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Unbranched Riemann Domains over Q-Complete Spaces

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Abstract- It is proved that if $\Pi : X \to \Omega$ is an unbranched Riemann domain and locally r-complete morphism over a q-complete space Ω, then X is cohomo- logically $(q + r - 1)$-complete, if $q \geq 2$. We have shown in [1] that if $\Pi : X \to \Omega$ is an unbranched Riemann domain and locally q-complete morphism over a Stein space Ω, then X is cohomologically q-complete with respect to the struc- ture sheaf. In section 4 of this article, we prove by means of a counterexample that that there exists for each integer $n \geq 3$ an open subset $\Omega \subset Cn$ which is locally $(n - 1)$-complete but Ω is not $(n - 1)$-complete. The counterexample we give is obtained by making a slight modification of a recent example given by the author [2].

In 1962, Andreotti and Grauert [3] showed finiteness and vanishing the- orems for cohomology groups of analytic spaces under geometric conditions of q-convexity. Since then the question whether the reciprocal statements of these theorems are true have been subject to extensive studies.

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2. Y. Alaoui, A Counter-Example to the AndreotiGrauert Conjecture, Vladikavkaz Math. J., 2022, vol. 24, no. 1, pp. 1424. DOI: 10.46698/a8931-0543-3696-o

Unbranched Riemann Domains over Q-Complete Spaces

Youssef Alaoui

Abstract

It is proved that if $\pi : X \to \Omega$ is an unbranched Riemann domain and locally r-complete morphism over a q-complete space Ω, then X is cohomologically $(q + r - 1)$-complete if $q \geq 2$. We have shown in [1] that if $\pi : X \to \Omega$ is an unbranched Riemann domain and locally q-complete morphism over a Stein space Ω, then X is cohomologically q-complete for the structure sheaf $\mathcal{O}_X$. In section 4 of this article, we prove using a counter-example that there exists for each integer $n \geq 3$ an open subset $\Omega \subset \mathbb{C}^n$ which is locally $(n-1)$-complete, but Ω is not $(n-1)$-complete. The counter-example we give is based on a recent example given by the author [2].

By the theory of Andreotti and Grauert [3] it is known that a q-complete complex space is always cohomologically q-complete. A counter-example to the converse of this theorem was given in [2], where it is shown that there exists for each integer $n \geq 3$ a domain $\Omega \subset \mathbb{C}^n$ which is cohomologically $(n-1)$-complete but Ω is not $(n-1)$-complete. Since then, the question of whether the joint statements of these theorems are factual has been subject to extensive studies. For example, it was shown that if X is a Stein manifold and if $D \subset X$ is an open subset that has a C^2 boundary such that $H^p(D, \mathcal{O}_D) = 0$ for all $p \geq q$, then D is q-complete.

In this article, we prove that for any pair of integers (n, q), $2 \leq q < n$, there exists an open subset Ω of $\mathbb{C}^n$ which is cohomologically $(\tilde{q} - 1)$-complete but Ω is not $(\tilde{q} - 1)$-complete if $n = mq + 1$, where $m = [\frac{n}{q}]$ denotes as usual the integral part of $\frac{n}{q}$ and $\tilde{q} = n - m + 1$.

I. Introduction

Let $\pi : X \to Y$ be a holomorphic map of complex spaces. Then π is said to be locally r-complete if there exists for every $x \in Y$ an open neighborhood U in Y such that $\pi^{-1}(U)$ is r-complete.

A Riemann domain over a complex space Y is a pair (X, Π), where $\pi : X \to Y$ is a holomorphic map which is non-degenerate at every point of X, i.e., $\pi^{-1}(\pi(x))$ is a discrete set at each point $x \in X$. The pair (X, π) is called unbranched or unramified if $\pi : X \to Y$ is locally biholomorphic.

Let X and Y be complex spaces and $\pi : X \to Y$ an unbranched Riemann domain such that Y is q-complete and π a locally r-complete morphism.

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Does it follow that X is $(q+r-1)$-complete?

It was shown in [4] that this problem has a positive answer when $q=r=1$ and X and Y have isolated singularities.

It is known from [9] that if $\pi : X \to \Omega$ is an unbranched Riemann domain between two complex spaces with isolated singularities, Ω q-complete, and π is locally 1-complete, then X is q-complete.

We have shown in [1] that if $\pi : X \to \Omega$ is a locally q-complete unbranched Riemann domain over an n-dimensional Stein complex space Ω, then X is cohomologically q-complete with respect to the structure $\mathcal{O}_X$.

As a result, the author has provided a positive answer to the local Steiness problem : he has proved that if X is a Stein space and if $\Omega \subset X$ is a locally Stein open subset of X, then Ω is Stein. (See [1]).

In this article, we prove that if $\pi : X \to \Omega$ is a locally r-complete unbranched Riemann domain over a q-complete n-dimensional complex space Ω, then for any coherent analytic sheaf $\mathcal{F}$ on X, the cohomology group $H^l(X,\mathcal{F})$ vanishes for all $l \geq r+q-1$, if $q \geq 2$.

In particular, we obtain the interesting conclusion.

Corollary. *If X is a q-complete complex space of dimension n and if $\Omega \subset X$ is a locally r-complete open subset of X, then*

(a) Ω is cohomologically $(q+r-1)$-complete if $q \geq 2$.

(b) Ω is cohomologically r-complete with respect to the structure sheaf if X is a Stein space $(q=1)$.

It should be mentioned [13] that if Y is q-complete and if $\pi : X \to Y$ is a locally r-complete morphism, then the space X is cohomologically $(q+r)$-complete. But in general, $H^{q+r-1}(X,\mathcal{O}_X)$ does not vanish, even when $\pi : X \to Y$ is locally 1-complete and $q=1$ [12] (See also [6]).

The above question generalizes the following classical problem:

Is a locally q-complete open subset Ω of a Stein space X necessarily q-complete?

A counter-example to this problem is not known. One can easily verify that Ω is cohomologically $(q+1)$-complete. It is easy to see that a cohomologically q-complete open subset $\Omega \subset \mathbb{C}^n$ is always q-complete with corners. But it is unknown if these two conditions are equivalent.

By the theory of Andreotti and Grauert [3], it is known that if X is a q-complete complex space, then for every coherent analytic sheaf $\mathcal{F}$ on X, the cohomology group $H^p(X,\mathcal{F}) = 0$ for all $p \geq q$. But it is not known if these two conditions are equivalent except when X is a Stein manifold, $\Omega \subset X$ is cohomologically q-complete with respect to the structure sheaf $\mathcal{O}_\Omega$ and Ω has a smooth boundary [7]. In [2], we have shown that there exists for each $n \geq 3$ an open subset $\Omega \subset \mathbb{C}^n$ which is cohomologically $(n-1)$-complete, but Ω is not $(n-1)$-complete.

In section 4 of this article, we prove that for each $n \geq 3$, there exists an integer q with $2 \leq q < n$ such that for any coherent analytic sheaf $\mathcal{F}$, the cohomology group $H^p(\Omega,\mathcal{F})$ vanishes for all $p \geq q$ but Ω is not q-complete.

II. Preliminaries

We start by recalling some definitions and results concerning q-complete spaces.

Let Ω be an open set in $\mathbb{C}^n$ with complex coordinates $z_1, \cdots, z_n$. Then it is known that a function $\phi \in C^\infty(\Omega)$ is q-convex if for every point $z \in \Omega$, the Levi form.

$$L_z(\phi,\xi) = \sum_{i,j} \frac{\partial^2 \phi(z)}{\partial z_i \partial \bar{z}_j} \xi_i \bar{\xi}_j, \quad \xi \in \mathbb{C}^n$$

Has at most $q-1$ negative or zero eigenvalues.

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1. Y. Alaoui, On the cohomology groups of unbranched Riemann domains over Stein spaces. To appear in Rendiconti del Seminario Matematico. Politecnico di Torino

A smooth real-valued function ϕ on a complex space X is called q-convex if every point $x \in X$ has a local chart $U \to D \subset \mathbb{C}^n$ such that $\phi|_U$ has an extension $\hat{\phi} \in C^\infty(D, \mathbb{R})$ which is q-convex on D.

Two q-convex functions ϕ, ψ on X have the exact positivity directions if, for each point $x \in X$, there exists an open neighborhood U of x that can be identified to a closed analytic subset B of a domain D of some $\mathbb{C}^n$, and a complex vector subspace E of $\mathbb{C}^n$ of dimension $\geq n-q+1$ such that the Levi forms of $L_z(\phi, \xi)$ and $L_z(\psi, \xi)$, $z \in U$, are positive definite when restricted to E.

We say that X is q-complete if there exists a q-convex function $\phi \in C^\infty(X, \mathbb{R})$ which is exhaustive on X, i.e. $\{x \in X; \phi(x) < c\}$ is relatively compact for any $c \in \mathbb{R}$.

A complex space X is said to be cohomologically q-complete if the cohomology groups $H^p(X, \mathcal{F})$, $\mathcal{F} \in Coh(X)$, vanish for all $p \geq q$.

An open subset D of Ω is called q-Runge, if for every compact set $K \subset D$, there is a q-convex exhaustion function $\phi \in C^\infty(\Omega)$ such that

$$K \subset \{x \in \Omega : \phi(x) < 0\} \subset\subset D$$

This generalizes the classical notion of Runge pairs of Stein spaces.
It is shown in [3] that if D is q-Runge in Ω, then for every coherent analytic sheaf $\mathcal{F}$ on Ω, the cohomology groups $H^p(D, \mathcal{F})$ vanish for $p \geq q$ and the restriction map

$$H^p(\Omega, \mathcal{F}) \longrightarrow H^p(D, \mathcal{F})$$

has a dense image for all $p \geq q-1$.

A holomorphic map $\pi : X \to \Omega$ of complex spaces is called a q-complete morphism if there exists a q-convex function $\quad : X \to \mathbb{R}$ such that for every real number $\mu \in \mathbb{R}$, the restriction of Π from $\{x \in X; \quad (x) \leq \mu\}$ to Ω is proper. The canonical topologies on $H^p(X, \mathcal{F})$ are separated for all $p \geq q+1$ and for every coherent analytic sheaf $\mathcal{F}$ on X.

III. Unbrunched Riemann Domains Over Q-Complete Spaces

Theorem 1. Let X and Y be two n-dimensional complex spaces such that Y is q-complete and $\pi : X \to Y$ is an unbranched Riemann domain and locally r-complete morphism. Then X is cohomologically $(q+r-1)$-complete.

Proof. Since Y is q-complete, there exists, according to [14], a smooth q-convex function $\phi : Y \to]0, +\infty[$ such that for every real number λ, $Y(\lambda) = \{y \in Y : \phi(y) < \lambda\}$ is relatively compact in Y and $\{y \in Y : \phi(y) \leq \lambda\} \setminus \partial Y(\lambda)$ contains at most one point. Put $p = q+r-1$ and let $\mathcal{F}$ be a coherent analytic sheaf on X. We define $X(\lambda) = \Pi^{-1}(Y(\lambda))$ and consider the set A of all real numbers λ such that $H^p(X(\lambda), \mathcal{F}) = 0$.

To prove that $H^p(X(\lambda), \mathcal{F}) = 0$ for every $\lambda \in \mathbb{R}$, it will be sufficient to show that

(a) $A \neq \emptyset$ and, if $\lambda \in A$ and $\lambda' < \lambda$, then $\lambda' \in A$.

(b) if $\lambda_j \to \lambda$ and $\lambda_j \in A$ for all j, then $\lambda \in A$.

(c) if $\lambda_0 \in A$, there exists $\varepsilon_0 > 0$ such that $\lambda_0 + \varepsilon_0 \in A$.

We first prove assertion (a). Clearly, A is not empty. Indeed if $\lambda_0 = min\{\phi(x); x \in Y\}$, then $]-\infty, \lambda_0] \subset A$. Also, if $\lambda \in A$ and $\lambda' < \lambda$, then by theorem 1 of [13], the restriction map

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3. A. Andreotti and H. Grauert, Théorèmes de finitude de la cohomologie des espaces complexes. Bull. Soc. Math. France 90 (1962;) 193 – 259.

$$H^p(X(\lambda),\mathcal{F}) \xrightarrow{\rho} H^p(X(\lambda'),\mathcal{F})$$

has a dense range. Moreover, ρ is, in addition, injective. In fact, let

$$H^p(X(\lambda'),\mathcal{F}) \xrightarrow{\rho'} H^p(X(\mu),\mathcal{F})$$

be the restriction map, where μ is any real number with $\mu < Min(\lambda', \lambda_0)$. Then the compostion $\rho' o \rho$ is obviously injective. This implies that the restriction ρ is injective, which means that $H^p(X(\lambda),\mathcal{F}) = 0$ and $\lambda' \in A$.

To prove (c), we fix some $\lambda_0 \in A$ and suppose that $\{y \in Y : \phi(y) = \lambda_0\} \backslash \partial Y(\lambda_0) = \{y_0\}$ for some $y_0 \in Y$.

Let U be a Stein open neighborhood of y_0 such that $\Pi^{-1}(U)$ is r-complete and $\overline{U} \cap \overline{Y(\lambda_0)} = \emptyset$. There exist finitely many Stein open sets $U_i \subset\subset Y$, $1 \leq i \leq k$, disjoint from U such that $\partial Y(\lambda_0) \subset \bigcup_{i=1}^{k} U_i$ and $\Pi^{-1}(U_i)$ are r-complete. Let $\theta_i \in C_0^\infty(U_i, \mathbb{R}^+)$ be smooth compactly supported functions such that $\sum_{i=1}^{k} \theta_i(\xi)) > 0$ at every point $\xi \in \partial Y(\lambda_0)$. We can therefore choose sufficiently small numbers $c_i > 0$, $0 \leq i \leq k$, so that the functions $\phi_i : Y \to \mathbb{R}$, $1 \leq i \leq k$, defined by

$$\phi_0 = \phi, \quad \phi_i = \phi - \sum_{j=1}^{i} c_j \theta_j$$

Are q-convex with the same positivity directions. If we set

$$Y_i = \{x \in Y : \phi_i(x) < \lambda_0\} \text{ and } Y_0 = Y(\lambda_0), \text{ then}$$

$$Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_k, Y_0 \subset\subset Y_k, Y_i \setminus Y_{i-1} \subset\subset U_i \text{ for } 1 \leq i \leq k$$

Moreover, since ϕ is exhaustive, there exists $\varepsilon_0 > 0$ such that $Y(\lambda_0 + \varepsilon_0) \subset Y_k \cup U$. We define for an arbitrary real number λ with $\lambda_0 < \lambda < \lambda_0 + \varepsilon_0$ and integer $j = 0, \cdots, k$, the sets $Y_j(\lambda) = Y_j \cap Y(\lambda)$ and $X_j(\lambda) = \Pi^{-1}(Y_j(\lambda))$.

Since $Y(\lambda) = (Y(\lambda) \cap Y_k) \cup (Y(\lambda) \cap U)$, then $X(\lambda) = X_k(\lambda) \cup V(\lambda)$, where $V(\lambda) = \Pi^{-1}(Y(\lambda) \cap U) = \{x \in \Pi^{-1}(U) : \phi o \Pi(x) < \lambda\}$ is p-complete, because $\Pi^{-1}(U)$ is r-complete and $\phi o \Pi$ is q-convex. Moreover, $X_k(\lambda) \cap V(\lambda)$ is p-Runge in $V(\lambda)$. Therefore

$$H^p(X(\lambda),\mathcal{F}) = H^p(X_k(\lambda),\mathcal{F}) \oplus H^p(V(\lambda),\mathcal{F}) = H^p(X_k(\lambda),\mathcal{F})$$

To prove (c), we show inductively on j that $H^p(X_j(\lambda),\mathcal{F}) = 0$. For $j = 0$ this is clearly satisfied since $X_0(\lambda) = X(\lambda_0)$ and $\lambda_0 \in A$. Assume now that $j \geq 1$ and that $H^p(X_{j-1}(\lambda),\mathcal{F}) = 0$. Since $Y_j = Y_{j-1} \cup (Y_j \cap U_j)$, then $X_j(\lambda) = X_{j-1}(\lambda) \cup V_j(\lambda)$, where

Notes

$$V_j(\lambda) = \Pi^{-1}(U_j \cap Y_j(\lambda)) = \{x \in \Pi^{-1}(U_j) : \phi o\Pi(x) < \lambda, \phi_j o\Pi(x) < \lambda_0\}$$

is p-complete since $\Pi^{-1}(U_j)$ is r-complete and $\phi o\Pi$ and $\phi_j o\Pi$ are q-convex with the same positivity directions. Furthermore, as $X_{j-1}(\lambda) \cap V_j(\lambda)) = X_{j-1}(\lambda) \cap \Pi^{-1}(U_j) = \{x \in \Pi^{-1}(U_j) : \phi_{j-1} o\Pi(x) < \lambda_0, \phi o\Pi(x) < \lambda\}$ is clearly p-Runge in $\Pi^{-1}(U_j)$, then the restriction map

$$H^s(\Pi^{-1}(U_j), \mathcal{F}) \xrightarrow{\rho'} H^s(X_{j-1}(\lambda) \cap V_j(\lambda), \mathcal{F})$$

has a dense image for all $s \geq p-1$. Since ρ' is clearly injective and $p-1 \geq r$, then $H^{p-1}(X_{j-1}(\lambda) \cap V_j(\lambda), \mathcal{F}) = 0$. Therefore from the Mayer-Vietoris sequence for cohomology

$$\cdots \to H^{p-1}(X_{j-1}(\lambda) \cap V_j(\lambda), \mathcal{F}) \to H^p(X_j(\lambda), \mathcal{F}) \to H^p(X_{j-1}(\lambda), \mathcal{F}) \to \cdots,$$

we deduce that $H^p(X_j(\lambda), \mathcal{F}) = 0$.

To prove statement (b), it is sufficient to show that if $\lambda_j \nearrow \lambda$ and $\lambda_j \in A$ for all j, then

$$H^{p-1}(X(\lambda_{j+1}), \mathcal{F}) \longrightarrow H^{p-1}(X(\lambda_j), \mathcal{F})$$

has a dense image.

To complete the proof of theorem 1, it is, therefore, enough, according to (Cf. [3], p. 250), to show the following lemma.

Lemma 1. For every pair of real numbers $\mu < \lambda$, the restriction map

$$H^{p-1}(X(\lambda), \mathcal{F}) \to H^{p-1}(X(\mu), \mathcal{F})$$

has a dense range.

Proof. We consider the set T of all real numbers λ such that

$$H^{p-1}(X(\lambda), \mathcal{F}) \to H^{p-1}(X(\mu), \mathcal{F})$$

has a dense range for all $\mu \leq \lambda$.

To see that T is not empty, we choose $\lambda_0 = min\{\phi(y); y \in Y\}$. Then clearly $]-\infty, \lambda_0] \subset T$.

To prove that T is open in $]-\infty, +\infty[$ it is, therefore, sufficient to show that if $\lambda \in T$, there exists $\varepsilon > 0$ such that $\lambda + \varepsilon \in T$. For this, we consider a finite covering $(U_i)_{1 \leq i \leq k}$ of $\{y \in Y : \phi(z) = \lambda\}$ by Stein open sets $U_i \subset\subset Y$ and compactly supported functions $\theta_i \in C_o^\infty(U_i)$, $\theta_j \geq 0$, $j = 1, \cdots, k$ such that $\Pi^{-1}(U_i)$ is r-complete and $\sum_{i=1}^{k} \theta_i(x) > 0$ at any point of $\partial Y(\lambda)$. Define $Y_j = \{z \in Y : \phi_j(z) < \lambda\}$ where

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3. A. Andreotti and H. Grauert, Théorèmes de finitude de la cohomologie des espaces complexes. Bull. Soc. Math. France 90 (1962;) 193 – 259.

$\phi_j(z) = \phi(z) - \sum_{1}^{j} c_i\theta_i$, with $c_i > 0$ sufficiently small so that $\phi_j(z)$ are still q-convex With the same positivity directions for $1 \leq j \leq k$.

If we consider the following sets defined in the lemma 2

$Y(\lambda) = \{y \in Y : \phi(y) < \lambda\}$, $X(\lambda) = \Pi^{-1}(Y(\lambda))$, $Y_i = \{x \in Y : \phi_i(x) < \lambda_0\}$ and $Y_0 = Y(\lambda_0)$, then

$$Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_k, Y_0 \subset\subset Y_k, Y_i \setminus Y_{i-1} \subset\subset U_i \quad \text{for} \quad 1 \leq i \leq k$$

and $X_j(\lambda) = \Pi^{-1}(Y_j \cap Y(\lambda)) = X_{j-1}(\lambda) \cup V_j(\lambda)$, where

$$V_j(\lambda) = \Pi^{-1}(U_j \cap Y_j(\lambda)) = \{x \in \Pi^{-1}(U_j) : \phi o \Pi(x) < \lambda, \phi_j o \Pi(x) < \lambda_0\}$$

Now since $X_{j-1}(\lambda) \cap V_j(\lambda)$ is p-Runge in the p-complete set $V_j(\lambda)$ and $H^p(X_j(\lambda), \mathcal{F}) = 0$, it follows from the long exact sequence of cohomology

$$\cdots \to H^{p-1}(X_j(\lambda), \mathcal{F}) \to H^{p-1}(X_{j-1}(\lambda), \mathcal{F}) \oplus H^{p-1}(V_j(\lambda), \mathcal{F}) \to$$

$$H^{p-1}(X_{j-1}(\lambda) \cap V_j(\lambda), \mathcal{F}) \to H^p(X_j(\lambda), \mathcal{F}) \to \cdots$$

that the restriction map

$$H^{p-1}(X_j(\lambda), \mathcal{F}) \to H^{p-1}(X_{j-1}(\lambda), \mathcal{F})$$

has a dense range.

Moreover, since ϕ is exhaustive, there exists $\varepsilon > 0$ such that $Y(\lambda + \varepsilon) \subset Y_k$. We deduce that the restriction map

$$H^{p-1}(X(\lambda + \varepsilon), \mathcal{F}) \to H^{p-1}(X(\lambda), \mathcal{F})$$

has a dense image, which implies that $\lambda + \varepsilon \in T$.

Let now $\lambda_j \in T$, $j \geq 0$, such that $\lambda_j \nearrow \lambda$, and let $\mathcal{U} = (U_i)_{i \in I}$ be a countable base of Stein open covering of X. Then the restriction map between spaces of cocycles

$$Z^{p-1}(\mathcal{U}|_{X_{\lambda_{j+1}}}, \mathcal{F}) \to Z^{p-1}(\mathcal{U}|_{X_{\lambda_j}}, \mathcal{F})$$

has dense image for $j \geq 0$. Let $\lambda' < \lambda$ and $j \in \mathbb{N}$ such that $\lambda' < \lambda_j$. By [1, p.246], the restriction map $Z^{n-2}(\mathcal{U}|_{X_\lambda}, \mathcal{F}) \to Z^{n-2}(\mathcal{U}|_{X_{\lambda_j}}, \mathcal{F})$ has a dense image. Since $\lambda_j \in T$, then $Z^{n-2}(\mathcal{U}|_{X_{\lambda_j}}, \mathcal{F}) \to Z^{n-2}(\mathcal{U}|_{X_{\lambda'}}, \mathcal{F})$ has also a dense image, and hence $\lambda \in T$.

Now since $H^p(X(j), \mathcal{F}) = 0$ for all $j \in \mathbb{N}$ and $H^{p-1}(X(j+1), \mathcal{F})$ has a dense image in $H^{p-1}(X(j), \mathcal{F})$ for all $j \geq 0$, it follows from ([3], p. 250) that

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3. A. Andreotti and H. Grauert, Théorèmes de finitude de la cohomologie des espaces complexes. Bull. Soc. Math. France 90 (1962;) 193 – 259.

$$H^p(X,\mathcal{F}) \to H^p(X(0),\mathcal{F})$$

is bijective, which shows that $H^p(X,\mathcal{F}) = 0$.

IV. A Counter-Example to the Andreotti-Grauert Conjecture

Theorem 2. There exists for each integer $n \geq 3$ a cohomologically q-complete open subset $\Omega \subset \mathbb{C}^n$, $2 \leq q < n$, which is not q-complete.

We consider the following example due to Diederich and Forness [4]. Let (n,q) be a pair of integers with $2 \leq q < n$ and such that $n = mq+1$, where $m = [\frac{n}{q}]$ is the integral part of $\frac{n}{q}$. We define the functions.

$$\phi_j(z) = \sigma_j(z) + \sum_{i=1}^{m} \sigma_i(z)^2 + N||z||^4 - \frac{1}{4}||z||^2, \quad j = 1, \cdots, m,$$

and

$$\phi_{m+1}(z) = -\sigma_1(z) - \cdots - \sigma_m(z) + \sum_{i=1}^{m} \sigma_i(z)^2 + N||z||^4 - \frac{1}{4}||z||^2,$$

where $\sigma_j(z) = Im(z_j) + \sum_{i=m+1}^{n} |z_i|^2 - (m+1) \sum_{i=m+(j-1)(q-1)+1}^{m+j(q-1)} |z_i|^2$, for $j = 1, \cdots, m$

$z = (z_1, z_2, \cdots, z_n)$, and $N > 0$ a positive constant. Then, if N is large enough, the functions $\phi_1, \cdots \phi_2$ are q-convex on $\mathbb{C}^n$ and, if $\rho = Max(\phi_1, \cdots, \phi_{m+1})$, then, for $\varepsilon_o > 0$ small enough, the set $D_{\varepsilon_o} = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon_o\}$ is relatively compact in the unit ball $B = B(0,1)$ if N is sufficiently large. (See [4]).

We fix some $\varepsilon > \varepsilon_0$ and consider a covering $(U_i)_{i\in\mathbb{N}}$ of ∂D_ε, by Stein open subsets $U_i \subset\subset D_{\varepsilon_0}$ and functions $\theta_i \in C_0^\infty(\mathbb{C}^n, \mathbb{R})$ such that

$$\theta_j \geq 0, \quad Supp(\theta_j) \subset\subset U_j, \sum_{i=1}^{k} \theta_j(x) > 0 \text{ at any point } x \in \partial D_\varepsilon.$$

We can therefore choose sufficiently small positive numbers $c_1, \cdots, c_k$ so that the functions $\phi_{i,j} = \phi_i - \sum_{l=1}^{j} c_l\theta_l$ are q-convex for $i = 1, \cdots, m+1$ and $1 \leq j \leq k$.

We define $\phi_{i,0} = \phi_i$ for $i = 1, \cdots, m+1$, $D_0 = D_\varepsilon$ and $D_j = \{z \in D_{\varepsilon_0} : \rho_j(z) < -\varepsilon\}$, where $\rho_j(z) = \rho - \sum_{i=1}^{j} c_i\theta_i$ for $j = 1, \cdots, k$. Then ρ_j are q-convex with corners and it is clear that

$$D_0 \subset D_1 \subset \cdots \subset D_k, \; D_0 \subset\subset D_k \subset\subset D_{\varepsilon_0} \text{ and } D_j \setminus D_{j-1} \subset\subset U_j \text{ for } j = 1, \cdots = k.$$

Lemma 2. In the situation described above, for any coherent analytic sheaf $\mathcal{F}$ on D_{ε_0}, the restriction map $H^p(D_{j+1},\mathcal{F}) \to H^p(D_j,\mathcal{F})$ is surjective for all $p \geq \tilde{q}-1$ and all $0 \leq j \leq k-1$. In particular, $dim_\mathbb{C} H^p(D_j,\mathcal{F}) < \infty$, if $p \geq \tilde{q}-1$.

Ref

4. M. Coltoiu and K. Diederich, The levi problem for Riemann domains over Stein spaces with isolated singularities. Math. Ann. (2007) 338: 283 – 289.

Proof. We first prove that the cohomology group $H^p(D_j \cap U_l, \mathcal{F}) = 0$ for all $p \geq \tilde{q}-1$, $0 \leq j \leq k$, and $1 \leq l \leq k$. In fact, the set $D_j \cap U_l$ can be written in the form $D_j \cap U_l = D'_1 \cap \cdots \cap D'_{m+1}$, where $D'_i = \{z \in U_l : \phi_{i,j}(z) < -\varepsilon\}$ are clearly q-complete. Then for every $i_1, \cdots, i_m \in \{1, \cdots, m+1\}$, $D'_{i_1} \cap \cdots \cap D'_{i_m}$ are $(\tilde{q}-1)$-complete. Therefore, by using Proposition 1 of [11], we obtain

$$H^p(D_j \cap U_l, \mathcal{F}) \cong H^{p+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{F})$$

if $p \geq \tilde{q}-1$, which implies that $H^p(D_j \cap U_l, \mathcal{F}) = 0$ for all $p \geq \tilde{q}-1$.

Now since $D_{j+1} = D_j \cup (D_{j+1} \cap U_{j+1})$, it follows from the Mayer-Vietoris sequence for cohomology

$$\to H^p(D_{j+1}, \mathcal{F}) \to H^p(D_j, \mathcal{F}) \oplus H^p(D_{j+1} \cap U_{j+1}, \mathcal{F}) \to H^p(D_j \cap U_{j+1}, \mathcal{F}) \to H^{p+1}(D_{j+1}, \mathcal{F}) \to$$

that the restriction map

$$H^p(D_{j+1}, \mathcal{F}) \longrightarrow H^p(D_j, \mathcal{F})$$

is surjective when $p \geq \tilde{q}-1$.

Let now A be the set of all real numbers $\varepsilon \geq \varepsilon_0$ such that $H^p(D_\varepsilon, \mathcal{F}) = 0$ for all $p \geq \tilde{q}-1$.

Lemma 3. *-The set A is not empty and, if $\varepsilon \in A$, $\varepsilon > \varepsilon_0$, then there exits $\varepsilon' \in A$ such that $\varepsilon_0 \leq \varepsilon' < \varepsilon$.*

Proof. In fact, if $\mu_0 = Min_{z \in \overline{B}}\{\phi_i(z), i = 1, \cdots, m+1\}$, then one sees easily that $[-\mu_0, +\infty[\subset A$.

For the proof of the second assertion, if with the notations of lemma 1 we set $D_0 = D_\varepsilon$, we obtain $D_0 \subset D_1 \subset \cdots \subset D_k$, $D_0 \subset\subset D_k \subset\subset D_{\varepsilon_0}$ and $D_j \setminus D_{j-1} \subset\subset U_j$ for $j = 1, \cdots = k$.

We fix some $1 \leq j \leq k$ and $1 \leq l \leq k$, and set $D_j \cap U_l = D'_1 \cap \cdots D'_{m+1}$, where $D'_i = \{z \in U_l : \phi_{i,j}(z) < -\varepsilon\}$, then D'_i are q-complete and q-Runge in U_l. Therefore because of the proof of lemma 2, one obtains

$$H^p(D_j \cap U_l, \mathcal{F}) \cong H^{p+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{F}) = 0$$

for $p \geq \tilde{q}-1$ and, consequently, the restriction map

$$H^p(D_{j+1}, \mathcal{F}) \longrightarrow H^p(D_j, \mathcal{F})$$

is surjective for all $p \geq \tilde{q}-1$.

We now show inductively on j that $H^{\tilde{q}-1}(D_j, \mathcal{F}) = 0$. For $j = 0$, this is clearly satisfied since $D_0 = D_\varepsilon$ and $\varepsilon \in A$. Assume now that this property has already

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been proved for $j < k$. Since for every $i_1, \cdots, i_m$, in $\{1, \cdots, m+1\}$, the open set $D'_{i_1} \cap \cdots \cap D'_{i_m}$ is $(\tilde{q}-1)$-Runge in U_l, then the restriction map

$$H^p(U_l, \mathcal{F}) \longrightarrow H^p(D'_{i_m} \cap \cdots \cap D'_{i_m}, \mathcal{F})$$

has a dense range for $p \geq \tilde{q}-2$. Since the canonical topologies on $H^i(D'_{i_m} \cap \cdots \cap D'_{i_m}, \mathcal{F})$ are obviously separated for $i \geq 2$, then $H^p(D'_{i_m} \cap \cdots \cap D'_{i_m}, \mathcal{F}) = 0$ for all $p \geq \tilde{q}-2$. We know from Proposition 1 of [11] that $H^p(D_j \cap U_l, \mathcal{F}) \cong H^{p+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{F})$ for $p \geq \tilde{q}-2 = n-m-1$. We can choose the covering $(U_i)_{1 \leq i \leq k}$ of ∂D_ε such that $U_l \setminus D'_1 \cup \cdots \cup D'_{m+1}$ has no compact connected components, so it follows from the mean theorem of [5], that the restriction $H^p(U_l, \mathcal{F}) \longrightarrow H^p(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{F})$ has a dense image for $p \geq n-1$. This proves that

$$H^p(D_j \cap U_l, \mathcal{F}) \cong H^{p+m}(D'_1 \cup \cdots \cup D'_{m+1}, \mathcal{F}) = 0 \ \text{ for all } \ p \geq \tilde{q}-2.$$

Now since $H^{\tilde{q}-2}(D_j \cap U_{j+1}, \mathcal{F}) = H^{\tilde{q}-1}(D_{j+1} \cap U_{j+1}, \mathcal{F}) = H^{\tilde{q}-1}(D_j, \mathcal{F}) = 0$, it follows from the Mayer-Vietoris sequence for cohomolgy

$$\to H^{\tilde{q}-2}(D_j \cap U_{j+1}, \mathcal{F}) \to H^{\tilde{q}-1}(D_{j+1}, \mathcal{F}) \to H^{\tilde{q}-1}(D_j, \mathcal{F}) \oplus H^{\tilde{q}-1}(D_{j+1} \cap U_{j+1}, \mathcal{F}) \to$$

that $H^{\tilde{q}-1}(D_{j+1}, \mathcal{F}) = 0$.

On the other hand, since ρ is proper, there exists $\varepsilon' > 0$ such that $\varepsilon - \varepsilon' > \varepsilon_o$ and $D_{\varepsilon-\varepsilon'} = \{z \in D_{\varepsilon_o} : \rho(z) < \varepsilon' - \varepsilon\} \subset\subset D_k$.

Since $H^{\tilde{q}-1}(D_k, \mathcal{F}) \to H^{\tilde{q}-1}(D_{\varepsilon-\varepsilon'}, \mathcal{F})$ is surjective, $H^{\tilde{q}-1}(D_k, \mathcal{F}) = 0$ and $dim_{\mathbb{C}} H^{\tilde{q}-1}(D_{\varepsilon-\varepsilon'}, \mathcal{F}) < \infty$, then $H^{\tilde{q}-1}(D_{\varepsilon'-\varepsilon'}, \mathcal{F}) = 0$, whence $\varepsilon - \varepsilon' \in A$.

Lemma 4. The open set D_{ε_0} is cohomologically $(\tilde{q}-1)$-complete.

Proof. For this, we consider the set A of all real numbers $\varepsilon \geq \varepsilon_0$ such that $H^p(D_\varepsilon, \mathcal{F}) = 0$ for all $p \geq \tilde{q}-1$. Then by lemma 3, A is not empty and open in $[\varepsilon_0, \infty[$. Moreover, if $\varepsilon = Inf(A)$, there exists a decreasing sequence of real numbers $\varepsilon_j \in A$, $j \geq 1$, such that $\varepsilon_j \searrow \varepsilon$. Since $H^p(D_{\varepsilon_j}, \mathcal{F}) = 0$ for $p \geq \tilde{q}-1$ and, by lemma 1, the restriction map $H^p(D_{\varepsilon_{j+1}}, \mathcal{F}) \to H^p(D_{\varepsilon_j}, \mathcal{F})$ is surjective for all $p \geq \tilde{q}-2$, then by ([3], p. 250), the restriction map

$$H^p(D_\varepsilon, \mathcal{F}) \longrightarrow H^p(D_{\varepsilon_1}, \mathcal{F})$$

is an isomorphism for $p \geq \tilde{q}-1$, which shows that $\varepsilon \in A$.

Assume now that $\varepsilon > \varepsilon_0$. Then there exists, according to lemma 1, $\varepsilon' \in A$ such that $\varepsilon_0 < \varepsilon' < \varepsilon$, which contradicts the fact that $\varepsilon = Inf(A)$. We conclude that $\varepsilon = \varepsilon_0 \in A$, and hence D_{ε_0} is cohomologically $(\tilde{q}-1)$-complete.

End of the proof of theorem 2

We have shown that D_{ε_0} is cohomologically $(\tilde{q}-1)$-complete. We are now going to prove that for a good choice of the contants ε_0 and N, we can find an $\varepsilon > \varepsilon_0$ such that D_ε is cohomologically $(\tilde{q}-1)$-complete but Ω not $(\tilde{q}-1)$-complete.

Ref

11. K. Matsumoto. On the cohomological completeness of q-complete domains with corners. Nagoya Math. J. Vol. 165 (2002), 105-112.

In fact, it was shown by Diederich-Forness [4] that if $\delta > 0$ is small enough, then the topological sphere of real dimension $n + \tilde{q} - 2$

$$S_\delta = \{z \in \mathbb{C}^n : x_1^2 + \cdots + x_m^2 + |z_{m+1}|^2 + \cdots + |z_n|^2 = \delta,$$

$$y_j = -\sum_{i=m+1}^{n} |z_i|^2 + (m+1) \sum_{i=m+(j-1)(q-1)+1}^{m+j(q-1)} |z_i|^2 \ \text{ for } \ j = 1, \cdots, m\}$$

is not homologous to 0 in D_{ε_0}. This follows from the fact that the set $E = \{z \in \mathbb{C}^n : x_1 = z_2 = \cdots = z_n = 0\}$ does not intersect D_{ε_0}, since on E

$$\phi_j = y_j + \frac{3}{4}\sum_{i=1}^{m} y_i^2 + N(\sum_{i=1}^{m} y_i^2)^2 \ \text{ for } \ j = 1, \cdots, m$$

and

$$\phi_{m+1} = -y_1 - \cdots - y_m + \frac{3}{4}\sum_{i=1}^{m} y_i^2 + N(\sum_{i=1}^{m} y_i^2)^2$$

such that $\rho \geq 0$ on E. So the following real form of degree $n + \tilde{q} - 2$

$$\omega = (\sum_{i=1}^{n} x_i^2 + \sum_{i=m+1}^{n} y_i^2)^{-2n+m}(\sum_{i=1}^{n}(-1)^i x_i dx_1 \wedge \cdots \widehat{dx_i} \wedge \cdots \wedge dx_n \wedge dy_{m+1} \wedge \cdots \wedge$$

$$dy_n + \sum_{i=1}^{n-m}(-1)^{n+i} y_{m+i} dx_1 \wedge \cdots \wedge dx_n \wedge dy_{m+1} \wedge \cdots \wedge \widehat{dy_{m+i}} \wedge \cdots \wedge dy_n)$$

is well-defined and d-closed on D_{ε_0}. Since ω does not depend on $y_1, \cdots, y_m$, then by the standard argument $\int_{S_\delta} \omega \neq 0$. Therefore S_δ is not homologous to 0 in D_{ε_0}.

Let $\mathcal{E}_q$ be the sheaf of germs of C^∞ q-forms on $\mathbb{C}^n$ and $\mathcal{T}_q$ the sheaf of germs of C^∞ d-closed q-forms. Then we have an exact sequence of sheaf homomorphisms

$$0 \to \mathcal{T}_q \to \mathcal{E}_q \overset{d}{\to} \mathcal{T}_{q+1} \to 0$$

Since by the de Rham theorem for every $p \geq 1$, the cohomology group $H^p(D_{\varepsilon_0}, \mathbb{C})$ is isomorphic to

$$\frac{\{\omega \in \Gamma(D_{\varepsilon_0}, \mathcal{E}_p) : df = 0\}}{\{d\omega : \omega \in \Gamma(D_{\varepsilon_0}, \mathcal{E}_{p-1})\}},$$

it follows from Stokes formula that $H^{n+\tilde{q}-2}(D_{\varepsilon_0}, \mathbb{C})$ does not vanish.

We are going to show that $H^r(D_{\varepsilon_0}, \mathcal{O}_{D_{\varepsilon_0}}) = 0$ for all r with $1 \leq r \leq \tilde{q} - 3$.

We first assert that we can choose N, ε_0, and $\varepsilon > \varepsilon_0$ such that, if, with the notations of Proposition 1, we set

$$\phi_j(z) = \sigma_j(z) + \sum_{i=1}^{m} \sigma_i(z)^2 + N||z||^4 - \frac{1}{4}||z||^2, \ \ j = 1, \cdots, m,$$

and

$$\phi_{m+1}(z) = \sigma(z) + \sum_{i=1}^{m} \sigma_i(z)^2 + N||z||^4 - \frac{1}{4}||z||^2, \ \ \text{where} \ \ \sigma(z) = -\sum_{i=1}^{m} \sigma_i(z),$$

Ref

4. M. Coltoiu and K. Diederich, The levi problem for Riemann domains over Stein spaces with isolated singularities. Math. Ann. (2007) 338: 283 – 289.

$$\sigma_j(z) = Im(z_j) + \sum_{i=m+1}^{n} |z_i|^2 - (m+1) \sum_{i=m+(j-1)(q-1)+1}^{m+j(q-1)} |z_i|^2, \text{ for } j = 1, \cdots, m,$$

$$(z) = N||z||^4 - \frac{1}{4}||z||^2 + \varepsilon_0 \text{ and } \rho(z) = Max(\phi_1(z), \cdots, \phi_{m+1}) + \sum_{i=1}^{m} \sigma_i(z)^2 + (z) - \varepsilon_0,$$

then we obtain

$$D_\varepsilon = \{z \in D_\varepsilon : \phi(z) < \varepsilon_0 - \varepsilon\}$$

where $m' = Min_{z \in \overline{D}_{\varepsilon_0}} (z)$, and

$$\phi(z) = \sigma(z) + \sum_{i=1}^{m} \sigma_i(z)^2 + m'$$

In fact, we can choose $\varepsilon > \varepsilon_0$ sufficiently big and $\lambda > 0$ small enough so that $\varepsilon_0 - \varepsilon < m' < (1+\lambda).Min_{z \in \overline{D}_\varepsilon} (z)$ and $\lambda\varepsilon - (1+\lambda)\varepsilon_0 > 0$.

On the other hand, if $\delta = Min_{z \in \overline{D}_{\varepsilon_0}} ||z||^2$, then we have

$$0 < \delta \leq ||z||^2 < \frac{1}{4N} - \frac{\varepsilon_0}{N} \text{ for every } z \in \overline{D}_{\varepsilon_0}$$

Therefore by suitable choice of ε_0, ε and N we can also achieve that

$$(N||z||^4 - \frac{1}{4}||z||^2) - Min_{z \in \overline{D}_{\varepsilon_0}} (N||z||^4 - \frac{1}{4}||z||^2) < Min(\frac{\varepsilon - \varepsilon_0}{2}, \lambda\varepsilon - (1+\lambda)\varepsilon_0),$$

and

$$Max_{z \in \overline{D}_{\varepsilon_0}} (N||z||^4 - \frac{1}{4}||z||^2) - |(N||z||^4 - \frac{1}{4}||z||^2) < Min(\frac{\varepsilon - \varepsilon_0}{2}, \lambda\varepsilon - (1+\lambda)\varepsilon_0),$$

for every $z \in \overline{D}_\varepsilon$.

Because $(z) < \varepsilon_0 - \varepsilon$ on $\overline{D}_\varepsilon$, then clearly we obtain

$$\phi(z) = \sigma(z) + \sum_{i=1}^{m} \sigma_i(z)^2 + m' < \sigma(z) + \sum_{i=1}^{m} \sigma_i(z)^2 + (1+\lambda).\psi(z)) < (1+\lambda)(\varepsilon_0 - \varepsilon), \text{ if } z \in \overline{D}_\varepsilon,$$

which shows that

$$D_\varepsilon = \{z \in D_\varepsilon : \phi(z) < \varepsilon_0 - \varepsilon\}$$

We are now going to show that for every none-positive real number α with $\alpha < \varepsilon_0 - \varepsilon$, the open sets

$$B_\alpha = \{z \in D_\varepsilon : \phi(z) < \alpha\}$$

are relatively compact in D_ε.

To see this, we consider a sequence $(z_j)_{j \geq 0} \subset B_\alpha$, which converges to a point $z \in \overline{D}_\varepsilon$. Then one has for every sufficiently large integer j

$$\rho(z_j) = Max(\sigma_1(z_j), \cdots, \sigma_m(z_j), \sigma(z_j)) + \sum_{i=1}^{m} \sigma_i(z_j)^2 + N||z_j||^4 - \frac{1}{4}||z_j||^2 < -\varepsilon$$

Since

$$\phi(z_j) < \varepsilon_0 - \varepsilon + \lambda\psi(z_j) < (1+\lambda)(\varepsilon_0 - \varepsilon)$$

and

$$N||z_j||^4 - \frac{1}{4}||z_j||^2 - Min_{z\in\overline{D}_{\varepsilon_0}}(N||z||^4 - \frac{1}{4}||z||^2) < \lambda\varepsilon - (1+\lambda)\varepsilon_0$$

then

$$\rho(z_j) = \phi(z_j) + N(||z_j||^4 - \frac{1}{4}||z_j||^2) - m < \varepsilon_0 - \varepsilon + \lambda\psi(z_j) + \lambda\varepsilon - (1+\lambda)\varepsilon_0$$

A passage to the limit shows that

$$\rho(z) \leq \varepsilon_0 - \varepsilon + \lambda\psi(z) + \lambda\varepsilon - (1+\lambda)\varepsilon_0 < (1+\lambda)(\varepsilon_0 - \varepsilon) + \lambda\varepsilon - (1+\lambda)\varepsilon_0 = -\varepsilon,$$

because $(z) < \varepsilon_0 - \varepsilon$, which implies that $z \in D_\varepsilon$. We conclude that with such a choice of ε_0, N, and ε the limit $z \in D_\varepsilon$, and hence the open set

$$B_\alpha = \{z \in D_\varepsilon : \phi(z) < \alpha\}$$

is relatively compact in D_ε for all real numbers α, with $\alpha < \varepsilon_0 - \varepsilon$.

Now since ϕ is in addition $(m+2)$-convex, then a similar proof of theorem 15 of [3] shows that, if Ω^i is the sheaf of germs of holomorphic i-forms on $\mathbb{C}^n$, $i \geq 0$, $(\Omega^0 = \mathcal{O}_{\mathbb{C}^n})$, and $B_c = \{z \in D_\varepsilon : \phi(z) < c\}$ for $c \leq \varepsilon_0 - \varepsilon$, then the map

$$H^r(D_\varepsilon, \Omega^i) \longrightarrow H^r(D_\varepsilon \setminus B_c, \Omega^i)$$

is injective for every $r < n-m-1$ and $c < \varepsilon_0 - \varepsilon$. Then obviously $H^r(D_\varepsilon, \Omega^i) = 0$ for $1 \leq r \leq n-m-2$ and $i \geq 0$. In fact, let $c_0 = Max_{z\in\overline{D}_\varepsilon}\phi(z)$. Then there exists $z_1 \in \partial D_\varepsilon$ such that $\phi(z_1) = c_0$. Since $c_0 = \phi(z_1) = \sigma(z_1) + \sum_{i=1}^{m}\sigma_i(z_1)^2 + m' < \rho(z_1) + \varepsilon_0 \leq \varepsilon_0 - \varepsilon$, then $B_{c_0} = D_\varepsilon$, and hence $H^r(D_\varepsilon, \Omega^i) = 0$ for $1 \leq r \leq n-m-2$.

Now if we suppose that D_ε is $(\tilde{q}-1)$-complete, then there exists a C^∞ strictly $(\tilde{q}-1)$-convex function $: D_\varepsilon \to \mathbb{R}$ such that $D_{\varepsilon,c} = \{z \in D_\varepsilon : (z) < c\}$ is relatively compact in D_ε for every $c \in \mathbb{R}$.

We now consider the resolution of the constant sheaf $\mathbb{C}$ on D_ε

$$0 \to \mathbb{C} \to \mathcal{O} \stackrel{d}{\to} \Omega^1 \stackrel{d}{\to} \cdots \to \stackrel{d}{\to} \Omega^n \to 0$$

If we set $Z^j = Im(\Omega^{j-1} \stackrel{d}{\to} \Omega^j)$ for $1 \leq j \leq n-1$, then we get short exact sequences

$$0 \to \mathbb{C} \to \mathcal{O} \to Z^1 \to 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$0 \to Z^j \to \Omega^j \to Z^{j+1} \to 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$0 \to Z^{n-2} \to \Omega^{n-2} \to Z^{n-1} \to 0$$

$$0 \to Z^{n-1} \to \Omega^{n-1} \to \Omega^n \to 0$$

Since, by Proposition 1, D_ε is cohomologically $(\tilde{q}-1)$-complete, then $H^r(D_\varepsilon, \Omega^i) = 0$ for all $r \geq \tilde{q}-1$ and $i \geq 0$. So we obtain the isomorphisms

$$H^{\tilde{q}-1}(D_\varepsilon, Z^{n-1}) \cong \cdots \cong H^{2n-m-2}(D_\varepsilon, Z^1) \cong H^{2n-m-1}(D_\varepsilon, \mathbb{C})$$

and the exact sequence

$$\cdots \to H^{\tilde{q}-2}(D_\varepsilon, \Omega^n) \to H^{\tilde{q}-1}(D_\varepsilon, Z^{n-1}) \to H^{\tilde{q}-1}(D_\varepsilon, \Omega^{n-1}) = 0$$

Ref

3. A. Andreotti and H. Grauert, Théorèmes de finitude de la cohomologie des espaces complexes. Bull. Soc. Math. France 90 (1962;) 193 – 259.

We deduce that the map

$$H^{\tilde{q}-2}(D_\varepsilon, \Omega^n) \xrightarrow{\varphi} H^{n+\tilde{q}-2}(D_\varepsilon, \mathbb{C})$$

is surjective. The map φ is defined as follows : If a differential form $\omega \in C^\infty_{n,\tilde{q}-2}(D_\varepsilon)$ satisfies the equation $\overline{\partial}\omega = 0$, then ω is also d-closed and therefore defines a cohomology class in $H^{n+\tilde{q}-2}(D_\varepsilon, \mathbb{C})$.

Moreover, since, by theorem 1 in [8], every d-closed differential form $\omega \in C^\infty_{n,\tilde{q}-2}(D_\varepsilon)$ is cohomologous to a $\overline{\partial}$-closed $(n, \tilde{q}-2)$ differntial form $\omega' \in C^\infty_{n,\tilde{q}-2}(D_\varepsilon)$, it follows that the map

$$H^{\tilde{}-2}(D_\varepsilon, \Omega^n) \xrightarrow{\varphi} H^{n+\tilde{q}-2}(D_\varepsilon, \mathbb{C})$$

is bijective.

Now if we suppose that D_ε is $(\tilde{q}-1)$-complete, then there exists a C^∞ strictly $(\tilde{q}-1)$-convex function $: D_\varepsilon \to \mathbb{R}$ such that $D_{\varepsilon,c} = \{z \in D_\varepsilon : \ (z) < c\}$ is relatively compact in D_ε for every $c \in \mathbb{R}$.

Notice that for the given ε, if $\delta > 0$ is small enough, the topological sphere

$$S_\delta = \{z \in \mathbb{C}^n : x_1^2 + |z_2|^2 + \cdots + |z_n|^2 = \delta, \ \sigma_1(z) = 0\} \subset D_\varepsilon$$

Since is exhaustive on D_ε, there exists $c' > 0$ such that S_δ is not homologuous to 0 in $D_{\varepsilon,c'}$. Let $c > c'$. Then $D_{\varepsilon,c}$ and $D_{\varepsilon,c'}$ are $(\tilde{q}-1)$-complete and, similarly $H^p(D_{\varepsilon,c}, \Omega^i) = H^p(D_{\varepsilon,c'}, \Omega^i) = 0$ for $1 \leq p \leq n-m-2$ and $i \geq 0$. Also the maps $H^{\tilde{q}-2}(D_{\varepsilon,c}, \Omega^n) \to H^{n+\tilde{q}-2}(D_{\varepsilon,c}, \mathbb{C})$ and $H^{\tilde{q}-2}(D_{\varepsilon,c'}, \Omega^n) \to H^{n+\tilde{q}-2}(D_{\varepsilon,c'}, \mathbb{C})$ are bijective. Moreover, since the levi form of has at least $m+1$ strictly positive eingenvalues, then by using Morse theory (See for instance [7]) we find that

$$H^{n+\tilde{q}-2}(D_{\varepsilon,c}, \mathbb{C}) \cong H^{n+\tilde{q}-2}(D_{\varepsilon,c'}, \mathbb{C})$$

It follows from the commutative diagram of continuous maps

$$\begin{array}{ccc} H^{\tilde{q}-2}(D_{\varepsilon,c}, \Omega^n) & \to & H^{n+\tilde{q}-2}(D_{\varepsilon,c}\mathbb{C}) \\ \downarrow & & \downarrow \\ H^{\tilde{q}-2}(D_{\varepsilon,c'}, \Omega^n) & \to & H^{n+\tilde{q}-2}(D_{\varepsilon,c'}, \mathbb{C}) \end{array}$$

that the restriction homomorphism

$$H^{\tilde{q}-2}(D_{\varepsilon,c}, \Omega^n) \to H^{\tilde{q}-2}(D_{\varepsilon,c'}, \Omega^n)$$

is bijective. Since in addition $D_{\varepsilon,c'}$ is relatively compact in $D_{\varepsilon,c}$, the function being exhaustive on D_ε, then, according to theorem 11 of [1], one obtains

$$dim_{\mathbb{C}} H^{\tilde{q}-2}(D_{\varepsilon,c'}, \Omega^n) < \infty$$

Since the sheaf Ω^n is isomorphic to $\mathcal{O}_{D_\varepsilon}$, then we have also $dim_{\mathbb{C}} H^{\tilde{q}-2}(D_{\varepsilon,c'}, \mathcal{O}_{D_\varepsilon}) < \infty$. Furthermore, since $D_{\varepsilon,c'}$ is cohomologically $(\tilde{q}-1)$-complete and $H^r(D_\varepsilon, \mathcal{O}_{D_\varepsilon}) = 0$ for $1 \leq r \leq n-m-2$, it follows from theorem 1 of [6] that $D_{\varepsilon,c'}$ is Stein, which is in contradiction with the fact that $H^{n+\tilde{q}-2}(D_{\varepsilon,c'}, \mathbb{C}) \neq 0$, since $S_\delta \subset D_{\varepsilon,c'}$ is not homologous to 0 in $D_{\varepsilon,c'}$. We conclude that D_ε is cohomologically $(\tilde{q}-1)$-complete but not $(\tilde{q}-1)$-complete.

Ref

7. H. Diederich, J. E. Fornaess, Smoothing q-convex functions and vanishing theorems. Invent. Math. 82. 291-305 (1985).

Theorem 3. *There exists for each integer $n \geq 3$ a cohomologically $(n-1)$-complete open subset Ω of $\mathbb{C}^n$ which is locally $(n-1)$-complete in $\mathbb{C}^n$ but Ω is not $(n-1)$-complete.*

Proof. We consider for $n \geq 3$ the functions $\phi_1, \phi_2 : \mathbb{C}^n \to \mathbb{R}$ defined by

$$\phi_1(z) = \sigma_1(z) + \sigma_1(z)^2 + N||z||^4 - \frac{1}{4}||z||^2,$$

$$\phi_2(z) = -\sigma_1(z) + \sigma_1(z)^2 + N||z||^4 - \frac{1}{4}||z||^2,$$

where $\sigma_1(z) = Im(z_1) + \sum_{i=3}^{n} |z_i|^2 - |z_2|^2$, $z = (z_1, z_2, \cdots, z_n)$, and $N > 0$ a positive constant. Then, if N is large enough, the functions ϕ_1 and ϕ_2 are $(n-1)$-convex on $\mathbb{C}^n$ and, if $\rho = Max(\phi_1, \phi_2)$, then, for $\varepsilon_o > 0$ small enough, the set $D_{\varepsilon_o} = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon_o\}$ is relatively compact in the unit ball $B = B(0,1)$, if N is sufficiently large.

According to ([2], p. 20), we can choose $\varepsilon_0 > 0$ such that if $\delta = Min_{z \in \overline{D}_{\varepsilon_0}} ||z||^2$, then we have

$$0 < \delta \leq ||z||^2 < \frac{1}{4N} - \frac{\varepsilon_0}{N} \quad \text{for every} \quad z \in \overline{D}_{\varepsilon_0}$$

and that by a suitable choice of $\varepsilon > \varepsilon_0$,

$$D_\varepsilon = \{z \in \mathbb{C}^n : \rho(z) < -\varepsilon\}$$

is cohomologically $(n-1)$-complete but not $(n-1)$-complete.

Now if we suppose that at a boundary point $z_0 \in \partial D_\varepsilon$, we have $\phi_1(z_0) = \phi_2(z_0)$, then $\sigma_1(z_0) = 0$ and, hence $N|z_0|^4 - \frac{|z_0|^2}{4} = \varepsilon$. This implies $|z_0|^2 = \frac{1}{8N}(1 + \sqrt{1 + 64N\varepsilon} < \frac{1}{4N}$. Therefore $\frac{1}{2}\sqrt{1 + 64N\varepsilon} < \frac{1}{2}$, which is a contradiction. This implies that $\phi_1(z) \neq \phi_2(z)$ at every boundary point $z \in \partial D_\varepsilon$. We conclude that with such a choice of ε_0 , N and ε, D_ε is obviously locally $(n-1)$-complete in $\mathbb{C}^n$.

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