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A New Subclass of Multivalent Function Defined by using Jackson Derivative Operator

By Shivani Indora & S. K. Bissu

Abstract- In this paper the authors have used Jackson Derivative operator to form a new subclass of multivalent function and derived some results for a function belonging to new subclass of multivalent functions. The main emphasis is on coefficient estimate of functions belonging to new subclass of multivalent function, the radii of starlikeness, convexity and close to convexity properties of a function have also been discussed. The results reduces to the earlier known results of Silverman, Srivastava, Altintas and Khosravianarab by assuming some particular values of the parameters.

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Notes

A New Subclass of Multivalent Function Defined by using Jackson Derivative Operator

Shivani Indora [α] & S. K. Bissu [σ]

Abstract- In this paper the authors have used Jackson Derivative operator to form a new subclass of multivalent function and derived some results for a function belonging to new subclass of multivalent functions. The main emphasis is on coefficient estimate of functions belonging to new subclass of multivalent function, the radii of starlikeness, convexity and close to convexity properties of a function have also been discussed. The results reduces to the earlier known results of Silverman, Srivastava, Altintas and Khosravianarab by assuming some particular values of the parameters.

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I. Introduction

Let $\mathcal{H}(p)$ be the class of analytic and p-valent function $f(z)$. The function $f(z)$ can be expressed as

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \tag{1.1}$$

where p is some natural number, $n \in \mathbb{N}$

The function $f(z)$ defined in (1.1) isan analytic function and p - valent function in the open unit disc

$$U_1 = \{z : |z| < 1\}$$

If a function $f(z) \in \mathcal{H}(p)$ satisfies the following condition

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \varphi \qquad z \in U_1 \ , \ 0 \leq \varphi < p \ , p \in \mathbb{N} \tag{1.2}$$

then $f(z)$ is a $p-$ valent starlike function of order φ

and if a function $f(z) \in \mathcal{H}(p)$ satisfies the following condition

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$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \varphi \quad z \in U_1\,,\ 0 \leq \ \varphi < p \ \ ,p\ \in \mathbb{N} \tag{1.3}$$

then $f(z)$ is a $p-$ valent convex function of order φ

To define a new subclass of multivalent function by using Jackson derivative, we use the following definitions

Definition 1: Let $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ *and* $g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$ are the members of the class $\mathcal{H}(p)$, then their convolution product or Hadamard product is defined as

$$(f * g)(z) = (g * f)(z) \ = z^p - \sum_{k=n+p}^{\infty} a_k b_k z^k \tag{1.4}$$

and generally the convolution product of functions $f(z)$ and $g(z)$ is denoted by $(f * g)(z)$ or $(g * f)(z)$.

Definition 2: The Jackson q- derivative of a function $f(z)$ is denoted by $D_q f(z)\ or\ D_{q,z} f(z)$ and it is defined as

$$D_{q,z} f(z) = \frac{f(z)-f(zq)}{z-zq}\, z \neq 0 \ and\ q \ \neq \ 1 \tag{1.5}$$

The Jackson's q – derivative tends to ordinary derivative when q tends to 1.
The Jackson q- derivative can also be written as

$$D_{q,z}^{\mathrm{m}} z^r \ = \frac{\Gamma_q(1+\mathrm{r})}{\Gamma_q(1+\mathrm{r}-\mathrm{m})} z^{r-\mathrm{m}} \qquad\qquad where\ \ m\ \geq 0\,, r\ > -1 \tag{1.6}$$

A new class of multivalent function form by using Jackson Derivative Operator is defined in the following definition.

Definition 5.3: A function $f(z) \in \ \mathcal{H}(p)$ is also belongs to new subclass $\Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, \mathrm{q})$ if it follow the following condition

$$Re\left\{\frac{(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right) + \lambda(z^2+\beta z)\left(D_{q,z}^{m+\xi+2} f(z)\right)}{(1-\lambda)\left(D_{q,z}^{m+\xi} f(z)\right) + \lambda(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right)}\right\} > \alpha \tag{1.7}$$

where $z \in U_1, m \in \mathbb{N} \cup \{0\}\,, 0 \leq \alpha < p,\ 0 \leq \beta < 1, 0 \leq \lambda \leq 1\ and\ 0 \leq \xi < 1$

By taking particular values of the parameters, $n, p, q, \beta, \lambda, \xi$ we get the previously defined subclasses of univalent and multivalent function. These classes were studied by Silverman [14], Srivastava [15], Altintas et. al [2] and Khosravianarb et. al [7].

Particular Cases:

1. If $m = 0, \beta = 0, \xi = 0, q \to 1$ then from (1.7) we get

$$Re\left\{\frac{z\left(D_{1,z} f(z)\right) \ + \ \lambda\, z^2\left(D_{1,z}^2 f(z)\right)}{(1-\lambda)\left(D_{1,z}^0 f(z)\right) \ + \ \lambda z\left(D_{1,z} f(z)\right)}\right\} > \alpha$$

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14. Silverman, H. (1975), *Univalent functions with negative coefficients,* Proceedings of the American mathematical society, 51(1), pp109-116. doi.org/10.1090/S0002-9939-1975-0369678-0

which is equivalent to

$$Re\left\{\frac{zf'(z)\ +\ \lambda z^2 f''(z)}{(1-\lambda)f(z)\ +\ \lambda zf'(z)}\right\} > \alpha$$

sowe get $\Psi_{0,n,p}(\alpha, 0, \Lambda, 0,1) \equiv T(n, p, \lambda, \alpha)$ and this classwas studied byAltintas et al. [2]

2. If $m = 0, \beta = 0, q \to 1$then from (1.7) we get

$$Re\left\{\frac{z\left(D_{1,z}^{\xi+1}f(z)\right)\ +\ \lambda\, z^2\left(D_{1,z}^{\xi+2}f(z)\right)}{(1-\lambda)\left(D_{1,z}^{\xi}f(z)\right)\ +\ \lambda z\left(D_{1,z}^{\xi+1}f(z)\right)}\right\} > \alpha$$

so $\Psi_{0,n,p}(\alpha, 0, \lambda, \xi, 1) \equiv T(n, p, \lambda, \alpha, \xi)$ and this class was studied by Khosravianarb et al.[7]

3. If $m = 0, \beta = 0, \xi = 0, q \to 1, \lambda = 0$ then from (1.7) we get

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad 0 \le \alpha < p$$

so $\Psi_{0,n,p}(\alpha, 0, 0{,}0{,}1) \equiv T^*(p, \alpha)$ and $T^*(p, \alpha)$ is the class of p valent starlike function of order α.

4. If $m = 0, \beta = 0, \xi = 0, q \to 1, \lambda = 0, p = 1$ then from (1.7) we get

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad 0 \le \alpha < 1$$

so $\Psi_{0,n,1}(\alpha, 0, 0{,}0{,}1) \equiv T^*(1, \alpha)$, which was earlier studied by Srivastava et al. [15].

5. If $m = 0, \beta = 0, \xi = 0, q \to 1, \lambda = 0, p = 1, n = 1$ then from (1.7) we get

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad 0 \le \alpha < 1$$

Then we get a class which was earlier discussed by Silverman [14].

6. If $m = 0, \beta = 0, \xi = 0, q \to 1, \lambda = 1$then from (1.7) we get

$$Re\left\{\frac{zf'(z)\ +\ z^2 f''(z)}{z\, f'(z)}\right\} > \alpha \qquad 0 \le \alpha < p$$

which is equivalent to $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad 0 \le \alpha < p$

so $\Psi_{0,n,p}(\alpha, 0, 1{,}0{,}1) \equiv C^*(p, \alpha)$ and $C^*(p, \alpha)$ represent a class of p valent convex function of order α.

Ref

2. Altintaş, O., Irmak, H., & Srivastava, H. M. (1995), *Fractional calculus and certain starlike functions with negative coefficients*, Computers & Mathematics with Applications, 30(2), pp 9-15. doi: 10.1016/0898-1221(95)00073-8

7. If $m = 0, \beta = 0, \xi = 0, q \to 1, \lambda = 1, p = 1$then from (1.7) we get

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad 0 \le \alpha < 1$$

sowe get $\Psi_{0,n,1}(\alpha, 0, 1,0,1) \equiv C^*(1,\alpha)$,which was earlier by studied Srivastava et al. [15].

8. If $m = 0, \beta = 0, \xi = 0, q \to 1, \lambda = 1, p = 1, n = 1$ then from (1.7) we get

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad 0 \le \alpha < 1$$

and this class of convex function was first introduced by Silverman [14].

II. Coefficient Estimate

In this part of the paper we derive the coefficient estimate of function $f(z)$, $f(z) \in \Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$

Theorem 1: A function $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ and $f(z) \in \mathcal{H}(p)$ then $f(z)$ belong to the class $\Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$ if and only if

$$\sum_{k=n+p}^{\infty} E_{p,k}^{m,\xi} \left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\} a_k \le 1 \tag{2.1}$$

where $E_{p,k}^{m,\xi} = \frac{\Gamma_q(1+k)\Gamma_q(1+p-(m+\xi))}{\Gamma_q(1+p)\Gamma_q(1+k-(m+\xi))}$

$$z \in U_1, m \in \mathbb{N} \cup \{0\}, 0 \le \alpha < p, \qquad 0 \le \beta < 1, 0 \le \lambda \le 1 \text{ and } 0 \le \xi < 1$$

Proof: Let us consider that $f(z) \in \Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$ so we have

$$Re\left\{\frac{(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right) + \lambda\,(z^2+\beta z)\left(D_{q,z}^{m+\xi+2} f(z)\right)}{(1-\lambda)\left(D_{q,z}^{m+\xi} f(z)\right) + \lambda(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right)}\right\} > \alpha$$

Since $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ and

$$D_{q,z}^{m+\xi} f(z) = \frac{\Gamma_q(1+p)}{\Gamma_q(1+p-(m+\xi))} z^{p-(m+\xi)} - \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q(1+k-(m+\xi))} a_k z^{k-(m+\xi)} \tag{2.2}$$

so we have

$$D_{q,z}^{m+\xi+1} = \frac{\Gamma_q(1+p)}{\Gamma_q(p-(m+\xi))} z^{p-(m+\xi+1)} - \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q(k-(m+\xi))} a_k z^{k-(m+\xi+1)} \tag{2.3}$$

$$D_{q,z}^{m+\xi+2} = \frac{\Gamma_q(1+p)}{\Gamma_q(p-(m+\xi+1))} z^{p-(m+\xi+2)} - \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q(k-(m+\xi+1))} a_k z^{k-(m+\xi+2)} \tag{2.4}$$

Ref

14. Silverman, H. (1975), *Univalent functions with negative coefficients*, Proceedings of the American mathematical society, 51(1), pp109-116. doi.org/10.1090/S0002-9939-1975-0369678-0

By using (2.2), (2.3) and (2.4) in (2.1) then we get numerator and denominator of (2.1) as numerator is denoted by N and denominator by D

$$\mathrm{N} = (z+\beta z^2)\left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-(m+\xi))} z^{p-(m+\xi+1)} - \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q(k-(m+\xi))} a_k\, z^{k-(m+\xi+1)}\right] +$$

$$\lambda(z^2+\beta z)\left[\frac{\Gamma_q(1+p)}{\Gamma_q(p-(m+\xi+1))} z^{p-(m+\xi+2)} - \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q(k-(m+\xi+1))} a_k\, z^{k-(m+\xi+2)}\right]$$

$$\mathrm{D} = (1-\lambda)\left[\frac{\Gamma_q(1+p)}{\Gamma_q(1+p-(m+\xi))} z^{p-(m+\xi)} - \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q(1+k-(m+\xi))} a_k\, z^{k-(m+\xi)}\right] +$$

$$\lambda\,(z+\beta z^2)\left[\frac{\Gamma_q(1+p)}{\Gamma_q\big(p-(m+\xi)\big)} z^{p-(m+\xi+1)} - \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q\big(k-(m+\xi)\big)} a_k\, z^{k-(m+\xi+1)}\right]$$

solve above by using $[n]_q = \frac{\Gamma_q(1+n)}{\Gamma_q(n)}$ and on considering the value of z to be real and let $z \to 1$ then we get

$$\frac{\Gamma_q(1+p)}{\Gamma_q\big(1+p-(m+\xi)\big)}\left[(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)\right]$$

$$\geq \sum_{k=n+p}^{\infty} \frac{\Gamma_q(1+k)}{\Gamma_q\big(1+k-(m+\xi)\big)} a_k\left[(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right] - \alpha(1-\lambda)\right]$$

on simplifying we get,

$$\sum_{k=n+p}^{\infty} E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\} a_k \leq 1$$

where $E_{p,k}^{m,\xi} = \frac{\Gamma_q(1+k)\Gamma_q(1+p-(m+\xi))}{\Gamma_q(1+p)\Gamma_q(1+k-(m+\xi))}$

Conversely: Let us assume the inequality (2.1) is true

To Prove: $f(z) \in \Psi_{m,n,p}(\alpha,\beta,\lambda,\xi,q)$, for this we have to show that

$$Re\left\{\frac{(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right) + \lambda\,(z^2+\beta z)\left(D_{q,z}^{m+\xi+2} f(z)\right)}{(1-\lambda)\left(D_{q,z}^{m+\xi} f(z)\right) + \lambda(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right)}\right\} > \alpha$$

According to Lemma [4]

Ref

4. Aqlan, E.S. (2004), *Some Problems Connected with Geometric Function Theory*, Ph.D. Thesis, Pune University, Pune.

if $\quad w = u + iv$ then $Re\, w \geq \alpha \Leftrightarrow |w - (1+\alpha)| \leq |w + (1-\alpha)| \quad (2.5)$

Let $L = |w - (1+\alpha)|$

and
$$w = \frac{(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right) + \lambda\,(z^2+\beta z)\left(D_{q,z}^{m+\xi+2} f(z)\right)}{(1-\lambda)\left(D_{q,z}^{m+\xi} f(z)\right) + \lambda(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right)} \quad (2.6)$$

$$L = \left|\frac{(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right) + \lambda\,(z^2+\beta z)\left(D_{q,z}^{m+\xi+2} f(z)\right)}{(1-\lambda)\left(D_{q,z}^{m+\xi} f(z)\right) + \lambda(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right)} - (1+\alpha)\right| \quad (2.7)$$

and $K = |w + (1-\alpha)|$

$$K = \left|\frac{(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right) + \lambda\,(z^2+\beta z)\left(D_{q,z}^{m+\xi+2} f(z)\right)}{(1-\lambda)\left(D_{q,z}^{m+\xi} f(z)\right) + \lambda\,(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right)} + (1-\alpha)\right| \quad (2.8)$$

From (2.7) and (2.8), $K - L > 0$
i.e.$|w + (1-\beta)| - |w - (1+\beta)| > 0$ which implies $Re(w) > \alpha$

Hence the inequality
$$Re\left\{\frac{(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right) + \lambda\,(z^2+\beta z)\left(D_{q,z}^{m+\xi+2} f(z)\right)}{(1-\lambda)\left(D_{q,z}^{m+\xi} f(z)\right) + \lambda(z+\beta z^2)\left(D_{q,z}^{m+\xi+1} f(z)\right)}\right\} > \alpha$$

which implies $f(z) \in \Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$
so, the proof of theorem 1 is completed

Corollary 1: Let the function $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ is a member of new subclass $\Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$ of multivalent function then

$$a_k \leq \left\{\frac{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda + \lambda[p-(m+\xi+1)]_q\right] - \alpha(1-\lambda)}{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda + \lambda[k-(m+\xi+1)]_q\right] - \alpha(1-\lambda)}\right\} \frac{1}{E_{p,k}^{m,\xi}} \quad (2.9)$$

where $k = n + p$, p is some natural number, n is a natural number.

III. Property of New Subclass Related to Radii of Star Likeness, Convexity and Close to Convexity

In this part of the paper, we derive some results related to Radii of starlikeness, convexity and close to convexity for the function $f(z)$ belonging to the new subclass $\Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$

Theorem 2: Let the function $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ and $f(z)$ belong to $\Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$ then the function $f(z)$ is p-valent close to convex of order φ ; $0 \leq \varphi < p$ in $|z| < r_1^*$, where

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$$r_1^* = \underset{k \geq n+p}{inf}\left\{\left(\frac{p-\varphi}{k}\right)\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\}E_{p,k}^{m,\xi}\right\}^{\frac{1}{k-p}} \quad (3.1)$$

Proof: Let $f(z) \in \Psi_{m,n,p}(\alpha,\beta,\lambda,\xi,q)$ *and* $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$

To prove $f(z)$ is p-valent close to convex of order $\varphi\,; 0 \leq \varphi < p$ in $|z| < r_1^*$ for this we have to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \varphi \qquad |z| < r_1^* \quad (3.2)$$

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|\frac{pz^{p-1} - \sum_{k=n+p}^{\infty} k a_k z^{k-1}}{z^{p-1}} - p\right|$$

$$= \left|\frac{\sum_{k=n+p}^{\infty} k a_k z^{k-1}}{z^{p-1}}\right|$$

$$\leq \sum_{k=n+p}^{\infty} k a_k |z|^{k-p} \quad (3.3)$$

The inequality (3.2) is less than or equal to $p-\varphi$ if

$$\sum_{k=n+p}^{\infty}\left(\frac{k}{p-\varphi}\right) a_k |z|^{k-p} \leq 1 \quad (3.4)$$

we know that $f(z) \in \Psi_{m,n,p}(\alpha,\beta,\lambda,\xi,q)$ if and only if

$$\sum_{k=n+p}^{\infty} E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\} a_k \leq 1$$

The inequality (3.2) is hold true if

$$\left(\frac{k}{p-\varphi}\right)|z|^{k-p}$$

$$\leq E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\}$$

or, we have

$$|z|^{k-p} \leq \left(\frac{p-\varphi}{k}\right) E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\} \quad (3.5)$$

so we get the required result

$|z| < r_1^*$

$$= \inf_{k \geq n+p} \left\{ \left(\frac{p-\varphi}{k} \right) \left\{ \frac{(1+\beta)[k-(m+\xi)]_q \left[1-\alpha\lambda + \lambda[k-(m+\xi+1)]_q - \alpha(1-\lambda)\right]}{(1+\beta)[p-(m+\xi)]_q \left[1-\alpha\lambda + \lambda[p-(m+\xi+1)]_q - \alpha(1-\lambda)\right]} \right\} E_{p,k}^{m,\xi} \right\}^{\frac{1}{k-p}}$$

Hence, the given function $f(z)$ is p-valent close to convex of order φ

Theorem 3: Let the function $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ and $f(z) \in \Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$ then the function $f(z)$ is a p-valent starlike of order φ ; $0 \leq \varphi < p$ in $|z| < r_2^*$,where

$$r_2^* = \inf_{k \geq n+p} \left\{ \left(\frac{p-\varphi}{k-\varphi} \right) \left\{ \frac{(1+\beta)[k-(m+\xi)]_q [1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q [1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q]-\alpha(1-\lambda)} \right\} E_{p,k}^{m,\xi} \right\}^{\frac{1}{k-p}} \tag{3.6}$$

Proof: Let $f(z) \in \Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$ *and* $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$

To prove the function $f(z)$ is p-valent starlike of order $\varphi; 0 \leq \varphi < p$ in $|z| < r_2^*$ for this we have to show that

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \varphi \qquad |z| < r_2^* \tag{3.7}$$

Now we take the L.H.S. part of the inequality (3.7)

$$\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \frac{z(pz^{p-1} - \sum_{k=n+p}^{\infty} k a_k z^{k-1})}{z^p - \sum_{k=n+p}^{\infty} a_k z^k} - p \right|$$

$$= \left| \frac{\sum_{k=n+p}^{\infty} (k-p) a_k z^k}{z^p - \sum_{k=n+p}^{\infty} a_k z^k} \right|$$

$$\leq \frac{\sum_{k=n+p}^{\infty} (k-p) a_k |z|^{k-p}}{1 - \sum_{k=n+p}^{\infty} a_k |z|^{k-p}} \tag{3.8}$$

The inequality (3.7) is less than or equal to $p - \varphi$ if

$$\sum_{k=n+p}^{\infty} \frac{(k-\varphi)}{(p-\varphi)} a_k |z|^{k-p} \leq 1 \tag{3.9}$$

we know that $f(z) \in \Psi_{m,n,p}(\alpha, \beta, \lambda, \xi, q)$ if and only if

$$\sum_{k=n+p}^{\infty} E_{p,k}^{m,\xi} \left\{ \frac{(1+\beta)[k-(m+\xi)]_q \left[1-\alpha\lambda + \lambda[k-(m+\xi+1)]_q\right] - \alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q \left[1-\alpha\lambda + \lambda[p-(m+\xi+1)]_q\right] - \alpha(1-\lambda)} \right\} a_k \leq 1$$

The inequality (3.9) is hold true if

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$$\left(\frac{k-\varphi}{p-\varphi}\right)|z|^{k-p}$$

$$\leq E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\}$$

or, we have

$$|z|^{k-p}\leq\left(\frac{p-\varphi}{k-\varphi}\right)E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\} \quad (3.10)$$

so we get the required result

$|z| < r_2^*$

$$=\inf_{k\geq n+p}\left\{\left(\frac{p-\varphi}{k-\varphi}\right)\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\}E_{p,k}^{m,\xi}\right\}^{\frac{1}{k-p}}$$

Hence, the given function $f(z)$ is p-valent starlike of order φ

Theorem 4: Let the function $f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k$ and $f(z)\in\Psi_{m,n,p}(\alpha,\beta,\lambda,\xi,q)$ then the given function $f(z)$ is a p-valent convex function of order $\varphi\,;0\leq\varphi<p$ in $|z|<r_3^*$, where

$$r_3^*=\inf_{k\geq n+p}\left\{\frac{p}{k}\left(\frac{p-\varphi}{k-\varphi}\right)\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\}E_{p,k}^{m,\xi}\right\}^{\frac{1}{k-p}} \quad (3.11)$$

Proof: Let $f(z)\in\Psi_{m,n,p}(\alpha,\beta,\lambda,\xi,q)$ *and* $f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k$

To prove the function $f(z)$ is p-valent convex function of order $\varphi\,;0\leq\varphi<p$ in $|z|<r_3^*$ for this we have to show that

$$\left|\frac{zf''(z)}{f'(z)}+(1-p)\right|\leq p-\varphi \qquad |z|<r_3^* \quad (3.12)$$

Taking the L.H.S. part of the inequality (3.12)

$$\left|\frac{zf''(z)}{f'(z)}+(1-p)\right|=\left|\frac{z(p(p-1)z^{p-2}-\sum_{k=n+p}^{\infty}k(k-1)a_kz^{k-2})}{pz^{p-1}-\sum_{k=n+p}^{\infty}ka_kz^{k-1}}+(1-p)\right|$$

$$=\left|\frac{\sum_{k=n+p}^{\infty}k(k-p)a_kz^{k-p}}{p-\sum_{k=n+p}^{\infty}k\alpha_kz^{k-p}}\right|$$

$$\leq \frac{\sum_{k=n+p}^{\infty} k(k-p)a_k|z|^{k-p}}{p-\sum_{k=n+p}^{\infty} ka_k|z|^{k-p}}$$

The inequality (3.12) is less than or equal to $p-\varphi$ if

$$\sum_{k=n+p}^{\infty} \frac{k(k-\lambda)}{p(p-\lambda)} a_k\,|z|^{k-p} \leq 1 \qquad (3.13)$$

we know that $f(z) \in \Psi_{m,n,p}(\alpha,\beta,\lambda,\xi,q)$ if and only if

$$\sum_{k=n+p}^{\infty} E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\} a_k \leq 1$$

The inequality (3.12) is hold true if

$$\frac{k}{p}\left(\frac{k-\varphi}{p-\varphi}\right)|z|^{k-p}$$

$$\leq E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\}$$

or, we have

$$|z|^{k-p} \leq \frac{p}{k}\left(\frac{p-\varphi}{k-\varphi}\right) E_{p,k}^{m,\xi}\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\} \qquad (3.14)$$

so we get the required result

$$|z| < r_3^*$$

$$= \inf_{k \geq n+p}\left\{\frac{p}{k}\left(\frac{p-\varphi}{k-\varphi}\right)\left\{\frac{(1+\beta)[k-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[k-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}{(1+\beta)[p-(m+\xi)]_q\left[1-\alpha\lambda+\lambda[p-(m+\xi+1)]_q\right]-\alpha(1-\lambda)}\right\}E_{p,k}^{m,\xi}\right\}^{\frac{1}{k-p}}$$

Hence, the given function $f(z)$ is p-valent convex function of order φ

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