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- Linear Multistep Hybrid Block
- Ordinary Differential Equations
- Ratio Estimator for Population
- Maximum Distance Separable Codes

Discovering Thoughts, Inventing Future

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Maximum Distance Separable Codes to Order

By Ted Hurley, Donny Hurley & Barry Hurley
Institute of Technology

Abstract- Maximum distance separable (MDS) are constructed to required specifications. The codes are explicitly given over finite fields with efficient encoding and decoding algorithms. Series of such codes over finite fields with ratio of distance to length approaching $(1 - R)$ for given R, $0 < R < 1$ are derived. For given rate $R =$ r n, with p not dividing n, series of codes over finite fields of characteristic p are constructed such that the ratio of the distance to the length approaches $(1 - R)$. For a given field *GF(q)* MDS codes of the form $(q-1,\ r)$ are constructed for any r. The codes are encompassing, easy to construct with efficient encoding and decoding algorithms of complexity $\max\{O(n \log n), t^2\}$, where t is the error-correcting capability of the code.

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6. Ted Hurley and Donny Hurley, "Coding theory: the unit-derived methodology", Int. J. Information and Coding Theory, Vol. 5, no.1, 55-80, 2018.

Maximum Distance Separable Codes to Order

Ted Hurley [α], Donny Hurley [σ] & Barry Hurley [ρ]

Abstract- Maximum distance separable (MDS) are constructed to required specifications. The codes are explicitly given over finite fields with efficient encoding and decoding algorithms. Series of such codes over finite fields with ratio of distance to length approaching $(1-R)$ for given R, $0 < R < 1$ are derived. For given rate $R = \frac{r}{n}$, with p not dividing n, series of codes over finite fields of characteristic p are constructed such that the ratio of the distance to the length approaches $(1-R)$. For a given field *GF(q)* MDS codes of the form $(q-1, r)$ are constructed for any r. The codes are encompassing, easy to construct with efficient encoding and decoding algorithms of complexity max $\{O(n \log n), t^2\}$, where t is the error-correcting capability of the code.

I. Introduction

Coding theory is at the heart of modern day communications. Maximum distance separable, MDS, codes are at the heart of coding theory. Data needs to be transmitted *safely* and sometimes securely. Best rate and error-correcting capabilities are the aim, and MDS codes can meet the requirements; they correct the maximum number of errors for given length and dimension.

General methods for constructing MDS codes over finite fields are given in Section 2 following [6, 7, 15]. The codes are explicitly constructed over finite fields with efficient encoding and decoding algorithms of complexity $\max\{O(n \log n), O(t^2)\}$, where t is the error-correcting capability. These are exploited. For given $\{n, r\}$ MDS (n, r) codes are constructed over finite fields with characteristics not dividing n, section 3.1. For given rate and given error-correcting capability series of MDS codes to these specifications are constructed over finite fields, section 3.2. For given rate R, $0 < R < 1$, series of MDS codes are constructed over finite fields in which the ratio of the distance by the length approaches $(1-R)$, section 3.3.

For a given finite field $GF(q)$, MDS $(q-1, r)$ codes of different types are constructed over $GF(q)$ for any given r, $1 \leq r \leq (q-1)$, section 3.7. The codes are explicit with efficient encoding and decoding algorithms as noted. In addition for each $n/(q-1)$, MDS codes of length n and dimension r are constructed over $GF(q)$ for any given r, $1 \leq r \leq n$. In particular for p a prime, MDS $(p-1, r)$ codes are constructed in $GF(p) = \mathbb{Z}_p$ in which case the arithmetic is modular arithmetic which works smoothly and very efficiently.

For given $R = \frac{r}{n}$, $0 < R < 1$, with $p \not| n$, series of codes over finite fields of characteristic p are constructed in which the ratio of the distance to the length approaches $(1-R)$, section 3.4. Note $0 < R < 1$ if and only if $0 < (1-R) < 1$. In particular such series are constructed in fields of characteristic 2 for cases where the denominator n of the given rate is odd.

Author α: National University of Ireland Galway. e-mail: Ted.Hurley@NuiGalway.ie
Author σ: Institute of Technology, Sligo. e-mail: hurley.donny@itsligo.ie
Author ρ: e-mail: barryj 2000@yahoo.co.uk

Series of MDS codes over prime fields $GF(p) = \mathbb{Z}_p$ are constructed such that the ratio of the distance to the length approaches $(1 - R)$ for given $R, 0 < R < 1$; in these cases the arithmetic is modular arithmetic which is extremely efficient and easy to implement, section 3.5.

Samples are given in the different sections and an example is given on the workings of the decoding algorithms in section 3.6.1. The explicit examples given need to be of reasonably small size for display here but in general there is no restriction on the length or dimension in practice.

Explicit efficient encoding and decoding algorithms of complexity $\max\{O(n \log n), O(t^2)\}$ exist for the codes and this is explained in section 2.3.

The codes are encompassing and excel known used and practical codes. See for example section 3.8 for the following: MDS codes of the form $(255, r)$ for any $r, 1 \leq r \leq 255$ are constructed over $GF(2^8)$. They are constructed explicitly and have efficient encoding and decoding algorithms which reduce to finding a solution of a Hankel $t \times (t + 1)$ system, where t is the error-correcting capability, and matrix multiplications by a Fourier matrix. These can be compared to the Reed-Solomon codes over $GF(2^8)$. The method extends easily to the formation of MDS codes of the form $(511, r)$ for any $r, 1 \leq r \leq 511$ over $GF(2^9)$, and then further to MDS codes $(2^k - 1, r)$ over $GF(2^k)$. Codes over prime fields are particularly nice and as an example $(256, r)$ codes are constructed over $GF(257) = \mathbb{Z}_{257}$. The arithmetic is modular arithmetic over $\mathbb{Z}_{257}$; these perform better than the $(255, r)$ RS codes over $GF(2^8)$. These can also easily be extended for larger primes as for example $(10008, r)$ MDS codes over $GF(10009)$.

In general: For any prime p, $(p-1, r)$ codes over $GF(p) = \mathbb{Z}_p$ are constructed for any $r, 1 \leq r \leq (p-1)$; for any k, $(2^k - 1, r)$ codes are constructed over $GF(2^k)$ and any $r, 1 \leq r \leq (2^k - 1)$. As already noted the constructed codes have (very) efficient encoding and decoding algorithms.

The encoding and decoding methods involve multiplications by a Fourier matrix and finding a solution to a Hankel $t \times (t + 1)$ system, where t is the error-correcting capability of the MDS code.

Background on coding theory and field theory may be found in [1], [17] or [18]. An (n, r) linear code is a linear code of length n and dimension r; the *rate* of the code is $\frac{r}{n}$. An (n, r, d) linear code is a code of length n, dimension r and (minimum) distance d. The code is an MDS code provided $d = (n - r + 1)$, which is the maximum distance an (n, r) code can attain. The error-capability of (n, r, d) is $t = \lfloor \frac{d-1}{2} \rfloor$ which is the maximum number of errors the code can correct successfully. The finite field of order q is denoted by $GF(q)$ and of necessity q is a power of a prime.

The codes are generated by the unit-derived method – see [9, 11, 16] – by choosing rows in sequence of Fourier/Vandermonde matrices over finite fields following the methods developed in [6, 7]. They are easy to implement, explicit and with efficient encoding and decoding algorithms of complexity $\max\{(O \log n), O(t^2)$ where t is the error-correcting capability.

a) Particular types of MDS codes

Different *types* of MDS codes, such as Quantum or Linearly complementary dual (LCD) codes, can be constructed based on general schemes; see section 3.9.1 for references on these developments. This section also notes a reference to using these types of error-correcting codes in solving underdetermined systems of equations for *compressed sensing* applications.

II. Constructions

a) Background material

In [9, 16] systems of *unit-derived codes* are developed; a suitable version in book chapter form is available at [11]. In summary the unit-derived codes are obtained as follows. Let $UV = I_n$ in a ring. Let G be the $r \times n$ matrix generated by choosing any r rows of U and let H^{T} be the $n \times (n - r)$ matrix obtained from V by eliminating the corresponding columns of V. Then G generates an (n, r) code and H is the the check matrix of the code. The system can be considered in format as $GH^{\mathrm{T}} = 0_{r \times (n-r)}$.

When the first rows are chosen as generator matrix, the process may be presented as follows. Let $UV = I_n$ with $U = \begin{pmatrix} A \\ B \end{pmatrix}$, $V = (C, D)$ where A is an $r \times n$ matrix, B is an $(n - r) \times n$ matrix, C is an $n \times r$ matrix and D is an $n \times (n - r)$ matrix. Then $UV = I$ gives $\begin{pmatrix} A \\ B \end{pmatrix} (C, D) = \begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix}$ and so in

Ref

1. Richard E. Blahut, *Algebraic Codes for data transmission*, Cambridge University Press, 2003.

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particular this gives $AD = 0_{r\times(n-r)}$. The matrices have full rank. Thus with A as the generating matrix of an (n,r) code it is seen that D^{T} is the check matrix of the code.

By explicit row selection, the process is as follows. Denote the rows of U in order by $\langle e_0, e_1, \ldots, e_{n-1}\rangle$ and the columns of V in order by $\langle f_0, f_1, \ldots, f_{n-1}\rangle$. Then $\begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1}\end{pmatrix}(f_0, f_1, \ldots, f_{n-1}) = I_n$. From this it is seen that $e_i f_i = 1, e_i f_j = 0, i \neq j$.

Thus if $G = \begin{pmatrix} e_{i_1} \\ e_{i_2} \\ \vdots \\ e_{i_r}\end{pmatrix}$ (for distinct e_{i_k}) and $H^{\mathrm{T}} = (f_{j_1}, f_{j_2}, \ldots, f_{j_{n-r}})$ where $\{j_1, j_2, \ldots, j_{n-r}\} = \{0, 1, \ldots, n-1\}/\{i_1, i_2, \ldots, i_r\}$. Then $GH^{\mathrm{T}} = 0_{r\times(n-r)}$.

Both G and H have full rank.

When the first r rows chosen this gives $\begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_r\end{pmatrix}(f_0, f_{n-1}, f_{n-2}, \ldots, f_{n-r}) = 0_{r\times(n-r)}$ for the code system expressing the generator and check matrices.

b) Vandermonde/Fourier matrices

When the rows are chosen from Vandermonde/Fourier matrices and taken in arithmetic sequence with arithmetic difference k satisfying $\gcd(n,k) = 1$ then MDS codes are obtained. In particular when $k = 1$, that is when the rows are taken consecutively, MDS codes are obtained. This follows from results in [6] and these are explicitly recalled in Theorems 2.1, 2.2 below.

The $n \times n$ Vandermonde matrix $V(x_1, x_2, \ldots, x_n)$ is defined by

$$V = V(x_1, x_2, \ldots, x_n) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{pmatrix}$$

As is well known, the determinant of V is $\prod_{i<j}(x_i - x_j)$. Thus $\det(V) \neq 0$ if and only the x_i are distinct.

A primitive n^{th} root of unity ω in a field $\mathbb{F}$ is an element ω satisfying $\omega^n = 1_{\mathbb{F}}$ but $\omega^i \neq 1_{\mathbb{F}}, 1 \leq i < n$. Often $1_{\mathbb{F}}$ is written simply as 1 when the field is clearly understood.

The field $GF(q)$ (where q is necessarily a power of a prime) contains a primitive $(q-1)$ root of unity, see [1, 18] or any book on field theory, and such a root is referred to as a *primitive element in the field* $GF(q)$. Thus also the field $GF(q)$ contains a primitive n^{th} roots of unity for any $n/(q-1)$.

A Fourier $n \times n$ matrix over $\mathbb{F}$ is a special type of Vandermonde matrix in which $x_i = \omega^{i-1}$ and ω is a primitive n^{th} root of unity in $\mathbb{F}$. Thus:

$$F_n = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

is a Fourier matrix over $\mathbb{F}$ where ω is a primitive n^{th} root of unity in $\mathbb{F}$.

Ref

6. Ted Hurley and Donny Hurley, "Coding theory: the unit-derived methodology", Int. J. Information and Coding Theory, Vol. 5, no.1, 55-80, 2018.

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Then

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \\ 1 & \omega^{n-2} & \omega^{2(n-2)} & \dots & \omega^{(n-1)(n-2)} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & \omega & \omega^2 & \dots & \omega^{(n-1)} \end{pmatrix} = nI_n$$

Hence $F_nF_n^* = nI_n$ where F_n^* denotes the second matrix on the left of the equation. Replacing ω by ω^{n-1} in F_n is seen to give this F_n^* which itself is a Fourier matrix. Refer to section 3.9 for results on which fields contain an n^{th} root of unity but in any case an n^{th} root of unity can only exist in a field whose characteristic does not divide n.

The following theorem on deriving MDS codes from Fourier matrices by unit-derived scheme is contained in [6]:

Theorem 2.1 *[6]*

(i) Let F_n be a Fourier $n\times n$ matrix over a field $\mathbb{F}$. Let $\mathcal{C}$ be the unit-derived code obtained by choosing in order r rows of V in arithmetic sequence with arithmetic difference k and $\gcd(n,k) = 1$. Then $\mathcal{C}$ is an MDS $(n,r,n-r+1)$ code. In particular this is true when $k = 1$, that is when the r rows are chosen in succession.

(ii) Let $\mathcal{C}$ be as in part (i). Then there exist efficient encoding and decoding algorithms for $\mathcal{C}$.

There is a similar, more general in some ways, theorem for Vandermonde matrices:

Theorem 2.2 *[6] Let $V = V(x_1, x_2, \dots, x_n)$ be a Vandermonde $n\times n$ matrix over a field $\mathbb{F}$ with distinct and non-zero x_i. Let $\mathcal{C}$ be the unit-derived code obtained by choosing in order r rows of V in arithmetic sequence with difference k. If $(x_ix_j^{-1})$ is not a k^{th} root of unity for $i \neq j$ then $\mathcal{C}$ is an $(n,r,n-r+1)$ mds code over $\mathbb{F}$.*

In particular the result holds for consecutive rows as then $k = 1$ and $x_i \neq x_j$ for $i \neq j$.

These are fundamental results.

For 'rows in sequence' in the Fourier matrix cases, Theorem 2.1, and for some Vandermonde cases, it is permitted that rows may *wrap around* and then e_k is taken to mean $e_{(k \mod n)}$. Thus for example Theorem 2.1 could be applied to a code generated by $\langle e_r, \dots, e_{n-1}, e_0, e_1, \dots, e_s\rangle$ where $\langle e_0, e_1, \dots, e_{n-1}\rangle$ are the rows in order of a Fourier matrix.

The general Vandermonde case is more difficult to deal with in practice but in any case using Fourier matrices is sufficient for coding purposes.

Decoding methods for the codes produced are given in the algorithms in [6] and in particular these are particularly nice for the codes from Fourier matrices. The decoding methods are based on the decoding schemes derived in [15] in connection with *compressed sensing* for solving underdetermined systems using error-correcting codes. These decoding methods themselves are based on the error-correcting methods due to Pellikaan [13] which is a method of finding error-correcting pairs – error-correcting pairs are shown to exist for the constructed codes and efficient decoding algorithms are derived from this. These decoding algorithms are explicitly written down in detail in [6]. In addition the encoding itself is straightforward.

The complexity of encoding and decoding is $\max\{O(n\log n), O(t^2)\}$ where $t = \lfloor \frac{n-r}{2} \rfloor$; t is the error-correcting capability of the code. The complexity is given in Section 2.3 and is derived in [6].

Let F_n^* denote the matrix with $F_nF_n^* = nI_{n\times n}$ for the Fourier matrix F_n. Denote the rows of F_n in order by $\{e_0, e_1, \dots, e_{n-1}\}$ and denote the columns of F_n^* in order by $\{f_0, f_1, \dots, f_{n-1}\}$. *Then it is important to note that $f_i = {e_{n-i}}^{\mathrm{T}}, e_i = {f_{n-i}}^{\mathrm{T}}$* with the convention that suffices are taken modulo n. Also note $e_if_i = n$ and $e_if_j = 0, i \neq j$.

Ref

15. T. Hurley, "Solving underdetermined systems with error correcting codes", Intl. J. Information and Coding Theory, Vol 4, no. 4, 201-221, 2017.

Thus

$$\begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix} (f_0, f_1, f_2, \ldots, f_{n-1}) = \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix} ({e_0}^{\mathrm{T}}, {e_{n-1}}^{\mathrm{T}}, {e_{n-2}}^{\mathrm{T}}, \ldots, {e_1}^{\mathrm{T}}) = nI_n$$

c) Complexity

Efficient encoding and decoding algorithms exist for these codes by the methods/algorithms developed in [6] which follow from those developed in [15]. In general the complexity is $\max\{O(n \log n), O(t^2)\}$ where n is the length and t is the error-correcting capability, that is, $t = \lfloor \frac{d-1}{2} \rfloor$ where d is the distance. See the algorithms in [6] for details; there the decoding algorithms are derived and are written down precisely in suitable format. The decoding algorithms reduce to finding a solution to a Hankel $t \times (t+1)$ systems, which can be done in $O(t^2)$ time at worst, and the other encoding and decoding algorithms are matrix multiplications which can be reduced to multiplication by a Fourier matrix which takes $O(n \log n)$ time.

III. Maximum Distance Separable Codes

a) Given n, r

Suppose it is required to construct MDS (n, r) codes for given n and r. First construct a $n \times n$ Fourier matrix over a finite field. A Fourier $n \times n$ matrix is constructible over a finite field of characteristic p where $p \nmid n$, see section 3.9. Take r rows in sequence with arithmetic difference k satisfying $\gcd(n, k) = 1$ from this Fourier matrix. Then by Theorem 2.1, see [6] for details, the code generated by these rows is an (n, r) MDS code. There are many different ways for constructing the (n, r) code from the Fourier $n \times n$ matrix – one could start at any row with $k = 1$ and could also start at any row for any k satisfying $(n, k) = 1$. A check matrix may be read off immediately from section 2 and a direct decoding algorithm of complexity $\max\{O(n \log n), O(t^2)\}$ is given in [6], where t is the error-correcting capability.

b) MDS to required rate and error-correcting capability

Suppose it is required to construct an MDS code of rate R and to required error-correcting capability. The required code is of the form (n, r) with $(n - r + 1) \geq (2t + 1)$ where t is the required error-correcting capability. Now $(n - r + 1) \geq (2t + 1)$ requires $n(1 - R) \geq 2t$. *Thus require* $n \geq \frac{2t}{1-R}$. With these requirements construct the Fourier $n \times n$ and from this take $r \geq nR$ rows in arithmetic sequence with arithmetic difference k satisfying $\gcd(n, k) = 1$. The code constructed has the required parameters. The finite fields over which this Fourier matrix can be constructed is deduced from section 3.9.

Samples It is required to construct a rate $R = \frac{7}{8}$ code which can correct 25 errors. Thus, from general form $n \geq \frac{2t}{1-R}$, require $n \geq \frac{50}{\frac{1}{8}}$ and so $n \geq 400$.

Consider $n = 400$. Construct a Fourier 400×400 matrix F_{400} over a suitable finite field. Then $r = 350$ for rate $\frac{7}{8}$. Now take any 350 rows in sequence from F_{400} with arithmetic difference k satisfying $\gcd(400, k) = 1$. Now $k = 1$ starting at first row works in any case but there are many more which are suitable. The code generated by these rows is an $(400, 350, 51)$ code, Theorem 2.1, which can correct 25 errors.

Over which fields can the Fourier 400×400 matrix exist? The characteristic of the field must not divide 400 but finite fields of any other characteristic exist over which the Fourier 400×400 matrix is constructible. For example: the order of $3 \mod 400$ is 20 so $GF(3^{20})$ is suitable; the order of $7 \mod 400$ is 4 so $GF(7^4)$ is suitable. Exercise: Which other fields are suitable?

However 401 is prime and the order of $401 \mod 400$ is 1 and thus the prime field $GF(401)$ is suitable. It is also easy to find a primitive 400 root of unity in $GF(401)$; indeed the order of $\omega = (3 \mod 401)$ is 400 in $GF(401)$ and this element may be used to generate the 400×400 Fourier matrix over $GF(401)$. The arithmetic is modular arithmetic in $\mathbb{Z}_{401} = GF(401)$. [1]

A field of characteristic 2 close to the requirements may be prescribed. Then let $n = 399$ and note that the order of $2 \mod 399$ is 18. Thus use the field $GF(2^{18})$ over which the Fourier 399×399 matrix

Ref

6. Ted Hurley and Donny Hurley, "Coding theory: the unit-derived methodology", Int. J. Information and Coding Theory, Vol. 5, no.1, 55-80, 2018.

may be constructed. Take 348 rows of this Fourier matrix in sequence with arithmetic difference k satisfying $(399, k) = 1$ to form a $(399, 348, 51)$ code which can correct 25 errors. Rate is 0.8746.. which is close to required rate $\frac{7}{8}$.

Exercise: How many (different) (n, r) MDS codes may be formed from this Fourier $n \times n$ matrix over the finite field? Note the sequence may 'wrap over' and then the numbering is mod n.

c) *Infinite series with given rate*

Construct an infinite series of codes with given rate R such that the limit of the distance by the length approaches $(1 - R)$.

Let $R = \frac{r}{n}$ be given. Construct the Fourier $n \times n$ matrix and from this derive the (n, r) MDS code as in section 2. Let $n_i = i * n, r_i = i * r$ for an increasing set of positive integers $\{i\}$. Construct the Fourier $n_i \times n_i$ matrix and from this derive an (n_i, r_i) MDS code. The rate of the code is $\frac{r_i}{n_i} = \frac{r}{n} = R$. The distance of the code is $d_i = (n_i - r_i + 1)$. The ratio of the length by the distance is $\frac{n_i - r_i + 1}{n_i} = 1 - R + \frac{1}{n_i}$. Now as $i \to \infty$ it is seen that the ratio of the distance by the length approaches $(1 - R)$.

Note that $0 < R < 1$ if and only if $0 < (1 - R) < 1$ so could start off with a requirement that the limit approaches a certain fraction.

There are many choices by this method giving different series. At each stage there are many different $n_i \times n_i$ Fourier matrices to choose from and within each of these are many choices of r_i rows for obtaining (n_i, r_i) MDS codes.

By methods/algorithms of [6] the codes have efficient encoding and decoding algorithms of complexity $\max\{O(n \log n), t^2\}$ where t is the error-correcting capability.

d) *Series in characteristic p with given rate*

Suppose codes over fields of characteristic 2 are required. Now a Fourier matrix of even size in characteristic 2 cannot exist. It is necessary to consider rates of the form $\frac{r}{n}$ where n is odd in order for the general method of section 3.3 to work in characteristic 2. The method of section 3.3 is then applied by taking the increasing sequence $\{i\}$ to consist of odd elements only. Then construct the Fourier $(n * i) \times (n * i)$ matrix for odd i (and odd n) in a field of characteristic 2 – see section 3.9 on method to form such a Fourier $n * i \times n * i$ matrix in a finite field of characteristic 2. From this Fourier matrix construct an MDS $(n * i, r * i)$ code with rate $\frac{r}{n}$ by method of Theorem 2.1; there are choices for this code as noted.

As a sample consider the rate $\frac{7}{9}$. Then Fourier $(9 * i) \times (9 * i)$ matrices are constructible over fields of characteristic 2 for odd i. From this $(9 * i, 7 * i, 2 * i + 1)$ codes are constructed.

Thus $(9, 7, 3)$ code over $GF(2^6)$, $(27, 21, 7)$ code over $GF(2^{18})$, $(45, 35, 11)$ code over $GF(2^{12})$, $(63, 49, 15)$ over $GF(2^6)$, and so on, are constructed. The fields of characteristic 2 used depend on the order of 2 modulo the required length. The ratio of the distance by the length approaches $(1 - R) = \frac{2}{9}$.

Similarly infinite series of codes over fields of characteristic p are constructed with given rate $\frac{r}{n}$ where $p \nmid n$.

Sample For example consider rate $R = \frac{7}{10}$ for characteristic 3. Then the method constructs MDS $\{(10, 7, 4), (20, 14, 7), (40, 28, 13), (50, 35, 16), (70, 49, 22), (80, 56, 25), \ldots\}$ codes in fields of characteristic 3.

Now OrderMod$(3, 10) = 4$, OrderMod$(3, 20) = 4$, OrderMod$(3, 40) = 4$, OrderMod$(3, 50) = 20$, OrderMod$(3, 70) = 12$, OrderMod$(3, 80) = 4$, …so these codes can be constructed respectively over $\{GF(3^4), GF(3^4), GF(3^4), GF(3^{20}), GF(3^{12}), GF(3^4), \ldots\}$. It is seen that $(80, 56, 25)$ is over a relatively small field $GF(81)$ and can correct 12 errors. The limit of the distance over the length is $(1 - R) = \frac{3}{10}$.

[1] The Computer Algebra system GAP [5] has the command Order Mod(r, m) which is useful. This system also has the coding package GUAVA with which experiments can be made.

Ref

6. Ted Hurley and Donny Hurley, "Coding theory: the unit-derived methodology", Int. J. Information and Coding Theory, Vol. 5, no.1, 55-80, 2018.

i. *Note*

In characteristic p the rates $\frac{r}{n}$ attainable require $p \not| n$ so that the Fourier matrix $n \times n$ can be constructed in characteristic p. This is not a great restriction. For any given fraction R and any given $\epsilon > 0$ there exists a fraction with numerator not divisible by p between R and $R + \epsilon$. The details are omitted. For example suppose in characteristic 2 the rate required is $\frac{3}{4}$ and $\epsilon > 0$ is given. Say $\frac{1}{32} < \epsilon$ and then need a fraction of the required type between $\frac{3}{4}$ and $\frac{3}{4} + \frac{1}{32} = \frac{25}{32}$. Now $\frac{24}{31}$ will do and we can proceed with this fraction to construct the codes over characteristic 2; the Fourier 31×31 matrix exists over $GF(2^5)$.

Notes

e) *Infinite series in prime fields with given rate*

Arithmetic in prime fields is particularly nice. Here we develop a method for constructing series of MDS codes over prime fields.

Suppose a rate R is required, $0 < R < 1$. Let p be a prime and consider the field $GF(p) = \mathbb{Z}_p$. This has an element of order $(p-1)$ and thus construct the Fourier $(p-1) \times (p-1)$ matrix F_{p-1} over $GF(p) = \mathbb{Z}_p$. For this it is required to find a primitive $(p-1)$ root of unity in $GF(p) = \mathbb{Z}_p$. [2] Let $r = \lfloor (p-1) * R \rfloor$. Now p must be large enough so that $r \geq 1$. Form the $(p-1, r)$ MDS code over F_{p-1}. This has rate close to R.

Let $\{p_1, p_2, \ldots, p_i, \ldots\}$ be an infinite increasing set of primes such that $(p_1 - 1) * R \geq 1$ in which case $(p_i - 1) * R \geq 1$ for each i. Form the Fourier $(p_i - 1) \times (p_i - 1)$ matrix over $GF(p_i)$. Let $r_i = \lfloor (p_i - 1) * R \rfloor$. Form the $(p_i - 1, r_i)$ MDS code over $GF(p_i)$. The ratio of the distance to the length is $\frac{p_i - 1 - r_i + 1}{p_i - 1} = 1 - \frac{r_i}{p_i - 1} + \frac{1}{p_i - 1}$. Now as $i \to \infty$ this ratio approaches $(1 - R)$.

Sample Let $\{p_1, p_2, \ldots\}$ be the primes of the form $(4n+1)$ and let $R = \frac{3}{4}$. Now $p_1 = 5, p_2 = 13, p_3 = 17, \ldots$. Let $r_i = (p_i - 1) * R = 4 * j * \frac{3}{4} = j * 3$ for some j. Form the $(p_i - 1, r_i)$ code from Fourier $(p_i - 1) \times (p_i - 1)$ matrix over $GF(p_i)$.

Get codes $\{(4,3,2), (12,9,4), (16,12,5), (28,21,8), (36,27,10), \ldots\}$ over, respectively, the following fields

$$\{GF(5), GF(13), GF(17), GF(29), GF(37), \ldots\}.$$

f) *Sample of the workings*

Here is an example of MDS codes in $GF(13) = \mathbb{Z}_{13}$. A primitive element in $GF(13)$ is $\omega = (2 \mod 13)$. The Fourier 12×12 matrix with this ω as the element of order 12 in $GF(13) = \mathbb{Z}_{13}$ is:

$$F_{12} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 \\ 1 & 4 & 3 & 12 & 9 & 10 & 1 & 4 & 3 & 12 & 9 & 10 \\ 1 & 8 & 12 & 5 & 1 & 8 & 12 & 5 & 1 & 8 & 12 & 5 \\ 1 & 3 & 9 & 1 & 3 & 9 & 1 & 3 & 9 & 1 & 3 & 9 \\ 1 & 6 & 10 & 8 & 9 & 2 & 12 & 7 & 3 & 5 & 4 & 11 \\ 1 & 12 & 1 & 12 & 1 & 12 & 1 & 12 & 1 & 12 & 1 & 12 \\ 1 & 11 & 4 & 5 & 3 & 7 & 12 & 2 & 9 & 8 & 10 & 6 \\ 1 & 9 & 3 & 1 & 9 & 3 & 1 & 9 & 3 & 1 & 9 & 3 \\ 1 & 5 & 12 & 8 & 1 & 5 & 12 & 8 & 1 & 5 & 12 & 8 \\ 1 & 10 & 9 & 12 & 3 & 4 & 1 & 10 & 9 & 12 & 3 & 4 \\ 1 & 7 & 10 & 5 & 9 & 11 & 12 & 6 & 3 & 8 & 4 & 2 \end{pmatrix}$$

Let the rows of F_{12} in order be denoted by $\{e_0, e_1, \ldots, e_{11}\}$.

Various MDS codes over $GF(13)$ may be constructed from F_{12}.

Two of the $(12, 6, 7)$ codes are as follows:

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 \\ 1 & 4 & 3 & 12 & 9 & 10 & 1 & 4 & 3 & 12 & 9 & 10 \\ 1 & 8 & 12 & 5 & 1 & 8 & 12 & 5 & 1 & 8 & 12 & 5 \\ 1 & 3 & 9 & 1 & 3 & 9 & 1 & 3 & 9 & 1 & 3 & 9 \\ 1 & 6 & 10 & 8 & 9 & 2 & 12 & 7 & 3 & 5 & 4 & 11 \end{bmatrix}, L = \begin{bmatrix} 1 & 2 & 4 & 8 & 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 \\ 1 & 12 & 1 & 12 & 1 & 12 & 1 & 12 & 1 & 12 & 1 & 12 \\ 1 & 7 & 10 & 5 & 9 & 11 & 12 & 6 & 3 & 8 & 4 & 2 \\ 1 & 3 & 9 & 1 & 3 & 9 & 1 & 3 & 9 & 1 & 3 & 9 \\ 1 & 5 & 12 & 8 & 1 & 5 & 12 & 8 & 1 & 5 & 12 & 8 \\ 1 & 4 & 3 & 12 & 9 & 10 & 1 & 4 & 3 & 12 & 9 & 10 \end{bmatrix}$$

[2] It seems there is no known algorithm for finding a generator of $(\mathbb{Z}_p/\{0\})$ that is substantially better than a brute force method - see Keith Conrad's notes [4]. Note however there are precisely $\phi(p-)$ generators.

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The first matrix takes the first 6 rows of F_{12}; the second matrix takes rows $\{e_1, e_6, e_{11}, e_4, e_9, e_2\}$ which are 6 rows in sequence with arithmetic difference 5, $\gcd(12,5) = 1$, starting with the second row. These are generator matrices for $(12,6,7)$ codes over $GF(13) = \mathbb{Z}_{13}$ and each can correct 3 errors.

i. *Correcting errors sample*

Efficient decoding algorithms for the codes are established in [6]. Here is an example to show how the algorithms work in practice. The matrix K as above, formed from the first 6 rows of Fourier matrix F_{12}, is the generator matrix of a $(12,6,7)$ code. Apply Algorithm 6.1 from [6] to correct up to 3 errors of the code as follows. Note the work is done in $\mathbb{Z}_{13} = GF(13)$ using modular arithmetic.

1. The word $\underline{w} = (8,9,2,6,3,3,10,8,4,1,5,7)$ is received.
2. Apply check matrix to $\underline{w}$ and get $\underline{e} = (2,9,12,10,11,11)$. Thus there are errors and $\underline{w}$ is not a codeword. (The check matrix $({e_1}^{\mathrm{T}}, {e_2}^{\mathrm{T}}, {e_3}^{\mathrm{T}}, {e_4}^{\mathrm{T}}, {e_5}^{\mathrm{T}}, {e_6}^{\mathrm{T}})$ is immediate, see section 2.)
3. Find a non-zero element of the kernel of $\begin{pmatrix} 2 & 9 & 12 & 10 \\ 9 & 12 & 10 & 11 \\ 12 & 10 & 11 & 11 \end{pmatrix}$. This is a 3×4 Hankel matrix, formed from $\underline{e}$; the first row consists of elements $(1-4)$ of $\underline{e}$, the second row consists of elements $(2-5)$ of $\underline{e}$, and the third row consists of elements $(3-6)$ of $\underline{e}$. A non-zero element of the kernel is $\underline{x} = (7,1,7,1)^{\mathrm{T}}$.
4. Now $\underline{a} = (e_1, e_2, e_3, e_4) * \underline{x} = (3,12,7,0,1,0,1,2,4,0,10,12)$. Thus errors occur at $4^{th}, 6^{th}, 10^{th}$ positions (which are the positions of the zeros of $\underline{a}$).
5. Solve (from $4^{th}, 6^{th}, 10^{th}$ columns of $(2-7)$ rows of F_{12}):

$$\begin{pmatrix} 8 & 6 & 5 \\ 12 & 10 & 12 \\ 5 & 8 & 8 \\ 1 & 9 & 1 \\ 8 & 2 & 5 \\ 12 & 12 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 9 \\ 12 \\ 10 \\ 11 \\ 11 \end{pmatrix}$$

   In fact only the first three equations need be solved; answer is $(10,1,4)^{\mathrm{T}}$. Thus error vector is $\underline{k} = (0,0,0,10,0,1,0,0,0,4,0,0)$.
6. Correct codeword is $\underline{c} = \underline{w} - \underline{k} = (8,9,2,9,3,2,10,8,4,10,5,7)$.
7. If required, the original data word can be obtained directly by multiplying by the right inverse of the generator matrix; the right inverse is read off as $K = (e_0, e_{11}, e_{10}, e_9, e_8, e_7)^{\mathrm{T}} * 12$. Then $\underline{c} * K = (1,2,3,4,5,6)$ which is the original data word to be safely transmitted.

The equations to be solved are Hankel matrices of size the order of $t \times t$ where t is the error-correcting capability.

g) *Length $(2^q - 1)$ MDS codes in $GF(2^q)$*

$2^3 - 1 = 7, 2^4 - 1 = 15, 2^5 - 1 = 31, 2^6 - 1 = 63, \ldots$.

In general consider the characteristic 2 field $GF(2^q)$. In this field acquire an element of order $n = (2^q - 1)$ and construct the Fourier $n \times n$ matrix over $GF(2^q)$. From this, MDS (n,r) codes are constructed for $1 \le r \le n$. It is better to take odd r from consideration of the error-correcting capability.

1. $2^3 - 1 = 7$. From the Fourier 7×7 matrix over $GF(2^3)$ construct the MDS $\{(7,5,3),(7,3,5),(7,1,7)\}$ codes which can correct respectively $\{1,2,3\}$ errors.
2. $2^4 - 1 = 15$. From the Fourier 15×15 matrix over $GF(2^4)$ construct $\{(15,13,3),(15,11,5),(15,9,7),(15,7,9),(15,5,11),(15,3,13),(15,1,15)\}$ MDS codes over $GF(2^4)$ which can correct respectively $\{1,2,3,4,5,6,7\}$ errors.

Ref

6. Ted Hurley and Donny Hurley, "Coding theory: the unit-derived methodology", Int. J. Information and Coding Theory, Vol. 5, no.1, 55-80, 2018.

Notes

3. $2^5 - 1 = 31$. From the Fourier 31×31 matrix over $GF(2^5)$ construct the MDS $\{(31,29,3),(31,27,5),(31,25,7),\ldots,(31,3,29),(31,1,31)\}$ codes which can respectively correct $\{1,2,3,\ldots,14,15\}$ errors.

4. $2^8 - 1 = 255$. Thus MDS codes $(255, r)$ are constructed over $GF(2^8)$ for all r. These could be compared to Reed-Solomon codes used in practice and perform better.

   Even further consider $2^9 - 1 = 511$. Then MDS codes $(511, r)$ are constructed over $GF(2^9)$. For example $(511,495,17),(511,487,25)$ codes are constructed over $GF(2^9)$; the decoding algorithm involves finding a solution to 9×8, 13×12 (respectively) Hankel systems of equations, and matrix Fourier multiplication.

   The codes over prime fields in section 3.8 of length 256 over $GF(257) = \mathbb{Z}_{257}$ and of length 508 over $GF(509) = \mathbb{Z}_{509}$ perform better.

5. ......

6. General $2^q - 1 = n$. From the Fourier $n \times n$ matrix over $GF(2^q)$ construct the MDS $(n, n-2, 3), (n, n-4, 5), (n, n-6, 7), \ldots, (n, n-2m, 2m+1), \ldots, (n, 3, n-2), (n, 1, n)$ codes which can correct respectively $\{1, 2, 3, \ldots, m, \ldots, \frac{n-3}{2}, \frac{n-1}{2}\}$ errors.

It is clear that similar series of relatively large length MDS codes may be constructed over finite fields of characteristics other than 2.

h) Length (p − 1) codes in prime field GF(p) $= \mathbb{Z}_p$

Construct large length MDS codes over prime fields. This is a particular general case of section 3.7 but is singled out as the arithmetic involved, modular arithmetic, is smooth and very efficient and the examples are nice and practical. For any prime p the Fourier $(p-1) \times (p-1)$ matrix exists over $GF(p) = \mathbb{Z}_p$. A primitive $(p-1)$ root of unity is required in $GF(p)$ [3]. The arithmetic is modular arithmetic in $\mathbb{Z}_p$ which is nice. The general method then allows the construction of MDS $(p-1, r)$ codes over $GF(p)$ for any $1 \le r \le (p-1)$. It is better to use even r, so that the distance is then odd – for $p > 2$.

Here are samples:

1. $p = 11$. Then MDS codes of the form $\{(10,8,3),(10,6,5),(10,4,7),(10,2,9)\}$ are constructed over $GF(11) = \mathbb{Z}_{11}$. They can respectively correct $\{1,2,3,4\}$ errors. A primitive 10^{th} root of unity is $(2 \mod 11)$; also $(7 \mod 11)$ is a primitive 10^{th} root of unity. The method allows the construction of (at least) $\phi(11) = 10$ MDS $(12, r)$ codes for each r.

2. $p = 13$. Then MDS codes of the forms $\{(12,10,3),(12,8,5),(12,6,7),(12,4,9),(12,2,11)\}$ are constructed over $GF(13) = \mathbb{Z}_{13}$ which can correct respectively $\{1,2,3,4,5\}$ errors. A primitive 12^{th} root of unity is $(2 \mod 13)$ or $(7 \mod 13)$.

3. $p = 17$. Then MDS codes of the forms

   $\{(16,14,3),(16,12,5),(16,10,7),(16,8,9),(16,6,11),(16,4,13),(16,2,15)\}$ which can correct respectively $\{1,2,3,4,5,6,7\}$ errors are constructed over $GF(17) = \mathbb{Z}_{17}$. A primitive 16^{th} root of unity in $GF(17)$ is $(3 \mod 17)$ or $(5 \mod 17)$ and there are $\phi(16) = 8$ such generators.

4. ......

5. Relatively large sample with modular arithmetic: for comparison. Consider $GF(257) = \mathbb{Z}_{257}$ and 257 is prime. Construct the Fourier matrix F_{256} with a primitive 256^{th} root of unity ω in $GF(257)$. Since the order of $3 \mod 257$ is 256 then a choice for ω is $(3 \mod 257)$. Denote the rows of F_{256} in order by $\{e_0, e_1, \ldots, e_{255}\}$.

   Suppose a dimension r is required. Choose $\mathcal{C} = \langle e_0, e_1, \ldots, e_{r-1} \rangle$ to get an MDS $(256, r)$ code. The arithmetic is modular arithmetic, $\mod 257$, and work is done with powers of $(3 \mod 257)$.

[3] It seems there is no known algorithm in which to find a generator of $(\mathbb{Z}_p/\{0\})$ that is substantially better than a brute force method - see Keith Conrad's notes [4]. Note however there are precisely $\phi(p-1)$ primitive $(p-1)$ roots of unity in $GF(p) = \mathbb{Z}_p$.

In addition (5 mod 257) or (7 mod 257) could be used to generate the Fourier 256×256 matrix over $GF(257) = \mathbb{Z}_{257}$; indeed there exist $\phi(256) = 128$ generators that could be used to generate the Fourier matrix.

Note that $(256, 240, 17)$ and $(256, 224, 23)$ codes over $GF(257)$ are constructed as well as other rate codes. These particular ones could be compared to the Reed-Solomon $(255, 239, 17)$ and $(255, 223, 23)$ codes which are in practical use; the ones from $GF(257)$ perform better and faster. There is a much bigger choice for rate and error-correcting capability.

Bigger primes could also be used. Taking $p = 509$ gives $(508, r)$ MDS codes for any $1 < r < 508$. Thus for example $(508, 486, 23)$ MDS codes over $GF(509) = \mathbb{Z}_{509}$ are constructed.

The method allows the construction of $\phi(256) = 128$ such MDS $(256, r)$ codes with different generators for the Fourier matrix. For larger primes the number that could be used for the construction of the Fourier matrix is substantial and cryptographic methods could be devised from such considerations. For example for the prime $p = 2^{31} - 1$ the Fourier $(p-1) \times (p-1)$ matrix exists over $GF(p)$ and $\phi(p-1) = 534600000$ elements could be used to generate the Fourier matrix.

6. The p can be very large and the arithmetic is still doable. For example $p = 10009$ allows the construction of $(10008, r)$ MDS codes over $GF(10009) = \mathbb{Z}_{10009}$. If 100 errors are required to be corrected the scheme supplies $(10008, 9808, 201)$ MDS codes over $GF(10009) = \mathbb{Z}_{10009}$ which have large rate $\approx .98$ and can correct 100 errors. The arithmetic is modular arithmetic. The order of $\omega = (11 \mod 10009)$ is 10008 so this ω could be used to generate the Fourier 10008×10008 matrix over $GF(10009) = \mathbb{Z}_{10009}$; indeed there are $\phi(10008) = 3312$ different elements in $GF(10009) = \mathbb{Z}_{10009}$ that could be used to generate the Fourier 10008×10008 matrix.

7. General p. Then MDS codes of the form $(p-1, p-3, 3), (p-1, p-5, 5), (p-1, p-7, 7), \ldots, (p-1, p-(2i+1), 2i+1), \ldots, (p-1, 2, p-2)$ are constructed which can respectively correct $\{1, 2, 3, \ldots, i, \ldots \frac{p-3}{2}\}$ errors are constructed. A primitive modular element (of order $(p-1)$) is obtained in $GF(p) = \mathbb{Z}_p$ with which to construct the Fourier matrix; as already noted it seems a brute force method for obtaining such seems to be as good as any.

i) *The fields*

Suppose n is given and it is required to find finite fields over which a Fourier $n \times n$ matrix exists. The following argument is essentially taken from [6]. It is included for clarity and completeness and is necessary for deciding on the relevant fields to be used in cases.

Note first of all that the field must have characteristic which does not divide n in order for the Fourier $n \times n$ matrix to exist over the field.

Proposition 3.1 *There exists a finite field of characteristic p containing an n^{th} root of unity for given n if and only if $p \nmid n$.*

Proof: Let p be a prime which does not divide n. Hence $p^{\phi(n)} \equiv 1 \mod n$ by Euler's theorem where ϕ denotes the Euler ϕ function. More specifically let β be the least positive integer such that $p^\beta \equiv 1 \mod n$. Consider $GF(p^\beta)$. Let δ be a primitive element in $GF(p^\beta)$. Then δ has order $(p^\beta - 1)$ in $GF(p^\beta)$ and $(p^\beta - 1) = sn$ for some s. Thus $\omega = \delta^s$ has order n in $GF(p^\beta)$.

On the other hand if p/n then $n = 0$ in a field of characteristic p and so no n^{th} root of unity can exist in the field. □

The proof is constructive. Let n be given and $p \nmid n$. Let β be the least power such that $p^\beta \equiv 1 \mod n$; it is known that $p^{\phi(n)} \equiv 1 \mod n$ and thus β is a divisor of $\phi(n)$. Then the Fourier $n \times n$ matrix over $GF(p^\beta)$ exists.

Sample Suppose $n = 52$. The prime divisors of n are $2, 13$ so take any other prime p and then there is a field of characteristic p which contains a 52^{nd} root of unity. For example take $p = 3$. Know $3^{\phi(52)} \equiv 1 \mod 52$ and $\phi(52) = 24$ but indeed $3^6 \equiv 1 \mod 52$. Thus the field $GF(3^6)$ contains a primitive 52^{nd} root of unity and the Fourier 52×52 matrix exists in $GF(3^6)$. Also $5^4 \equiv 1 \mod 52$, and so $GF(5^4)$ can be used. Now $5^4 = 625 < 729 = 3^6$ so $GF(5^4)$ is a smaller field with which to work.

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Ref
6. Ted Hurley and Donny Hurley, "Coding theory: the unit-derived methodology", Int. J. Information and Coding Theory, Vol. 5, no.1, 55-80, 2018.

Even better though is $GF(53) = \mathbb{Z}_{53}$ which is a prime field. This has an element of order 52 from which the Fourier 52×52 matrix can be formed. Now $\omega = (2 \mod 53)$ is an element of order 52 in $GF(53)$. Work and codes with the resulting Fourier 52×52 matrix can then be done in modular arithmetic, within $\mathbb{Z}_{53}$, using powers of $(2 \mod 53)$.

i. *Developments on different types of MDS codes that can be constructed*

This section is for information on developments and is not required subsequently.

Particular *types* of MDS codes may be required. These are not dealt with here but the following is noted.

- A quantum MDS code is one of the form $[[n, r, d]]$ where $2d = n - r + 2$, see [20] for details. In [12] the methods are applied to construct and develop MDS quantum codes of different types and to required specifications. This is done by requiring the constructed codes to be *dual-containing MDS codes* from which *quantum MDS error-correcting codes* are constructed from the CSS construction developed in [2, 3].

  This is further developed for the construction and development of *Entanglement assisted quantum error-correcting codes*, EAQECC, of different types and to required specifications in [8].

- In [10] Linear complementary dual (LCD), MDS codes are constructed based on the general constructions. An LCD code $\mathcal{C}$ is a code such that $\mathcal{C} \cap \mathcal{C}^{\perp} = 0$. These have found use in security, in data storage and communications' systems. In [10] the rows are chosen according to a particular formulation so as to derive LCD codes which are also MDS codes.
- In [15] error-correcting codes, similar to ones here, are used for solving *underdetermined systems of equations* for use in *compressed sensing*.
- By using rows of the Fourier matrix as matrices for polynomials, MDS convolutional codes, achieving the *generalized Singleton bound* see [21], are constructed and analysed in [14].
- The codes developed here seem particularly suitable for use in McEliece type encryption/decryption, [19]; this has yet to be investigated.

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On the Generalized Power Transformation of Left Truncated Normal Distribution

By Okoli Christian O., Nwosu Dozie F., Osuji George A. & Nsiegbe Nelson A.

Chukwuemeka Odumegwu Ojukwu University

Abstract- The purpose of this study is to establish a unified approach to the transformation problems of certain type of random variable and their associated probability density functions in the generalized setting. The results presented in this research, trivialized the results obtained by several researchers in the literature, in particular for a random variable that follows a left-truncated normal distribution.

Keywords: *truncated distribution, normal distribution, transformation, moments.*

GJSFR-F Classification: *FOR Code: 11J71*

ONTHEGENERALIZEDPOWERTRANSFORMATIONOFLEFTTRUNCATEDNORMALDISTRIBUTION

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On the Generalized Power Transformation of Left Truncated Normal Distribution

Okoli Christian O. [α], Nwosu Dozie F. [σ], Osuji George A. [ρ] & Nsiegbe Nelson A. [ω]

Abstract- The purpose of this study is to establish a unified approach to the transformation problems of certain type of random variable and their associated probability density functions in the generalized setting. The results presented in this research, trivialized the results obtained by several researchers in the literature, in particular for a random variable that follows a left-truncated normal distribution.

Keywords: truncated distribution, normal distribution, transformation, moments.

I. Introdution and Preliminary

Let ωbe an element of an appropriate non-empty sample space Ω and $X:\Omega\longrightarrow \mathrm{R}(R=(-\infty,\infty))$ a real-valued function (random variable) defined on Ω. To each element of the event

$$\Gamma_X=\{\omega\in\Omega:\ \mathrm{X}(\omega)=\mathrm{x}\}\in 2^{\Omega} \tag{1.1}$$

is associated with a probability measure $P:2^{\Omega}\longrightarrow[0,1]$ in the measure space $(\Omega,2^{\Omega},P)$ and then denotes the probability density function (pdf) f associated with the real-valued function (random variable) X by $f(x)$. Where $f:\mathrm{X}(\Omega)\longrightarrow[0,1]$.

Let α be an arbitrary but fixed point of a scalar field $\mathcal{F}$ $(i.e\ \alpha\in\mathcal{F})$, then we consider a continuous bijective function or transformation $h_\alpha: X(\Omega)\longrightarrow \mathrm{R}$ define by

$$h_\alpha(x)=x^{\alpha}\quad\forall\ \alpha\in D \tag{1.2}$$

If f_{h_α} is the function induced by h_α on f, then we denoted the probability density function (pdf) g associated with the real-valued function (random variable) h_α by $f_{h_\alpha}(x)$; $f_{h_\alpha}$is the probability density function induced by h_α on f such that

$$g:\mathrm{X}(\Omega)=f_{h_\alpha}:\mathrm{X}(\Omega)=f:h_\alpha\big(\mathrm{X}(\Omega)\big)\longrightarrow[0,1] \tag{1.3}$$

Remark 1.1

1) If $\alpha=0$, then h_α $(i.e.h_0)$reduces to a constant function. Hence at this point the domain of $g$reduces to a singleton set which is not of interest (in terms of data transformations).

Author α ω: Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University Uli, Nigeria. e-mail: fedocon2003@gmail.com
Author σ: Department of Mathematics & Statistics, Federal Polythenic Nekede, Owerri, Imo State Nigeria.
Author ρ: Department of Statistics, Nnamdi Azikiwe University, Awka Nigeria.

2) If $\alpha = 1$, then h_α $(i.e. h_1)$ reduces to a identity function so that $g(x) = f(x) \; \forall \, x \in \mathrm{X}(\Omega)$.

Hence in this research, we require that $\alpha \neq 0$, as such we consider the following propositions:

Proposition 1.2 If $\alpha \leq -1$, then g is an inverse α-power transform of f.
Proof.
This easily follows from the fact that $h_\alpha(x) = \frac{1}{x^\alpha} \;\; \forall \, \alpha \geq 1$.

Proposition 1.3 If $\alpha \geq 1$, then g is an α-power transform of f.

Proof.
This easily follows from the fact that $h_\alpha(x) = x^\alpha \;\; \forall \, \alpha \geq 1$.

Proposition 1.4 If $0 < \alpha < 1$, then there exist a positive constant$c$ such that g is a $(c+1)\, th\, root$ power transform of f.

Proof.
If $0 < \alpha < 1$, then it follows that $\frac{1}{\alpha} > 1; \Longrightarrow \frac{1}{\alpha} = 1 + c,$ for some $c > 0$;

$\Longrightarrow \alpha = \frac{1}{1+c}$, for some $c > 0$, so that $h_\alpha(x) = x^{\frac{1}{1+c}} \;\; \forall \, c > 0$ which is as stated.

Proposition 1.5 If $-1 < \alpha < 0$, then there exist a positive constant$c$ such that g is an inverse $(c+1) th\, root$ power transform of f.

Proof.

If $-1 < \alpha < 0$, then it follows that $0 < -\alpha < 1; \Longrightarrow 0 < \beta < 1$, where $\beta =- \alpha$.
Thus by proposition 1.4 $\beta = \frac{1}{1+c}$, for some $c > 0$;

$\Longrightarrow \alpha = \frac{-1}{1+c}$, for some $c > 0$ which is as stated.

Remark 1.6
Now, observe in particular;

1) In proposition 1.2, if $\alpha = -1, -2$, then g is an inverse, inverse square, transform of f respectively.
2) In proposition 1.3, if $\alpha = 1, 2$, then g is the identity, square, transform of f respectively.
3) In proposition 1.4, if $c = 1, \Longrightarrow \alpha = \frac{1}{2}$, then g is a square root transform of f.
4) In proposition 1.5, if $c = 1, \Longrightarrow \alpha = -\frac{1}{2}$, then g is an inverse square root transform of f.

II. The Left Truncated Normal Distribution

Definition 2.1 Let X be a random variable that follow a normal distribution with $\mu (\mu \neq 0)$ and variance $\sigma^2 (\sigma^2 > 0) (i.e.\ X \sim (\mu, \sigma^2))$ then the probability distribution function (pdf)[4] is given by

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \in R \qquad (2.1)$$

Notes

14. M.K. Okasha and Iyad M. A. Alqanoo, (2014) Inference on The Doubly Truncated Gamma Distribution For Lifetime Data, *International Journal Of Mathematics And Statistics Invention (IJMSI) E-ISSN: 2321 – 4767, Volume 2 Issue 11.*

Lifetime data pertain to the lifetimes of units, either industrial or biological, an industrial or a biological unit cannot be in operation forever. Such a unit cannot continue to operate in the same condition forever. Any random variable is said to be truncated if it can be observed over part of its range. Truncation occurs in various situations. For example, right truncation occurs in the study of life testing and reliability of items such as an electronic component, light bulbs, etc. Left truncation arises because, in many situations, failure of a unit is observed only if it fails after a certain period (for more on this, see [14-15] and the references therein). Unfortunately, often time in practice, the random variable Xwhich follow a $\mathrm{N}(\mu,\sigma^2)$ distribution do not take values that are less than or equal to zero $(X \leq 0)$. As such, it naturally calls for one to truncate the pdf in (2.1) to take care of the restriction of the random variable in the region X ¿ 0 without alteration to the properties of thepdf. Hence we seek for such truncated normal distribution of f and then denote it by f_T. It suffices to find a constant M such that $\int_0^\infty Mf(x)dx = 1$, where M is the so-called normalizing constant and then define $f_T(x) = Mf(x)$.

Now, we solve for such M by evaluating the integral $\int_0^\infty f(x)dx$. Observe that If we take $z = \frac{x-\mu}{\sigma}$, then

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{\frac{-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}z^2} dz = \Phi\left(\frac{-\mu}{\sigma}\right)$$

It then follows that $M = \frac{1}{\Phi\left(\frac{-\mu}{\sigma}\right)}$. Hence, the left truncated normal distribution of f is given by

$$f_T(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \in R_+ \tag{2.2}$$

Observe that $0 \leq f_T(x;\mu,\sigma) \leq 1\ \forall\, x \in R_+$ $(R_+ = (0,\infty))$ and by the method of derivation of $f_T(x;\mu,\sigma)$, we have that $\int_0^\infty f_T(x;\mu,\sigma)dx = 1$. Thus $f_T(x;\mu,\sigma)$ is a proper pdf.

III. Distribution associated with Truncated Normal Distribution under Arbitrary α-Power Transformation

Let α be an arbitrary but fixed point of a scalar field $\mathcal{F}$ $(i.e\ \alpha \in \mathcal{F})$ and $h_\alpha(x) = x^\alpha\ \ \forall\, \alpha \in \mathcal{F}$ as in equation (1.2). There is no loss of generality if we put $y = h_\alpha(x)$ and $\alpha = n;\ \Rightarrow y = x^n$. Hence by standard result in classical calculus [2], the transformed function g induced by h_α on f is given by

$$g(y;\mu,\sigma,n) = f_T(x;\mu,\sigma)\left|\left|\frac{dx}{dy}\right|\right| \tag{3.1}$$

Where $\left|\left|\frac{dx}{dy}\right|\right|$is the absolute value of the Jacobian (determinant) of the transformation [2]. If $y = x^n$, then

$$dy = nx^{n-1}\mathrm{dx}; \Rightarrow \left|\left|\frac{dx}{dy}\right|\right| = \frac{1}{|n|x^{n-1}}$$

By substituting appropriately into equation (3.1) and simplifying, we have

$$g(y;\mu,\sigma,n) = \begin{cases} \dfrac{y^{\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2}, & y \in R_+, n \in \mathcal{F} \\ 0 & otherwise \end{cases} \quad (3.2)$$

It now remain to show that $g(y;\mu,\sigma,n)$ given in equation (3.2) is a well-defined pdf. It suffices to show that $\int_0^\infty g(y;\mu,\sigma,n)dy = 1$. To see this we proceed as follows:

$$\int_0^\infty g(y;\mu,\sigma,n)dy = \int_0^\infty \frac{y^{\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy$$

$$= \int_0^\infty K y^{\frac{1}{n}-1} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy; \quad K = \frac{1}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

Let $u = y^{\frac{1}{n}}$; $\Rightarrow dy = ny^{1-\frac{1}{n}}$ du, substituting into the integral above gives

$$\int_0^\infty K y^{\frac{1}{n}-1} e^{\frac{-1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} n y^{1-\frac{1}{n}} \mathrm{du} = \int_0^\infty nK e^{\frac{-1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} \mathrm{du}$$

Let $z = \frac{u-\mu}{\sigma}$; $\Rightarrow \sigma dz = du$, substituting into the integral above gives

$$\int_{\frac{-\mu}{\sigma}}^\infty n\sigma K e^{\frac{-1}{2}z^2} \mathrm{dz} = \int_{\frac{-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}z^2} \mathrm{dz}$$

$$= \left(\frac{1}{\Phi\left(\frac{-\mu}{\sigma}\right)}\right)\left(\frac{1}{\sqrt{2\pi}} \int_{\frac{-\mu}{\sigma}}^\infty e^{\frac{-1}{2}z^2} \mathrm{dz}\right) = \frac{\Phi\left(\frac{-\mu}{\sigma}\right)}{\Phi\left(\frac{-\mu}{\sigma}\right)} = 1$$

This is as required.

IV. The *jth* Moment about the Mean and the Origin

In this section, for all fixed $n \in R$, we solved for the jth moment of the random variable Y about the mean μ, which is also called the jth central moment is defined as $\mu_j(\mu,\sigma,n) = E\left[(Y-\mu)^j; \mu,\sigma,n\right]$ ($\mu_j(n)$ for short). This implies that

Notes

$$\mu_j(n) = \int_0^\infty (y-\mu)^j \frac{y^{\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy$$

$$= \int_0^\infty \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{j-p} \mu^{j-p} y^p \frac{y^{\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy$$

$$= \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{j-p} \mu^{j-p} \int_0^\infty \frac{y^{p+\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy$$

$$= \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{j-p} \mu^{j-p}\, E[Y^P;\mu,\sigma,n] \qquad (4.1)$$

and we proceed to compute the pth moment about the origin $E[Y^P;\mu,\sigma,n]$ which is given by

$$E[Y^P;\mu,\sigma,n] = \int_0^\infty y^p \frac{y^{\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy$$

$$= K\int_0^\infty y^{p+\frac{1}{n}-1} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy$$

Let $u = y^{\frac{1}{n}}$; $\Rightarrow dy = ny^{1-\frac{1}{n}}\,\mathrm{du}$, substituting into the integral above and simplifying, we have

$$K\int_0^\infty y^{p+\frac{1}{n}-1} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} ny^{1-\frac{1}{n}}\,\mathrm{du} = nK\int_0^\infty u^{\mathrm{n}p} e^{\frac{-1}{2\sigma}(u^2-2u+1)}\,\mathrm{du}$$

$$= nKe^{\frac{-1}{2\sigma^2}} \int_0^\infty u^{\mathrm{n}p} e^{\frac{-u^2}{2\sigma^2}} e^{\frac{u}{\sigma^2}}\,\mathrm{du} = nKe^{\frac{-1}{2\sigma}} \int_0^\infty u^{\mathrm{n}p} e^{\frac{-u^2}{2\sigma^2}} \sum_{r\geq 0} \frac{\left(\frac{u}{\sigma^2}\right)^r}{r!}\,\mathrm{du}$$

Observe that the series $\sum_{r\geq 0} \frac{\left(\frac{u}{\sigma^2}\right)^r}{r!}$ converges uniformly (by ratio test) [3,13], hence by Taylors series expansion, for some positive constant k (sufficiently large enough) [3,13], there exists a number $\delta(r_k)$ between 0 and $\frac{u}{\sigma^2}$ such that $\delta(r_k) \longrightarrow 0$ as $r \longrightarrow \infty$, it then follows that as $r \longrightarrow \infty$

$$nKe^{\frac{-1}{2\sigma^2}} \int_0^\infty u^{\mathrm{n}p} e^{\frac{-u^2}{2\sigma^2}} \sum_{r=0}^{k} \left(\frac{1}{r!}\left(\frac{u}{\sigma^2}\right)^r + \frac{1}{r!}\left(\frac{u}{\sigma^2}\right)\left(\delta(r_k)\right)^r\right) \mathrm{du}$$

Ref

3. U. A. Osisiogu, (1998). An Introduction to Real Analysis, Bestsort Educational Book. Nigeria, 1998.

can be approximated by

$$nKe^{\frac{-1}{2\sigma^2}}\int_0^\infty u^{np}\, e^{\frac{-u^2}{2\sigma^2}}\sum_{r=0}^{k}\frac{1}{r!}\left(\frac{u}{\sigma^2}\right)^r du = nKe^{\frac{-1}{2\sigma^2}}\sum_{r=0}^{k}\frac{1}{\sigma^{2r}r!}\int_0^\infty u^{r+np}\, e^{\frac{-u^2}{2\sigma^2}}\, du$$

Let $w=\frac{u^2}{2\sigma^2}$; $\Longrightarrow \sigma^2 dw = udu$, then substituting appropriately into the integral above and simplifying, we have

$$nKe^{\frac{-1}{2\sigma^2}}\sum_{r=0}^{k}\frac{1}{\sigma^{2r}r!}\sigma^2\int_0^\infty \sigma^{r+np-1}(2w)^{\frac{r+np-1}{2}}e^{-w}\, du$$

$$= nKe^{\frac{-1}{2\sigma^2}}\sum_{r=0}^{k}\frac{2^{\frac{r+np-1}{2}}}{\sigma^{r-np-1}r!}\int_0^\infty w^{\left(\frac{r+np+1}{2}\right)-1}e^{-w}\, du$$

$$= \frac{e^{\frac{-1}{2\sigma^2}}\sum_{r=np}^{k}\frac{2^{\frac{r+np+1}{2}}}{\sigma^{r}r!}\Gamma\left(\frac{r+np+1}{2}\right)}{2\sigma^{np+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

Thus,

$$E[Y^P;\mu,\sigma,n] = \frac{e^{\frac{-1}{2\sigma^2}}\sum_{r=\lfloor -np\rfloor}^{k}\frac{2^{\frac{r+np+1}{2}}}{\sigma^{r}r!}\Gamma\left(\frac{r+np+1}{2}\right)}{2\sigma^{np+1}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} \qquad (4.2)$$

And

$$\mu_j(\mu,\sigma,n) = E\left[(Y-\mu)^j;\mu,\sigma,n\right] = \sum_{p=0}^{j-1}(-1)^{j-p}\binom{j}{j-p}\mu^{j-p}\, E[Y^P;\mu,\sigma,n] + E\left[Y^j;\mu,\sigma,n\right]$$

$$= \sum_{p=0}^{j-1}(-1)^{j-p}\binom{j}{j-p}\mu^{j-p}\,\frac{e^{\frac{-1}{2\sigma^2}}\sum_{r=\lfloor -np\rfloor}^{k}\frac{2^{\frac{r+np+1}{2}}}{\sigma^{r}r!}\Gamma\left(\frac{r+np+1}{2}\right)}{2\sigma^{np+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

$$+ \frac{e^{\frac{-1}{2\sigma^2}}\sum_{r=\lfloor -jn\rfloor}^{k}\frac{2^{\frac{r+jn+1}{2}}}{\sigma^{r}r!}\Gamma\left(\frac{r+jn+1}{2}\right)}{2\sigma^{jn+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

$$= \sum_{p=0}^{j}(-1)^{j-p}\binom{j}{j-p}\mu^{j-p}\,\frac{e^{\frac{-1}{2\sigma^2}}\sum_{r=\lfloor -np\rfloor}^{k}\frac{2^{\frac{r+np+1}{2}}}{\sigma^{r}r!}\Gamma\left(\frac{r+np+1}{2}\right)}{2\sigma^{np+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} \qquad (4.3)$$

Where $\lfloor x\rfloor$ is the greatest integer function less than x.

It is important to observe that in particular, in equation (4.2), if we take $n=-1$, then g is an inverse transform of f and by putting $k=7, \mu=1$ and evaluating $E[Y^P;1,\sigma,-1]$ at $p=1,2$ respectively, we obtain the result in [6].

Ref

6. Nwosu C. R, Iwueze I.S. and Ohakwe J. (2010). Distribution of the Error Term of the Multiplicative Time Series Model Under Inverse Transformation. Advances and Applications in Mathematical Sciences. Volume 7, Issue 2, 2010, pp. 119 – 139.

Remark 4.1 Further more observe that;

1) Iwueze (2007), for $\mu = 1,\ n = 1$, the authors expressed $E[Y]$ in terms of cumulative distribution function of the standard normal distribution and $E[(Y-1)^2]$ in terms of cumulative distribution function of the standard normal distribution and Chi-square distribution function,
2) Nwosu, Iwueze and Ohakwe (2010), for $\mu = 1, n = -1$, the authors expressed $E[Y]$ and $E[(Y-1)^2]$in terms of cumulative distribution function of the standard normal distribution and Gamma distribution function,
3) Ohakwe, Dike and Akpanta (2012), for $\mu = 1, n = 2$, the authors expressed $E[Y]$ and $E[(Y-1)^2]$in terms of cumulative distribution function of the standard normal distribution,
4) Nwosu, Iwueze, and Ohakwe. (2013), for $\mu = 1,\ n = -1$, the authors expressed $E[Y]$ and $E[(Y-1)^2]$in terms of cumulative distribution function of the standard normal distribution and Chi-square distribution function,
5) Ibeh and Nwosu(2013), for $\mu = 1,\ n = -2$, the authors expressed $E[Y]$ and $E[(Y-1)^2]$in terms of cumulative distribution function of the standard normal distribution and Chi-square distribution function,
6) Ajibade, Nwosu and Mbegdu (2015), for $\mu = 1,\ n = \frac{-1}{2}$, the authors expressed $E[Y]$ and $E[(Y-1)^2]$in terms of cumulative distribution function of the standard normal distribution and Chi-square distribution function.

Hence, it suffices to say that the expression for the moments is by no means unique. Furthermore, the aforementioned authors above seems to be somewhat restrictive in their estimation of moments; they all estimated only for the first moment about the origin (mean) and the second central moment (variance). Hence, in this paper such restriction is dispensed with.

V. The Moment Generating Function Associated with $g(y;\mu,\sigma,n)$ and $f_T(x;\mu,\sigma)$

The moment generating function of Y is given by

$$M_Y(t;\mu,\sigma,n) = E(e^{tY};\mu,\sigma,n) = \int_0^\infty e^{ty} g(y;\mu,\sigma,n)dy = \int_0^\infty \sum_{i\geq 0} \frac{(ty)^i}{i!} g(y;\mu,\sigma,n)dy$$

Observe that the series $\sum_{r=0}^{\infty} \frac{(ty)^i}{i!}$ converges uniformly (by ratio test) [3,13], hence by Taylors series expansion, for some positive constant l (sufficiently large enough), there exists a number $\rho(i_l)$ between 0 and ty such that $\rho(i_l) \longrightarrow 0$ as $i \longrightarrow \infty$[3,13], it then follows that as $i \longrightarrow \infty$

$$\int_0^\infty \sum_{i=0}^{k} \left(\frac{1}{i!}(ty)^i + \frac{1}{i!}(ty)(\rho(i_l))^i\right) g(y;\mu,\sigma,n)dy$$

can be approximated by

Ref

3. U. A. Osisiogu, (1998). An Introduction to Real Analysis, Bestsort Educational Book. Nigeria, 1998.

$$\int_0^\infty \sum_{i=0}^{l} \frac{1}{i!}(ty)^i g(y;\mu,\sigma,n)dy = \sum_{i=0}^{l} \frac{t^i}{i!}\int_0^\infty y^i g(y;\mu,\sigma,n)dy$$

$$= \sum_{i=0}^{l} \frac{t^i}{i!}\int_0^\infty \frac{y^{i+\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2} dy = \sum_{i=0}^{l} \frac{t^i}{i!} E\left[Y^i;\mu,\sigma,n\right]$$

$$= \sum_{i=0}^{l} \frac{t^i}{i!} \frac{e^{\frac{-1}{2\sigma^2}} \sum_{r=\lfloor -ni \rfloor}^{k} \frac{2^{\frac{r+ni+1}{2}}}{\sigma^r r!}\Gamma\left(\frac{r+ni+1}{2}\right)}{2\sigma^{ni+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

For the moment generating function of X, recall that at $n=1, y=x$, it follows that $g(y;\mu,\sigma,1) = f_T(x;\mu,\sigma)$. Hence

$$M_Y(t;\mu,\sigma,1) = \int_0^\infty e^{ty} g(y;\mu,\sigma,1)dy = \int_0^\infty e^{tx} f_T(x;\mu,\sigma)dx$$

$$= E(e^{tX};\mu,\sigma) = M_X(t;\mu,\sigma) = \sum_{i=0}^{l} \frac{t^i}{i!} \frac{e^{\frac{-1}{2\sigma^2}} \sum_{r=\lfloor -i \rfloor}^{k} \frac{2^{\frac{r+i+1}{2}}}{\sigma^r r!}\Gamma\left(\frac{r+i+1}{2}\right)}{2\sigma^{i+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

VI. Existence of the Bell-Shape Curve Associated with $g(y;\mu,\sigma,n)$ and $f_T(x;\mu,\sigma)$

Recall that $f_T(x;\mu,\sigma)$, the left truncated normal distribution of f, which is given by

$$f_T(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, x \in R_+$$

Is normal distribution in the region $X > 0$ with mean $\mu_1(\mu,\sigma,1)$ and variance $\mu_2(\mu,\sigma,1)$, where

$$\mu_1(\mu,\sigma,1) = \frac{e^{\frac{-1}{2\sigma^2}} \sum_{r=\lfloor -jn \rfloor}^{k} \frac{2^{\frac{r+2}{2}}}{\sigma^r r!}\Gamma\left(\frac{r+2}{2}\right)}{2\sigma^3\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

$$\mu_2(\mu,\sigma,1) = \sum_{p=0}^{2} (-1)^{2-p}\binom{2}{2-p}\mu^{2-p} \frac{e^{\frac{-1}{2\sigma^2}} \sum_{r=\lfloor -p \rfloor}^{k} \frac{2^{\frac{r+p+1}{2}}}{\sigma^r r!}\Gamma\left(\frac{r+p+1}{2}\right)}{2\sigma^{p+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

Notes

If we denote this mean and variance of the truncated normal distribution $f_T(x;\mu,\sigma)$ by μ_T and σ_T^2 (i.e. $\mu_T=\mu_1(\mu,\sigma,1)$ and $\sigma_T^2=\mu_2(\mu,\sigma,1)$). It is well known that the shape of $f_T(x;\mu,\sigma)$ varies as the value of σ_T^2 varies (consequently as σ varies since σ_T^2 depend on σ), hence σ is also the shape parameter for $f_T(x;\mu,\sigma)$.

Also recall that $g(y;\mu,\sigma,n)$, the generalized power transformation of $f_T(x;\mu,\sigma)$, which is given by

$$g(y;\mu,\sigma,n)=\begin{cases}\dfrac{y^{\frac{1}{n}-1}}{|n|\sigma\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n}}-\mu}{\sigma}\right)^2}, & y\in R_+, n\in\mathcal{F}\\ 0 & otherwise\end{cases}$$

Is normal distribution in the region $X>0$ with mean $\mu_1(\mu,\sigma,n)$ and variance $\mu_2(\mu,\sigma,n)$, where

$$\mu_1(\mu,\sigma,n)=\frac{e^{\frac{-1}{2\sigma^2}}\sum_{r=\lfloor -jn\rfloor}^{k}\frac{2^{\frac{r+jn+1}{2}}}{\sigma^r r!}\Gamma\left(\frac{r+jn+1}{2}\right)}{2\sigma^{jn+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

$$\mu_2(\mu,\sigma,n)=\sum_{p=0}^{2}(-1)^{2-p}\binom{2}{2-p}\mu^{2-p}\frac{e^{\frac{-1}{2\sigma^2}}\sum_{r=\lfloor -np\rfloor}^{k}\frac{2^{\frac{r+np+1}{2}}}{\sigma^r r!}\Gamma\left(\frac{r+np+1}{2}\right)}{2\sigma^{np+2}\sqrt{2\pi}\,\Phi\left(\frac{-\mu}{\sigma}\right)}$$

If we denote this mean and variance of the generalized n-power transform of$f_T(x;\mu,\sigma)$ by $\mu_T(n)$ and $\sigma_T^2(n)$ (i.e. $\mu_T(n)=\mu_1(\mu,\sigma,n)$ and $\sigma_T^2(n)=\mu_2(\mu,\sigma,n)$). It follows that for every fixed $n\in R$, the shape of $g(y;\mu,\sigma,n)$ varies as the value of $\sigma_T^2(n)$ varies (consequently as σ varies since $\sigma_T^2(n)$ depend on σ), hence σ is also the shape parameter for $g(y;\mu,\sigma,n)$. Observe that. $\mu_T(1)=\mu_1(\mu,\sigma,1)=\mu_T$ and $\sigma_T^2(1)=\mu_2(\mu,\sigma,1)=\sigma_T^2$.

Now, we observe that $\sigma_T^2(n)$ (and σ_T^2) depend on σ. A common research interest of several authors (see [5-12]) is to find the value of σ for which $\mu_T(1)=\mu_T(n)$ for every fixed $n\neq 1$ $(n\in R)$. This is the so-called normality condition. Furthermore, It is expected that at this point $\sigma_T^2(1)=\sigma_T^2(n)$ for every fixed $n\neq 1$ $(n\in R)$. Observe that $g(y;\mu,\sigma,n)$ and $f_T(x;\mu,\sigma)$are strictly monotone and have one turning point, furthermore $g(y;\mu,\sigma,n)>0$ and $f_T(x;\mu,\sigma)>0$ for every $x,y\in R_+$, $and\ for\ a\ fixed\ n\in\mathcal{F}$. Which implies that the values of x,y at these turning points maximizes $f_T(x;\mu,\sigma)$, $g(y;\mu,\sigma,n)$ respectively. Consequently by classical calculus, it is well known that these values of x,y at this turning point coincide with the mode of $f_T(x;\mu,\sigma),g(y;\mu,\sigma,n)$ respectively. We shall determine this values of x,y using the Rolle' s theorem. Now we state the following theorem which is equivalent to the (so-called) normality condition.

Ref

5. Iwueze Iheanyi S. (2007). Some Implications of Truncating the N (1,σ2) Distribution to the left at Zero. Journal of Applied Sciences. 7(2) (2007) pp 189-195.

Theorem 6.1

Let $f_T(x;\mu_T,\sigma_{\dot{T}})$ be a truncated normal distribution and $g(y;\mu_T(n),\sigma_{\dot{T}}(n_0),n_0)$the generalized n_0-power transformation of$f_T(x;\mu_T,\sigma_{\dot{T}})$ induced by $y=x^{n_0}$, then $g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)$ has a Bell-shape that coincide with$f_T(x;\mu_T,\sigma_{\dot{T}})$ if thereexists a sequence $\{\sigma_j\}_{j=1}^{\infty}\subset(\beta_1,\beta_2)\subset R_+$ and at least one point $\sigma_0\in(\beta_1,\beta_2)$ such that the $\{\sigma_j\}_{j=1}^{\infty}$ converges to $\sigma_0\in(\beta_1,\beta_2)$ (i.e. $\sigma_j\longrightarrow\sigma_0\ as\ j\longrightarrow\infty$) and σ_0 is a zero solution to the problem

$$maximizemaximize\colon\ g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0) \tag{6.1}$$

$$at\ the\ point\colon\ y=x_0 \tag{6.2}$$

provided $f_T(x;\mu_T,\sigma_{\dot{T}})\le f_T(x_0;\mu_T,\sigma_{\dot{T}})\ \forall\ x\in R_+$.

Proof.

Observe that $f_T(x;\mu_T,\sigma_{\dot{T}})$ is bounded above and continuous, hence by boundedness above it follows there exist a positive constant C such that

$$f_T(x;\mu_T,\sigma_{\dot{T}})\le C\ \ \forall\ x\in R_+$$

and by continuity in R_+, it follows that there exists a constant $u_0\in R_+$such that$C=Sup f_T(u_0;\mu_T,\sigma_{\dot{T}})$, hence we must have $u_0=x_0$. This justify the existence of such x_0. Hence the problem is equivalent to

$$maximizemaximize\colon g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0) \tag{6.3}$$

$$at\ the\ point\colon y=u_0 \tag{6.4}$$

Now, suppose for contradiction that there is no such $\sigma\in R_+$ (recall that $\sigma_{\dot{T}}\ is\ a\ function\ of\ \sigma,\ i.e.\ \sigma_{\dot{T}}\ depend\ on\ \sigma$) that satisfies the maximization problem. This implies that for every $\sigma\in R_+$, the maximization problem becomes

$$maximizemaximize\colon g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)$$

$$at\ the\ point\colon y\neq u_0$$

If $y\neq u_0$, it implies that there is an $\varepsilon\neq 0$ such that $y=u_0\pm\varepsilon$, hence the maximization problem becomes

$$maximizemaximize\colon g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)$$

$$at\ the\ point\colon y=u_0\pm\varepsilon$$

It then follows that

$$C=Sup\{f_T(u_0\pm\varepsilon;\mu_T,\sigma_{\dot{T}})\colon\ \forall\ \varepsilon\neq 0\}\Longrightarrow\Longleftarrow.$$

Observe that this is a contradiction to the maximality of C at u_0 since $\varepsilon\neq 0$. And converly, if the maximality condition of C holds, it

$$\Longrightarrow\{f_T(u_0\pm\varepsilon;\mu_T,\sigma_{\dot{T}})\colon\ \forall\ \varepsilon\neq 0\}<C$$

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$$\Longrightarrow f_T(u_0;\mu_T,\sigma_{\dot{T}}) \le C \;\; for\; \varepsilon = 0$$

$$\Longrightarrow Sup f_T(u_0;\mu_T,\sigma_{\dot{T}}) = C$$

This contradict the fact that $\varepsilon \neq 0$.

Thus we must have that there is at least one $\sigma \in R_+$ (for such $\sigma \in R_+$, $\varepsilon = 0$) that satisfies the maximization problem. This completes the proof.

We now proceed to solve the maximization problem of equation (6.3) and equation (6.4) which is equivalent to the maximization problem of equation (6.1) and equation (6.2).

Clearly $g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)$ is differentiable in the given subset D of R_+ and by classical optimization theory of calculus, a necessary condition for existence of maximum (extreme) point of $g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)$ is that the derivatives of $g(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)$must be equal to zero [3,4,13]. This implies that

$$\frac{dg\,(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)}{dy} = 0 \qquad (6.6)$$

We now proceed to solve for equation (6.6). Observe that

$$\frac{dg(y;\mu_T(n_0),\sigma_{\dot{T}}(n_0),n_0)}{dy} =$$

$$K\left[\left(\frac{1}{n_0}-1\right)y^{\frac{1}{n_0}-2}e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma}\right)^2} - y^{\frac{2}{n_0}-2}\,\frac{1}{n_0}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma^2}\right)e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma}\right)^2}\right] =$$

$$Ky^{\frac{1}{n_0}-2}e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma}\right)^2}\left[\left(\frac{1}{n_0}-1\right) - y^{\frac{1}{n_0}}\frac{1}{n_0}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma^2}\right)\right]$$

By equation (6.6) it follows that

$$Ky^{\frac{1}{n_0}-2}e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma}\right)^2}\left[\left(\frac{1}{n_0}-1\right) - y^{\frac{1}{n_0}}\frac{1}{n_0}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma^2}\right)\right] = 0.$$

Since $Ky^{\frac{1}{n_0}-2}e^{\frac{-1}{2}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma}\right)^2} > 0 \;\forall\, y \in R_+$, we must have that

$$\left(\frac{1}{n_0}-1\right) - y^{\frac{1}{n_0}}\frac{1}{n_0}\left(\frac{y^{\frac{1}{n_0}}-\mu}{\sigma^2}\right) = 0$$

By simplifying the above equation we have

$$\sigma^2(1-n_0) - y^{\frac{2}{n_0}} + \mu y^{\frac{1}{n_0}} = 0$$

Ref

3. U. A. Osisiogu, (1998). An Introduction to Real Analysis, Bestsort Educational Book. Nigeria, 1998.

Now if we take $v = y^{\frac{1}{n_0}}$, we obtain

$$v^2 - \mu v - \sigma^2(1-n_0) = 0 \tag{6.8}$$

and if we take $v = y^{\frac{-1}{n_0}}$, we obtain

$$\sigma^2(1-n_0)v^2 + \mu v - 1 = 0 \tag{6.9}$$

Thus, the solution to equation (6.8) and equation (6.9) is given by

$$v = \begin{cases} \frac{\mu \pm \sqrt{\mu^2 - 4\sigma^2(n_0-1)}}{2} \\ \frac{\mu \pm \sqrt{\mu^2 - 4\sigma^2(n_0-1)}}{2\sigma^2(n_0-1)} \end{cases} \tag{6.10}$$

Where $\left(\frac{\mu}{2\sigma}\right)^2 > n_0 - 1$.

Solutions relating to equation (6.9) have been given by virtually all the authors mentioned above for specific value of n_0 and μ. Using equation (6.8), we have that $v = y_{max}^{\frac{1}{n_0}}$. Now, by equation (6.4) it follows that $u_0 = y_{max} = \mu$. Thus,

$$v^2 - u_0 v - \sigma^2(1-n_0) = 0 \quad if\ v = u_0^{\frac{1}{n_0}}$$

And

$$\sigma^2(1-n_0)v^2 + u_0 v - 1 = 0 \quad if\ v = u_0^{\frac{-1}{n_0}}$$

If we put $z_0 = u_0^{\frac{1}{n_0}}$ and $w_0 = u_0^{\frac{-1}{n_0}}$, then we have

$$G(\sigma) = 0;\ G(\sigma) = z_0^2 - u_0 z_0 + \sigma^2(n_0-1)$$

And

$$H(\sigma) = 0;\ H(\sigma) = -\sigma^2(n_0-1)w_0^2 + u_0 w_0 - 1$$

This reduces to solving for the zero of the functions $G(\sigma)$ and $H(\sigma)$.

For $G(\sigma)$, this implies that given $0 \le \delta_1 < \delta_2$, if we take $\sigma_a = \sqrt{\frac{u_0 z_0 - z_0^2 - \delta_1}{n_0 - 1}}$ and $\sigma_b = \sqrt{\frac{u_0 z_0 - z_0^2 + \delta_2}{n_0 - 1}}$, then $G\left(\sqrt{\frac{u_0 z_0 - z_0^2 - \delta_1}{n_0 - 1}}\right) = -\delta_1 \le 0$ and $G\left(\sqrt{\frac{u_0 z_0 - z_0^2 + \delta_2}{n_0 - 1}}\right) = \delta_2 > 0$

It follows that

$$G\left(\sqrt{\frac{u_0 z_0 - z_0^2 - \delta_1}{n_0 - 1}}\right) G\left(\sqrt{\frac{u_0 z_0 - z_0^2 + \delta_2}{n_0 - 1}}\right) = -\delta_1\delta_2 < 0\ if\ \delta_1 \ne 0$$

This implies that there exists a sequence $\{\sigma_j\}_{j=1}^{\infty} \subset (\sigma_a, \sigma_b)$ and at least one point $\sigma_0 \in (\sigma_a, \sigma_b)$ such that the $\{\sigma_j\}_{j=1}^{\infty}$ converges to $\sigma_0 \in (\sigma_a, \sigma_b)$ (i.e. $\sigma_j \longrightarrow \sigma_0\ as\ j \longrightarrow \infty$) and $G(\sigma_0) = 0$

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For $H(\sigma)$, this implies that given $\gamma_1 = 0 \; and \; \gamma_2 > 0$, if we take $\sigma_p = \sqrt{\frac{(u_0w_0+\gamma_1)}{(n_0-1){w_0}^2}}$ and $\sigma_q = \sqrt{\frac{(u_0w_0+\gamma_2)}{(n_0-1){w_0}^2}}$, then $H\left(\sqrt{\frac{(u_0w_0+\gamma_1)}{(n_0-1){w_0}^2}}\right) = -1 < 0$ and $H\left(\sqrt{\frac{(u_0w_0-1-\gamma_2)}{(n_0-1){w_0}^2}}\right) = \gamma_2 > 0$

It follows that

$$H\left(\sqrt{\frac{u_0w_0+\gamma_1}{(n_0-1){w_0}^2}}\right) H\left(\sqrt{\frac{u_0w_0-1-\gamma_2}{(n_0-1){w_0}^2}}\right) = -\gamma_2 < 0$$

This implies that there exists a sequence $\{\sigma_i\}_{i=1}^{\infty} \subset (\sigma_p, \sigma_q)$ and at least one point $\sigma_0 \in (\sigma_p, \sigma_q)$ such that the sequence $\{\sigma_i\}_{i=1}^{\infty}$ converges to $\sigma_0 \in (\sigma_p, \sigma_q)$ (i.e. $\sigma_i \longrightarrow \sigma_0 \; as \; i \longrightarrow \infty$) and $H(\sigma_0) = 0$ [1]. This completes the proof.

(σ_a, σ_b) and (σ_p, σ_q) are intervals of normality corresponding to equation (6.8) and equation (6.9). This is the so-called interval of normality estimated by above mentioned authors using the Monte carol simulation method.

Furthermore, it follows from equation (6.10), that we can define the functions G and H as such

$$G(\sigma) = \mu - 2z_0 + \sqrt{\mu^2 - 4\sigma^2(n_0-1)} \tag{6.11}$$

$$H(\sigma) = \mu - 2\sigma^2(n_0-1)w_0 + \sqrt{\mu^2 - 4\sigma^2(n_0-1)} \tag{6.12}$$

Also equation (6.11) and equation (6.12) are nonlinear problems of finding the zero(s) of G and H for every given value of μ, which can be solved using any of the iteration formula for finding the zero(s) (i.e. root) of a nonlinear equations [1].

In particular, in equation (6.10), if we take $\mu = 1, n_0 = -2, \frac{-1}{2}$; as assumed by the authors in [10, 11] for a multiplicative time series model. We obtain the corresponding expressions for their y_{max} respectively.

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Difference Sequence Spaces of Second order Defined by a Sequence of Moduli

By Khalid Ebadullah & Kibreab Gebreselassie

Mai Nefhi College of Science

Abstract- In this article we introduce the sequence spaces $c_0(u, \triangle^2, F, p), c(u, \triangle^2, F, p)$ and $\ell_\infty(u, \triangle^2, F, p)$ for $F = (f_k)$ a sequence of moduli, $p = (p_k)$ sequence of positive reals and $u \in U$ the set of all sequences and establish some inclusion relations.

Keywords: *paranorm, sequence of moduli, difference sequence spaces.*

GJSFR-F Classification: *MSC 2010: 11B50*

DIFFERENCESEQUENCESPACESOFSECONDORDERDEFINEDBYASEQUENCEOFMODULI

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Difference Sequence Spaces of Second order Defined by a Sequence of Moduli

Khalid Ebadullah [α] & Kibreab Gebreselassie [σ]

Abstract- In this article we introduce the sequence spaces $c_0(u,\triangle^2,F,p)$, $c(u,\triangle^2,F,p)$ and $\ell_\infty(u,\triangle^2,F,p)$ for $F=(f_k)$ a sequence of moduli, $p=(p_k)$ sequence of positive reals and $u \in U$ the set of all sequences and establish some inclusion relations.

Keywords: paranorm, sequence of moduli, difference sequence spaces.

I. Introduction

Let N, R and C be the sets of all natural, real and complex numbers respectively. We write

$$\omega = \{x=(x_k) : x_k \in R \ or \ C\},$$

the space of all real or complex sequences. Let ℓ_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences respectively.

The following subspaces of ω were first introduced and discussed by Maddox [10-12].

$$l(p) = \{x \in \omega : \sum_k |x_k|^{p_k} < \infty\}$$

$$\ell_\infty(p) = \{x \in \omega : \sup_k |x_k|^{p_k} < \infty\}$$

$$c(p) = \{x \in \omega : \lim_k |x_k - l|^{p_k} = 0, \quad \text{for} \quad \text{some} \ l \in C\}$$

$$c_0(p) = \{x \in \omega : \lim_k |x_k|^{p_k} = 0\}$$

where $p=(p_k)$ is a sequence of strictly positive real numbers.

The idea of difference sequence sets

$$X_\triangle = \{x=(x_k) \in \omega : \triangle x = (x_k - x_{k+1}) \in X\}$$

where $X = \ell_\infty, \ c$ or c_0 was introduced by Kizmaz [6].

Kizmaz [6] defined the following sequence spaces,

Author α σ: Department of Mathematics, Mai Nefhi College of Science, Eritrea. e-mails: khalidebadullah@gmail.com, kgebreselassie@gmail.com

$$\ell_\infty(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in \ell_\infty\}$$

$$c(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c\}$$

$$c_0(\triangle) = \{x = (x_k) \in \omega : (\triangle x_k) \in c_0\}$$

where $\triangle x = (x_k - x_{k+1})$. These are Banach spaces with the norm

$$||x||_\triangle = |x_1| + ||\triangle x||_\infty.$$

Mikail [14] defined the sequence spaces

$$\ell_\infty(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in \ell_\infty\}$$

$$c(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in c\}$$

$$c_0(\triangle^2) = \{x = (x_k) \in \omega : (\triangle^2 x_k) \in c_0\}$$

Where $(\triangle^2 x) = (\triangle^2 x_k) = (\triangle x_k - \triangle x_{k+1})$.

The sequence spaces $\ell_\infty(\triangle^2), c(\triangle^2)$ and $c_0(\triangle^2)$ are Banach spaces with the norm

$$||x||_\triangle = |x_1| + |x_2| + ||\triangle^2 x||_\infty.$$

Mikail and Colak [15] defined the sequence spaces

$$\ell_\infty(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in \ell_\infty\}$$

$$c(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in c\}$$

$$c_0(\triangle^m) = \{x = (x_k) \in \omega : (\triangle^m x_k) \in c_0\}$$

where $m \in N$,

$$\triangle^0 x = (x_k),$$

$$\triangle x = (x_k - x_{k+1}),$$

$$\triangle^m x = (\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}),$$

and so that

$$\triangle^m x_k = \sum_{i=0}^{m} (-1)^i \begin{bmatrix} m \\ i \end{bmatrix} \quad x_{k+i}.$$

Ref

14. Mikail. On some difference sequence spaces, *Dogra-Tr:J:Math.*,17,(1993)18-24.

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and showed that these are Banach spaces with the norm

$$||x||_{\triangle} = \sum_{i=1}^{m} |x_i| + ||\triangle^m x||_{\infty}.$$

Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ $(k = 1, 2, 3....)$. Malkowsky[13] defined the following sequence spaces

$$\ell_{\infty}(u, \triangle) = \{x = (x_k) \in \omega : (u_k \triangle x_k) \in \ell_{\infty}\}$$

$$c(u, \triangle) = \{x = (x_k) \in \omega : (u_k \triangle x_k) \in c\}$$

$$c_0(u, \triangle) = \{x = (x_k) \in \omega : (u_k \triangle x_k) \in c_0\}$$

where $u \in U$.

The concept of paranorm (see[12]) is closely related to linear metric spaces. It is a generalization of that of absolute value.

Let X be a linear space. A function $g : X \longrightarrow R$ is called paranorm, if for all $x, y \in X$,

(PI) $g(x) = 0 \; if \; x = 0$,

(P2) $g(-x) = g(x)$,

(P3) $g(x + y) \leq g(x) + g(y)$,

(P4) If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and $x_n, a \in X$ with $x_n \to a$ $(n \to \infty)$, in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space.

The idea of modulus was structured by Nakano[16].

A function f : $[0,\infty)\longrightarrow[0,\infty)$ is called a modulus if

(P1) $f(\mathrm{t}) = 0$ if and only if t = 0,

(P2) f(t+u)≤ f(t)+ f(u) for all t,u≥0,

(P3) f is increasing, and

(P4) f is continuous from the right at zero.

Ruckle [17-19] used the idea of a modulus function f to construct the sequence space

$$X(f) = \{x = (x_k) : \sum_{k=1}^{\infty} f(|x_k|) < \infty\}$$

Ref

12. Maddox, I.J. Some properties of paranormed sequence spaces, *J. London. Math.* Soc.1 (1969), 316-322.

This space is an FK space. Ruckle[17-19] proved that the intersection of all such $X(f)$ spaces is ϕ, the space of all finite sequences.

The space $X(f)$ is closely related to the space l_1 which is an $X(f)$ space with $f(x) = x$ for all real $x \geq 0$. Thus Ruckle[17-19] proved that, for any modulus f.

$$X(f) \subset l_1 \text{ and } X(f)^\alpha = \ell_\infty$$

The space $X(f)$ is a Banach space with respect to the norm

$$||x|| = \sum_{k=1}^{\infty} f(|x_k|) < \infty.$$

Spaces of the type $X(f)$ are a special case of the spaces structured by Gramsch in[4]. From the point of view of local convexity, spaces of the type $X(f)$ are quite pathological. Symmetric sequence spaces, which are locally convex have been frequently studied by Garling[2-3], Köthe[9] and Ruckle[17-19].

Kolk [7-8] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$ and defined the sequence space

$$X(F) = \{x = (x_k) : (f_k(|x_k|)) \in X\}$$

Khan and Lohani [5] defined the following sequence spaces

$$\ell_\infty(u, \triangle, F) = \{x = (x_k) \in \omega : \sup_{k \geq 0} f_k(|u_k \triangle x_k|) < \infty\}$$

$$c(u, \triangle, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k \triangle x_k - l|) = 0, l \in C\}$$

$$c_0(u, \triangle, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k \triangle x_k|) = 0\}$$

where $u \in U$.

If we take x_k instead of $\triangle x$,then we have the following sequence spaces

$$\ell_\infty(u, F) = \{x = (x_k) \in \omega : \sup_{k \geq 0} f_k(|u_k x_k|) < \infty\}$$

$$c(u, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k x_k - l|) = 0, l \in C\}$$

$$c_0(u, F) = \{x = (x_k) \in \omega : \lim_{k \to \infty} f_k(|u_k x_k|) = 0\}$$

Ref

9. Köthe, G. *Topological Vector spaces*.1.(Springer,Berlin,1970.)

where $u \in U$.

Asma and Colak[1] defined the following sequence spaces

$$\ell_\infty(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|)^{p_k} \in \ell_\infty(p)\}$$

$$c(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|)^{p_k} \in c(p)\}$$

$$c_0(u, \triangle, p) = \{x = (x_k) \in \omega : (|u_k \triangle x_k|)^{p_k} \in c_0(p)\}$$

where $u \in U$, $p = (p_k)$ be any sequence of positive reals.

Khan and Lohani [5] defined the following sequence spaces

$$\ell_\infty(u, \triangle, F, p) = \{x = (x_k) \in \omega : \sup_{k \geq 0} (f_k(|u_k \triangle x_k|))^{p_k} < \infty\}$$

$$c(u, \triangle, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty} (f_k(|u_k \triangle x_k - l|))^{p_k} = 0, l \in C\}$$

$$c_0(u, \triangle, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty} (f_k(|u_k \triangle x_k|))^{p_k} = 0\}$$

which are paranormed spaces paranormed with

$$Q(x) = \sup_{k \geq 0} (f_k(|u_k \triangle x_k|))^{p_k})^{\frac{1}{H}} \leq a$$

where $H = max(1, \sup_{k \geq 0} p_k)$ and $a = f_k(l)$, $l = \sup_{k \geq 0}(|u_k \triangle x_k|)$.

II. Main Results

In this article we introduce the following class of sequence spaces.

$$\ell_\infty(u, \triangle^2, F, p) = \{x = (x_k) \in \omega : \sup_{k \geq 0} (f_k(|u_k \triangle^2 x_k|))^{p_k} < \infty\}$$

$$c(u, \triangle^2, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty} (f_k(|u_k \triangle^2 x_k - l|))^{p_k} = 0, l \in C\}$$

$$c_0(u, \triangle^2, F, p) = \{x = (x_k) \in \omega : \lim_{k \to \infty} (f_k(|u_k \triangle^2 x_k|))^{p_k} = 0\}$$

Theorem 2.1. $\ell_\infty(u, \triangle^2, F)$ is a Banach space with norm

$$||x||_{\triangle^2} = \sup_{k \geq 0} (f_k(|u_k \triangle^2 x_k|)) \leq \alpha,$$

where $\alpha = f_k(l)$ and $l = \sup_{k \geq 0}(|u_k \triangle^2 x_k|)$.

Ref

1. Asma, C., Colak, R. On the Kӧthe-Toeplitz duals of generalised sets of difference sequences. *Demonstratio Math;* 33(4), (2000)797-803.

Proof. Let(x^i) be a cauchy sequence in $\ell_\infty(u, \triangle^2, F)$ for each $i \in N$.
Let r, x_0 be fixed.Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer N such that

$$||x^i - x^j||_{\triangle^2} < \frac{\epsilon}{rx_0} \quad \text{for all i, j} \geq N$$

Using the definition of norm, we get

$$\sup_{k\geq 0} f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{||x^i - x^j||_{\triangle^2}}) \leq \alpha, \quad \text{for all i, j} \geq N$$

ie,

$$f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{||x^i - x^j||_{\triangle^2}}) \leq \alpha, \quad \text{for all i, j} \geq N$$

Hence we can find $r > 0$ with $f_k(\frac{rx_0}{2}) \geq \alpha$ such that

$$f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{||x^i - x^j||_{\triangle^2}}) \leq f_k(\frac{rx_0}{2})$$

$$\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{||x^i - x^j||_{\triangle^2}} \leq \frac{rx_0}{2}$$

This implies that

$$|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)| \leq \frac{rx_0}{2}\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$$

Since $u_k \neq 0$ for all k, we have

$$|\triangle^2 x_k^i - \triangle^2 x_k^j| \leq \frac{\epsilon}{2} \quad \text{for all i, j} \geq N$$

Hence $(\triangle^2 x_k^i)$ is a cauchy sequence in R.
For each $\epsilon > 0$ there exists a positive integer N such that $|\triangle^2 x_k^i - \triangle^2 x_k| < \epsilon$ for all $i > N$.
Using the continuity of $F = (f_k)$ we can show that

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x_k^i - \lim_{j\to\infty} \triangle^2 x_k^j)|) \leq \alpha,$$

Thus

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x_k^i - \triangle^2 x_k)|) \leq \alpha,$$

since $(x^i) \in \ell_\infty(u, \triangle^2, F)$ and $F = (f_k)$ is continuous it follows that $x \in \ell_\infty(u, \triangle^2, F)$
Thus $\ell_\infty(u, \triangle^2, F)$ is complete.

Notes

Notes

Theorem 2.2. $\ell_\infty(u, \triangle^2, F, p)$ is a complete paranormed space with

$$Q_u(x) = \sup_{k\geq 0}(f_k(|u_k\triangle^2 x_k|)^{p_k})^{\frac{1}{H}} \leq \alpha$$

where $H = max(1, \sup_{k\geq 0} p_k)$ and $\alpha = f_k(l)$, $l = \sup_{k\geq 0}(|u_k\triangle^2 x_k|)$.

Proof. Let (x^i) be a cauchy sequence in $\ell_\infty(u, \triangle^2, F, p)$ for each $i \in N$.
Let $r > 0, x_0$ be fixed.Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer N such that

$$Q_u(x^i - x^j)_{\triangle^2} < \frac{\epsilon}{rx_0} \quad \text{for all i, j} \geq N$$

Using the definition of paranorm, we get

$$\sup_{k\geq 0} f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}})^{\frac{p_k}{H}} \leq \alpha, \quad \text{for all i, j} \geq N$$

ie,

$$f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}})^{p_k} \leq \alpha, \quad \text{for all i, j} \geq N$$

Hence we can find $r > 0$ with $f_k(\frac{rx_0}{2}) \geq \alpha$ such that

$$f_k(\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}}) \leq f_k(\frac{rx_0}{2})$$

$$\frac{|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)|}{Q_u(x^i - x^j)_{\triangle^2}} \leq \frac{rx_0}{2}$$

This implies that

$$|u_k(\triangle^2 x_k^i - \triangle^2 x_k^j)| \leq \frac{rx_0}{2}\frac{\epsilon}{rx_0} = \frac{\epsilon}{2}$$

Since $u_k \neq 0$ for all k, we have

$$|\triangle^2 x_k^i - \triangle^2 x_k^j| \leq \frac{\epsilon}{2} \quad \text{for all i, j} \geq N$$

Hence $(\triangle^2 x_k^i)$ is a cauchy sequence in R.
For each $\epsilon > 0$ there exists a positive integer N such that $|\triangle^2 x_k^i - \triangle^2 x_k| < \epsilon$ for all $i > N$.
Using the continuity of $F = (f_k)$ we can show that

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x_k^i - \lim_{j\to\infty} \triangle^2 x_k^j)|)^{\frac{p_k}{H}} \leq \alpha,$$

Thus

$$\sup_{k\geq N} f_k(|u_k(\triangle^2 x_k^i - \triangle^2 x_k)|)^{\frac{p_k}{H}} \leq \alpha,$$

since $(x^i) \in \ell_\infty(u, \triangle^2, F, p)$ and $F = (f_k)$ is continuous it follows that $x \in \ell_\infty(u, \triangle^2, F, p)$ Thus $\ell_\infty(u, \triangle^2, F, p)$ is complete.

Theorem 2.3. Let $0 < p_k \leq q_k < \infty$ for each k. Then we have

$$c_0(u, \triangle^2, F, p) \subseteq c_0(u, \triangle^2, F, q)$$

Proof. Let $x \in c_0(u, \triangle^2, F, p)$ that is

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

This implies that

$$f_k(|u_k(\triangle^2 x_k)|) \leq 1$$

for sufficiently large k, since modulus function is non decreasing.
Hence we get

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{q_k} \leq \lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

Therefore $x \in c_0(u, \triangle^2, F, q)$.

Theorem 2.4. (a) Let $0 < \inf p_k \leq p_k \leq 1$. Then we have

$$c_0(u, \triangle^2, F, p) \subseteq c_0(u, \triangle^2, F).$$

(b) Let $1 \leq p_k \leq \sup_k p_k < \infty$. Then we have

$$c_0(u, \triangle^2, F) \subseteq c_0(u, \triangle^2, F, p).$$

Proof. (a) Let $x \in c_0(u, \triangle^2, F, p)$, that is

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

Since $0 < \inf p_k \leq p_k \leq 1$,

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|)) \leq \lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} = 0$$

Notes

Hence $x \in c_0(u, \triangle^2, F)$.

(b) Let $p_k \geq 1$ for each k and $\sup_k p_k < \infty$.

Suppose that $x \in c_0(u, \triangle^2, F)$.

Then for each $\epsilon > 0$ there exists a positive integer N such that

$$f_k(|u_k(\triangle^2 x_k)|) \leq \epsilon \quad \text{for all } k \geq N$$

Since $1 \leq p_k \leq \sup_k p_k < \infty$, we have

$$\lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|))^{p_k} \leq \lim_{k\to\infty} (f_k(|u_k(\triangle^2 x_k)|)) \leq \epsilon < 1$$

Therefore $x \in c_0(u, \triangle^2, F, p)$.

Notes

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Towards the Efficiency of the Ratio Estimator for Population Median in Survey Sampling

By Matthew Joshua Iseh
Akwa Ibom State University

Abstract- This study leveraged on the fact that researchers in survey sampling sometimes do not take into consideration the tool that will be most appropriate in the measure of location. As a result, users of statistics often go for the mean or total, which has been widely discussed in the finite population sampling literature, unlike the median, which is more complicated to deal with given that it has to do with ordered data. This study suggests an estimator of population median in single and double sampling technique with two auxiliary variables. From the result obtained, it is established that the proposed estimators perform better when the considered variables are from a highly skewed distribution, such as income, expenditure, scores. In addition, it was observed that the proposed estimators were less bias and more efficient than the existing estimators of their class.

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TOWARDSTHEEFFICIENCYOFTHERATIOESTIMATORFORPOPULATIONMEDIANINSURVEYSAMPLING

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Notes

Towards the Efficiency of the Ratio Estimator for Population Median in Survey Sampling

Matthew Joshua Iseh

Absract- This study leveraged on the fact that researchers in survey sampling sometimes do not take into consideration the tool that will be most appropriate in the measure of location. As a result, users of statistics often go for the mean or total, which has been widely discussed in the finite population sampling literature, unlike the median, which is more complicated to deal with given that it has to do with ordered data. This study suggests an estimator of population median in single and double sampling technique with two auxiliary variables. From the result obtained, it is established that the proposed estimators perform better when the considered variables are from a highly skewed distribution, such as income, expenditure, scores. In addition, it was observed that the proposed estimators were less bias and more efficient than the existing estimators of their class.

I. Introduction

Most times in survey sampling, some researchers do not take into consideration the tool that will be most appropriate in the measure of location. As a result, users of Statistics often go for the mean or total which has been widely discussed in the finite population sampling literature unlike the median which is more complicated to deal with since it has to do with ordered data. However, it has been established that the median unlike the mean performs better when the considered variables are from a highly skewed distribution. In surveys involving the estimation of income, expenditure, scores, etc., it is very reasonable to assume that the population median unlike the population mean is known, hence the possibility of incorporating auxiliary information in the formulation of such estimators.

Authors like Gross (1980), Kuk and Mak (1989), Singh et al. (2003), Singh and Solanki (2013), Aladag and Cingi (2015), have made useful contributions in estimation of population median. Works by Enang et.al. (2016) on alternative exponential median estimator, Shabbir, and Gupta (2017) on a generalized difference type estimator for population median and Iseh (2020)on enhancing efficiency of ratio estimator of population median by calibration techniques were added advantage in this area. However, it should also be noted that not much has been done in estimating population median.On further improvement of the median estimator, Singh, Joarder, and Tracy (2001), extended the ratio estimator to two-phase sampling, whileSingh, Singh, and Upadhyaya (2007) suggested the ratio-type estimator in two-phase sampling using two auxiliary variables. In addition, Jhajj, Kaur, and Jhajj (2016), defined a ratio exponential-type estimator in two phase sampling with two auxiliary variables. On the lines of Shabbir and Gupta (2017), Baig, Masood and Tarray (2019) suggested an

Author: Ph.D., Department of Statistics, AkwaIbom State University, MkpatEnin, Nigeria. e-mail: eeseaglechild@gmail.com

improved class of difference-type estimators for population median using two auxiliary variables under simple random scheme and two phase sampling scheme.

So far, the shortcoming of the existing estimators is that, while some of these estimators are less biased with large mean square error (MSE), others are highly biased with less MSE. Based on these developments as a benchmark, this study proposes a separate ratio exponential-type estimator in simple random sampling and two phase sampling schemes with two auxiliary variables that will be more precise with greater gains in efficiency to estimate the median for finite population.

Notes

a) Notations

Consider a finite population $U = \{u_1, u_2, \dots, u_N\}$ with size N. Let Y, X,and Z be the study and auxiliary variables respectively. Let y_i represents the samples of the interest variable and $x_i$and $z_i$represents the samples of the auxiliary variablesknown for every unit in the population for the i^{th} element drawn under SRSWOR. Let $f_Y(M_Y)$,$f_X(M_X)$, and $f_Z(M_Z)$represent the density functions of the random variables with $\widehat{M}_y$,$\widehat{M}_x$, and $\widehat{M}_z$ being the samples from the population median M_Y, M_X and M_Z respectively, with correlation coefficient $\rho_{M_Y M_X} = 4(P_{11} - 0.25)$, where $P_{11} = P(Y \le M_Y \cap X \le M_X)$, $\rho_{M_Y M_Z} = 4(P_{11} - 0.25)$, where $P_{11} = P(Y \le M_Y \cap Z \le M_Z)$, and $\rho_{M_X M_Z} = 4(P_{11} - 0.25)$, where $P_{11} = P(X \le M_X \cap Z \le M_Z)$, (considering the continuous distributions of all variablesy, x, and x with their marginal densities respectively as $N \to \infty$).

For large sample approximations, the following are obtainable:

$$\widehat{M}_y = M_Y(1 + e_0), \qquad \widehat{M}_x = M_X(1 + e_1), \qquad \widehat{M}_z = M_Z(1 + e_2),$$

$$e_0 = \frac{\widehat{M}_y - M_Y}{M_Y} \qquad e_1 = \frac{\widehat{M}_x - M_X}{M_X} \qquad e_2 = \frac{\widehat{M}_z - M_Z}{M_Z}$$

$$E(e_0) = E(e_1) = E(e_2) = 0 \qquad \lambda = \frac{1-f}{4n}$$

$$E(e_0^2) = \lambda C_{M_Y}^2 \qquad E(e_1^2) = \lambda C_{M_X}^2 \qquad E(e_2^2) = \lambda C_{M_Z}^2$$

$$E(e_0 e_1) = \lambda C_{M_Y} C_{M_X} \rho_{M_Y M_X} \qquad E(e_0 e_2) = \lambda C_{M_Y} C_{M_Z} \rho_{M_Y M_Z}$$

$$C_{M_Y} = \{M_Y f_Y(M_Y)\}^{-1} \qquad C_{M_X} = \{M_X f_X(M_X)\}^{-1} \qquad C_{M_{Xh}} = \{M_{Xh} f_X(M_{Xh})\}^{-1}$$

$$k_1 = \frac{C_{M_Y} \rho_{M_Y M_X}}{C_{M_X}} \qquad k_2 = \frac{C_{M_Z} \rho_{M_X M_Z}}{C_{M_X}}$$

where, it is also assumed that the distribution function$f_Y(M_Y)$, $f_X(M_X)$, and $f_X(M_Z)$are nonnegative.

II. Existing Estimators under Simple Random Sampling

This section considers some existing estimators in simple random sampling in estimating population median and the expression of bias and MSE up to the first order approximation as follows;

Notes

A: The median estimator (per unit) due to Gross (1980) is given by

$$\widehat{M}_G = \widehat{M}_y$$

$$var(\widehat{M}_G) = \lambda M_Y^2 C_{M_Y}^2 \qquad (1)$$

B: The median estimator (classical ratio) by Kuk and Mak (1989) is given by

$$\widehat{M}_R = \widehat{M}_y \left(\frac{M_X}{\widehat{M}_x}\right)$$

$$B(\widehat{M}_R) = \lambda M_Y C_{M_X}^2 (1 - k_1)$$

$$MSE(\widehat{M}_R) = \lambda M_Y^2 [C_{M_Y}^2 + C_{M_X}^2 (1 - 2k_1)] \qquad (2)$$

C: The median estimator (exponential ratio) following Bahl and Tuteja (1991) is given by

$$\widehat{M}_{ER} = \widehat{M}_y exp \left[\frac{M_X - \widehat{M}_x}{M_X + \widehat{M}_x}\right]$$

$$B(\widehat{M}_{ER}) = \frac{\lambda M_Y C_{M_X}^2 (3 - 4k_1)}{8}$$

$$MSE(\widehat{M}_{ER}) = \lambda M_Y^2 \left[C_{M_Y}^2 + \frac{C_{M_X}^2}{4} (1 - 4k_1)\right] \qquad (3)$$

D: The median estimator (chain ratio-type) by Kadilar and Cingi (2003) is given by

$$\widehat{M}_{CR} = \widehat{M}_y \left(\frac{M_X}{\widehat{M}_x}\right)^2$$

$$B(\widehat{M}_{CR}) = \lambda M_Y C_{M_X}^2 (1 + 2k_1)$$

$$MSE(\widehat{M}_{CR}) = \lambda M_Y^2 [C_{M_Y}^2 + 4C_{M_X}^2 (1 + k_1)] \qquad (4)$$

E: The median estimator (product-type) following Robson (1957) and Murthy (1964) is given by

$$\widehat{M}_P = \widehat{M}_y \left(\frac{\widehat{M}_x}{M_X}\right)$$

$$B(\widehat{M}_P) = \lambda M_Y C_{M_X}^2 k_1$$

$$MSE(\widehat{M}_P) = \lambda M_Y^2 [C_{M_Y}^2 + C_{M_X}^2 (1 + 2k_1)] \qquad (5)$$

F: The median estimator (exponential product-type) following Bahl and Tuteja (1991) is given by

$$\widehat{M}_{EP} = \widehat{M}_y exp\left[\frac{\widehat{M}_x - M_X}{M_X + \widehat{M}_x}\right]$$

$$B(\widehat{M}_{EP}) = \frac{\lambda M_Y C_{M_X}^2 (4k_1 - 1)}{8}$$

$$MSE(\widehat{M}_{EP}) = \lambda M_Y^2 \left[C_{M_Y}^2 + \frac{C_{M_X}^2}{4}(1 + 4k_1)\right] \quad (6)$$

G: The median estimator (alternative exponential) due to Enang et.al. (2016) is given by

$$\widehat{M}_{AE} = \alpha_1 \left[\widehat{M}_y exp\left[\frac{M_X - \widehat{M}_x}{M_X + \widehat{M}_x}\right]\right] + \alpha_2 \left[\widehat{M}_y exp\left[\frac{\widehat{M}_x - M_X}{M_X + \widehat{M}_x}\right]\right]$$

$$B(\widehat{M}_{AE}) = \lambda M_Y C_{M_X}^2 (4k_1 - 8{k_1}^2 + 1)$$

$$MSE(\widehat{M}_{AE}) = \lambda M_Y^2 C_{M_Y}^2 (1 - {\rho_{M_Y M_X}}^2) \quad (7)$$

H: Shabbir and Gupta (2017) suggested generalized difference-type estimator for population median as

$$\widehat{M}_{PP}^G = [m_1 \widehat{M}_y + m_2(M_X - \widehat{M}_x)]\left[\left(\frac{aM_X + b}{a\widehat{M}_x + b}\right) exp\left\{\frac{\alpha_2 a(M_X - \widehat{M}_x)}{a\{(\gamma - 1)M_X + \widehat{M}_x\} + 2b}\right\}\right]$$

where m_1 and m_2 are unknown constants whose values are to be determined.

Let a and b are defined to be unknown population parameters and α_1, α_2 and γ are scalar quantities which can take different values like $\alpha_1 = b = 0$ and $\alpha_2 = a = \gamma = 1$

At the optimum values of $m_{1(opt)} = \frac{1 - \frac{1}{2}\lambda M_X^2}{1 + \lambda M_Y^2 (1 - {\rho_{M_Y M_X}}^2)}$ and

$$m_{2(opt)} = \frac{M_Y}{M_X}\left[1 + m_{1(opt)}\left\{\frac{\rho_{M_Y M_X} C_{M_Y}}{C_{M_X}} - 2\right\}\right]$$

The expressions for the bias and the mean square error up to the first order of approximation are as follows:

$$B(\widehat{M}_{PP}^G) \cong (m_{1(opt)} - 1)M_Y + m_{2(opt)}\left\{\lambda M_Y\left(\frac{3}{8}C_{M_X}^2 - C_{M_Y}\right) + \lambda M_X C_{M_X}^2\right\} \text{ and}$$

$$MSE(\widehat{M}_{PP}^G) \cong \frac{\lambda M_Y^2}{1 + \lambda M_Y^2 (1 - {\rho_{M_Y M_X}}^2)}\left[C_{M_Y}^2 \left(1 - {\rho_{M_Y M_X}}^2\right)(1 - \lambda C_{M_X}^2) - \frac{1}{4}C_{M_X}^4\right] \quad (8)$$

Notes

I: Baig, Masood and Tarray (2019) suggested an improved class of difference-type estimators for population median using two auxiliary variables

$$\widehat{M}_P^I = [\widehat{M}_y + m_1(M_X - \widehat{M}_x)]\left[m_2 exp\left(\frac{M_Z - \widehat{M}_z}{M_Z + \widehat{M}_z}\right) + (1 - m_2)exp\left(\frac{\widehat{M}_z - M_Z}{M_Z + \widehat{M}_z}\right)\right]$$

Where $m_1$and m_2 are unknown constant.

$$B(\widehat{M}_P^I) = \lambda\left[m_1 M_X C_{M_{XZ}}\left(m_2 - \frac{1}{2}\right) + M_Y C_{M_{YZ}}\left(\frac{1}{2} - m_2\right)\right] M_Y$$

where $m_{1(opt)} = \frac{M_Y C_{M_Y}(\rho_{M_X M_Z}\rho_{M_Y M_Z} - \rho_{M_Y M_X})}{M_X C_{M_X}\left(1-\rho_{M_X M_Z}^2\right)}$ and

$$m_{2(opt)} = \frac{C_{M_Z}\left(\rho_{M_X M_Z}^2 - 1\right) + 2C_{M_Y}(\rho_{M_X M_Z}\rho_{M_Y M_X} - \rho_{M_Y M_Z})}{2C_{M_Z}\left(\rho_{M_X M_Z}^2 - 1\right)}$$

$$MSE(\widehat{M}_P^I) = \frac{\lambda M_Y^2 C_{M_Y}^2}{\left(1-\rho_{M_X M_Z}{}^2\right)}\left[\left(1 - \rho_{M_X M_Z}^2 - \rho_{M_Y M_X}^2 - \rho_{M_Y M_Z}^2 + 2\rho_{M_X M_Z}\rho_{M_Y M_X}\rho_{M_Y M_Z}\right)\right] \quad (9)$$

III. The Proposed Estimator under Simple Random Sampling with One Variable

$$\widehat{M}_{srs}(\alpha) = \widehat{M}_y\left[\alpha\frac{M_X}{\widehat{M}_x} + (1-\alpha)\frac{\widehat{M}_x}{M_X}\right] exp\left[\frac{(M_X - \widehat{M}_x)}{(M_X + \widehat{M}_x)}\right] \quad (10)$$

$$\widehat{M}_{srs}(\alpha) = M_Y(1 + e_0)[1 + e_1 - 2\alpha e_1 + \alpha e_1^2] exp\left\{\frac{-e_1}{2}\left[1 - \frac{e_1}{2} + \frac{e_1^2}{4}\right]\right\}$$

$$\widehat{M}_{srs}(\alpha) = M_Y\left[1 + e_0 + \left(\frac{1}{2} - 2\alpha\right)e_1 + \left(\frac{1}{2} - 2\alpha\right)e_0 e_1 + \left(2\alpha - \frac{1}{8}\right)e_1^2\right]$$

$$\widehat{M}_{srs}(\alpha) - M_Y = M_Y\left[e_0 + \left(\frac{1}{2} - 2\alpha\right)e_1 + \left(\frac{1}{2} - 2\alpha\right)e_0 e_1 + \left(2\alpha - \frac{1}{8}\right)e_1^2\right] \quad (11)$$

$$Bias\left(\widehat{M}_{srs}(\alpha)\right) = M_Y\left[\left(2\alpha - \frac{1}{8}\right)\lambda C_{M_X}^2 + \left(\frac{1}{2} - 2\alpha\right)\lambda C_{M_Y} C_{M_X}\rho_{M_Y M_X}\right] \quad (12)$$

Squaring both sides of (11), retaining terms to the second-degree and taking expectations, the MSE of $\widehat{M}_{st}(\alpha)$ to the first order of approximation is obtained as;

$$MSE\left(\widehat{M}_{srs}(\alpha)\right) = M_Y^2\left[\lambda C_{M_Y}^2 + \left(\frac{1}{2} - 2\alpha\right)^2\lambda C_{M_X}^2 + 2\left(\frac{1}{2} - 2\alpha\right)\lambda C_{M_Y} C_{M_X}\rho_{M_Y M_X}\right] \quad (13)$$

minimizing (13) with respect to α gives

$$\alpha = \frac{2k_1 + 1}{4}$$

$$MSE_{opt}\left(\widehat{M}_{srs}(\alpha)\right) = \lambda M_Y^2 C_{M_Y}^2(1 - \rho_{M_Y M_X}^2) \quad (14)$$

And the optimum bias becomes

$$Bias_{opt}\left(\widehat{M}_{srs}(\alpha)\right) = \lambda M_Y\left[\frac{3}{8}C_{M_X}^2 - C_{M_Y}^2\rho_{M_YM_X}^2 + C_{M_Y}C_{M_X}\rho_{M_YM_X}\right] \qquad (15)$$

a) A Case of Two Variables under Simple Random Sampling

$$\widehat{M}_{srs}^*(\alpha) = \widehat{M}_y\left[\alpha\frac{M_X}{\widehat{M}_x} + (1-\alpha)\frac{\widehat{M}_x}{M_X}\right]exp\left[\frac{(\widehat{M}_z - M_Z)}{(M_Z + \widehat{M}_z)}\right] \qquad (16)$$

$$Bias\left(\widehat{M}_{srs}^*(\alpha)\right) = \lambda M_Y\left[\alpha C_{M_X}^2 + \frac{3}{8}C_{M_Z}^2 + (1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_XM_Y} - (1-2\alpha)\frac{C_{M_X}C_{M_Z}\rho_{M_XM_Z}}{2} - \frac{C_{M_Y}C_{M_Z}\rho_{M_YM_Z}}{2}\right] \qquad (17)$$

$$MSE\left(\widehat{M}_{srs}^*(\alpha)\right) = \lambda M_Y^2\left[C_{M_Y}^2 + (1-2\alpha)^2C_{M_X}^2 + \frac{C_{M_Z}^2}{4} - (1-2\alpha)C_{M_X}C_{M_Z}\rho_{M_XM_Z} + 2(1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_XM_Y} - C_{M_Y}C_{M_Z}\rho_{M_YM_Z}\right] \qquad (18)$$

Minimizing (18) with respect to α gives $\alpha = \frac{2C_{M_X}^2 + 2C_{M_X}C_{M_Y}\rho_{M_XM_Y} - C_{M_X}C_{M_Z}\rho_{M_XM_Z}}{4C_{M_X}^2}$

Substituting into (18) gives

$$MSE_{opt}\left(\widehat{M}_{srs}^*(\alpha)\right) = \lambda M_Y^2\left[C_{M_Y}^2\left(1-\rho_{M_YM_X}^2\right) + \frac{C_{M_Z}^2}{4}\left(1-\rho_{M_XM_Z}^2\right) + C_{M_Y}C_{M_Z}\left(\rho_{M_XM_Z}\rho_{M_YM_X} - \rho_{M_YM_Z}\right)\right]$$

$$MSE_{opt}\left(\widehat{M}_{srs}^*(\alpha)\right) = \lambda M_Y^2\left[C_{M_Y}^2 + \frac{C_{M_Z}^2}{4} - \left(\frac{k_2}{2} - k_1\right)^2 - C_{M_Y}C_{M_Z}\rho_{M_YM_Z}\right] \qquad (19)$$

And the minimum bias given as

$$Bias_{opt}\left(\widehat{M}_{srs}^*(\alpha)\right) = \lambda M_Y\left[\frac{C_{M_X}^2}{2} + \frac{3}{8}C_{M_Z}^2 + \frac{C_{M_X}C_{M_Y}\rho_{M_XM_Y}}{2} - \frac{C_{M_X}C_{M_Z}\rho_{M_XM_Z}}{4} - C_{M_Y}^2\rho_{M_XM_Y}^2 - \frac{C_{M_Z}^2\rho_{M_XM_Z}^2}{4} - \frac{C_{M_Y}C_{M_Z}\rho_{M_YM_Z}}{2} + C_{M_Y}C_{M_Z}\rho_{M_XM_Z}\rho_{M_XM_Y}\right] \qquad (20)$$

b) Application

The bias and MSE values of the existing and proposed estimators are computed using two different populations under simple random sampling. The percent relative efficiencies of the estimators are obtained as follows:

$$\%RE = \frac{MSE(\widehat{M}_G)}{MSE(.)} \times 100$$

where $MSE(\widehat{M}_G)$ is the MSE of classical median estimator while $MSE(.)$ denotes the MSE of estimators mentioned here. The population statistics and the results of analyses are shown as follows:

Notes

Population 1: Let Population 1: Let y; x and z respectively be the number of fish caught by the marine recreational fisherman in years 1995, 1994 and 1993 in USA given by Singh (2003a)

$N = 69$; $n = 17$; $M_Y = 2068$; $M_X = 2011$; $M_Z = 2307$; $\rho_{M_X M_Y} = 0.1505$; $\rho_{M_X M_Z} = 0.1431$; $\rho_{M_Y M_Z} = 0.3166$; $f_Y(M_Y) = 0.00014$; $f_X(M_X) = 0.00014$; and $f_Z(M_Z) = 0.00013$.

Notes

Population 2: Let y as the U.S. exports to Singapore in billions of Singapore dollars, x as the money supply figures in billions of Singapore dollars and z is the local supply in U.S. dollars given by Aczel and Sounderpandian (2004).

$N = 67$; $n = 23$; $M_Y = 4.8$; $M_X = 7.0$;$M_Z = 151$; $\rho_{M_X M_Y} = 0.6624$; $\rho_{M_X M_Z} = 0.7592$; $\rho_{M_Y M_Z} = 0.8624$;$f_Y(M_Y) = 0.0763$;$f_X(M_X) = 0.0526$; and$f_Z(M_Z) = 0.0024$;

Table 1: Results for Simple Random Sampling

Estimator	Population I			Population II		
	Absolute bias	MSE	PRE	Absolute bias	MSE	PRE
$\widehat{M}_G$	0	565443.6	100	0	1.23	100
$\widehat{M}_R$	246.3	988372.8	57.2	0.08	0.82	150
$\widehat{M}_{ER}$	87.27	627420.2	90.1	0.01	0.72	170.8
$\widehat{M}_{CR}$	373.78	3307296	17.1	0.59	9.31	13.2
$\widehat{M}_P$	42.32	1338418	42.2	0.17	4.06	30.3
$\widehat{M}_{EP}$	14.98	802442.8	70.5	0.05	2.34	52.6
$\widehat{M}_{AE}$	408.87	552636.1	102.3	0.03	0.69	178.3
$\widehat{M}_{PP}^G$	3373.88	491568.8	115.0	1.54	0.52	236.5
$\widehat{M}_P^I$	46089.73	502378.1	112.6	0.48	0.31	396.8
$\widehat{M}_{srs}(\alpha)$	144.55	552636.1	102.3	0.15	0.69	178.3
$\widehat{M}_{srs}^*(\alpha)$	207.98	520612.8	108.6	0.13	0.38	323.7

c) *Median Estimators under Two Phase Sampling*

Consider a finite population with Nunits $U = \{u_1, u_2, \ldots, u_N\}$. The$i^{th}$ unit ofthe population values forthe study variable y, auxiliary variables x and z are y_i; x_i andz_i respectively. Under two-phase sampling design, a sample of size $n'$is drawn using simple random sampling without replacement at first phase and the values on x and$z$are obtained on the units of the sample. In second phase, a simple random samplingwithout replacement criteria is used for drawing sample of size n from the first phasesample and the values on the variables y; x and z are taken on selected units.To obtain the properties of the proposed median estimator under two-phase samplingscheme the following existing estimators are summarized as;

i) Singh, Joarder, and Tracy suggested a ratio estimator for median in two phase Sampling

$$\widehat{M}_{SA} = \frac{\widehat{M}_y}{\widehat{M}_x}\widehat{M}_x'$$

$$B(\widehat{M}_{SA}) = \left(\frac{1}{n} - \frac{1}{n'}\right)\frac{(1 - \rho_{M_X M_Y})}{4 f_Y(M_Y}$$

$$MSE(\widehat{M}_{SA}) = \frac{\{f_Y(M_Y)\}^{-2}}{4}\left[\left(\frac{1}{n}-\frac{1}{N}\right)+\left(\frac{1}{n}-\frac{1}{n'}\right)\left(\frac{M_Y f_Y(M_Y)}{M_X f_X(M_X)}\right)\left\{\left(\frac{\{M_Y f_Y(M_Y)\}}{\{M_X f_X(M_X)\}}\right)-2\rho_{M_X M_Y}\right\}\right] \quad (21)$$

Singh, Singh, and Upadhyay (2007) studied a ratio-type estimator of median using two auxiliary variables

$$\widehat{M}_S = \widehat{M}_y\left(\frac{\widehat{M}'_x}{\widehat{M}_x}\right)^{\alpha_1}\left(\frac{M_Z}{\widehat{M}'_z}\right)^{\alpha_2}\left(\frac{M_Z}{\widehat{M}_z}\right)^{\alpha_3}$$

Where

$$\alpha_{1opt} = \left(\frac{\{M_X f_X(M_X)\}}{\{M_Y f_Y(M_Y)\}}\right)\left(\frac{\rho_{M_X M_Z}\rho_{M_Y M_Z}-\rho_{M_Y M_X}}{\rho^2_{M_X M_Z}-1}\right),$$

$$\alpha_{2opt} = \left(\frac{\{M_Z f_Z(M_Z)\}}{\{M_Y f_Y(M_Y)\}}\right)\rho_{M_X M_Z}\left(\frac{\rho_{M_X M_Z}\rho_{M_Y M_Z}-\rho_{M_Y M_X}}{\rho^2_{M_X M_Z}-1}\right),$$

$$\alpha_{3opt} = \left(\frac{\{M_Z f_Z(M_Z)\}}{\{M_Y f_Y(M_Y)\}}\right)\left(\frac{\rho_{M_X M_Z}\rho_{M_Y M_X}-\rho_{M_Y M_Z}}{\rho^2_{M_X M_Z}-1}\right)$$

$$B(\widehat{M}_S) \cong \frac{\{f_Y(M_Y)\}^{-2}}{8M_Y\left(1-\rho^2_{M_X M_Z}\right)^2}\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\left(1-\rho^2_{M_X M_Z}\right)\left\{\left(\rho_{M_Y M_X}-\rho_{M_X M_Z}\rho_{M_Y M_Z}\right)^2\right.\right.$$

$$-2\rho_{M_X M_Y}\left(\rho_{M_Y M_X}-\rho_{M_X M_Z}\rho_{M_Y M_Z}\right)+\left(\frac{\{M_Y f_Y(M_Y)\}}{\{M_X f_X(M_X)\}}\right)\left(\rho_{M_Y M_X}-\rho_{M_X M_Z}\rho_{M_Y M_Z}\right)\Big\}$$

$$+\left(\frac{1}{n}-\frac{1}{N}\right)\left(\rho_{M_Y M_Z}-\rho_{M_X M_Z}\rho_{M_Y M_X}\right)\Big\{\left(\rho_{M_Y M_Z}-\rho_{M_X M_Z}\rho_{M_Y M_X}\right)$$

$$+2\rho_{M_X M_Z}\left(\rho_{M_Y M_X}-\rho_{M_X M_Z}\rho_{M_Y M_Z}\right)+\left(\frac{\{M_Y f_Y(M_Y)\}}{\{M_X f_X(M_X)\}}\right)\left(1-\rho^2_{M_X M_Z}\right)\Big\}$$

$$+\left(\frac{1}{n'}-\frac{1}{N}\right)\left(\rho_{M_Y M_X}-\rho_{M_X M_Z}\rho_{M_Y M_Z}\right)\Big\{\rho^2_{M_X M_Z}\left(\rho_{M_Y M_X}-\rho_{M_X M_Z}\rho_{M_Y M_Z}\right)$$

$$\left.\left.-2\rho_{M_Y M_Z}\rho_{M_X M_Z}\left(1-\rho^2_{M_X M_Z}\right)+\left(\frac{\{M_Y f_Y(M_Y)\}}{\{M_X f_X(M_X)\}}\right)\rho_{M_X M_Z}\left(1-\rho^2_{M_X M_Z}\right)\right\}\right]$$

$$MSE(\widehat{M}_S) \cong \frac{\{f_Y(M_Y)\}^{-2}}{4}\left[\left(\frac{1}{n}-\frac{1}{N}\right)-\left(\frac{1}{n'}-\frac{1}{N}\right)\rho^2_{M_X M_Z}-\left(\frac{1}{n}-\frac{1}{n'}\right)\frac{\rho^2_{M_Y M_X}+\rho^2_{M_Y M_Z}-2\rho_{M_Y M_X}\rho_{M_X M_Z}\rho_{M_Y M_Z}}{\left(1-\rho^2_{M_X M_Z}\right)}\right] \quad (22)$$

ii) Jhajj, Kaur, and Jhajj (2016) defined ratio-exponential-type estimator as

$$\widehat{M}_{YH} = \widehat{M}_y\left(\frac{M_Z}{\widehat{M}'_z}\right)^{v_1}\left(\frac{M_Z}{\widehat{M}_z}\right)^{v_2} exp\left[\left(\frac{v_3\left(\widehat{M}_x - M_X\right)}{\left(M_X+\widehat{M}_x\right)}\right)\right]$$

Notes

Where $v_{1opt} = \left(\frac{\{M_Z f_Z(M_Z)\}}{\{M_Y f_Y(M_Y)\}}\right) \rho_{M_X M_Z} \left(\frac{\rho_{M_X M_Z}\rho_{M_Y M_Z} - \rho_{M_Y M_X}}{\rho^2_{M_X M_Z} - 1}\right)$,

$$v_{2opt} = \left(\frac{\{M_Z f_Z(M_Z)\}}{\{M_Y f_Y(M_Y)\}}\right)\left(\frac{\rho_{M_X M_Z}\rho_{M_Y M_X} - \rho_{M_Y M_Z}}{\rho^2_{M_X M_Z} - 1}\right), \quad v_{3opt} = \left(\frac{2\{M_X f_X(M_X)\}}{\{M_Y f_Y(M_Y)\}}\right)\left(\frac{\rho_{M_X M_Z}\rho_{M_Y M_Z} - \rho_{M_Y M_X}}{\rho^2_{M_X M_Z} - 1}\right)$$

Notes

$$\begin{aligned} B(\widehat{M}_{YH}) &\cong \frac{\{f_Y(M_Y)\}^{-2}}{8M_Y\left(1-\rho^2_{M_X M_Z}\right)^2}\left[\left(\frac{1}{n}-\frac{1}{n'}\right)\left\{\left(\rho_{M_Y M_X} - \rho_{M_X M_Z}\rho_{M_Y M_Z}\right)^2\right.\right. \\ &- 2\rho_{M_X M_Y}\left(\rho_{M_Y M_X} - \rho_{M_X M_Z}\rho_{M_Y M_Z}\right)\left(1-\rho^2_{M_X M_Z}\right) \\ &+ \left.\left(\frac{\{M_Y f_Y(M_Y)\}}{\{M_X f_X(M_X)\}}\right)\left(\rho_{M_Y M_X} - \rho_{M_X M_Z}\rho_{M_Y M_Z}\right)\left(1-\rho^2_{M_X M_Z}\right)\right\} \\ &+ \left(\frac{1}{n}-\frac{1}{N}\right)\left(\rho_{M_Y M_Z} - \rho_{M_X M_Z}\rho_{M_Y M_X}\right)\left\{\left(\rho_{M_Y M_Z} - \rho_{M_X M_Z}\rho_{M_Y M_X}\right)\right. \\ &+ 2\rho_{M_X M_Z}\left(\rho_{M_Y M_X} - \rho_{M_X M_Z}\rho_{M_Y M_Z}\right) - 2\rho_{M_Y M_Z}\left(\rho_{M_Y M_Z} - \rho_{M_X M_Z}\rho_{M_Y M_X}\right)\left(1-\rho^2_{M_X M_Z}\right) \\ &+ \left.\left(\frac{\{M_Y f_Y(M_Y)\}}{\{M_Z f_Z(M_Z)\}}\right)\left(\rho_{M_Y M_Z} - \rho_{M_X M_Z}\rho_{M_Y M_X}\right)\left(1-\rho^2_{M_X M_Z}\right)\right\} \\ &+ \left(\frac{1}{n'}-\frac{1}{N}\right)\left(\rho_{M_Y M_X} - \rho_{M_X M_Z}\rho_{M_Y M_Z}\right)\rho_{M_X M_Z}\left\{\rho_{M_X M_Z}\left(\rho_{M_Y M_X} - \rho_{M_X M_Z}\rho_{M_Y M_Z}\right)\right. \\ &- \left.2\rho_{M_Y M_Z}\left(1-\rho^2_{M_X M_Z}\right)\right\} \\ &+ \left.\left(\frac{\{M_Y f_Y(M_Y)\}}{\{M_Z f_Z(M_Z)\}}\right)\rho_{M_X M_Z}\left(\rho_{M_Y M_X} - \rho_{M_X M_Z}\rho_{M_Y M_Z}\right)\left(1-\rho^2_{M_X M_Z}\right)\right] \end{aligned}$$

$$MSE(\widehat{M}_{YH}) \cong \frac{\{f_Y(M_Y)\}^{-2}}{4}\left[\left(\frac{1}{n}-\frac{1}{N}\right) - \left(\frac{1}{n'}-\frac{1}{N}\right)\rho^2_{M_Y M_Z} - \left(\frac{1}{n}-\frac{1}{n'}\right)\frac{\rho^2_{M_Y M_X} + \rho^2_{M_Y M_Z} - 2\rho_{M_Y M_X}\rho_{M_X M_Z}\rho_{M_Y M_Z}}{\left(1-\rho^2_{M_X M_Z}\right)}\right] \quad (23)$$

iii) Baig, Masood and Tarray (2019) suggested an improved class of difference-type estimators for population median under two phase sampling with two auxiliary variables

$$\widehat{M}^{I}_{P} = \left[\widehat{M}_y + m_1\left(\widehat{M}'_x - \widehat{M}_x\right)\right]\left[m_2 exp\left(\frac{M_Z - \widehat{M}'_z}{M_Z + \widehat{M}'_z}\right) + (1-m_2) exp\left(\frac{\widehat{M}'_z - M_Z}{M_Z + \widehat{M}'_z}\right)\right]$$

where m_1 and m_2 are unknown constant.

$$B\left(\widehat{M}^{I}_{P}\right) = M_Y \frac{1}{4}\left(\frac{1}{n}-\frac{1}{N}\right)\left(\frac{1}{2} - m_2\right)\rho_{M_Y M_Z} C_{M_Y} C_{M_Z}$$

Where $m_{1(opt)} = \frac{M_Y C_{M_Y}\rho_{M_Y M_X}}{M_X C_{M_X}}$ and $m_{2(opt)} = \frac{1}{2} + \frac{M_Y \rho_{M_Y M_Z}}{C_{M_Z}}$

$$MSE\left(\widehat{M}_P^I\right) = M_Y^2 \frac{C_{M_Y}^2}{4}\left[\left(\frac{1}{n'}-\frac{1}{N}\right)+\left(\frac{1}{n}-\frac{1}{n'}\right)\rho_{M_Y M_X}^2-\left(\frac{1}{n}-\frac{1}{N}\right)\rho_{M_Y M_Z}^2\right] \quad (24)$$

d) Proposed Median Estimator Under Two Phase Sampling

$$\widehat{M}_{srs}^D(\alpha) = \widehat{M}_y\left[\alpha\frac{\widehat{M}_x'}{\widehat{M}_x}+(1-\alpha)\frac{\widehat{M}_x}{\widehat{M}_x'}\right]exp\left[\frac{(M_Z-\widehat{M}_z')}{(M_Z+\widehat{M}_z')}\right] \quad (25)$$

$$\widehat{M}_{srs}^D(\alpha) = M_Y(1+e_0)[\alpha(1+e_1')(1+e_1)^{-1}+(1-\alpha)(1+e_1)(1+e_1')^{-1}]exp\left\{\frac{-e_2'}{2}\left[1-\frac{e_2'}{2}+\frac{e_2'^2}{4}\right]\right\}$$

$$Bias\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y\left[\alpha(\lambda-\lambda_1)C_{M_X}^2+\frac{3}{8}\lambda_1 C_{M_Z}^2+(\lambda-\lambda_1)(1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_X M_Y}-\lambda_1\frac{C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}}{2}\right] \quad (26)$$

$$MSE\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y^2\left[(\lambda-\lambda_1)(4\alpha^2-4\alpha+1)C_{M_X}^2+\lambda_1\frac{C_{M_Z}^2}{4}+\lambda C_{M_Y}^2+2(\lambda-\lambda_1)(1-2\alpha)C_{M_X}C_{M_Y}\rho_{M_X M_Y}-\lambda_1 C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}\right] \quad (27)$$

For optimum value of the MSE of $\widehat{M}_{srs}^D(\alpha)$, the value of α is given as $\alpha=\frac{k_1+1}{2}$, then (27) becomes

$$MSE_{opt}\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y^2\left[\lambda C_{M_Y}^2-(\lambda-\lambda_1)C_{M_Y}^2\rho_{M_X M_Y}^2+\lambda_1\frac{C_{M_Z}^2}{4}-\lambda_1 C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}\right] \quad (28)$$

And the optimum bias becomes

$$Bias_{opt}\left(\widehat{M}_{srs}^D(\alpha)\right) = M_Y\left[(\lambda-\lambda_1)\frac{C_{M_X}^2}{2}+(\lambda-\lambda_1)\frac{C_{M_X}C_{M_Y}\rho_{M_X M_Y}}{2}+\frac{3}{8}\lambda_1 C_{M_Z}^2-(\lambda-\lambda_1)C_{M_Y}^2\rho_{M_X M_Z}^2-\lambda_1\frac{C_{M_Y}C_{M_Z}\rho_{M_Y M_Z}}{2}\right] \quad (29)$$

e) Numerical study under two-phase sampling

Here three different populations will be considered to validate the theoretical claims of both the existing and the proposed estimators. The population statistics and the results of analyses are shown as follows:

Population 3: Let Population 1: Let y; x and z respectively be the number of fish caught by the marine recreational fisherman in years 1995, 1994 and 1993 in USA given by Singh (2003a).

$N = 69; n' = 24 n = 17$; $M_Y = 2068$; $M_X = 2011$; $M_Y = 2307$; $\rho_{M_X M_Y} = 0.1505$; $\rho_{M_X M_Z} = 0.1431$; $\rho_{M_Y M_Z} = 0.3166$; $f_Y(M_Y) = 0.00014$; $f_X(M_X) = 0.00014$; and $f_Z(M_Z) = 0.00013$.

Notes

Notes

Population 4: Let y as the U.S. exports to Singapore in billions of Singapore dollars, x as the money supply figures in billions of Singapore dollars and z is the local supply in U.S. dollars given by Aczel and Sounder pandian (2004).

$N = 67; n^{'} = 23; n = 15;\ M_Y = 4.8;\ M_X = 70; M_Z = 151;\ \rho_{M_X M_Y} = 0.6624;\ \rho_{M_X M_Z} = 0.7592;\ \rho_{M_Y M_Z} = 0.8624; f_Y(M_Y) = 0.0763; f_X(M_X) = 0.0526;$ and $f_Z(M_Z) = 0.0024;$

Population 5: Let y be the district-wise tomato production (tones) in 2003, x as a district-wise tomato production (tones) in 2002 and z as a district-wise tomato production (tones) in 2001 given by MFA (2004).

$N = 97;\ n^{'} = 46; n = 33;\ M_Y = 1242;\ M_X = 1233; M_Z = 1207;\ \rho_{M_X M_Y} = 0.2096;\ \rho_{M_X M_Z} = 0.15;\ \rho_{M_Y M_Z} = 0.123; f_Y(M_Y) = 0.00021; f_X(M_X) = 0.0002;$ and $f_Z(M_Z) = 0.0002;$

From the numerical study of three real life data sets, the following remarks are deduced;

Table 2: Results for Two Phase Sampling

	Population III			Population IV			Population V		
Estimator	AB	MSE	PRE	AB	MSE	PRE	AB	MSE	PRE
$\widehat{M}_G$	0	346606	100	0	1.23	100	0	64795.1	100
$\widehat{M}_{SA}$	0	729125.3	47.54	0.00	1.89	65.08	0	146126.9	44.3
$\widehat{M}_S$	58.72	391934.3	88.43	0.35	-85.12	-1.45	8.49	69573.62	93.1
$\widehat{M}_{YH}$	259.20	363711.8	95.30	0.09	-85.60	-1.44	286.51	70062.2	92.5
$\widehat{M}_P^I$	32818.5	294885.2	117.54	1.21	0.01	12300	894.49	65213.1	99.4
$\widehat{M}_{srs}^D(\alpha)$	8.20	310481.2	111.64	0.07	0.29	424.14	8.57	84943.4	76.3

N/B: AB=Absolute bias

IV. Discussion

From the result in Table 1, it is observed that the proposed estimator $\widehat{M}_{srs}^*(\alpha)$ has outperformed other existing estimators considered in this study and as a result considered the more preferred estimator with respect to this set of data. However, the existing estimator $\widehat{M}_P^I$ had greater efficiency but heavily biased in both population I and II respectively. Again, as shown in Table 2, the proposed estimator $\widehat{M}_{srs}^D(\alpha)$has an overwhelming performance in terms of higher gains in efficiency and minimum bias as compared to the existing estimators considered in this study. This superiority in the gain in efficiency and minimum bias of the proposed estimator is because of the endearing properties of the separate ratio product–type estimator. However, $\widehat{M}_P^I$, notice ably from population III, IV, and V had slight gain in efficiency over the proposed estimator but highly biased, and not recommended for estimation of population median.

It is important to note that, although the existing estimator $\widehat{M}_{SA}$ is asymptotically unbiased from the sets of data considered in this study, it was less efficient, and should not form a bet for median estimation. It becomes imperative to seek for an estimator that isless bias with minimum MSEthat will enhance the improvement of the estimation of the median of a finite population of which the proposed estimator has bridged the gap.

V. Conclusion

This study was towards formulating an improved exponential-type estimator for population median using both simple random and two-phase sampling with two auxiliary variables. The bias and mean square error were obtained. From the theoretical derivations and numerical illustrations, it is evident to say that among the existing and proposed estimators, the proposed estimators under simple random sampling and two-phase sampling exhibit superior performance both in negligible bias and in gain in efficiency. It suffices to conclude that the proposed estimators perform better than existing estimators considered in this work in estimating population median when the population median of the auxiliary variables are known and positively correlated with the study variable. However, though the suggested estimator performed poorly with less efficiency in population V, probably because of the data type, it was still shown to have less bias compared to other estimators of its class. The idea of using separate ratio-product exponential-type estimator in this case has really paid off in improving on the efficiency of the median estimator under stratified random sampling. In addition, the proposed estimator will be suitable and highly recommended when the variable considered is from a distribution that is highly skewed.

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Half-Step Implicit Linear Multistep Hybrid Block Third Derivative Methods of order Four for the Solution of Third order Ordinary Differential Equations

By Skwame, Yusuf, Raymond, Dominic, Tumba, Pius & Adiku, Lydia
Adamawa State University

Abstract- This paper proposes a half-step third derivative hybrid block method with two cases of order four for solution of third Order Ordinary Differential Equations. Method of interpolation and collocation of power series approximate solution was used to generate the continuous hybrid linear multistep method, which was then evaluated at non-interpolating points to give a continuous block method. The discrete block method was recovered when the continuous block was evaluated at all step points. The basic properties of the methods were investigated and were found to be zero-stable, consistent and convergent. The develop half-step method is applied to solve some third order initial value problems of ordinary differential equations and from the numerical results obtained, it is observed that our methods gives better approximation than the existing method compared with.

Keywords: *half-step, hybrid block method, third derivative.*

GJSFR-F Classification: *MSC 2010: 33E30*

Notes

Half-Step Implicit Linear Multistep Hybrid Block Third Derivative Methods of order Four for the Solution of Third order Ordinary Differential Equations

Skwame, Yusuf [α], Raymond, Dominic [σ], Tumba, Pius [ρ] & Adiku, Lydia [ω]

Abstract- This paper proposes a half-step third derivative hybrid block method with two cases of order four for solution of third Order Ordinary Differential Equations. Method of interpolation and collocation of power series approximate solution was used to generate the continuous hybrid linear multistep method, which was then evaluated at non-interpolating points to give a continuous block method. The discrete block method was recovered when the continuous block was evaluated at all step points. The basic properties of the methods were investigated and were found to be zero-stable, consistent and convergent. The develop half-step method is applied to solve some third order initial value problems of ordinary differential equations and from the numerical results obtained, it is observed that our methods gives better approximation than the existing method compared with.

Keywords: half-step, hybrid block method, third derivative.

I. Introduction

This paper consider third order initial value problems of the form

$$y''' = f\left(x, y(x), y'(x)\right),\ y(x_0) = y_0,\ y'(x_0) = y'_0 \tag{1}$$

Where f is continuous within the interval of integration, solving higher order derivatives method by reducing them to a system of first-order approach involves more functions to evaluate which then leads to a computational burden as in Kayode and Obuarhua (2013) and James et al. (2013). Various approaches have been proposed to find the analytic solution of (1) ranging from predictor-corrector method to hybrid methods. Despite the success recorded by the predictor-corrector methods, its major setback is that the predictor are in reducing order of accuracy especially when the value of the step-length is high and moreover the result are at overlapping interval. The hybrid method was established to circumvent the Dahlquist barrier theorem which gives a better approximation to solutions of initial value problems of stiff ordinary differential equations than the k-step method.

Scholars who recently adopted the hybrid method other than the direct method in approximating (1) include among others Adesanya et al. (2013), Adebayo and

Author α: Department of Mathematical Sciences, Faculty of Science, Adamawa State University, Mubi-Nigeria
Author σ ω: Department of Mathematics and Statistics, Faculty of Pure and Applied Sciences, Federal University Wukari Nigeria.
e-mails: domehbazza@yahoo.co.uk, raymond@fuwukari.edu.ng
Author ρ: Department of Mathematics, Faculty of Science, Federal University Gashua, Nigeria.

Adebola (2016), Adoghe and Omole (2019), Kuboye and Omar (2015), Olabode and Momoh (2016), Kayode and Adeyeye (2011) and Mohammed and Adeniyi (2014). We adopted the method of collocation and interpolation of the power series approximation as the basis function to generate continuous linear multistep method as other scholars in solving (1). The derivation of continuous linear multistep methods for direct solution of ordinary differential equations have been discussed over the years in literature and these include, among others collocation, interpolation, integration, interpolating polynomials and basis functions such as, Chebyshev polynomials, trigonometric functions, exponential functions.

Notes

In this paper, we developed half-step third derivative hybrid block method of order four with two cases for direct solution of third order ordinary differential equation which is implemented in block. The method developed evaluates less function per step and circumventing the Dahlquist barrier's by introducing a hybrid points.

The paper is organized as follows: First is a discussion of the new half-step third derivative hybrid block method and the materials for the development of the method. This is followed by a consideration of analysis of the basic properties of the new half-step third derivative hybrid block method, which include convergence, stability region, numerical experiments where the efficiency of the derived method is tested on some stiff numerical examples and discussion of results. Lastly, the study concludes by comparing the results obtained with an existing work of Adeyeye and Omar (2018), Adebayo and Adebola (2016), Adoghe and Omole (2019) and Mohammed and Adeniyi (2014).

II. Derivation of the New Half-Step Hybrid Block Method

In this section we intend to develop a family of half-step third derivative hybrid method with three hybrid points u, v and w, which are all rational numbers $u,v,w \in \left[0,\frac{1}{2}\right]$ of the form

$$y_{n+t} = \sum_{i=0,u,v} \left(\alpha_i(t) y_{n+i}\right) + h^3 \left[\sum_{j=0}^{k} \beta_j(t) f_{n+j} + \beta_u(t) f_{n+u} + \beta_v(t) f_{n+v} + \beta_w(t) f_{n+w}\right] \tag{2}$$

$\alpha_0(t), \alpha_u(t), \alpha_v(t), \beta_j(t), \beta_u(t), \beta_v(t), \beta_w(t)$ are polynomials, $y_{n+j} = y\left(x_{n+j}\right)$, $f_{n+j} = f\left(x_{n+j}, y_{n+j}\right)$

$t = \frac{x - x_n}{h}$ Consider the power series approximate solution of the form

$$y(x) = \sum_{j=0}^{s+r-1} a_i \left(\frac{x - x_n}{h}\right)^j \tag{3}$$

where $r = 3$ and $s = 5$ are the numbers of interpolation and collocation points respectively, is considered to be a solution to (1).

The third derivative of (3) gives

$$y'''(x) = \sum_{j=3}^{s+r-1} \frac{a_j j!}{h^3 (j-3)} \left(\frac{x - x_n}{h}\right)^{j-3} \tag{4}$$

Substituting (4) into (1) gives

$$f\left(x,y,y''\right)=\sum_{j=3}^{s+r-1}\frac{a_j j!}{h^3\left(j-3\right)}\left(\frac{x-x_n}{h}\right)^{j-3} \tag{5}$$

Collocating (5) at all points $x_{n+s}, s=0,u,v,w,\frac{1}{2}$ and Interpolating Equation (3) at $x_{n+r}, r=0,u,v$, gives a system of non linear equation of the form

Notes

$$AX=U \tag{6}$$

Where

$$A=\left[a_0,a_1,a_2,a_3,a_4,a_5,a_6,a_7\right]^T,$$

$$U=\left[y_n, y_{n+u}, y_{n+v}, f_n, f_{n+u}, f_{n+v}, f_{n+w}, f_{n+\frac{1}{2}}\right]^T,$$

$$X=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & u & u^2 & u^3 & u^4 & u^5 & u^6 & u^7 \\ 1 & v & v^2 & v^3 & v^4 & v^5 & v^6 & v^7 \\ 0 & 0 & 0 & \frac{3!}{0!h^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3!u}{0!h^3} & \frac{4!u^2}{1!h^3} & \frac{5!u^3}{2!h^3} & \frac{6!u^4}{3!h^3} & \frac{7!u^5}{4!h^3} \\ 0 & 0 & 0 & \frac{3!v}{0!h^3} & \frac{4!v^2}{1!h^3} & \frac{5!v^3}{2!h^3} & \frac{6!v^4}{3!h^3} & \frac{7!v^5}{4!h^3} \\ 0 & 0 & 0 & \frac{3!w}{0!h^3} & \frac{4!w^2}{1!h^3} & \frac{5!w^3}{2!h^3} & \frac{6!w^4}{3!h^3} & \frac{7!w^5}{4!h^3} \\ 0 & 0 & 0 & \frac{3!\frac{1}{2}}{0!h^3} & \frac{4!(\frac{1}{2})^2}{1!h^3} & \frac{5!(\frac{1}{2})^3}{2!h^3} & \frac{6!(\frac{1}{2})^4}{3!h^3} & \frac{7!(\frac{1}{2})^5}{4!h^3} \end{bmatrix}$$

Solving (6) for $a_i's$ using Gaussian elimination method, gives a continuous hybrid linear multistep method of the form

$$y(x)=\sum_{j=0,u,v}\alpha_j y_{n+j}+h^3\left[\sum_{i=0}^{1/2}\beta_j f_{n+j}+\beta_k f_{n+k}\right], k=u,v,w \tag{7}$$

Differentiating (7) twice yields

$$p''(x)=\frac{1}{h^2}\sum_{j=0,u,v}\alpha_j y_{n+j}+h\left[\sum_{j=u,v,w}\beta_j f_{n+j}+\sum_{j=0}^{1/2}\beta_j f_{n+j}\right] \tag{8}$$

We then impose the following conditions on $y(x)$ in (3) for

$$y\left(x_{n+j}\right)=y_{n+j} \text{ and } y'''\left(x_{n+j}\right)=f_{n+j}$$

The coefficient of $y_{n+j}, j=0,u,v$ and $f_{n+j}, j=0,u,v,w,\frac{1}{2}$ gives

$$y_{n+t}=\sum_{i=0,u,v}\left(\alpha_i(t)y_{n+i}\right)+h^3\left[\beta_0(t)f_n+\beta_u(t)f_{n+u}+\beta_v(t)f_{n+v}+\beta_w(t)f_{n+w}+\beta_{\frac{1}{2}}(t)f_{n+\frac{1}{2}}\right]$$

Where

$$\alpha_0 = 1 - \frac{(u+v)\left(-x_n + x\right)}{u\,v\,h} + \frac{\left(-x_n + x\right)^2}{h^2\,u\,v}$$

$$\alpha_u = \frac{\left(-x_n + x\right)^2}{u\,(u-v)\,h^2} - \frac{v\left(-x_n + x\right)}{u\,(u-v)\,h}$$

$$\alpha_v = \frac{u\left(-x_n + x\right)}{v\,(u-v)\,h} - \frac{\left(-x_n + x\right)^2}{v\,(u-v)\,h^2}$$

$$\beta_0 = -\frac{1}{840}\,\frac{1}{w}\Big(\big(6u^4 - 8u^3 v - 14u^3 w - 8u^2 v^2 + 28u^2 v w - 8u v^3 + 28 u v^2 w + 6 v^4$$
$$- 14 v^3 w - 7u^3 + 14u^2 v + 21 u^2 w + 14 u v^2 - 84 u v w - 7 v^3 + 21 v^2 w\big)\big(-x_n$$
$$+ x\big)h^2\Big) + \frac{1}{840}\,\frac{1}{u\,v\,w}\Big(\big(6u^5 - 8u^4 v - 14u^4 w - 8u^3 v^2 + 28 u^3 v w - 8u^2 v^3$$
$$+ 28 u^2 v^2 w - 8 u v^4 + 28 u v^3 w + 6 v^5 - 14 v^4 w - 7u^4 + 14 u^3 v + 21 u^3 w + 14 u^2 v^2$$
$$- 84 u^2 v w + 14 u v^3 - 84 u v^2 w - 7 v^4 + 21 v^3 w\big)\left(-x_n + x\right)^2 h\Big) + \frac{1}{6}\left(-x_n + x\right)^3$$
$$- \frac{1}{24}\,\frac{(2uvw + uv + uw + vw)\left(-x_n + x\right)^4}{u\,v\,w\,h}$$
$$+ \frac{1}{60}\,\frac{(2uv + 2uw + 2vw + u + v + w)\left(-x_n + x\right)^5}{u\,v\,w\,h^2}$$
$$- \frac{1}{120}\,\frac{(2u + 2v + 2w + 1)\left(-x_n + x\right)^6}{u\,v\,w\,h^3} + \frac{1}{105}\,\frac{\left(-x_n + x\right)^7}{u\,v\,w\,h^4}$$

$$\beta_u = \frac{1}{840}\,\frac{1}{(2u-1)(u-v)(u-w)}\Big(v\big(8u^4 - 6u^3 v - 14 u^3 w - 6u^2 v^2 + 14 u^2 v w$$
$$- 6 u v^3 + 14 u v^2 w - 6 v^4 + 14 v^3 w - 7u^3 + 7u^2 v + 14 u^2 w + 7 u v^2 - 21 u v w + 7 v^3$$
$$- 21 v^2 w\big)\left(-x_n + x\right) h^2\Big) - \frac{1}{840}\,\frac{1}{(2u-1)\,u\,(u-v)(u-w)}\Big(\big(8u^5 - 6u^4 v$$
$$- 14 u^4 w - 6 u^3 v^2 + 14 u^3 v w - 6 u^2 v^3 + 14 u^2 v^2 w - 6 u v^4 + 14 u v^3 w - 6 v^5$$
$$+ 14 v^4 w - 7u^4 + 7u^3 v + 14 u^3 w + 7 u^2 v^2 - 21 u^2 v w + 7 u v^3 - 21 u v^2 w + 7 v^4$$
$$- 21 v^3 w\big)\left(-x_n + x\right)^2 h\Big) - \frac{1}{24}\,\frac{v\,w\left(-x_n + x\right)^4}{(2u-1)\,u\,(u-v)(u-w)\,h}$$
$$+ \frac{1}{60}\,\frac{(2vw + v + w)\left(-x_n + x\right)^5}{(2u-1)\,u\,(u-v)(u-w)\,h^2} - \frac{1}{120}\,\frac{(2v + 2w + 1)\left(-x_n + x\right)^6}{(2u-1)\,u\,(u-v)(u-w)\,h^3}$$
$$+ \frac{1}{105}\,\frac{\left(-x_n + x\right)^7}{(2u-1)\,u\,(u-v)(u-w)\,h^4}$$

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$$\beta_v = \frac{1}{840}\,\frac{1}{(2v-1)(u-v)(v-w)}\Big(u\big(6u^4+6u^3v-14u^3w+6u^2v^2-14u^2vw$$

$$+6uv^3-14uv^2w-8v^4+14v^3w-7u^3-7u^2v+21u^2w-7uv^2+21uvw+7v^3$$

$$-14v^2w\big)\big(h^2x-x_n\big)\Big)-\frac{1}{840}\,\frac{1}{(2v-1)(u-v)v(v-w)}\Big(\big(6u^5+6u^4v$$

$$-14u^4w+6u^3v^2-14u^3vw+6u^2v^3-14u^2v^2w+6uv^4-14uv^3w-8v^5$$

$$+14v^4w-7u^4-7u^3v+21u^3w-7u^2v^2+21u^2vw-7uv^3+21uv^2w+7v^4$$

$$-14v^3w\big)\big(-x_n+x\big)^2h\Big)+\frac{1}{24}\,\frac{uw\big(-x_n+x\big)^4}{(2v-1)(u-v)v(v-w)h}$$

$$-\frac{1}{60}\,\frac{(2uw+u+w)\big(-x_n+x\big)^5}{(2v-1)(u-v)v(v-w)h^2}+\frac{1}{120}\,\frac{(2u+2w+1)\big(-x_n+x\big)^6}{(2v-1)(u-v)v(v-w)h^3}$$

$$-\frac{1}{105}\,\frac{\big(-x_n+x\big)^7}{(2v-1)(u-v)v(v-w)h^4}$$

$$\beta_w = -\frac{1}{840}\,\frac{1}{(2w-1)(v-w)(u-w)w}\Big(uv\big(6u^4-8u^3v-8u^2v^2-8uv^3+6v^4$$

$$-7u^3+14u^2v+14uv^2-7v^3\big)\big(-x_n+x\big)h^2\Big)$$

$$+\frac{1}{840}\,\frac{1}{(2w-1)(v-w)(u-w)w}\Big(\big(6u^5-8u^4v-8u^3v^2-8u^2v^3-8uv^4+6v^5$$

$$-7u^4+14u^3v+14u^2v^2+14uv^3-7v^4\big)\big(-x_n+x\big)^2h\Big)$$

$$-\frac{1}{24}\,\frac{uv\big(-x_n+x\big)^4}{(2w-1)(v-w)(u-w)wh}+\frac{1}{60}\,\frac{(2uv+u+v)\big(-x_n+x\big)^5}{(2w-1)(v-w)(u-w)wh^2}$$

$$-\frac{1}{120}\,\frac{(2u+2v+1)\big(-x_n+x\big)^6}{(2w-1)(v-w)(u-w)wh^3}+\frac{1}{105}\,\frac{\big(-x_n+x\big)^7}{(2w-1)(v-w)(u-w)wh^4}$$

$$\beta_{\frac{1}{2}} = \frac{2}{105}\,\frac{1}{(2w-1)(2v-1)(2u-1)}\Big(uv\big(3u^4-4u^3v-7u^3w-4u^2v^2+14u^2vw$$

$$-4uv^3+14uv^2w+3v^4-7v^3w\big)\big(-x_n+x\big)h^2\Big)$$

$$-\frac{2}{105}\frac{1}{(2w-1)(2v-1)(2u-1)}\Big(\big(3u^5-4u^4v-7u^4w-4u^3v^2+14u^3vw$$

$$-4u^2v^3+14u^2v^2w-4uv^4+14uv^3w+3v^5-7v^4w\big)\left(-x_n+x\right)^2h\Big)$$

$$+\frac{2}{3}\frac{uvw\left(-x_n+x\right)^4}{(2w-1)(2v-1)(2u-1)h}-\frac{4}{15}\frac{(uv+uw+vw)\left(-x_n+x\right)^5}{(2w-1)(2v-1)(2u-1)h^2}$$

$$+\frac{2}{15}\frac{(u+v+w)\left(-x_n+x\right)^6}{(2w-1)(2v-1)(2u-1)h^3}-\frac{8}{105}\frac{\left(-x_n+x\right)^7}{(2w-1)(2v-1)(2u-1)h^4}$$

a) Specification of the New Half-step Third Derivative Method

The family of hybrid half-step method with three off-grid hybrid points u,v and w which are rational numbers. Equation (7) is evaluated at the non-interpolating points $\left\{x_{n+w},x_{n+\frac{1}{2}}\right\}$ and (8) at all points $\left\{x_n,x_{n+u},x_{n+v},x_{n+w},x_{n+\frac{1}{2}}\right\}$, which produces the following general equations in block form

$$AY_L=BR_1+CR_2+DR_3 \tag{9}$$

Case One:

We consider the case where $\left(u=\frac{1}{8},v=\frac{1}{7},w=\frac{1}{6}\right)$ equation (9) becomes

$$A=\begin{bmatrix}\frac{16}{9} & -\frac{49}{18} & 1&0&0&0&0&0&0&0&0&0\\ 80 & \frac{147}{2} & 0&1&0&0&0&0&0&0&0&0\\ -\frac{64}{h} & \frac{49}{h} & 0&0&0&0&0&0&0&0&0&0\\ \frac{48}{h} & -\frac{49}{h} & 0&0&1&0&0&0&0&0&0&0\\ \frac{64}{h} & -\frac{63}{h} & 0&0&0&1&0&0&0&0&0&0\\ \frac{256}{3h} & -\frac{245}{3h} & 0&0&0&0&1&0&0&0&0&0\\ \frac{384}{h} & -\frac{343}{h} & 0&0&0&0&0&1&0&0&0&0\\ \frac{896}{h^2} & -\frac{784}{h^2} & 0&0&0&0&0&0&0&0&0&0\\ \frac{896}{h^2} & -\frac{784}{h^2} & 0&0&0&0&0&0&1&0&0&0\\ \frac{896}{h^2} & -\frac{784}{h^2} & 0&0&0&0&0&0&0&1&0&0\\ \frac{896}{h^2} & -\frac{784}{h^2} & 0&0&0&0&0&0&0&0&1&0\\ \frac{896}{h^2} & -\frac{784}{h^2} & 0&0&0&0&0&0&0&0&0&1\end{bmatrix},\ y_L=\begin{bmatrix}y_{n+\frac{1}{8}}\\ y_{n+\frac{1}{7}}\\ y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{2}}\\ y'_{n+\frac{1}{8}}\\ y'_{n+\frac{1}{7}}\\ y'_{n+\frac{1}{6}}\\ y'_{n+\frac{1}{2}}\\ y''_{n+\frac{1}{8}}\\ y''_{n+\frac{1}{7}}\\ y''_{n+\frac{1}{6}}\\ y''_{n+\frac{1}{2}}\end{bmatrix},\ B=\begin{bmatrix}\frac{1}{18}&0&0\\ \frac{15}{2}&0&0\\ -\frac{15}{h}&1&0\\ -\frac{1}{h}&0&0\\ \frac{1}{h}&0&0\\ \frac{11}{3h}&0&0\\ \frac{41}{h}&0&0\\ \frac{112}{h^2}&0&1\\ \frac{112}{h^2}&0&0\\ \frac{112}{h^2}&0&0\\ \frac{112}{h^2}&0&0\\ \frac{112}{h^2}&0&0\end{bmatrix},\ R_1=\begin{bmatrix}y_n\\ y'_n\\ y''_n\end{bmatrix},\ C=\begin{bmatrix}\frac{764401h^3}{669139107840}\\ -\frac{833821h^3}{550731776}\\ \frac{411933h^3}{688414720}\\ -\frac{21251h^3}{1032622080}\\ \frac{297119h^3}{14456709120}\\ \frac{9470009h^3}{125463582720}\\ -\frac{3132947h^3}{147517440}\\ -\frac{32189281h^3}{1032622080}\\ \frac{1191359h^3}{516311040}\\ \frac{158321h^3}{68841472}\\ \frac{64551877h^3}{27880796160}\\ -\frac{38277037h^3}{206524416}\end{bmatrix},\ R_2=\begin{bmatrix}f_n\end{bmatrix}$$

Notes

$$
D=\begin{bmatrix}
\frac{816523h^3}{11762210880} & -\frac{1827557h^3}{29262643200} & \frac{4340411h^3}{223046369280} & -\frac{310657h^3}{30111259852800} \\
\frac{140041h^3}{1075648} & -\frac{1810517h^3}{8028160} & \frac{59340195h^3}{550731776} & \frac{703301h^3}{2753658880} \\
\frac{61669h^3}{4033680} & -\frac{22313h^3}{1204224} & \frac{774009h^3}{137682944} & -\frac{2369h^3}{688414720} \\
-\frac{39563h^3}{32269440} & \frac{4601h^3}{3763200} & -\frac{239733h^3}{688414720} & \frac{1921h^3}{10326220800} \\
\frac{35261h^3}{28235760} & -\frac{83539h^3}{70246400} & \frac{332721h^3}{963780608} & -\frac{13399h^3}{72283545600} \\
\frac{3356641h^3}{735138180} & -\frac{291271h^3}{73156608} & \frac{6156091h^3}{4646799360} & -\frac{51451h^3}{75278149632} \\
\frac{150989h^3}{96040} & -\frac{5895259h^3}{2150400} & \frac{12481155h^3}{9834496} & \frac{1174009h^3}{245862400} \\
-\frac{2673353h^3}{6050520} & \frac{2792877h^3}{5017600} & -\frac{59549067h^3}{344207360} & \frac{1743611h^3}{15489331200} \\
\frac{100712h^3}{756315} & -\frac{209213h^3}{1505280} & \frac{6755481h^3}{172103680} & -\frac{6473h^3}{309786624} \\
\frac{853303h^3}{6050520} & -\frac{1928953h^3}{15052800} & \frac{13220469h^3}{344207360} & -\frac{320773h^3}{15489331200} \\
\frac{22476493h^3}{163364040} & -\frac{4982251h^3}{45158400} & \frac{9878035h^3}{206524416} & -\frac{8856199h^3}{418211942400} \\
\frac{74773303h^3}{6050520} & -\frac{322920953h^3}{15052800} & \frac{3309380469h^3}{344207360} & \frac{1176351227h^3}{15489331200}
\end{bmatrix},\quad
R_3=\begin{bmatrix} f_{n+\frac{1}{8}} \\ f_{n+\frac{1}{7}} \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{2}} \end{bmatrix}
$$

The modified form of (9) gives the half-step hybrid block for case one in the form

$$
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{8}}\\ y_{n+\frac{1}{7}}\\ y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{2}}\\
y'_{n+\frac{1}{8}}\\ y'_{n+\frac{1}{7}}\\ y'_{n+\frac{1}{6}}\\ y'_{n+\frac{1}{2}}\\
y''_{n+\frac{1}{8}}\\ y''_{n+\frac{1}{7}}\\ y''_{n+\frac{1}{6}}\\ y''_{n+\frac{1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
1 & \frac{h}{8} & \frac{h^2}{128}\\
1 & \frac{h}{7} & \frac{h^2}{98}\\
1 & \frac{h}{6} & \frac{h^2}{72}\\
1 & \frac{h}{2} & \frac{h^2}{8}\\
0 & 1 & \frac{h}{8}\\
0 & 1 & \frac{h}{7}\\
0 & 1 & \frac{h}{6}\\
0 & 1 & \frac{h}{2}\\
0 & 0 & 1\\
0 & 0 & 1\\
0 & 0 & 1\\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} y_n\\ y'_n\\ y''_n \end{bmatrix}
+
\begin{bmatrix}
\frac{1327h^3}{7864320}\\
\frac{22987h^3}{98825160}\\
\frac{13h^3}{38880}\\
\frac{h^3}{480}\\
\frac{537h^2}{163840}\\
\frac{977h^2}{252105}\\
\frac{109h^2}{23328}\\
-\frac{h^2}{160}\\
\frac{2057h}{61440}\\
\frac{2411h}{72030}\\
\frac{217h}{6480}\\
-\frac{37h}{240}
\end{bmatrix}
\begin{bmatrix} f_n \end{bmatrix}
+
\begin{bmatrix}
\frac{71h^3}{46080} & \frac{79919h^3}{39321600} & \frac{1701h^3}{2621440} & -\frac{53h^3}{117964800}\\
\frac{86144h^3}{37059435} & -\frac{4369h^3}{1440600} & \frac{15849h^3}{16470860} & -\frac{487h^3}{741188700}\\
\frac{8h^3}{2187} & -\frac{343h^3}{72900} & \frac{77h^3}{51840} & -\frac{7h^3}{6998400}\\
\frac{8h^3}{45} & -\frac{343h^3}{1200} & \frac{81h^3}{640} & \frac{7h^3}{28800}\\
\frac{223h^2}{5760} & -\frac{40817h^2}{819200} & \frac{513h^2}{32768} & -\frac{77h^2}{7372800}\\
\frac{7424h^2}{151263} & -\frac{457h^2}{7350} & \frac{3267h^2}{168070} & -\frac{97h^2}{7563150}\\
\frac{688h^2}{10935} & -\frac{45619h^2}{583200} & \frac{53h^2}{2160} & -\frac{7h^2}{437400}\\
\frac{16h^2}{9} & -\frac{2401h^2}{800} & \frac{27h^2}{20} & \frac{17h^2}{3600}\\
\frac{23h}{40} & -\frac{213689h}{307200} & \frac{4347h}{20480} & -\frac{41h}{307200}\\
\frac{20992h}{36015} & -\frac{719h}{1050} & \frac{2538h}{12005} & -\frac{8h}{60025}\\
\frac{704h}{1215} & -\frac{2401h}{3600} & \frac{53h}{240} & -\frac{13h}{97200}\\
\frac{64h}{5} & -\frac{26411h}{1200} & \frac{783h}{80} & \frac{91h}{1200}
\end{bmatrix}
\begin{bmatrix} f_{n+\frac{1}{8}}\\ f_{n+\frac{1}{7}}\\ f_{n+\frac{1}{6}}\\ f_{n+\frac{1}{2}} \end{bmatrix}
\tag{10}
$$

Case Two:

We consider the case where $\left(u=\frac{1}{5}, v=\frac{1}{4}, w=\frac{1}{3}\right)$ equation (9) becomes

$$A=\begin{bmatrix}
\frac{25}{9} & -\frac{32}{9} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{25}{2} & -12 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\frac{25}{h} & \frac{16}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{15}{h} & -\frac{16}{h} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{25}{h} & -\frac{24}{h} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{125}{3h} & -\frac{112}{3h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
\frac{75}{h} & -\frac{64}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
\frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
\frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
\frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
\frac{200}{h^2} & -\frac{160}{h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},\quad
y_L=\begin{bmatrix}
y_{n+\frac{1}{5}}\\ y_{n+\frac{1}{4}}\\ y_{n+\frac{1}{3}}\\ y_{n+\frac{1}{2}}\\ y'_{n+\frac{1}{5}}\\ y'_{n+\frac{1}{4}}\\ y'_{n+\frac{1}{3}}\\ y'_{n+\frac{1}{2}}\\ y''_{n+\frac{1}{5}}\\ y''_{n+\frac{1}{4}}\\ y''_{n+\frac{1}{3}}\\ y''_{n+\frac{1}{2}}
\end{bmatrix},\quad
B=\begin{bmatrix}
\frac{2}{9} & 0 & 0\\
\frac{3}{2} & 0 & 0\\
-\frac{9}{h} & 1 & 0\\
-\frac{1}{h} & 0 & 0\\
\frac{1}{h} & 0 & 0\\
\frac{13}{3h} & 0 & 0\\
\frac{11}{h} & 0 & 0\\
\frac{40}{h^2} & 0 & 1\\
\frac{40}{h^2} & 0 & 0\\
\frac{40}{h^2} & 0 & 0\\
\frac{40}{h^2} & 0 & 0\\
\frac{40}{h^2} & 0 & 0
\end{bmatrix},\quad
R_1=\begin{bmatrix} y_n\\ y'_n\\ y''_n\end{bmatrix},\quad
C=\begin{bmatrix}
\frac{24127h^3}{1166400000}\\
\frac{1693h^3}{12800000}\\
\frac{36887h^3}{22400000}\\
-\frac{31321h^3}{336000000}\\
\frac{3107h^3}{33600000}\\
\frac{1331209h^3}{3265920000}\\
\frac{13723h^3}{16800000}\\
-\frac{247967h^3}{4800000}\\
\frac{18017h^3}{4800000}\\
\frac{2943h^3}{800000}\\
\frac{56099h^3}{14400000}\\
-\frac{2033h^3}{4800000}
\end{bmatrix},\quad
R_2=\begin{bmatrix} f_n\end{bmatrix}$$

$$D=\begin{bmatrix}
\frac{42181h^3}{559872000} & -\frac{1723h^3}{6075000} & \frac{34679h^3}{259200000} & -\frac{6103h^3}{874800000}\\
\frac{1093h^3}{204800} & -\frac{371h^3}{200000} & \frac{67149h^3}{25600000} & \frac{41h^3}{3200000}\\
\frac{24461h^3}{1075200} & -\frac{22667h^3}{1050000} & \frac{262791h^3}{44800000} & -\frac{5743h^3}{16800000}\\
-\frac{3469h^3}{1075200} & \frac{11447h^3}{5250000} & -\frac{125091h^3}{224000000} & \frac{2603h^3}{84000000}\\
\frac{229h^3}{67200} & -\frac{1347h^3}{700000} & \frac{6021h^3}{11200000} & -\frac{169h^3}{5600000}\\
\frac{2262227h^3}{156764160} & -\frac{201223h^3}{51030000} & \frac{759131h^3}{241920000} & -\frac{353201h^3}{2449440000}\\
\frac{4041h^3}{89600} & -\frac{6281h^3}{262500} & \frac{389313h^3}{11200000} & \frac{2217h^3}{1400000}\\
-\frac{89701h^3}{230400} & \frac{9883h^3}{25000} & -\frac{355077h^3}{3200000} & \frac{23863h^3}{3600000}\\
\frac{5407h^3}{46080} & -\frac{6959h^3}{75000} & \frac{73467h^3}{3200000} & -\frac{4553h^3}{3600000}\\
\frac{16087h^3}{115200} & -\frac{2363h^3}{37500} & \frac{33399h^3}{1600000} & -\frac{2131h^3}{1800000}\\
\frac{778073h^3}{6220800} & \frac{532h^3}{2025000} & \frac{534769h^3}{9600000} & -\frac{155699h^3}{97200000}\\
\frac{60299h^3}{230400} & -\frac{20351h^3}{75000} & \frac{994923h^3}{3200000} & \frac{173863h^3}{3600000}
\end{bmatrix},\quad
R_3=\begin{bmatrix} f_{n+\frac{1}{5}}\\ f_{n+\frac{1}{4}}\\ f_{n+\frac{1}{3}}\\ f_{n+\frac{1}{2}}\end{bmatrix}$$

Notes

The modified form of (9) gives the half-step hybrid block for case two in the form

$$\begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{5}}\\ y_{n+\frac{1}{4}}\\ y_{n+\frac{1}{3}}\\ y_{n+\frac{1}{2}}\\ y'_{n+\frac{1}{5}}\\ y'_{n+\frac{1}{4}}\\ y'_{n+\frac{1}{3}}\\ y'_{n+\frac{1}{2}}\\ y''_{n+\frac{1}{5}}\\ y''_{n+\frac{1}{4}}\\ y''_{n+\frac{1}{3}}\\ y''_{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{h}{5} & \frac{h^2}{50}\\ 1 & \frac{h}{4} & \frac{h^2}{32}\\ 1 & \frac{h}{3} & \frac{h^2}{18}\\ 1 & \frac{h}{2} & \frac{h^2}{8}\\ 0 & 1 & \frac{h}{5}\\ 0 & 1 & \frac{h}{4}\\ 0 & 1 & \frac{h}{3}\\ 0 & 1 & \frac{h}{2}\\ 0&0&1\\ 0&0&1\\ 0&0&1\\ 0&0&1 \end{bmatrix} \begin{bmatrix} y_n\\ y'_n\\ y''_n \end{bmatrix} + \begin{bmatrix} \frac{4619h^3}{6562500}\\ \frac{2069h^3}{1720320}\\ \frac{239h^3}{102060}\\ \frac{31h^3}{5376}\\ \frac{537h^2}{62500}\\ \frac{349h^2}{30720}\\ \frac{233h^2}{14580}\\ -\frac{h^2}{40}\\ \frac{1039h}{18750}\\ \frac{85h}{1536}\\ \frac{h}{18}\\ -\frac{5h}{96} \end{bmatrix} \begin{bmatrix} f_n \end{bmatrix} + \begin{bmatrix} \frac{2039h^3}{630000} & -\frac{5888h^3}{1640625} & \frac{9153h^3}{8750000} & -\frac{316h^3}{4921875}\\ \frac{13375h^3}{2064384} & -\frac{187h^3}{26880} & \frac{459h^3}{229376} & -\frac{157h^3}{1290240}\\ \frac{3625h^3}{244944} & -\frac{128h^3}{8505} & \frac{197h^3}{45360} & -\frac{4h^3}{15309}\\ \frac{1375h^3}{32256} & -\frac{17h^3}{420} & \frac{243h^3}{17920} & -\frac{13h^3}{20160}\\ \frac{467h^2}{9000} & -\frac{864h^2}{15625} & \frac{1971h^2}{125000} & -\frac{134h^2}{140625}\\ \frac{2875h^2}{36864} & -\frac{19h^2}{240} & \frac{459h^2}{20480} & -\frac{31h^2}{23040}\\ \frac{2125h^2}{17496} & -\frac{416h^2}{3645} & \frac{37h^2}{1080} & -\frac{22h^2}{10935}\\ \frac{125h^2}{576} & -\frac{h^2}{5} & \frac{27h^2}{320} & -\frac{h^2}{720}\\ \frac{38h}{75} & -\frac{4576h}{9375} & \frac{837h}{6250} & -\frac{74h}{9375}\\ \frac{1625h}{3072} & -\frac{11h}{24} & \frac{135h}{1024} & -\frac{h}{128}\\ \frac{125h}{243} & -\frac{32h}{81} & \frac{h}{6} & -\frac{2h}{243}\\ \frac{125h}{192} & -\frac{2h}{3} & \frac{27h}{64} & \frac{h}{24} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{5}}\\ f_{n+\frac{1}{4}}\\ f_{n+\frac{1}{3}}\\ f_{n+\frac{1}{2}} \end{bmatrix} \tag{11}$$

III. Analysis of basic Properties of the Method

a) *Order of the Block*

According to fatunla (1991) and lambert (1973) the truncation error associated with (9) is defined by

$$L\left[y(x);h\right]=\sum_{j=0,u,v}\left(\alpha_j(t)y_{n+j}\right)-h^3\beta_0 y'''\left(x+jh\right)-h^3\beta_u y'''\left(x+uh\right)-h^3\beta_v(y'''\left(x+vh\right)-h^3\beta_w y'''\left(x+wh\right)-h^3\beta_{\frac{1}{2}}y'''\left(x+\tfrac{1}{2}h\right) \tag{12}$$

Assumed that $y(x)$ can be differentiated. Expanding (12) in Taylor's series and comparing the coefficient of h gives the expression

$$L\{y(x):h\}=C_0 y(x)+C_1 y'(x)+,\ldots+C_p h^p y^p(x)+C_{p+1}h^{p+1}y^{p+1}(x)+C_{p+2}h^{p+2}y^{p+2}(x)+\ldots$$

Where the constant coefficients are given below

$$C_0=\sum_{j=0}^{k}\alpha_j\ ,\ C_1=\sum_{j=1}^{k}j\alpha_j$$

$$C_q=\frac{1}{q!}\sum_{j=0}^{k}j^q\alpha_j-q(q-1)(q-2)\left(\sum_{j=0}j^{q-3}\beta_j+u^{q-3}\beta_u+v^{q-3}\beta_v+w^{q-3}\beta_w+\left(\frac{1}{2}\right)^{q-3}\beta_{\frac{1}{2}}\right),\quad q=2,3,\ldots$$

Definition: Linear operator L and associated block formula are said to be of order p, *if* $C_0 = C_1 = , \ldots = C_p = C_{p+1} = C_{p+2} = 0.$ *and* $C_{p+3} \neq 0.$ C_{p+3} is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+3} h^{p+3} y^{p+3}(x) + O\left(h^{p+4}\right)$

For case one of our the new half-step third derivative method,

Notes

$$L\{y(x):h\} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{8}}\\ y_{n+\frac{1}{7}}\\ y_{n+\frac{1}{6}}\\ y_{n+\frac{1}{2}}\\ y'_{n+\frac{1}{8}}\\ y'_{n+\frac{1}{7}}\\ y'_{n+\frac{1}{6}}\\ y'_{n+\frac{1}{2}}\\ y''_{n+\frac{1}{8}}\\ y''_{n+\frac{1}{7}}\\ y''_{n+\frac{1}{6}}\\ y''_{n+\frac{1}{2}} \end{bmatrix} - \begin{bmatrix} 1 & \frac{h}{8} & \frac{h^2}{128}\\ 1 & \frac{h}{7} & \frac{h^2}{98}\\ 1 & \frac{h}{6} & \frac{h^2}{72}\\ 1 & \frac{h}{2} & \frac{h^2}{8}\\ 0 & 1 & \frac{h}{8}\\ 0 & 1 & \frac{h}{7}\\ 0 & 1 & \frac{h}{6}\\ 0 & 1 & \frac{h}{2}\\ 0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_n\\ y'_n\\ y''_n \end{bmatrix} - \begin{bmatrix} \frac{1327h^3}{7864320}\\ \frac{22987h^3}{98825160}\\ \frac{13h^3}{38880}\\ \frac{h^3}{480}\\ \frac{537h^2}{163840}\\ \frac{977h^2}{252105}\\ \frac{109h^2}{23328}\\ -\frac{h^2}{160}\\ \frac{2057h}{61440}\\ \frac{2411h}{72030}\\ \frac{217h}{6480}\\ -\frac{37h}{240} \end{bmatrix} \left[f_n\right] - \begin{bmatrix} \frac{71h^3}{46080} & \frac{79919h^3}{39321600} & \frac{1701h^3}{2621440} & -\frac{53h^3}{117964800}\\ \frac{86144h^3}{37059435} & -\frac{4369h^3}{1440600} & \frac{15849h^3}{16470860} & -\frac{487h^3}{741188700}\\ \frac{8h^3}{2187} & -\frac{343h^3}{72900} & \frac{77h^3}{51840} & -\frac{7h^3}{6998400}\\ \frac{8h^3}{45} & -\frac{343h^3}{1200} & \frac{81h^3}{640} & \frac{7h^3}{28800}\\ \frac{223h^2}{5760} & -\frac{40817h^2}{819200} & \frac{513h^2}{32768} & -\frac{77h^2}{7372800}\\ \frac{7424h^2}{151263} & -\frac{457h^2}{7350} & \frac{3267h^2}{168070} & -\frac{97h^2}{7563150}\\ \frac{688h^2}{10935} & -\frac{45619h^2}{583200} & \frac{53h^2}{2160} & -\frac{7h^2}{437400}\\ \frac{16h^2}{9} & -\frac{2401h^2}{800} & \frac{27h^2}{20} & \frac{17h^2}{3600}\\ \frac{23h}{40} & -\frac{213689h}{307200} & \frac{4347h}{20480} & -\frac{41h}{307200}\\ \frac{20992h}{36015} & -\frac{719h}{1050} & \frac{2538h}{12005} & -\frac{8h}{60025}\\ \frac{704h}{1215} & -\frac{2401h}{3600} & \frac{53h}{240} & -\frac{13h}{97200}\\ \frac{64h}{5} & -\frac{26411h}{1200} & \frac{783h}{80} & \frac{91h}{1200} \end{bmatrix} \begin{bmatrix} f_{n+\frac{1}{8}}\\ f_{n+\frac{1}{7}}\\ f_{n+\frac{1}{6}}\\ f_{n+\frac{1}{2}} \end{bmatrix} \qquad (13)$$

expanding (13) in Taylor series and comparing the coefficient of h gives $C_0 = C_1 = C_2 = C_3 = \ldots = C_7 = 0$ and

$$C_8 = \begin{bmatrix} 3.8985e{-11}, 5.6772e{-11}, 8.6066e{-11}, 5.5361e{-09}, 1.5509e{-07}, 1.551e{-07},\\ 1.5508e{-07}, 9.0422e{-08}, 1.1252e{-08}, 1.124e{-08}, 1.127e{-08}, 8.7839e{-07} \end{bmatrix}^T$$

Hence our method is of order four (4).

Notes

For case two of the new half-step third derivative method,

$$L\{y(x):h\}=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{5}}\\y_{n+\frac{1}{4}}\\y_{n+\frac{1}{3}}\\y_{n+\frac{1}{2}}\\y'_{n+\frac{1}{5}}\\y'_{n+\frac{1}{4}}\\y'_{n+\frac{1}{3}}\\y'_{n+\frac{1}{2}}\\y''_{n+\frac{1}{5}}\\y''_{n+\frac{1}{4}}\\y''_{n+\frac{1}{3}}\\y''_{n+\frac{1}{2}}\end{bmatrix}-\begin{bmatrix}1&\frac{h}{5}&\frac{h^2}{50}\\1&\frac{h}{4}&\frac{h^2}{32}\\1&\frac{h}{3}&\frac{h^2}{18}\\1&\frac{h}{2}&\frac{h^2}{8}\\0&1&\frac{h}{5}\\0&1&\frac{h}{4}\\0&1&\frac{h}{3}\\0&1&\frac{h}{2}\\0&0&1\\0&0&1\\0&0&1\\0&0&1\end{bmatrix}\begin{bmatrix}y_n\\y'_n\\y''_n\end{bmatrix}-\begin{bmatrix}\frac{4619h^3}{6562500}\\\frac{2069h^3}{1720320}\\\frac{239h^3}{102060}\\\frac{31h^3}{5376}\\\frac{537h^2}{62500}\\\frac{349h^2}{30720}\\\frac{233h^2}{14580}\\-\frac{h^2}{40}\\\frac{1039h}{18750}\\\frac{85h}{1536}\\\frac{h}{18}\\-\frac{5h}{96}\end{bmatrix}\left[f_n\right]-\begin{bmatrix}\frac{2039h^3}{630000}&-\frac{5888h^3}{1640625}&\frac{9153h^3}{8750000}&-\frac{316h^3}{4921875}\\\frac{13375h^3}{2064384}&-\frac{187h^3}{26880}&\frac{459h^3}{229376}&-\frac{157h^3}{1290240}\\\frac{3625h^3}{244944}&-\frac{128h^3}{8505}&\frac{197h^3}{45360}&-\frac{4h^3}{15309}\\\frac{1375h^3}{32256}&-\frac{17h^3}{420}&\frac{243h^3}{17920}&-\frac{13h^3}{20160}\\\frac{467h^2}{9000}&-\frac{864h^2}{15625}&\frac{1971h^2}{125000}&-\frac{134h^2}{140625}\\\frac{2875h^2}{36864}&-\frac{19h^2}{240}&\frac{459h^2}{20480}&-\frac{31h^2}{23040}\\\frac{2125h^2}{17496}&-\frac{416h^2}{3645}&\frac{37h^2}{1080}&-\frac{22h^2}{10935}\\\frac{125h^2}{576}&-\frac{h^2}{5}&\frac{27h^2}{320}&-\frac{h^2}{720}\\\frac{38h}{75}&-\frac{4576h}{9375}&\frac{837h}{6250}&-\frac{74h}{9375}\\\frac{1625h}{3072}&-\frac{11h}{24}&\frac{135h}{1024}&-\frac{h}{128}\\\frac{125h}{243}&-\frac{32h}{81}&\frac{h}{6}&-\frac{2h}{243}\\\frac{125h}{192}&-\frac{2h}{3}&\frac{27h}{64}&\frac{h}{24}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{5}}\\f_{n+\frac{1}{4}}\\f_{n+\frac{1}{3}}\\f_{n+\frac{1}{2}}\end{bmatrix} \qquad (14)$$

expanding (14) in Taylor series and comparing the coefficient of h gives $C_0=C_1=C_2=C_3=\ldots=C_7=0$ and

$$C_8=\begin{bmatrix}1.4857e{-09},2.7803e{-09},5.897e{-09},1.5501e{-08},2.1577e{-08},3.0195e{-08},\\4.4681e{-08},6.7171e{-08},1.7296e{-07},1.718e{-07},1.7623e{-07},7.2338e{-07}\end{bmatrix}^T$$

Hence our method is of order four (4).

b) Consistency

The hybrid block method (13) and (14) is said to be consistent if it has an order more than or equal to one.

Therefore, the new half-step third derivative hybrid block method is consistent.

c) Zero Stability of Our Method

Definition 2: A block method (10) and (11) is said to be zero-stable if as $h \to 0$, the root $z_i, i=1(1)k$ of the first characteristic polynomial $\rho(z)=0$ that is $\rho(z)=\det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right]=0$ Satisfies $|z_i|\le 1$ and for those roots with $|z_i|=1$, multiplicity must not exceed two.

Hence, the new half-step third derivative hybrid block method is zero-stable.

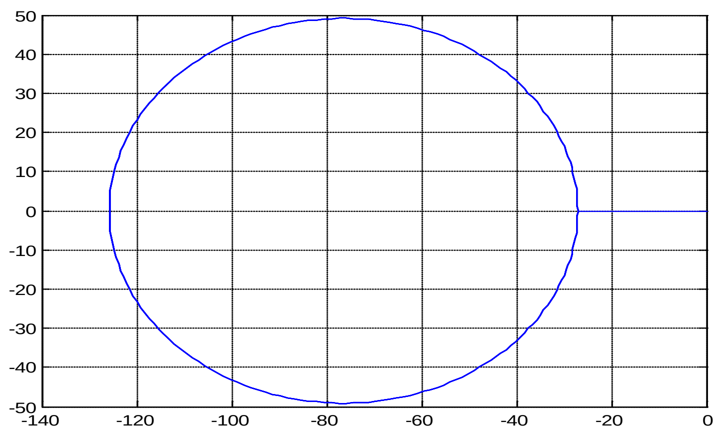
d) Regions of Absolute Stability

For case one: The stability polynomial for half step with three off step point gives

$$\left(-\frac{13}{440484715560960}w^3+\frac{1}{11012117889024000}w^4\right)h^{12}+\left(\frac{31}{7283146752000}w^4\right.$$

$$\left.-\frac{38147}{22658678784000}w^3\right)h^9+\left(\frac{1007}{159318835200}w^4-\frac{16978247}{955913011200}w^3\right)h^6+\left(\right.$$

$$\left.-\frac{235423}{995742720}w^4-\frac{37143145}{1593188352}w^3\right)h^3+w^4-\frac{13}{8}w^3$$



Stability Region for Case Two

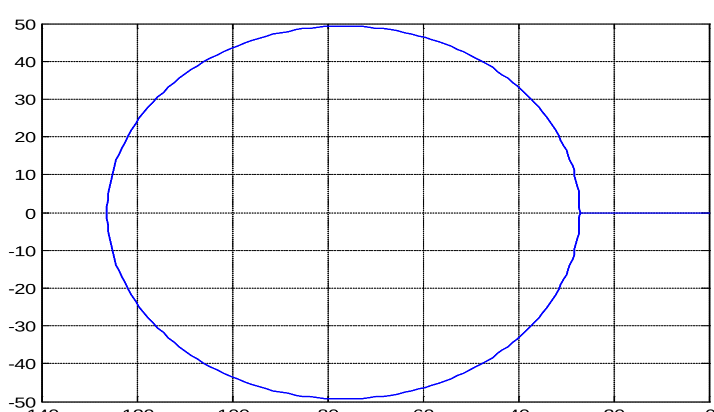
For case two:

The stability polynomial for half step with three offstep point gives

$$\left(-\frac{1}{2239488000000}w^3+\frac{1}{62705664000000}w^4\right)h^{12}+\left(\frac{1}{6220800000}w^4\right.$$

$$\left.-\frac{8983}{1393459200000}w^3\right)h^9+\left(-\frac{525359}{21772800000}w^3+\frac{97}{580608000}w^4\right)h^6$$

$$+\left(\frac{287}{12960000}w^4-\frac{2460731}{103680000}w^3\right)h^3+w^4-\frac{13}{8}w^3$$



Stability Region for Case One

Notes

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e) Numerical Example

Problem I We consider a highly stiff problem

$$y'''-y''+y'-y=0\,,y(0)=1,\,y'(0)=0,\,y''(0)=-1,$$

Exact Solution: $y(x)=\cos x,\ \ h=\frac{1}{100}$

Notes

Table 1: Comparison of the proposed new half-step method with Adoghe & Omole (2019)

x-values	Error in case one of our method	Error in case two our method	Error in Adoghe & Omole 2019
0.01	6.100e-20	3.0000e-20	0.0000e+00
0.02	1.200e-19	6.0000e-20	1.1102e-16
0.03	1.900e-19	1.2000e-19	4.4409e-16
0.04	2.500e-19	1.6000e-19	5.8842e-15
0.05	3.200e-19	1.9000e-19	2.6201e-14
0.06	3.900e-19	2.5000e-19	8.3822e-14
0.07	4.500e-19	2.8000e-19	2.0750e-13
0.08	5.100e-19	3.2000e-19	4.4142e-13
0.09	5.600e-19	3.5000e-19	8.4743e-13
0.10	6.300e-19	4.0000e-19	1.5086e-12

Problem II We consider a highly stiff problem

$$y'''+5y''+7y'+3y=0\,,y(0)=1,\,y'(0)=0,\,y''(0)=-1,$$

Exact Solution: $y(x)=e^{-x}+xe^{-x},\ \ h=\frac{1}{10}$

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Table 2: Comparison of the proposed new half-step method with Mohammed & Adeniyi 2014

x-values	Error in case one of our method	Error in case two our method	Error in Mohammed & Adeniyi 2014
0.1	1.0434e-14	4.1832e-15	1.0000e-10
0.2	9.8731e-14	1.8599e-14	3.0000e-10
0.3	3.1317e-13	4.4245e-14	7.0000e-10
0.4	6.6668e-13	8.0431e-14	7.0000e-10
0.5	1.1507e-12	1.2552e-13	6.0000e-10
0.6	1.7445e-12	1.7743e-13	2.0000e-10
0.7	2.4220e-12	2.3395 e-13	9.0000e-10
0.8	3.1554e-12	2.9298e-13	2.8000e-09
0.9	3.9178e-12	3.5259e-13	5.4000e-09
1.0	4.6852e-12	4.1112e-13	3.5000e-09

Problem III: We consider the third order ODE

$$y'''=-4y'+x,\ y(0)=1, y'(0)=0, y''(0)=1$$

Exact Solution: $y(x)=\frac{3}{16}(1-\cos 2x)+\frac{x^2}{8},\ h=\frac{1}{10}$

Table 3: Comparison of the proposed new half-step method with Adebayo & Adebola (2016)

x-values	Error in case one of our method	Error in case two our method	Error in Adebayo &Adebola (2016)
0.1	7.9512e-14	3.1484e-14	2.970e-08
0.2	8.6717e-13	1.5843e-13	1.988e-07
0.3	3.1385e-12	4.2379e-13	6.508e-07
0.4	7.5504e-12	8.5820e-13	1.5480e-06
0.5	1.4585e-11	1.4765e-12	3.062e-06
0.6	2.4504e-11	2.2752e-12	5.3625e-06
0.7	3.7317e-11	3.2313e-12	8.6068e-06
0.8	5.2765e-11	4.3022e-12	1.2926e-05
0.9	7.0321e-11	5.4266e-12	1.8118e-05
1.0	8.9206e-11	6.5277e-12	2.5129e-05

Problem IV: $y'''=\exp(x),\ \ y(0)=3, y'(0)=1, y''(0)=5$, $0\le x\le 1.$

Exact Solution: $y(x)=2+2x^2+e^x \ with\ h=\frac{1}{10}$

Notes

Table 4: Comparison of the proposed new half-step method with Adeyeye & Omar (2018)

x-values	Error in case one of our method	Error in case two our method	Error in Adeyeye & Omar (2018)
0.1	1.70091e-15	4.54792e-15	6.34270e-13
0.2	1.91790e-14	3.06193e-14	2.32882e-12
0.3	7.25153e-14	9.80503e-14	5.44348e-12
0.4	1.83912e-13	2.28761-13	9.85317e-12
0.5	3.77884e-13	4.46995e-13	1.59974e-11
0.6	6.81541e-13	7.79496e-13	2.37223e-11
0.7	1.12480e-12	1.25594e-12	3.35679e-11
0.8	1.74091e-12	1.90890e-12	4.53443e-11
0.9	2.56633e-12	2.77460e-12	5.97084e-11
1.0	3.64152e-12	3.89290e-12	7.64322e-11

Notes

IV. Conclusions

It is evident from the above tables that our proposed new half-step third derivative hybrid block methods are converging faster than the existing method and can handle stiff problems.

Comparing the new method with the existing method respectively shows that the new half-step third derivative hybrid block method performs better on stiff and highly stiff problems than the existing method which is of high Order as compared to ours which is of Order four.

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Notes

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Career | Credibility | Financial

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We accept the manuscript submissions in any standard (generic) format.

We typeset manuscripts using advanced typesetting tools like Adobe In Design, CorelDraw, TeXnicCenter, and TeXStudio. We usually recommend authors submit their research using any standard format they are comfortable with, and let Global Journals do the rest.

Alternatively, you can download our basic template from https://globaljournals.org/Template.zip

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Acknowledgments

Contributors to the research other than authors credited should be mentioned in Acknowledgments. The source of funding for the research can be included. Suppliers of resources may be mentioned along with their addresses.

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Authors can submit papers and articles in an acceptable file format: MS Word (doc, docx), LaTeX (.tex, .zip or .rar including all of your files), Adobe PDF (.pdf), rich text format (.rtf), simple text document (.txt), Open Document Text (.odt), and Apple Pages (.pages). Our professional layout editors will format the entire paper according to our official guidelines. This is one of the highlights of publishing with Global Journals—authors should not be concerned about the formatting of their paper. Global Journals accepts articles and manuscripts in every major language, be it Spanish, Chinese, Japanese, Portuguese, Russian, French, German, Dutch, Italian, Greek, or any other national language, but the title, subtitle, and abstract should be in English. This will facilitate indexing and the pre-peer review process.

The following is the official style and template developed for publication of a research paper. Authors are not required to follow this style during the submission of the paper. It is just for reference purposes.

Manuscript Style Instruction (Optional)

- Microsoft Word Document Setting Instructions.
- Font type of all text should be Swis721 Lt BT.
- Page size: 8.27" x 11'", left margin: 0.65, right margin: 0.65, bottom margin: 0.75.
- Paper title should be in one column of font size 24.
- Author name in font size of 11 in one column.
- Abstract: font size 9 with the word "Abstract" in bold italics.
- Main text: font size 10 with two justified columns.
- Two columns with equal column width of 3.38 and spacing of 0.2.
- First character must be three lines drop-capped.
- The paragraph before spacing of 1 pt and after of 0 pt.
- Line spacing of 1 pt.
- Large images must be in one column.
- The names of first main headings (Heading 1) must be in Roman font, capital letters, and font size of 10.
- The names of second main headings (Heading 2) must not include numbers and must be in italics with a font size of 10.

Structure and Format of Manuscript

The recommended size of an original research paper is under 15,000 words and review papers under 7,000 words. Research articles should be less than 10,000 words. Research papers are usually longer than review papers. Review papers are reports of significant research (typically less than 7,000 words, including tables, figures, and references)

A research paper must include:

a) A title which should be relevant to the theme of the paper.
b) A summary, known as an abstract (less than 150 words), containing the major results and conclusions.
c) Up to 10 keywords that precisely identify the paper's subject, purpose, and focus.
d) An introduction, giving fundamental background objectives.
e) Resources and techniques with sufficient complete experimental details (wherever possible by reference) to permit repetition, sources of information must be given, and numerical methods must be specified by reference.
f) Results which should be presented concisely by well-designed tables and figures.
g) Suitable statistical data should also be given.
h) All data must have been gathered with attention to numerical detail in the planning stage.

Design has been recognized to be essential to experiments for a considerable time, and the editor has decided that any paper that appears not to have adequate numerical treatments of the data will be returned unrefereed.

i) Discussion should cover implications and consequences and not just recapitulate the results; conclusions should also be summarized.
j) There should be brief acknowledgments.
k) There ought to be references in the conventional format. Global Journals recommends APA format.

Authors should carefully consider the preparation of papers to ensure that they communicate effectively. Papers are much more likely to be accepted if they are carefully designed and laid out, contain few or no errors, are summarizing, and follow instructions. They will also be published with much fewer delays than those that require much technical and editorial correction.

The Editorial Board reserves the right to make literary corrections and suggestions to improve brevity.

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It is necessary that authors take care in submitting a manuscript that is written in simple language and adheres to published guidelines.

All manuscripts submitted to Global Journals should include:

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The title page must carry an informative title that reflects the content, a running title (less than 45 characters together with spaces), names of the authors and co-authors, and the place(s) where the work was carried out.

Author details

The full postal address of any related author(s) must be specified.

Abstract

The abstract is the foundation of the research paper. It should be clear and concise and must contain the objective of the paper and inferences drawn. It is advised to not include big mathematical equations or complicated jargon.

Many researchers searching for information online will use search engines such as Google, Yahoo or others. By optimizing your paper for search engines, you will amplify the chance of someone finding it. In turn, this will make it more likely to be viewed and cited in further works. Global Journals has compiled these guidelines to facilitate you to maximize the web-friendliness of the most public part of your paper.

Keywords

A major lynchpin of research work for the writing of research papers is the keyword search, which one will employ to find both library and internet resources. Up to eleven keywords or very brief phrases have to be given to help data retrieval, mining, and indexing.

One must be persistent and creative in using keywords. An effective keyword search requires a strategy: planning of a list of possible keywords and phrases to try.

Choice of the main keywords is the first tool of writing a research paper. Research paper writing is an art. Keyword search should be as strategic as possible.

One should start brainstorming lists of potential keywords before even beginning searching. Think about the most important concepts related to research work. Ask, "What words would a source have to include to be truly valuable in a research paper?" Then consider synonyms for the important words.

It may take the discovery of only one important paper to steer in the right keyword direction because, in most databases, the keywords under which a research paper is abstracted are listed with the paper.

Numerical Methods

Numerical methods used should be transparent and, where appropriate, supported by references.

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Authors must list all the abbreviations used in the paper at the end of the paper or in a separate table before using them.

Formulas and equations

Authors are advised to submit any mathematical equation using either MathJax, KaTeX, or LaTeX, or in a very high-quality image.

Tables, Figures, and Figure Legends

Tables: Tables should be cautiously designed, uncrowned, and include only essential data. Each must have an Arabic number, e.g., Table 4, a self-explanatory caption, and be on a separate sheet. Authors must submit tables in an editable format and not as images. References to these tables (if any) must be mentioned accurately.

Figures

Figures are supposed to be submitted as separate files. Always include a citation in the text for each figure using Arabic numbers, e.g., Fig. 4. Artwork must be submitted online in vector electronic form or by emailing it.

Preparation of Eletronic Figures for Publication

Although low-quality images are sufficient for review purposes, print publication requires high-quality images to prevent the final product being blurred or fuzzy. Submit (possibly by e-mail) EPS (line art) or TIFF (halftone/ photographs) files only. MS PowerPoint and Word Graphics are unsuitable for printed pictures. Avoid using pixel-oriented software. Scans (TIFF only) should have a resolution of at least 350 dpi (halftone) or 700 to 1100 dpi (line drawings). Please give the data for figures in black and white or submit a Color Work Agreement form. EPS files must be saved with fonts embedded (and with a TIFF preview, if possible).

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Tips for Writing a Good Quality Science Frontier Research Paper

Techniques for writing a good quality Science Frontier Research paper:

1. Choosing the topic: In most cases, the topic is selected by the interests of the author, but it can also be suggested by the guides. You can have several topics, and then judge which you are most comfortable with. This may be done by asking several questions of yourself, like "Will I be able to carry out a search in this area? Will I find all necessary resources to accomplish the search? Will I be able to find all information in this field area?" If the answer to this type of question is "yes," then you ought to choose that topic. In most cases, you may have to conduct surveys and visit several places. Also, you might have to do a lot of work to find all the rises and falls of the various data on that subject. Sometimes, detailed information plays a vital role, instead of short information. Evaluators are human: The first thing to remember is that evaluators are also human beings. They are not only meant for rejecting a paper. They are here to evaluate your paper. So present your best aspect.

2. Think like evaluators: If you are in confusion or getting demotivated because your paper may not be accepted by the evaluators, then think, and try to evaluate your paper like an evaluator. Try to understand what an evaluator wants in your research paper, and you will automatically have your answer. Make blueprints of paper: The outline is the plan or framework that will help you to arrange your thoughts. It will make your paper logical. But remember that all points of your outline must be related to the topic you have chosen.

3. Ask your guides: If you are having any difficulty with your research, then do not hesitate to share your difficulty with your guide (if you have one). They will surely help you out and resolve your doubts. If you can't clarify what exactly you require for your work, then ask your supervisor to help you with an alternative. He or she might also provide you with a list of essential readings.

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7. Revise what you wrote: When you write anything, always read it, summarize it, and then finalize it.

8. Make every effort: Make every effort to mention what you are going to write in your paper. That means always have a good start. Try to mention everything in the introduction—what is the need for a particular research paper. Polish your work with good writing skills and always give an evaluator what he wants. Make backups: When you are going to do any important thing like making a research paper, you should always have backup copies of it either on your computer or on paper. This protects you from losing any portion of your important data.

9. Produce good diagrams of your own: Always try to include good charts or diagrams in your paper to improve quality. Using several unnecessary diagrams will degrade the quality of your paper by creating a hodgepodge. So always try to include diagrams which were made by you to improve the readability of your paper. Use of direct quotes: When you do research relevant to literature, history, or current affairs, then use of quotes becomes essential, but if the study is relevant to science, use of quotes is not preferable.

10. Use proper verb tense: Use proper verb tenses in your paper. Use past tense to present those events that have happened. Use present tense to indicate events that are going on. Use future tense to indicate events that will happen in the future. Use of wrong tenses will confuse the evaluator. Avoid sentences that are incomplete.

11. Pick a good study spot: Always try to pick a spot for your research which is quiet. Not every spot is good for studying.

12. Know what you know: Always try to know what you know by making objectives, otherwise you will be confused and unable to achieve your target.

13. Use good grammar: Always use good grammar and words that will have a positive impact on the evaluator; use of good vocabulary does not mean using tough words which the evaluator has to find in a dictionary. Do not fragment sentences. Eliminate one-word sentences. Do not ever use a big word when a smaller one would suffice.

Verbs have to be in agreement with their subjects. In a research paper, do not start sentences with conjunctions or finish them with prepositions. When writing formally, it is advisable to never split an infinitive because someone will (wrongly) complain. Avoid clichés like a disease. Always shun irritating alliteration. Use language which is simple and straightforward. Put together a neat summary.

14. Arrangement of information: Each section of the main body should start with an opening sentence, and there should be a changeover at the end of the section. Give only valid and powerful arguments for your topic. You may also maintain your arguments with records.

15. Never start at the last minute: Always allow enough time for research work. Leaving everything to the last minute will degrade your paper and spoil your work.

16. Multitasking in research is not good: Doing several things at the same time is a bad habit in the case of research activity. Research is an area where everything has a particular time slot. Divide your research work into parts, and do a particular part in a particular time slot.

17. Never copy others' work: Never copy others' work and give it your name because if the evaluator has seen it anywhere, you will be in trouble. Take proper rest and food: No matter how many hours you spend on your research activity, if you are not taking care of your health, then all your efforts will have been in vain. For quality research, take proper rest and food.

18. Go to seminars: Attend seminars if the topic is relevant to your research area. Utilize all your resources.

19. Refresh your mind after intervals: Try to give your mind a rest by listening to soft music or sleeping in intervals. This will also improve your memory. Acquire colleagues: Always try to acquire colleagues. No matter how sharp you are, if you acquire colleagues, they can give you ideas which will be helpful to your research.

20. Think technically: Always think technically. If anything happens, search for its reasons, benefits, and demerits. Think and then print: When you go to print your paper, check that tables are not split, headings are not detached from their descriptions, and page sequence is maintained.

21. Adding unnecessary information: Do not add unnecessary information like "I have used MS Excel to draw graphs." Irrelevant and inappropriate material is superfluous. Foreign terminology and phrases are not apropos. One should never take a broad view. Analogy is like feathers on a snake. Use words properly, regardless of how others use them. Remove quotations. Puns are for kids, not grunt readers. Never oversimplify: When adding material to your research paper, never go for oversimplification; this will definitely irritate the evaluator. Be specific. Never use rhythmic redundancies. Contractions shouldn't be used in a research paper. Comparisons are as terrible as clichés. Give up ampersands, abbreviations, and so on. Remove commas that are not necessary. Parenthetical words should be between brackets or commas. Understatement is always the best way to put forward earth-shaking thoughts. Give a detailed literary review.

22. Report concluded results: Use concluded results. From raw data, filter the results, and then conclude your studies based on measurements and observations taken. An appropriate number of decimal places should be used. Parenthetical remarks are prohibited here. Proofread carefully at the final stage. At the end, give an outline to your arguments. Spot perspectives of further study of the subject. Justify your conclusion at the bottom sufficiently, which will probably include examples.

23. Upon conclusion: Once you have concluded your research, the next most important step is to present your findings. Presentation is extremely important as it is the definite medium though which your research is going to be in print for the rest of the crowd. Care should be taken to categorize your thoughts well and present them in a logical and neat manner. A good quality research paper format is essential because it serves to highlight your research paper and bring to light all necessary aspects of your research.

Informal Guidelines of Research Paper Writing

Key points to remember:

- Submit all work in its final form.
- Write your paper in the form which is presented in the guidelines using the template.
- Please note the criteria peer reviewers will use for grading the final paper.

Final points:

One purpose of organizing a research paper is to let people interpret your efforts selectively. The journal requires the following sections, submitted in the order listed, with each section starting on a new page:

The introduction: This will be compiled from reference matter and reflect the design processes or outline of basis that directed you to make a study. As you carry out the process of study, the method and process section will be constructed like that. The results segment will show related statistics in nearly sequential order and direct reviewers to similar intellectual paths throughout the data that you gathered to carry out your study.

The discussion section:

This will provide understanding of the data and projections as to the implications of the results. The use of good quality references throughout the paper will give the effort trustworthiness by representing an alertness to prior workings.

Writing a research paper is not an easy job, no matter how trouble-free the actual research or concept. Practice, excellent preparation, and controlled record-keeping are the only means to make straightforward progression.

General style:

Specific editorial column necessities for compliance of a manuscript will always take over from directions in these general guidelines.

To make a paper clear: Adhere to recommended page limits.

Mistakes to avoid:

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- Submitting a manuscript with pages out of sequence.
- In every section of your document, use standard writing style, including articles ("a" and "the").
- Keep paying attention to the topic of the paper.
- Use paragraphs to split each significant point (excluding the abstract).
- Align the primary line of each section.
- Present your points in sound order.
- Use present tense to report well-accepted matters.
- Use past tense to describe specific results.
- Do not use familiar wording; don't address the reviewer directly. Don't use slang or superlatives.
- Avoid use of extra pictures—include only those figures essential to presenting results.

Title page:

Choose a revealing title. It should be short and include the name(s) and address(es) of all authors. It should not have acronyms or abbreviations or exceed two printed lines.

Abstract: This summary should be two hundred words or less. It should clearly and briefly explain the key findings reported in the manuscript and must have precise statistics. It should not have acronyms or abbreviations. It should be logical in itself. Do not cite references at this point.

An abstract is a brief, distinct paragraph summary of finished work or work in development. In a minute or less, a reviewer can be taught the foundation behind the study, common approaches to the problem, relevant results, and significant conclusions or new questions.

Write your summary when your paper is completed because how can you write the summary of anything which is not yet written? Wealth of terminology is very essential in abstract. Use comprehensive sentences, and do not sacrifice readability for brevity; you can maintain it succinctly by phrasing sentences so that they provide more than a lone rationale. The author can at this moment go straight to shortening the outcome. Sum up the study with the subsequent elements in any summary. Try to limit the initial two items to no more than one line each.

Reason for writing the article—theory, overall issue, purpose.

- Fundamental goal.
- To-the-point depiction of the research.
- Consequences, including definite statistics—if the consequences are quantitative in nature, account for this; results of any numerical analysis should be reported. Significant conclusions or questions that emerge from the research.

Approach:

- o Single section and succinct.
- o An outline of the job done is always written in past tense.
- o Concentrate on shortening results—limit background information to a verdict or two.
- o Exact spelling, clarity of sentences and phrases, and appropriate reporting of quantities (proper units, important statistics) are just as significant in an abstract as they are anywhere else.

Introduction:

The introduction should "introduce" the manuscript. The reviewer should be presented with sufficient background information to be capable of comprehending and calculating the purpose of your study without having to refer to other works. The basis for the study should be offered. Give the most important references, but avoid making a comprehensive appraisal of the topic. Describe the problem visibly. If the problem is not acknowledged in a logical, reasonable way, the reviewer will give no attention to your results. Speak in common terms about techniques used to explain the problem, if needed, but do not present any particulars about the protocols here.

The following approach can create a valuable beginning:

- Explain the value (significance) of the study.
- Defend the model—why did you employ this particular system or method? What is its compensation? Remark upon its appropriateness from an abstract point of view as well as pointing out sensible reasons for using it.
- Present a justification. State your particular theory(-ies) or aim(s), and describe the logic that led you to choose them.
- Briefly explain the study's tentative purpose and how it meets the declared objectives.

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Use past tense except for when referring to recognized facts. After all, the manuscript will be submitted after the entire job is done. Sort out your thoughts; manufacture one key point for every section. If you make the four points listed above, you will need at least four paragraphs. Present surrounding information only when it is necessary to support a situation. The reviewer does not desire to read everything you know about a topic. Shape the theory specifically—do not take a broad view.

As always, give awareness to spelling, simplicity, and correctness of sentences and phrases.

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This part is supposed to be the easiest to carve if you have good skills. A soundly written procedures segment allows a capable scientist to replicate your results. Present precise information about your supplies. The suppliers and clarity of reagents can be helpful bits of information. Present methods in sequential order, but linked methodologies can be grouped as a segment. Be concise when relating the protocols. Attempt to give the least amount of information that would permit another capable scientist to replicate your outcome, but be cautious that vital information is integrated. The use of subheadings is suggested and ought to be synchronized with the results section.

When a technique is used that has been well-described in another section, mention the specific item describing the way, but draw the basic principle while stating the situation. The purpose is to show all particular resources and broad procedures so that another person may use some or all of the methods in one more study or referee the scientific value of your work. It is not to be a step-by-step report of the whole thing you did, nor is a methods section a set of orders.

Materials:

Materials may be reported in part of a section or else they may be recognized along with your measures.

Methods:

- Report the method and not the particulars of each process that engaged the same methodology.
- Describe the method entirely.
- To be succinct, present methods under headings dedicated to specific dealings or groups of measures.
- Simplify—detail how procedures were completed, not how they were performed on a particular day.
- If well-known procedures were used, account for the procedure by name, possibly with a reference, and that's all.

Approach:

It is embarrassing to use vigorous voice when documenting methods without using first person, which would focus the reviewer's interest on the researcher rather than the job. As a result, when writing up the methods, most authors use third person passive voice.

Use standard style in this and every other part of the paper—avoid familiar lists, and use full sentences.

What to keep away from:

- Resources and methods are not a set of information.
- Skip all descriptive information and surroundings—save it for the argument.
- Leave out information that is immaterial to a third party.

Results:

The principle of a results segment is to present and demonstrate your conclusion. Create this part as entirely objective details of the outcome, and save all understanding for the discussion.

The page length of this segment is set by the sum and types of data to be reported. Use statistics and tables, if suitable, to present consequences most efficiently.

You must clearly differentiate material which would usually be incorporated in a study editorial from any unprocessed data or additional appendix matter that would not be available. In fact, such matters should not be submitted at all except if requested by the instructor.

Content:

- Sum up your conclusions in text and demonstrate them, if suitable, with figures and tables.
- In the manuscript, explain each of your consequences, and point the reader to remarks that are most appropriate.
- Present a background, such as by describing the question that was addressed by creation of an exacting study.
- Explain results of control experiments and give remarks that are not accessible in a prescribed figure or table, if appropriate.
- Examine your data, then prepare the analyzed (transformed) data in the form of a figure (graph), table, or manuscript.

What to stay away from:

- Do not discuss or infer your outcome, report surrounding information, or try to explain anything.
- Do not include raw data or intermediate calculations in a research manuscript.
- Do not present similar data more than once.
- A manuscript should complement any figures or tables, not duplicate information.
- Never confuse figures with tables—there is a difference.

Approach:

As always, use past tense when you submit your results, and put the whole thing in a reasonable order.

Put figures and tables, appropriately numbered, in order at the end of the report.

If you desire, you may place your figures and tables properly within the text of your results section.

Figures and tables:

If you put figures and tables at the end of some details, make certain that they are visibly distinguished from any attached appendix materials, such as raw facts. Whatever the position, each table must be titled, numbered one after the other, and include a heading. All figures and tables must be divided from the text.

Discussion:

The discussion is expected to be the trickiest segment to write. A lot of papers submitted to the journal are discarded based on problems with the discussion. There is no rule for how long an argument should be.

Position your understanding of the outcome visibly to lead the reviewer through your conclusions, and then finish the paper with a summing up of the implications of the study. The purpose here is to offer an understanding of your results and support all of your conclusions, using facts from your research and generally accepted information, if suitable. The implication of results should be fully described.

Infer your data in the conversation in suitable depth. This means that when you clarify an observable fact, you must explain mechanisms that may account for the observation. If your results vary from your prospect, make clear why that may have happened. If your results agree, then explain the theory that the proof supported. It is never suitable to just state that the data approved the prospect, and let it drop at that. Make a decision as to whether each premise is supported or discarded or if you cannot make a conclusion with assurance. Do not just dismiss a study or part of a study as "uncertain."

Research papers are not acknowledged if the work is imperfect. Draw what conclusions you can based upon the results that you have, and take care of the study as a finished work.

- o You may propose future guidelines, such as how an experiment might be personalized to accomplish a new idea.
- o Give details of all of your remarks as much as possible, focusing on mechanisms.
- o Make a decision as to whether the tentative design sufficiently addressed the theory and whether or not it was correctly restricted. Try to present substitute explanations if they are sensible alternatives.
- o One piece of research will not counter an overall question, so maintain the large picture in mind. Where do you go next? The best studies unlock new avenues of study. What questions remain?
- o Recommendations for detailed papers will offer supplementary suggestions.

Approach:

When you refer to information, differentiate data generated by your own studies from other available information. Present work done by specific persons (including you) in past tense.

Describe generally acknowledged facts and main beliefs in present tense.

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Introduction	Containing all background details with clear goal and appropriate details, flow specification, no grammar and spelling mistake, well organized sentence and paragraph, reference cited	Unclear and confusing data, appropriate format, grammar and spelling errors with unorganized matter	Out of place depth and content, hazy format
Methods and Procedures	Clear and to the point with well arranged paragraph, precision and accuracy of facts and figures, well organized subheads	Difficult to comprehend with embarrassed text, too much explanation but completed	Incorrect and unorganized structure with hazy meaning
Result	Well organized, Clear and specific, Correct units with precision, correct data, well structuring of paragraph, no grammar and spelling mistake	Complete and embarrassed text, difficult to comprehend	Irregular format with wrong facts and figures
Discussion	Well organized, meaningful specification, sound conclusion, logical and concise explanation, highly structured paragraph reference cited	Wordy, unclear conclusion, spurious	Conclusion is not cited, unorganized, difficult to comprehend
References	Complete and correct format, well organized	Beside the point, Incomplete	Wrong format and structuring

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