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Hybrid Quaternions of Pell and Jacobsthal

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Abstract- Knowing that the Pell and Jacobsthal sequences are second-order linear recursive sequences and that they have similarities between them, this study aims to explore these sequences. Thus, an investigation will be carried out on the Pell and Jacobsthal numbers based on the hybrid numbers and their quaternions. In this way, it will be presented as a great among these hybrid themes of Pell and will be presented as a formula of hybrids of Pell Jacobsthal and will be presented, transforming function and its extension to the indices.

Keywords: pell sequence, jacobsthal sequence, pell and jacobsthal hybrid quaternions, hybrid numbers.

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HYBRIDQUATERNIONSOFPELLANDJACOBSTHAL

Hybrid Quaternions of Pell and Jacobsthal

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Abstract- Knowing that the Pell and Jacobsthal sequences are second-order linear recursive sequences and that they have similarities between them, this study aims to explore these sequences. Thus, an investigation will be carried out on the Pell and Jacobsthal numbers based on the hybrid numbers and their quaternions. In this way, it will be presented as a great among these hybrid themes of Pell and will be presented as a formula of hybrids of Pell Jacobsthal and will be presented, transforming function and its extension to the indices.

Keywords: pell sequence, jacobsthal sequence, pell and jacobsthal hybrid quaternions, hybrid numbers.

I. Introduction

In this research we will address two second-order linear recursive sequences, they are: Pell and Jacobsthal. The Pell sequence came from the English mathematician John Pell (1611 - 1685), known for being one of the most enigmatic mathematicians of the century XVII. The Jacobsthal sequence is named after the mathematician Ernest Erich Jacobsthal (1882-1965), specialist in Number Theory.

These sequences have similarity in their recurrence, the Pell sequence is defined by:

$$Pe_{n+1} = 2Pe_n + Pe_{n-1}, n \geq 2, \tag{1.1}$$

being $Pe_0 = 0$ $Pe_1 = 1$ its initial conditions.

As for the Jacobsthal sequence, it is presented by the recurrence:

$$J_{n+1} = J_n + 2J_{n-1}, n \geq 2, \tag{1.2}$$

being $J_0 = 0$ $J_1 = 1$ its initial conditions.

On the other hand, we have the quaternions numbers, it is believed that the quaternions arose in an attempt to transform the complex number $z = a + bi$ in three dimensions [1]. Quaternions are presented as formal sums of scalars with usual vectors of three-dimensional space, existing four dimensions and is described by: $q = a+bi+cj+dk$, where a, b, c and d are real numbers and i, j, k the orthogonal part at the base $\mathbf{R}^3$. In the work of [3] the authors presents the quaternionic product as $i^2 = j^2 = k^2 = -1, ij = k = -ij, jk = i = -kj$ and $ki = j = -ik$.

Ref

1. M. J. Menon, “Sobre as origens das definições dos produtos escalar e vetorial,” *Revista Brasileira de Ensino de Física*, vol. 3117, no. 2, 2009.

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On the other hand, there are the hybrid numbers, presented initially by [4], which he studied and lists three sets of numbers, they are: the complex, hyperbolic and dual. A hybrid number is defined as:

$$\mathbf{K} = \{z = a + bi + c\varepsilon + dh : a, b, c, d \in \mathbf{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}.$$

The hybridization of linear sequences consists of relating the hybrid numbers with linear sequences, researches around this hybridization are found in the mathematical literature, as we can find in [10, 11, 12, 13, 14, 15, 16]. Regarding the study on quaternions, we find work on the Padovan and Perrin quaternions in [5], Pell-Padovan in [6] and Fibonacci and Fibonacci Complexes in [2, 7, 8, 9].

Finally, in this research, we take as a basis the works of [17, 18] to present, in the following sections, the hybrid quaternions of Pell and Jacobsthal.

II. Pell Hybrid Quaternions Numbers

In this section, we will present Pell's hybrid quaternions based on the work of [17, 18].

Definition 2.1. Pell's hybrid number, denoted by HPe_n, is defined by:

$$HPe_n = Pe_n + Pe_{n+1}i + Pe_{n+2}\varepsilon + Pe_{n+3}h.$$

Definition 2.2. The Recurrence Relationship for Pell's Hybrids, $n \geq 2$, is defined by:

$$HPe_{n+1} = 2HPe_n + HPe_{n-1}, \tag{2.1}$$

with $HPe_0 = i + 2\varepsilon + 5h$ and $HPe_1 = 1 + 2i + 5\varepsilon + 12h$ its initial terms.

Definition 2.3. Pell's quaternion number, denoted by QPe_n, it is given by:

$$QPe_n = Pe_n + Pe_{n+1}i + Pe_{n+2}j + Pe_{n+3}k.$$

Definition 2.4. The recurrence relation for Pell's quaternions, $n \geq 2$, is defined by:

$$QPe_{n+1} = 2QPe_n + QPe_{n-1}, \tag{2.2}$$

with $QPe_0 = i + 2j + 5k$ and $QPe_1 = 1 + 2i + 5j + 12k$ its initial terms.

Now, from what was seen above, we will approach Pell's hybrid quaternions.

Definition 2.5. Pell's hybrid quaternion number is defined as:

$$\tilde{Pe}_n = HPe_n + HPe_{n+1}i + HPe_{n+2}j + HPe_{n+3}k.$$

where i, j, k are the units of the quaternions and HPe_n it's the n-th Pell hybrid number. Thus, Pell's hybrid quaternions can be rewritten by:

$$\begin{aligned}
\tilde{Pe}_n &= (Pe_n + Pe_{n+1}i + Pe_{n+2}\varepsilon + Pe_{n+3}h) + \\
&\quad (Pe_{n+1} + Pe_{n+2}i + Pe_{n+3}\varepsilon + Pe_{n+4}h)i + \\
&\quad (Pe_{n+2} + Pe_{n+3}i + Pe_{n+4}\varepsilon + Pe_{n+5}h)j + \\
&\quad (Pe_{n+3} + Pe_{n+4}i + Pe_{n+5}\varepsilon + Pe_{n+6}h)k \\
&= \widehat{HPe}_n + \widehat{HPe}_{n+1}i + \widehat{HPe}_{n+2}\varepsilon + \widehat{HPe}_{n+3}h.
\end{aligned}$$

where i, ε and h are the imaginary units of the hybrid numbers and $\widehat{HPe}_n = Pe_n + Pe_{n+1}i + Pe_{n+2}j + Pe_{n+3}k$.

Ref

6. D. Tasci, "Padovan and Pell-Padovan quaternions," *Journal of Science and Arts*, vol. 42, no. 1, pp. 125-132, 2018.

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Definition 2.6. The recurrence relationship for Pell's hybrid quaternions, $n \geq 2$, is defined by:

$$\tilde{Pe}_{n+1} = 2\tilde{Pe}_n + \tilde{Pe}_{n-1}, \tag{2.3}$$

with the following initial terms: $\tilde{Pe}_0 = HPe_0 + HPe_1 i + HPe_2 j + HPe_3 k$ *and* $\tilde{Pe}_1 = HPe_1 + HPe_2 i + HPe_3 j + HPe_4 k$.

And yet, extending these non-positive integer indices, we have:

Definition 2.7. The recurrence relation for the hybrid quaternions of non-positive Pell indices, $n \geq 0$, is defined by:

$$\tilde{Pe}_{-n} = \tilde{Pe}_{-n+2} - 2tildePe_{-n+1}.$$

Theorem 2.1. Pell's hybrid quaternion, $\tilde{Pe}_n$, satisfies the following recurrence:

$$\tilde{Pe}_{n+1} = 2\tilde{Pe}_n + \tilde{Pe}_{n-1}. \tag{2.4}$$

Proof.

$$\begin{aligned}
2\tilde{Pe}_n + \tilde{Pe}_{n-1} &= 2HPe_n + 2HPe_{n+1}i + 2HPe_{n+2}j + 2HPe_{n+3}k + \\
&\quad HPe_{n-1} + HPe_n i + HPe_{n+1}j + HPe_{n+2}k \\
&= (2HPe_n + HPe_{n-1}) + (2HPe_{n+1} + HPe_n)i + \\
&\quad (2HPe_{n+2} + HPe_{n+1})j + (2HPe_{n+3} + HPe_{n+2})k \\
&= HPe_{n+1} + HPe_{n+2}i + HPe_{n+3}j + HPe_{n+4}k \\
&= \tilde{Pe}_{n+1}.
\end{aligned}$$

According to the recurrence relationship, $\tilde{Pe}_{n+1} = 2\tilde{Pe}_n + \tilde{Pe}_{n-1}$, one can present its characteristic equation, defined by $x^2 - 2x - 1 = 0$ where is a quadratic equation having two real roots $x_1 = 1 + \sqrt{2}$ and $x_2 = 1 - \sqrt{2}$.

Definition 2.8. Pell's hybrid quaternion conjugate can be defined in three different types for $\tilde{Pe}_n = \widehat{HPe}_n + \widehat{HPe}_{n+1}i + \widehat{HPe}_{n+2}\varepsilon + \widehat{HPe}_{n+3}h$:

- *Quaternion conjugate,* $\overline{\tilde{Pe}_n}$: $\overline{\tilde{Pe}_n} = \overline{\widehat{HPe}_n} + \overline{\widehat{HPe}_{n+1}i} + \overline{\widehat{HPe}_{n+2}\varepsilon} + \overline{\widehat{HPe}_{n+3}h}$;
- *Hybrid conjugate,* $(\tilde{Pe}_n)^C$: $(\tilde{Pe}_n)^C = \widehat{HPe}_n - \widehat{HPe}_{n+1}i - \widehat{HPe}_{n+2}\varepsilon - \widehat{HPe}_{n+3}h$;
- *Total conjugate,* $(\tilde{Pe}_n)^T$: $(\tilde{Pe}_n)^T = \overline{(\tilde{Pe}_n)^C} = \overline{\widehat{HPe}_n} - \overline{\widehat{HPe}_{n+1}i} - \overline{\widehat{HPe}_{n+2}\varepsilon} - \overline{\widehat{HPe}_n{}^{+3}h}$.

Theorem 2.2. The generating function of Pell's hybrid quaternions, denoted by $G_{\tilde{Pe}_n}(x)$, with $n \in \mathbf{N}$, is presented by:

$$G_{\tilde{Pe}_n}(x) = \frac{\tilde{Pe}_0 + (\tilde{Pe}_1 - \tilde{Pe}_0)x}{(1 - 2x - x^2)}.$$

Proof. To define the generating function of Pell's hybrid quaternions we will write a sequence in which each term of the sequence corresponds to the coefficients.

$$G_{\tilde{Pe}_n}(x) = \sum_{n=0}^{\infty} \tilde{Pe}_n x^n.$$

Making algebraic manipulations due to the recurrence relation we can write this sequence as:

$$\begin{aligned}
\tilde{P}e_n(x) &= \tilde{P}e_0 + \tilde{P}e_1 x + \tilde{P}e_2 x^2 + \cdots + \tilde{P}e_n x^n + \cdots \\
-2x\tilde{P}e_n(x) &= -\tilde{P}e_0 2x - \tilde{P}e_1 2x^2 - \tilde{P}e_2 2x^3 - \cdots - \tilde{P}e_n 2x^{n+1} - \cdots \\
-x^2\tilde{P}e_n(x) &= -\tilde{P}e_0 x^2 - \tilde{P}e_1 x^3 - \tilde{P}e_2 x^4 - \cdots - \tilde{P}e_n x^{n+2} - \cdots
\end{aligned}$$

Adding each member, we have:

$$\begin{aligned}
(1-2x-x^2)\tilde{P}e_n(x) &= \tilde{P}e_0 + (\tilde{P}e_1 - \tilde{P}e_0)x \\
\tilde{P}e_n(x) &= \frac{\tilde{P}e_0 + (\tilde{P}e_1 - \tilde{P}e_0)x}{1-2x-x^2}
\end{aligned}$$

Theorem 2.3. For $n \geq 0$, we have that the Binet formula for the Pell's hybrid quaternion is given by:

$$\tilde{P}e_n = \frac{(\tilde{P}e_1 - \tilde{P}e_0 x_2)x_1^n - (\tilde{P}e_1 - \tilde{P}e_0 x_1)x_2^n}{x_1 - x_2},$$

where x_1 and x_2 are the real roots of the characteristic equation.

Proof. The Binet formula can be represented as follows:

$$\tilde{P}e_n = Ax_1^n + Bx_2^n.$$

For $n = 0$, there is: $A + B = \tilde{P}e_0$ a, for $n = 1$, we have $Ax_1 + Bx_2 = \tilde{P}e_1$.
Thus, we have the following linear system:

$$\begin{cases} A + B = \tilde{P}e_0 \\ Ax_1 + Bx_2 = \tilde{P}e_1 \end{cases}$$

Solving the linear system, we have that the coefficients found were: $A = \frac{\tilde{P}e_1 - \tilde{P}e_0 x_2}{x_1 - x_2}$ and $B = \frac{\tilde{P}e_0 x_1 - \tilde{P}e_1}{x_1 - x_2}$.
Making the appropriate substitutions in the Binet formula, we have:

$$\tilde{P}e_n = \frac{(\tilde{P}e_1 - \tilde{P}e_0 x_2)x_1^n - (\tilde{P}e_1 - \tilde{P}e_0 x_1)x_2^n}{x_1 - x_2}.$$

III. Jacobsthal Hybrid Quaternions Numbers

Jacobsthal's hybrid quaternion numbers are defined from below.

Definition 3.1. Jacobsthal's hybrid number, denoted by HJ_n, is defined by:

$$HJ_n = J_n + J_{n+1}i + J_{n+2}\varepsilon + J_{n+3}h.$$

Definition 3.2. The Recurrence Relationship for Jacobsthal's hybrid, $n \geq 2$, is defined by:

$$HJ_{n+1} = HJ_n + 2HJ_{n-1}, \tag{3.1}$$

with $HJ_0 = i + \varepsilon + 3h$ and $HJ_1 = 1 + i + 3\varepsilon + 5h$ its initial terms.

Definition 3.3. Jacobsthal's quaternion number, denoted by QJ_n, it is given by:

$$QJ_n = J_n + J_{n+1}i + J_{n+2}j + J_{n+3}k.$$

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Definition 3.4. *The recurrence relation for Jacobsthal's quaternions, $n \geq 2$, is defined by:*

$$QJ_{n+1} = QJ_n + 2QJ_{n-1}, \tag{3.2}$$

with $QJ_0 = i + j + 3k$ and $QJ_1 = 1 + i + 3j + 5k$ its initial terms.

Now, from what was seen above, we will approach Jacobsthal's hybrid quaternions.

Definition 3.5. *Jacobsthal's hybrid quaternion number is defined as:*

$$\tilde{J}_n = HJ_n + HJ_{n+1}i + HJ_{n+2}j + HJ_{n+3}k.$$

where i, j, k are the units of the quaternions and HJ_n it's the n-th Jacobsthal hybrid number. Thus, Jacobsthal's hybrid quaternions can be rewritten by:

$$\begin{aligned}
\tilde{J}_n &= (J_n + J_{n+1}i + J_{n+2}\varepsilon + J_{n+3}h) + \\
&\quad (J_{n+1} + J_{n+2}i + J_{n+3}\varepsilon + J_{n+4}h)i + \\
&\quad (J_{n+2} + J_{n+3}i + J_{n+4}\varepsilon + J_{n+5}h)j + \\
&\quad (J_{n+3} + J_{n+4}i + J_{n+5}\varepsilon + J_{n+6}h)k \\
&= \widehat{HJ}_n + \widehat{HJ}_{n+1}i + \widehat{HJ}_{n+2}\varepsilon + \widehat{HJ}_{n+3}h.
\end{aligned}$$

where i, ε and h are the imaginary units of the hybrid numbers and $\widehat{HJ}_n = J_n + J_{n+1}i + J_{n+2}j + J_{n+3}k$.

Definition 3.6. *The recurrence relationship for Jacobsthal's hybrid quaternions, $n \geq 2$, is defined by:*

$$\tilde{J}_{n+1} = \tilde{J}_n + 2\tilde{J}_{n-1}, \tag{3.3}$$

with the following initial terms: $\tilde{J}_0 = HJ_0 + HJ_1 i + HJ_2 j + HJ_3 k$ and $\tilde{J}_1 = HJ_1 + HJ_2 i + HJ_3 j + HJ_4 k$.

And yet, extending these non-positive integer indices, we have:

Definition 3.7. *The recurrence relation for the hybrid quaternions of non-positive Jacobsthal indices, $n \geq 0$, is defined by:*

$$\tilde{J}_{-n} = \frac{\tilde{J}_{-n+2} + \tilde{J}_{-n+1}}{2}.$$

Theorem 3.1. *Jacobsthal's hybrid quaternion, $\tilde{J}_n$, satisfies the following recurrence:*

$$\tilde{J}_{n+1} = \tilde{J}_n + 2\tilde{J}_{n-1}. \tag{3.4}$$

Proof.

$$\begin{aligned}
\tilde{J}_n + 2\tilde{J}_{n-1} &= HJ_n + HJ_{n+1}i + HJ_{n+2}j + HJ_{n+3}k + \\
&\quad 2HJ_{n-1} + 2HJ_n i + 2HJ_{n+1}j + 2HJ_{n+2}k \\
&= (HJ_n + 2HJ_{n-1}) + (HJ_{n+1} + 2HJ_n)i + \\
&\quad (HJ_{n+2} + 2HJ_{n+1})j + (HJ_{n+3} + 2HJ_{n+2})k \\
&= HJ_{n+1} + HJ_{n+2}i + HJ_{n+3}j + HJ_{n+4}k \\
&= \tilde{J}_{n+1}.
\end{aligned}$$

According to the recurrence relationship, $\tilde{J}_{n+1} = \tilde{J}_n + 2\tilde{J}_{n-1}$, one can present its characteristic equation, defined by $k^2 - k - 2 = 0$ where is a quadratic equation having two real roots $k_1 = 2$ and $k_2 = -1$.

Definition 3.8. *Jacobsthal's hybrid quaternion conjugate can be defined in three different types for* $\tilde{J}_n = \widehat{HJ}_n + \widehat{HJ}_{n+1}i + \widehat{HJ}_{n+2}\varepsilon + \widehat{HJ}_{n+3}h$:

- *Quaternion conjugate,* $\overline{\tilde{J}_n}$: $\overline{\tilde{J}_n} = \overline{\widehat{HJ}_n} + \overline{\widehat{HJ}_{n+1}}i + \overline{\widehat{HJ}_{n+2}}\varepsilon + \overline{\widehat{HJ}_{n+3}}h$;
- *Hybrid conjugate,* $(\tilde{J}_n)^C$: $(\tilde{J}_n)^C = \widehat{HJ}_n - \widehat{HJ}_{n+1}i - \widehat{HJ}_{n+2}\varepsilon - \widehat{HJ}_{n+3}h$;
- *Total conjugate,* $(\tilde{J}_n)^T$: $(\tilde{J}_n)^T = \overline{(\tilde{J}_n)^C} = \overline{\widehat{HJ}_n} - \overline{\widehat{HJ}_{n+1}}i - \overline{\widehat{HJ}_{n+2}}\varepsilon - \overline{\widehat{HJ}_{n+3}}h$.

Theorem 3.2. *The generating function of Jacobsthal's hybrid quaternions, denoted by* $G_{\tilde{J}_n}(x)$, *with* $n \in \mathbf{N}$, *is presented by:*

$$G_{\tilde{J}_n}(k) = \frac{\tilde{J}_0 + (\tilde{J}_1 - \tilde{J}_0)k}{1 - k - 2k^2}.$$

Proof. To define the generating function of Jacobsthal's hybrid quaternions we will write a sequence in which each term of the sequence corresponds to the coefficients.

$$G_{\tilde{J}_n}(k) = \sum_{n=0}^{\infty} \tilde{J}_n k^n.$$

Making algebraic manipulations due to the recurrence relation we can write this sequence as:

$$\begin{aligned} G_{\tilde{J}_n}(k) &= \tilde{J}_0 + \tilde{J}_1 k + \tilde{J}_2 k^2 + \cdots + \tilde{J}_n k^n + \cdots \\ -kG_{\tilde{J}_n}(k) &= -k\tilde{J}_0 - k^2\tilde{J}_1 - k^3\tilde{J}_2 - \cdots - k^{n+1}\tilde{J}_n - \cdots \\ -2k^2 G_{\tilde{J}_n}(k) &= -2k^2\tilde{J}_0 - 2k^3\tilde{J}_1 - 2k^4\tilde{J}_2 - \cdots - 2k^{n+2}\tilde{J}_n - \cdots \end{aligned}$$

Adding each member, we have:

$$\begin{aligned} (1 - k - 2k^2)G_{\tilde{J}_n}(k) &= \tilde{J}_0 + (\tilde{J}_1 - \tilde{J}_0)k + (\tilde{J}_2 - \tilde{J}_1 - 2\tilde{J}_0)k^2 + \ldots \\ (1 - k - 2k^2)G_{\tilde{J}_n}(k) &= \tilde{J}_0 + (\tilde{J}_1 - \tilde{J}_0)k \\ G_{\tilde{J}_n}(k) &= \frac{\tilde{J}_0 + (\tilde{J}_1 - \tilde{J}_0)k}{(1 - k - k^2)}. \end{aligned}$$

Theorem 3.3. *For* $n \geq 0$, *we have that the Binet formula for the Jacobsthal's hybrid quaternion is given by:*

$$\tilde{J}_n = \frac{\tilde{J}_0 - \tilde{J}_1}{3}k_1^n + \frac{2\tilde{J}_0 - \tilde{J}_1}{3}k_2^n,$$

where k_1 *and* k_2 *are the real roots of the characteristic equation.*

Proof. The Binet formula can be represented as follows:

$$\tilde{J}_n = Ax_1^n + Bx_2^n.$$

Notes

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For $n = 0$, there is: $A + B = \tilde{J}_0$ a, for $n = 1$, we have $Ak_1 + Bk_2 = \tilde{J}_1$.
Thus, we have the following linear system:

$$\begin{cases} A + B = \tilde{J}_0 \\ Ak_1 + Bk_2 = \tilde{J}_1 \end{cases}$$

Solving the linear system, we have that the coefficients found were:

$$A = \frac{\tilde{J}_0 - \tilde{J}_1}{3}$$

$$B = \frac{2\tilde{J}_0 - \tilde{J}_1}{3}$$

Making the appropriate substitutions in the Binet formula, we have:

$$\tilde{J}_n = \frac{\tilde{J}_0 - \tilde{J}_1}{3}k_1^n + \frac{2\tilde{J}_0 - \tilde{J}_1}{3}k_2^n.$$

IV. Conclusion

This research was carried out around the numbers of Pell and Jacobsthal, which was made an investigation into the process of complexification and hybridization of these numbers, this research was based on the works of Dagdeviren and Kürüz (2020) and Mangueira, Alves and Catarino (2022). From the definition of the hybrid quaternions of Pell and Jacobsthal, it was possible to present their characteristic equations that have two real roots, their generating function and Binet's formula. Thus, it is expected that this research has contributed with important theorems for the studies of Pell and Jacobsthal sequences.

For future work, we can explore more properties around the Pell and Jacobsthal hybrid quaternions, as well as carry out investigation into the hybridization and complexification process in other numerical sequences.

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Resumo. Sabendo que as sequências de Pell e Jacobsthal são sequências lineares recursivas de segunda ordem e que apresentam similiariedades entre elas, neste estudo, tem-se o intuito de explorar essas sequências. Assim, será realizado uma investigação sobre os números de Pell e Jacobsthal com base nos números híbridos e os seus quaternions. Dessa forma, será realizado uma junção entre esses temas e será apresentado os quaternions híbridos de Pell e Jacobstahl, abordando sua equação característica, formula de Binet, função geradora e sua extensão para índices negativos.

Palavras-chave. Sequência de Pell, sequência de Jacobsthal, quaténions híbridos de Pell e Jacobsthal, números híbridos.

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