.1796	11.4196	101.1856	20.0006	5.6108	50.5207	13.274	3.674	33.6324
		0.8	3.3826	1.7471	6.8894	1.6021	0.7746	3.3698	1.0086	0.4504	2.1969
		2.5	0.0964	0.0587	0.1796	-0.0409	-0.0597	0.0365	-0.0867	-0.1124	0.0623
		5	-0.0228	-0.0334	0.0199	-0.1006	-0.1182	0.0852	-0.1630	-0.1448	0.1127
	0.8	0.5	53.7698	20.9489	123.6435	26.7355	10.2928	61.7434	17.7240	6.7407	41.1101
		0.8	4.8107	3.2056	8.3631	4.0964	1.4211	2.2559	1.4043	0.8263	2.6748
		2.5	0.1383	0.1077	0.2196	-0.0802	-0.1200	0.0608	-0.1531	-0.2063	0.0899
		5	-0.044	-0.0671	0.0332	-0.1718	-0.217	0.1118	-0.2142	-0.2658	0.1455
	2.5	0.5	77.5903	41.6750	154.4153	-0.3633	-0.4321	0.1223	-0.2030	-0.2482	0.0960
		0.8	7.5588	6.3782	10.3526	38.4897	20.4759	77.1446	25.4562	13.4095	51.3878
		2.5	0.2047	0.2143	0.2922	3.4740	2.8275	5.0995	2.1124	1.6439	3.3496
		5	0.2047	0.2143	0.2922	-0.1159	-0.1368	0.05306	-0.3389	-0.4053	0.1065
	5	0.5	-0.4458	-0.5291	0.1601	-0.1581	-0.1632	0.0657	-0.4600	-0.5050	0.0834
		0.8	85.2087	48.9973	162.5146	42.2233	24.0734	81.2140	27.8949	15.7655	54.1138
		2.5	8.4942	7.4992	10.8653	3.8661	3.3244	5.3776	2.3234	1.9327	3.5490
		5	0.2146	0.2520	0.3287	-0.2736	-0.2952	0.1119	-0.4364	-0.4752	0.0812

Source: Simulated data of TIIHLEL Distribution

Table 3: The Skewness and Kurtosis of TIIHLEL Distribution for $\beta = 2$, 4 and 6 for n=10

α	λ	θ	β=2		β=4		β=6	
			Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
0.5	0.5	0.5	3.5464	11.9068	-0.0399	-0.4102	4.1274	15.0420
		0.8	4.1293	15.0517	4.1293	15.0518	4.1293	15.052
		2.5	4.1253	15.0308	4.1273	15.0411	-1.0789	0.2465
		5	4.0212	14.4969	-0.8080	0.5781	-0.7710	-0.4931
	0.8	0.5	-0.7710	-0.4931	0.2994	-0.5633	-0.3013	-1.3714
		0.8	4.1292	15.0514	4.1292	15.0516	4.1293	15.0517
		2.5	4.1237	15.0220	4.1261	15.0348	4.1259	15.0341
		5	3.9634	14.1964	0.4488	-0.0229	-0.4095	-1.2322
	2.5	0.5	-0.0479	-0.6149	1.1663	0.0356	1.1663	0.0356
		0.8	2.5930	4.8830	-0.9425	0.7085	2.5043	4.5680
		2.5	2.5668	4.8055	1.3855	0.5407	0.9324	-0.7940
		5	0.9981	0.4693	1.9109	2.4284	1.2256	0.1210
	5	0.5	1.0998	-0.0543	1.6549	1.6896	1.5630	1.1823
		0.8	2.5547	4.7666	2.6107	4.9391	2.5879	4.8580
		2.5	2.4724	4.5225	2.3406	3.9807	1.5861	1.0848
		5	1.3736	0.1968	1.6657	1.6172	1.6161	1.3417
5	0.5	0.5	2.6423	5.0353	2.6428	5.0370	2.6434	5.0387
		0.8	2.5618	4.7875	2.5767	4.8328	2.5910	4.8764
		2.5	2.4756	4.5230	0.3896	-1.6530	0.2895	-1.1503
		5	0.3994	-1.6633	-0.6531	0.1630	-0.8860	0.5991
	0.8	0.5	2.6187	4.9623	2.6197	4.9656	2.62083	4.9688
		0.8	2.4570	4.4742	4.5589	2.4858	4.6407	2.5134
		2.5	2.2635	3.9013	0.5666	-1.4616	0.8356	-1.0938
		5	-1.4278	0.5930	-0.9532	0.3123	-0.7628	0.0933
	2.5	0.5	2.5637	4.7951	1.4043	0.6012	1.5816	1.0399
		0.8	3.7904	2.2164	38.4897	20.4759	2.5669	4.8047
		2.5	1.5624	2.0875	2.2603	3.9102	4.0278	2.3027
		5	2.0875	1.5624	1.6408	1.3026	1.3312	0.6895
	5	0.5	1.2398	0.2087	1.7410	2.6698	2.2363	1.8416
		0.8	4.7364	2.5441	4.7402	2.5454	4.7440	2.5467
		2.5	3.5655	2.1331	2.5454	4.7402	2.2019	3.7486
		5	1.2211	1.1634	3.6576	2.1680	1.3464	0.0667

Source: Simulated data of TIIHLEL Distribution

Notes

Notes

Table 4: The Mean, Median and Standard of TIIHLEL Distribution for β = 2, 4 and 6 for n=100

α	λ	θ	β=2			β=4			β=6		
			Mean	Median	SD(σ)	Mean	Median	SD(σ)	Mean	Median	SD(σ)
0.5	0.5	0.5	-0.5481	-0.5647	0.0866	-0.0599	-0.0232	0.0714	-0.1157	-0.0426	0.1413
		0.8	20.8214	-0.0022	197.9756	10.3249	-0.0124	98.9463	6.8260	-0.0189	65.9366
		2.5	1.0874	-0.0049	8.6688	0.4579	-0.0215	4.2889	0.2480	-0.0272	2.8305
		5	-0.0196	-0.0169	0.1671	-0.0956	-0.0351	0.1206	-0.1209	-0.0472	0.1448
	0.8	0.5	-0.1343	-0.0476	0.1683	-0.1027	-0.0723	0.0855	-0.1859	-0.1244	0.1626
		0.8	21.2299	-0.01517	200.310	10.4804	-0.0548	100.118	6.8972	-0.0717	66.7204
		2.5	1.12112	-0.02616	8.7772	0.4259	-0.0764	4.3465	0.1942	-0.0957	2.8718
		5	-0.05502	-0.0574	0.1772	-0.1620	-0.1094	0.1430	-0.1977	-0.1274	0.1699
	2.5	0.5	-0.2474	-0.2515	0.0827	-0.1865	-0.2321	0.2013	-0.4139	-0.4190	0.1417
		0.8	21.6878	-0.0906	202.9944	10.5537	-0.2745	101.4752	6.8423	-0.3432	67.6354
		2.5	1.09779	-0.1650	8.9141	0.2586	-0.3251	4.4306	-0.0210	-0.3913	2.9379
		5	-0.2136	-0.1389	0.1920	-0.3834	-0.39937	0.1384	-0.4491	-0.4471	0.1529
	5	0.5	10.51839	-0.1848	95.5571	-0.3265	-0.3458	0.0694	-0.5363	-0.5473	0.0956
		0.8	21.7714	-0.1782	203.6345	10.5126	-0.4272	101.8038	6.7597	-0.5033	67.8602
		2.5	1.0558	-0.2301	8.9529	0.1548	-0.4592	4.4596	-0.1454	-0.5473	2.9627
		5	-0.2613	-0.3182	0.2144	-0.5036	-0.5268	0.1151	-0.5844	-0.6068	0.1096
5	0.5	0.5	-0.6062	-0.6182	0.1136	-0.0020	-0.010	0.0850	-0.0868	-0.0370	0.099
		0.8	2047.148	0.9040	19528.85	1023.488	0.4232	9764.385	682.2681	0.2629	6509.56
		2.5	20.9390	0.2003	154.3769	10.3837	0.0714	77.1404	6.8652	0.0284	51.395
		5	0.18371	0.0005	0.5985	0.0060	-0.0171	0.2319	-0.0531	-0.0320	0.1384
	0.8	0.5	-0.1150	-0.0439	0.1383	-0.0207	-0.0347	0.0920	-0.1449	-0.0982	0.1169
		0.8	2088.702	2.6374	19758.93	1044.216	1.2348	9879.42	696.0546	0.7672	6586.26
		2.5	22.5256	0.5845	156.2336	11.1282	0.2083	78.0714	7.3291	0.0829	52.0175
		5	0.2173	0.0014	0.6242	-0.0258	-0.0604	0.243	0.1069	-0.0906	0.153
	2.5	0.5	-0.1863	-0.1281	0.1593	-0.1020	-0.1394	0.1166	-0.3412	-0.3527	0.1003
		0.8	2143.59	8.8757	20022.4	1071.505	4.1554	10011.19	-0.3020	0.3438	0.1745
		2.5	25.2476	1.9671	158.325	12.3336	0.7011	79.1334	8.0289	0.2791	52.7363
		5	0.2546	0.0049	0.6716	-0.1629	-0.2382	0.2800	-0.3020	-0.3438	0.1745
	5	0.5	-0.4209	-0.4160	0.1352	0.6716	-0.1394	0.1166	-0.3412	-0.3527	0.1003
		0.8	2143.59	8.8757	20022.43	1071.505	4.1554	10011.19	714.1431	2.5819	6674.113
		2.5	25.2476	1.9671	158.325	12.3336	0.7011	79.1334	8.0289	0.2791	52.7363
		5	0.25460	0.0049	0.6716	-0.1629	-0.2382	0.2800	-0.3020	-0.3020	0.1745

Source: Simulated data of TIIHLEL Distribution

Table 5: The Skewness and Kurtosis of TIIHLEL Distribution for β = 2, 4 and 6 for n=100

α	λ	θ	β=2		β=4		β=6	
			Skewness	Kurtosis	Skewness	Kurtosis	Skewness	Kurtosis
0.5	0.5	0.5	1.1736	0.7588	-0.6676	-0.1583	-1.2238	0.2005
		0.8	9.8457	94.9622	9.8460	94.9670	9.8464	94.9714
		2.5	9.7617	93.853	9.7968	94.3173	9.8155	94.5665
		5	8.9969	84.1612	-0.5674	0.5139	-1.1980	0.1487
	0.8	0.5	-1.3157	0.5088	-0.1719	-0.8034	-0.7097	-0.7971
		0.8	9.8447	94.9494	9.8451	94.9550	9.8455	94.9601
		2.5	9.7357	93.5077	9.7761	94.0432	9.7947	94.2936
		5	8.3285	75.7668	-0.1286	-0.2565	-0.6729	-0.8766
	2.5	0.5	1.5456	3.8865	7.4744	64.3324	0.3906	-0.8094
		0.8	9.8432	94.9296	9.8435	94.9342	9.8438	94.9384
		2.5	9.6867	92.8574	9.7221	93.3273	9.7405	93.5753
		5	-0.7875	-0.6494	1.6503	5.0171	0.5092	-0.7558
	5	0.5	9.8342	94.8121	3.5722	20.1740	0.9609	0.2476
		0.8	9.8427	94.9235	9.8429	94.9266	9.8432	94.9295
		2.5	9.6668	92.5942	9.6934	92.9465	9.7117	93.1910
		5	7.1334	59.3632	3.9842	24.2980	1.5016	2.6275
5	0.5	0.5	0.0257	0.8155	8.3230	74.7721	-1.0862	-0.1474
		0.8	9.8463	94.9701	9.8463	94.9702	9.8463	94.9702
		2.5	9.7379	93.5376	93.5752	9.74078	9.7435	93.6121
		5	7.0477	57.7457	8.9977	83.6560	6.8053	58.3758
	0.8	0.5	-1.1905	0.1152	7.3446	62.3746	-0.5338	-1.0873
		0.8	9.8455	94.9598	9.8455	94.9598	9.8455	94.95993
		2.5	9.7074	93.1331	9.7108	93.1786	9.71424	93.223
		5	6.3999	49.7371	8.4331	76.2919	5.8779	47.5133
	2.5	0.5	-0.6689	-0.8557	36.3640	5.2628	0.8337	-0.2018
		0.8	9.844	94.9448	9.84	94.9448	0.8337	-0.2018
		2.5	9.6638	92.5563	9.6668	92.5952	9.6697	92.6338
		5	5.3529	37.72	9.8443	94.9449	6.4528	52.3660
	5	0.5	0.4957	-0.6774	5.2628	36.3640	0.8337	-0.2018
		0.8	9.8443	94.9448	9.8443	94.9448	9.8443	94.9449
		2.5	92.5563	9.6638	9.6668	92.5952	9.6697	92.6335
		5	5.3529	37.7232	6.9774	57.2513	6.4528	52.3660

Source: Simulated data of TIIHLEE Distribution

From the above table 2, and 4 present simulation study for sample 10 and 100 respectively, this observed that the mean, standard deviation, and median are increasing, and decreasing functions of the shape parameters θ when the other parameters are held constant, an increase in scale parameter β increases the mean, standard deviation and median for fixed α, λ, and θ. Also, Table 3 and 5, Skewness, and kurtosis is an increasing function as the scale parameter β increases when the shape parameters are held constant. An increase in the shape parameters reduces the values of skewness and kurtosis when the scale parameter is held constant.

Notes

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VII. Applications of TIIHLEL Distribution

In this section, the flexibility of the TIIHLEL Distribution is illustrated employing two (2) real data sets are demonstrated. We compare the fits of the TIIHLEL distribution with some distribution namely; Type II half Burr X (TIIHLBX), Type I half Burr X (TIHLBX), Type II half exponential (TIIHLE), Exponential distribution, Exponentiated Exponential (EE), Pareto Distribution.

The Log-likelihood (*LL*), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (A^*), Cramér-von Mises (W^*) and Kolmogorov-Smirnov (K-S) are computed to compare the fitted model, the smaller the value of these statistics the better the fit to the data.

Table 6: First data set: Here apply the data consist the remission times (in months) of a random sample of 128 bladder cancer patients studied by Lee et al. (2003) reported by Cordeiro and Lemonte (2013) and Oluyede et al. (2014) as presented in Table 4.5

0.080	0.200	0.400	0.500	0.510	0.810	0.900	1.050	1.190	1.260	1.350	1.400	1.460	1.760
2.020	2.020	2.070	2.09	2.230	2.260	2.460	2.540	2.620	2.640	2.690	2.690	2.750	2.830
2.870	3.020	3.25	3.310	3.360	3.360	3.480	3.520	3.570	3.640	3.700	3.820	3.880	4.180
4.230	4.260	4.330	4.340	4.400	4.500	4.510	4.870	4.980	5.060	5.090	5.170	5.320	5.320
5.340	5.410	5.410	5.490	5.620	5.710	5.850	6.540	6.760	6.930	6.94	6.970	7.090	7.260
7.280	7.320	7.390	7.590	7.620	7.630	7.660	7.870	7.930	8.260	8.370	8.530	8.650	8.660
9.020	9.220	9.470	9.740	10.06	10.34	10.66	10.75	11.25	11.64	11.79	11.98	12.02	12.03
12.07	12.63	13.11	13.29	13.80	14.24	14.76	14.77	14.83	15.96	16.62	17.14	18.10	19.13
21.73	22.69	23.63	25.74	25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05		

Source: Lee et al. (2003)

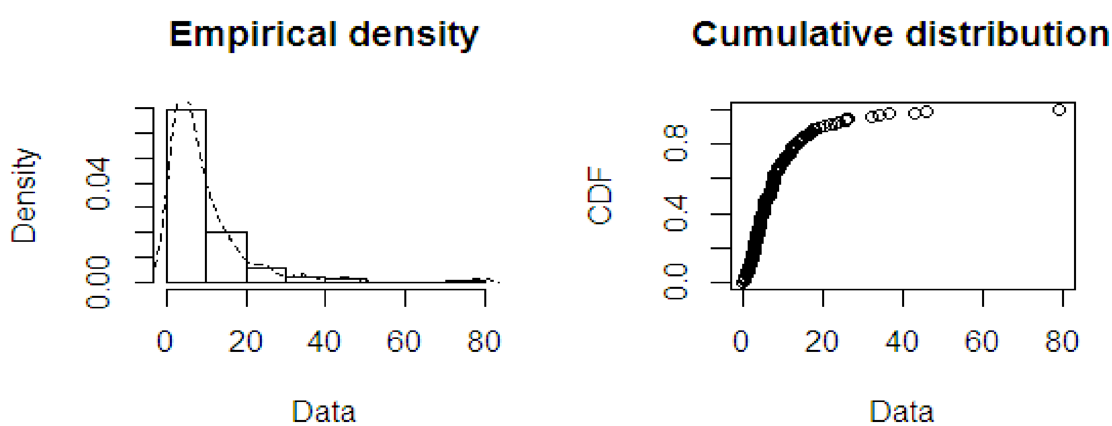

Fig. 5: Histogram and CDF Plots of an Empirical Distribution for bladder cancer patient

The empirical density plot in figure 5 shows that the data set is right skewed.

Fig. 4.1: Histogram and CDF Plots of an Empirical Distribution for Bladder Cancer Patient

Table 7: Descriptive Statistics Bladder Cancer Patient

Min	Max	Mean	SD	Median	Q1	Q3	Skewness	Kurtosis
0.080	79.050	9.366	10.5083	6.395	3.348	11.840	3.2866	15.4831

The Summary statistics of data in table 6 displayed in table 7 shows that the first data set on a bladder cancer patient is over-dispersed and right-skewed,

Correlation matrix of TIIHLEL

$$\begin{pmatrix} 1.0000000 & 0.008513745 & -0.999555798 & 0.006030967 \\ 0.008513745 & 1.0000000 & 0.006280632 & -0.978044473 \\ -0.999555798 & 0.006280632 & 1.0000000 & -0.004450433 \\ -0.006030967 & -0.006030967 & -0.004450433 & 1.0000000 \end{pmatrix}$$

The above correlation matrix indicates the pairs which have negative and positive correlation coefficient depending on the combination of the parameters.

Table 8: Maximum Likelihood Estimates of Parameters and their Standard Errors for Bladder Cancer Patient

	Estimated Parameter			
Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
TIIHLEL	8.0874 (34.3684)	0.0161 (0.0108)	0.2017 (0.8573)	7.4603 (4.2817)
TIIHLBX	0.1758 (0.6831)	0.0432 (0.0039)	2.7817 (10.8111)	
TIHLBX	0.1758 (0.6831)	0.0432 (0.0039)	2.7817 (10.8111)	
TIIHLE	1.5988 (0.2032)	0.1454 (0.0149)		
EE	1.2186 (0.1489)	8.2466 (0.9228)		
E	6.0891 (0.3999)			

Table 8 shows the parameters estimates obtained from the proposed distribution and it sub distributions

Table 9: Goodness-of-fit Statistics, Log-likelihood and Information Criteria for Bladder Cancer Patient

Distribution	AIC	BIC	LL	A^*	W^*	K-S
TIIHLEL	**828.4315**	**839.8397**	**-410.2158**	**0.1971**	**0.0279**	**0.0385**
TIIHLBX	847.7955	856.3516	-420.8978	2.8260	0.5545	0.1242
TIHLBX	847.7955	856.3516	-420.8978	45.9030	9.8390	0.5000
TIIHLE	973.9203	979.6243	-484.9601	5.6365	0.9516	0.1365
EE	830.8943	835.8593	-413.0776	0.7110	0.1272	0.0723
E	1023.348	1026.2000	-510.6739	11.2831	1.9938	0.1792

Table 9 revealed that the TIIHLEE distribution provides a better fit to the bladder cancer patient data than other competitive models, the TIIHLE, HLEE, HLE, and the E distribution. The TIIHLEE distribution has the highest log-likelihood and the smallest K-S, W^*, A^*, AIC, and BIC values compared to the other models. Although the TIIHLEE distribution provides the best fit to the data and the EE distribution is alternatively good model for the data since it measures of fit value are close to that of the TIIHLEE distribution.

Notes

The estimated asymptotic variance-covariance matrix of the TIIHLEL distribution for the bladder cancer patient data is given by

$$I_{ij}^{-1} = \begin{pmatrix} 1.181187e+03 & 3.178176e-03 & -2.945031e+01 & -0.88750073 \\ 3.178176e-03 & 1.179766e-04 & 5.848234e-05 & -0.04548612 \\ -2.945031e+01 & 5.848234e-05 & 7.349316e-01 & -0.01633609 \\ -8.875007e-01 & -4.548612e-02 & 1.633609e-02 & 18.33346935 \end{pmatrix}$$

However, the confidence interval at 95% for the estimate $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\theta}$ are (-59.2747, 75.44947), (0.0052, 0.0379), (-1.4786, 1.8819) and (-0.9319, 11.7421) respectively. the confidence interval for the parameters contains zero. Thus, all the estimated parameters of the TIIHLEL distribution were significant at the 5% significance level

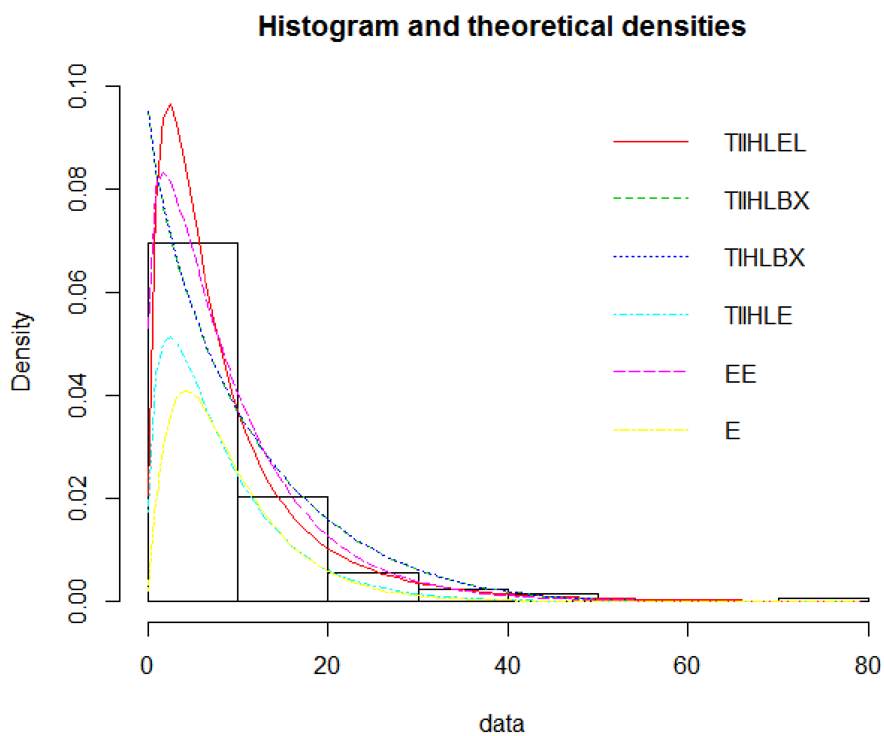

Fig. 6: Fit Plot of TIIHLEE Model with other Models for Bladder Cancer Patient

Fig 6 shows the histogram of the bladder cancer data set along with the fitted model. The theoretical density of TIIHLEL distribution has a better spread to the right than other existing models on the data set.

Table 10: Second Data Set

17.88	28.92	33.00	41.52	42.12	45.60	48.80	51.84
51.96	54.12	55.56	67.80	68.44	68.64	68.88	84.12
93.12	98.64	105.12	105.84	127.92	128.04	173.40	

Source: Lawless (1982)

Table 10 Represents the number of million revolutions before failure for each of twenty-three (23) deep groove ball bearings in the life tests. The data set was given by Lawless (1982).

Table 11: Summary Statistics of Deep Groove Ball Bearings

Min	Max	Mean	SD	Median	Q1	Q3	Skewness	Kurtosis
17.88	173.40	72.23	37.4804	67.8	47.20	95.88	0.9419	0.4889

Table 11 gives a summary statistics of data in Table 10 which indicates is under dispersed and right-skewed

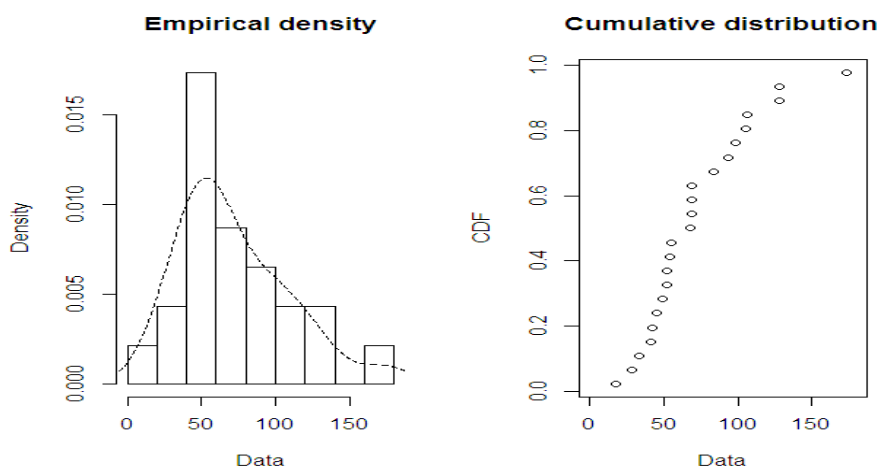

Fig. 7: Histogram and CDF Plots of an Empirical Distribution for deep groove ball bearings

Figure 7 shows the empirical plot of data set in table 10 which is right skewed

Correlation matrix of TIIHLEL Distribution

$$\begin{pmatrix} 1.0000000 & 0.06631924 & -0.93493249 & 0.05851001 \\ 0.06631924 & 1.0000000 & 0.06631901 & -0.70226965 \\ -0.93493249 & 0.06631901 & 1.0000000 & 0.05850993 \\ 0.05851001 & -0.9368186 & 0.05850993 & 1.0000000 \end{pmatrix}$$

The above correlation matrix indicates the pairs which have negative and positive correlation coefficient depending on the combination of the parameters.

Table 12: Maximum Likelihood Estimates of Parameters and their Standard Errors for Deep Groove Ball Bearings

	Estimated Parameter			
Distributions	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
TIIHLEL	2.6919 (2.9647)	0.0056 (0.0018)	2.5214 (2.7769)	6.3149 (1.8623)
TIIHLBX	0.5337 (3.1771)	0.0119 (0.0017)	2.7063 (16.1097)	
TIHLBX	0.5337 (3.1771)	0.0119 (0.0017)	2.7063 (16.1097)	
TIIHLE	5.7321 (2.2575)	0.0332 (0.0065)		
EE	5.2936 (2.0542)	30.9302 (6.1506)		
PARETO	0.2399 (0.0500)			

Table 12 shows the parameters estimates obtained from the proposed distribution and it sub distributions.

Notes

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Table 13: Goodness-of-fit Statistics, Log-likelihood and Information Criteria for Deep Groove Ball Bearings

Distribution	AIC	BIC	LL	A*	W*	K-S
TIIHLEL	438.8069	446.2917	-215.4034	0.1681	0.0241	0.0740
TIIHLBX	452.346	457.9596	-223.1730	1.5815	0.2769	0.1572
TIHLBX	452.346	457.9596	-223.1730	15.6856	3.3333	0.5000
TIIHLE	439.8658	448.6082	-217.9329	0.5730	0.0798	0.0963
EE	442.8943	446.6367	-219.4471	0.8629	0.1301	0.1169
PARETO	593.7705	595.6417	-295.8853	16.9144	3.6482	0.5483

The TIIHLEL distribution provides a better fit to the data set compared to the other models. From Table 13, the TIIHLE distribution has the highest log-likelihood and the smallest K-S, W*, A*, AIC, and BIC values compared to the other fitted models.

The asymptotic variance covariance matrix for the estimated parameters is

$$I_{ij}^{-1} = \begin{pmatrix} 8.7894149234 & 3.583368e-04 & 7.6968938753 & 0.32304036 \\ 0.0003583368 & 3.321573e-06 & 0.0003356335 & -0.00238354 \\ -7.6968938753 & 3.356335e-04 & 7.7109968035 & 0.30257389 \\ 0.3230403634 & -2.383540e-03 & -0.3025738897 & 3.46811291 \end{pmatrix}$$

The confidence interval at 95% for the estimate $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\theta}$ are (-3.1189, 8.5028), (0.0055, 0.0167), (-2.92126, 7.9641) and (2.6648, 9.9649)respectively. . It can be observed that the confidence intervals for the parameters do not contain zero, this shows that all the parameters of the TIIHLEL distribution were significant at the 5% significance level.

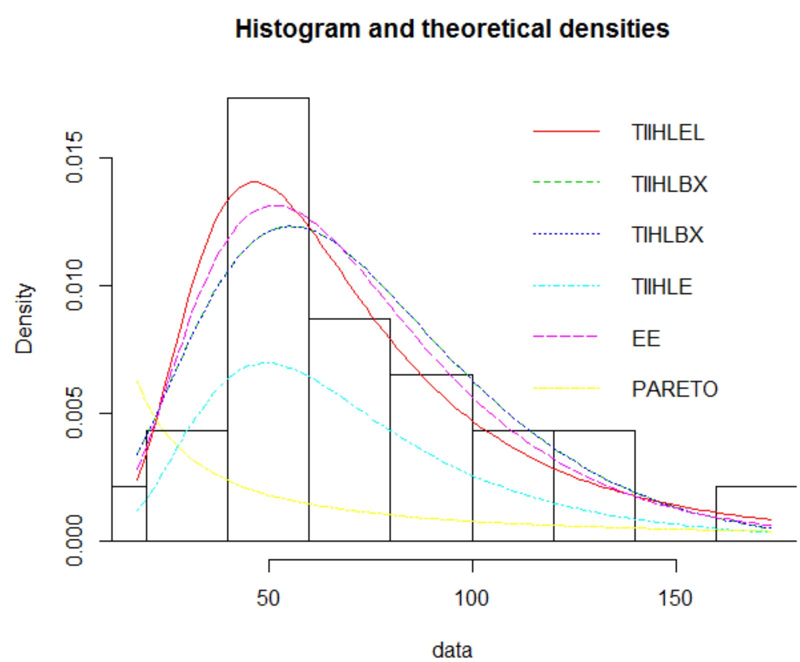

Fig. 8: Fit Plot of TIIHLEL Model with other Models for Deep Groove Ball Bearings Discharge

Fig 8 also shows the histogram of the Deep Groove Ball Bearings data set along with the fitted model. The theoretical density of TIIHLEL distribution has a better spread to the right than other competing models on the data set.

VIII. Conclusions

In this study, we have derived, studied the properties and applications Type II Half Logistic Exponentiated Lomax (TIIHLEL). Some structural mathematical properties; Moment Incomplete moments, Probability Weighted Moment, Order Statistic, and Rényi entropy of the derived model are investigated. A simulation study is carried out to estimate the behavior of the shape and scale model parameters. Also, maximum likelihood estimators were investigated. The application of two real-life data set shows that the TIIHLEE strong and better fit as compare to other existing distribution. However, we hope that this distribution will attract wider applications in the areas of sciences and applied sciences.

Notes

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Development of Fuzzy Membership Functions and Meteorological Drought Classification as Per SPI

By Md. Anisur Rahman
Islamic University

Abstract- Drought -as an environmental occurrence, is an essential part of climatic variability. Droughts are the product of keen water shortage causing severe and sometimes disastrous economic and social consequences. The Standardized Precipitation Index (SPI) provides the forecasting of drought. The SPI also detects moisture shortage more rapidly which has a reply time scale of approximately 3, 6, 12, 24, 48 months. On the other hand, fuzzy logic can be used to focus on modeling problems characterized by imprecise or ambiguous information. It also needs a complete understanding of the drought causing factors, severity classification and to interpret the drought forecasted output variables. In this study, the number of linguistic terms referred to as fuzzy sets, is assign to variable rainfall. The degree of membership (from 0 to 1) of a real valued input (SPI) to a particular fuzzy set A (ED, SD, MD, N, MW, SW, EW) is specified by a membership function $\mu_A(x)$. Fuzzification of linguistic variables is classified into linguistic labels by transfer membership functions for each of the variable.

Keywords: *fuzzy set, membership functions, SPI, drought, climate, forecast.*

GJSFR-F Classification: *MSC 2010: 03B52*

DEVELOPMENTOFFUZZYMEMBERSHIPFUNCTIONSANDMETEOROLOGICALDROUGHTCLASSIFICATIONASPERSPI

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Md. Anisur Rahman

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Keywords: fuzzy set, membership functions, SPI, drought, climate, forecast.

I. Introduction

In recent decades, rapid climate change has created a new challenge to obstruct expansion development throughout the world. It is extensively recognized that the impacts of climate-change intensify unfavorable climatic and environmental situations, particularly in developing countries. Drought studies are important because which manipulate on the society and the economy of any country. Drought is a weather-related natural disaster. It affects vast regions for months or years. It is a regular characteristic of the climate and occurs in virtually all-climatic zones. These characteristics vary significantly in different regions. Drought is linked to shortage of precipitation over an extended period of time, usually for a season or more. This scarcity results in a water shortage for some activity, cluster or ecological division. Drought is too linked to the timing of precipitation. Otherwise climatic factors such as high temperature, high wind speed and low humidity are often related with drought.

The most vulnerable countries to climate change which is facing climate change induced hazards such as rising sea levels and storm surges, heat stress, extreme precipitation, inland and coastal flooding, landslides, salinity intrusion, drought,

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increased aridity, water scarcity and air pollution[1]. It is the fifth most affected country by extreme weather events in the last 20 years. That means, it is the worst hit victim of global climate change and most probably will continue being so in the future. Among the different types of natural disasters that the country experiences, the most frequent events are: (i) floods and river erosion (ii) cyclones and tropical storms (iii) extreme temperature and drought. Climate change is already a painful reality in different country: farmers and peasants are facing losses of their harvest every year due to continual reduced rainfall. Different physical effects of climate change include increased temperature and precipitation, increased salinity and extreme weather events such as floods, cyclones and droughts have profound negative impacts on different country water resources [2]. The variability of weather patterns can lead to flooding and drought as direct results and indirect impacts such as reduced availability and quality of freshwater. In the northern part of Bangladesh floods in the monsoon season and drought in the dry season and other associated stressors like upstream river diversion and damming, can have severe implications for water resources and agricultural yields [3]. The socio-economic effects of climate change arise from interactions between climate and society and which in turn affect the both natural and managed environments. Traditionally, climatic variations have provided opportunities (resources) and imposed costs (hazards), depending on how society modified to the environment. Thus, a bountiful floodplain rice-growing system, finely tuned to seasonal climate variations, is often disrupted by floods, droughts, and cyclones. Agriculture is still the main source of economic activities in the most of different country in the world. Agricultural sectors are frequently affected by natural disasters such as droughts, excessive rainfall, floods, flashflood, heavy fog, earthquakes, storm, cyclones, salinity and landslides [4]. In order to increase crop production and protecting crops, human life, and ecosystem there is an increasing demand from the policy makers for a reliable prediction, and in particular, the rainfall. The crop yielding depends on the rainfall amount and rain duration. The crop production uncertainty is generally defined by the variability of climate and especially for rainfall variability. So the rainfall variability, regional distribution and predictions and to adopt needful events is extreme importance to alleviate the problems due to rainfall in an agrarian country. Otherwise, drought -as an environmental phenomenon, is an integral part of climatic variability. Droughts are the result of acute water shortage causing severe and sometimes disastrous economic and social consequences. Out of all common natural disasters, droughts affect more people and larger areas than any other. It is generally considered to be occurring when the rainfall in monsoon is insufficient. Shortage of rainfall in the monsoon may reduce crop productions, creates shortage of drinking water and also may affect the socio-economic life. Generally, the amount of rainfall absorbed in any region provides idea about the occurrence of drought of that region.

In the last 50 years, many countries have suffered about 20 or more drought conditions. A part from loss to agriculture, droughts have significant effect on land dreadful conditions, live reserve population, employment and health. Past droughts have typically affected about 47 percent of the country and 53 percent of the population [5]. Climate variability and droughts are commonly known important stress factors in developing counties, where rural households have adapted to such factors for decades and in extreme dry regions households have even moved beyond climate dependence. Many countries have already showed an increased incidence of droughts in recent years [2]. Meteorological drought is directly related to the weather parameter rainfall, but agricultural drought in is the consequence of meteorological drought [6]. The drought

Ref

1. A. Das and N. Hossain, Appraising Climate Change Impact Mitigation Standards to Ground Realities: The Lessons from Bangladesh Climate Change Trust Funded Projects, *International Conference on Disaster Risk Mitigation, Dhaka*, 2017: 1-4.

condition in some counties in the recent decades had led to a short fall of crop production. The biophysical, environmental and health issues were concerning drought occurrence in many countries. The analysis exposed that, during the drought period, rainfall as the main factor of supplying surface water and normalizing the dryness of the nature was almost 46% lower than the previous (normal) years [7]. Many countries and regions are particularly vulnerable to droughts [8]. However the drought has attracted less scientific attraction than flood or cyclone, several authors found that the impact of drought can be more unprotected than flood and cyclone [9].

The Standardized Precipitation Index (SPI) provides the forecasting of drought. The primary reason for using SPI is that SPI is based on rainfall only so that drought assessment is probable even if other hydro meteorological measurements are not accessible. The SPI is distinct more than different timescales, which allows it to explain the condition of drought over a range of meteorological, hydrological and agricultural applications. Another advantage of SPI comes from its standardization, which ensures that the frequencies of extreme events at any location and on any time scale are consistent. The SPI also detects wetness scarcity more rapidly which has a response time scale of approximately 3, 6, 12, 24, 48 months. On the other hand, fuzzy logic can be used to focus on modeling problems characterized by imprecise or ambiguous information. The underlying power of fuzzy set theory is that linguistic variables are used in it rather than quantitative variables to represent imprecise concepts. Fuzzy logic is successful in two kinds of situation: (i) very complex models where understanding is strictly limited or in fact quite judgmental and (ii) processes where human reasoning, human perception or human decision making is inextricably involved. Drought is a creeping occurrence where the above illustrated factors play vital roles. It also needs a complete understanding of the drought causing factors, severity classification and to interpret the drought forecasted output variables. Several studies used SPI for real time monitoring and analysis of drought [10]. SPI was applied to monitor the concentration and spatial extension of droughts at different time scales in different country [11]. In a study, severity and spatial pattern of meteorological drought are analyzed by using multi-temporal SPI. The maximum SPI value are found -2.27 for 6 month time scale, the -2.17 for 12 months time scale and -1.85 for 3 months time scale respectively [12]. Fuzzification is a method for determining the degree of membership that a value has to a exacting fuzzy set. This is determined by evaluating the membership function of the fuzzy set for the value [13]. In standard models variables have real number values, the relationships are defined in terms of mathematical functions and the outputs are crisp numerical values [14]. The Mamdani fuzzy inference system has been used to estimate the average rainfall behaviour in the many countries. The rules of m^n fuzzy-logic principles were used to make operations for both the cases of fuzzification operation and defuzzification operation. In study fuzzy logic based rainfall prediction method by using the Mamdani fuzzy inference system may be successively used for different environmental problem estimation to mitigate unexpected meteorological problems [15]. Fuzzy rule based system is use to predict rainfall. Fuzzy inference is the actual procedure of mapping with a given set of input and output from side to side a set of fuzzy systems. Fuzzy levels and membership functions obtained after minimum composition of inference part of the fuzzifications done for temperature and wind speed were considered as they represent the environmental condition enhance a rainfall occurrence [16]. [17] used fuzzy inference model for predicting rainfall. The ability of fuzzy logic model predicted outputs were compared with the actual rainfall data [18]. [19] used a model based on fuzzy rules and

Ref

8. S.I. Anik, M. H. Kabir and S. Ray, Climate Change and Food Security, Elusive Progress: State of Food Security in Bangladesh, *Shrabon Prokashani*, Dhaka, 2012.

neural networks using large-scale climatic signals to predict rainfall in the western Iran. [20] used fuzzy logic for rainfall prediction. The fuzzy logic method is used to model and to predict confined rainfall data. Since drought is closely associated with crop productions and thus food security; hence, the study on drought hazards, particularly drought monitoring is important to decrease its impact in any country. Therefore, the study aims at developing a drought assessment procedure in meteorological and agricultural contexts and to develop a fuzzy rule based drought forecasting method.

Notes

II. Methods

The SPI calculation for any location is based on the long-term rainfall record for a desired period.

$$SPI = \frac{R_{ij} - R_{im}}{\sigma}$$

Where, R_{ij} is the seasonal precipitation at the *i*-th rain gauge station and *j*-th observation, R_{im} is the long-term seasonal mean and σ is its standard deviation. A fuzzy logic based drought forecasting method was developed employing the SPI. The fuzzy ranks from 1 to 7 were assigned based on SPI ranges with respective drought classification. The ranges and the ranks assigned for each drought classification are shown in the table 1 below.

Table 1: Fuzzy ranks assigned for drought classification

Drought Classification	SPI Values	Fuzzy Ranks
Extremely dry	-2 and less	1
Severely dry	-1.5 to -1.99	2
Moderately dry	-1 to -1.49	3
Normal	-0.99 to 0.99	4
Moderately wet	1 to 1.49	5
Very wet	1.5 to 1.99	6
Extremely wet	2 and above	7

Fuzzy Set: Let X be a universal set. Then A is called a (fuzzy) subset of X if A is a set of ordered pairs.

$$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}$$

where μ_A is the membership function of *A*, $\mu_A(x)$ is the grade of the membership of x in A.

The linguistic expression for the variables and their membership functions are evaluated from the following triangular membership functions and it is defined over $[a,b]$.

$$\mu_A(x) = \begin{cases} 0 & when \quad x \le a \\ \frac{x-a}{m-a} & when \quad a < x \le m \\ \frac{x-b}{m-b} & when \quad m < x < b \\ 0 & when \quad x \ge b \end{cases}$$

where m is the midpoint of $[a,b]$.

Notes

The value of the membership function $\mu(x)$, ranges from 0 to 1, with 0 denoting no membership, 1 for full membership and values in between has partial membership as shown in Figure 1.

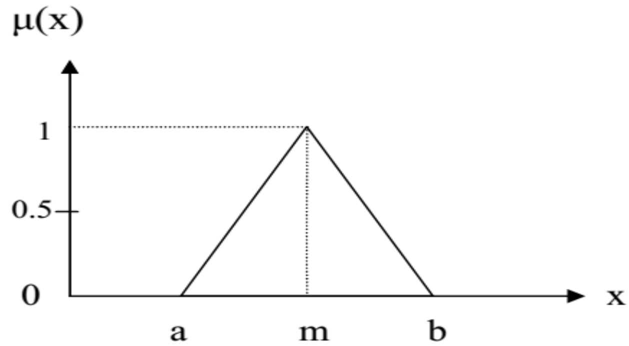

Figure 1: Fuzzy Membership Function

At the aggregation stage, output fuzzy sets of each rule are aggregated to form a single fuzzy set. The fuzzy max function is presented for use of forecasting of drought:

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)]$$

III. Results

In this study, the number of linguistic terms extremely dry (ED), severely dry (SD), moderately dry (MD), normal (N), moderately wet (MW), severely wet (SW), and extremely wet (EW) referred to as fuzzy sets, is assigned to variable rainfall. These fuzzy sets overlap and cover the necessary range of variation for that variable. The degree of membership (from 0 to 1) of a real valued input (SPI) to a particular fuzzy set A (ED, SD, MD, N, MW, SW, EW) is given by a membership function $\mu_A(x)$. This transformation of real valued inputs into a degree of membership in a particular fuzzy set is called fuzzification. Fuzzification of linguistic variables is classified into linguistic labels by assigning membership functions for each of the variable.

$$Var.(x) = \begin{cases} ED & if & SPI \le -2.0 \\ SD & if & -1.99 \le SPI \le -1.5 \\ MD & if & -1.49 \le SPI \le -1.0 \\ N & if & -0.99 \le SPI \le 0.99 \\ MW & if & 1.0 \le SPI \le 1.49 \\ SW & if & 1.50 \le SPI \le 1.9 \\ EW & if & SPI \ge 2 \end{cases}$$

The definite membership functions of these linguistic values are given as follows:

$$\mu_{ED}(x) = \begin{cases} 1 & when & x \le -2.0 \\ -(2x+3) & when & -2.0 < x < -1.5 \\ 0 & when & x \ge -1.5 \end{cases}$$

$$\mu_{SD}(x)=\begin{cases} 0 & when \quad x\leq -2.0 \\ 2(x+2) & when \quad -2.0<x\leq -1.5 \\ -2(x+1) & when \quad -1.5<x<-1.0 \\ 0 & when \quad x\geq -1.0 \end{cases}$$

$$\mu_{MD}(x)=\begin{cases} 0 & when \quad x\leq -1.5 \\ 2x+3 & when \quad -1.5<x\leq -1.0 \\ -x & when \quad -1.0<x<0.0 \\ 0 & when \quad x\geq 0.0 \end{cases}$$

$$\mu_{N}(x)=\begin{cases} 0 & when \quad x\leq -1.0 \\ x+1 & when \quad -1.0<x\leq 0.0 \\ 1-x & when \quad 0.0<x<1.0 \\ 0 & when \quad x\geq 1.0 \end{cases}$$

$$\mu_{MW}(x)=\begin{cases} 0 & when \quad x\leq 0.0 \\ x & when \quad 0.0<x\leq 1.0 \\ 3-2x & when \quad 1.0<x<1.5 \\ 0 & when \quad x\geq 1.5 \end{cases}$$

$$\mu_{SW}(x)=\begin{cases} 0 & when \quad x\leq 1.0 \\ 2(x-1) & when \quad 1.0<x\leq 1.5 \\ -2(x-2) & when \quad 1.5<x<2.0 \\ 0 & when \quad x\geq 2.0 \end{cases}$$

$$\mu_{EW}(x)=\begin{cases} 0 & when \quad x\leq 1.5 \\ 2x-3 & when \quad 1.5<x<2.0 \\ 1 & when \quad x\geq 2.0 \end{cases}$$

The computed SPI values are converted into fuzzy membership values ranging from 0 to 1. This fuzzy set is denoted as Set ‘A’. The fuzzy set ‘A’ can be divided into ‘7’ number of subsets according to the drought index considered in the study. Based on SPI drought classifications, fuzzy set ‘A’ is defined using triangular fuzzy function as shown in Figure 2.

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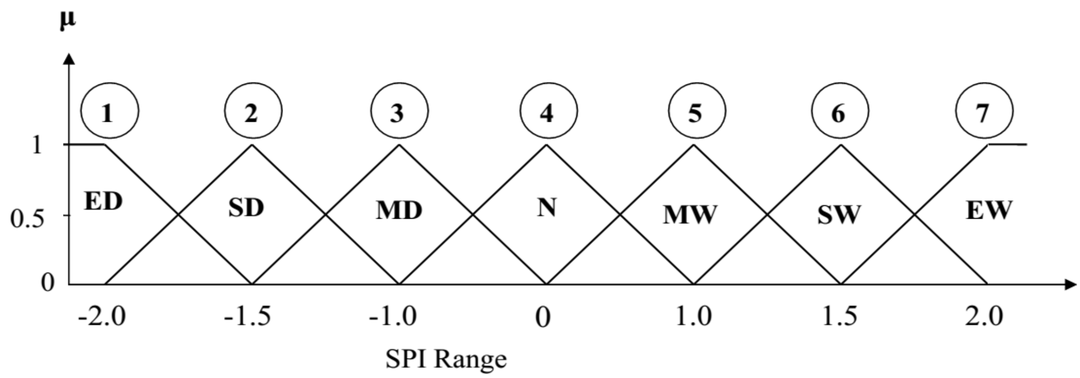

Figure 2: Assigning Fuzzy Membership Values

The drought forecasting measures in the form of fuzzy ranks were identified for each year. These values were obtained after the SPI values were transferred to the triangular fuzzy membership function (figure 2). The particular SPI value falling on the respective set(s) was identified. If a value falling in the SPI range from -2.0 to -1.5, the maximum value of fuzzy membership function will be considered and the value is assigned that particular set rank. Similarly, all the fuzzified ranks were identified for each SPI values. The fuzzified ranks were used for forecasting the drought severity class for different years.

IV. Discussion and Conclusion

The forecasting of drought is performed employing SPI and fuzzy logic. The analysis and drought classification using SPI and forecasting the SPI for immediate time step by applying the proposed drought forecasting methods using fuzzy logic are discussed. The SPI values are calculated based on monthly rainfall values with annual time scale for all the rain gauge stations of Bangladesh. The actual fuzzy membership functions are determined on the basis of SPI. Fuzzy ranks are assigned on the basis of SPI ranges and membership values as shown in Figure 2. The SPI values and fuzzy ranks for each year of the study are identified as the drought forecasting measures for different years. Finally, it can be concluded that the fuzzy logic based drought forecasting method using SPI can be successively used for drought analysis, forecasting and to mitigate unexpected meteorological problems.

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A Modified Taylor Series Expansion Method for Solving Fredholm Integro-Differential Equations

By Itthithep Navarasuchitr
Srinakharinwirot University

Abstract- In this paper, we use a modified Taylor series expansion method for solving the linear Fredholm integro-differential equations. This method transfroms the equation to linear system equations that can be solved easily with reduced row echelon method. Finally, we show the efficiency of this method with numerical examples by comparing the approximate solutions with exact solutions.

Keywords: fredholm integro-differential equations, numerical method, taylor, series.

GJSFR-F Classification: MSC 2010: 45D05; 45E10; 65M20

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Strictly as per the compliance and regulations of:

A Modified Taylor Series Expansion Method for Solving Fredholm Integro-Differential Equations

Itthithep Navarasuchitr

Abstract- In this paper, we use a modified Taylor series expansion method for solving the linear Fredholm integro-differential equations. This method transfroms the equation to linear system equations that can be solved easily with reduced row echelon method. Finally, we show the efficiency of this method with numerical examples by comparing the approximate solutions with exact solutions.

Keywords: fredholm integro-differential equations, numerical method, taylor, series.

I. Introduction

Mathematical modeling of real-life problems usually results in functional equations, e.g. differential equations, integro-differential equations, stochastic equations and others. Integro-differential equation is a hybrid of integral and differential equations which have found extensive applications in sciences and engineering.

In particular, integro-differential equations arise in fluid dynamics, biological models and chemical kinetics. The analytical solutions of some integro-differential equations (IDEs) cannot be found, thus numerical method are required. The numerical methods for linear integro-differential equations have been extensively studied by many authors [4, 6, 9]. There is an alternative method for approximating the solution of IDEs that is a Taylor series expansion. The Taylor series expansion is one of the methods used to calculate the solution of differential equations(DEs) and integral equations (IEs) since it is easy to compute and efficient [1–3, 5, 9]. Those who started to use taylor in solving IEs were Y.Ren et al. [10] for Fredholm integral equation and Pallop et al. [11] have modified Y.ren's method for more accurate results and used for wider class of Fredholm integral equation and Itthithep et al. [8] used this method for the solution of Volterra integro-differential equation.

In this research, we use [8] methods to approximate the solution of Fredholm integro-differential equations (FIDEs), given in the form

$$y'(x) - \int_a^b k(x,t)y(t)\mathbf{d}t = f(x), \quad y(0) = y_0. \tag{1}$$

where the functions $f(x)$ and the kernel $k(x,t)$ are known.

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II. Modified Taylor-Series Expansion Method

We consider the FIDE in form (1)

$$y'(x) - \int_a^b k(x,t)y(t)\mathbf{d}t = f(x), \quad y(0) = y_0.$$

The Taylor series approximation can be made for the solution $y(t)$ in the (1):

$$y(t) \approx y(x) + y'(x)(t-x) + \frac{y''(x)}{2!}(t-x)^2 + \frac{y'''(x)}{3!}(t-x)^3 + \cdots + \frac{y^{(n)}}{n!}(x)(t-x)^n. \tag{2}$$

Substituting (2) for $y(t)$ in the integral in (1),

$$y'(x) - \int_a^b k(x,t)\Big[y(x) + y'(x)(t-x) + \frac{y''(x)}{2!}(t-x)^2 + \cdots + \frac{y^{(n)}(x)}{n!}(t-x)^n\Big]\,\mathbf{d}t = f(x). \tag{3}$$

We obtain,

$$\Big[-\int_a^b k(x,t)\,\mathbf{d}t\Big]y(x) + \Big[1 - \int_a^b k(x,t)(t-x)\,\mathbf{d}t\Big]y'(x) + \cdots + \Big[-\frac{1}{n!}\int_a^b k(x,t)(t-x)^n\,\mathbf{d}t\Big]y^{(n)}(x) \approx f(x). \tag{4}$$

Next, we integrate both sides of (1) with respect to t from 0 to x,

$$\int_0^x y'(t)\,\mathbf{d}t - \int_0^x \int_a^b k(s,t)y(t)\,\mathbf{d}t\mathbf{d}s = \int_0^x f(t)\,\mathbf{d}t. \tag{5}$$

Application of integration by parts yields

$$\int_0^x P_k(t)y^{(k)}(t)\,\mathbf{d}t = P_k(t)y^{(k-1)}(t)|_0^x - P_k'(t)y^{(k-2)}(t)|_0^x + P_k''(t)y^{(k-3)}(t)|_0^x + \cdots + (-1)^{(k-1)}P_k^{(k-1)}(t)y(t)|_0^x + (-1)^{(k)}\int_0^x P_k^{(k)}(t)y(t)\,\mathbf{d}t. \tag{6}$$

Where $P_k(t) = 1$ and $k = 1$. So $P_k^{(i)}(t) = 0$ for $i = 1, 2, \ldots, k$.
So that,

$$\int_0^x y'(t)\,\mathbf{d}t = y(t)|_0^x = y(x) - y(0) \tag{7}$$

Notes

Substituting (7) for $\int_0^x y'(t)\,\mathbf{d}t$ in the integral in (5), we obtain

$$y(x) - y(0) - \int_0^x \int_a^b k(s,t)y(t)\,\mathbf{d}t\mathbf{d}s = \int_0^x f(t)\,\mathbf{d}t. \tag{8}$$

Similarly, Substitute $y(t)$ is replaced by $y(x)$ in (8) by the right sides of (2) to obtain

$$\begin{aligned} y(x) - y(0) - \int_0^x \int_a^b k(s,t)\Big[y(x) + y'(x)(t-x) + \frac{y''(x)}{2!}(t-x)^2 + \cdots \\ + \frac{y^{(n)}(x)}{n!}(t-x)^n\Big]\,\mathbf{d}t\mathbf{d}s = \int_0^x f(t)\,\mathbf{d}t. \end{aligned} \tag{9}$$

We obtain,

$$\begin{aligned} &\Big[1 - \int_0^x \int_a^b k(s,t)\,\mathbf{d}t\,\mathbf{d}s\Big]y(x) - \Big[\int_0^x \int_a^b k(s,t)(t-x)\,\mathbf{d}t\,\mathbf{d}s\Big]y'(x) \\ &-\Big[\frac{1}{2!}\int_0^x \int_a^b k(s,t)(t-x)^2\,\mathbf{d}t\,\mathbf{d}s\Big]y''(x) - \cdots \\ &-\Big[\frac{1}{n!}\int_0^x \int_a^b k(s,t)(t-x)^n\,\mathbf{d}t\,\mathbf{d}s\Big]y^{(n)}(x) = y(0) + \int_0^x f(t)\,\mathbf{d}t. \end{aligned} \tag{10}$$

Next, we differentiate both sides of (1) n times, one obtains

$$y''(x) - \frac{\partial}{\partial x}\Big[\int_a^b k(x,t)y(t)\,\mathbf{d}t\Big] = f'(x), \tag{11}$$

$$y'''(x) - \frac{\partial^2}{\partial x^2}\Big[\int_a^b k(x,t)y(t)\,\mathbf{d}t\Big] = f''(x), \tag{12}$$

$$\vdots$$

$$y^{(n)}(x) - \frac{\partial^n}{\partial x^n}\Big[\int_a^b k(x,t)y(t)\,\mathbf{d}t\Big] = f^{(n-1)}(x). \tag{13}$$

Using Leibnitz rule, we find that

$$\frac{\partial^n}{\partial x^n}\Big[\int_a^b k(x,t)y(t)\,\mathbf{d}t\Big] = \int_a^b \frac{\partial^n}{\partial x^n}\Big[k(x,t)\Big]y(t)\,\mathbf{d}t = \int_a^b \Big[k_x^{(n)}(x,t)\Big]y(t)\,\mathbf{d}t$$

Therefore,

$$y''(x) - \int_a^b k'_x(x,t)y(t)\,\mathbf{d}t = f'(x), \tag{14}$$

Notes

$$y'''(x) - \int_a^b k''_x(x,t)y(t)\,\mathbf{d}t = f''(x), \tag{15}$$

$$\vdots$$

$$y^{(n)}(x) - \int_a^b k_x^{(n-1)}(x,t)y(t)\,\mathbf{d}t = f^{(n-1)}(x). \tag{16}$$

Notes

Substitute $y(t)$ is replaced by $y(x)$ in (14)-(16) by the right sides of (2) to obtain

$$\begin{aligned} y''(x) - \int_a^b k'_x(x,t)\Big[y(x) + y'(x)(t-x) + \frac{y''(x)}{2!}(t-x)^2 + \cdots \\ + \frac{y^{(n)}(x)}{n!}(t-x)^n\Big]\,\mathbf{d}t = f'(x), \end{aligned} \tag{17}$$

$$\begin{aligned} y'''(x) - \int_a^b k''_x(x,t)\,ds\Big[y(x) + y'(x)(t-x) + \frac{y''(x)}{2!}(t-x)^2 + \cdots \\ + \frac{y^{(n)}(x)}{n!}(t-x)^n\Big]\,\mathbf{d}t = f''(x), \end{aligned} \tag{18}$$

$$\vdots$$

$$\begin{aligned} y^{(n)}(x) - \int_a^b k_x^{(n)}(x,t)\Big[y(x) + y'(x)(t-x) + \frac{y''(x)}{2!}(t-x)^2 + \cdots \\ + \frac{y^{(n)}(x)}{n!}(t-x)^n\Big]\,\mathbf{d}t = f^{(n-1)}(x). \end{aligned} \tag{19}$$

We obtain,

$$\begin{aligned} &\Big[-\int_a^b k'_x(x,t)\,\mathbf{d}t\Big]y(x) + \Big[-\int_a^b k'_x(x,t)(t-x)\,\mathbf{d}t\Big]y'(x) \\ &+\Big[1 - \frac{1}{2!}\int_a^b k'_x(x,t)(t-x)^2\,\mathbf{d}t\Big]y''(x) + \cdots \\ &+\Big[-\frac{1}{n!}\int_a^b k'_x(x,t)(t-x)^n\,\mathbf{d}t\Big]y^{(n)}(x) = f'(x) \end{aligned} \tag{20}$$

$$\begin{aligned} &\Big[-\int_a^b k''_x(x,t)\,\mathbf{d}t\Big]y(x) + \Big[-\int_a^b k''_x(x,t)(t-x)\,\mathbf{d}t\Big]y'(x) \\ &+\Big[-\frac{1}{2!}\int_a^b k''_x(x,t)(t-x)^2\,\mathbf{d}t\Big]y''(x) + \cdots \\ &+\Big[-\frac{1}{n!}\int_a^b k''_x(x,t)(t-x)^n\,\mathbf{d}t\Big]y^{(n)}(x) = f''(x) \end{aligned} \tag{21}$$

$$\ddots$$

Notes

$$\left[-\int_a^b k_x^{(n-1)}(x,t)\,\mathbf{d}t\right]y(x)+\left[-\int_a^b k_x^{(n-1)}(x,t)(t-x)\,\mathbf{d}t\right]y'(x)$$

$$+\left[-\frac{1}{2!}\int_a^b k_x^{(n-1)}(x,t)(t-x)^2\,\mathbf{d}t\right]y''(x)+\cdots$$

$$+\left[1-\frac{1}{n!}\int_a^b k_x^{(n-1)}(x,t)(t-x)^n\,\mathbf{d}t\right]y^{(n)}(x)=f^{(n-1)}(x) \tag{22}$$

Combining equations (4), (10), (20)-(22) , we obtain

$$\begin{pmatrix} 1-\int_0^x\int_a^b k(s,t)\,\mathbf{d}t\,\mathbf{d}s & -\int_0^x\int_a^b k(s,t)(t-x)\,\mathbf{d}t\,\mathbf{d}s & \cdots & -\frac{1}{n!}\int_0^x\int_a^b k(s,t)(t-x)^n\,\mathbf{d}t\,\mathbf{d}s \\ -\int_a^b k(x,t)\,\mathbf{d}t & 1-\int_a^b k(x,t)(t-x)\,\mathbf{d}t & \cdots & -\frac{1}{n!}\int_a^b k(x,t)(t-x)^n\,\mathbf{d}t \\ -\int_a^b k'_x(x,t)\,\mathbf{d}t & -\int_a^b k'_x(x,t)(t-x)\,\mathbf{d}t & \cdots & -\frac{1}{n!}\int_a^b k'_x(x,t)(t-x)^n\,\mathbf{d}t \\ -\int_a^b k''_x(x,t)\,\mathbf{d}t & -\int_a^b k''_x(x,t)(t-x)\,\mathbf{d}t & \cdots & -\frac{1}{n!}\int_a^b k''_x(x,t)(t-x)^n\,\mathbf{d}t \\ \vdots & \vdots & \ddots & \vdots \\ -\int_a^b k_x^{(n-1)}(x,t)\,\mathbf{d}t & -\int_a^b k_x^{(n-1)}(x,t)(t-x)\,\mathbf{d}t & \cdots & 1-\frac{1}{n!}\int_a^b k_x^{(n-1)}(x,t)(t-x)^n\,\mathbf{d}t \end{pmatrix}\times$$

$$\begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \\ y'''(x) \\ \vdots \\ y^{(n)}(x) \end{pmatrix}=\begin{pmatrix} y(0)+\int_0^x f(t)\,\mathbf{d}t \\ f(x) \\ f'(x) \\ f''(x) \\ \vdots \\ f^{(n-1)}(x) \end{pmatrix} \tag{23}$$

Equation (23) becomes a linear systems of $n+1$ algebraic equation for $n+1$ unknowns $y(x), y'(x), y''(x), \ldots, y^{(n)}(x)$, which can be solved easily use of initial condition.

III. Numerical Examples

We present in this section numerical result for some examples to show efficient and accuracy of the modified Taylor-series expansion method, and the corresponding absolute errors between their values as $e_n(x) = |exact_n(x) - app_n(x)|$.

Example 3.1. Consider

$$y'(x)-\int_0^1 (xt)y(t)\,\mathbf{d}t = 3+6x, \qquad y(0)=0 \tag{24}$$

such that $k(x,t)=xt$, $f(x)=3+6x$, $a=0$, $b=1$ and exact solution is $y(x)=4x^2+3x$.

Let $n = 2$. We apply equation (23) to approach the equation (24) that is,

$$\begin{pmatrix} 1-\int_0^x\int_0^1 (st)\,\mathbf{d}t\,\mathbf{d}s & -\int_0^x\int_0^1 (st)(t-x)\,\mathbf{d}t\,\mathbf{d}s & -\frac{1}{2}\int_0^x\int_0^1 (st)(t-x)^2\,\mathbf{d}t\,\mathbf{d}s \\ -\int_0^1 (xt)\,\mathbf{d}t & 1-\int_0^1 (xt)(t-x)\,\mathbf{d}t & -\frac{1}{2}\int_0^1 (xt)(t-x)\,\mathbf{d}t \\ -\int_0^1 (t)\,\mathbf{d}t & -\int_0^1 (t)(t-x)\,\mathbf{d}t & 1-\frac{1}{2}\int_0^1 (t)(t-x)\,\mathbf{d}t \end{pmatrix} \times$$

$$\begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \end{pmatrix} = \begin{pmatrix} \int_0^x (3+6t)\,\mathbf{d}t \\ 3+6x \\ 6 \end{pmatrix}. \tag{25}$$

We obtain,

$$\begin{pmatrix} 1-\frac{1}{4}x^2 & -\frac{1}{6}x^2+\frac{1}{4}x^3 & -\frac{1}{16}x^2+\frac{1}{6}x^3-\frac{1}{8}x^4 \\ -\frac{1}{2}x & 1-\frac{1}{3}x+\frac{1}{2}x^2 & -\frac{1}{8}x+\frac{1}{3}x^2-\frac{1}{4}x^3 \\ -\frac{1}{2} & -\frac{1}{3}+\frac{1}{2}x & \frac{7}{8}+\frac{1}{3}x-\frac{1}{4}x^2 \end{pmatrix} \times$$

$$\begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \end{pmatrix} = \begin{pmatrix} 3x^2+3x \\ 3+6x \\ 6 \end{pmatrix}. \tag{26}$$

And approximation is

$$y(x) = 4x^2 + 3x.$$

Table 1: Numerical approximation for $y(x)$ in Example 3.1 with $n = 2$.

x	$y(x)$		Absolute error	
	Exact	Our approx.	Exact	Our approx.
0.0	0.00000	0.00000	0.00000	0.00000
0.1	0.34000	0.34000	0.00000	0.00000
0.2	0.76000	0.76000	0.00000	0.00000
0.3	1.26000	1.26000	0.00000	0.00000
0.4	1.84000	1.84000	0.00000	0.00000
0.5	2.50000	2.50000	0.00000	0.00000
0.6	3.24000	3.24000	0.00000	0.00000
0.7	4.06000	4.06000	0.00000	0.00000
0.8	4.96000	4.96000	0.00000	0.00000
0.9	5.94000	5.94000	0.00000	0.00000
1.0	7.00000	7.00000	0.00000	0.00000

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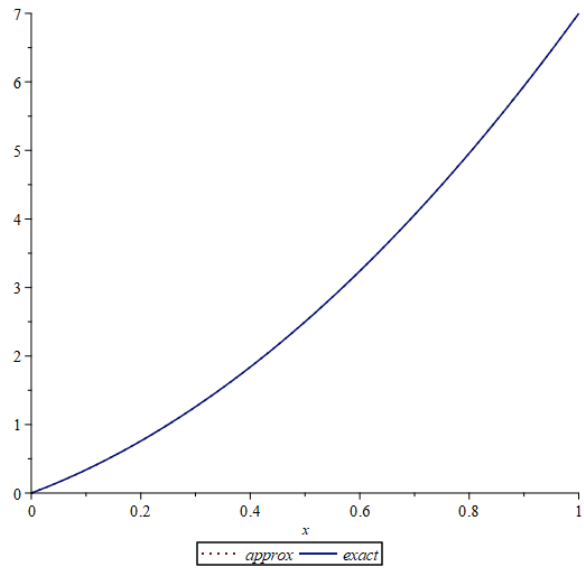

Figure 1: Comparison of approximations and exact solution with $n = 2$.

Example 3.2. As a second example, we solve the following Fredholm integro-differential equation

$$y'(x) - \int_{-1}^{1} (1 - x^2t^2)y(t)\, dt = 4, \qquad y(0) = -2 \tag{27}$$

such that $k(x,t) = 1 - x^2t^2$, $f(x) = 4$, $a = -1$, $b = 1$ and exact solution is $y(x) = -2 + \frac{4}{9}x^3$.

Let $n = 3$. We apply equation (23) to approach the equation (27) that is,

$$\begin{pmatrix} 1 - \int_0^x \int_{-1}^1 (1 - s^2t^2)\, dt\, ds & \cdots & -\frac{1}{3!} \int_0^x \int_{-1}^1 (1 - s^2t^2)(t-x)^3\, dt\, ds \\ -\int_{-1}^1 (1 - x^2t^2)\, dt & \cdots & -\frac{1}{3!} \int_{-1}^1 (1 - x^2t^2)(t-x)^3\, dt \\ -\int_{-1}^1 (-2t^2x)\, dt & \cdots & -\frac{1}{3!} \int_{-1}^1 (-2t^2x)(t-x)^3\, dt \\ -\int_{-1}^1 (-2t^2)\, dt & \cdots & 1 - \frac{1}{3!} \int_{-1}^1 (-st^2)(t-x)^3\, dt \end{pmatrix} \times$$

$$\begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \\ y'''(x) \end{pmatrix} = \begin{pmatrix} -2 + \int_0^x (4)\, dt \\ 4 \\ 0 \\ 0 \end{pmatrix}. \tag{28}$$

We obtain,

$$\begin{pmatrix} 1 + \frac{2}{9}x^3 - 2x & -\frac{2}{9}x^4 + 2x^2 & \frac{1}{9}x^5 - \frac{14}{15}x^3 - \frac{1}{3}x & -\frac{1}{27}x^6 + \frac{4}{15}x^4 + \frac{1}{3}x^2 \\ \frac{2}{3}x^2 - 2 & 1 - \frac{2}{3}x^3 + 2x & \frac{1}{3}x^4 - \frac{4}{5}x^2 - \frac{1}{3} & -\frac{1}{9}x^5 + \frac{2}{15}x^3 + \frac{1}{3}x \\ \frac{4}{3}x & -\frac{4}{3}x^2 & \frac{2}{3}x^3 + \frac{2}{5}x + 1 & -\frac{2}{9}x^4 - \frac{2}{5}x^2 \\ \frac{4}{3} & -\frac{4}{3}x & \frac{2}{3}x^2 + \frac{2}{5} & 1 - \frac{2}{9}x^3 - \frac{2}{5}x \end{pmatrix} \times$$

$$\begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \\ y^{(3)}(x) \end{pmatrix} = \begin{pmatrix} -2+4x \\ 4 \\ 0 \\ 0 \end{pmatrix}. \tag{29}$$

And approximation is

$$y(x) = -2 + \frac{4}{9}x^3.$$

Example 3.3. Next, we discuss the Fredholm integro-differential equation

$$y'(x) - \int_0^{\frac{\pi}{2}} (t)y(t)\,dt = \; -1 + \cos(x), \qquad y(0) = 0 \tag{30}$$

such that $k(x,t) = t$, $f(x) = -1 + \cos(x)$, $a = 0$, $b = \frac{\pi}{2}$ and exact solution is $y(x) = \sin(x)$.

Table 2: Numerical approximation for $y(x)$ in Example 3.2 with $n = 3$.

x	$y(x)$		Absolute error	
	Exact	Our approx.	Exact	Our approx.
-1.0	-2.44444	-2.44444	0.00000	0.00000
-0.8	-2.22755	-2.22755	0.00000	0.00000
-0.6	-2.09600	-2.09600	0.00000	0.00000
-0.4	-2.02844	-2.02844	0.00000	0.00000
-0.2	-2.00356	-2.00356	0.00000	0.00000
0.0	-2.00000	-2.00000	0.00000	0.00000
0.2	-1.99644	-1.99644	0.00000	0.00000
0.4	-1.97156	-1.97156	0.00000	0.00000
0.6	-1.90400	-1.90400	0.00000	0.00000
0.8	-1.77244	-1.77244	0.00000	0.00000
1.0	-1.55556	-1.55556	0.00000	0.00000

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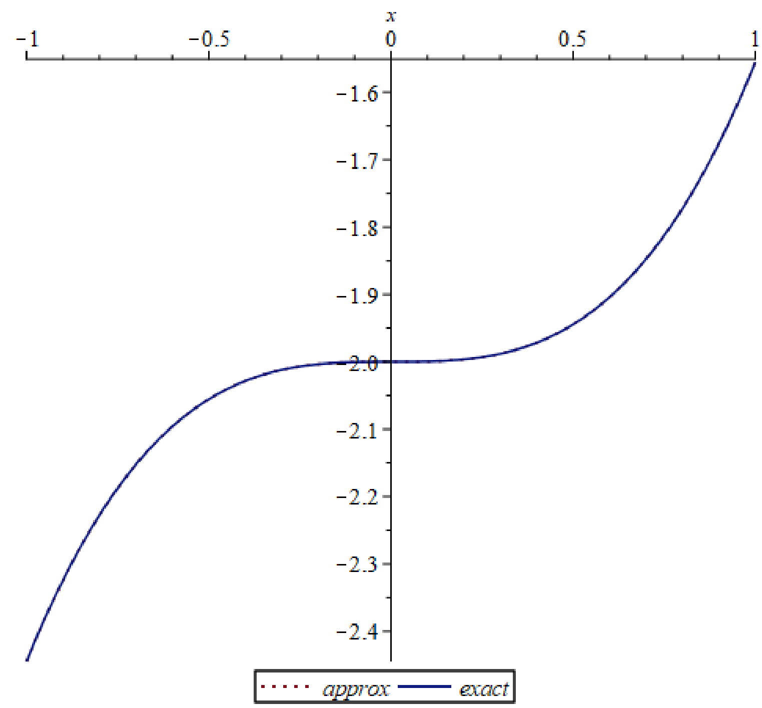

Figure 2: Comparison of approximations and exact solution with $n = 3$.

Let $n = 8$. We apply equation (23) to approach the equation (30) that is,

$$\begin{pmatrix} 1 - \int_0^x \int_0^{\frac{\pi}{2}} (t)\, dt\, ds & \cdots & -\frac{1}{8!} \int_0^x \int_0^{\frac{\pi}{2}} (t)(t-x)^8\, dt\, ds \\ -\int_0^{\frac{\pi}{2}} (t)\, dt & \cdots & -\frac{1}{8!} \int_0^{\frac{\pi}{2}} (t)(t-x)^8\, dt \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \times$$

$$\begin{pmatrix} y(x) \\ y'(x) \\ y''(x) \\ \vdots \\ y^{(8)}(x) \end{pmatrix} = \begin{pmatrix} \int_0^x (-1+\cos(t))\, dt \\ -1+\cos(x) \\ -\sin(x) \\ \vdots \\ \sin(x) \end{pmatrix} \tag{31}$$

Table 3: Numerical approximation for $y(x)$ in Example 3.3 with $n = 8$.

x	$y(x)$		Absolute error	
	Exact	Our approx.	Exact	Our approx.
0	0.00000	0.00000	0.00000	0.00000
$\frac{\pi}{20}$	0.15643	0.15644	0.00000	0.00000
$\frac{2\pi}{20}$	0.30901	0.30902	0.00000	0.00000
$\frac{3\pi}{20}$	0.45400	0.45400	0.00000	0.00000
$\frac{4\pi}{20}$	0.58778	0.58778	0.00000	0.00000
$\frac{5\pi}{20}$	0.70711	0.70711	0.00000	0.00000
$\frac{6\pi}{20}$	0.80902	0.80902	0.00000	0.00000
$\frac{7\pi}{20}$	0.89101	0.89101	0.00000	0.00000
$\frac{8\pi}{20}$	0.95106	0.95106	0.00000	0.00000
$\frac{9\pi}{20}$	0.98769	0.98769	0.00000	0.00000
$\frac{10\pi}{20}$	1.00000	1.00000	0.00000	0.00000

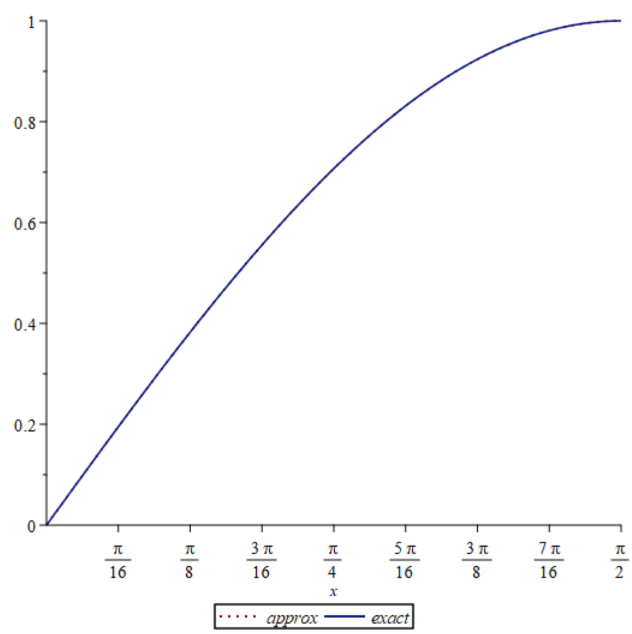

Figure 3: Comparison of approximations and exact solution with $n = 8$.

IV. Conclusion

As illustrated in the examples of this paper, the modified Taylor-series method is a powerful procedure for solving FIDEs. Using the proposed method in solving integral equation shows the high capability of this method compared to other methods.

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Notes

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A Polynomial Composites and Monoid Domains as Algebraic Structures and their Applications

By Łukasz Matysiak & Magdalena Jankowska

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Abstract- This paper contains the results collected so far on polynomial composites in terms of many basic algebraic properties. Since it is a polynomial structure, results for monoid domains come in here and there. The second part of the paper contains the results of the relationship between the theory of polynomial composites, the Galois theory and the theory of nilpotents. The third part of this paper shows us some cryptosystems. We find generalizations of known ciphers taking into account the infinite alphabet and using simple algebraic methods. We also find two cryptosystems in which the structure of Dedekind rings resides, namely certain elements are equivalent to fractional ideals. Finally, we find the use of polynomial composites and monoid domains in cryptology.

Keywords: *cryptology, domain, field, monoid, monoid domain, polynomial composites.*

GJSFR-F Classification: *MSC 2010: 08A40*

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Łukasz Matysiak [α] & Magdalena Jankowska [σ]

Abstract- This paper contains the results collected so far on polynomial composites in terms of many basic algebraic properties. Since it is a polynomial structure, results for monoid domains come in here and there. The second part of the paper contains the results of the relationship between the theory of polynomial composites, the Galois theory and the theory of nilpotents. The third part of this paper shows us some cryptosystems. We find generalizations of known ciphers taking into account the infinite alphabet and using simple algebraic methods. We also find two cryptosystems in which the structure of Dedekind rings resides, namely certain elements are equivalent to fractional ideals. Finally, we find the use of polynomial composites and monoid domains in cryptology.

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I. Introduction

Let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. By a ring we mean a commutative ring with unity. Let R be a ring. Denote by R^* the group of all invertible elements of R. The set of all irreducible elements in R will be denoted by $\operatorname{Irr} R$. By a domain we mean a commutative ring with unity without zero divisors. An element $r \in R$ is called nilpotent if there is $n \in \mathbb{N}$ such that $r^n = 0$.

The most important motivation for writing this paper is to quote the most important results related to polynomial composites, their algebraic place in mathematics and their application in cryptology.

D.D. Anderson, D.F. Anderson, M. Zafrullah in [2] called object $A + XB[X]$ as a composite, where A be a subdomain of the field B. If B be a domain and M be an additive cancellative monoid (a semigroup with neutral element and cancellative property) we can define a monoid domain $B[M] = \{a_0 X^{m_0} + \cdots + a_n X^{m_n} : a_0, \ldots, a_n \in B, m_1, \ldots m_n \in M\}$. If $M = \mathbb{N}_0$, then $B[M] = B[X]$. Monoid domains appear in many works, for example [8], [16].

There are a lot of works where composites are used as examples to show some properties. But the most important works are presented below.

In 1976 [4] authors considered the structures in the form $D + M$, where D is a domain and M is a maximal ideal of ring R, where $D \subset R$. Later

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1. D.D. Anderson, D.F. Anderson, M. Zafrullah, Factorization in integral domains, Journal of Pure and Applied Algebra, 69, (1990) 1–19.

(2.6), we could prove that in composite in the form $D+XK[X]$, where D is a domain, K is a field with $D\subset K$, that $XK[X]$ is a maximal ideal of $K[X]$. Next, Costa, Mott and Zafrullah ([5], 1978) considered composites in the form $D+XD_S[X]$, where D is a domain and D_S is a localization of D relative to the multiplicative subset S. In 1988 [3] Anderson and Ryckaert studied classes groups $D+M$. Zafrullah in [17] continued research on structure $D+XD_S[X]$ but he showed that if D is a GCD-domain, then the behaviour of $D^{(S)}=\{a_0+\sum a_iX^i \mid a_0\in D, a_i\in D_S\}=D+XD_S[X]$ depends upon the relationship between S and the prime ideals P od D such that D_P is a valuation domain (Theorem 1, [17]). Fontana and Kabbaj in 1990 ([7]) studied the Krull and valuative dimensions of composite $D+XD_S[X]$. In 1991 there was an article ([2]) that collected all previous composites and the authors began to create a theory about composites creating results. In this paper, the structures under consideration were officially called as composites. After this article, various minor results appeared. But the most important thing is that composites have been used in many theories as examples. In [10] we have a general definition of composite as polynomial composite.

In the second section we can find many results about polynomial composites and monoid domains. Basic algebraic properties such as irreducible elements, nilpotents and ideals have been examined. Theorem 2.6 is especially worth noting. In this theorem, for $A\subset B$ be fields, we can note that every nonzero prime ideal polynomial composites is maximal, every prime ideal different from some maximal ideal of polynomial composite is principal and every polynomial composites are atomic (every element of polynomial composites be a product of finite irreducibles(atoms)). In Theorem 2.9 we have an iformation about irreducibles of monoid domain. In the second part of the second section we have results about ACCP and atomic properties. Recall, a property: for any ascending chain of principal ideals of ring R: $I_1\subset I_2\subset I_3\subset\dots$, there is $n\in\mathbb{N}$ such that $I_n=I_{n+1}=\dots$. A domain with ACCP property is called ACCP-domain. Every ACCP-domain be atomic. In Theorem 2.23 it turns out that the polynomial composite of the form $K+XL[X]$ (where $K\subset L$ be fields) is a Dedekind ring. This is a very important class of rings in algebra.

In the third section we can find relationships between a theory of polynomial composites and Galois theory. Galois theory contributed greatly to the development of many fields, not only in mathematics. Particularly noteworthy is the solution of three ancient problems of construction with a compass and a straightedge in the 19th century. The results in this section, under different assumptions, boil down to the relationship between field extensions and Noetherian rings. Recall a Noetherian ring is called a ring with ACCP-property. Equivalence, a ring such that every ideal be a finite generated. In Theorems 3.13 and 3.14 we combine the Magid's results with the current results to create a complete characterization of field extensions using polynomial composites and idempotents. Recall, an element $e\in R$ is called idempotent if $e^2=e$ holds. For example, in $\mathbb{Z}$ we have two trivial idempotents 0 and 1.

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2. D.D. Anderson, D.F. Anderson, and M. Zafrullah, Rings between D[X] and K[X], Houston J. of Mathematics, 17, (1991) 109–129.

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12. Ł. Matysiak, Generalized RSA cipher and Diffie-Hellman protocol, J. Appl. Math. & Informatics Vol.39 (2021), No. 1 - 2, pp. 93 – 103.

Sections four and five are reminder from [12] a generalized RSA cipher and a Diffie-Hellman protocol key exchange. Such a reminder is purposeful because we want to draw attention to the replacement of the finite alphabet with the infinite one and the replacement of classical prime numbers with prime ideals. Such a swap will be extremely difficult for third person to break.

In sections six and seven we have cryptosystems which use the structure Dedekind. The former uses this structure in the key, and the latter uses it in two different alphabets. Of course, these ciphers can be generalized to infinite alphabets and ideals.

Section eight shows a cryptosystem based on polynomial composites. Section nine shows a cryptosystem based on monoid domains. Note that in the last cryptosystem, in order to break it, the discrete logarithm calculation should be used. At the moment, there is no mathematical way to facilitate the computation of discrete logarithms. We can count using computers, but here the algorithm would consist in checking each successive number, not on a specific indication of the number. And this is a great difficulty in breaking the last cryptosystem.

II. Polynomial Composites and Monoid Domains

In this section we introduce the most important facts about polynomial composites and monoid domains in math.

Let $T = A + XB[X]$, $T_n = A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ where A, B, A_0, A_1, ..., A_n be domains such that $A \subset B$, $A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset B$."

Let's start from the following Lemma which is very easy to proof.

Lemma 2.1. *T_n, T be rings, $T_n \subset T$.*

Now let's look at invertible and nilpotent elements.

Proposition 2.2. *Let $f = a_0 + a_1X + \cdots + a_nX^n \in T$ for any $n \in \mathbb{N}_0$. Then $f \in T^*$ if and only if $a_0 \in A^*$ and $a_1, a_2, \ldots, a_n$ are nilpotents.*

Proof. We know that if R is a ring then $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]^*$ if and only if $a_0 \in R^*$ and $a_1, a_2, \ldots, a_n$ are nilpotents. In our Proposition we have $a_1, a_2, \ldots, a_n$ are nilpotents. Of course we get $a_0 \in A^*$.

Proposition 2.3. *Let $f = a_0 + a_1X + \ldots a_{n-1}X^{n-1} + a_nX^n + \cdots + a_mX^m \in T_n$, where $0 \leq n \leq m$ and $a_i \in A_i$ for $i = 0, 1, \ldots, n$ and $a_j \in B$ for $j = n, n+1, \ldots, m$.*

(i) $f \in T_n^*$ if and only if $a_0 \in A_0^*$ and $a_1, a_2, \ldots, a_m$ are nilpotents.

(ii) f be a nilpotent if and only if $a_0, a_1, \ldots, a_m$ are nilpotents.

Proof. Analogous proof like in Proposition 2.2.

Proposition 2.4. *Let B be a domain and $f = a_{m_1}X^{m_1} + a_{m_2}X^{m_2} + \cdots + a_{m_n}X^{m_n} \in B[M]$, where $m_1, m_2, \ldots, m_n \in M$ and $a_{m_1}, a_{m_2}, \ldots, a_{m_n} \in B$.*

(i) $f \in B[M]^*$ *if and only if there exist* $m_i \in M$ *such that* $a_{m_i} \in B^*$ *and* $m_i = 0$ *and for every* $m_k \neq m_i$ *we have* a_{m_k} *be nilpotents.*

(ii) f *be a nilpotent if and only if* $a_{m_1}, a_{m_2}, \ldots, a_{m_n}$ *are nilpotents.*

Proof. (i) Assume $f \in B[M]^*$. Then there exists $g = b_{m'_1}X^{m'_1} + b_{m'_2}X^{m'_2} + \cdots + b_{m'_n}X^{m'_n}$, where $m'_1, m'_2, \ldots, m'_n \in M$ and $b_{m'_1}, b_{m'_2}, \ldots, b_{m'_n} \in B$ such that $fg = 1$. Hence there exist $m_i, m_j \in M$ such that $a_{m_i}b_{m_j}X^{m_i+m_j} = 1$. We have $a_i \in B^*$ and $m_i, m_j = 0$. The rest of coefficients are nilpotents. On the other side of the proof it is easy.
(ii) Obvious.

Let's recall Theorem from [2] (Theorem 2.9) in a different form.

Theorem 2.5. *Let* A *be a subfield of* B. *Consider* $D = A + XB[X]$. *Then* $\operatorname{Irr} D = \{aX, a \in B\} \cup \{a(1 + Xf(X)), a \in A, f \in B[X], 1 + Xf(X) \in \operatorname{Irr} B[X]\}$.

Theorem 2.6. *Consider* $T = A + XB[X]$, *where* A *be a subfield of* B; $T_n = A_0 + A_1X + A_2X^2 \cdots + A_{n-1}X^{n-1} + X^nB[X]$, *where* $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset B$ *be fields. Then*

(i) *every nonzero prime ideal of* T *(*T_n*, respectively) is maximal;*

(ii) *every prime ideal* P *different from* $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ *(in* T_n*) is principal;*

(iii) *every prime ideal* P *different from* $XB[X]$ *(in* T*) is principal;*

(iv) T_n *is atomic, i.e., every nonzero nonunit of* T *is a finite product of irreducible elements (atoms);*

(v) T *is atomic.*

Proof. *(i)*. We proof for T_n. The proof for T will be a corollary.

First note that $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ is maximal since $T_n/A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X] \cong A_0$. Let P be a nonzero prime ideal of T_n. Now $X \in P$ implies $(T_n/A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X])^2 \subseteq P$ and hence $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X] \subseteq P$ so $P = A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$. So suppose that $X \notin P$. Then for $N = \{1, X, X^2, \ldots\}$, P_N is a prime ideal in the PID $B[X, X^{-1}] = T_{n,N}$. (In fact, $B[X, X^{-1}] \subseteq R_P$ and R_P is a DVR (discrete valuation ring)). So P is minimal and is also maximal unless $P \subsetneq A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$. But let $k_nX^n + \cdots + k_sX^s \in P$ with $k_n \in \mathbb{N}_0$, where $k_n, \ldots, k_s \in B$ for any n, s. Then $X^{n+1} + k_n^{-1}k_{n+1}X^{n+2} + \cdots + k_n^{-1}k_sX^s \in P$, so $X \notin P$ implies that $1 + k_n^{-1}k_{n+1}X + \cdots + k_n^{-1}k_sX^{s-n} \in P$, a contradiction. So every nonzero prime ideal is maximal.

(ii), (similarly *(iii)*). If P is different from $A_1X + A_2X^2 + \cdots + A_{n-1}X^{n-1} + X^nB[X]$, then it contains an element of the form $1 + a_1X + a_2X^2 + \cdots +$

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2. D.D. Anderson, D.F. Anderson, and M. Zafrullah, Rings between D[X] and K[X], Houston J. of Mathematics, 17, (1991) 109–129.

Notes

$a_{n-1}X^{n-1}+X^nf(X)$, where $a_i \in A_i$ for $i=1,2,\ldots,n-1$ and $f(X)\in B[X]$. Now if $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)$ can be factored in $A_1X+A_2X^2+\cdots+A_{n-1}X^{n-1}+X^nB[X]$ it can be written as $(1+b_1X+b_2X^2+\cdots+b_{n-1}X^{n-1}+X^ng(X))(1+c_1X+c_2X^2+\cdots+c_{n-1}X^{n-1}+X^nh(X))$, where $b_i,c_i \in A_i$ for $i=1,2,\ldots,n-1$ and $g(X),h(X)\in B[X]$. Hence $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)$ is irreducible in T_n if and only if it is irreducible in $A_1X+A_2X^2+\cdots+A_{n-1}X^{n-1}+X^nB[X]$.

Now let $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)$ be irreducible in T_n and suppose that $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)\mid k(X)l(X)$ in T_n. Then $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)\mid k(X)l(X)$ in $A_1X+A_2X^2+\cdots+A_{n-1}X^{n-1}+X^nB[X]$, and so in $A_1X+A_2X^2+\cdots+A_{n-1}X^{n-1}+X^nB[X]$ we have, say $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)\mid k(X)$. Then, in $A_1X+A_2X^2+\cdots+A_{n-1}X^{n-1}+X^nB[X]$, $k(X)=(1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X))d(X)$. Now $d(X)$ can be written as $d(X)=aX^r(1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^np(X))$. If $r>0, d(X)\in T_n$, while if $r=0, k(X)=(1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X))(b(1+b_1X+b_2X^2+\cdots+b_{n-1}X^{n-1}+X^np(X))$ and $b\in A_0$ because $k(X)\in T_n$. In either case, $d(X)\in T_n$ and so $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)\mid k(X)$ in T_n. Consequently, in T_n every irreducible element of the type $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)$ is prime.

Now since every element of the form $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)$ is a product of irreducible elements of the same form and hence is a product of prime elements, it follows that every prime ideal of P different from $A_1X+A_2X^2+\cdots+A_{n-1}X^{n-1}+X^nB[X]$ contains a principal prime and hence is actually principal.

(iv) (similarly v). From *(ii)* a general element of T_n can be written as $aX^r(1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X))$, where $a\in B$ (with $a\in A_0$ if $r=0$) and $1+a_1X+a_2X^2+\cdots+a_{n-1}X^{n-1}+X^nf(X)$ is a product of primes.

Now, We give some basic information related to ideals.

Corollary 2.7. *(i) If A be a field, then $XB[X]$ be an maximal ideal in T.*

(ii) If A be an integral domain, then $XB[X]$ be an prime ideal in T.

(iii) $T/(X)\cong A$.

(iv) $T/B\cong\{0\}$.

(v) Let $A\subset B$ be fields in T. $T/(aX)$ be a field for any $a\in B$.

(vi) Let $A\subset B$ be fields in T. $T/(a(1+Xf(X)))$ be a field for any $a\in A, f\in B[X]$ such that $1+Xf(X)\in \mathrm{Irr}\,B[X]$.

Proof. (i) Let A be a field. The proof follows from $T/XB[X]\cong A$. We have $XB[X]$ is a maximal ideal in T.

(ii) – (iv) Obvious.

(v), (vi) From Theorem 2.9 in [2] aX for any $a \in B$ is an irreducible element. We get $T/(aX)$ be a field. We also have $a(1 + Xf(X))$ for any $a \in A, f \in B[X]$ such that $1 + Xf(X) \in \operatorname{Irr} B[X]$ is a irreducible element. We have $T/(a(1 + Xf(X)))$ be a field.

Corollary 2.8. *(i) If $A_0 + A_1X + \cdots + A_{n-1}X^{n-1}$ be a field (where $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset B$), then $X^nB[X]$ be an maximal ideal in T_n.*

(ii) If $A_0 + A_1X + \cdots + A_{n-1}X^{n-1}$ be a domain, then $X^nB[X]$ be an prime ideal in T_n.

(iii) $T_n/(X) \cong A_0$.

(iv) $T_n/B \cong \{0\}$.

(v) Let $A_0 \subset A_1 \subset \cdots \subset B$ be fields in T_n. $T_n/(aX)$ be a field for any $a \in B$.

(vi) Let $A_0 \subset A_1 \subset \cdots \subset B$ be fields in T_n. $T_n/(a(1 + a_1X + a_2X^2 + \cdots + a_{n-1}X^{n-1} + X^nf(X)))$ be a field for any $a \in B, a_i \in A_i (i = 1, 2, \dots, n-1), f \in B[X]$ such that $1 + Xf(X) \in \operatorname{Irr} B[X]$.

Proof. The proof is similarly to proof of Corollary 2.7.

Theorem 2.9. *Consider $T = A + XB[X]$, where A be a subfield of B; $T_n = A_0 + A_1X + A_2X^2 \cdots + A_{n-1}X^{n-1} + X^nB[X]$, where $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset B$ be fields. Then*

(i) $f \in \operatorname{Irr} T$ if and only if $f \in \operatorname{Irr} B[X], f(0) \in A$.

(ii) $f \in \operatorname{Irr} T_n$ if and only if $f \in \operatorname{Irr} B[X], a_i \in A_i$, where $f = a_0 + a_1X + \dots a_{n-1}X^{n-1} + a_nX^n + \cdots + a_mX^m$ with $a_i \in A_i$ for $i = 0, 1, \dots n-1$ and $a_n, a_{n+1}, \dots, a_m \in B (n < m)$.

Proof. (i). Suppose that $f \notin \operatorname{Irr} B[X]$ or $f(0) \notin A$. If $f(0) \notin A$, then $f \notin T$, so $f \notin \operatorname{Irr} B[X]$. Now, assume that $f \notin \operatorname{Irr} B[X]$. Then $f = gh$, where $g, h \in B[x] \setminus B$. Let $g = a_0 + a_1X + \cdots + a^nX^n, h = b_0 + b_1X + \cdots + b_mX^m$. We have $f = (a_0 + a_1X + \cdots + a^nX^n)(b_0 + b_1X + \cdots + b_mX^m)$. Then $f = \left(1 + \frac{a_1}{a_0}X + \cdots + \frac{a_n}{a_0}X^n\right)(a_0b_0 + a_0b_1X + \cdots + a_0b_mX^m)$, where $a_0b_0 = f(0) \in A$. Now, suppose that $f \notin \operatorname{Irr} T$. If $f \notin T$, then $f(0) \notin A$. Now, assume that $f \in T$. Then we have $f = gh$, where $g, h \in T \setminus A$. This implies $g, h \in B[x] \setminus B$.

(ii) Occur in the same way as in *(i)*.

In [8], Lemma 6.4 we have informations about irreducible element in monoid domain $D[S]$, where D be a domain, and S be a submonoid of $\mathbb{Q}_+$. I present a generalized Proposition.

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2. D.D. Anderson, D.F. Anderson, and M. Zafrullah, Rings between D[X] and K[X], Houston J. of Mathematics, 17, (1991) 109–129.

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

Notes

Proposition 2.10. *Let B be an integral domain with quotient field K and M a monoid with quotient group $G \neq M$. Assume that B contains prime elements $p_1, p_2, \dots, p_{r-1}$. Assume that M is integrally closed and each nonzero element of G is type $(0, 0, \dots)$ (G satisfies the ascending chain condition on cyclic subgroups). Consider $m_1, m_2, \dots, m_r \in M$ such that $m_1 \in \operatorname{Irr} M$ and $m_2, m_3, \dots, m_r \notin m_1 + M$. Then $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} \in \operatorname{Irr} B[M]$.*

Proof. Let $\leq$ be a total order on G. We may assume that $m_r < m_{r-1} < \dots m_2 < m_1$. Suppose that $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} = fg$ with $f, g \in B[M]$. Write $f = a_1X^{t_1} + \dots a_mX^{t_m}$ and $g = b_1X^{k_1} + \dots + b_nX^{k_n}$ in canonical form, where $t_1 < \dots < t_m$ and $k_1 < \dots < k_n$. First assume that either f or g is a monomial, say $f = aX^t$.Then $a \in B^*, m_1 = t + k_n, m_2 = t + k_1, m_3 = t + k_2, \dots, m_r = t + k_{r-1}$. Since $m_1 \in \operatorname{Irr} M$, either t or k_n is invertible in M. If k_n is invertible, then $m_2 = t + k_1 = (m_1 - k_n) + k_1 \in m_1 + M, m_3 = t + k_2 = (m_1 - k_n) + k_2 \in m_1 + M, \dots, m_r \in m_1 + M$, a contradiction. Thus t is invertible in M, and hence f is a unit in $B[M]$. Thus we may assume that f and g are not monomials. Now consider the reduction of $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} = fg$ modulo the ideal $(p_1, p_2, \dots, p_{r-1})$. Then $(-1 + (p_1, p_2, \dots, p_{r-1}) = ((a_m + (p_1, p_2, \dots, p_{r-1}))X^{t_m})((b_n + (p_1, p_2, \dots, p_{r-1}))X^{k_n})$. This means that $a_1 + (p_1, p_2, \dots, p_{r-1}) = b_1 + (p_1, p_2, \dots, p_{r-1}) = (p_1, p_2, \dots, p_{r-1})$. In this case $c_1p_1 + \dots c_{r-1}pr - 1 = a_1b_1 \in (p_1, \dots, p_{r-1})^2$, a contradiction. Thus $p_{r-1}X^{m_r} - \dots - p_2X^{m_3} - p_1X^{m_2} - X^{m_1} \in \operatorname{Irr} B[M]$.

Proposition 2.11. *$B[M]/(p_{r-1}X^{m_r} - \dots - p_1X^{m_2} - X^{m_1})$ be a field, where B be a domain, $p_1, p_2, \dots, p_{m_r} \in B, m_1, m_2, \dots, m_r \in M$ with $m_1 \in \operatorname{Irr} M$, $m_2, m_3, \dots, m_r \notin m_1 + M$.*

Proof. It follows from Proposition 2.10.

Recall that Noetherian rings satisfy the ACCP condition. Almost every mathematician has encountered such rings. For example, $\mathbb{Z}$ is a Noetherian ring. Below are the results of ACCP properties in polynomial composites and monoid domains.

Proposition 2.12. *Let A be an integral domain, B be a field such that $A \subset B$. Let R be a ring with $A[X] \subseteq R \subseteq B[X]$. Then R has ACCP if and only if $R \cap B$ has ACCP and for each ascending chain of polynomials $f_1R \subseteq f_2R \subseteq f_3R \subseteq \dots$ where $f_i \in R$ all have the same degree, then there is $d \in (R \cap B) \setminus \{0\}$ such that $df_i \in (R \cap B)[X]$.*

Proof. [11], Proposition 2.1.

Proposition 2.13 shows that between $A[X]$ and $A + XB[X]$ we can find a structure which satysfying ACCP condition.

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

Proposition 2.13. *Let A be an integral domain, B be a field such that $A \subset B$. Let C be a domain such that $A[X] \subseteq C \subseteq A + XB[X]$. Suppose that for each $n \in \mathbb{N}_0$, there exists $a_n \in A \setminus \{0\}$ for all $f \in C$ with $\deg f \leq n$. Then C has ACCP if and only if A has ACCP.*

Proof. [11], Proposition 2.2.

The above Proposition is not obvious for arbitrary composition. This would be a valuable remark, as it would allow we to choose the smallest possible composite.

Question: For subdomains $A_0, A_1, \ldots, A_{n-1}$ of a field B, is the Proposition 2.13 valid for such domain C satysfying $A_0[X] \subseteq C \subseteq A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$, where the condition $A_0 \subset A_1 \subset \cdots \subset A_{n-1} \subset B$ holds or not?

Kim in [8] proved very serious fact about ACCP monoid domain.

Lemma 2.14. *Let A be a domain. Then A has ACCP if and only if $A[X]$ has ACCP.*

Proof. [8], Corollary 2.2. Can be easily proved by comparing degrees.

It also turn out that ACCP property moves between A and $A + XB[X]$. This is important because we do not have to choose a general polynomial, and we can limit the inclusion to the smallest composite needed. Such a significant limitation of a polynomial to a composite is important, e.g. in Galois theory in commutative rings.

Theorem 2.15. *Let A be an integral domain, B be a field such that $A \subset B$. An A has ACCP if and only if $A + XB[X]$ has ACCP.*

Proof. From Proposition 2.13 we have $A[X] \subseteq A + XB[X] \subseteq A + XK[X]$, where K be a qoutient field of B. We can to prove that for each $n \geq 0$, there exists $a_n \in A \setminus \{0\}$ for all $f \in A + XB[X]$ with $\deg f \leq n$. Because A has ACCP then from Proposition 2.13 we get $A + XB[X]$ has ACCP. Conversly, because $A + XB[X]$ has ACCP then A has ACCP.

The next facts are the conclusions of Theorem 2.15.

Corollary 2.16. *Let $A_0, A_1, \ldots, A_{n-1}$ be subdomains of a field B such that $A_0 \subset A_1 \subset \cdots \subset B$. Let C be a domain with $A_0[X] \subseteq C \subseteq A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$. Suppose that for each $n \geq 0$, there exists $a_n \in A_0 \setminus \{0\}$ for all $f \in C$ with $\deg f \leq n$. Then C has ACCP if and only if A_0 has ACCP.*

Corollary 2.17. *Let $A_0, A_1, \ldots, A_{n-1}$ be subdomains of a field B such that $A_0 \subset A_1 \subset \cdots \subset B$. An A_0 has ACCP if and only if $A_0 + A_1X + \cdots + A_{n-1}X^{n-1} + X^nB[X]$ has ACCP.*

Next Lemmas coming from Kim [8] are results about ACCP properties in monoid domains.

Ref

8. H. Kim, Factorization in monoid domains, Comm. Algebra, 29, (2001) 1853–1869.

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

Lemma 2.18. *Let $S \subseteq T$ be an extension of torsion-free cancellative monoids. If T satisfies ACCP and $T^* \cap S = S^*$, then S satisfies the ACCP.*

Proof. [8] Proposition 1.2. (1).

Lemma 2.19. *Let D be an integral domain, S a torsion-free cancellative additive monoid, and $D[S]$ the monoid domain. If $D[S]$ satisfies ACCP, then D and S satisfy ACCP.*

Proof. [8], Proposition 1.5.

Next Theorem is the answer about question from Kim [8] Question 1.6. In [8] Proposition 1.5 (1) we have an implication. Kim asked that are the sufficient conditions in [8] Proposition 1.5 (1) for the monoid domain to satisfy ACCP, necessary.

Theorem 2.20. *Let A be an integral domain and B be a field such that $A \subset B$ and $A[S]^* = B[S]^*$. Let S be a torsion-free cancellative monoid. Both A and $B[S]$ satisfy ACCP if and only if $A[S]$ satisfies ACCP.*

Proof. ($\Rightarrow$) The proof is similar to [8], Proposition 1.5.

($\Leftarrow$) From Lemma 2.19, since $A[S]$ has ACCP, then A has ACCP. Now, consider $f_1, f_2, \dots \in B[S]$ such that $\dots, f_3 \mid f_2, f_2 \mid f_1$. Without loss of generality, we can assume that $f_1, f_2, \dots \in \operatorname{Irr} B[S]$ because every ACCP-domain is atomic. Since $A^* = B^*$, so $f_1, f_2, \dots \in \operatorname{Irr} A[S]$. By assumption $A[S]$ has ACCP, so there exists $n \geqslant 1$ such that $f_n \mid f_{n-1}, \dots, f_3 \mid f_2, f_2 \mid f_1$. We get $(f_1) \subseteq (f_2) \subseteq \dots \subseteq (f_n) = (f_{n+1}) = \dots$ in $B[S]$ which is stationary.

Recall that each ACCP-domain is atomic. Hence, all previous results about the ACCP-domains hold for the atomic domains. We complete the knowledge about the atomicity condition in monoid domains.

Lemma 2.21. *Let D be an integral domain, S a torsion-free cancellative monoid, and $D[S]$ the monoid domain. If $D[S]$ be atomic, then D and S be atomic.*

Proof. [8], Proposition 1.4.

Next Theorem is similarly to 2.20.

Theorem 2.22. *Let A be an integral domain and B be a field such that $A \subset B$ with $A[S]^* = B[S]^*$. Let S be a torsion-free cancellative monoid. Both A and $B[S]$ be atomic if and only if $A[S]$ be atomic.*

Proof. ($\Rightarrow$) Since $B[S]$ be atomic, then consider $f = g_1 g_2 \dots g_n \in B[S]$, where $g_1, g_2, \dots, g_n \in \operatorname{Irr} B[S]$. Hence from assumption we have $g_1, g_2, \dots, g_n \in \operatorname{Irr} A[S]$. Then $A[S]$ is atomic.

($\Leftarrow$) From Lemma 2.21 since $A[S]$ be atomic, then A and S be atomic. Now consider $f = g_1 g_2 \dots g_n \in A[S]$, where $g_1, g_2, \dots, g_n \in \operatorname{Irr} A[S]$, because $A[S]$ be atomic. Then $g_1, g_2, \dots, g_n \in \operatorname{Irr} B[S]$, hence $B[S]$ be atomic.

Ref

8. H. Kim, Factorization in monoid domains, Comm. Algebra, 29, (2001) 1853–1869.

Anderson, Anderson and Zafrullah asked in [1] (Question 1) is $R[X]$ atomic when R is atomic. I say no. I have no example but we can deduce from well known facts:

Suppose that $R[X]$ is not atomic. We want to get R is not atomic. Since $R[X]$ is not atomic then $R[X]$ has no ACCP. Hence R has no ACCP which it does not imply R is not atomic because there exists an example atomic domain which is not ACCP.

Converse, if R is not atomic, then R has no ACCP. Hence $R[X]$ has no ACCP which it does not imply $R[X]$ is atomic.

In [13] we have another results about polynomial composites. Various properties have been investigated: BFD (bounded factorization domain), HFD (half factorial domain), idf (each nonzero element of domain has at most a finite number of nonassociate irreducible divisors), FFD (finite factorization domain), S-domain (for each height-one prime ideal P of domain, height of $P[X]$ is equal to 1 in polynomial ring over domain), Hilbert ring (every prime ideal of domain is an intersection of maximal ideals of that domain).

Theorem 2.23 says that, under some assumption, a polynomial composite has the structure of Dedekind rings. Dedekind rings are a very important class of rings in algebra. There are a lot of work, the results associated with it. On the basis of the Dedekind structure, I developed cryptosystems with the Dedekind structure in Sections 6 and 7.

Theorem 2.23. *Let $K \subset L$ be a finite fields extension. Then $K + XL[X]$ be a Dedekind domain.*

Proof. By [2], Theorem 2.7 every nonzero prime ideal is a maximal. By [13] Proposition 3.1 we have $K + XL[X]$ is integrally closed. By Theorem 3.2 [14] $K + XL[X]$ is noetherian domain. Hence $K + XL[X]$ be a Dedekind domain.

In the following Proposition, we provide the most important and fundamental facts about the structure of Dedekind.

Proposition 2.24. *Let $K \subset L$ be an extension fields and let $T = K+XL[X]$.*

(a) *If P be a nonzero prime ideal of T and $P' = \{x \in T_0; xP \subset T\}$, then $PP' = T$.*

(b) *Every nonzero ideal of T has an unambiguous representation in the form product of prime ideals.*

(c) *Every nonzero ideal of T is invertible.*

(d) *If I is an ideal of T, then T/I is a principal ideal domain.*

(e) *$Cl(T)$ (a group of class of invertible ideals) be isomorphic to $Pic(T)$ (a group of class of invertible modules).*

Ref

1. D.D. Anderson, D.F. Anderson, M. Zafrullah, Factorization in integral domains, Journal of Pure and Applied Algebra, 69, (1990) 1–19.

(f) *If M be a finite generated torsion-free T-module, then $M \cong I_1 \oplus I_2 \oplus \dots \oplus I_k$, where $I_1, I_2, \dots, I_k$ are nonzero ideals of T and k is a rang of M. Moreover*

$$M \cong T^{k-1} \oplus I_1 I_2 \dots I_k.$$

(g) *If M be a finite generated T-module, then*

$$M \cong T^{k-1} \oplus I \oplus \bigoplus_{(P_i, n_i)} T/P_i^{n_i},$$

where $k = \dim_{T_0}(M \otimes_T T_0)$, $I \subset T$, I is unambiguously, with the accuracy to isomorphism, a designated ideal, P_i are nonzero prime ideals of T, $n_i > 0$, and a finite set of pair (P_i, n_i) is designated unambiguously.

III. Relationships between Polynomial Composites and Certain types of Fields Extensions

Let $K \subset L$ be a fields extension. Let's build a polynomial composites $K + XL[X]$. In this section, we will answer the question of whether there are relationships between field extensions and polynomial composites.

All my considerations began with the Theorem 3.1 below. This Proposition motivated me to further consider polynomial composites $K + XL[X]$ in a situation where the extension of fields $K \subset L$ is algebraic, separable, normal and Galois, respectively.

Theorem 3.1. *Let $K \subset L$ be a field extension. Put $T = K + XL[X]$. Then T is Noetherian if and only if $[L\colon K] < \infty$.*

Proof. ($\Rightarrow$) Since $XL[X]$ is a finitely generated ideal of $K + XL[X]$, it follows from [14] Lemma 3.1 that $[L\colon K] < \infty$. Thus, $L[X]$ is module-finite over the Noetherian ring $K + XL[X]$.

($\Leftarrow$) $L[X]$ is Noetherian ring and module-finite over the subring $K + XL[X]$. This is the situation covered by P.M. Eakin's Theorem [6].

Every Propositions and Theorems of this section we can find in [14].

Proposition 3.2. *Let $K \subset L$ be a fields extension such that $L^{G(L|K)} = K$. Put $T = K + XL[X]$. T is Noetherian if and only if $K \subset L$ be an algebraic extension.*

Proposition 3.3. *Let $K \subset L$ be fields extension such that K be a perfect field and assume that any K-isomorphism $\varphi\colon M \to M$, where $\varphi(L) = L$ holds for every field M such that $L \subset M$. Put $T = K + XL[X]$. T be a Noetherian if and only if $K \subset L$ be a separable extension.*

Proposition 3.4. *Let $K \subset L$ be fields extension. Assume that if a map $\varphi\colon L \to a(K)$ is K-embedding, then $\varphi(L) = L$. Put $T = K + XL[X]$. T be a Noetherian if and only if $K \subset L$ be a normal extension.*

Ref

14. Ł. Matysiak, Polynomial composites and certain types of fields extensions, arXiv: 2011.09904, (2021).

Proposition 3.5. *Let $K \subset L$ be fields extension such that $L^{G(L|K)} = K$. Put $T = K + XL[X]$. T be a Noetherian if and only if $K \subset L$ be a normal extension.*

Proposition 3.6. *Let $T = K + XL[X]$ be Noetherian, where $K \subset L$ be fields. Assume $|G(L \mid K)| = [L\colon K]$ and any K-isomorphism $\varphi\colon M \to M$, where $\varphi(L) = L$ holds for every field M such that $L \subset M$. T be a Noetherian if and only if $K \subset L$ be a Galois extension.*

Proposition 3.7. *Let $T = K + XL[X]$, where $K \subset L$ be fields such that $K = L^{G(L|K)}$. T be a Noetherian if and only if $K \subset L$ be a Galois extension.*

Proposition 3.8. *Let $K \subset L \subset M$ be fields such that K be a perfect field. If $K + XL[X]$ and $L + XM[X]$ be Noetherian then $K \subset M$ be separable fields extension.*

Moreover, if we assume that any K-isomorphism $\varphi\colon M' \to M'$, where $\varphi(M) = M$ holds for every field M' such that $M \subset M'$, then $K + XM[X]$ be a Noetherian.

Proposition 3.9. *Let $K \subset L \subset M$ be fields such that $M^{G(M|K)} = K$. If $K + XM[X]$ be Noetherian then $L \subset M$ be a normal fields extension. Moreover, $L + XM[X]$ be Noetherian.*

Proposition 3.10. *Let $K \subset L$ be extension fields such that $[L\colon K] = 2$. Then $K + XL[X]$ be Noetherian. Moreover, if $L^{G(L|K)} = K$, then $K \subset L$ be a normal.*

Theorem 3.11 ([9], Theorem 1.2.). *Let M be an algebraically closed field algebraic over K, and let L such that $K \subseteq L \subseteq M$ be an intermediate field. Then the following are equivalent:*

(a) *L is separable over K.*

(b) *$M \otimes_K L$ has no nonzero nilpotent elements.*

(c) *Every element of $M \otimes_K L$ is a unit times an idempotent.*

(d) *As an M-algebra $M \otimes_K L$ is generated by idempotents.*

Theorem 3.12 ([9], Theorem 1.3.). *Let M be an algebraically closed field containing K, and let L be a field algebraic over K. Then the following are equivalent:*

(a) *L is separable over K.*

(b) *$M \otimes_K L$ has no nonzero nilpotent elements.*

(c) *Every element of $M \otimes_K L$ is a unit times an idempotent.*

(d) *As an M-algebra $M \otimes_K L$ is generated by idempotents.*

Below we have conclusions from the above results.

Theorem 3.13. *In Theorems 3.11 and 3.12 if assume $L^{G(L|K)} = K$, then conditions (a) – (d) are equivalent to*

Notes

(e) $K + XL[X]$ *be a Noetherian.*

(f) $[L\colon K] < \infty$

(g) $K \subset L$ *be an algebraic extension.*

(h) $K \subset L$ *be a Galois extension.*

Notes

Proof. (h)⇒(a) – Obvious.

(a)⇒(g)⇒(e)⇒(h) If $K \subset L$ be a separable extension, then be an algebraic extension. By Proposition 3.2 $K + XL[X]$ be a Noetherian. By Proposition 3.7 $K \subset L$ be a Galois extension.

(e)⇒(f) – Theorem 3.1.

Theorem 3.14. In Theorem 3.13 if assume K be a perfect field and $L^{G(L|K)} = K$, then conditions (a) – (h) are equivalent to
(g) $K \subset L$ be a normal extension.

Proof. (g)⇒(a) If $K \subset L$ be a normal extension, then be an algebraic extension. By definition perfect field $K \subset L$ be a separable extension.

(h)⇒(g) Obvious.

Proposition 3.7, Theorems 3.13 and 3.14 can be used to solve the inverse Galois problem. The inverse Galois problem concerns whether or not every finite group appears as the Galois group of some Galois extension of the rational numbers $\mathbb{Q}$. This problem, first posed in the early 19th century, is unsolved.

There is a lot of work. And it is enough to solve the problem for non-abelian groups. Thus, the following question arises:

Question:
Can all the statements of this sections operate in noncommutative structures?

And another question also arises regarding polynomial composites:

Question:
Under certain assumptions for any type of $K \subset L$, we get that $K+XL[X]$ be a Noetherian ring. When can $K + XL[X]$ be isomorphic to any Noetherian ring?

IV. Generalized RSA Cipher

In [12] we have an information about how can we make a finite alphabet to an infinite alphabet?

We can assign an appropriate number to each letter of the alphabet: $A = 0, B = 1, C = 2, D = 3, E = 4, F = 5, G = 6, H = 7, I = 8, J = 9, K = 10, L = 11, M = 12, N = 13, O = 14, P = 15, Q = 16, R = 17, S = 18, T = 19, U = 20, V = 21, W = 22, X = 23, Y = 24, Z = 25$. So the alphabet is a finite set. The opposite side can easily decipher using the length of

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

the alphabet. What if we extend this alphabet to an infinite set? In this situation, we can stay with the alphabet, but extend the length to infinity. So we have $A = 0+26k_0, B = 1+26k_1, C = 2+26k_2, \ldots, Y = 24+26k_{24}, Z = 25 + 26k_{25}$, where $k_0, k_1, \ldots, k_{25} \in \mathbb{N}_0$. So, for example, the text $ABACAB$ can be converted to 0 1 0 2 0 1, but also to 0 1 26 54 26 53. And we can give this number sequence to encrypt.

a) Generating keys

Let's choose distinct prime ideals $P = (p)$ and $Q = (q)$ (p, q are distinct primes) such that $N = PQ$ such that $|N| < |(x)|$, where x is the length of the alphabet.

$$\text{Compute } \Phi(N) = (\varphi(n)) := (P-1)(Q-1) = (p-1)(q-1).$$

Let's choose the ideal $E = (e)$ such that e and $\varphi(n)$ are relatively primes ($\gcd(e, \varphi(n) = 1)$) and $|\Phi(N)| < |E| \subsetneq (1) = \mathbb{N}_0$.

We find the ideal $D = (d)$ such that $ED \equiv 1 (\text{mod } \Phi(N))$.

The public key is defined as the pair of ideals (N, E), while the private key is the pair (N, D).

b) Encryption and decryption

We encrypt the message $M = M_0 M_1 \ldots M_r$ by calculation

$$C_i \equiv M_i E (\text{mod } \Phi(N))$$

The encrypted message $C = C_0 C_1 \ldots C_r$ is decrypted by formula

$$M_1 \equiv C_i D (\text{mod } \Phi(N)).$$

V. Generalized Diffie - Hellman Key Exchange

From [12] recall a generalized Diffie-Hellman key exchange.

First person F and second person S agree on the prime ideals (p) and (g) in $\mathbb{N}_0$ such that $|(p)| < |(g)|$.

Person F chooses any secret (a) in $\mathbb{N}_0$ and sends to person S

$$(A) \equiv (g)(a) (\text{mod } (p)).$$

Person S chooses any secret (b) in $\mathbb{N}_0$ and sends to person F

$$(B) \equiv (g)(b) (\text{mod } (p)).$$

Person F compute $(s) \equiv (B)(a) (\text{mod } (p))$.

Person S compute $(s) \equiv (A)(b) (\text{mod } (p))$.

Person F and person S share a secret ideal (s). This is because

$$(s) \equiv (g)(a)(b) \equiv (g)(b)(a) (\text{mod } (p)).$$

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

Ref

12. Ł. Matysiak, Generalized RSA cipher and Diffie-Hellman protocol, J. Appl. Math. & Informatics Vol.39 (2021), No. 1 - 2, pp. 93 – 103.

VI. A Key that is a Fractional Ideal

In section 6 and 7 we have cryptosystems that use the Dedekind structure ([15] in cooperation with M. Jankowska). My goal was not to create an entire cryptosystem based on the Dedekind structure. The first cryptosystem has a Dedekind structure in the key. The second cryptosystem has a Dedekind structure in two different alphabets. It is essential. This increases the security of our data. First of all, we use the fractional ideal structure. The definition itself is very interesting and motivated to apply.

Ref

15. M. Jankowska and L. Matysiak, A structure of Dedekind in the cryptosystem, http://lukmat.ukw.edu.pl/files/A-structure-of-Dedekind-in-the-cryptosystem.pdf, (2021).

Let $A = \{a_0, a_1, \ldots, a_n\}$ be an alphabet such that $|A|$ be a prime number. Let $x \in \{2, 3, \ldots, |A|\}$ be the value of one of the letters of the alphabet, $k \geqslant 2$ be an key. Then

$$y = xk \pmod{|A|},$$

where y be the value of one of the letters of the alphabet be an encrypted letter.

Now, assume we have encrypted letter y. Then we get a decrypted letter x by a formula

$$x = (y + (k - d) \cdot |A|) \cdot k^{-1},$$

where d be the remainder of dividing y by k.

Proof.

$$x = \frac{y + (k - y \pmod{k})|A|}{k} =$$

$$= \frac{xk \pmod{|A|} + ((k - (xk \pmod{|A|})) \pmod{k})|A|}{k} = x$$

As proposed in [12] (Introduction of section 3), this cipher can be generalized to a complete algebraic structure. It is enough to adopt the infinite alphabet as in [12], x be transformed into the principal ideal (x), k be transformed into the principal ideal (k), y into the principal ideal (y). This way we get algebraic encryption where the key (k) be the fractional ideal in the Dedekind's ring, in this case $\mathbb{Z}$.

VII. The Alphabet as a Fractional Ideal

Let A be a set of characters. Assume $|A|$ is equal to any prime number.

Secretly establish a second alphabet A' such that $A' \subset A$ with a prime length.

Let $m_1 m_2 m_3 \ldots m_n$ be a message, we want to encrypt.

A secret short alphabet A' divides a large public alphabet into zones. We skip the extra characters such that 0, 1. So we have a clean alphabet from 2. Let's move one over, so we have 1. Suppose $p = |A|$, $q = |A'|$. We have

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

$\lceil \frac{p}{q} \rceil$ zones. Zero zone, includes the alphabet from 1 to q. The first zone, i.e. the alphabet from $q+1$ to $2q$ and so on. The last zone $(\lceil \frac{p}{q} \rceil - 1)$ includes the alphabet from $\lceil \frac{p}{q} \rceil q$ to p.

Let's extend the message values with random numbers informing us about a given zone of a given letter (this information denote by z_i):

$$z_1 m_1 z_2 m_2 \ldots z_n m_n$$

Denote by k the key. Multiply each value of the message (not the information about the zone) by k and use the modulo q.

Hence ciphertext is:

$$z_1 d_1 z_2 d_2 \ldots z_n d_n,$$

where $d_1 d_2 \ldots d_n$ be a encrypted message.

Now let's decode the message.

$$z_1 d_1 z_2 d_3 \ldots z_n d_n$$

by dividing it into blocks (each block contains a zone and a message).

Let's apply the formula:

$$m_i = \frac{d_i + (z_i + t_i \cdot k)|A|}{k},$$

where m_i is the decoded letter, d_i encrypted letter, z is a number satisfies a congruence $|A|^{-1} z_i \equiv d_i \pmod{k}$, k be the key, t be a zone.

Of course, this cryptosystem can also be easily generalized by turning individual elements into ideals.

VIII. Applications of Polynomial Composites in Cryptology

Finally, we will show cryptosystems based on polynomial composites and monoid domains.

Lemma 8.1. Let $f = a_0 + a_1 X + \cdots + a_{n-1} X^{n-1} + \sum_{j=n}^{m} a_j X^j$, $g = b_0 + b_1 X + \cdots + b_{n-1} X^{n-1} + \sum_{j=n}^{m} b_j X^j$, *where* $a_i, b_i \in A_i$ *for* $i = 0, 1, \ldots, n-1$ *and* $a_j, b_j \in B$ *for* $j = n, n+1, \ldots, m$. *Then*

$$fg \in A_0 + XB[X].$$

Put A_i, B_j $(i, j = 0, 1, \ldots, n-1)$ be different encryption systems. Then we have f and g are composition of encryption systems. No consider B.

Notes

To improve security, let's fix that $\deg f = n-1$, $\deg g = n-k$, where $k \in \{2, \ldots, n-1\}$. And such f, g Alice and Bob agree before the message is sent.

Alice and Bob multiply these composites to form one. We have $fg = (A_0 + A_1X + \ldots A_kX^k)(B_0 + B_1X + \cdots + B_lX^l) = A_0B_0 + (A_0B_1 + A_1B_0)X + \cdots + A_kB_lX^{k+l}$.

Notes

Note that the sum and product of the encryption systems must be defined in the formula above. Definitions we leave Alice and Bob. But in this section we can put $S_iS_j : x \to (x)_{S_i}(x)_{S_j}$ and $S_i + S_j : x \to ((x)_{S_i})_{S_j}$. We can define the product and the sum of cryptosystems completely differently.

So in the product we encrypt the letter as two letters, the first in the first system and the second in the second system. And in the sum we encrypt the letter using the first system and then the second system. Of course, we can define completely different, at our discretion.

Assume that degree of fg is m and text to encrypt consists of more letters then $m+1$. Then we divide the text into blocks of length $m+1$. We can assume that $fg(0)$ encrypts the first letter of each block. Expression at X of fg encrypts the second letter of each block, and expression at X^2 of fg encrypts the third letter and so on.

Now, let's see how to decrypt in this idea.

Assume that we have an encrypted message $M_0M_1 \ldots M_n$. If our key is degree m, then we divide message on $m+1$ partition. And every partion divide to two. Every two letters are one letter of message.

Earlier we define $S_iS_j : x \to (x)_{S_i}(x)_{S_j}$ and $S_i + S_j : x \to ((x)_{S_i})_{S_j}$. Then decryption of two letters M_lM_{l+1} $(l = 0, 2, 4, \ldots)$ are $M_lM_{l+1} = (M_l)_{S_i}(M_{l+1})_{S_j} = N_{l,l+1}$ (one letter) and $M_l = ((M_l)_{S_i})_{S_j} = (N_l)_{ij}$ (one letter).

The use of many cryptosystems in various configurations in a polynomial composite increases our security. The security here lies in the fact that the encrypted message is resistant to breaking under many cryptanalyst criteria.

It is very easy to decrypt the message when you know the key.

IX. Applications of Monoid Domains in Cryptology

Any alphabet of characters creates a finite set. Most ciphers are based on finite sets. But we can have the idea of using the infinite alphabet $\mathbb{A}$, although in reality they can be cyclical sets with an index that would mean a given cycle. For example, A_0 - 0, B_0 - 1, ... , Z_0 - 25, A_1 - 0, B_1 - 1, ... , where A_i=A, ..., Z_i=Z for $i = 0, 1, \ldots$. We see that this is isomorphic to a monoid $\mathbb{N}_0$ non-negative integers by a formula

$$f\colon \mathbb{A} \to \mathbb{N}, f(m_i) = i.$$

Then we can use a monoid domain by a map

$$\varphi\colon \mathbb{A} \to F[\mathbb{A}], \varphi(m_0, m_1, \ldots, m_n) = a_0X^{m_0} + \ldots a_nX^{m_n}.$$

Global Journal of Science Frontier Research (F) Volume XXI Issue III Version I Year 2021

We want to encrypt the message $m_0 m_1 m_2 \dots m_n$ (the letters transform to numbers by a function φ). We establish the secret key X. Let F be a field. We determine any coefficients from this field: a_0, a_1, ..., a_n. Then the message $m_0 m_1 m_2 \dots m_n$ be transformed into a polynomial of the form:

$$a_0 X^{m_0} + a_1 X^{m_1} + \cdots + a_n X^{m_n}.$$

We compute for $i = 0, 1, \dots, n$: $d_i = a_i X^{m_i} \pmod{|\mathbb{A}|}$ ($|\mathbb{A}|$ must be prime) and then we have a decrypt message $d_0 d_1 \dots d_n$.

To decrypt it we need to use a formula (for $i = 0, 1, \dots, n$):

$$m_i = \log_X \frac{d_i}{a_i} \pmod{|\mathbb{A}|}.$$

Proof.

$$\log_X \frac{a_i X^{m_i}}{a_i} = m_i \pmod{|\mathbb{A}|}.$$

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16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

- Insertion of a title at the foot of a page with subsequent text on the next page.
- Separating a table, chart, or figure—confine each to a single page.
- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- o Explain the value (significance) of the study.
- o Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- o Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- o Briefly explain the study's tentative purpose and how it meets the declared objectives.

Approach:

Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

Procedures (methods and materials):

This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- o Report the method and not the particulars of each process that engaged the same methodology.
- o Describe the method entirely.
- o To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- o Simplify—detail how procedures were completed, not how they were performed on a particular day.
- o If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- o Resources and methods are not a set of information.
- o Skip all descriptive information and surroundings—save it for the argument.
- o Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- Give details of all of your remarks as much as possible, focusing on mechanisms.
- Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

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Topics	Grades		
	A-B	**C-D**	**E-F**
Abstract	Clear and concise with appropriate content, Correct format. 200 words or below	Unclear summary and no specific data, Incorrect form Above 200 words	No specific data with ambiguous information Above 250 words
Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

Index

A

Alleviate · 81
Allosteric · 29, 30
AnIdempotent · 113
Aridity · 81

B

Bountiful · 81

C

Ciphers · 102, 104, 118
Cryptanalyst · 118

I

Inclusion · 109
Inferring · 1, 2

K

Kurtosis · 65, 70

L

Lattice · 5, 8, 24
Linguistic · 79, 82, 83, 84

O

Obstruct · 79

P

Probabilistic · 65

S

Slant · 4, 6, 8, 9, 11, 12, 14
Spiraling · 11
Stepping · 29

V

Verbatim · 44

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